EKR Sets for Large n and r

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Abstract. Let $\mathcal{A} \subset \binom{[n]}{r}$ be a compressed, intersecting family and let $X \subset [n]$. Let $\mathcal{A}(X) = \{A \in \mathcal{A} : A \cap X \neq \emptyset\}$ and $\mathcal{S}_{n,r} = \binom{[n]}{r}(\{1\})$. Motivated by the Erdős-Ko-Rado theorem, Borg asked for which $X \subset [2, n]$ do we have $|\mathcal{A}(X)| \leq |\mathcal{S}_{n,r}(X)|$ for all compressed, intersecting families $\mathcal{A}$? We call X that satisfy this property *EKR*. Borg classified EKR sets X such that $|X| \geq r$. Barber classified X, with $|X| \leq r$, such that X is EKR for sufficiently large n, and asked how large n must be. We prove n is sufficiently large when n grows quadratically in r. In the case where $\mathcal{A}$ has a maximal element, we sharpen this bound to $n > \varphi^2 r$ implies $|\mathcal{A}(X)| \leq |\mathcal{S}_{n,r}(X)|$. We conclude by giving a generating function that speeds up computation of $|\mathcal{A}(X)|$ in comparison with the naïve methods.

Key words. Erdős-Ko-Rado Theorem, Intersecting Family, Compressed Family

1. Introduction

The main objects of study in this paper are compressed, intersecting families. We begin by defining these terms. Let $\binom{[n]}{r}$ denote the set of r element subsets of $[n] = \{1, \ldots, n\}$. We label elements of $\binom{[n]}{r}$ in increasing order i.e. for $B = \{b_1, \ldots, b_r\} \in \binom{[n]}{r}$ we have $b_i < b_{i+1}$. A *family* $\mathcal{A}$ is a subset $\mathcal{A} \subset \binom{[n]}{r}$. Call $\mathcal{A}$ *intersecting* if $B, C \in \mathcal{A}$ implies $B \cap C \neq \emptyset$. Notice that $\mathcal{A}$ is trivially intersecting if $n < 2r$ by the pigeonhole principle. We define a partial order known as the *compression order* on $\binom{[n]}{r}$, as follows. For $A = \{a_1, \ldots, a_r\}$, and $B = \{b_1, \ldots, b_r\}$, define $A \leq B$ if $a_i \leq b_i$ for all $1 \leq i \leq r$. A family $\mathcal{A}$ is *compressed* if $A \in \mathcal{A}$ implies $B \in \mathcal{A}$ for $B \leq A$. We extend this partial order to $2^{[n]}$: for $C = \{c_1, \ldots, c_k\}$, we say $A \prec C$ if $r \geq k$ and $a_i \leq c_i$ for $1 \leq i \leq k$. For example, there are $\binom{n-2}{r-2} + \binom{n-3}{r-2}$ elements $A \in \binom{[n]}{r}$ with $A \prec \{1, 3\}$, since the first two elements must be $\{1, 2\}$ or $\{1, 3\}$.

Let $\mathcal{A}$ be a family and $X \subset [n]$. Define $\mathcal{A}(X) = \{A \in \mathcal{A} : A \cap X \neq \emptyset\}$. An important example of such a family is $\mathcal{S}_{n,r}$, defined by

$$\mathcal{S}_{n,r} = \binom{[n]}{r}(\{1\}) = \{A \in \binom{[n]}{r} : 1 \in A\}.$$

If n and r are clear, then $\mathcal{S}_{n,r}$ will be simplified to $\mathcal{S}$. It is easy to check that $\mathcal{S}$ is compressed and intersecting.

The following theorem is one of the fundamental results about intersecting families.

Theorem 1 *(Erdős-Ko-Rado)[4] (See also [6]) Let $n \geq 2r$ and let $\mathcal{A} \subset \binom{[n]}{r}$ be an intersecting family. Then $|\mathcal{A}| \leq |\mathcal{S}|$.*

In [3], Borg considered a variant of the Erdős-Ko-Rado theorem. Borg asked which sets $X \subset [2, n] = \{2, 3, \ldots, n\}$ have the property that $|\mathcal{A}(X)| \leq |\mathcal{S}(X)|$ for all compressed, intersecting families $\mathcal{A}$. Call X with this property *EKR*. We assume $X \subset [2, n]$, because if $1 \in X$, then $\mathcal{S}(X) = \mathcal{S}$ and X is trivially EKR by the Erdős-Ko-Rado theorem. There are many X which are not EKR. For example, consider the Hilton-Milner family $\mathcal{N} = \mathcal{S}([2, r + 1]) \cup \{[2, r + 1]\}$, see [7] . Then for $X = [2, r + 1]$, we have $|\mathcal{N}(X)| = |\mathcal{S}(X)| + 1$.

The motivation for considering compressed families is twofold. Firstly, the question is uninteresting without the requirement that $\mathcal{A}$ be compressed, since for any $x \in X$, we have $\mathcal{A} = \binom{[n]}{r}(\{x\})$ maximizes $|\mathcal{A}(X)|$ for intersecting families $\mathcal{A}$. Secondly, arbitrary sets lack structure, and by imposing more conditions, we may gain more information. In fact, compressed families and the shifting technique (see [5] for a survey) are powerful techniques in extremal set theory and can be used to give a simple proof of the Erdős-Ko-Rado theorem.

In [3], Borg classified X that are EKR for $|X| \geq r$ and gave a partial solution in the case $|X| < r$. Barber continued with Borg's work in [2] by considering $|X| \leq r$. To describe his results, we introduce the notion of *eventually EKR* sets, which are finite sets $X \subset \mathbb{Z}_{\geq 2}$ such that for fixed r, the set X is EKR for sufficiently large n. Notice that X does not vary with r or n, but whether or not X is EKR may depend on r and n.

Theorem 2 *(Barber) Let $r \geq 3$, $n \geq 2r$ and $X \subset [2, n]$ with $|X| \leq r$. If $X \not\subseteq [2, r+1]$, then X is eventually EKR if and only if one of the following holds*

1. $|X| = 1$
2. $|X| = 2$ *and* $\{2,3\} \cap X = \emptyset$
3. $|X| = 3$ *and* $\{2,3\} \not\subset X$
4. $|X| \geq 4$.

Barber asked which n are sufficiently large to imply X is EKR. This paper provides bounds on n.

Based on numerical results for small n and r stated in [2], Barber speculated that $n \geq 2r + 2$ was sufficient to imply X is EKR. However, as will be seen in Section 2, this bound does not hold in general. We replace the suggested bound of $n \geq 2r + 2$ with the following conjecture, which is supported by computer evidence for $r \leq 5$.

Conjecture 3 *Let $r \geq 2$ and $X \subset [2, n]$ be eventually EKR. Then $n > \varphi^2 r$ implies X is EKR, where $\varphi = \frac{1+\sqrt{5}}{2}$.*

Notice that in the case $r = 2$, there are only two maximal compressed, intersecting families, namely $\mathcal{S}$ and the family $\{\{1,2\},\{1,3\},\{2,3\}\}$, so the result is easy in this case, and we assume $r \geq 3$.

In order to describe results towards Conjecture 3, we need the notion of generating sets. These were introduced by Ahlswede and Khachatrian in [1], and Barber considered a variant definition, which is more useful in the present context. Let $\mathcal{G} \subset 2^{[n]}$ and define

$$F(n, r, \mathcal{G}) = \{A \in \binom{[n]}{r} : A \prec G \text{ for some } G \in \mathcal{G}\}.$$

Observe that such families are naturally compressed. We may now state the main theorems of this paper.

Theorem 4 *Let $r \geq 3$ and let X be eventually EKR with $|X| \leq r$. For each $c > \frac{1}{\log 2}$, there exists an r_c such that $r \geq r_c$ implies that X is EKR. Furthermore, for $r \geq 6$, X is EKR for $n > 2r^2$.*

In the case of a single generator, we have a sharper bound, which provides evidence for Conjecture 3.

Theorem 5 *Let $r \geq 3$, let $\mathcal{A} = F(n, r, \mathcal{G})$ be an intersecting family with $|\mathcal{G}| = 1$, and let X be eventually EKR with $|X| \leq r$. Then $n > \varphi^2 r$ implies $|\mathcal{A}(X)| \leq |\mathcal{S}(X)|$.*

The outline of the paper is as follows. In Section 2 we give an example of a family $\mathcal{A}$ and set X showing that the coefficient φ^2 in Conjecture 3 cannot be made any smaller. Section 3 gives a necessary and sufficient condition for a compressed family to be intersecting. This will be useful in the sections that follow. The proofs of Theorems 5 and 4 occur in Sections 4 and 5, respectively. Finally, in Section 6 we give a generating function that greatly speeds up numerical computations for $|F(n, r, \mathcal{G})(X)|$ in comparison with the naïve methods.

2. A family for which $|\mathcal{A}(X)| > |\mathcal{S}(X)|$

In this section, we exhibit a family that shows tightness of the coefficient φ^2 in the bound $n > \varphi^2 r$ of Conjecture 3.

We remark that for compressed, intersecting families $\mathcal{A}$ that are not subfamilies of $\mathcal{S}$, the quantity $|\mathcal{A}(X)|$ is usually maximized by a family of the form $F(n, r, \mathcal{G})$ with $|\mathcal{G}| = 1$ (cf. Theorem 5). For such families, the family and choice of X given in Proposition 6 appears to be the largest. These observations motivate Conjecture 3.

Proposition 6 *Let $X = \{2, 4, r+2\}$, $\mathcal{A} = F(n, r, \{\{2, 3\}\})$. For $r \geq 4$, $n \geq 2r$, if $|\mathcal{A}(X)| > |\mathcal{S}(X)|$ then $n < \frac{3r+1+\sqrt{5r^2-22r+25}}{2}$.*

Proof. We begin by computing $|\mathcal{A}(X) \backslash \mathcal{S}(X)|$ and $|\mathcal{S}(X) \backslash \mathcal{A}(X)|$. Consider the intersection of an element of one of these families with the set $\{1, 2, 3, 4, r+2\}$. For an element of $\mathcal{S}(X) \backslash \mathcal{A}(X)$, the intersection must be $\{1, 4\}, \{1, r+2\}$, or $\{1, 4, r+2\}$. For an element of $\mathcal{A}(X) \backslash \mathcal{S}(X)$, the intersection must be $\{2, 3\}, \{2, 3, 4\}, \{2, 3, r+2\}$, or $\{2, 3, 4, r+2\}$. Given one of these sets I, the number of elements of the family that have that intersection with $\{1, 2, 3, 4, r+2\}$ is $\binom{n-5}{r-|I|}$. Thus

$$|\mathcal{A}(X) \backslash \mathcal{S}(X)| - |\mathcal{S}(X) \backslash \mathcal{A}(X)| = \binom{n-5}{r-3} + \binom{n-5}{r-4} - \binom{n-5}{r-2}.$$

Multiplying by $\frac{(r-2)!(n-r-1)!}{(n-5)!}$ and expanding, we get that $|\mathcal{A}(X) \backslash \mathcal{S}(X)| - |\mathcal{S}(X) \backslash \mathcal{A}(X)| > 0$ if and only if

$$n^2 + (-3r-1)n + r^2 + 7r - 6 < 0,$$

i.e.

$$n < \frac{3r + 1 + \sqrt{5r^2 - 22r + 25}}{2},$$

as desired. □

Notice that $\frac{3r+1+\sqrt{5r^2-22r+25}}{2} = \varphi^2 r + o(r)$, so the bound given in Conjecture 3 is tight, up to lower order terms.

3. Conditions for a family to be intersecting

This section determines necessary and sufficient conditions for a family to be intersecting that will be useful later in the paper. The purpose of this section is to prove the following proposition.

Proposition 7 *A compressed family $\mathcal{A}$ is intersecting if and only if for any $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots b_r\} \in \mathcal{A}$ there exists a pair i, j, with $1 \leq i, j \leq r$ such that $i + j > \max\{a_i, b_j\}$.*

Note that this is most useful in the case where $\mathcal{A} = F(n, r, \mathcal{G})$. In this case assume that for some $A, B \in \mathcal{A}$, we have an i, j such that $i + j > \max\{a_i, b_j\}$. For any $C = \{c_1, \ldots, c_r\} \leq A$ and $D = \{d_1, \ldots, d_r\} \leq B$, we have $i + j > \max\{c_i, d_j\}$, thus it is sufficient to find such a pair i, j for each pair of generators.

Proof. Assume that there exists i, j with $i + j > \max\{a_i, b_j\}$. Without loss of generality, assume $a_i \leq b_j$. Then $a_1, \ldots, a_i, b_1, \ldots, b_j \leq b_j$, so we have $i + j > b_j$ elements that are less than or equal to b_j, so two must be the same by the pigeonhole principle, so $A \cap B \neq \emptyset$.

The opposite direction is a direct consequence of Proposition 8.1 of [5] in the case of 2 families. When this proposition is rephrased using the notation of this paper, and the result is restricted to the present conditions, it states that if $\mathcal{A}_1, \mathcal{A}_2$ are compressed, intersecting families, and $F_i \in \mathcal{A}_i$ is fixed, then there exists an ℓ such that $|F_1 \cap [1, \ell]| + |F_2 \cap [1, \ell]| > \ell$. To apply this result take $\mathcal{A}_1 = F(n, r, \{A\})$, $\mathcal{A}_2 = F(n, r, \{B\})$, $F_1 = A$ and $F_2 = B$, then let $i = |A \cap [1, \ell]|$, and $j = |B \cap [1, \ell]|$. □

This has the following corollary.

Corollary 8 *Let $A = \{a_1, \ldots, a_r\}$. There exists an s such that $A \prec [s, 2s-1]$ if and only if $F(n, r, \{A\})$ is an intersecting family.*

Proof. If $A \prec [s, 2s-1]$ and $\{b_1, \dots, b_r\}, \{c_1, \dots, c_r\} \leq A$, let $i = j = s$, and we have $s+s > \max\{a_s, a_s\} \geq \max\{b_s, c_s\}$, so $F(n, r, \{A\})$ is intersecting. Conversely, assume there exists i and j such that $i+j > a_i, a_j$. Without loss of generality $i \geq j$, so

$$2i \geq i + j > a_i.$$

This is sufficient to show $A \prec [i, 2i-1]$, since $a_k \leq a_{k+1} - 1$. □

4. Proof of Theorem 5

Let $\mathcal{A}_{n,r,s} = F(n, r, \{[s, 2s-1]\})$ for $s \geq 2$. Notice that by Corollary 8, for the proof of Theorem 5 it is sufficient to consider families of this form. In [3], Borg proved that for a compressed family $\mathcal{A}$, and $X, X' \subset [n]$ with $X' \geq X$ we have $|\mathcal{A}(X)| \geq |\mathcal{A}(X')|$. Notice that $\{r+2\}$, $\{4, r+2\}$, $\{2, 4, r+2\}$ and $\{2, \dots, |X|, r+2\}$, are the minimal elements in each of the cases of Theorem 2, thus it suffices to assume X is one of these sets. This is important, because it greatly reduces the number of cases we need to consider. The proof will break into several cases depending on various possibilities for X, which is uniquely determined by $t := |X|$. We assume throughout that $t \leq r$ and $s \geq 2$.

For $t \geq 2$, we will show that $|\mathcal{A}_{n,r,s}(X)|$ is decreasing in s, so it is sufficient to show that $|\mathcal{S}(X)| \geq |\mathcal{A}_{n,r,s}(X)|$ for small values of s. The case $t = 1$ is slightly more complicated, since $|\mathcal{A}_{n,r,s}(X)|$ is not decreasing in s. For this case we may write $|\mathcal{A}_{n,r,s}(X)| = D(n,r,s) + E(n,r,s)$ where $D(n,r,s)$ is decreasing in s, and $E(n,r,s)/|\mathcal{S}(X)|$ goes to 0 as $r \to \infty$. By considering the ratio $|\mathcal{A}_{n,r,s}(X)|/|\mathcal{S}(X)|$, we will show that it is less than 1 for large r, and then use a computer to check the result for small values of r. We begin by considering the case $t \geq 2$.

4.1. The case $t \geq 2$

Lemma 9 *If $t \geq 2$, $|\mathcal{A}_{n,r,s}(X)|$ is decreasing in s for $s \geq 3$. Also, if $n > \varphi^2 r$ and $t \geq 3$ then $|\mathcal{A}_{n,r,2}(X)| \geq |\mathcal{A}_{n,r,3}(X)|$.*

Proof. First address the case $2s-1 \geq r+2$. Let $B_{n,r,s} = \mathcal{A}_{n,r,s+1}(X) \backslash \mathcal{A}_{n,r,s}(X)$. Observe that if $A = \{a_1, \dots, a_r\} \in B_{n,r,s}$, then $a_s = 2s$ and $a_{s+1} = 2s+1$. Notice that for an element of $\mathcal{A}_{n,r,s+1} \backslash \mathcal{A}_{n,r,s}$ there are $2s-1$ possibilities for the first $s-1$ elements and $n-(2s+1)$ possibilities for the last $r-s-1$ elements, hence $|\mathcal{A}_{n,r,s+1} \backslash \mathcal{A}_{n,r,s}| = \binom{2s-1}{s-1}\binom{n-2s-1}{r-s-1}$ (we may assume that $n \geq r+s+1$, so $\binom{n-2s-1}{r-s-1} \neq 0$). Similarly since $2s-1 \geq r+2$, there are $\binom{2s-1-t}{s-1}\binom{n-2s-1}{r-s-1}$ elements of $\mathcal{A}_{n,r,s+1} \backslash \mathcal{A}_{n,r,s}$ that do not contain an element of X (notice this is 0 if $t \geq 2s-1$). Subtracting gives that

$$|B_{n,r,s}| = \left(\binom{2s-1}{s-1} - \binom{2s-1-t}{s-1}\right)\binom{n-2s-1}{r-s-1}.$$

Now consider $C_{n,r,s} = \mathcal{A}_{n,r,s}(X) \backslash \mathcal{A}_{n,r,s+1}(X)$. Notice that if $A = \{a_1, \dots, a_r\} \in C_{n,r,s}$, then $\{a_1, \dots, a_s\} \leq [s, 2s-1]$ and $a_{s+1} > 2s+1$. We use the same counting method as used above, namely, we first count the number of elements of $\mathcal{A}_{n,r,s} \backslash \mathcal{A}_{n,r,s+1}$ by considering possibilities for the first s elements of a set and the last $r-s$ elements to find that there are $\binom{2s-1}{s}\binom{n-2s-1}{r-s}$ such elements (Note that this differs slightly for the method used to count $B_{n,r,s}$ since in that case we new the values of the elements in positions s and $s+1$). Then we count the number of elements of $\mathcal{A}_{n,r,s} \backslash \mathcal{A}_{n,r,s+1}$ that do not contain an element of X by considering possibilities for the first s and last $r-s$ elements, and multiplying. We find there are $\binom{2s-1-t}{s}\binom{n-2s-1}{r-s}$. We then subtract to get

$$|C_{n,r,s}| = \left(\binom{2s-1}{s} - \binom{2s-1-t}{s}\right)\binom{n-2s-1}{r-s}.$$

This same method of counting $B_{n,r,s}$ and $C_{n,r,s}$ will be repeated several times below with only slight variations, so we omit the explanations of those computations.

We wish to show that $|C_{n,r,s}| \geq |B_{n,r,s}|$. Notice that $\binom{2s-1}{s-1} - \binom{2s-1-t}{s} \geq \binom{2s-1}{s} - \binom{2s-1-t}{s-1}$ since $s-1 \geq \frac{1}{2}(2s-1-t)$. Also, $\binom{n-2s-1}{r-s} \geq \binom{n-2s-1}{r-s-1}$, and combining these shows that $|C_{n,r,s}| \geq |B_{n,r,s}|$, hence $|\mathcal{A}_{n,r,s}|$ is decreasing in s when $2s-1 \geq r+2$.

Now consider the case that $r+2 \in \{2s, 2s+1\}$. Using the counting method described above, modified for the fact that now for an element of $B_{n,r,s}$, we have $r+2 = a_s$ or a_{s+1}, we find that

$$|B_{n,r,s}| = \binom{2s-1}{s-1}\binom{n-2s-1}{r-s-1}$$

$$|C_{n,r,s}| = \left(\binom{2s-1}{s} - \binom{2s-t}{s}\right)\binom{n-2s-1}{r-s}.$$

Let $\alpha = \binom{2s-1}{s}$ and $\beta = \binom{2s-t}{s}$. In this case, to show $|C_{n,r,s}| \geq |B_{n,r,s}|$, by dividing both sides of this inequality by $\binom{n-2s-1}{r-s-1}$ and rearranging, it is sufficient to show that

$$\frac{n}{r} \geq 1 + \frac{\alpha}{\alpha-\beta} + \frac{s}{r}\left(1 - \frac{\alpha}{\alpha-\beta}\right)$$

Since the last term is negative, it is enough to show $\frac{n}{r} \geq 1 + \frac{\alpha}{\alpha-\beta}$. Notice that $\frac{\alpha}{\alpha-\beta}$ is largest in the case $t = 2$, and in this case it simplifies to $\frac{\alpha}{\alpha-\beta} = 1 - \frac{1}{s} \leq \frac{3}{2}$ for $s \geq 2$. This gives $|C_{n,r,s}| \geq |B_{n,r,s}|$ for $\frac{n}{r} > 2.5$.

Now address the case that $2s + 1 < r + 2$. Using the counting methods given above, we find that if $t \geq 4$ and $s \geq 2$, or $t \in \{2, 3\}$ and $s \geq 3$, then

$$|B_{n,r,s}| = \binom{2s-1}{s-1}\binom{n-2s-1}{r-s-1} - \binom{2s-t}{s-1}\binom{n-2s-2}{r-s-1},$$

$$|C_{n,r,s}| = \binom{2s-1}{s}\binom{n-2s-1}{r-s} - \binom{2s-t}{s}\binom{n-2s-2}{r-s}.$$

We wish to show that $|C_{n,r,s}| \geq |B_{n,r,s}|$. By rearranging terms and using the identities $\binom{a}{b-1} = \frac{b}{a-b+1}\binom{a}{b}$, $\binom{a-1}{b-1} = \frac{b}{a}\binom{a}{b}$, we find that showing $|C_{n,r,s}| \geq |B_{n,r,s}|$ is equivalent to showing

$$\binom{2s-1}{s}\binom{n-2s-1}{r-s}\left(1 - \frac{r-s}{n-r-s}\right) \geq$$
$$\binom{2s-t}{s}\binom{n-2s-2}{r-s}\left(1 - \frac{s}{s-t+1}\cdot\frac{r-s}{n-r-s-1}\right).$$

Notice that we may assume that $s > t - 1$, because otherwise the $\binom{2s-t}{s}$ in the expression for $|C|$ is 0, in which case it is easy to see that $|C_{n,r,s}| \geq |B_{n,r,s}|$. From the expression above, we see that to show $|C_{n,r,s}| \geq |B_{n,r,s}|$, it is sufficient to show that $1 - \frac{r-s}{n-r-s} \geq 1 - \frac{s(r-s)}{(s-t+1)(n-r-s-1)}$, which is easy to show by simple rearrangements of the inequality. Thus $|C_{n,r,s}| \geq |B_{n,r,s}|$.

It remains to check the case that $t = 3, s = 2$. Notice that in this case $|C_{n,r,s}| = 3\binom{n-5}{r-2} - \binom{n-6}{r-3} = 2\binom{n-5}{r-2} + \binom{n-6}{r-3}$ and $|B_{n,r,s}| = 3\binom{n-5}{r-3}$. Observe that $\binom{n-6}{r-3} = \frac{(n-r-2)(n-r-3)}{(n-5)(r-2)}\binom{n-5}{r-3}$, and that for $n > \varphi^2 r$ we have that $\frac{(n-r-2)(n-r-3)}{(n-5)(r-2)} \geq 1$ (this can be shown by clearing denominators, using the quadratic formula in $a = \frac{n}{r}$, and then maximizing with respect to r). Combining this inequality with $2\binom{n-5}{r-2} \geq 2\binom{n-5}{r-3}$ gives $|C_{n,r,s}| \geq |B_{n,r,s}|$. □

We now prove Theorem 5 in the case $t \geq 2$.

Proof. By Lemma 9, it is sufficient to show that $|\mathcal{S}(X)| \geq |\mathcal{A}_{n,r,s}(X)|$ in the cases $(t, s) = (2, 2), (2, 3), (3, 2)$ and the case $t \geq 4$, $s = 2$. We first address the case $t \geq 4$, $s = 2$. Using the same counting methods as in Lemma 9, we count $|\mathcal{S}(X)\backslash\mathcal{A}_{n,r,2}(X)| = \binom{n-3}{r-1} - \binom{n-t-1}{r-1}$ and $|\mathcal{A}_{n,r,2}(X)\backslash\mathcal{S}(X)| = \binom{n-3}{r-2}$. Since $|\mathcal{S}(X)\backslash\mathcal{A}_{n,r,2}(X)|$ is decreasing in t, we may assume that $t = 4$. Thus it is sufficient to show $\binom{n-3}{r-1} - \binom{n-5}{r-1} - \binom{n-3}{r-2} \geq 0$. Multiply this expression by $\frac{(r-1)!(n-r-1)!}{(n-5)!}$ to get that this inequality is equivalent to

$$(n-3)(n-4)(n-r-1) - (r-1) - (n-r-3)(n-r-2)(n-r-1) \geq 0.$$

By letting $n = ar$ and expanding, we get a quadratic expression in a. By using the quadratic formula, and then maximizing with respect to r, we find that $|\mathcal{S}(X)| \geq |\mathcal{A}_{n,r,s}(X)|$ for $a > \varphi^2$.

In the case $t = 2, s = 2$, we have $|\mathcal{S}(X)\backslash\mathcal{A}_{n,r,2}(X)| = \binom{n-3}{r-1} - \binom{n-5}{r-1}$ and $|\mathcal{A}_{n,r,2}(X)\backslash\mathcal{S}(X)| = \binom{n-3}{r-2} - \binom{n-5}{r-2}$. We may use the same method as in the case $t \geq 4$ to get $|\mathcal{S}(X)| \geq |\mathcal{A}_{n,r,s}(X)|$ for $n > 2r$.

The case $t=3$, $s=2$ was addressed in Section 2, so it only remains to check the case $t=2, s=3$. In this case we have $|\mathcal{S}(X)\backslash\mathcal{A}_{n,r,3}(X)| = 2\binom{n-5}{r-2}+2\binom{n-6}{r-3}$ and $|\mathcal{A}_{n,r,3}(X)\backslash\mathcal{S}(X)| = \binom{n-4}{r-3}+2\binom{n-5}{r-3}+\binom{n-6}{r-4}$. We use the same method, as in the case $t\geq 4$, but this time we get a polynomial that is cubic in a. Using the cubic formula and maximizing, we find that $|\mathcal{S}(X)| \geq |\mathcal{A}_{n,r,2}(X)|$ for $a \geq 2.46$. □

4.2. The case $t=1$

Lemma 10 *For $t=1$, we have $|\mathcal{A}_{n,r,s}(X)| = D(n,r,s)+E(n,r,s)$ where*

$$D(n,r,s) = \sum_{i=s}^{\min(r+1,2s-1)} \binom{i-1}{s-1}\binom{n-i-1}{r-s-1}$$

and

$$E(n,r,s) = \begin{cases} \binom{r+1}{s-1}\binom{n-r-2}{r-s} + \sum_{i=r+3}^{2s-1}\binom{i-2}{s-2}\binom{n-i}{r-s} & 2s-1\geq r+2 \\ 0 & \textit{otherwise.}\end{cases}$$

Proof. We count the number of elements in $\mathcal{A}_{n,r,s}(X)$ with the s-th element being i. For $i \leq \min(r+1, 2s-1)$, there are $i-1$ possibilities for the first $s-1$ elements, one of the elements must be $r+2$, and there are $n-i-1$ possibilities for the remaining $r-s-1$ elements. If $2s-1 < r+2$, all elements of $\mathcal{A}_{n,r,s}$ have the s-th element less than $r+2$, so we have counted all of $\mathcal{A}_{n,r,s}(X)$. Otherwise, using similar logic we get the expression for $E(n,r,s)$. □

We address $D(n,r,s)$ and $E(n,r,s)$ separately. We first consider $D(n,r,s)$.

Lemma 11 *$D(n,r,s)$ is decreasing in s.*

Proof. In the case that $\min(r+1,2s-1) = r+1$, notice that for $s\leq i\leq r+1$ we have that $2(s-1)\geq i-1$, so by increasing s, each term $\binom{i-1}{s-1}\binom{n-i-1}{r-s-1}$ decreases. Also, increasing s decreases the total number of terms, so in this case it is easy to see that $D(n,r,s)$ is decreasing in s.

Otherwise, consider $D(n,r,s)-D(n,r,s+1)$. Let $C_s = \{A \in F(n,r,\{[s,2s-1]\}) : r+2 \in A, r+2 \text{ is in position } s+1 \text{ or greater}\}$. By reviewing the proof of Lemma 10 it is easy to see that $D(n,r,s)$ counts $|C_s|$. Thus we have $D(n,r,s)-D(n,r,s+1) = |C_s\backslash C_{s+1}| - |C_{s+1}\backslash C_s|$. We first address the case that $r+2\geq 2s+2$. Notice that $C_s\backslash C_{s+1}$ consists of elements that have the first s elements less that $2s-1$, and the $s+1$-st element is at least $2s+2$. Also, notice that $C_{s+1}\backslash C_s$ consists of elements that have s-th element equal to $2s$, and the $s+1$-st element is $2s+1$. By considering such elements, we see that

$$|C_s\backslash C_{s+1}| - |C_{s+1}\backslash C_s| = \binom{2s-1}{s}\binom{n-2s-2}{r-s-1} - \binom{2s-1}{s-1}\binom{n-2s-2}{r-s-2}.$$

It is easy to see that that this is greater than 0. The only remaining case is when $2s+1 = r+2$, and in this case we have $|C_s\backslash C_{s+1}| - |C_{s+1}\backslash C_s| = \binom{2s-1}{s}\binom{n-2s-1}{r-s-1}$. Thus we see $D(n,r,s)$ is decreasing in s. □

We now consider $E(n,r,s)$. Notice that when $t=1$, we have $|\mathcal{S}(X)| = \binom{n-2}{r-2}$, so we begin by showing that $E(n,r,s)/\binom{n-2}{r-2}$ is negligibly small.

Lemma 12 *Let $n/r = a$, and $r\geq 4$. Let $Q(n,r,s) = E(n,r,s)/\binom{n-2}{r-2}$. For any fixed $a \geq 2.2$, $\max\{Q(n,r,s)\}_{s=2}^{r}$ goes to 0 exponentially in r as $r\to\infty$, for any s. For $a\geq\varphi^2$, we have $Q(ar,r,s)\leq 3.25r^{3/2}(.954)^r$.*

Proof. We address each term in $E(n,r,s)$ individually. We begin by using Stirling's approximation, and then use a computer to find the maximal term.

We first address $\dfrac{\sum_{i=r+3}^{2s-1}\binom{i-2}{s-2}\binom{n-i}{r-s}}{\binom{n-2}{r-2}}$ for $s<r$. We will consider each term in the sum individually, so the same computations apply to the first term of $E(n,r,s)$, and to the case $s=r$, so we omit the analysis of those cases.

Define $q(n,r,i,s)$ by

$$q(n,r,i,s) = \frac{\binom{i-2}{s-2}\binom{n-i}{r-s}}{\binom{n-2}{r-2}} = \frac{s(s-1)n(n-1)}{i(i-1)r(r-1)}\cdot\frac{i!(n-i)!r!(n-r)!}{s!(i-s)!(r-s)!(n-i-r+s)!n!}.$$

Recall Stirling's approximation $n! \sim \sqrt{2\pi n}\,(n/e)^n$. Although this only holds for large n, in general we have $\sqrt{2\pi n}\,(n/e)^n \leq n! \leq e\sqrt{n}\,(n/e)^n$. Applying this gives

$$q(n,r,i,s) \leq \frac{s(s-1)n(n-1)e^4}{i(i-1)r(r-1)(2\pi)^2} \cdot \sqrt{\frac{i(n-i)r(n-r)}{2\pi s(i-s)(r-s)(n-i-r+s)n}}$$
$$\cdot \frac{i^i(n-i)^{n-i}r^r(n-r)^{n-r}}{s^s(i-s)^{i-s}(r-s)^{r-s}(n-i-r+s)^{n-i-r+s}n^n}.$$

Notice that many of the powers of e from Stirling's approximation have canceled. We substitute $n = ar$. We wish to find for which a this term goes to 0 exponentially in r. To do this, divide numerator and denominator by r^{2ar}, and pull out an r-th root. We also substitute $I = i/r$, and $S = s/r$. We have also used the fact $(s-1)/(i-1) \leq s/i$. This gives

$$q(n,r,i,s) \leq T(a,r,i,s) = \frac{s^2a(ar-1)e^4}{i^2(r-1)(2\pi)^2} \cdot \sqrt{\frac{i(ar-i)(ar-r)}{2\pi s(i-s)(r-s)(ar-i-r+s)a}}$$
$$\cdot \left(\frac{I^I(a-I)^{a-I}(a-1)^{a-1}}{S^S(I-S)^{I-S}(1-S)^{1-S}(a-I-1+S)^{a-I-1+S}a^a}\right)^r. \tag{1}$$

We first address $B(I,S,a) = \dfrac{I^I(a-I)^{a-I}(a-1)^{a-1}}{S^S(I-S)^{I-S}(1-S)^{1-S}(a-I-1+S)^{a-I-1+S}a^a}$. Notice that we have the bounds $1/2 \leq S \leq 1$ and $1 \leq I \leq 2S$. Using a computer we compute the maximum of $B(I,S,a)$ for $1/2 \leq S \leq 1$, $1 \leq I \leq 2S$, and $a \geq 2.2$. This can be done using the function `minimized_constrained` in SAGE. The maximum occurs at $S = \frac{1}{2}$, $I = 1$, $a = 2.2$, for $a \geq 2.2$ we have $B(I,S,a) \leq B(1,\frac{1}{2},2.2) = 0.992388$. In the case that we restrict $a \geq \varphi^2$, we get $B(I,S,a) \leq .954$. Thus each term in $Q(n,r,s)$ goes to 0 exponentially in r for $a \geq 2.2$. To finish the proof of the first part of the proposition, notice that we have shown each term in the sum goes to 0 exponentially in r. Since there are at most $r-4$ terms, the sum still goes to 0 exponentially in r.

We now wish to evaluate how quickly this term goes to 0 when $a \geq \varphi^2$. We first use a computer to show that $T(a,r,i,s)$ is decreasing in a. This can be done by computing the partial derivative $\frac{\partial}{\partial a}T(a,r,i,s)$ and then computing the maximum of the partial derivative, using the same commands as above. We find that the maximum is -7.5454×10^{-23}, which occurs at $(a,r,i,s) = (15.0, 19.1, 18.2, 18.1)$, so $T(a,r,i,s)$ is decreasing in a. Maximizing the first term and using the value for $B(I,S,a)$ computed above, we find $T(\varphi^2 r, r, i, s) \leq 3.25\sqrt{r}(.954)^r$. Since $q(n,r,i,s) \leq T(a,r,i,s)$ and there are less than r terms, we get $Q(ar,r,s,t) \leq 3.25r^{3/2}(.954)^r$ for $a \geq \varphi^2$. □

We need one more lemma.

Lemma 13 *For constants $b, c \in \mathbb{N}$, with $b \geq c$ and $n = ar$, we have*

$$\lim_{r\to\infty} \frac{\binom{n-b}{r-c}}{\binom{n-2}{r-2}} = \frac{(a-1)^{b-c}}{a^{b-2}}.$$

Proof. We expand the binomial coefficients.

$$\frac{\binom{n-b}{r-c}}{\binom{n-2}{r-2}} = \frac{(r-2)\cdots(r-c+1)\cdot(n-r)\cdots(n-r-b+c-1)}{(n-2)\cdots(n-b+1)}$$
$$= \frac{r^{c-2}(n-r)^{b-c}}{n^{b-2}} \prod_{i=2}^{c-1}\frac{r-i}{r} \prod_{j=0}^{b+c-1}\frac{n-r-j}{n-r} \prod_{k=2}^{b-1}\frac{n}{n-k}$$

As r goes to infinity, the products go to 1, hence

$$\lim_{r\to\infty} \frac{\binom{n-b}{r-c}}{\binom{n-2}{r-2}} = \lim_{r\to\infty} \frac{r^{c-2}(n-r)^{b-c}}{n^{b-2}} = \frac{(a-1)^{b-c}}{a^{b-2}},$$

as desired. □

We may now finish the proof of Theorem 5 in the case $t = 1$.

Proof. Notice $D(n, r, 2) = \binom{n-3}{r-3} + 2\binom{n-4}{r-3}$, so by substituting $n = ar$,

$$\frac{D(n,r,2)}{\binom{n-2}{r-2}} = \frac{r-2}{ar-2} + \frac{(ar-r)(r-2)}{(ar-2)(ar-3)}.$$

Taking the derivative with respect to a, it is possible to show for any r that this is decreasing in a when $a > \varphi^2$. By Lemmas 11 and 12, we have that

$$\begin{aligned}\frac{|\mathcal{A}_{n,r,s}(X)|}{|\mathcal{S}(X)|} = \frac{D(n,r,s)+E(n,r,s)}{\binom{n-2}{r-2}} &\leq \frac{D(n,r,2)}{\binom{n-2}{r-2}} + 3.25r^{3/2}(.954)^r \\ &\leq \frac{r-2}{\varphi^2 r-2} + \frac{2(r-2)(\varphi^2 r - r)}{(\varphi^2 r-2)(\varphi^2 r-3)} + 3.25r^{3/2}(.954)^r\end{aligned}$$

By using calculus and a computer, we check that this is less than 1 for $r \geq 217$.

Thus $|\mathcal{A}_{n,r,s}(X)| \leq |\mathcal{S}(X)|$ for $n \geq \varphi^2 r$ when $r \geq 217$. Using the formula given in Lemma 10, we can check that $n \geq \varphi^2 r$ implies $|\mathcal{A}_{n,r,s}(X)| \leq |\mathcal{S}(X)|$ for $4 \leq r \leq 217$, which completes the proof of Theorem 5. □

5. Proof of Theorem 4

To prove Theorem 4, we construct a large compressed family $\mathcal{B}$ that contains most intersecting families. It is not intersecting, but it still satisfies $|\mathcal{B}(X)| \leq |\mathcal{S}(X)|$ for sufficiently large n, which implies $|\mathcal{A}(X)| \leq |\mathcal{S}(X)|$ for these values of n. Recall that we may assume that when $t = |X| = 1$ we have $X = \{r+2\}$, when $t = 2$ we have $X = \{4, r+2\}$, when $t = 3$ we have $X = \{2, 4, r+2\}$, and when $t \geq 4$ then $X = \{2, \ldots, t, r+2\}$.

Proposition 14 *Let*

$$\mathcal{B} = F(n, r, \{\{1, r+1\}\}) \cup F(n, r, \{\{2, 3, r+2\}\}) \cup \bigcup_{s=3}^{r} \mathcal{A}_{n,r,s}.$$

Let $\mathcal{A}$ be an intersecting family. If $\mathcal{A} \not\subset \mathcal{S}$ and $\mathcal{A} \not\subset \mathcal{A}_{n,r,2}$, then $\mathcal{A} \subset \mathcal{B}$.

Proof. Consider an element $A = \{a_1, \ldots, a_r\} \in \mathcal{A}$. We begin by assuming $a_1 = 1$, and we consider the maximal possible value for a_2. For the sake of contradiction, assume that $a_2 \geq r+2$. Since $a_i > a_{i-1}$, we have $a_i \geq r+i$ for $i \geq 2$. Since $\mathcal{A} \not\subset \mathcal{S}$, there exists some $B = \{b_1, \ldots, b_r\} \in \mathcal{A}$ such that $b_1 \geq 2$. By Proposition 7, there exists a pair i, j such that $i + j > \max\{a_i, b_j\}$. If $i = 1$, then $1 + j > b_j$, which implies $b_j = j$, since for any $B \in \binom{[n]}{r}$, we have $b_j \geq j$. This implies $b_1 = 1$, which is a contradiction. So $i \geq 2$ and we have $i + j > a_i \geq r + i$, hence $j > r$, which is impossible. Thus we cannot have $a_2 \geq r+2$, so all $A \in \mathcal{A}$ with smallest element 1 are contained in $F(n, r, \{\{1, r+1\}\})$.

Now assume that $A \prec [s, 2s-1]$ for some $s \geq 2$. If $s \geq 3$, then $A \in \mathcal{A}_{n,r,s} \subset \mathcal{B}$. Thus we may assume $s = 2$. If $a_1 = 1$, we know $A \in \mathcal{B}$ by the previous paragraph, so we may assume $a_1 = 2$, which implies $a_2 = 3$. To show $A \in F(n, r, \{\{2, 3, r+2\}\})$, we argue as in the previous paragraph. If $a_3 \geq r+3$, then $a_i \geq r+i$ for $i \geq 3$. Since $\mathcal{A} \notin \mathcal{A}_{n,r,2}$, there exists some $B \in \mathcal{A}$ with $B \not\prec [2, 3]$. This implies $b_2 \geq 4$. By Proposition 7, there exists a pair i, j with $i + j > \max\{a_i, b_j\}$. As in the previous paragraph, if $i \geq 3$, then $j > r$ which is impossible, so $i \in \{1, 2\}$.

We first show we cannot have $i = 1$. If $i = 1$, then we cannot have $j = 1$ since $a_2 = 2$. If $j \geq 2$, then notice that since $i = 1$, we have $1 + j > b_j$, but we always have $b_j \geq j$, so $b_j = j$. Since $b_j < b_{j+1}$, this implies $b_2 = 2$, which contradicts $b_2 \geq 4$. Thus $i \neq 1$. Now we show we cannot have $i = 2$. Assume that $i = 2$. If $j = 1$, then we have $3 > a_2 = 3$ which is impossible, hence $j \geq 2$. However, if $j \geq 2$, then we have $2 + j > b_j \geq j$, so $b_j \in \{j, j+1\}$. Since $b_{j-1} \leq b_j - 1$, this implies $b_2 \in \{2, 3\}$, which is false. Thus $i \notin \{1, 2\}$. This is impossible if $a_3 \geq r+3$, so we have $a_3 \leq r+2$, so $A \in F(n, r, \{2, 3, r+2\})$. □

We now prove Theorem 4.

Proof. As in the proof of Theorem 5, we first consider the case $t \geq 2$, then the case $t = 1$. We consider $|\mathcal{B}(X)\backslash\mathcal{S}(X)|$ and $|\mathcal{S}(X)\backslash\mathcal{B}(X)|$. For all t, we have $|\mathcal{S}(X)\backslash\mathcal{B}(X)| = \binom{n-r-2}{r-2}$, since the smallest element of a set in $\mathcal{S}\backslash\mathcal{B}$ must be 1 and the second largest must be at least $r+2$, and if it intersects X, then it must be $r+2$. There are $\binom{n-r-2}{r-2}$ choices for the remaining elements. To count $|\mathcal{B}(X)\backslash\mathcal{S}(X)|$, notice that $F(n,r,\{\{1,r+1\}\})\backslash\mathcal{S}(X) = \emptyset$. Also, $F(n,r,\{\{2,3,r+2\}\}) \subset F(n,r,\{\{2,3\}\})$, and $|F(n,r,\{\{2,3\}\})\backslash\mathcal{S}(X)| = \binom{n-4}{r-3}+\binom{n-5}{r-3}$. To count the remaining elements of $\mathcal{B}(X)\backslash\mathcal{S}(X)$ it is possible to show that $|\mathcal{A}_{n,r,s}(X)\backslash\mathcal{S}(X)|$ is decreasing in s by using the same argument as in Lemma 9 (i.e. Consider $B_{n,r,s} = (\mathcal{A}_{n,r,s+1}(X)\backslash\mathcal{S}(X))\backslash(\mathcal{A}_{n,r,s}(X)\backslash\mathcal{S}(X))$ and $C_{n,r,s} = (\mathcal{A}_{n,r,s}(X)\backslash\mathcal{S}(X))\backslash(\mathcal{A}_{n,r,s+1}(X)\backslash\mathcal{S}(X))$, and count $|C_{n,r,s}|$ and $|B_{n,r,s}|$ using the method of Lemma 9, and check that $|C_{n,r,s}| \geq |B_{n,r,s}|$). This gives

$$\begin{aligned}|\mathcal{B}(X)\backslash\mathcal{S}(X)| &\leq \tbinom{n-4}{r-3} + \tbinom{n-5}{r-3} + |\mathcal{A}_{n,r,3}(X)\backslash\mathcal{S}(X)| + r|\mathcal{A}_{n,r,4}(X)\backslash\mathcal{S}(X)| \\ &\leq \tbinom{n-4}{r-3} + 3\tbinom{n-5}{r-3} + \\ &\quad r\left(\tbinom{n-5}{r-4} + 3\tbinom{n-6}{r-4} + \tbinom{n-7}{r-5} + 6\tbinom{n-7}{r-4} + 3\tbinom{n-8}{r-5}\right).\end{aligned}$$

Divide by $\binom{n-2}{r-2}$. Notice that by Lemma 13, for large r the right hand side is approximately

$$\frac{a-1}{a^2} + \frac{3(a-1)^2}{a^3} + r\left(\frac{a-1}{a^3} + \frac{3(a-1)^2}{a^4} + \frac{(a-1)^2}{a^5} + \frac{6(a-1)^3}{a^5} + \frac{3(a-1)^3}{a^6}\right).$$

Notice that if we take $a = cr$, then this goes to 0, hence $|\mathcal{B}(X)\backslash\mathcal{S}(X)|$ is arbitrarily small for large r.

Also, it is easy to check that $\binom{n-r-2}{r-2}/\binom{n-2}{r-2}$ is increasing in n for any fixed r. So it is sufficient to find an r for which the inequality holds. Notice that

$$\frac{\binom{n-r-2}{r-2}}{\binom{n-2}{r-2}} = \frac{(n-r-2)(n-r-3)\cdots(n-2r+1)}{(n-2)(n-3)\cdots(n-r+1)}.$$

Since $\frac{n-r-b}{n-b} = 1 - \frac{r}{n-b} \geq 1 - \frac{r}{n-r}$, for $2 \leq b \leq r-1$, we get that

$$\frac{\binom{n-r-2}{r-2}}{\binom{n-2}{r-2}} \geq \left(1 - \frac{r}{n-r}\right)^{r-2}.$$

Take $n = cr^2$ for some constant c. Then some simple manipulations give that

$$\left(1 - \frac{r}{n-r}\right)^{r-2} = \left(\left(1 - \frac{1}{cr-1}\right)^{cr-1}\left(1 - \frac{1}{cr-1}\right)^{1-2c}\right)^{1/c}.$$

As $r \to \infty$, this goes to $e^{-1/c}$. Thus for $n = cr$, $|\mathcal{S}(X)/\mathcal{B}(X)|/\binom{n-2}{r-2}$ does not go to 0, but $|\mathcal{B}(X)/\mathcal{S}(X)| \to 0$ as $r \to \infty$, so for any c, there exists a value r_c such that for $n > cr^2$, we have that $|\mathcal{S}(X)| \geq |\mathcal{A}(X)|$ for all compressed, intersecting families $\mathcal{A}$. In particular, by using calculus and a computer (i.e. by checking that $|\mathcal{B}(X)\backslash\mathcal{S}(X)|/\binom{n-2}{r-2}$ and $(1-\frac{r}{n-r})^{r-2}$ are decreasing in r for sufficiently large r, then finding when their sum is less than 1) we find that when $c = 1$ we have $r_1 = 36$ and for $c = 2$ we have $r_2 = 6$.

The cases for $t = 3$ and $t \geq 4$ are very similar to the case $t = 2$. We still have $|\mathcal{S}(X)\backslash\mathcal{B}(X)| = \binom{n-r-2}{r-2}$, but the number of elements in $F(n,r,\{2,3,r+2\})(X)$ that aren't in $F(n,r,\{1,r+2\})(X)$ is now $\binom{n-3}{r-2} - \binom{n-r-2}{r-2}$, and using the same methods as above, this goes to $1-e^{-1/c}$ as $r \to \infty$. As above, $(|\mathcal{A}_{n,r,3}(X)\backslash\mathcal{S}(X)|+r|\mathcal{A}_{n,r,4}(X)\backslash\mathcal{S}(X)|)/\binom{n-2}{r-2} \to 0$, so we just need to choose c so that $e^{-1/c} > 1-e^{-1/c}$. This is satisfied for $c > \frac{1}{\log 2}$. Thus for $n > cr^2$, we have that $|\mathcal{S}(X)| \geq |\mathcal{A}(X)|$ for all compressed, intersecting families $\mathcal{A}$. In particular, when $t \geq 3$, $c = 2$ we use a computer to find that it holds with $r_2 = 4$.

The case $t = 1$ is similar to the previous cases, though we also have to address the contribution from the $E(n,r,s)$ term. However, as shown in Lemma 12, the term $E(n,r,s)/\binom{n-2}{r-2}$ is negligibly small. Using a similar method to the one used in Lemma 12, we find $E(2r^2,r,s) \leq .3e^r 2^{-2r^2+r+3} r^{-4r^2+2r+6}$, which allows us to use a computer to find that $r_2 = 4$. ☐

6. A generating function for $|\mathcal{A}(X)|$

In this section, we introduce a generating function that can be used to compute $|\mathcal{A}(X)|$ for any compressed family $\mathcal{A}$ and $X \subset [n]$. This method of calculating $|\mathcal{A}(X)|$ is much faster in practice than enumerating elements of $\mathcal{A}$ and checking if each intersects X. In our experiments, the method given below was approximately 40 times as fast.

Before beginning we note that for any non-empty set of generators $G \subset 2^{[n]}$, we may obtain a set of generators $\mathcal{G}' \subset \binom{[n]}{r}$ such that $F(n,r,\mathcal{G}) = F(n,r,\mathcal{G}')$ and $|\mathcal{G}'| \leq |\mathcal{G}|$. Indeed, consider $G = \{g_1, \dots, g_k\} \in \mathcal{G}$ with $k \leq r$. Notice that if $g_i \geq n-r+i$, then for $A = \{a_1, \dots, a_r\} \prec G$ the inequality $a_i \leq g_i$ is trivially satisfied. So we may replace g_i by $n-r+i$ in G without changing the set of elements generated by G. Then if $k < r$ we replace G by $G \cup [n-(r-k)+1, n]$ to get a set with size r. Let $\mathcal{G}'$ be the family of sets obtained in this way, along with the elements of $\mathcal{G}$ with size r. Notice we have $F(n,r,\mathcal{G}) = F(n,r,\mathcal{G}')$. Thus we may assume any set of generators is a family.

For a family $\mathcal{A}$, consider the function $f_{\mathcal{A}}$ defined by

$$f_{\mathcal{A}}(x_1, \dots, x_n) = \sum_{B \in \mathcal{A}} \prod_{i \in B} x_i.$$

Notice that $f_{\mathcal{A}}(1, \dots, 1) = |\mathcal{A}|$. Let

$$\delta_{i,X} = \begin{cases} 1 & i \notin X \\ 0 & i \in X \end{cases}.$$

We define $f_{\mathcal{A}}(x_1, \dots x_n)|_{X=0} = f_{\mathcal{A}}(\delta_{1,X}x_1, \delta_{2,X}x_2, \dots, \delta_{n,X}x_n)$. Notice that

$$f_{\mathcal{A}}(x_1, \dots, x_n) - f_{\mathcal{A}}(x_1, \dots x_n)|_{X=0} = f_{\mathcal{A}(X)}(x_1, \dots, x_n).$$

For $\mathcal{A}$ of the form $\mathcal{A} = F(n,r,\{\{a_1, \dots, a_r\}\})$ we denote $f_{\mathcal{A}}$ by $f_{a_1,\dots,a_r}$. Proposition 15 gives a recursive method for computing $f_{a_1,\dots,a_n}$.

Proposition 15 *For a compressed family $\mathcal{A} = F(n,r,\{\{a_1, \dots, a_r\}\})$, we have*

$$f_{a_1,\dots,a_r}(x_1, \dots, x_n) = \sum_{i=r}^{a_r} x_i f_{\{\min\{a_j, i+j-r\}\}_{j=1}^{r-1}}(x_1, \dots, x_{n-1}),$$

where $\{\min\{a_j, i+j-r\}\}_{j=1}^{r-1}$ denotes the $r-1$ element set where the j-th element is $\min\{a_j, i+j-r\}$.

Proof. Consider $B \in \mathcal{A} = F(n,r,\{\{a_1, \dots, a_r\}\})$ with largest element i. Notice we must have $r \leq i \leq a_r$, and $b_j \leq a_j$ because $B \leq \{a_1, \dots, a_r\}$. Since $b_j \leq b_{j+1} - 1$, and the largest element of B is i, the j-th element is at most $i+j-r$. Combining these observations, we have $\{b_1, \dots, b_{r-1}\} \leq \{\min\{a_j, i+j-r\}\}_{j=1}^{r-1}$. Conversely, every B such that $b_r = i$ and $\{b_1, \dots, b_{r-1}\} \leq \min\{a_j, i+j-r\}$ is in $\mathcal{A}$ since $\mathcal{A}$ is compressed, which gives the desired expression. □

Remark 16 *Observe that $B \leq G = \{g_1, \dots, g_r\}$ and $B \leq H = \{h_1, \dots, h_r\}$ if and only if $B \leq \{\min\{g_i, h_i\}\}_{i=1}^{r}$. Thus by the preceding proposition, we may obtain $f_{\mathcal{A}}$ with $\mathcal{A} = F(n,r,\mathcal{G})$ for any set of generators $\mathcal{G}$ using the principle of inclusion-exclusion. For example with $\mathcal{G} = \{G, H\}$,*

$$f_{F(n,r,\{G,H\})} = f_{F(n,r,\{G\})} + f_{F(n,r,\{H\})} - f_{F(n,r,\{\{\min\{g_i,h_i\}\}_{i=1}^{r}\})}.$$

7. Concluding Remarks

Theorems 4 and 5 each serve an important purpose. Previously, there was no known relation between n and r that guarantees that one of the eventually EKR sets classified by Barber is EKR. Theorem 4 gives such a bound. However, it is unlikely that the bound given in Theorem 4 is tight, and Theorem 5 gives a suggestion for the optimal bound.

One possible method for proving Conjecture 3 is by answering the following question, a slight variant of one posed in [2].

Question 17 *Given X, is there a short list of families, one of which maximizes $|\mathcal{A}(X)|$?*

A natural choice for such a list is $\mathfrak{A} = \{\mathcal{A}_{n,r,s}\}_{s=1}^{r}$, and this list would be especially useful because of Theorem 5. However, this does not hold in general, for example when $X = \{4, r+2\}$, $r = 5$, and $n = 11$, we have

$$\begin{aligned} |\mathcal{A}_{n,r,1}(X)| = |\mathcal{S}(X)| &= 140, \\ |\mathcal{A}_{n,r,2}(X)| &= 121, \\ |\mathcal{A}_{n,r,3}(X)| &= 136, \\ |\mathcal{A}_{n,r,4}(X)| &= 140, \\ |\mathcal{A}_{n,r,5}(X)| &= 105, \end{aligned}$$

but

$$|F(n, r, \{\{2, 3, 4\}, \{3, 4, 6, 7\}\})(X)| = 142.$$

One may still hope that $\mathfrak{A}$ provides such a list for certain values of n and r.

Another possible direction for research concerns t-intersecting families. We say a family $\mathcal{A}$ is *t-intersecting* if for any $A, B \in \mathcal{A}$, we have $|A \cap B| \geq t$.

Question 18 *Can our results be generalized to t-intersecting families?*

For t-intersecting families, [2] suggests considering

$$\mathcal{A}(s, X) = \{A \in \mathcal{A} : |A \cap X| \geq s\}$$

and asks for which X do we have $|\mathcal{A}(s, X)| \leq |\mathcal{S}_{n,r}^{t}(s, X)|$ for all compressed and t-intersecting $\mathcal{A}$, where $\mathcal{S}_{n,r}^{t} = \{A \in \binom{[n]}{r} : [t] \subset A\}$. We suspect it is possible to use similar techniques to those used in [2] and this paper to obtain partial results in this more general case.

Acknowledgements. This research was done at the University of Minnesota Duluth math REU, which is run by Joe Gallian, to whom the author is thankful. The REU was supported by the National Science Foundation and the Department of Defense (grant number 1062709) and the National Security Agency (grant number H98230-11-1-0224). The author would like to thank program advisors Eric Riedl, Davie Rolnick, and Adam Hesterberg for many helpful discussions, as well as for their comments on this paper. The author would also like to thank Duluth visitor Jonathan Wang for comments on this paper, and an anonymous referees for helpful suggestions, which simplified the proofs of Theorems 5 and 4.

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Received: January 2013