

# **Physical Meaning of Zeros and Transmission Zeros from Bond Graph Models**

by

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## **Abstract**

To achieve high performance, the integration of control and structural design of new products is necessary. Key issues for the performance are not only the closed loop poles, and the gain factor, but also the zeros of the controlled plant. Therefore, the integration requires good insight into the relationship between the zeros of the controlled plant (single input single output as well as multiple input multiple output) and the individual parameters of the structural design.

The power flow within a physical system determines its dynamic behaviour, of which zeros are an important characteristic. Therefore, the bond graph representation of the system is used to visualize the paths of power flow and the energy interacting elements. The properties of a zero are then translated to the bond graph : zeros disturb the power flow by introducing energy blocking nodes. As a consequence, some parts of the system are energetically isolated; the nodes also create constraints between some variables of the system.

Zeros turn out to be the eigenvalues of such isolated subsystems. If the bond graph contains loops, the constraints at the end of loop branches create a second category of zeros, which can also include unstable zeros. Rules are presented to identify the zero-providing subsystems from the bond graph. The knowledge about eigenvalues, together with this theory determining which part of the system provides a certain zero, enable the designer to accurately devise the structural parameters to obtain the most appropriate zeros.

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# Chapter 1

## Introduction

---

### 1.1 Purpose of Thesis

System theory often characterizes the dynamic behaviour of a linear system by its poles and zeros. Until now, poles have been of major interest in literature. They are recognized in the eigenmodes of the system. The influence of their location in the complex plane on the system's behaviour is well understood. There are two reasons for this major interest in poles. First, the capability of simple proportional control strategy to change the open-loop system poles to more appropriate locations in the closed-loop system makes the poles very accessible design parameters. The knowledge of the relationship between the poles and the overall system behaviour enables the control engineer to calculate the right control system in order to meet the required specifications. Second, the physical meaning of the poles is clear. They represent the behaviour of the autonomous system, and are often visible as the eigenfrequencies of a vibrating system. Strong mathematical methods like the modal decomposition of the system dynamics are often applied and highlight the importance of the poles for the system.

In contrast to the poles, the role of the zeros in the system and its control strategy remained more mysterious. Since the growth of interest in Multiple Input Multiple Output (MIMO) systems and their state space representation, much effort has been made to define and calculate the “transmission zeros” for such systems. Unlike the poles, a clear generalization of the Single Input Single Output (SISO) zeros, which

are the roots of the numerator of the transfer function, to MIMO zeros was not obvious. Therefore, many authors use purely mathematical concepts such as rank deficiency or generalized eigenvalue problems to define the so-called transmission zeros for MIMO systems. Though all these different definitions are consistent and give zeros the same mathematical meaning, until now there is nothing more than that abstract interpretation. All formulas state that the transmission zeros characterize the input signals that are blocked by the system; in other words, those inputs have no visible effect on the output. A more physical interpretation is lacking. Moreover, the relationship between those zeros given by the formal definitions and the physical parameters of the system, such as a spring constant or the presence of damping, is still a major issue.

On the other hand, the control system design needs a clear understanding of the influence of the zeros on the system behaviour. Presently, zeros still play a secondary role in control theory. For instance, although zeros are important to shape the trajectories in the root locus plot of a feedback system, such plots are finally meant to visualize the evolution of the pole locations under an increasing feedback gain. Many designers tend to concentrate only on these pole locations, whereas the dynamic behaviour depends on both the pole and zero locations.

The main purpose of this thesis is to present a physical interpretation of the zeros and transmission zeros of SISO and MIMO systems. That physical insight is supported by the use of “bond graphs”, which offer a graphical representation of the system based on energy flow. This thesis will show that zeros actually block the energy flow within the system at certain points, thereby creating a set of energetically separated subsystems (see chapter 3). The dynamic behaviour of these subsystems determines the zeros and the zero-dynamics. Therefore, a clear energetic representa-



tion of the system, such as the bond graphs, turns out to be a major tool in identifying the right division into subsystems that provide zeros and zero-dynamics. Rules for this subdivision will be presented.

The thesis will also study the relationship between the zeros and the overall dynamic behaviour. Zeros affect both the input of the system, which is in most physical systems some actuator effort, such as a motor torque, and the output. Some zero locations and zero dynamics can cause severe problems in control.

## 1.2 Motivation of Thesis

The control theory literature shows the importance of the zeros of the original system in different aspects. First, systems with unstable zeros (zero locations in the right half complex plane) make the control design particularly difficult. Those so-called “non minimum phase systems” have fundamental limitations on the performance of every possible controller, no matter how complex it is. Many studies now come up with exact relationships between the unstable zero location and the performance limitation [1]. For instance, Freudenberg and Looze [2] use the sensitivity function  $S(j\omega)$  as criterion for the quality of the feedback system :  $S(j\omega)$  represents the sensitivity of the output for proces noise and should be as small as possible. They show that its integral over the frequency range is equal to a certain number, determined by the unstable zero  $z = x + jy$  :

$$\pi \log \left| \prod_{i=1}^{N_p} \frac{\bar{p}_i + z}{p_i - z} \right| = \int_{-\infty}^{\infty} \log |S(j\omega)| \frac{x}{x^2 + (y - \omega)^2} d\omega \geq 0, \quad (1.1)$$

with  $p_i$  : unstable poles;  $\bar{p}_i$  : complex conjugate;  $N_p$  : number of unstable poles.

This means that, if the closed loop performance at some frequency is very good ( $S(j\omega) \ll 1$ , that must be traded off against some bad performance in other frequency ranges.

Second, most widely used control strategies are not able to change the zeros [17]. So, zeros are completely determined by the physical properties of the original system. Because of this characteristic of zeros, physical system design is the only way to choose the optimal zeros in order to achieve the desired performance of the overall system. Zeros are, therefore, one of the key issues that require an integration of both the physical system design and the control design. Both designs should interact permanently, instead of being performed in a rigid sequence (mostly the control design after the system design), assuming all results of the previous design stage as fixed. The basic insight into the meaning and the role of zeros in system dynamics, presented in this thesis, enables the designer to go back and forth much more easily between the control and the physical design aspects. The physical meaning of the zeros also provides a method to break down the system in more tractable and easier-to-calculate subsystems, which guarantees a much faster procedure to achieve a global optimum in the integrated physical and control design.

### 1.3 Previous Related Research

Almost all studies on zeros start from the definitions of zeros, based on the mathematical model of the system [13, 14, 15, 16]. For linear MIMO systems the two most often-used models are the transfer function matrix and the state space representation. Besides the comparison of different generalisations of SISO zeros to MIMO systems, those studies mainly focused on the relation between “unstable zeros” and the performance limitations on one hand [2], and on different algorithms to perform the numerically very poorly conditioned calculation of the transmission zeros of a MIMO system on the other hand [22, 23].

Recently, a few studies have tried to develop some more physical insight into the zero dynamics. In his Ph.D thesis [10], Shang-Teh Wu defined a set of rules to

derive the relative degree of a transfer function based on the causality relationship between the variables in the system (more on relative degree and its importance in subsection 2.4.5). Because the bond graph representation provides a quick and easy method to find out that causal relationship between the input, output and intermediate variables of the system, those rules are expressed in terms of that graphical representation. Together with the order of the system, the relative degree indicates the number of zeros for that particular transfer function. Based on the same causal relationship, he also worked out some sufficient conditions on the system's energetical structure and parameters in order to have a minimum-phase system. Both properties, relative degree and minimum phase, are important as most control algorithms make initial assumptions about them.

Miu [11] presented some ideas about the physical meaning of the zeros in simple mechanical systems based on frequency domain considerations. For those systems, zeros are eigenfrequencies of some subsystem. The definition of the subsystems is, however, rather heuristic, and merely uses the structural topology of the system (how the different parts of the system are physically interconnected with each other). This procedure is somewhat superficial : it has some good applications for finding the zeros of a complex structure using a finite element model, but for control purposes, the presented theory often does not work, is incomplete, or gives wrong results (see subsection 2.5.2). Nevertheless, the main idea of separating different subsystems is very worthwhile, and definitely a good contribution to the understanding of the real physical meaning of the zeros.

## 1.4 Overview of Thesis

In the second chapter, this thesis will first give an overview of the existing definitions and interpretations of zeros. Attention is paid to the relationship between the SISO

zeros and the MIMO transmission zeros. Further, the chapter focuses on the influence of control strategies on zeros, and on consequences of the zeros for the behaviour of the final system. The chapter finishes with a set of practical design problems, emphasizing the importance of good physical insight into the zeros to solve those problems.

Chapter three covers the physical interpretation for SISO systems. It proves the role of subsystems in the zero dynamics, and sets up a general procedure and rules to perform the appropriate subdivision of the original system. Some worked-out examples support the presented theory.

Chapter four generalizes the results of the previous chapter to MIMO systems. Although the same basic philosophy concerning subsystems is again applicable, the identification of the subsystems is more involved because of the multiple actuators and sensors. The special case of pole-zero cancellation turns out to have a much more natural explanation, using the presented MIMO physical interpretation, than using the mathematical definitions of poles and zeros.

Chapter five deals with systems with a loop in their energetic topology (visible in the bond graphs). Loops are one of the reasons for systems to become non minimum phase. Methods to cope with different loop configurations are presented.

## Chapter 2

# The Importance of Zeros

---

### 2.1 Introduction

Most studies in system theory and control theory concentrate on the role of the poles or eigen-frequencies in linear dynamic systems. Zeros and zero dynamics are often treated in a short side remark, so that much misunderstanding about them exists. Therefore, this chapter will present a quick overview of the current definitions and knowledge of zeros in the literature. It will also highlight the important role of the zeros of the *physical system* for different control strategies; the *physical* zeros turn out to be as important as the *physical* poles of the system. So, the design of the physical system should already take them into consideration. An example at the end of the chapter shows that recalculating the mathematical model for every alternative design, and then from that model the zeros of the system, is cumbersome. This introduces the necessity for a theory that directly relates the physical properties to the zeros of the system; such theory can form a good guideline for the integration of the physical and control design.

### 2.2 Classical Definitions of Zeros in SISO Systems

Single Input Single Output (SISO) systems are often represented by a transfer function relating the input to the output. For linear systems this transfer function can be

proved to be a rational function in the Laplace variable  $s$  :

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}. \quad (2.1)$$

Often the zeros of the system are simply defined as the roots of the numerator of the transfer function. This is a straightforward definition, easy to calculate, but without any insight in the role of the zeros in the system's dynamics.

A better understanding of the numerator's importance is possible, if we rewrite the transfer function in its original form with the Laplace transform of the output  $y(s)$  at the left hand side, and the Laplace transform of the input  $u(s)$  at the right hand side :

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0).y(s) = (b_m s^m + b_{m-1} s^{m-1} + \dots + b_0).u(s) = \mathcal{F}(s). \quad (2.2)$$

Normal mathematical studies of differential equations deal with only one forcing function  $\mathcal{F}(s)$  (right hand side of the differential equation). Equation (2.2), however, shows that in system theory the forcing function itself is a combination of the input and its derivatives. The relationship between the input  $u(s)$  and the mathematical forcing function  $\mathcal{F}(s)$  is called the “zero dynamics” of the system. These zero dynamics act as a prefilter on the input  $u(s)$  before it is fed to the actual system's dynamics.

Whereas the notion of zero dynamics is also applicable to non-linear systems, the term “zero” makes sense only in the linear case. A short paragraph about the solutions of linear differential equations shows this.

The basic property of a linear differential equation (LDE) is the very simple form of its eigen-functions; all exponentials are eigen-functions for any LDE :

$$\begin{aligned} (b_m s^m + b_{m-1} s^{m-1} + \dots + b_0).Ae^{\alpha t} &= (b_m \alpha^m + b_{m-1} \alpha^{m-1} + \dots + b_0).Ae^{\alpha t} \\ (b_m s^m + b_{m-1} s^{m-1} + \dots + b_0).Ae^{\alpha t} &= \lambda.Ae^{\alpha t} \end{aligned} \quad (2.3)$$

$$(b_m \alpha^m + b_{m-1} \alpha^{m-1} + \dots + b_0) = \lambda. \quad (2.4)$$

These exponential eigen-functions can span the whole space of functions, which in system theory always represent some real signal. In fact, the meaning of the Fourier and the Laplace transform is nothing more than a decomposition of a function  $f(t)$  into this function-basis of exponentials :

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega).e^{j\omega t}.d\omega \quad (2.5)$$

$$= \frac{1}{2\pi j} \int_{-\infty}^{\infty} F(s).e^{st}.ds. \quad (2.6)$$

This decomposition is simple, as exponential signals form an orthogonal basis for the total signal space. Therefore, from now on all signals are considered exponential when studying linear systems.

On each of these exponential signals, the zero dynamics will have a different effect, uniquely characterized by the eigenvalue  $\lambda(z)$ . The output of the zero dynamics under an input  $e^{zt}$  is a constant factor times the input :  $\lambda(z)e^{zt}$ . Of particular importance is the finite number of exponential signals  $Ae^{zt}$  with a zero eigenvalue :

$$\lambda = b_m z^m + b_{m-1} z^{m-1} + \dots + b_0 = 0. \quad (2.7)$$

Their exponent  $z$  is called a “zero” of the zero dynamic. As equation (2.7) shows, this definition is compatible with the original one considering the roots of the numerator. But it explains much more clearly the meaning of the “zeros” for the considered input and output signals : zeros identify those exponential signals that are completely blocked by the zero dynamics. The resulting forcing function is identically zero. If in addition the system is stable, and all initial states are zero, then also the output of the system will be identically zero.

## 2.3 Classical Definitions of Zeros in MIMO Systems

Multiple Input Multiple Output (MIMO) systems are dynamic systems with  $m$  inputs and  $l$  outputs. The two most common mathematical representations of linear MIMO systems are the transfer function matrix and the state equations. Each element of the  $l \times m$  transfer function matrix,  $\mathbf{H}(s)$ , is a rational transfer function (with numerator and denominator) :

$$\mathbf{H}(s) = \begin{bmatrix} \frac{\alpha_{11}(s)}{\beta_{11}(s)} & \cdots & \frac{\alpha_{1m}(s)}{\beta_{1l}(s)} \\ \vdots & & \vdots \\ \frac{\alpha_{l1}(s)}{\beta_{l1}(s)} & \cdots & \frac{\alpha_{lm}(s)}{\beta_{lm}(s)} \end{bmatrix}. \quad (2.8)$$

The whole transfer function matrix completely shows the mapping of the input signals into the output signals, considering the dynamic system as a black box :

$$\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s), \quad (2.9)$$

where  $\mathbf{u}(t)$  is the input vector and  $\mathbf{y}(t)$  is the output vector.

In contrast, the state equations start from the dynamics of the system themselves and describe how these are coupled to the  $m$  inputs and  $l$  outputs :

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned} \quad (2.10)$$

where  $\mathbf{x}(t)$  is the state vector;  $\mathbf{A}$  is an  $n \times n$  matrix;  $\mathbf{B}$  an  $n \times m$  matrix;  $\mathbf{C}$  an  $l \times n$  matrix; and  $\mathbf{D}$  an  $l \times m$  matrix.

The relationship between the state equations and the transfer function matrix is :

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}. \quad (2.11)$$

Subsections 2.3.1 to 2.3.4 present the theory about zeros of MIMO systems, using the transfer function matrix description. The following subsection starts from the state space representation.



### 2.3.1 The Basic Rank-Deficiency Zero Definition

The discussion in the previous section about SISO zeros first formulates the practical rule for SISO zeros and then works out the deeper meaning of that rule from the signal level point of view. Zeros are characteristics for those input signals that can be totally blocked by the system's dynamics. The extension of the practical rule to the MIMO case can serve as a first trial for a MIMO zeros definition.

In that approach, a partial MIMO zero  $z_p$  is a root of one numerator  $\alpha_{ij}(s)$  of the transfer function matrix of equation (2.8). A global MIMO zero  $z_g$  is a common root of all numerators  $\alpha_{ij}(s)$ , making the transfer function matrix  $\mathbf{H}(s)$  equal to the zero matrix. Neither definition is of a practical use : the former denies the MIMO character of the system by focusing on one input-output pair, whereas the latter seems to be far too strong, so that almost no system has a global zero.

To find an “intermediate” definition, a reasoning on signal level, similar to that in the SISO case, is needed. For multiple input systems, however, an exponential signal

$$\mathbf{u}(t) = \mathbf{u}e^{zt} \quad (2.12)$$

is characterized not only by the time exponent  $z$ , but also by the “direction” of the vector  $\mathbf{u}$ . That direction represents the proportion in amplitude of the individual input signals. Since a “zero” should completely characterize an input signal blocked by the internal dynamics, it encompasses both the time exponent  $z$  as the direction vector  $\mathbf{u}$ . The Laplace transform of such an input is :

$$\mathcal{L}(\mathbf{u}(t)) = \frac{\mathbf{u}}{s - z}. \quad (2.13)$$

The Laplace transform of the output, when the initial state is zero, is :

$$Y(s) = \mathbf{H}(s) \frac{\mathbf{u}}{s - z}. \quad (2.14)$$

Partial fraction expansion into the  $n$  factors  $(s - p_i)$  of the common denominator of  $\mathbf{H}(s)$  and  $(s - z)$  yields :

$$Y(s) = \frac{\mathbf{H}(s)|_{s=z} \mathbf{u}}{s - z} + \sum_{i=1}^n \frac{\alpha_i}{s - p_i}, \quad (2.15)$$

where  $\alpha_i$  are the residues of the transients. The time domain expression of the output is :

$$\mathbf{y}(t) = \mathbf{H}(s)|_{s=z} \mathbf{u} e^{zt} + \sum_{i=1}^n \alpha_i e^{p_i t}. \quad (2.16)$$

A transmission zero  $z$  is defined as those values  $z$  for which the output only contains transients. That means that the time evolution of the input,  $e^{zt}$ , is not visible at the output. Therefore, the requirement for  $z$  and  $\mathbf{u}$  to form a zero input (further written as  $(z, \mathbf{u})$ ) is :

$$0 = \mathbf{H}(s)|_{s=z} \mathbf{u}, \quad (2.17)$$

since exponentials are always positive. Linear Algebra theory interprets this requirement as the matrix  $\mathbf{H}(s)|_{s=z}$  having a non zero nullspace, or equivalently,  $\mathbf{H}(s)|_{s=z}$  having a rank  $< m$ .

A special remark is needed for systems with more inputs than outputs ( $m > l$ ), for which  $\mathbf{H}(s)|_{s=z}$  has a non zero nullspace for all  $z \in \mathcal{C}$ . Though such systems are uncommon, the general definition of a zero will remove this triviality by requiring the rank<sup>1</sup> of  $\mathbf{H}(s)|_{s=z}$  to be lower than the maximal possible rank of  $\mathbf{H}(s)$  for any  $s \in \mathcal{C}$ .

Thus, the rank-deficiency definition is

$\mathbf{u} e^{zt}$  is a zero input of a MIMO system  $\mathbf{H}(s)$  if and only if

1.  $\mathbf{H}(s)|_{s=z}$  is rank deficient.

---

<sup>1</sup>The rank of  $\mathbf{H}(s)|_{s=z}$ , i.e. when  $\mathbf{H}(s)$  is evaluated at  $s = z$ , is often referred to as the “local rank”

2.  $\mathbf{u}$  belongs to the nullspace of  $\mathbf{H}(s)|_{s=z}$ , which means :

$$\mathbf{H}(s)|_{s=z} \cdot \mathbf{u} = [\mathbf{h}_1 \mathbf{h}_2 \dots \mathbf{h}_m] \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad (2.18)$$

$$= u_1 \mathbf{h}_1 + u_2 \mathbf{h}_2 + \dots + u_m \mathbf{h}_m, \quad (2.19)$$

$$= \mathbf{0}. \quad (2.20)$$

This is the definition of the so-called “transmission zeros”  $(z, \mathbf{u})$  of MIMO systems, describing input signals that are not “transmitted” to the output by the system’s dynamics. Obviously, the higher defined global zero  $z_g$  is a special case of transmission zero, with  $\text{rank}(\mathbf{H}(s)|_{s=z_g})=0$ . The multiplicity of a transmission zero  $z$  is equal to the difference between the local rank of  $\mathbf{H}(s)|_{s=z}$  and the maximal rank of  $\mathbf{H}(s)$  [19]

The calculation of the zero inputs of a system consists of two steps, like the definition above :

1. Find the different complex values  $z$  for which  $\mathbf{H}(s)|_{s=z}$  is rank deficient.
2. Calculate the basis vectors of the nullspace of  $\mathbf{H}(s)|_{s=z}$ .

Of both steps, the first one is the more difficult, so that the literature mainly concentrates on that problem within the calculation of zeros. The following paragraphs give some methods for the calculation of  $z \in \mathcal{C}$  making  $\mathbf{H}(s)|_{s=z}$  rank deficient.

### 2.3.2 Calculation via the Smith Mc Millan Form

Like the eigenvalue decomposition and the singular value decomposition of real matrices, the Smith Mc Millan Form is a canonical form for rational function matrices. Similar to the other decompositions, the composing matrices provide a great deal of information about the original matrix or system, such as the transmission zeros.

The decomposition of a rational transfer function matrix  $\mathbf{H}(s)$  consists of two steps :

1. Find the least common denominator  $d(s)$  of all the elements of  $\mathbf{H}(s)$  :

$$\mathbf{H}(s) = \frac{1}{d(s)} \mathbf{N}(s), \quad (2.21)$$

where  $d(s)$  is a polynomial in  $s$  and  $\mathbf{N}(s)$  a polynomial matrix in  $s$ .

2. Decompose  $\mathbf{N}(s)$  in :

$$\mathbf{N}(s) = \mathbf{L}^{-1}(s) \mathbf{S}(s) \mathbf{R}^{-1}(s), \quad (2.22)$$

where  $\mathbf{L}^{-1}(s)$  and  $\mathbf{R}^{-1}(s)$  are the inverses appropriate unimodular matrices and  $\mathbf{S}(s)$  has only diagonal non zero elements. Unimodular matrices  $\mathbf{X}(s)$  are matrices that do not lose rank for any value of  $s$ .

The Smith Mc Millan Form is then defined as :

$$\mathbf{M}(s) = \frac{\mathbf{S}(s)}{d(s)}, \quad (2.23)$$

where  $\mathbf{M}(s)$  has the form

$$\mathbf{M}(s) = \begin{bmatrix} \bar{\mathbf{M}}(s) & \mathbf{0}_{r,l-r} \\ \mathbf{0}_{m-r,r} & \mathbf{0}_{m-r,l-r} \end{bmatrix} \quad (2.24)$$

with

$$\bar{\mathbf{M}}(s) = \text{diag} \left\{ \frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, \dots, \frac{\varepsilon_r(s)}{\psi_r(s)} \right\}. \quad (2.25)$$

This diagonal matrix  $\bar{\mathbf{M}}(s)$  reveals both the poles and zeros of the system represented by  $\mathbf{H}(s)$ . Define the “pole polynomial”  $p(s)$  as :

$$p(s) = \prod_{i=1}^r \psi_i(s), \quad (2.26)$$

and the “zero polynomial”  $z(s)$  as

$$z(s) = \prod_{i=1}^r \varepsilon_i(s). \quad (2.27)$$

The roots of these polynomials are respectively the poles and zeros of the system with appropriate multiplicity.

The following equations give an example of the Smith Mc Millan Form for an arbitrary transfer function matrix :

$$\mathbf{H}(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{s^2+s-4}{(s+1)(s+2)} & \frac{2s^2-s-8}{(s+1)(s+2)} \\ \frac{s^2-4}{(s+1)(s+2)} & \frac{2s^2-8}{(s+1)(s+2)} \end{bmatrix} \quad (2.28)$$

$$\mathbf{M}(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{(s-2)}{(s+1)} \\ 0 & 0 \end{bmatrix}. \quad (2.29)$$

So, this transfer function matrix  $\mathbf{H}(s)$  represents a system with one zero  $z = 2$  and two poles  $p_1 = -1$ , with multiplicity 2, and  $p_2 = -2$ , with multiplicity 1.

Though the Smith Mc Millan Form seems a good alternative for the original rank deficiency definition of transmission zeros, the practical use of it is rather limited, since the calculation of the matrices  $\mathbf{R}(s)$  and  $\mathbf{L}(s)$  is not straightforward.

### 2.3.3 Calculation via Minors

Another method to calculate the polynomials  $p(s)$  and  $z(s)$  from equations (2.26) and (2.27) uses the minors of  $\mathbf{H}(s)$ . A minor of order  $n$  is the determinant of a square  $n \times n$  matrix obtained by striking out an appropriate number of columns and rows in  $\mathbf{H}(s)$ . The results are:

1. The pole polynomial  $p(s)$  is the least common denominator of all non-identically zero minors of order 1 to  $\max(l, m)$  of  $\mathbf{H}(s)$ .

2. The zero polynomial  $z(s)$  is the greatest common divisor of the numerators of all minors of  $\mathbf{H}(s)$  of order  $r$  (the normal rank of  $\mathbf{H}(s)$ ), provided that these minors have all been adjusted in such a way that the pole polynomial  $p(s)$  is their common denominator.

In the following example, the normal rank  $r$  of  $\mathbf{H}(s)$  is 2,

$$\mathbf{H}(s) = \frac{1}{(s+1)(s+2)(s-1)} \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix}. \quad (2.30)$$

The minors of order 2 are :

$$m_1 = \frac{1}{(s+1)(s+2)} \quad m_2 = \frac{2}{(s+1)(s+2)} \quad m_3 = \frac{-(s-1)}{(s+1)(s+2)^2}. \quad (2.31)$$

Consequently, the pole polynomial is

$$p(s) = (s+1)(s+2)^2(s-1). \quad (2.32)$$

The adjusted minors of order 2 are :

$$m_1 = \frac{(s-1)(s+2)}{p(s)} \quad m_2 = \frac{2(s-1)(s+2)}{p(s)} \quad m_3 = \frac{-(s-1)^2}{p(s)}, \quad (2.33)$$

so that the zero polynomial is :

$$z(s) = (s-1). \quad (2.34)$$

### 2.3.4 Desoer-Schulman Definition of Zeros

Desoer and Schulman [14] suggest another decomposition of the transfer function matrix, which comes up with a closer relationship between the MIMO and the SISO case.

$\mathbf{H}(s)$  is decomposed into a product of a polynomial matrix  $\mathbf{N}_l(s)$  (or  $\mathbf{N}_r(s)$ ), with similar dimensions  $l \times m$  and the inverse of a square  $l \times l$  matrix  $\mathbf{D}_l(s)$  (or square

$m \times m$  matrix  $\mathbf{D}_r(s)$  :

$$\mathbf{H}(s) = \mathbf{D}_l(s)^{-1} \cdot \mathbf{N}_l(s) = \mathbf{N}_r(s) \cdot \mathbf{D}_r(s)^{-1}. \quad (2.35)$$

Both  $\mathbf{N}$  and  $\mathbf{D}$  matrices are left, respectively right, coprime. This means that there exist polynomials  $\mathbf{P}(s)$  and  $\mathbf{Q}(s)$  so that :

$$\mathbf{N}_l(s) \cdot \mathbf{P}(s) + \mathbf{D}_l(s) \cdot \mathbf{Q}(s) = \mathbf{I}_m, \quad (2.36)$$

or

$$\mathbf{P}(s) \cdot \mathbf{N}_r(s) + \mathbf{Q}(s) \cdot \mathbf{D}_r(s) = \mathbf{I}_l. \quad (2.37)$$

Apparently, in these definitions  $\mathbf{N}_l(s)$  or  $\mathbf{N}_r(s)$  play the role of numerator and  $\mathbf{D}_l(s)$  and  $\mathbf{D}_r(s)$  the role of denominator from the SISO case.

Using this decomposition, a transmission zero is the complex value  $z$  making

$$\text{rank}(\mathbf{N}_l(s)|_{s=z}) < \min(m, l). \quad (2.38)$$

The right hand side is the maximal rank of  $\mathbf{N}_l(s)$ , as the authors assume that all the inputs and all the outputs are independent. Though this definition maintains a rank deficiency as well, the matrices involved are only polynomial matrices instead of rational matrices. The importance is rather the interpretation the authors give for the zero input. Especially for the uncommon case with more inputs than outputs, they give following explanation of the zero input : as explained before, it makes no sense for such systems to require the output to be identically zero, since that is achievable for every value  $z \in \mathcal{C}$ . For the  $z$ 's making the transfer function matrix rank deficient, however, there exists an linear combination of the output  $\mathbf{y}(t)$ .

$$\psi(t) = \mathbf{c}^T \cdot \mathbf{D}_l(z) \cdot \mathbf{y}(t) \equiv 0, \quad (2.39)$$

which is identically zero for every input of the form

$$\mathbf{u}(t) = \mathbf{g}e^{zt} \quad (2.40)$$

with an *arbitrary* vector  $\mathbf{g}$ . The required initial conditions depend on the chosen vector  $\mathbf{g}$ .

The similarity with the SISO case makes the Desoer Schulman representation of zeros interesting : if the role of inputs and outputs is reversed, the zeros play an important role in the dynamics of the system, like the poles do in the regular system.

### 2.3.5 State Space Definition of MIMO Zeros

The state space description of equations (2.10) of a system provides another way to calculate a zero input  $(z, \mathbf{u})$ . Moreover, it will also yield the initial state  $\mathbf{x}_0$  necessary to suppress the transients in the output. So, the zero input is defined by its exponent  $z$ , the input vector  $\mathbf{u}$  and the initial state  $\mathbf{x}_0$  (written  $(z, \mathbf{u} \ \mathbf{x}_0)$ ).

The basic requirement in the calculation is the identically zero output vector  $\mathbf{y}(t)$ . The derivation below considers a proper<sup>2</sup> system  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$ . Mc Farlane [13] and Wolovich [18] do a similar derivation for a non proper system. Suppose the input is exponential (section 2.2 in this chapter explains why an exponential signal is general enough for a linear system) with the following form :

$$\mathbf{u}(t) = \mathbf{u}e^{zt}. \quad (2.41)$$

Its Laplace transform is :

$$U(s) = \frac{\mathbf{u}}{s - z}. \quad (2.42)$$

To obtain an identically zero output, the following equations must hold :

1. at  $t = 0$  the state  $\mathbf{x} = \mathbf{x}_0$ , so

$$\mathbf{C}.\mathbf{x}_0 = \mathbf{0}. \quad (2.43)$$

---

<sup>2</sup>proper means that the number of poles  $n \geq m$  the number of zeros; strictly proper means  $n > m$  and non-proper means  $n < m$



2. for  $t > 0$  :  $y(t) \equiv 0 \Rightarrow Y(s) = 0$ ,

$$0 = C.(sI - A)^{-1}.x_0 + C.(sI - A)^{-1}.B.\frac{u}{s - z} \quad (2.44)$$

$$= C.(sI - A)^{-1}.[(sI - zI).x_0 + Bu].\frac{1}{s - z} \quad (2.45)$$

$$= C.(sI - A)^{-1}.[(sI - A + A - zI).x_0 + Bu].\frac{1}{s - z} \quad (2.46)$$

$$= \underbrace{\frac{C.x_0}{s - z}}_{=0} + C.(sI - A)^{-1}.\underbrace{[(A - zI).x_0 + Bu]}_{\text{independent from } s}.\frac{1}{s - z} \quad (2.47)$$

$$= (A - zI).x_0 + Bu. \quad (2.48)$$

Equations (2.43) and (2.48) are often combined in a generalized eigenvalue problem :

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u \end{pmatrix} = z \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u \end{pmatrix}. \quad (2.49)$$

This generalized eigenvalue problem can also be reformulated as a rank deficiency condition, by subtracting the left from the right hand side. This is also the definition derived in [17] :

$$\text{rank} \begin{bmatrix} A - zI & B \\ C & 0 \end{bmatrix} < n + \min(l, m). \quad (2.50)$$

Sometimes this matrix is called the “system matrix” [16]. From this rank deficiency condition, the transmission zeros are the roots of the greatest common divisor of all  $(n + \min(l, m)) \times (n + \min(l, m))$  minors of the system matrix.

Moreover, Davison [19] shows that the multiplicity of the transmission zeros in these state space methods is the same as that in the zero polynomial  $z(s)$  (equation (2.27)) of the Smith Mc Millan method.

### 2.3.6 The Numerical Calculation of Transmission Zeros

The numerical computation of zeros is very unstable. For instance, only small changes in the numerical values of the matrices in the state space representation can change the number of zeros. The calculation of the rank of a matrix has to cope with similar

problems. Zeros at infinity constitute a particular problem in all zero calculations. Therefore, most calculation schemes first transform the given system in another form with the same finite transmission zeros, but which does not cause problems with zeros at infinity. Maciejowski [22] presents complicated algorithms that avoid most numerical difficulties. Another strategy of Van Dooren [23] works in two phases : first perform symbolic operations to obtain a polynomial, then calculate numerically the zeros as its roots .

All these calculations, however, start from the mathematical representation, in most cases the state space matrices. The more physical methods of next chapters provide a complete different path, using the well-known and stable algorithms for pole calculation on subsystems of the system. Consequently, a better physical insight can even have a contribution in a very abstract mathematical field !

## **2.4 Zeros and Control**

### **2.4.1 Relationship of Zeros with Input-Output Response**

Control theory studies how to activate a certain system to obtain first stable dynamics and second a nice input-output relationship. For the stability issue, poles are of major importance : all system poles should lie in the left half complex plane. However, the input-output relationship is determined by both the poles and the zeros, although also in this field most attention is paid to the role of the poles.

Studies of the input-output relationship of a system often use the step response as sample. The optimal response should follow the input step as close as possible, so that the effect of the internal dynamics is almost invisible. To qualify different step responses, three parameters are used as gauge : the rise time, the settling time and the overshoot. They have following definitions :

**rise time** the time needed for the response to reach the steady state value for the first time. Sometimes, the definition is slightly modified to the time needed to go from 10 % of the initial condition to 90 % of the steady state value.

**settling time** the time needed for the response before it completely stays within a narrow band of 2 % around the steady state value

**overshoot** the maximum peak above the steady state value

Figure 2.1 shows these parameters on a sample step response. A good step response behaviour is a trade-off between those criteria.

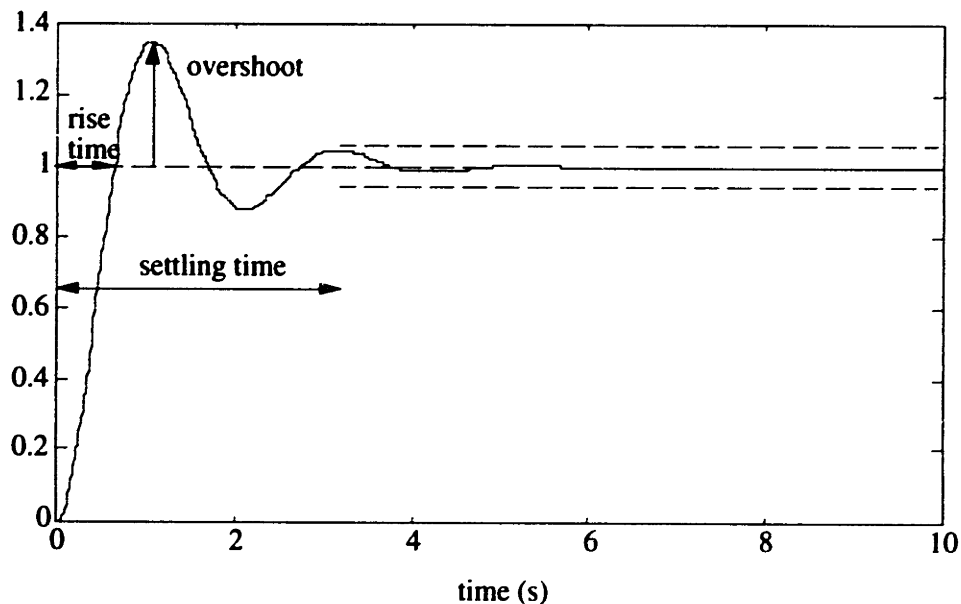


Figure 2.1: Definition of rise time, settling time and overshoot of a step response

A decomposition of the step response in different “modes” will help to derive basic rules for how the relative location of the poles and zeros in the complex plane influence the step response criteria. Consider first a system without zeros and with

distinct, stable poles. Its transfer function has following form :

$$H(s) = \frac{a}{(s - p_1)(s - p_2) \dots (s - p_n)}. \quad (2.51)$$

In a normalised system, the steady state value of the step response is 1. That requires the numerator to be :

$$a = \prod_{i=1}^n p_i. \quad (2.52)$$

The Laplace transform of the step response  $y(t)$  is

$$Y(s) = \frac{a}{s(s - p_1)(s - p_2) \dots (s - p_n)}, \quad (2.53)$$

which can be decomposed into a sum, using the partial fraction expansion,

$$Y(s) = \frac{1}{s} + \sum_{i=1}^n \frac{\alpha_i}{s - p_i}. \quad (2.54)$$

The numerators  $\alpha_i$  are the “residues” of the corresponding poles  $p_i$

$$\alpha_i = \frac{a}{p_i(p_i - p_1) \dots (p_i - p_{i-1})(p_i - p_{i+1}) \dots (p_i - p_n)}. \quad (2.55)$$

If necessary, the terms of complex conjugate poles can be combined into a second order term to avoid complex numbers in the expression. Those second order terms are then characterized by their natural frequency  $\omega_n$  and the damping ratio  $\zeta$ .

The partial fraction expansion of equation (2.54) leads to following conclusions :

1. The output consists of a unit step and  $n$  exponential terms. The slowest pole (closest to the origin)  $p_j$  is dominant in the output, as it will still have effect when all the other terms are already damped out. The output is then  $\approx 1 + \alpha_j \cdot e^{-p_j t}$ . As long as there is a much slower pole, the output is sluggish, and speeding up the fast poles will result in only a slight improvement of the response.
2. The residue of the slowest pole is always negative and in absolute value larger than 1. Consequently, the steady state response is approached from below.

Figure 2.2 shows the step response of a system with three poles : -1, -5 and -6.

The response  $y(t)$  can be written as :

$$y(t) = (1 - \frac{3}{2}e^{-t}) + \frac{3}{2}e^{-5t} - e^{-6t}. \quad (2.56)$$

The first term contains the steady state value and the transient of the slowest pole and is called  $q(t)$ , so

$$y(t) = q(t) + \frac{3}{2}e^{-5t} - e^{-6t}. \quad (2.57)$$

Figure 2.2 shows that  $q(t)$  (dashed line) is very close to  $y(t)$  (full line) when the faster transients  $\frac{3}{2}e^{-5t}$  and  $e^{-6t}$  are damped out (after 1 second).

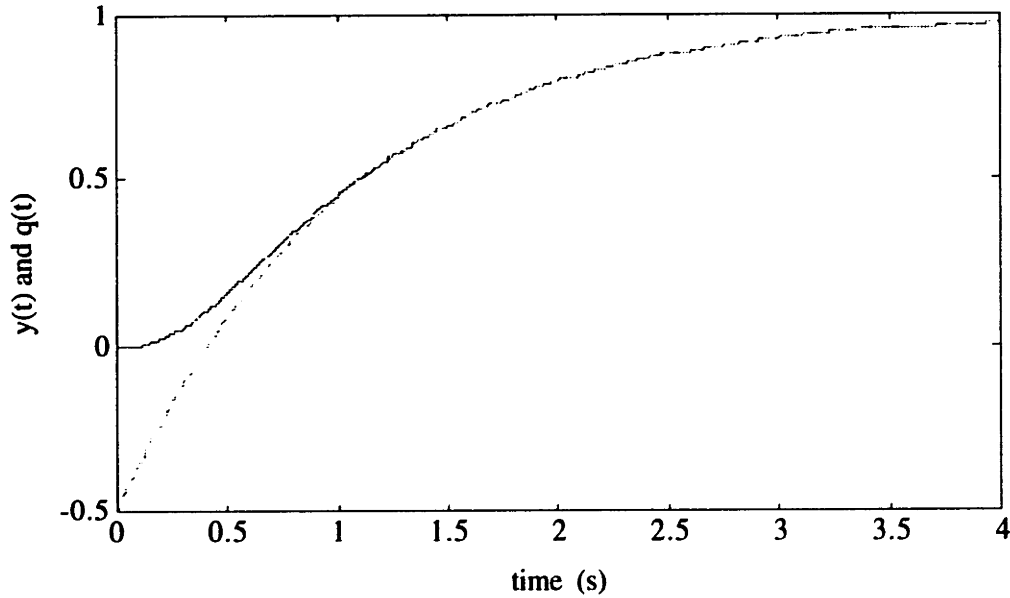


Figure 2.2: Step response : total response  $y(t)$  : full line;  $q(t)$  : dashed line

The introduction of a zero  $z$  in the system will not introduce other exponential functions in the step response, but changes the residues  $\alpha_i$ . The transfer function looks like :

$$H(s) = \frac{a.(s - z)}{(s - p_1)(s - p_2)...(s - p_n)}. \quad (2.58)$$

Equation (2.54) still gives the composition of the step response, but with :

$$\alpha_i = \frac{a.(p_i - z)}{p_i(p_i - p_1) \dots (p_i - p_{i-1})(p_i - p_{i+1}) \dots (p_i - p_n)}. \quad (2.59)$$

The discussion above shows that of all the residues, that of the slowest pole (take  $p_j$ ) is of major interest for the step response : after a certain time, the step response  $y(t) \approx 1 + \alpha_j e^{p_j t}$ . Whereas  $\alpha_j$  in a system without zeros is always smaller than -1, it can now take every value, even 0, depending on the relative location of the zero. Equation (2.59) has the following meaning : if the zero  $z$  coincides with the slowest pole  $p_j$  (pole-zero cancellation) or is very close to it, its sluggish influence on the step response is almost completely reduced. Also, the sign of  $\alpha_j$  depends on the relative location : if the zero is left from pole  $p_j$ , then  $\alpha_j < 0$ , if the zero is right from  $p_j$ ,  $\alpha_j > 0$ . The sign of  $\alpha_j$  tells the approach direction : if  $\alpha_j < 0$ , the response, close to  $1 + \alpha_j e^{p_j t}$ , tends to 1 from values such as 0.95. In contrast, a positive value of  $\alpha_j$  makes the response  $1 + \alpha_j e^{p_j t}$  settle down to 1 from higher values such as 1.05. Consequently, some overshoot will take place.

Besides the capability to cancel the sluggish influence of the slowest pole, overshoot is the second major feature of a zero. One way to show the overshoot effect of a zero is to write the step response as the linear combination of the step response  $y(t)$  of the system without the zero and its derivative  $\dot{y}(t)$  (equation (2.60) substitutes the denominator of Laplace transform (2.53) to  $d(s)$ ) :

$$\frac{(z + s).a}{d(s)} = \frac{(1 + \frac{s}{z}).z.a}{d(s)} = y(s) + s \cdot \frac{a}{d(s)} = \mathcal{L} \left( y(t) + \frac{1}{z} \cdot \dot{y}(t) \right). \quad (2.60)$$

Figure 2.3 shows a typical step response and its derivative (which is actually the impulse response of the system without the zero). Since the response  $y(t)$  is mainly determined by its slowest pole, the peak value of the impulse response has magnitude of order  $|p_j|$ ; that makes the added peak value to the original step response approximately equal to  $\left| \frac{p_j}{z} \right|$ . Like the discussion about the residues, this result shows that

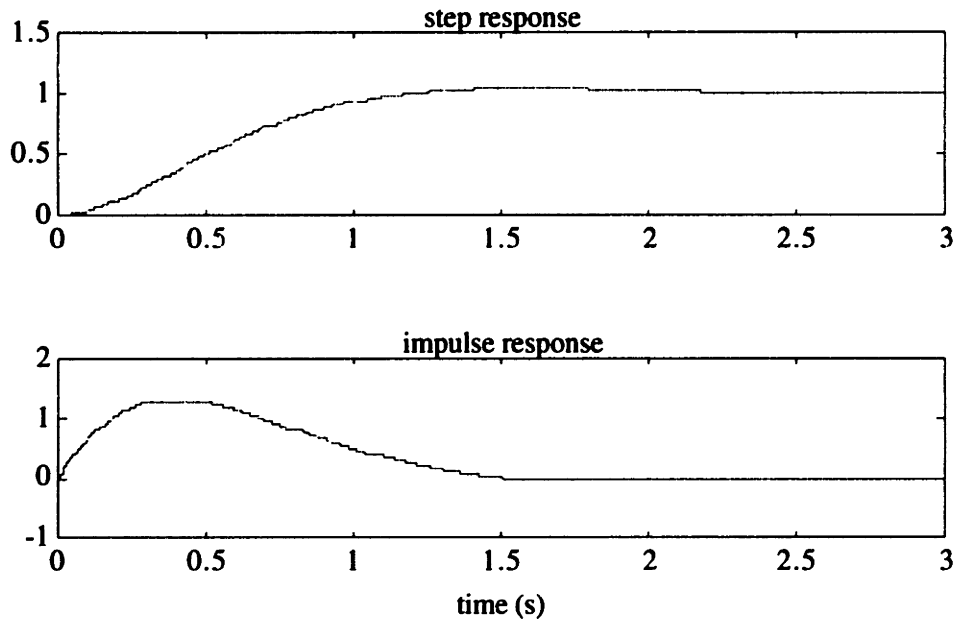


Figure 2.3: A typical step response and its derivative (poles  $2 \pm 2j$ )

a zero right of the slowest pole yields high overshoots. Figure 2.4 gives the resultant step response by adding a zero -1 to the system of figure 2.3 :

Finally, following rules of thumb give some insight in the influence of the zero location on the step response :

1. A zero close to the slowest pole makes the step response faster
2. A zero far to the right in the complex left half plane compared to the poles yields high overshoots and consequently worse settling times.
3. A zero far to the left compared to the poles has almost no influence on the step response

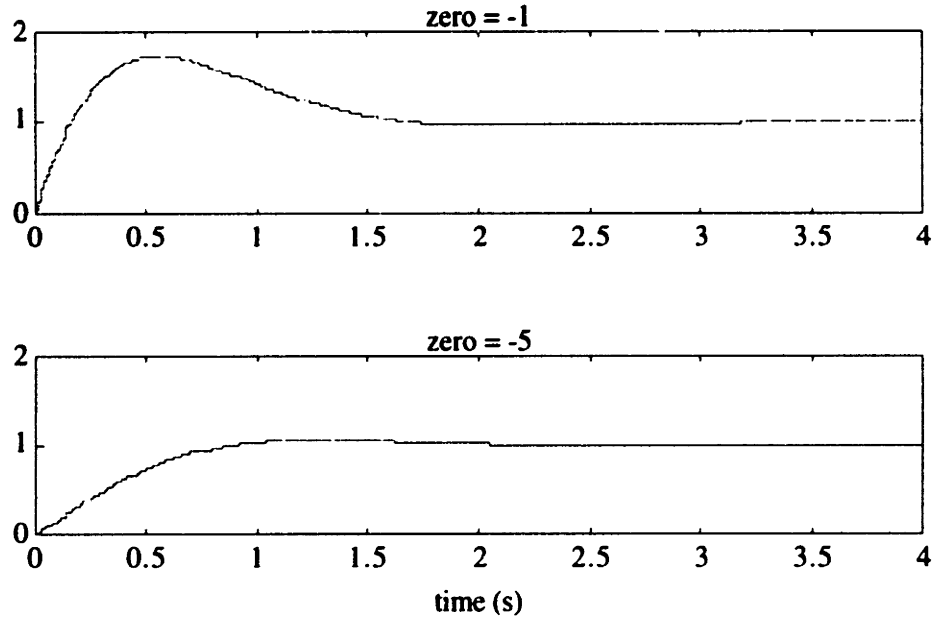


Figure 2.4: The step response for system with poles  $2 \pm 2j$  and zero a)  $-1$  and b)  $-5$

### 2.4.2 Feedback and Zeros

Feedback is one of the major control techniques and has different variants. After the previous discussion about the impact of the zeros of a given system on its input-output characteristics, this subsection looks into the interaction of the zeros of the original system and different feedback techniques.

#### State Feedback and Output Feedback

Since output feedback is a special case of state feedback, both can be considered at the same time. The previous section derives a formula for the transmission zeros of a MIMO system in state space representation, involving the rank of the so-called system matrix

$$\begin{bmatrix} \mathbf{A} - z\mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}. \quad (2.61)$$



The overall state space representation of a system  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$  under state feedback with gain-matrix  $\mathbf{K}$  is  $(\mathbf{A} - \mathbf{BK}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ . Davison and Wang [17] proved that the rank of the system matrix does not alter under state feedback. In general they showed that :

$$\text{rank} \begin{bmatrix} \mathbf{A} - z\mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{T}(\mathbf{A} + \mathbf{BK} - z\mathbf{I})\mathbf{T}^{-1} & \mathbf{TBG} \\ (\mathbf{C} + \mathbf{DK})\mathbf{T}^{-1} & \mathbf{DG} \end{bmatrix}. \quad (2.62)$$

with :  $\mathbf{T} : n \times n$  nonsingular transformation matrix

$\mathbf{G} : m \times m$  nonsingular transformation matrix

$\mathbf{K} : m \times n$  feedback gain matrix

This means that transmission zeros  $z$  are not affected by :

1. state coordinate transformations  $\mathbf{T}$ ,  $\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$ .
2. control input transformations  $\mathbf{G}$ ,  $\mathbf{u} = \mathbf{G}\bar{\mathbf{u}}$ .
3. state feedback  $\mathbf{K}$ .

Consequently, the state feedback control designer should be aware that, in contrast to the flexibility to change the poles of the system, the transmission zeros will stay where they are. So, state feedback is not a solution if some zeros are in bad positions (unstable zeros or lightly damped zeros). Moreover, the control designer should pay attention to the relative location of poles and zeros, while moving around the poles, since previous subsection shows that plays an important role in the behaviour of the system as well.

The following example considers a system with poles at -3 and -4 and a zero at -2. As the control designer wants to speed up the response of this system, he suggests to move the poles via state feedback more to the left in the complex plane, e.g. new poles at -7 and -7. Figure 2.5, however, shows that the result on the step response is rather disappointing. Since the zero remains at its original location, the poles are much further to the left of the zero, which introduces a big overshoot.

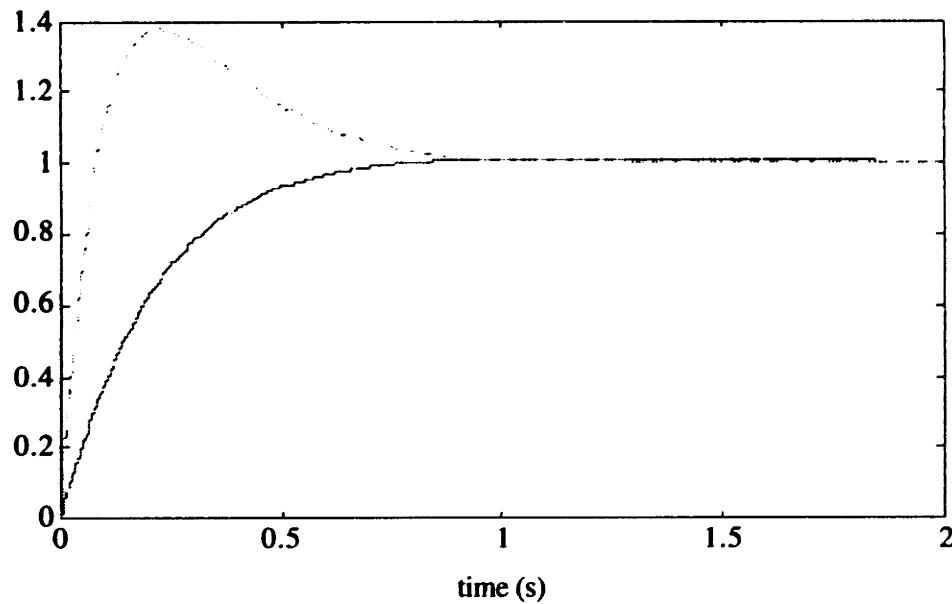


Figure 2.5: Step responses of : zero = -2.5 full line: poles = -3,-4 dashed line : poles = -7,-7

### Classical Controller Design

The configuration of the classical feedback control is different from that of the state feedback : it first compares the actual output with the desired output and then sends the error through a dynamic compensator or controller. The output of the controller is the actuator signal  $u$ . Figure 2.6 shows this setup.

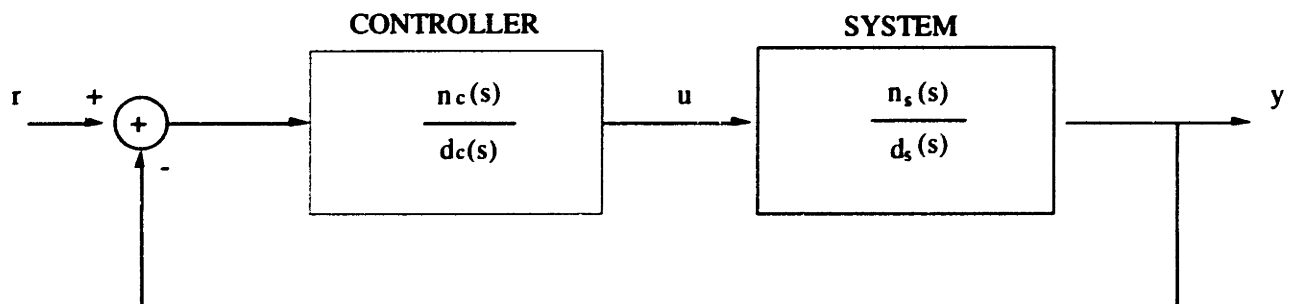


Figure 2.6: The classical control configuration

As the controller is now in the forward loop between  $r$  and  $y$ , it can add zeros to the closed loop transfer function between  $r$  and  $y$  :

$$H_d(s) = \frac{n_c(s).n_s(s)}{n_c(s).n_s(s) + d_c(s).d_s(s)}, \quad (2.63)$$

with :  $n_c(s)$  and  $d_c(s)$  : numerator and denominator of the controller

$n_s(s)$  and  $d_s(s)$  : numerator and denominator of the system

$H_d(s)$  : closed loop transfer function

This controller configuration can also remove stable zeros (in Left Half Complex Plane) from the closed loop transfer function. To that end, a pole zero cancellation should take place between the controller and the system. For SISO systems such cancellation only requires a same pole in the controller as the zero in the system :

$$H_d(s) = \frac{n_c(s).\tilde{n}_s(s).(s - z)}{n_c(s).[\tilde{n}_s(s).(s - z)] + [\tilde{d}_c(s).(s - z)].d_s(s)}. \quad (2.64)$$

In MIMO systems the pole-zero cancellation also requires the matching of the following “directions” :

$$\mathbf{C}.\mathbf{v}_p = \mathbf{u}_z, \quad (2.65)$$

with :  $\mathbf{C}$  : state-to-output matrix in state space representation

$\mathbf{v}_p$  : right eigenvector of the pole in the controller

$\mathbf{u}_z$  : input direction of the zero in the system

Subsection 3.7.1 will study the pole-zero cancellation in more detail.

In summary, the classical feedback control configuration has the power to both add and remove zeros from the closed loop transfer function. However, following subsections will show the drawbacks of this classical control, which make it less interesting for practical problems.

### 2.4.3 Duality between Zeros and Poles

The transfer function of a linear SISO system is a rational function. Pure mathematically, the inversion of that rational function changes the role of poles and zeros and makes a strictly proper system (having more poles than zeros) improper. This observation raises the question whether there is a similar duality between the zeros and the poles in a real physical system. However, as such systems are always strictly proper, the simple inversion does not physically exist. Yet, the inversion can be simulated by choosing the output as independent (= input of the fictitious inverse system) variable and investigating the system's behaviour and the required input.

To that end, the study of the forward (input  $\rightarrow$  output) relationship in the system is indispensable. This study uses the state space representation in order to obtain information about the internal behaviour of the system. Consider an exponential input with the following form :

$$\mathbf{u}(t) = \mathbf{u}.e^{s_0 t}. \quad (2.66)$$

Its Laplace transform is :

$$U(s) = \frac{\mathbf{u}}{s - s_0}. \quad (2.67)$$

The Laplace transform of the state, starting from the initial state  $\mathbf{x}_0$  and under the input of equation (2.66) can be derived from the state space equations :

$$sX(s) - \mathbf{x}_0 = \mathbf{A}X(s) + \frac{\mathbf{B}\mathbf{u}}{s - s_0} \quad (2.68)$$

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1} \cdot \left( \mathbf{x}_0 + \frac{\mathbf{B}\mathbf{u}}{s - s_0} \right). \quad (2.69)$$

The modal decomposition of  $\mathbf{A}$  yields a better insight in this result. In the following derivations  $\lambda_i$  is the  $i$ -th eigenvalue of  $\mathbf{A}$ ,  $\mathbf{v}_i$  the corresponding right eigenvector and  $\mathbf{w}_i$  the left eigenvector. Those eigenvectors assemble respectively the matrices  $\mathbf{V}$  and  $\mathbf{W}$ , which are inverses of each other. Further, all eigenvalues are considered distinct.

Using the eigenvalue decomposition,  $\mathbf{A}$  can be written as :

$$\mathbf{A} = \mathbf{V}.\mathbf{\Lambda}.\mathbf{W} \quad (2.70)$$

$$= \sum_{i=1}^n \lambda_i (\mathbf{v}_i \mathbf{w}_i^T), \quad (2.71)$$

and therefore,

$$(s\mathbf{I} - \mathbf{A})^{-1} = (\mathbf{V}.s\mathbf{I}.\mathbf{W} - \mathbf{V}.\mathbf{\Lambda}.\mathbf{W})^{-1} \quad (2.72)$$

$$= \mathbf{V}.(s\mathbf{I} - \mathbf{\Lambda})^{-1}.\mathbf{W} \quad (2.73)$$

$$= \sum_{i=1}^n \frac{1}{s - \lambda_i} (\mathbf{v}_i \mathbf{w}_i^T). \quad (2.74)$$

The substitution of this decomposition of the resolvent matrix  $(s\mathbf{I} - \mathbf{A})^{-1}$  into the partial fraction expansion of equation (2.69) yields the following expression for the state :

$$X(s) = \frac{(s_0\mathbf{I} - \mathbf{A})^{-1}.\mathbf{B}.\mathbf{u}}{(s - s_0)} + \sum_{i=1}^n \frac{1}{s - \lambda_i} \left[ \mathbf{v}_i \mathbf{w}_i^T . (\mathbf{x}_0 + \frac{\mathbf{B}\mathbf{u}}{\lambda_i - s_0}) \right], \quad (2.75)$$

and for the output, considering  $\mathbf{D} = \mathbf{0}$  :

$$Y(s) = \frac{\mathbf{C}(s_0\mathbf{I} - \mathbf{A})^{-1}.\mathbf{B}.\mathbf{u}}{(s - s_0)} + \sum_{i=1}^n \frac{\mathbf{C}\mathbf{v}_i}{s - \lambda_i} \left[ \mathbf{w}_i^T . (\mathbf{x}_0 + \frac{\mathbf{B}\mathbf{u}}{\lambda_i - s_0}) \right]. \quad (2.76)$$

Thus, the output also consists of exponential signals : one has the same exponent as the input, whereas the  $n$  other terms represent the modal dynamics of the system. The numerators  $\mathbf{C}\mathbf{v}_i \mathbf{w}_i^T . (\mathbf{x}_0 + \frac{\mathbf{B}\mathbf{u}}{\lambda_i - s_0})$  are the residues  $\alpha_i$  of the  $i$ -th mode, as introduced in the partial fraction expansion of equation (2.15) on page 22. In general, an input composed of  $m$  exponential signals

$$U(s) = \sum_{j=1}^m \frac{\mathbf{u}_j}{s - s_j} \quad (2.77)$$

causes the output

$$Y(s) = \sum_{j=1}^m \frac{\mathbf{C}(s_j\mathbf{I} - \mathbf{A})^{-1}.\mathbf{B}.\mathbf{u}_j}{(s - s_j)} + \sum_{i=1}^n \frac{\mathbf{C}\mathbf{v}_i}{s - \lambda_i} \left[ \mathbf{w}_i^T . \left( \mathbf{x}_0 + \mathbf{B} . \left( \sum_{j=1}^m \frac{\mathbf{u}_j}{\lambda_i - s_j} \right) \right) \right], \quad (2.78)$$

or,

$$Y(s) = \sum_{j=1}^m \frac{\mathbf{H}(s_j)\mathbf{u}_j}{(s - s_j)} + \sum_{i=1}^n \frac{\mathbf{C}\mathbf{v}_i}{s - \lambda_i} \underbrace{\left[ \mathbf{w}_i^T \cdot \left( \mathbf{x}_0 + \mathbf{B} \cdot \left( \sum_{j=1}^m \frac{\mathbf{u}_j}{\lambda_i - s_j} \right) \right) \right]}_{\gamma_i}, \quad (2.79)$$

with  $\mathbf{H}(s_j)$  the transfer function matrix evaluated at  $s_j$ . Equation (2.79) gives the decomposition of the output into  $n$  modal terms and  $m$  terms corresponding to the  $m$  different exponentials in the input. The sequel will use this expression to find the necessary input to obtain a certain output.

A full statement of the inverse problem is : “given an output

$$Y(s) = \frac{\mathbf{y}}{s - s_0}, \quad (2.80)$$

which input is needed to create it, what are the possible initial states  $\mathbf{x}_0$  and how do the states evolve internally ?” There are 3 cases depending on the value of  $s_0$  :

1.  $s_0$  is a zero. Since the amplitude  $\mathbf{y}$  of the output exponential  $\sim e^{s_0 t}$  equals  $\mathbf{H}(s_0)\mathbf{u}_j$ , and  $\mathbf{H}(s_0)$  is rank deficient, not every  $\mathbf{y}$  is reachable with a finite input. This is a first indication that zeros play the role of poles in the inverse problem, as a finite output ‘creates’ or ‘requires’ an infinite input.
2.  $s_0$  is a pole. No input is required if the system starts from initial state  $\mathbf{x}_0 = \mathbf{v}_j$ , the corresponding right eigenvector of that pole. So, a finite output ‘creates’ or ‘requires’ a zero input, exactly like the zero behaviour in the “forward system”.
3.  $s_0$  is neither a zero nor a pole. In this case, the input should at least contain an exponential  $\sim e^{s_0 t}$ . Its amplitude can be calculated from the transfer function matrix  $\mathbf{H}(s_0)$ . On top of that, all transients from the second summation in equation (2.79) should be suppressed, which is only possible if

$$\gamma_i = \left[ \mathbf{w}_i^T \cdot \left( \mathbf{x}_0 + \mathbf{B} \cdot \left( \sum_{j=1}^m \frac{\mathbf{u}_j}{\lambda_i - s_j} \right) \right) \right] = 0 \quad \forall i. \quad (2.81)$$

If the input exists only of the exponential  $\sim e^{s_0 t}$ , this equation completely determines the unique initial state  $\mathbf{x}_0$ , so that no mode dynamics are visible at the output. However, if the system has  $k$  zeros, a much larger set of inputs and initial states  $\mathbf{x}_0$  is possible :

$$U(s) = \frac{\mathbf{u}_0}{s - s_0} + \sum_{i=1}^k \frac{\alpha_i \mathbf{u}_i}{s - z_i}, \quad (2.82)$$

with :  $\mathbf{u}_i$  : the input vector corresponding to zero  $z_i$

$\alpha_i$  : a real scaling factor.

Putting all  $n$  equations of (2.81) below each other gives :

$$\mathbf{W} \cdot \mathbf{x}_0 = - \begin{bmatrix} \mathbf{w}_1^T \mathbf{B} \left( \frac{\mathbf{u}_0}{\lambda_1 - s_0} + \sum_{i=1}^k \frac{\alpha_i \mathbf{u}_i}{\lambda_1 - z_i} \right) \\ \vdots \\ \mathbf{w}_n^T \mathbf{B} \left( \frac{\mathbf{u}_0}{\lambda_n - s_0} + \sum_{i=1}^k \frac{\alpha_i \mathbf{u}_i}{\lambda_n - z_i} \right) \end{bmatrix}. \quad (2.83)$$

Premultiplying both sides by  $\mathbf{V}$  yields the general formulation of the required initial state :

$$\mathbf{x}_0 = -\mathbf{V} \cdot \begin{pmatrix} \frac{\mathbf{w}_1^T \mathbf{B} \mathbf{u}_0}{\lambda_1 - s_0} \\ \vdots \\ \frac{\mathbf{w}_n^T \mathbf{B} \mathbf{u}_0}{\lambda_n - s_0} \end{pmatrix} - \mathbf{V} \cdot \begin{bmatrix} \frac{\mathbf{w}_1^T \mathbf{B} \mathbf{u}_1}{\lambda_1 - z_1} & \cdots & \frac{\mathbf{w}_1^T \mathbf{B} \mathbf{u}_k}{\lambda_1 - z_k} \\ \vdots & & \vdots \\ \frac{\mathbf{w}_n^T \mathbf{B} \mathbf{u}_1}{\lambda_n - z_1} & \cdots & \frac{\mathbf{w}_n^T \mathbf{B} \mathbf{u}_k}{\lambda_n - z_k} \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}. \quad (2.84)$$

So, the initial state is not arbitrary, but an element of a translated  $k$ -dimensional subspace of  $\mathcal{R}^n$ . This restriction on the initial state is due to the fact that a strictly proper system has no physical inverse.

The similarity between equation (2.76) and equation (2.82) is striking and interesting. While the former represents the output of the 'forward' system, the latter can be interpreted as the 'output' of the 'inverse' system. In both cases the first term shows the direct effect of the external signal ( $\frac{\mathbf{u}}{s - s_0}$  in the first case,  $\frac{\mathbf{y}}{s - s_0}$  in the second); the following summation describes the evolution of the states and its effect on the 'output'.

The important conclusion is that zeros in fact have the meaning of poles, describing the internal dynamics of the 'inverse system'. But, since 'forward' and 'inverse' only involves the decision about what is considered as 'output' and which signals are independently determined as 'inputs', it does not alter the physical system itself at all. Therefore, both poles and zeros describe dynamical properties of the same system or at least a subsystem of it. And that is the major viewpoint in the following chapters, which try to discover the zeros immediately from a graphical representation of the system.

The inverse problem, as stated before, actually represents the tracking problem : "which input is needed to follow a desired output exactly ?". In non-minimum-phase (NMP) systems, at least one zero is unstable, so the internal dynamics while following the desired trajectory are unstable. That makes perfect tracking for NMP-systems impossible and explains why these systems are so difficult to control.

#### **2.4.4 Effect of Zeros on the Actuator Input**

This subsection investigates the practical consequences of the zero-pole duality in previous subsection. The theoretical derivations there prove that the zeros determine the internal dynamics of the system and also show up in the input (actuator signals), if a desired output is tracked perfectly (cf. equation (2.82)).

The aim of many control systems now is to achieve a good reference input tracking behaviour. So, whatever the control strategies are, built around the physical system, the zeros of the physical system will have some influence on the dynamic behaviour and the actuator signals. In an electromechanical system, for instance, the actuator is often an electric motor, so the actuator signals are the motor torque or motor velocity. Except for unstable zeros, which are almost disastrous for control, it seems that lightly damped stable zeros have a lot of drawbacks too.



The following example shows the impact of lightly damped zeros on two control strategies. Figure 2.7 pictures a mechanical system with 3 masses. In between them

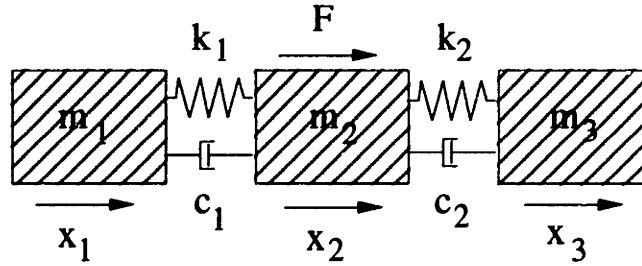


Figure 2.7: Model of a mechanical system with three masses

are springs (spring constants  $k_i$ ) and dampers (damping constants  $c_i$ ). The example can have as well a translational as a rotational interpretation. The latter is the actuation of a shaft mechanism, with located inertias  $J_i$ . The force (or torque) applied to the middle mass is the input of the system, and the position  $x_3$  of the third mass is the output. The general form of the transferfunction is :

$$\frac{X_3(s)}{F(s)} = \frac{(m_1 s^2 + c_1 s + k_1)(c_2 s + k_2)}{M_1 s^6 + (M_2 c_2 + M_3 c_1) s^5 + (M_2 k_2 + M_3 k_1 + M_4 c_1 c_2) s^4 + M_4 (c_1 k_2 + c_2 k_1) s^3 + M_4 k_1 k_2 s^2}, \quad (2.85)$$

where  $M_1 = m_1 m_2 m_3$ ,  $M_2 = m_1 m_2 + m_1 m_3$ ,  $M_3 = m_1 m_3 + m_2 m_3$ , and  $M_4 = m_1 + m_2 + m_3$ . Results of the control for two different sets of parameters show the influence of the zero.

**case 1:**  $m_1 = 3\text{kg}$ ,  $m_2 = 2\text{kg}$ ,  $m_3 = 5\text{ kg}$ ;  $k_1 = 1\text{ N/m}$ ,  $k_2 = 10\text{ N/m}$ ;  $c_1 = 0$ ,  $c_2 = 3\text{ N/(m/s)}$

zeros :  $\pm 0.57j$ ,  $-3.3$

poles :  $0, 0, -1.05 \pm 2.5j, -0.004 \pm 0.67j$

**case 2:**  $m_1 = 3\text{kg}$ ,  $m_2 = 2\text{kg}$ ,  $m_3 = 5\text{ kg}$ ;  $k_1 = 10\text{ N/m}$ ,  $k_2 = 10\text{ N/m}$ ;  $c_1 = 3$ ,  $c_2 = 3\text{ N/(m/s)}$

zeros :  $-0.5 \pm 1.7j$ ,  $-3.3$

poles :  $0, 0, -1.9 \pm 3j, -0.4 \pm 1.6j$

First, simple PD control is performed; figures 2.8 and 2.9 give the positions of the three masses and the actuator input for the first and the second set of parameter values respectively. In both cases, the response of the controlled position  $x_3$  is nice and smooth. In the undamped zero case, however, the actuator effort oscillates for a long time, with the frequency of the undamped zero (period  $T = \frac{2\pi}{0.57} = 11s$ ). In real systems, oscillations of the motor torque are very harmful for the hardware : excessive wear and high heat losses are some major disadvantages.

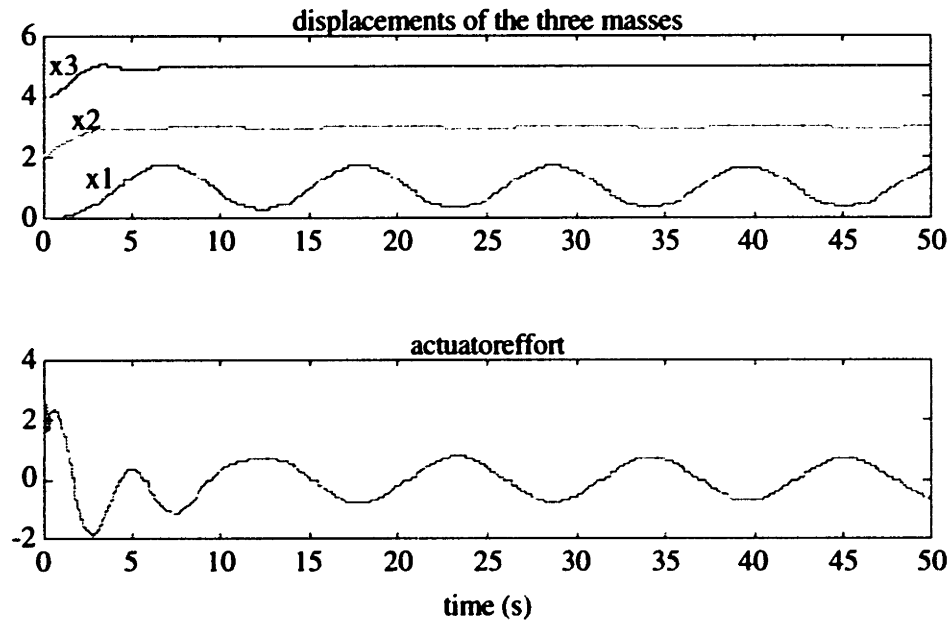


Figure 2.8: Displacements and actuator input for case 1

A vain attempt to diminish these bad effects of the low damped zero is to use the capability of the classical control setup to cancel the zero (cfr. 2.4.2). This, indeed

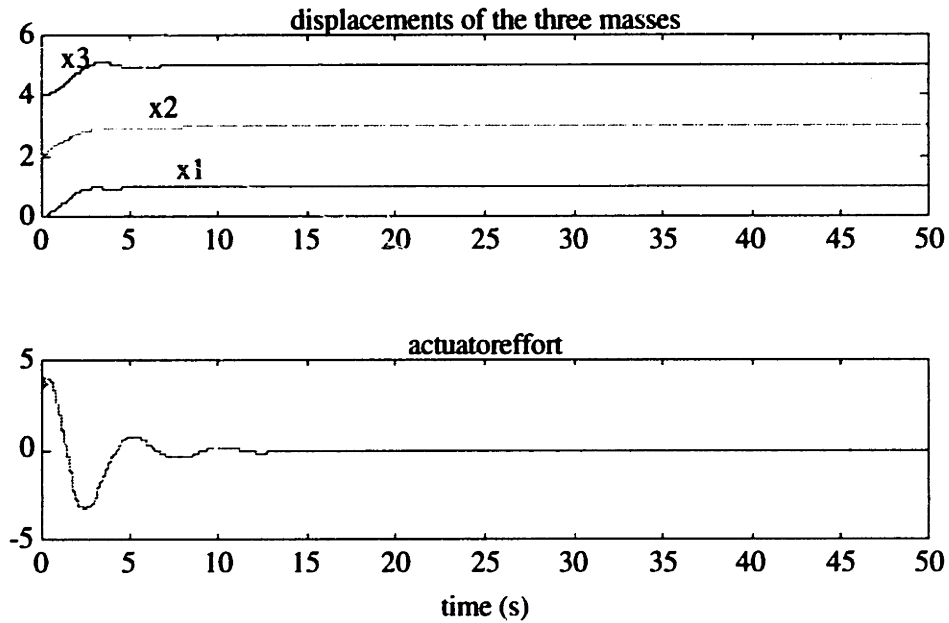


Figure 2.9: Displacements and actuator input for case 2

removes the zero from the closed loop relationship between the reference input and the position of the third mass, but of course cannot change the zero in the physical system. To that end, consider a system with four poles  $-1 \pm j$  and  $-5 \pm 2j$ , and two lightly damped zeros  $-0.1 \pm j$ . Two controllers are designed : a proportional controller and a zero cancelling controller with the following form

$$G_c(s) = \frac{k}{(s + 0.1 + j)(s + 0.1 - j)}. \quad (2.86)$$

Figure 2.10 shows in the plot on top the output (full line) and the actuator effort (dashed line) for the zero cancelling controller and in the bottom plot the same signals for the proportional controller. Compared to the proportional control, the pole-zero cancelling controller has a much better input output performance. In contrast, the actuator effort is much worse, seen the practical considerations about actuators mentioned above.

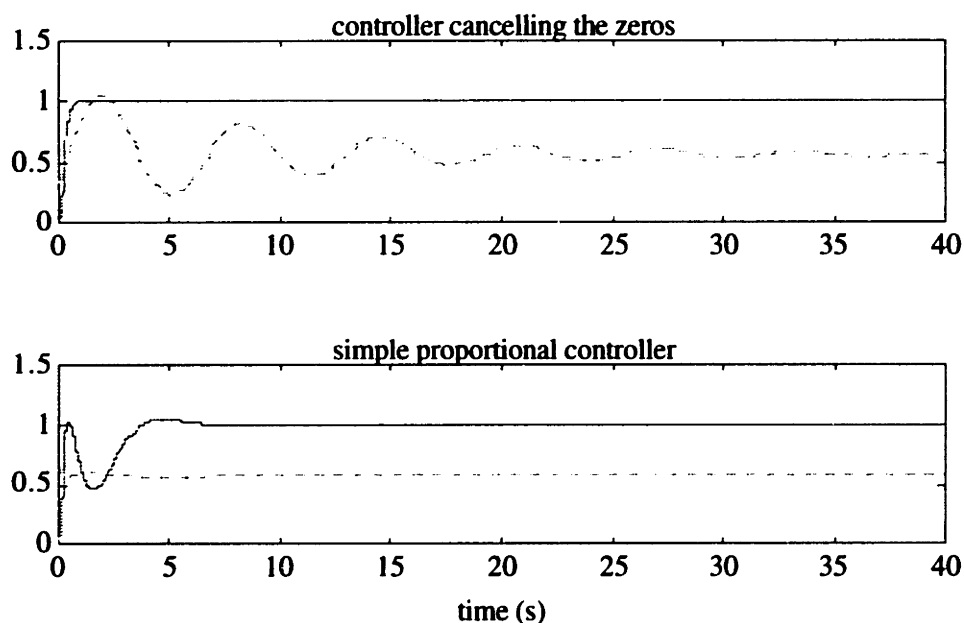


Figure 2.10: Bad influence on actuator effort of controller cancelling the zeros; output in full line and actuator effort in dashed line

The conclusion of this example is that the zeros *of the physical system itself* impose some trade-offs between output performance and actuator input behaviour, especially when they are unstable or lightly damped. Therefore, it is important to know how the *physical system* should be designed or altered to achieve more acceptable zeros. Then, modifying the hardware together with a simple and clear control design can often reach better results than implementing very complicated control strategies on the inappropriate physical system.

### 2.4.5 Zeros and the Relative Degree

The relative degree  $r$ , defined as the difference in the number of poles and zeros of a system, is an important number for the control design. It often determines the complexity of the controller and the achievable performance of the resulting controlled

system.

In most cases, the order of the system (number of poles) is fixed. Slight changes to the physical (damping or not) or the control design (which measurements are made), however, can alter the number of zeros, and thus the relative degree. This is another reason why it is important to have a physical insight into how some modifications to the system involve the number and values of the zeros.

In general, a higher relative degree makes the control design more difficult. A small relative degree (1 or 2), on the other hand, yields the system unconditionally stable under high proportional gain feedback. A simple root locus plot shows this. The number of asymptotes in the plot equals the relative degree, and their directions are under an angle  $\alpha$  :

$$\alpha = \frac{180^\circ}{r} + l \cdot \frac{360^\circ}{r} \quad l = 0..r-1 \quad (2.87)$$

For  $r > 2$ , there is at least one asymptote in the right half plane, so that the proportional feedback gain  $k$  must be limited to avoid instability.

Of course, the classical control configuration (see figure 2.6 on page 38) can diminish the relative degree of the controller-system combination, as the relative degree of a concatenation of two systems is the sum of the individual relative degrees. That, however, requires a non proper controller (i.e. more zeros than poles) : it takes some derivatives of the error signal. The presence of noise in the sensor signal makes its derivatives highly inaccurate. Consequently, the high relative degree remains a structural problem, and this alleged solution only adds more complexity to the controller.

The complexity of some special control techniques, such as the “Time Delay Control” of Wu [10] is directly correlated with the relative degree of the original system. The author therefore presents a method to derive the relative degree from a graphical representation (“Bond Graphs”, see the following chapter) of the system. This

method makes it unnecessary to go through the whole mathematical derivation of the transfer function to know its relative degree; the location of the input and output in the system are sufficient for this method. Similar procedures are developed in following chapters, yielding not only the number of zeros, and from that the relative degree, but also the actual values of the zeros. So, these methods have a double function : first they help us design the physical system to obtain a relative degree suitable for the control design, second, they also make it possible to get certain values for the zeros through the physical design.

## 2.5 Zeros in Simple Control Design

Previous sections have shown the importance of insight in the relationship of zeros versus physical design. This section first gives an example of how these questions rise in a practical design problem. Then it studies the results of a paper, which suggests some rules to determine zeros from the physical structure. Some examples will confront the paper with its inaccuracies and incompletenesses.

### 2.5.1 A Practical Design Problem

Consider following model of a drive system in figure 2.11 taken from [24]. Two bearings suspend a shaft which transmits the torque from the rotor of the electric motor to the arm link e.g. of a robot. A spring represents the shaft's compliance. All inertia is centralized in either the rotor or the arm link.

The input of this system is the motor torque exerted on the rotor; the output is the arm link position and the rotor position. The transfer functions of this system are :

$$\frac{\Theta_l(s)}{\tau(s)} = \frac{\frac{k}{J_r \cdot J_l}}{s^2[s^2 + (\frac{1}{J_r} + \frac{1}{J_l})k]} = \frac{\frac{k}{J_r \cdot J_l}}{D(s)} \quad (2.88)$$

$$\frac{\Theta_r(s)}{\tau(s)} = \frac{\frac{1}{J_r}(s^2 + \frac{k}{J_l})}{s^2[s^2 + (\frac{1}{J_r} + \frac{1}{J_l})k]} = \frac{\frac{1}{J_r}(s^2 + \frac{k}{J_l})}{D(s)} \quad (2.89)$$

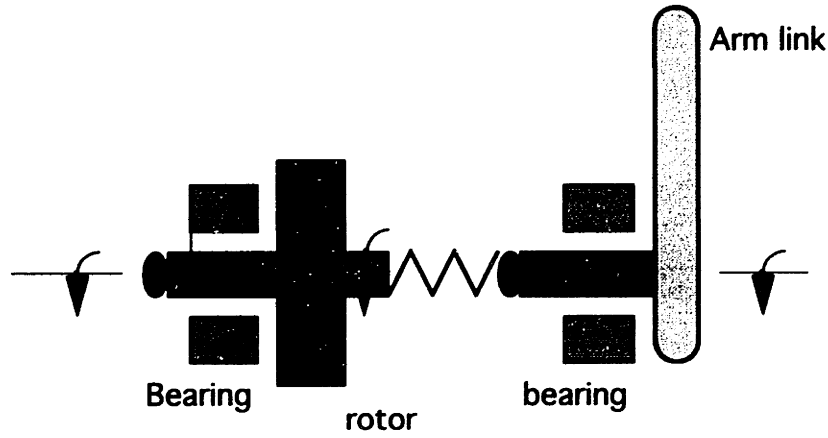


Figure 2.11: Schematic of a drive system

(2.90)

So, there is

- no zero for the transfer function  $\frac{\Theta_l(s)}{\tau(s)}$
- zeros  $\pm j\sqrt{\frac{k}{J_l}}$  for the transfer function  $\frac{\Theta_r(s)}{\tau(s)}$

An interesting question is how the transfer function and especially the zeros change by adding some linear damping to the system. The damping can occur in different places :

1. In bearing 1  $b_1$ : the transfer functions change to

$$\frac{\Theta_l(s)}{\tau(s)} = \frac{\frac{k}{J_r J_l}}{s[s^3 + \frac{b_1}{J_r} s^2 + (\frac{1}{J_r} + \frac{1}{J_l}) k s \frac{b_1 k}{J_r J_l}]} \quad (2.91)$$

$$\frac{\Theta_r(s)}{\tau(s)} = \frac{\frac{1}{J_r} (s^2 + \frac{k}{J_l})}{s[s^3 + \frac{b_1}{J_r} s^2 + (\frac{1}{J_r} + \frac{1}{J_l}) k s \frac{b_1 k}{J_r J_l}]} \quad (2.92)$$

(2.93)

with :

- no zero for the transfer function  $\frac{\Theta_l(s)}{\tau(s)}$

- same zeros  $\pm j\sqrt{\frac{k}{J_l}}$  for the transfer function  $\frac{\Theta_r(s)}{\tau(s)}$

2. In bearing 1  $b_2$ : the transfer functions change to

$$\frac{\Theta_l(s)}{\tau(s)} = \frac{\frac{k}{J_r J_l}}{s[s^3 + \frac{b_2}{J_l}s^2 + (\frac{1}{J_r} + \frac{1}{J_l})ks\frac{b_2 k}{J_r J_l}]} \quad (2.94)$$

$$\frac{\Theta_r(s)}{\tau(s)} = \frac{\frac{1}{J_r}(s^2 + \frac{b_2}{J_l} + \frac{k}{J_l})}{s[s^3 + \frac{b_2}{J_l}s^2 + (\frac{1}{J_r} + \frac{1}{J_l})ks\frac{b_2 k}{J_r J_l}]} \quad (2.95)$$

$$(2.96)$$

so that,

- no zero for the transfer function  $\frac{\Theta_l(s)}{\tau(s)}$
- different zeros  $\frac{-b_2}{2J_l} \pm j\sqrt{\left(\frac{-b_2}{2J_l}\right)^2 - \frac{k}{J_l}}$  for the transfer function  $\frac{\Theta_r(s)}{\tau(s)}$

3. In shaft  $b_3$  : the transfer functions change to

$$\frac{\Theta_l(s)}{\tau(s)} = \frac{\frac{b_3}{J_r J_l}(s + \frac{k}{b_3})}{s^2[s^2 + (\frac{1}{J_r} + \frac{1}{J_l})b_3 s + (\frac{1}{J_r} + \frac{1}{J_l})k]} \quad (2.97)$$

$$\frac{\Theta_r(s)}{\tau(s)} = \frac{\frac{1}{J_r}(s^2 + \frac{b_3}{J_l} + \frac{k}{J_l})}{s^2[s^2 + (\frac{1}{J_r} + \frac{1}{J_l})b_3 s + (\frac{1}{J_r} + \frac{1}{J_l})k]} \quad (2.98)$$

$$(2.99)$$

so that eventually,

- there appears zero  $-\frac{k}{b_3}$  for the transfer function  $\frac{\Theta_l(s)}{\tau(s)}$
- similar zeros  $\frac{-b_3}{2J_l} \pm j\sqrt{\left(\frac{-b_3}{2J_l}\right)^2 - \frac{k}{J_l}}$  as in case  $b_2$  for the transfer function  $\frac{\Theta_r(s)}{\tau(s)}$

Apparently, the place where the damping is added to the system has much influence on the zeros of the transfer functions, first of all on the number of zeros. For instance, only a dissipative nature of the shaft yields a zero in the transfer function  $\frac{\Theta_l(s)}{\tau(s)}$  of the non-collocated system. In contrast, damping in the first bearing has no influence at all on the zeros.



This example is of course very simple, so it is possible to perform all the calculations for the transfer functions, in order to see the influence on the zeros of some change in the physical design. For more complex systems, however, this can be cumbersome. Moreover, the physical changes will always be some blind guesses. The theory in next chapters therefore will offer the designer more insight and a better guideline for this design process.

### 2.5.2 Zeros as Eigen-frequencies of Structural Subsystems

In his article [11], D.K. Miu presents a physical interpretation for the zeros of the transfer function of flexible mechanical systems. This interpretation is very interesting, but can be reformulated in a more general way. To that end, following chapters will use the “bond graph” representation of the system, and derive from it different rules to identify the zeros. This section will explain the ideas of the article and show their incompleteness in some simple examples.

The conclusion of Miu's article is : “the complex conjugate zeros are the resonant frequencies of a substructure constrained by the sensor and actuator”. Several examples of spring mass systems as in figure 2.12 illustrate this rule :

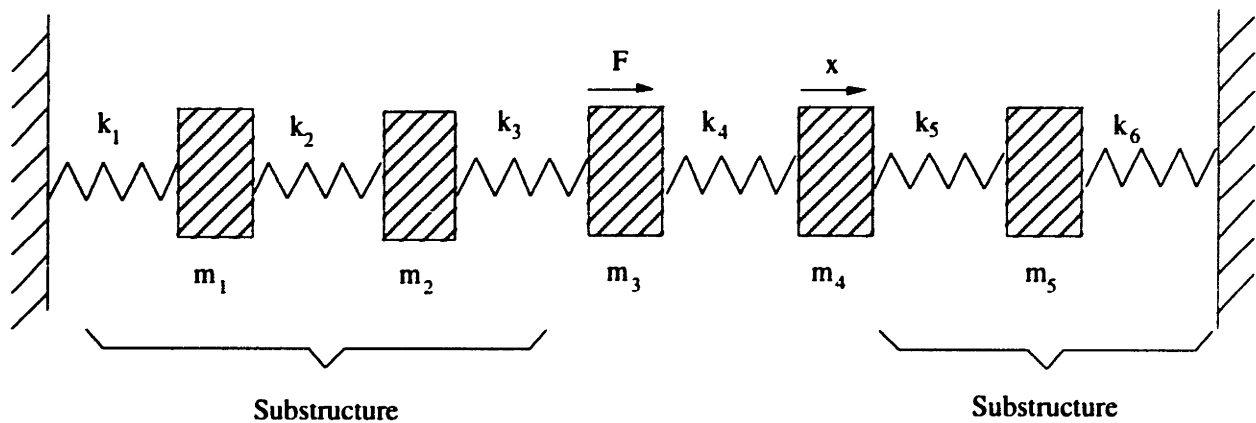


Figure 2.12: Substructures of a mass spring system by Miu

In other words : assume the output variable (position  $x$  of mass  $m_4$  in figure 2.12) and the position of mass  $m_3$ , on which the input force  $F$  acts, identically zero. Physically, that means that those two masses are standing still all the time. That assumption divides the structure into several substructures with the zero positions as boundaries. The resonance frequencies of these substructures which are *not located between the input and the output* are then the complex conjugate zeros of the system.

The formulation of this rule, however, raises some questions :

1. The identically zero condition on the output position is very clear : it represents the meaning of the zero as a certain input for which the output is constantly zero. On the other hand, it is not so clear why the position of the mass on which the input force acts, should be zero too.
2. Assuming that the boundaries between the substructures are given by the zero conditions above, which substructures will actually provide the zeros of the whole system and why ?
3. This rule can only provide complex conjugate zeros. What about real zeros ?
4. What happens if some damping is added to the system ? As a matter of fact, in real physical systems the presence of damping is more likely than its absence.

All these questions ask for a more general and deeper physical insight into the meaning of zeros. Therefore, next chapter does not start from the structural layout of the system (e.g. in figure 2.12 but rather from the energetical layout, represented in a “bond graph”. Energy is the only physical means by which subsystems can interact with each other.

This subsection ends with two examples, in which the basic rule of Miu’s paper does not apply.

Figure 2.13 shows a mechanical mass-spring-damper system similar to figure 2.12 from [11]. The input force acts on the first mass, whereas the position of the second mass is the output.

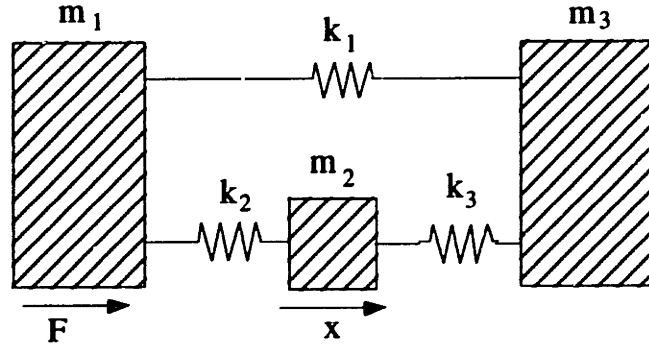


Figure 2.13: Mass spring systems for which Miu's rule does not apply

The application of the structural rule from the paper fixes mass 1 and mass 2. Two substructures result from these boundaries :

- mass 3 suspended by springs  $k_1$  and  $k_3$  in parallel
- only spring  $k_2$  between two fixed positions, so it cannot exhibit any dynamical behaviour.

The actual zeros of the system, calculated from the state space representation of the system, are :

$$z_{1,2} = \pm j \sqrt{\frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{m_3 k_2}} \quad (2.100)$$

Apparently, these zeros depend on the value of  $k_2$ , and can consequently not be provided by either of the substructures isolated by the rule of Miu's paper.

A second example introduces some damping in one of the mass spring systems used by Miu. Figure 2.14 shows the structural layout : the input force now acts on the first mass, whereas the position of the third mass forms the output. The damping  $c$  is located in the connection between the second and the third mass.

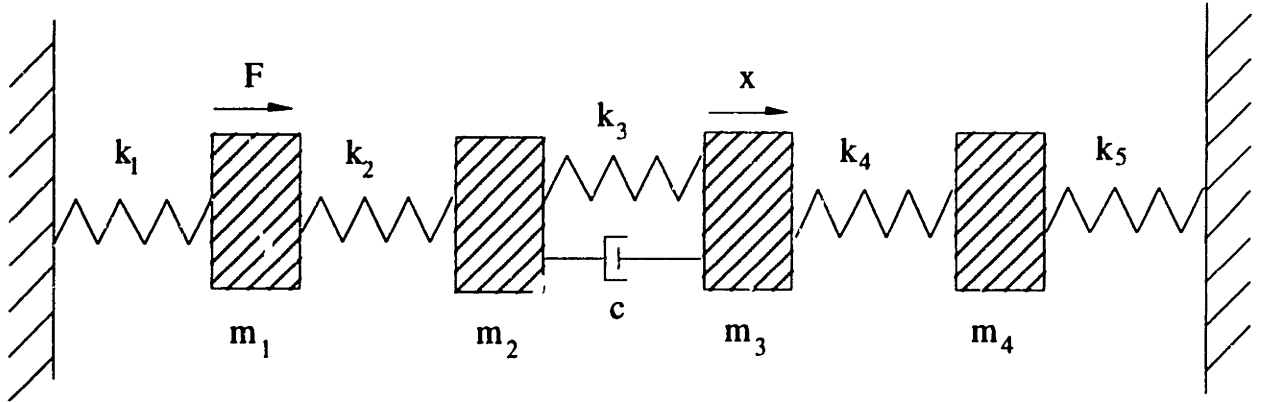


Figure 2.14: Mass spring system with damping for which Miu's rule does not apply

Since the damping is added in that part of the system between the position of the input force and the position of the output displacement measurement, there shouldn't be any effect on the zeros following Miu's rule (as discussed above). However, calculating the zeros from the state space representation of that systems yields following results :

1. There are two complex conjugate zeros,

$$z_{1,2} = \pm j \sqrt{\frac{k_4 + k_5}{m_4}}, \quad (2.101)$$

which are exactly the eigen-frequencies of the subsystem at the right side of  $m_3$ , namely the mass  $m_4$  suspended by the two springs  $k_4$  and  $k_5$  in series. These zeros agree with Miu's rule.

2. There is one additional real zero,

$$z_3 = -\frac{k_3}{c}, \quad (2.102)$$

which is not predicted at all in Miu's theory.

Similarly, Wu's method, mentioned in subsection 2.4.5, to derive the relative degree from the "Bond Graph" representation, also shows that adding the damper in this

example diminishes the relative degree of the transfer function between the input force and the output position  $x_3$  by one. As adding the damper does not change the order of a system (number of poles) in this example, this result implies that there is one additional zero.

## 2.6 Summary

This chapter first gives a literature survey of the existing definitions of zeros and transmission zeros in linear SISO and MIMO systems. The MIMO case makes a clear distinction between 'global' and 'transmission' zeros. Several computation schemes and definitions are presented : one group of definitions start from the transfer function matrix representation of the system, others use the state space equations.

The third section discusses the importance of the zeros of a system, first in the input-output relationship of the bare system, second in different control strategies, built around that system. It further discusses the duality between poles and zeros, and clarifies the results for the actuator input with an example.

The last section underlines the main conclusions : the zeros of the physical system are important in many aspects, and their influence cannot be cancelled by any control technique. Therefore, a real problem is depicted, raising some questions about the necessary hardware changes to the physical system to obtain more appropriate zeros for the control design. The ideas and deficiencies of an existing study about the relationship between a physical system configuration and the zeros form a springboard to next chapters.

# The Physical Meaning of Zeros in SISO Systems

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## 3.1 Introduction

This chapter presents a physical interpretation of zeros in SISO systems, based on bond graphs, a graphical representation of the power flow within the system. The second section introduces the bond graphs and shows the difference with the structural layout. It also mentions some other authors using that representation to study dynamic systems. Next section explains the meaning of zeros in bond graphs. It turns out that they create energetically isolated subsystems; therefore, it is followed by a thorough discussion on the characteristics of subsystems. The general procedure in section 3.5 sets up the rules to identify the right subsystems, whose eigenvalues are the zeros of the whole system. A few worked-out examples illustrate this procedure. The end of the chapter clarifies the relationship of this theory with the special case of pole-zero cancellation.

## 3.2 Bond Graphs : Energy and Powerflow

### 3.2.1 Difference between Bond Graphs and the Structural Layout

All the figures in previous chapter represent the structural layout of the system : they show how different parts of the system actually are connected to each other. In mechanical systems these connections can be at distance, using connection rods etc.,

or the two elements can be fixed by means of bolts or welds. In electrical systems, these connections are wires of any length. The use of such structural representation of the system to predict the zeros and their initial states, however, is rather limited, as shown by some counterexamples in subsection 2.5.2.

Therefore, this chapter will start from a different representation : the “bond graphs”. These bond graphs do not show the structural connection between parts of a system, but rather how they interact energetically. Also the identification of the parts or elements of a system is based on energy : an element can either store and release energy, dissipate energy (i.e. transform it into a type which is useless for the system, e.g. heat) or transform it into another useful type (e.g. an electric motor in an electromechanical system). Since systems in nature basically interact through energy transfer, this representation offers the best physical insight into the interaction processes within the system. Moreover, these processes form the actual dynamic behaviour of the system. Zeros are one aspect of that dynamic behaviour, so bond graphs should present some information about them too.

### 3.2.2 Meaning of Bonds and Junctions

The power of the bond graphs lies in their representation of how power flows through the system from one element to another. To that end, the elements are connected by a network of “bonds” and “junctions”. Every power flow results from the physical combination of a flow variable,  $f$ , and an effort variable,  $e$ . Both of them belong to the same energy field (e.g. mechanical, electrical,...) and determine the kind of power flow. A bond is a path of power flow of a certain type in the system. It consequently is described by the appropriate flow and effort variable. The graphical representation of the bond is a line; often the corresponding value of the effort variable is depicted above that line, the value of the flow variable below it. A half arrow at the end of

the line indicates the direction of the power flow.

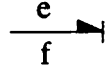


Figure 3.1: A bond with power flow  $P = e.f$  going to the right side

The junctions form the nodes in the network of power flow. There are two types of junctions : 1-junctions connect different bonds with the same flow variable  $f$ , which is the “junction variable” of that 1-junction. The 0-junction connects different bonds with the same effort variable  $e$ , similarly called the junction variable of the 0-junction. The second basic characteristic for every junction is that it cannot store energy. Therefore the algebraic sum of all the power flows into the junction should be zero (this sum should take into account the direction of the power flow of each bond, as indicated by its half arrow) :

$$\sum_i f_i e_i = 0 \quad (3.1)$$

$$f \left( \sum_i e_i \right) = 0 \quad \forall f \quad (1 - \text{junction}) \quad (3.2)$$

$$\left( \sum_i f_i \right) e = 0 \quad \forall e \quad (0 - \text{junction}) \quad (3.3)$$

$$\Rightarrow \left( \sum_i e_i \right) = 0 \quad (1 - \text{junction}) \quad (3.4)$$

$$\Rightarrow \left( \sum_i f_i \right) = 0 \quad (0 - \text{junction}) \quad (3.5)$$

In summary, bonds and junctions form a complex network for the power flow between all elements of the system. Bonds are the paths and junctions are the crossroads of a set of paths having a common flow or effort variable.



### 3.2.3 Other Information that can be derived from the Bond Graphs

Since the invention of the bond graphs by Professor Paynter (M.I.T.) in the late 1950's, a lot of authors have shown how this energy based graphical representation provides much dynamical information about the system. Many others extended the basic principles of the bond graphs to model more complex systems or physical phenomena, such as piezoelectric devices. This subsection cites a few of them.

Sueur and Dauphin-Tangy [3, 4] found some rules to derive the controllability and observability from the bond graphs. The most striking fact about those rules is that they do not need the actual values of the element parameters (e.g. the moment of inertia for an I-element or the resistance for an R-element). On the contrary, some bond graphs exhibit a “structural uncontrollability” (i.e. merely depending on the layout of the bond graph), and thus remain uncontrollable whatever the parameter values be. Such “structural properties” are interesting, since they encompass a whole class of systems.

Such a structural property of the layout of the bond graph corresponds to some relationship in the mathematical model. For instance, in a state space representation structural uncontrollability is translated into a relationship between the elements of the matrices **A**, **B**, **C**, and **D**, so that the controllability matrix  $\mathcal{C}_o$  is never of full rank. On the other hand, changing just one element of the **A** matrix often yields another system with a completely different structure.

Figure 3.2 shows a simple example of a SISO mass spring system that needs 9 matrix elements (**A**  $2 \times 2$ , **B**  $2 \times 1$ , **C**  $1 \times 2$  and **D**  $1 \times 1$ ), whereas the structure itself is completely determined by two parameters,  $k$  and  $m$ .

This shows another advantage of reasoning on bond graphs (besides the more natural energy topology) over mathematical models : bond graphs still contain a

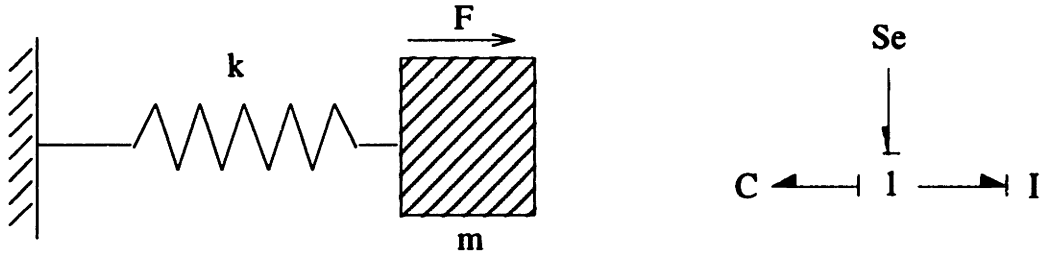


Figure 3.2: A mass spring system and its bondgraph representation

lot of information about the system's structure. That makes them the appropriate representation to reveal such structure-related properties.

Zeid and Rosenberg [8, 9] set up some rules to estimate the eigen spectra and stability conditions of systems, only based on the bonds and junctions layout of the system. This approach only uses the naked framework of the paths along which the energy can flow throughout the system.

Margolis [5, 6] uses bond graphs to model a distributed flexible system, such as a beam, and its dynamical interaction with a lumped system such as a spring attached at some position to the beam. The basic idea behind it consists of representing only those flexible modes of the system that lie in the frequency domain of interest.

Like all studies mentioned above, this thesis will also take advantage of the assets offered by the bond graph representation. Bond graphs have more physical meaning and are closer to the energetical structure of the system; therefore, they are the ideal tool to discover a more physical interpretation of zeros and their initial states.

### 3.3 Meaning of Zeros in Bond Graphs

This section introduces the concept of a zero in the bond graph representation of the system. The reasoning should be compared with that of Miu's article [11] in chapter 2. Bond graphs come up with some subdivision of the system into subsystems as well.

However, since the structural layout used in Miu's rules is different from the energetic layout of the bond graphs, in general, the resulting subsystems do not correspond either.

The next subsections follow the same logic as the discussion of zeros in sections 2.2 and 2.3 : they start from the mathematical definition and proceed gradually towards a more physical interpretation.

From this point on, the thesis makes one assumption about the input and output : the input is some effort source or flow source in the system, whereas the output is an effort junction variable of a 0-junction or a flow junction variable of a 1-junction. Nevertheless, the theory remains general. Some cases can easily be transformed to satisfy this condition, e.g. if the output is the position of a mass or the angle of an inertia. Simply taking the derivative (which is equivalent to adding one zero at the origin to the transfer function) yields the linear or angular velocity, which are the flow variables for mechanical power.

### 3.3.1 Zeros and their Initial States

Subsection 2.3.5 about the state space definition of zeros explains that the zero input is only completely defined by its exponent  $z$ , the input vector  $\mathbf{u}$  and the initial state  $\mathbf{x}_0$ . Since this chapter only deals with the SISO case, the input vector  $\mathbf{u}$  is just the scalar 1. The meaning of the exponent in the bond graph picture is clear : the flow source  $S_f$  or the effort source  $S_e$  provide power to the system so that its flow variable,  $f_s$ , effort variable,  $e_s$ , respectively evolve in time  $\sim e^{zt}$  :

$$f_s = f_0 e^{zt} \quad (3.6)$$

$$e_s = e_0 e^{zt} \quad (3.7)$$

The initial states (combined into one initial state vector  $\mathbf{x}_0$ ) play a key role in the development of the theory. The general definition of a state vector is : “all information needed to determine the output of the system, when the input is known as a function of time”. As mentioned above, in physical systems the dynamic behaviour is governed by the way how energy is stored and moved around the system. Therefore, the storage of energy tells all the needed information to derive the future evolution of the system. The energetic logic behind the bond graphs makes it possible to identify that energy storage, and thus the initial state vector  $\mathbf{x}_0$  easily : the causality assignment to the bond graph tells which energy storing elements are independent. The corresponding variable is a state of the system. The number of states is fixed for a certain system; which variables make up the states, however, are not.

Two other conclusions about the initial state vector are important for the sequel :

1. If all energy storage at a certain moment is known, together with the source input value, all other variables in the system can be written as a function of these states and the input. Most often, the input is not needed; compare with the second state equation  $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$  with  $\mathbf{D} = 0$ . This will be applied for the junction variables in figure 3.3.

$$\text{states} = (v_1, F_2, v_3)$$

$\Rightarrow$  junction variables as function of the states :

$$v_1 = v_1 \tag{3.8}$$

$$F_1 = F_2 + R.v_2 \tag{3.9}$$

$$v_3 = v_3 \tag{3.10}$$

2. If all energy storage is known, and if one part of the system is isolated, so that no energy can flow into or out of it, then that subsystem will evolve autonomously.

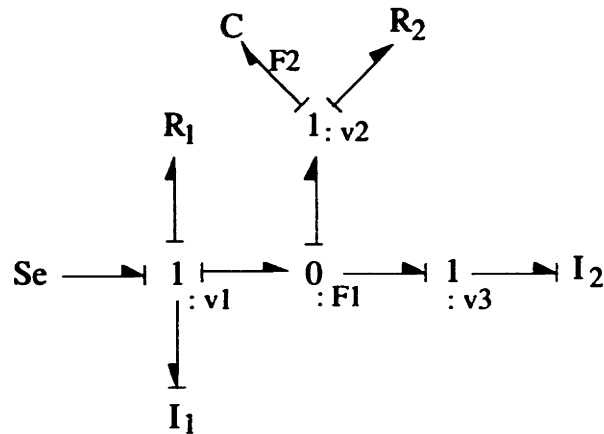


Figure 3.3: The state variables in the bond graph

The energy stored in the subsystem can be considered as its initial state. The subsystem has its own modal decomposition of the initial state, which determines the dynamic evolution completely. If it happens that the initial state is in the “direction of an eigenvector”, then the state’s evolution will even be colinear with that eigenvector.

### 3.3.2 The Energy Blocking Effect of a Zero

As mentioned above, the output variable is assumed to be a junction variable. All interpretations in chapter 2 define a zero input as that exponential input, together with the necessary initial state of the system, so that the output variable is identically zero. Instead of deriving a mathematical formula from this definition, this subsection looks into its meaning from an energetic point of view. The total power flowing into a junction is the algebraic sum of the power of all bonds towards the junction, taking into account the direction of the arrow. Since a junction cannot store nor dissipate energy, this total power flow is by definition always zero. If, however, the junction variable is identically zero, then the power through every bond will be identically zero too. It is as if those bonds do not exist at all. This is called the “energy blocking

effect” of a zero input; the junction forms a barrier for the energy and is called “a node” in the sequel.

**Definition 3.1 (Node)** *A node is a 1-junction whose junction flow variable  $f$  is identically zero, or a 0-junction whose junction effort variable  $e$  is identically zero.*

However, the junction has still an important function : though all bonds have power flow zero, the equation for the net power,

$$\sum_i f e_i = f \sum_i e_i = 0, \quad (3.11)$$

must still hold. Since this equation must hold for all values of  $f$  (not only  $f = 0$ ), it yields a condition on the algebraic sum of the complementary variables of the junction variable :

$$\sum_i e_i = 0 \quad (1 - \text{junction}). \quad (3.12)$$

$$\sum_i f_i = 0 \quad (0 - \text{junction}). \quad (3.13)$$

Therefore, a node blocks all energy transfer, but maintains some information transfer from one side to another. For example, figure 3.4 shows a mass, whose velocity is the output of the system, and the corresponding bond graph : The output variable  $f$  is identically zero. In addition, there is an “information equation”,

$$e_1 - e_2 - e_3 = 0, \quad (3.14)$$

(corresponding to the arrow directions).

**Definition 3.2 (Information equation)** *An information equation is the junction equation of a junction that is a node. It takes into consideration the arrow directions of the bonds towards the node. In a system, there is an information equation for every node.*

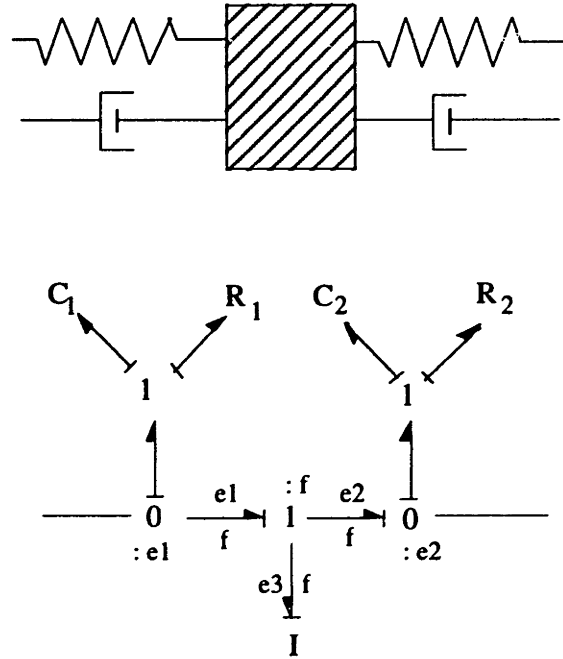


Figure 3.4: Bond graph representation of a node in the system

The constitutive law of the generalized inertia (= mass) indicates an additional relationship between  $f$  and  $e_3$  :

$$f = \frac{1}{m} \int_0^t e_3(\tau) \cdot d\tau \equiv 0 \quad \forall t. \quad (3.15)$$

This implies that also  $e_3 \equiv 0$ , which simplifies the information equation to

$$e_1 = e_2. \quad (3.16)$$

The following altered bond graph of figure 3.5 summarizes these results. This figure clearly indicates the double effect of a node : it energetically subdivides the system into subsystems and establishes an algebraic relationship between their variables.

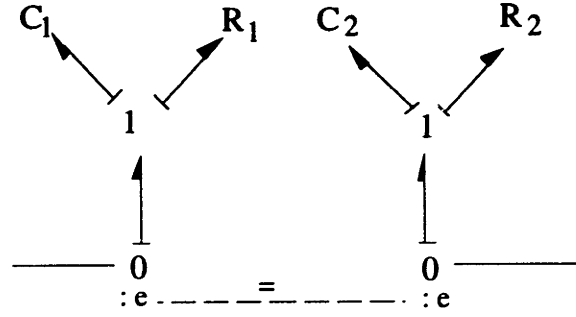


Figure 3.5: Bond graph showing the information equation of a node

### 3.3.3 The Zero Exponent $z$ as an Eigenvalue of a Subsystem

Previous subsection explains why a zero creates a node in the bond graph and, because of that node, isolates several subsystems. This subsection studies the evolution of the states when a zero input is applied and its implication for the isolated subsystems.

The equations (2.43) and (2.48) define the zero input  $\mathbf{u}e^{zt}$  with exponent  $z$ , input vector  $\mathbf{u}$ , and initial state vector  $\mathbf{x}_0$

$$\begin{aligned}(z\mathbf{I} - \mathbf{A})\mathbf{x}_0 &= \mathbf{B}\mathbf{u} \\ \mathbf{C}\mathbf{x}_0 &= \mathbf{0}.\end{aligned}$$

The Laplace transform of the input is :

$$\mathcal{L}(\mathbf{u}(t)) = \frac{\mathbf{u}}{s - z}. \quad (3.17)$$

Filling in these results in the state equations for the system yields the evolution of the state  $X(s)$  :

$$sX(s) - \mathbf{x}_0 = \mathbf{A}X(s) + \mathbf{B}\frac{\mathbf{u}}{s - z} \quad (3.18)$$

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1} \left( \mathbf{x}_0 + \mathbf{B}\frac{\mathbf{u}}{s - z} \right) \quad (3.19)$$

$$= (s\mathbf{I} - \mathbf{A})^{-1} \left( \mathbf{x}_0 + \frac{(z\mathbf{I} - \mathbf{A})\mathbf{x}_0}{s - z} \right) \quad (3.20)$$



$$= (s\mathbf{I} - \mathbf{A})^{-1} \left( \mathbf{x}_0 + \frac{(z\mathbf{I} - s\mathbf{I} + s\mathbf{I} - \mathbf{A})\mathbf{x}_0}{s - z} \right) \quad (3.21)$$

$$= (s\mathbf{I} - \mathbf{A})^{-1} \left( \mathbf{I} - \mathbf{I} + \frac{s\mathbf{I} - \mathbf{A}}{s - z} \right) \mathbf{x}_0 \quad (3.22)$$

$$= \frac{\mathbf{x}_0}{s - z}. \quad (3.23)$$

In the time domain this means :

$$\mathbf{x}(t) = \mathbf{x}_0 e^{zt}. \quad (3.24)$$

**Theorem 3.1 (Rectilinearity)** *If in a linear system with state space equation*

$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ , *and*

- *$z$  is a zero*
- *$\mathbf{u}$  is the corresponding input vector :  $\mathbf{u}(t) = \mathbf{u}e^{zt}$*
- *$\mathbf{x}_0$  is the initial state of the system such that the output is identically zero under the input  $\mathbf{u}e^{zt}$ ,*

*then the state of the system will evolve rectilinearly from  $\mathbf{x}_0$  with time exponent  $z$  :*

$$\mathbf{x}(t) = \mathbf{x}_0 e^{zt}$$

In an autonomous system, such a rectilinear evolution of the state is only possible if the initial state is along an eigenvector and the exponent is an eigenvalue . The total system is of course not autonomous, because of the zero input  $\mathbf{u}e^{zt}$ . Previous subsection, however, shows that the zero input creates nodes and consequently subsystems in the system, which are autonomous due to the energy blocking effect. Subsection 3.3.1 connects the initial state vector  $\mathbf{x}_0$  to the energy storage : it is possible to identify that part of  $\mathbf{x}_0$  responsible for the initial energy storage in the autonomous subsystem.

Therefore, if a zero input isolates a subsystem, the value  $z$  can only be an eigenvalue of that subsystem. This is a basic conclusion for the general procedure to find zeros and their initial states.

## 3.4 Subsystems

This section studies the different kinds of subsystems that may arise from the subdivision by “nodes”. They form the key point in the general procedure of next section. That procedure searches for the isolated subsystems, whose eigenvalues are the zeros  $z$ . Meanwhile, it calculates the corresponding initial state vector  $\mathbf{x}_0$  (often some assumptions about  $\mathbf{x}_0$  are already made while finding the subsystems, see below).

There are, however, some conditions on the subdivision and the resulting subsystems to be appropriate for yielding a zero and corresponding initial state. To find all zeros  $z$ , even more candidate subdivisions and subsystems have to be checked. Therefore, this section introduces the different properties of subsystems needed in that search for the “right” subsystems.

### 3.4.1 Zero Nodes and Artificial Nodes

As subsection 3.3.2 explains, “nodes” are junctions in the bond graph whose junction variable is considered identically zero. They make up the subdivision into subsystems. There are two types of nodes : “zero nodes” and “artificial nodes”.

**Definition 3.3 (Zero nodes)** *Zero nodes originate from the zero output condition in the definition of a zero.*

**Definition 3.4 (Artificial nodes)** *Artificial nodes are junctions whose junction variable is held deliberately zero, in order to create another subdivision, and from it other candidate subsystems for zeros.*

The reason for these artificial nodes is the following. While searching for an appropriate subdivision and subsystems, the corresponding initial state is still completely undetermined and can be chosen freely. Therefore, it is allowed to make some assumptions on the initial energy distribution (thus the initial state) first, and to find out afterwards whether there is a zero with an initial state  $\mathbf{x}_0$  according to that assumption.

Subsection 3.3.1 indicates that every variable in the system, so also the junction variables, can be written as a function of the states. Thus, choosing a certain junction variable to be identically zero is equivalent to setting up a relationship between the states or assuming a particular initial energy distribution. The extra created “artificial” node can reveal another subsystem and a new value for  $z$ .

The general procedure provides also some guidelines for this search process, so that the number of candidate artificial nodes is very small. That removes most overhead and the appropriate subsystems are found easily.

Following example in figure 3.6 features an artificial node with junction variable  $f_2$ , which is a linear combination of the states  $f_1, f_3, f_4, e_2, e_3$  in the system. The corresponding information equation is :

$$e_2 = e_3 + e_4. \quad (3.25)$$

From it,  $f_2$  can be written as a function of the states :

$$f_2 = \frac{k_1 f_1 + k_2 f_3 + k_3 f_4}{k_1 + k_2 + k_3}. \quad (3.26)$$

Figure 3.7 shows the resulting subsystems from these nodes.

### 3.4.2 Active and Passive Subsystems

Once the nodes have determined the resulting subsystems, these subsystems can be subdivided into different classes according to their properties. One difference is the

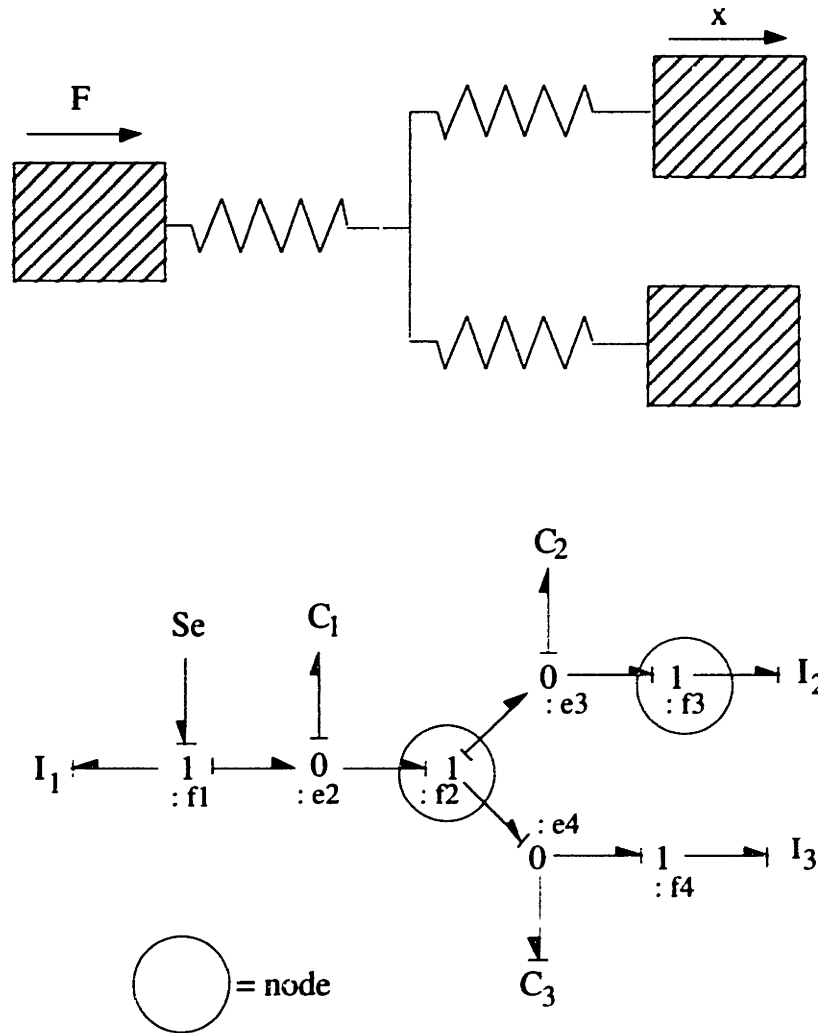


Figure 3.6: Bond graph showing a zero node and an artificial node

existence of a source (=active input) :

**Definition 3.5 (Active and passive subsystems)** *Subsystems containing a source are called active subsystems, others are passive subsystems.*

Passive subsystems behave autonomously. the only relation they have with the rest of the system are the “information equations” they have to comply with at all their boundary nodes. In active subsystems, however, the input source makes it still possible to steer the behaviour of the subsystem in a certain way. This will be useful to

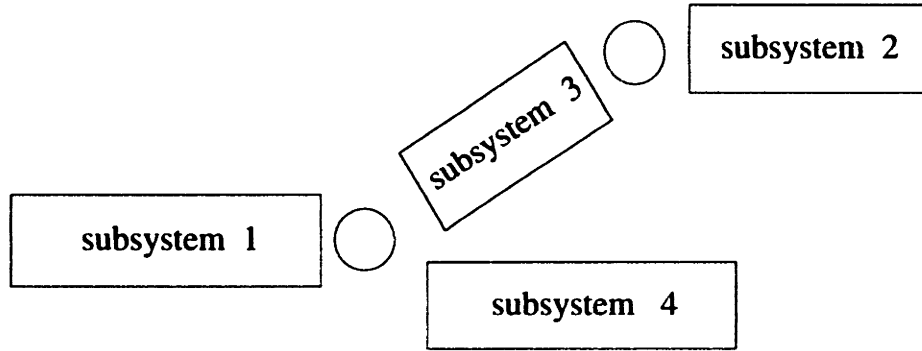


Figure 3.7: The subsystems resulting from the two nodes

satisfy the information equation at the boundary node.

Subsection 2.4.3 about the duality of poles and zeros proves that it is always possible to find a certain input  $\mathbf{u}(t) = \mathbf{u}e^{\alpha t}$  to a linear system and a certain initial state vector  $\mathbf{x}_0$  so that the output is  $\mathbf{y}(t) = \mathbf{y}e^{\alpha t}$ . If the system has zeros (transmission zeros for the MIMO case), all possible initial state vectors even form an  $n_z$ -subspace (with  $n_z$ , the number of zeros). This subsection applies that result to the active subsystem, which, indeed, is a linear system too. The outputs for an active subsystem are the variables complementary to the junction variables of the boundary nodes. For example, if the subsystem has as well a 1-junction node as a 0-junction node, it has two outputs, namely the effort variable  $e$  of its bond towards the 1-junction and the flow variable  $f$  of its bond towards the 0-junction.

Suppose  $\mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \mathbf{D}_s$  are the matrices of the state space representation of the active subsystem. Its outputs, as mentioned above, are to evolve as  $\mathbf{y}(t) = \mathbf{y}e^{\alpha t}$ . To that end, the input and state should have the following form :  $\mathbf{x}(t) = \mathbf{x}_0e^{\alpha t}$  and  $\mathbf{u}(t) = \mathbf{u}e^{\alpha t}$ . Filling this in in the state equations yields :

$$\begin{cases} \alpha \mathbf{x}_0 e^{\alpha t} = \mathbf{A}_s \mathbf{x}_0 e^{\alpha t} + \mathbf{B}_s \mathbf{u} e^{\alpha t} \\ \mathbf{y} e^{\alpha t} = \mathbf{C}_s \mathbf{x}_0 e^{\alpha t} + \mathbf{D}_s \mathbf{u} e^{\alpha t} \end{cases} \quad (3.27)$$

Omitting the  $e^{\alpha t}$  factors gives :

$$\begin{cases} \mathbf{y} = (\mathbf{C}_s(\alpha\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s + \mathbf{D}_s)\mathbf{u} \\ \mathbf{x}_{0s} = (\alpha\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s\mathbf{u} \end{cases}, \quad (3.28)$$

or,

$$\begin{cases} \mathbf{u} = (\mathbf{C}_s(\alpha\mathbf{I} - \mathbf{A}_s)^{-1}\mathbf{B}_s + \mathbf{D}_s)^{-1}\mathbf{y} \\ \mathbf{x}_{0s} = (\alpha\mathbf{I} - \mathbf{A}_s)^{-1}\mathbf{B}_s\mathbf{u} \end{cases}. \quad (3.29)$$

This can be solved for the vectors  $\mathbf{u}$  and  $\mathbf{x}_0$ . It is clear that, if  $\alpha$  is a transmission zero of the subsystem, the first equation cannot be solved for  $\mathbf{u}$ , because in that case  $(\mathbf{C}_s(\alpha\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s + \mathbf{D}_s)$  is singular. Moreover, if there are more outputs than inputs (in this context, more boundary nodes than sources in the subsystem), the equation for  $\mathbf{u}$  is normally unsolvable.

In this way, the active subsystem can solve the information equations in the system and match the requirements of the autonomously behaving passive subsystems. In addition, there should be enough degrees of freedom (sources) to solve all the information equations. To that end, there should be at least one active subsystem adjacent to every node. Next subsection, however, introduces another way to solve the information equation of a node.

Another result from equation (3.29) is the necessary initial state vector, or more physically, the necessary initial energy storage in the active subsystem to comply with all requirements of the zero input. Eventually, the general procedure will use this initial state vector of the active subsystem to assemble the total initial state vector corresponding to the zero input.

### 3.4.3 Relaxed Subsystems

Besides the difference active  $\leftrightarrow$  passive, there is another classification of the subsystems : every subsystem can be “relaxed” or not.

ACTIVE	PASSIVE	
energized	eigenmode	relaxed
$\mathbf{x}_{0s} = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s\mathbf{u}$ $\mathbf{u} = (\mathbf{C}_s(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s + \mathbf{D}_s)^{-1}\mathbf{y}$	$\mathbf{x}_{0s}$ =eigenvector $z$ = eigenvalue	$\mathbf{x}_{0s} = \mathbf{0}$
<u>remark</u> outputs are complementary variables of junction variables	<u>remark</u> subsystem evolves along an eigenmode	<u>remark</u> no energy storage

Table 3.1: The different conditions of subsystems

**Definition 3.6 (Relaxed subsystem)** *A relaxed subsystem contains no initial energy, and consequently has an initial state vector equal to zero. In addition, there should not be any extra power flow into the subsystem, so that its energy content remains zero all the time. The subsystem is virtually 'dead' and does not contribute to the dynamics of the whole system.*

Relaxation of subsystems is another way to solve information equations. If all adjacent subsystems to a certain node are relaxed, the information equation of that node is automatically satisfied. In a relaxed system, namely, all variables are zero, so that both sides of the information equation are identically zero,  $0 \equiv 0$ .

To end this section, table 3.1 summarizes the different conditions of subsystems, where  $\mathbf{x}_{0s}$  is the initial state corresponding to that particular subsystem and  $\mathbf{u}$  is the input vector.

## 3.5 General Procedure

### 3.5.1 Layout of the Algorithm

Finding the zeros of a system is equivalent to finding the different subsystems whose eigenvalues are the zeros. Following steps suggest some candidate subsystems and

check then whether they are valid or not.

1. Set up a scheme of nodes, starting with the “zero nodes” and if necessary adding some “artificial nodes” (see section 3.5.2).
2. Identify which subsystems are active and which are passive.
3. Choose one passive subsystem with at least two elements of which at least one energy storage element. Assume it to perform an autonomous modal motion along one of its eigenvectors. All the other passive subsystems are supposed to be relaxed. Exception is made if two subsystems have the same eigenvalue. Then they can both be considered in modal motion.
4. Set up all “information equations” of the nodes.
5. Check whether all information equations can be solved by having the right input source function  $ue^{zt}$  and initial states of the active subsystem.
6. If all information equations can be satisfied, assemble the initial state vector of the zero  $z$ , and go back to 3) if there is another mode of that subsystem. If the information equations cannot be satisfied, this subsystem won’t provide any zeros. Go back to 3) and choose another passive subsystem. If that is not possible, go back to 1) and set up another scheme of nodes.

### 3.5.2 Where to Choose Artificial Nodes ?

If there is only one power line<sup>1</sup> from the input to the output, the artificial nodes, if any, will always lie on that power line. We show this by negative demonstration. Suppose one chooses a junction not lying on the single power line to be an artificial node. That would cause the active subsystem to have more than one boundary node

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<sup>1</sup>A power line is a chain of bonds and junctions



and, therefore, more than one information equation to satisfy, which is an unsolvable problem.

Moreover, not all junctions on the single power line are valid candidates for artificial nodes. Zeros different from the origin can only be eigenvalues of subsystems with two or more elements of which at least one is an energy storing element. Zeros at the origin proceed from an  $I$ -element or  $C$ -element in integral causality attached to respectively a 0-junction or a 1-junction. Therefore, only those junctions on the power line that branch off one of these mentioned systems should be chosen as artificial node.

If there are several power lines between the source input and the zero output, more positions for the artificial nodes are possible. The last chapter will cover some features of that kind of system.

**Rule 3.1 (Where to choose artificial nodes)** *In case of a single power line connecting the input to the output, consider for artificial nodes only junctions on that power line that branch off a zero providing subsystem as explained above.*

### 3.5.3 Which Subsystems should be Relaxed ?

**Rule 3.2 (Relaxation of other passive subsystems)** *All passive subsystems with eigenvalues other than the proposed zero  $z$  should be relaxed.*

This is needed because all variables in the system should evolve with the same exponent  $\sim e^{zt}$ . If two passive subsystems performed a different modal motion, the information equations at the boundary nodes would exhibit different time evolutions, e.g.  $\sim e^{z_1 t}$  and  $\sim e^{z_2 t}$ . It is impossible to satisfy both equations for all time  $t$  with all inputs  $\sim e^{zt}$ .

This rule immediately determines a considerable part of the initial state vector  $\mathbf{x}_0$ , since no initial energy storage exists in relaxed subsystems. This is even more

clear, if the states are in close relationship to the energy storage in one particular part of the system. In that case, a lot of zeros appear in the  $\mathbf{x}_0$  vector.

Another result of this rule is the following situation :

**Rule 3.3 (Relaxation to solve an information equation)** *If a node has only two adjacent subsystems of which one is relaxed, then the other should be relaxed too.*

Again, we prove by negative demonstration. Suppose a node has only two adjacent subsystems of which one is in modal motion and the other is relaxed, its information equation is unsolvable, since the information equation will look like  $0 = ae^{zt}$  with  $a \neq 0$ .

### 3.5.4 The Assembly of the Initial State Vector

The initial state vector  $\mathbf{x}_0$  can be assembled from the information about the nodes and the subsystems. The use of natural states, representing the energy storage in all subparts, makes this assembly easier. Following rules apply :

1. If the junction variable of a node is a state, that state is always zero, so also at time  $t_0$ .
2. All states corresponding to a relaxed subsystem are zero.
3. The states of a subsystem in modal motion are "colinear" with the corresponding eigenvector.
4. The states of an active subsystem,  $\mathbf{x}_{0s}$ , can be calculated from the equation :

$$\mathbf{x}_{0s} = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s \left[ \mathbf{C}_s(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s + \mathbf{D}_s \right]^{-1} \mathbf{y}. \quad (3.30)$$

5. The magnitudes of the states for the different subsystems found in 3) and 4) are determined by the information equations of the nodes. It will turn out, however, that the final  $\mathbf{x}_0$  is always known up to a factor  $\mu$ .

The examples in next section will illustrate these rules. If linear combinations of the natural states are used in the vector  $\mathbf{x}_0$ , it is easier to calculate first the natural states and to convert that result afterwards, using the linear combination.

### 3.6 Worked-out Example

Consider following mechanical system consisting of masses, springs and dampers. Figure 3.9 gives its bond graph representation.

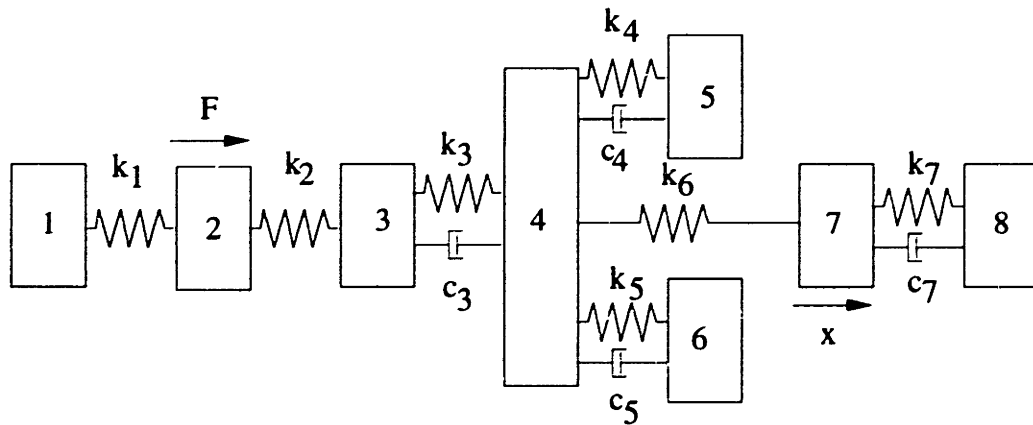


Figure 3.8: Example of mechanical system

The input force works on mass 2, whereas the position of mass 7 is the output. There are 16 states in the system : the position and velocity of each mass. This section will focus on the *procedure* to find the zeros. Therefore, it will avoid the numerical computations; however, part of it can be found in Appendix A. The logic steps below follow the general procedure in previous section.

#### First scheme of nodes

1. *Set up a scheme of nodes.* There is one zero node : the 1-junction representing the velocity of mass 7. This node must be present in every suggested scheme of

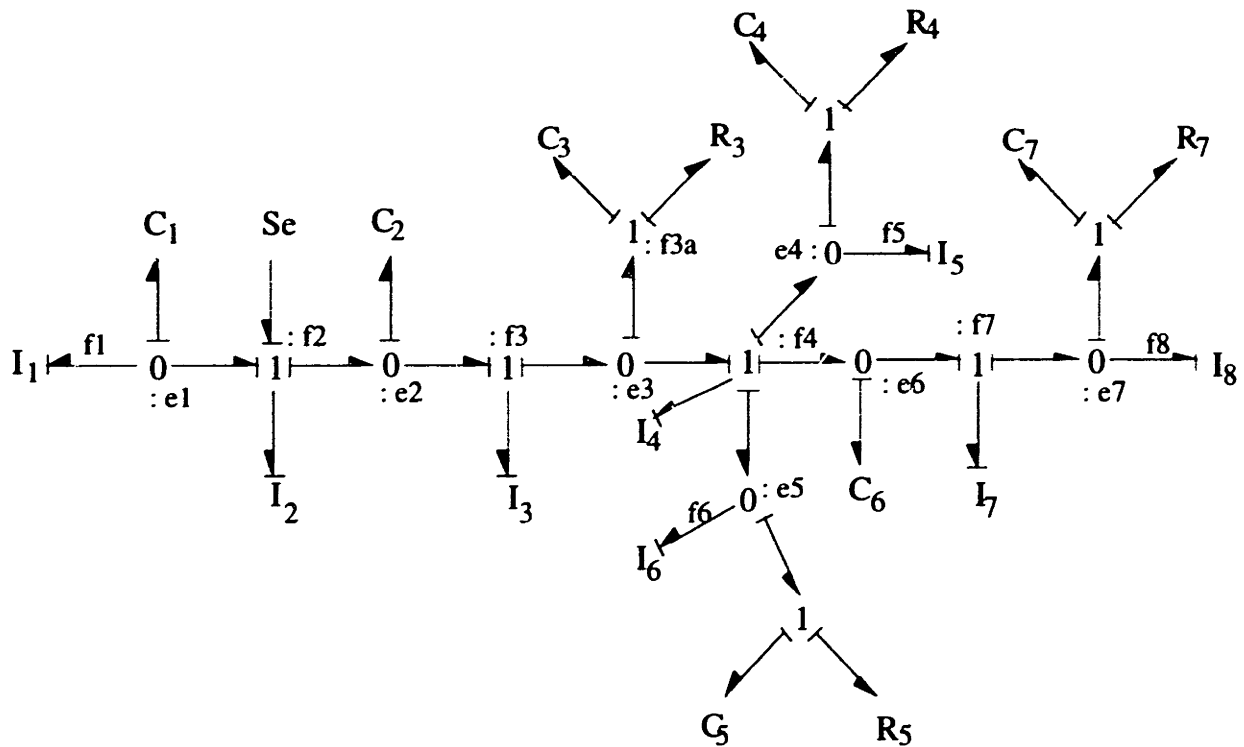


Figure 3.9: Bond graph of mechanical system

nodes. First, consider only this node.

2. *Identify the subsystems.* The single zero node divides the system into an active subsystem at the left side and a passive subsystem at the right side of mass 7. Figure 3.10 shows the subsystems as blocks and the nodes as circles.

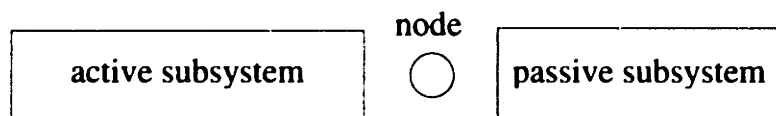


Figure 3.10: The subsystems resulting from the first scheme of nodes

3. *Assume autonomous modal motion of the passive subsystem.* The subsystem is a simple mass-spring-damper system with following bond graph. It has two eigenmodes :  $\mathbf{v}_1 e^{\lambda_1 t}$  and  $\mathbf{v}_2 e^{\lambda_2 t}$ , determined by  $I_8$ ,  $C_7$ , and  $R_7$ . Its eigenvalues

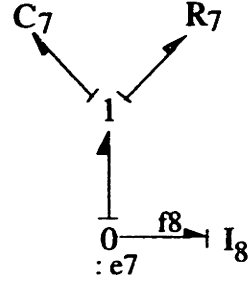


Figure 3.11: The bond graph of the passive subsystem

$\lambda_1$  and  $\lambda_2$  are the first two zeros of the whole system  $z_1$  and  $z_2$ . The vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the  $2 \times 1$  eigenvectors containing the two states of the subsystem. Note that the eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  indicate direction information only and not magnitude. Choose the first mode.

4. *Set up all information equations.* There is an information equation for every node. In this case, there is only one :

$$e_6 = e_{I_7} + e_7. \quad (3.31)$$

and since  $e_{I_7} = 0$ , this is equivalent to :

$$e_6 = e_7. \quad (3.32)$$

5. *Can all information equations be satisfied ?* In this case, it is possible :  $e_7$  can be considered as an output of the passive subsystem and has following form :

$$e_7 = ae^{z_1 t}, \quad (3.33)$$

in which the constant scalar  $a$  is a function of the elements of vector  $\mathbf{v}_1$ . The active subsystem can be forced with an certain input  $u_{a1}e^{z_1 t}$  and an initial state  $\mathbf{x}_{a1}$  in order to balance  $e_7$ . Therefore, consider  $e_6$  as the output of the active

subsystem and set up its state space equations with matrices  $\mathbf{A}_{a1}$ ,  $\mathbf{B}_{a1}$ ,  $\mathbf{C}_{a1}$ ,  $\mathbf{D}_{a1}$ . Previous section gives the equations to calculate the necessary input vector and initial state of the active subsystem :

$$u_{a1} = [\mathbf{C}_{a1}(z_1\mathbf{I} - \mathbf{A}_{a1})^{-1}\mathbf{B}_{a1} + \mathbf{D}_{a1}]^{-1} a \quad (3.34)$$

$$\mathbf{x}_{a1} = (z_1\mathbf{I} - \mathbf{A}_{a1})^{-1}\mathbf{B}_{a1} \cdot \mathbf{u}_{a1}. \quad (3.35)$$

6. *Assemble the total initial state  $\mathbf{x}_0$ .* The assembly has the following building blocks. The positions and velocities of masses 1 through 6 are given by the  $12 \times 1$  initial state vector of the active subsystem  $\mathbf{x}_{0s}$ . Since mass 7 is the node, its position and velocity are zero. The last two states, the position and velocity of mass 8, are determined by the  $2 \times 1$  eigenvector  $\mathbf{v}_1$

$$\mathbf{x}_0 = (x_{1o} \ \dot{x}_{1o} \ x_{2o} \ \dot{x}_{2o} \dots x_{8o} \ \dot{x}_{8o})^T = \begin{pmatrix} \mathbf{x}_{a1} \\ 0 \\ 0 \\ \mathbf{v}_1 \end{pmatrix} \quad (3.36)$$

$$= \begin{pmatrix} (z_1\mathbf{I} - \mathbf{A}_{a1})^{-1}\mathbf{B}_{a1} \cdot [\mathbf{C}_{a1}(z_1\mathbf{I} - \mathbf{A}_{a1})^{-1}\mathbf{B}_{a1} + \mathbf{D}_{a1}]^{-1} a \\ 0 \\ 0 \\ \mathbf{v}_1 \end{pmatrix}. \quad (3.37)$$

Following the general procedure, we can find another zero, by starting from the second mode of the same subsystem  $\mathbf{v}_2 e^{z_2 t}$ . If  $e_7 = b e^{z_2 t}$ , the according initial state for this zero  $z_2$  is :

$$\mathbf{x}_0 = \begin{pmatrix} (z_2\mathbf{I} - \mathbf{A}_{a1})^{-1}\mathbf{B}_{a1} \cdot [\mathbf{C}_{a1}(z_2\mathbf{I} - \mathbf{A}_{a1})^{-1}\mathbf{B}_{a1} + \mathbf{D}_{a1}]^{-1} b \\ 0 \\ 0 \\ \mathbf{v}_2 \end{pmatrix}. \quad (3.38)$$

Since there is no other passive subsystem for this scheme of nodes, the next step is to consider additional artificial nodes. In principle, every junction can be considered as an artificial node. However, as there is a single power line in this system connecting

the input and the output, all possible artificial nodes (i.e. providing a zero) lie on that powerline. According to subsection 3.5.2, only nodes at 1-junctions 2 and 4, and at 0-junction 3 can provide more zeros.

### Second scheme of nodes

1. First put an artificial node at 1-junction 4, together with the zero node at 1-junction 7.
2. The resulting subsystems are :

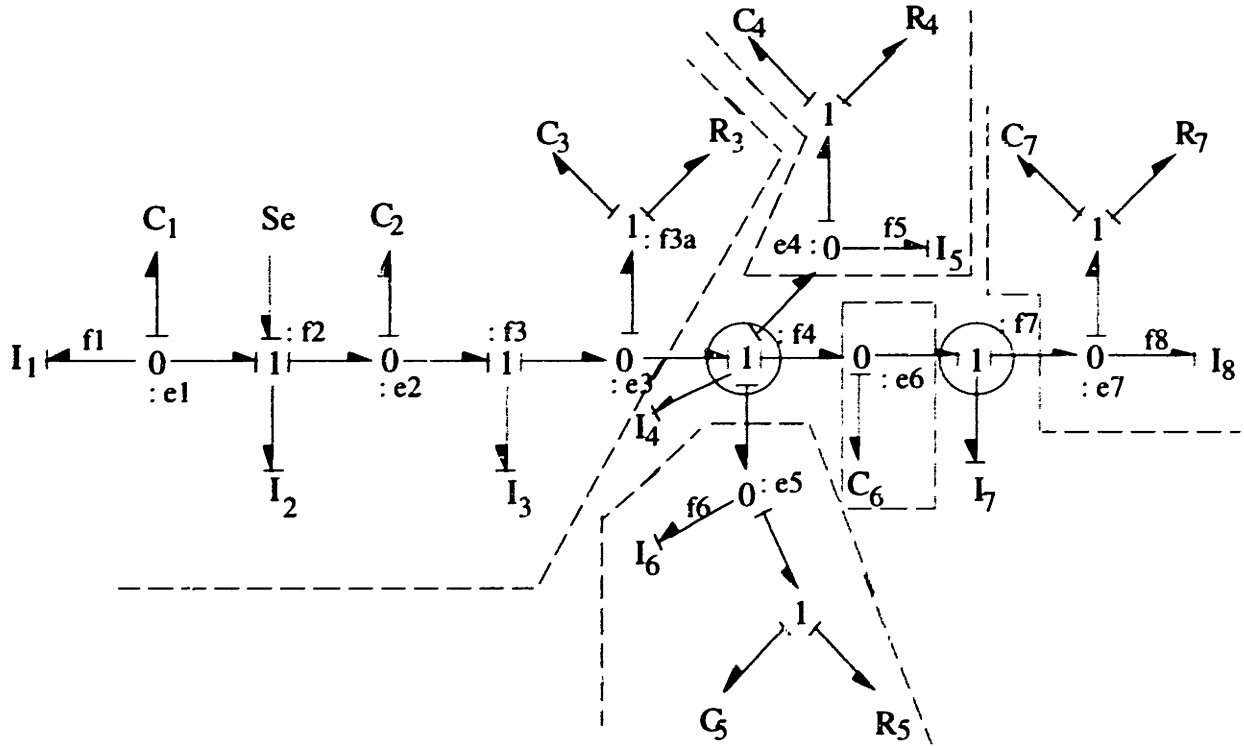


Figure 3.12: Bond graph of the second scheme of nodes

3. Assume modal motion of passive subsystem 1 (containing mass  $m_5$ , damper  $c_4$  and spring  $k_4$ ). It has two modes  $\mathbf{v}_3 e^{z_3 t}$  and  $\mathbf{v}_4 e^{z_4 t}$ .

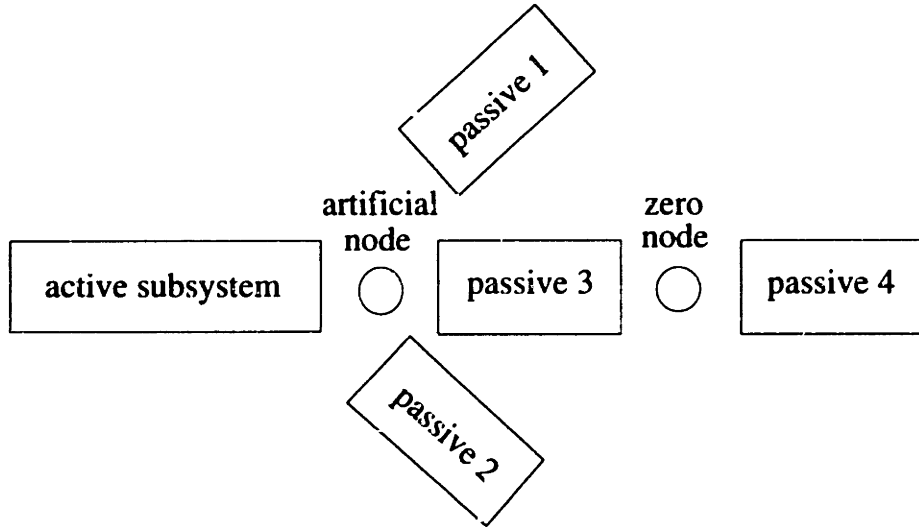


Figure 3.13: The subsystems resulting from second scheme of nodes

4. There are two information equations :

$$\begin{cases} e_3 = e_4 + e_5 + e_6 \\ e_6 = e_7 \end{cases} \quad (3.39)$$

Since the two subsystems adjacent to the zero node are passive, they are both relaxed. Moreover, because passive subsystems 2 has different eigenvalues than passive subsystem 1, it must be relaxed too. All the variables of relaxed subsystems are identically zero. Consequently, the only non trivial information equation is :

$$e_3 = e_4. \quad (3.40)$$

5. The information equation can be satisfied using the one degree of freedom (one source) of the active subsystem  $\mathbf{A}_{a2}, \mathbf{B}_{a2}, \mathbf{C}_{a2}, \mathbf{D}_{a2}$ , which has 6 states  $(x_{1o}, \dot{x}_{1o}, x_{2o}, \dot{x}_{2o}, x_{3o}, \dot{x}_{3o})$  and output  $e_3 = ce^{z_3 t}$ . In the assembly of the initial state, all states of the relaxed passive subsystems 2,3 and 4 are 0 (states  $x_{6o}, \dot{x}_{6o}, \dots, \dot{x}_{8o}$ ). Mass 4 is a node, so its position and velocity are also zero.



Finally  $x_{50}$  and  $\dot{x}_{50}$  are determined by the eigenvector  $\mathbf{v}_3$  :

$$\mathbf{x}_0 = \begin{pmatrix} (\lambda_3 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2} \cdot [\mathbf{C}_{a2}(\lambda_3 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2} + \mathbf{D}_{a2}]^{-1} c \\ 0 \\ 0 \\ \mathbf{v}_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.41)$$

Repeating the steps 3 to 5 for the other mode of passive subsystem 1 provides the initial state for the fourth zero  $z_4$ .

Another choice of passive subsystem is number 3. This subsystem, however, should always be relaxed, as mentioned before, and cannot perform a modal motion.

Finally, passive subsystem 2 has two modes  $\mathbf{v}_5 e^{z_5 t}$  and  $\mathbf{v}_6 e^{z_6 t}$ . From the structure, it is clear that this case is completely symmetric to the study of subsystem 1 in modal motion.

### Third scheme of nodes

1. Choose 0-junction 3 as artificial node, together with the zero node.
2. The resulting subsystems are given in figure 3.14
3. The active subsystem has state space matrices  $\mathbf{A}_{a3}$ ,  $\mathbf{B}_{a3}$ ,  $\mathbf{C}_{a3}$ ,  $\mathbf{D}_{a3}$ . Passive subsystems 2 and 3 should be relaxed. The passive subsystem 1 has only one real mode :  $z_7 = -\frac{k_3}{c_3}$ .
4. If the position and velocity of all masses are chosen as states, the junction variable of this artificial node,  $e_3$ , is not a state. Therefore, the node introduces

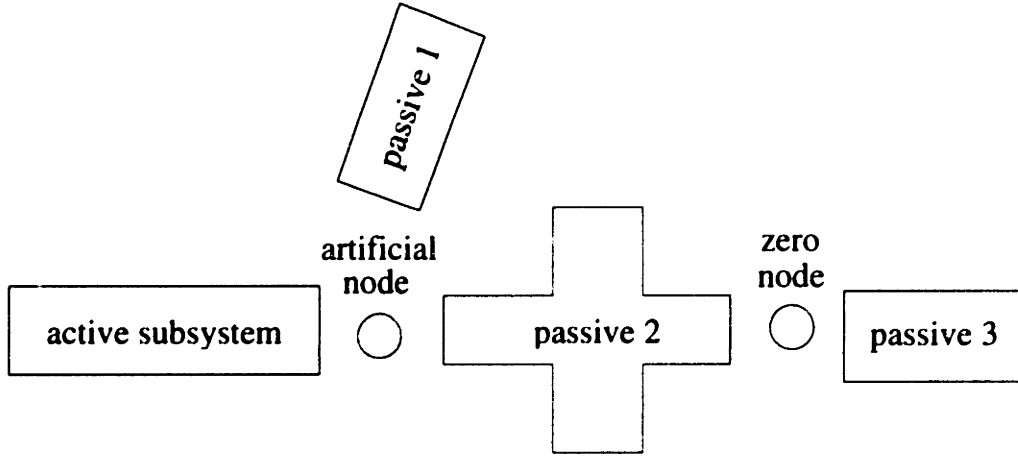


Figure 3.14: The subsystems resulting from the third scheme of nodes

a relationship between the states :

$$e_3 \equiv 0 \Leftrightarrow k_3(x_3 - x_4) + c_3(\dot{x}_3 - \dot{x}_4) \equiv 0. \quad (3.42)$$

Because subsystem 2 is relaxed, this simplifies to :

$$k_3 x_3 + c_3 \dot{x}_3 \equiv 0. \quad (3.43)$$

The corresponding information equation is :

$$f_3 = f_{3i} = g e^{-\frac{k_3}{c_3} t}, \quad (3.44)$$

where  $g$  is some constant. The resulting initial state is :

$$\mathbf{x}_0 = \begin{pmatrix} (z_7 \mathbf{I} - \mathbf{A}_{a3})^{-1} \mathbf{B}_{a3} \cdot [\mathbf{C}_{a3}(z_7 \mathbf{I} - \mathbf{A}_{a3})^{-1} \mathbf{B}_{a3} + \mathbf{D}_{a3}]^{-1} g \\ \mathbf{0}_{10 \times 1} \end{pmatrix}. \quad (3.45)$$

The question remains whether this initial state also satisfies the requirement (3.43).

This is however always true. In this case, all variables evolve  $\sim e^{-\frac{k_3}{c_3} t}$ . This means following relationship between the initial states  $x_3$  and  $\dot{x}_3$  :

$$\dot{x}_3 = -\frac{k_3}{c_3} x_3, \quad (3.46)$$

so that

$$k_3 x_3 + c_3 \dot{x}_3 = k_3 x_3 + c_3 \left( -\frac{k_3}{c_3} x_3 \right) = 0 \quad (3.47)$$

#### Fourth scheme of nodes

1. The last possible artificial node is 1-junction 2.
2. In this case, the active subsystem is in its most simple form : it is the bare effort source  $Se$ , as shown in figure 3.15 : Subsystem 1 performs the modal motion,

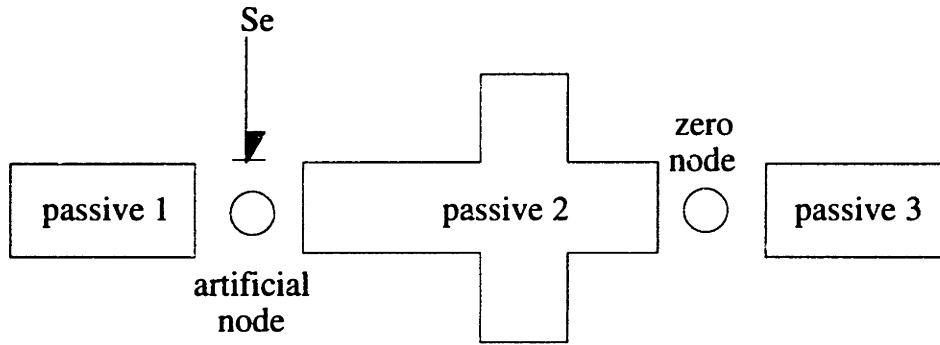


Figure 3.15: The subsystems resulting from fourth scheme of nodes

whereas all other subsystems are relaxed.

3. The setup of the information equation is not necessary; the source simply balances the reaction forces from the modal motion.
4. The initial states for both eigenvalues  $z_8$  and  $z_9$  are just the corresponding eigenvectors of subsystem 1 :

$$\mathbf{x}_0 = \begin{pmatrix} \mathbf{v}_{8,9} \\ \mathbf{0}_{14 \times 1} \end{pmatrix} \quad (3.48)$$

### 3.7 Pole-Zero Cancellation

The main result of the theory in previous sections is the identification of a zero of the system as the eigenvalue of one of its subsystems. One special case, but important for

control theory, happens when there are a pole and a zero in the system with the same value (in SISO systems). That case is called a pole-zero cancellation, because to the outside world neither the pole, nor the zero have visible effect. However, internally they are present and an unstable pole-zero cancellation (complex value in the right half plane) causes a lot of problems. Given the relationship between the zeros and the eigenvalues of subsystems, it is obvious that the pole-zero cancellation is embedded in a special bond graph structure.

### 3.7.1 Definition of Pole-Zero Cancellation

Suppose value  $a$  is as well a pole as a zero of system  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ . Being a pole means there is at least one vector  $\mathbf{v}$  in the null space of following matrix :

$$(a\mathbf{I} - \mathbf{A}) \mathbf{v} = 0. \quad (3.49)$$

so that an unforced (i.e. no actuation) modal motion of the whole system is possible :

$$\mathbf{x}(t) = \mathbf{v}e^{at}. \quad (3.50)$$

Being a zero requires (see equation (2.50)) :

$$\begin{bmatrix} a\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{pmatrix} \xi \\ \mathbf{u} \end{pmatrix} = 0 \quad (3.51)$$

for certain vectors  $\xi$  and  $\mathbf{u}$ . To exhibit a pole-zero cancellation, however, vectors  $\xi$  and  $\mathbf{v}$  should be colinear [21, pages 66–67] :

$$\xi = k\mathbf{v}. \quad (3.52)$$

That requirement simplifies equation (3.51), using equation (3.49), to :

$$\begin{cases} \mathbf{B}\mathbf{u} = 0 \\ \mathbf{C}\mathbf{v} + \mathbf{D}\mathbf{u} = 0 \end{cases} \quad (3.53)$$

In practice, the columns of the  $\mathbf{B}$  matrix (in MIMO systems) are linearly independent as linear dependence would imply that all actuators (e.g. motors) can work together in a certain way to achieve no net effect on the states. No real physical system will exhibit such behaviour, because it means pure loss of power. Therefore, the  $\mathbf{u}$  vector must be identically zero and the only remaining equation is :

$$\mathbf{C}\mathbf{v} = 0. \quad (3.54)$$

The conclusion is that in a system with a pole-zero cancellation, there is no zero input at all (the actuation itself is zero) and the corresponding initial state  $\mathbf{x}_0$  is the eigenvector  $\mathbf{v}$  of the same pole.

### 3.7.2 Pole-Zero Cancellations in Bond Graphs

Since the modal motion of the total system also behaves as the zero input motion, it must have the same characteristics : some junctions in the system are nodes and some subsystems are isolated from the rest. Consequently, all information equations of the nodes must be satisfied by the modal motion. Previous chapters state that solving an information equation requires at least one active subsystem adjacent to that node. This pole-zero cancellation case, however, is an exception to that rule, as all sources are inactive, which means that all subsystems are passive. The information equation of a node surrounded by nothing but passive subsystems is only solvable if and only if at least two of the adjacent subsystems have the same eigenvalue.

**Rule 3.4 (Pole-zero cancellation)** *A system exhibiting a pole-zero cancellation must have a scheme of nodes, so that at least one node has two adjacent subsystems with the same eigenvalue  $z$ , solving the information equation of that node. All other nodes are surrounded by relaxed subsystems.*

Proof : The input vector  $\mathbf{u}$  of a pole-zero cancellation should be  $\mathbf{0}$ . So, no active subsystem can be energized, nor can they solve the information equations. The two remaining ways to solve an information equation are :

- All subsystems surrounding the node are relaxed.
- Two or more subsystems adjacent to the node have the same eigenvalue and balance each other.

If all information equations in a pole-zero cancellation case were solved in the first way, the whole system would be relaxed. That is not acceptable, since the eigenvector or the initial state vector should be different from the null-vector. Therefore, at least one information equation should be solved in the second way, and there are at least two subsystems with the same eigenvalue. (q.e.d.)

Several possible schemes exist : just two subsystems in similar modal motion, adjacent to the same node, or a whole chain of such subsystems. Following figures give two an example of both possibilities.

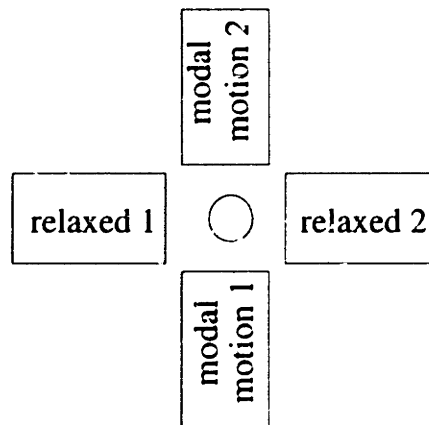


Figure 3.16: A possible structure of subsystems for a pole zero cancellation 1

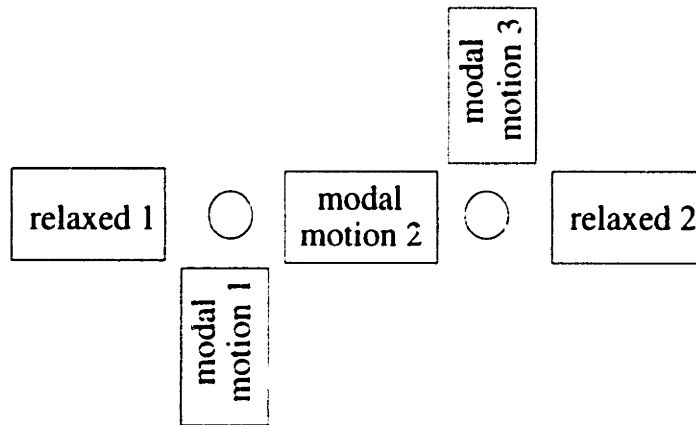


Figure 3.17: A possible structure of subsystems for a pole zero cancellation 2

### 3.8 Summary

This chapter gives a physical interpretation of zeros in SISO systems : zeros are the eigenvalues of subsystems that are energetically isolated from the rest of the system by the zero output condition (zero nodes) or by a chosen relationship between the states (artificial nodes). The identification of energetically isolated subsystems is easier using the bond graph representation, which displays the power flow paths in the system. The procedure also provides the necessary equations to obtain the according initial state for the considered zero, not by solving a generalized eigenvalue problem, but easy regular eigenvalue problems and linear algebra computations.

The presented procedure is not only helpful for calculating the zeros and their initial states of a given system, but also give insight into how changes to the parameters of the real design may influence the values of the zeros.

Finally, a small comment is made about the special case of pole-zero cancellation, which has a clear interpretation in this framework too.

## Chapter 4

# The Physical Meaning of Zeros in MIMO Systems

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### 4.1 Introduction

This chapter extends the ideas of previous chapter to physical systems with multiple inputs and multiple outputs (MIMO). Chapter 2 already covered in detail the more mathematical aspects and definitions of so-called transmission zeros in MIMO systems. The bond graph representation of a MIMO system is very similar to that of SISO systems; they only differ in the presence of more than one source input. The second section discusses the manifold similarities between the reasoning about transmission zeros and the zeros of SISO systems. The third section on the other hand introduces the differences of MIMO systems and how they affect the physical interpretation of the transmission zeros and their calculation. A description of the general procedure, adapted for MIMO systems, precedes a section with a worked-out example of the calculation, using the same mechanical structure as in chapter 3.

### 4.2 Similarity SISO-MIMO Systems

The basic ideas of the physical interpretation of zeros and the derived procedure to calculate the corresponding initial state vector are easily transferable to the MIMO case. Again, energy considerations form the starting point. Therefore, also the study of transmission zeros utilizes bond graphs as a major tool to identify the parts of the system responsible for every transmission zero. Another analogy is the explanation



for the special case of pole-zero cancellations.

### 4.2.1 Transmission Zeros as Eigenvalues of Subsystems

Also the MIMO case assumes that all outputs are junction variables. The inputs are flow or effort sources. Following steps repeat the logic behind the procedure and add some comments to show their applicability for MIMO systems.

1. A junction whose variable is one of the outputs is called a zero node. No energy can go from one side of the node to the other side : the energy blocking effect is exactly the same as in the SISO case. In addition, every zero node introduces an information equation.
2. The presence of (multiple) nodes in the system causes a subdivision into subsystems, of which at least one is energetically isolated and will behave autonomously.
3. The derivation of equation (3.23) on page 69 still holds; the evolution of the states under a zero input is rectilinear :

$$\mathbf{x}(t) = \mathbf{x}_0 e^{zt}. \quad (4.1)$$

(All state space derivations are applicable to  $l \times m$  MIMO systems by merely replacing the scalars  $u$  and  $y$  by vectors  $\mathbf{u}$  ( $m \times 1$ ) and  $\mathbf{y}$  ( $l \times 1$ ))

4. The states represent the energy storage. They can be allocated to certain parts of the system
5. An isolated subsystem can only perform a rectilinear motion in the state space, when it is in modal motion (with its corresponding eigenvalue as time exponent), or when it is relaxed. In the latter case, there is no initial energy stored and

no additional energy can enter the subsystem, so that all states are identically zero. Therefore, every transmission zero of a physical system without loops (see chapter 5) is also an eigenvalue of an energetically isolated subsystem.

6. The active subsystems offer the necessary actuation number to balance the information equations at the nodes. The outputs of the active subsystem, expressed by matrices  $\mathbf{C}_s$  and  $\mathbf{D}_s$ , are the complementary variables of the junction variables of their boundary nodes. They are collected in vector  $\mathbf{y}$ . Equation (3.30) represents the initial state of the active subsystem to obtain the required outputs  $\mathbf{y}_s(t) = \mathbf{y}_s e^{z t}$ .
7. The assembly of the total initial state vector has following building blocks : the eigenvectors of subsystems in modal motion, the initial state of the active subsystems and identically zero states for node variables and variables of relaxed subsystems.
8. The scheme of nodes can be extended by adding artificial nodes. That may reveal additional isolated subsystems whose eigenvalues are transmission zeros of the system

### 4.2.2 Pole-Zero Cancellations

The requirement for a pole-zero cancellation in a MIMO system is not only the coincidence of a pole and a transmission zero in the complex plane, but also the equality of their directions in the state space, i.e. the eigenvector for the pole and the initial state for the zero.

In fact, a pole-zero cancellation always involves a non-observable mode. In other words, there is a modal motion so that the output variables are zero. Moreover,

because of its modal character, the input vector  $\mathbf{u}$  for a transmission zero coinciding with a pole is identically zero.

The consequence of the nodes in the modal motion is an energetic subdivision in isolated subsystems, similar to that for ordinary transmission zeros, and the presence of information equations. To comply with both these features, the transmission zero must be the eigenvalue of at least two such isolated subsystems ! Thus, the main characteristic of a pole-zero cancellation is the same in the SISO and MIMO case.

### 4.3 Difference between SISO-MIMO Systems

Besides the striking similarities between SISO and MIMO systems in the basic ideas of the procedures, some differences make the calculations more difficult in the MIMO case.

#### 4.3.1 Multiple Outputs

Multiple outputs introduce multiple zero nodes in the node scheme. This can yield a more subdivided network of subsystems. However, interference between the different nodes can form additional implications. It is necessary to check for these interferences before identifying the isolated subsystems.

First, two nodes can work together in such a way that they imply a third node in the bond graph. This often happens when there is only one other junction between the two nodes (consequently, the nodes are junctions of the same type). Consider the following partial bond graph example. The encircled junctions are nodes. So,  $f_1$  and  $f_2$  are, therefore, identically zero. The junction equation of the 0-junction in between the two nodes requires :

$$f_1 = f_2 + f_3. \quad (4.2)$$

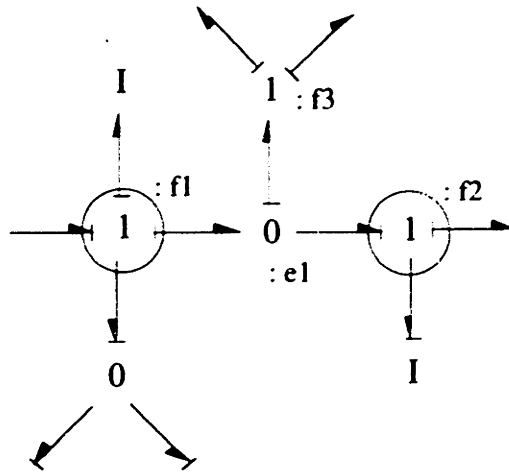


Figure 4.1: Bond graph example of interference of nodes 1

Combining both statements results in :

$$f_3 \equiv 0. \quad (4.3)$$

This implies that the 1-junction with variable  $f_3$  is also a node.

Second, if two nodes are next to each other, they form not only an energy barrier, but also an information barrier. It results in a complete separation of the system. Figure 4.2 shows an example of this kind of interference. The bond between the

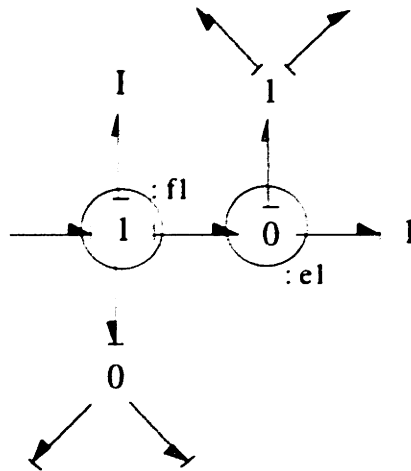


Figure 4.2: Bond graph example of interference of nodes 2

two nodes cannot carry any power, because not only one variable, but both its flow

variable,  $f_1$ , and effort variable,  $e_1$ , are identically zero. No physical contact through this bond is possible, and it is as if the bond is not existing at all. The system then divided into two separate smaller subsystems which should be studied individually.

### 4.3.2 Multiple Inputs

Whereas more than one node is in fact also possible in the SISO case by adding artificial nodes, multiple sources are a unique characteristic of MIMO systems.

Every active subsystem has a certain actuation number.

**Definition 4.1 (Actuation number of active subsystem)** *The actuation number of an active subsystem,  $\phi$ , is the column rank of the  $\mathbf{B}_s$  matrix in the state space description of the subsystem.*

The actuation number is only equal to the number of sources if all sources are independent. A dependency between the sources can also be detected from the bond graph as the following example explains. The effect of both effort sources  $Se_1$  and  $Se_2$

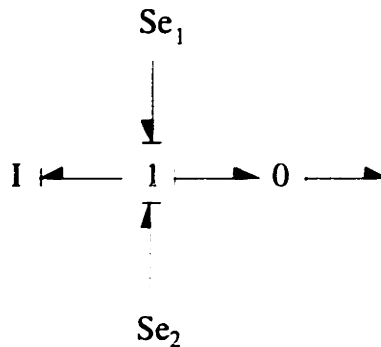


Figure 4.3: Bond graph example of dependent sources

on the states of the system (in this case the flow variable of the generalized inertia  $I$ ) is the same: it is impossible to determine from which source some effort actually originates. Therefore, this hypothetical active subsystem has actuation number equal to 1.

On the other hand, every node in the system contains an information equation, which can be solved in following ways :

1. Use an adjacent active subsystem.
2. All adjacent subsystems are relaxed.
3. All adjacent subsystems are passive and at least two of them perform a balancing modal motion (of course with same time exponent).

Once a scheme of nodes and the resulting subsystems are determined, every node will belong to one of these 3 groups. The equations of nodes in groups 2 and 3 are in fact trivial, as there is no need of inputs to solve them.

**Definition 4.2 (Non-trivial nodes)** *Non-trivial nodes are nodes whose information equation is solved by an adjacent active subsystem.*

An active subsystem can also be relaxed (see table 4.1 on page 103). To that end, the input is identically zero as well as all initial states. In the SISO case, that makes only little sense, since there is only one actuated subsystem. In the MIMO case, however, an active subsystem can perform a modal motion too, and so possibly provide another zero. This is the third condition of an active subsystem in MIMO systems, besides the regular energized and relaxed conditions.

**Definition 4.3 (Net actuation number)** *The “net actuation number” of an active subsystem is defined as the minimum of either the actuation number or the number of boundary nodes of that active subsystem.*

The meaning of the net actuation number is the actual number of independent variables that the considered active subsystem has in the set of information equations according to a particular node scheme. All previous definitions lead to following statement about the solvability of a scheme of nodes :

**Rule 4.1 (Solvability of a scheme of nodes )** *A scheme of nodes is solvable only if the sum of the net actuation numbers of all energized active subsystems is greater than or equal to the number of non trivial nodes.*

Proof : The proof will refer to the system in figure 4.4, which contains 3 passive subsystems P1, P2 and P3 and 3 active subsystems A1, A2 and A3. All 5 nodes are

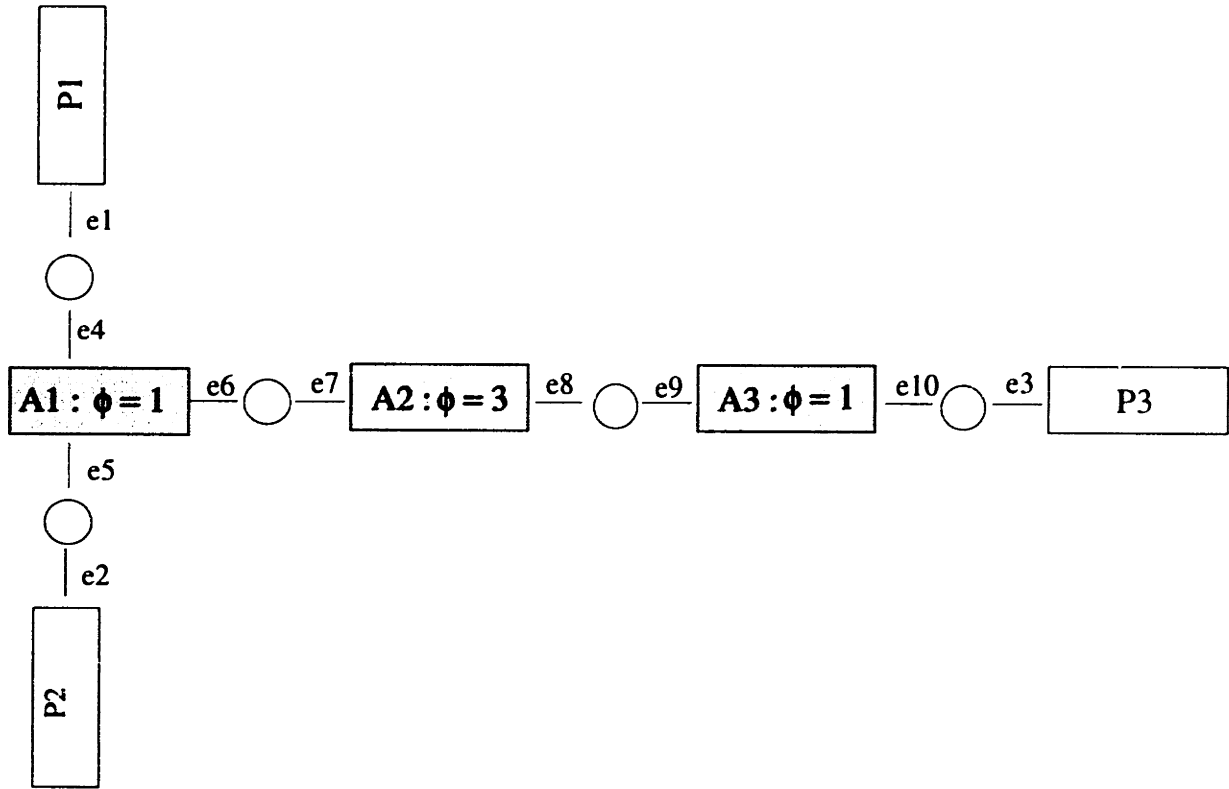


Figure 4.4: Example of a scheme of nodes and subsystems in MIMO system

considered 1-junctions. The node scheme is solvable if and only if the corresponding set of information equations is solvable. These information equations are all linear algebraic equations and contain the complementary variables of the node junction variable of all bonds towards that node. There is an information equation for every

non-trivial node. The set of information equations for this case is :

$$\begin{cases} e_1 = e_4 \\ e_2 = e_5 \\ e_6 = e_7 \\ e_8 = e_9 \\ e_{10} = e_3 \end{cases} \quad (4.4)$$

First, the passive subsystems P1, P2 and P3 are considered in a certain condition : either in modal motion or relaxed. That decision determines the values of  $e_1$ ,  $e_2$  and  $e_3$ . They all have the same time evolution  $e^{zt}$ . At this point, all 7 other variables are independent variables for this set of linear equations. Each of them can be assigned as output of a certain active subsystem. Variables  $e_4$ ,  $e_5$  and  $e_6$  belong to active subsystem A1; variables  $e_7$  and  $e_8$  belong to active subsystem A2; variables  $e_9$  and  $e_{10}$  belong to active subsystem A3. Consequently, no active subsystem can introduce more independent variables into the set of information equations than its number of boundary nodes. This fact is used in the definition of the net actuation number.

Second, the set of linear information equations will be solvable if at least 5 (the number of information equations and, therefore, the number of non-trivial nodes) of the variables  $e_4$  to  $e_{10}$  can be chosen independently. That is not always true, since these variables are determined by the inputs and initial states of every active subsystem. For each subsystem, the before mentioned outputs are combined into the corresponding output vector  $\mathbf{y}_s$ . The following equations give the necessary initial state  $\mathbf{x}_{0,s}$  and input vector  $\mathbf{u}_s$  to obtain the output  $\mathbf{y}_s$ .

$$\begin{cases} \mathbf{y}_s = (\mathbf{C}_s(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s + \mathbf{D}_s)\mathbf{u}_s \\ \mathbf{x}_{0,s} = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s\mathbf{u}_s \end{cases}, \quad (4.5)$$

If  $\mathbf{y}_s$  has more elements than  $\mathbf{u}_s$  (which is the number of independent sources and, therefore, the actuation number  $\phi$  of the active subsystem), the first equation of (4.5) is not solvable for all  $\mathbf{y}_s$ . In other words, only  $\phi$  elements of  $\mathbf{y}_s$  can be determined independently. That proves that the net actuation number cannot be larger than  $\phi$  of



that active subsystem. Moreover, if  $(\mathbf{C}_s(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s + \mathbf{D}_s)$  is rank deficient, which happens when the zero  $z$  of the whole system is also a zero of the considered active subsystem, not every  $\mathbf{y}_s$  is solvable either and that may further reduce the number of outputs that can be determined independently.

Third, combining previous reasonings explains the definition of the net actuation number as minimum of  $\phi$  and the number of boundary nodes, and its use in the rule. The set of information equations is solvable only if the number of independent variables of the set  $e_4$  to  $e_{10}$  is larger than the number of information equations. The former is the sum of all net actuation numbers of the active subsystems, whereas the latter is equal to the number of non-trivial nodes. (q.e.d.)

Application of the rule to the example in figure 4.4 yields the following result. No active subsystems is considered relaxed or in modal motion; they have respectively actuation number 1, 3 and 1. Therefore, all 5 nodes are non-trivial nodes. The net actuation number of subsystem A2, however, is only 2, since it has only 2 boundary nodes. The condition consequently does not hold for this example :

$$1(A1) + 2(A2) + 1(A3) = 4 < 5(\text{non trivial nodes}). \quad (4.6)$$

This scheme of nodes cannot provide any zero.

If the condition holds, the difference between the sum of all *regular* actuation numbers of the energized active subsystems and the number of non-trivial nodes indicates how many linear independent input vectors  $\mathbf{u}$  exist for every zero resulting from that scheme of nodes.

### 4.3.3 Subdivision using Artificial Nodes

Similar to the SISO case, artificial nodes can extend the scheme of zero nodes. In principle, every junction could be chosen as a node. However, only junctions with at least 3 bonds towards other junctions or towards a source, can yield a scheme of nodes with a zero different from the origin. Once different artificial nodes are added to the zero nodes, it is necessary to check whether there are no interferences between the nodes, as mentioned above.

## 4.4 General Procedure

### 4.4.1 Layout of the Algorithm

1. Set up a scheme of nodes, starting with the “zero nodes”, possibly adding some “artificial nodes”.
2. Identify which subsystems are active and which are passive.
3. Choose one passive subsystem *or an active subsystem* with at least two elements of which at least one energy storage element. Assume it to perform an autonomous modal motion along one of its eigenvectors. All the other passive subsystems are supposed to be relaxed. Exception is made if two subsystems have the same eigenvalue. Then they can be considered both in modal motion.
4. *Decide for every node how its information equation will be solved (3 categories mentioned in subsection 4.3.2). To that end, some active subsystems can be considered relaxed. Then check the solvability by comparing the sum of net actuation number and the number of non-trivial nodes. If the test fails, return to step 1*
5. Set up all information equations.

ACTIVE			PASSIVE	
energized	eigenmode	relaxed	eigenmode	relaxed
$\mathbf{x}_{0s} = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s\mathbf{u}_s$	$\mathbf{x}_{0s}=\text{eigenvector}$	$\mathbf{x}_{0s} = 0$	$\mathbf{x}_{0s}=\text{eigenvector}$	$\mathbf{x}_{0s} = \mathbf{0}$
$\mathbf{u}_s = [\mathbf{C}_s(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_s + \mathbf{D}_s]^{-1}\mathbf{y}_s$	$\mathbf{u}_s = 0$	$\mathbf{u}_s = 0$		

Table 4.1: The different building blocks for the assembly, where  $\mathbf{x}_{0s}$  is the initial state of the subsystem, and  $\mathbf{u}_s$  the input vector of the subsystem

6. Assemble the initial state vector, corresponding to that transmission zero, based on previous decisions for every subsystem (relaxed or not etc. ). *Assemble also the inputvector  $\mathbf{u}$  from all inputs.* Following tabel displays the necessary building blocks for the assemblies.

Go back to 3) if there is another mode of that subsystem. Otherwise, choose another passive subsystem from the same scheme of nodes. If that is not possible, go back to 1) and set up another scheme of nodes.

## 4.5 Worked-out Example

This example uses the same mechanical system as the example worked out for the SISO case. Only the location and number of the actuators and sensors differ : input forces act on mass 1 and mass 6, and the positions of mass 2 and mass 4 are the outputs. Again, this section will discuss the procedure, appendix B contains some numerical computations. Figures 4.5 and 4.6 depict the system and its bondgraph.

### First Scheme of Nodes

1. *Set up a scheme of nodes.* First consider only the zero nodes : 1-junctions 2 and 4 (with variables  $f_2$  and  $f_4$ ).

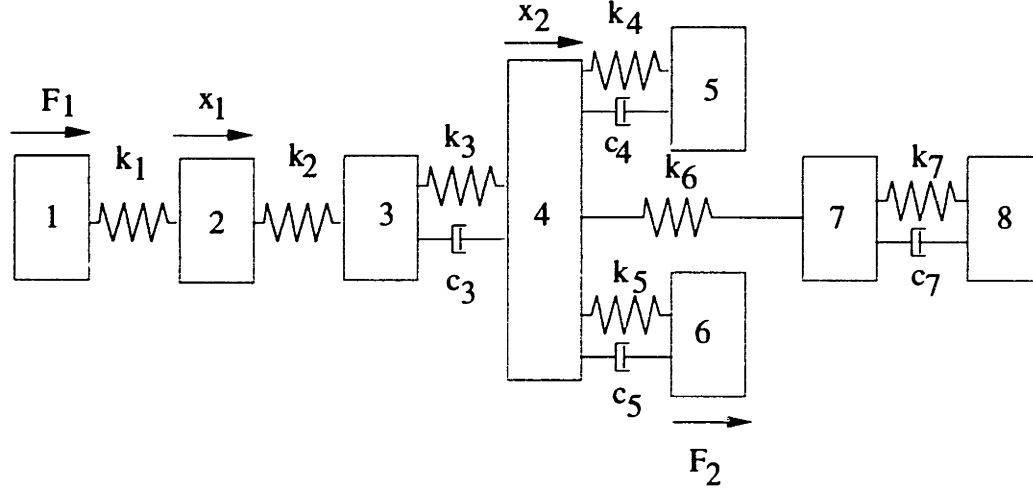


Figure 4.5: Example of mechanical system

2. *Identify the subsystems.* The two nodes create 2 active subsystems (labeled A1 and A2 in figure 4.7) and 3 passive subsystems (labeled P1, P2 and P3). For clarity, all active subsystems are depicted by gray rectangles.
3. *Choose a subsystem and assume it in modal motion.* First choice : take passive subsystem P1 in one of its two modes. The other passive subsystems P2 and P3 are relaxed.
4. *Decide for every node how its information equation will be solved.* Both nodes are adjacent to an active subsystem. It makes no sense to consider one of them relaxed. So, both nodes are non-trivial nodes. Subsystems A1 and A2 contain only one source and have 1 boundary node. Consequently, their net actuation number is 1. The sum of all net actuation numbers is 2 and equals the number of non-trivial nodes. This scheme of subsystems is solvable and will provide zeros.
5. *Set up all information equations :*

$$e_1 = e_2 \quad (4.7)$$

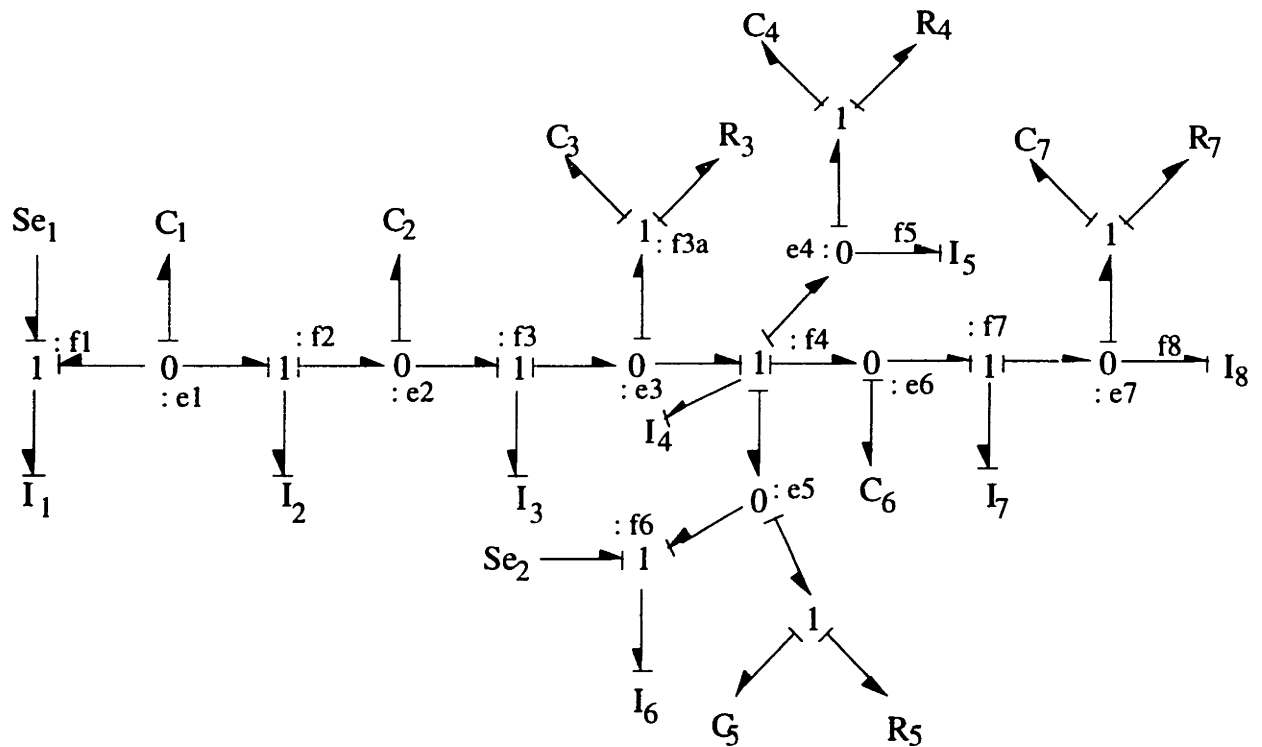


Figure 4.6: Bond graph of mechanical system

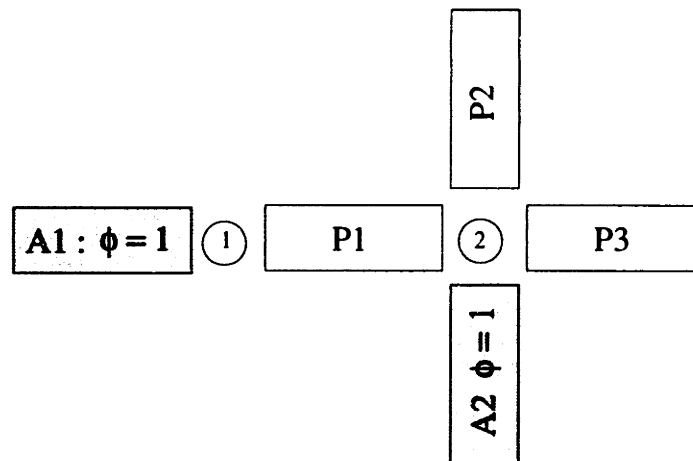


Figure 4.7: First scheme of nodes and subsystems for the worked-out example

$$e_3 = e_4 + e_5 + e_6. \quad (4.8)$$

Since P2 and P3 are relaxed,  $e_4$  and  $e_6$  are identically zero, which simplifies the second equation to :

$$e_3 = e_5. \quad (4.9)$$

6. *Assemble the initial state vector.* P1 has two modes  $\mathbf{v}_1 e^{z_1 t}$  and  $\mathbf{v}_2 e^{z_2 t}$ . The state space matrices of the active subsystems A1 and A2 are respectively :  $\mathbf{A}_{a1}, \mathbf{B}_{a1}, \mathbf{C}_{a1}, \mathbf{D}_{a1}$  and  $\mathbf{A}_{a2}, \mathbf{B}_{a2}, \mathbf{C}_{a2}, \mathbf{D}_{a2}$ . The efforts  $e_2$  and  $e_3$  under the modal motion of eigenvalue  $z_1$  are :

$$e_2 = a e^{z_1 t} \quad (4.10)$$

$$e_3 = b e^{z_1 t}, \quad (4.11)$$

with the values  $a$  and  $b$  function of the elements of  $\mathbf{v}_1$  and  $z_1$ .

From this, the total initial state vector can be constructed :

$$\begin{aligned} \mathbf{x}_0 &= (x_{1o} \ \dot{x}_{1o} \ x_{2o} \ \dot{x}_{2o} \dots x_{8o} \ \dot{x}_{8o})^T \quad (4.12) \\ &= \begin{pmatrix} (z_1 \mathbf{I} - \mathbf{A}_{a1})^{-1} \mathbf{B}_{a1} \cdot [\mathbf{C}_{a1}(z_1 \mathbf{I} - \mathbf{A}_{a1})^{-1} \mathbf{B}_{a1} + \mathbf{D}_{a1}]^{-1} a \\ \mathbf{0}_{2 \times 1} \\ \mathbf{v}_1 \\ \mathbf{0}_{4 \times 1} \\ (z_1 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2} \cdot [\mathbf{C}_{a2}(z_1 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2} + \mathbf{D}_{a2}]^{-1} b \\ \mathbf{0}_{4 \times 1} \end{pmatrix}. \quad (4.13) \end{aligned}$$

Similarly, the input vector  $\mathbf{u}$  is :

$$\mathbf{u} = \begin{pmatrix} [\mathbf{C}_{a1}(z_1 \mathbf{I} - \mathbf{A}_{a1})^{-1} \mathbf{B}_{a1} + \mathbf{D}_{a1}]^{-1} a \\ [\mathbf{C}_{a2}(z_1 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2} + \mathbf{D}_{a2}]^{-1} b \end{pmatrix}. \quad (4.14)$$

Go back to step 3. Other passive subsystems are P2 and P3. Assume P2 in modal motion and P1 and P3 relaxed. P2 has two modes.

4. *Decide for every node how its information equation will be solved.* Node 1 has an active and a relaxed subsystem adjoining. Its information equation can only be solved if the active subsystem A1 is also relaxed. Node 1 then becomes a

trivial node. Node 2, however, is non-trivial, since A2 should work to balance the modal motion of P2.

The sum of all net actuation numbers of energized active subsystems is therefore 1 and equals the number of non-trivial nodes. In fact, the resulting motion is similar to the SISO case, since only A2, which contains 1 source, and P2 are not relaxed. The calculation of the initial state vector follows exactly the same procedure :  $\mathbf{x}_0$  contains  $\mathbf{v}_3$ , the eigenvector of P2, and the initial state vector of A2. All other states are zero.

The input vector  $\mathbf{u}$  is :

$$\mathbf{u} = \begin{pmatrix} 0 \\ [\mathbf{C}_{a2}(z_1\mathbf{I} - \mathbf{A}_{a2})^{-1}\mathbf{B}_{a2} + \mathbf{D}_{a2}]^{-1}c \end{pmatrix}, \quad (4.15)$$

with  $c$ , the initial value of  $e_4$  under the modal motion of P2.

The last passive subsystem in this scheme of nodes is P3. That case is completely similar to the modal motion of P2 : A1, P1, P2 are relaxed and A2 balances the modal motion of P3, as if it were a SISO system.

In MIMO problems, also active subsystems can perform modal motion. All passive subsystems are then relaxed. In this example, neither A1 nor A2 can be in modal motion, since they are not adjacent to each other.

### Second Scheme of Nodes

The candidate junctions for artificial nodes are  $e_3$  and  $e_5$  . Only the 0-junction of  $e_5$  leads to additional transmission zeros. Suppose the junction of  $e_3$  is chosen as artificial node (node 2 in figure 4.8). This is an example of two adjacent nodes. The system falls apart into two separated parts I and II. Part II cannot yield new zeros, as passive subsystems P3 and P4 of figure 4.8 are the same as subsystems P2 and P3 in figure 4.7. Because of the rule "relaxation to solve an information equation"

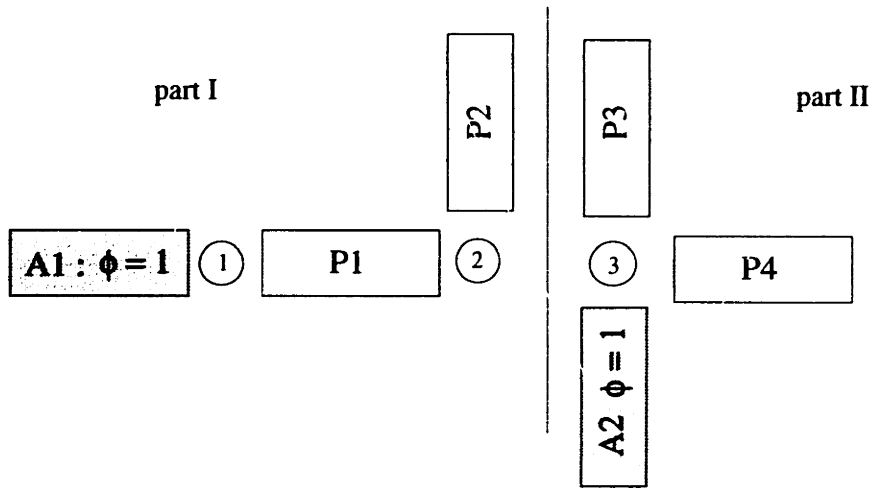


Figure 4.8: Failing scheme of nodes and subsystems for the worked-out example in chapter 3, both  $P1$  and  $P2$  should be relaxed. Consequently, part I cannot yield new zeros either. Therefore, this scheme of nodes does not provide any additional transmission zeros.

Figure 4.9 shows the resulting subsystems when the junction of  $e_5$  is chosen as artificial node. Again, there are two adjacent nodes. The system falls apart into

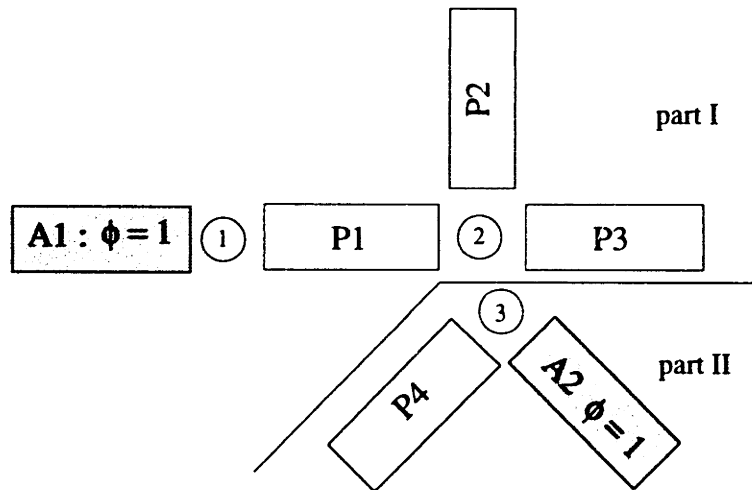


Figure 4.9: Second scheme of nodes and subsystems for the worked-out example two separated parts I and II. Only part II can yield new zeros (as it contains other subsystems than in previous subdivisions). The eigenvalue of  $P4$  (which is real) is



a 7th transmission zero for the system. As A2 has only one effort source, this case reduces to a simple SISO problem. All subsystems of part I should be relaxed.

Conclusion : This mechanical system with given inputs and outputs has 7 transmission zeros : 3 complex conjugated pairs and 1 real zero.

## 4.6 Summary

This chapter extended the procedure to find the zeros and their initial states from SISO to MIMO systems. Like in SISO systems, transmission zeros of MIMO systems are eigenvalues of some energetically isolated subsystems. Bond graphs are the main tool to identify these subsystems. In addition to the SISO case, this chapter gives a rule to decide whether a complex scheme of nodes, passive and active subsystems (often with more than one input source) is solvable or not. In a solvable scheme, all inputs and initial states can be chosen to satisfy all information equations. The eigenvalue of the subsystem in modal motion is then a transmission zero of the whole system. Some worked-out examples at the end of the chapter demonstrate the different steps and reasoning of the general procedure.

# Bond Graphs with Loops

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## 5.1 Introduction

The physical interpretation discussed in chapters 3 and 4 is valid for bond graphs with a tree structure. In some cases, however, there are loops in the energy paths of the bond graph. In other words, there is a circular power line. Figure 5.1 shows a bond graph with one loop.

Though the main ideas are still applicable to this kind of systems, the loops make the calculations more difficult. In some cases, they are responsible for additional zeros, which even may be unstable. Systems featuring unstable zeros are called non-minimum phase systems; they are very difficult to control, and have low limits of performance. Therefore, this chapter shows the origin of the additional zeros : they are not eigenvalues of subsystems, but result from a one-side actuated loop. The theory also indicates which part of the system determines the zero. This is the most important contribution : all effort to change the zero should be concentrated on that part. The calculation scheme, however, is often more complex than the eigenvalue problem encountered in the procedure for regular zeros.

## 5.2 Loops Solvable with the Classic Theory

### 5.2.1 The Validity of the Basic Ideas

This subsection summarizes those basic ideas of the theory for zeros of SISO and MIMO systems of previous chapters, that are still valid in the presence of loops in

the bond graph. First, a zero output creates a zero node in the system, which still exhibits the energy blocking effect. Also other junctions in the bond graph can be chosen as artificial nodes (every junction in the loop is a candidate artificial node). The resulting scheme of nodes yields a set of passive and active subsystems. Each subsystem has at least one boundary node. In bond graphs with loops, however, one node can bound two different branches of the same subsystem.

Second, every node introduces an information equation. These equations play an important role for loop configurations, especially when they contain several variables of the same subsystem.

Third, equation (3.23) on page 69 still holds : the state vector performs a rectilinear motion in the state space. As a consequence, all variables in the whole system have the same time evolution  $\sim e^{zt}$ . This means that their time derivatives can be written as a constant times the variable itself :

$$v(t) = \alpha e^{zt} \quad (5.1)$$

$$\dot{v}(t) = z\alpha e^{zt} \quad (5.2)$$

$$\ddot{v}(t) = z^2\alpha e^{zt} \quad (5.3)$$

This fact is helpful for the calculation of zeros resulting from loops.

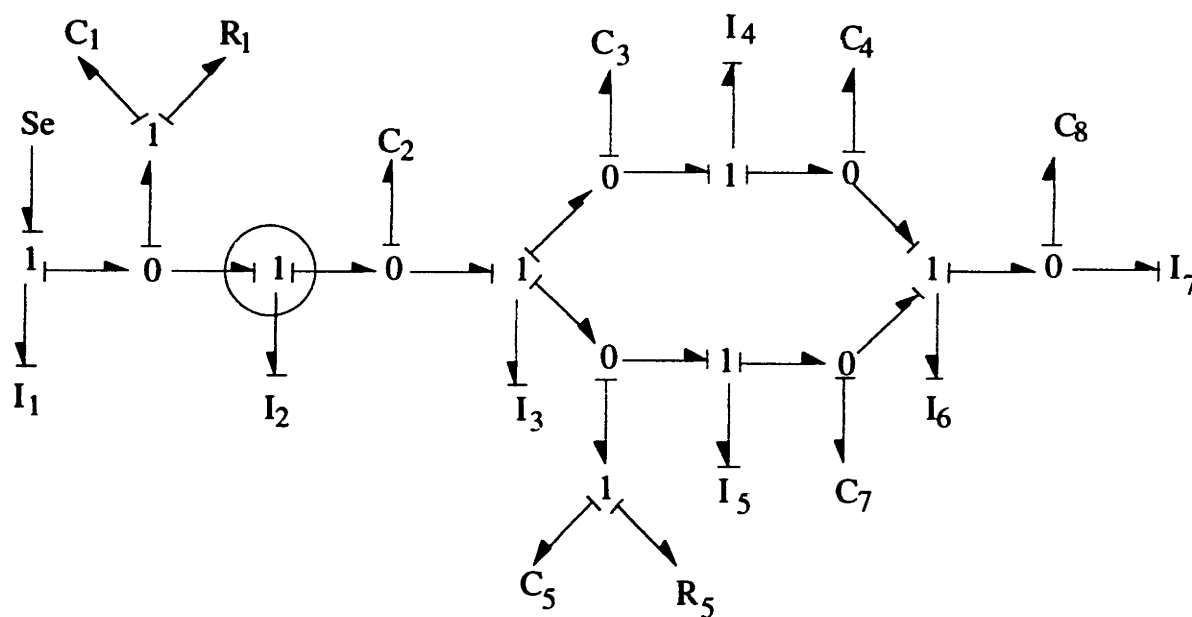
Fourth, the active subsystems offer the necessary degrees of freedom to solve the information equations of a certain node scheme.

### 5.2.2 A Loop within a Subsystem and Double Broken Loops

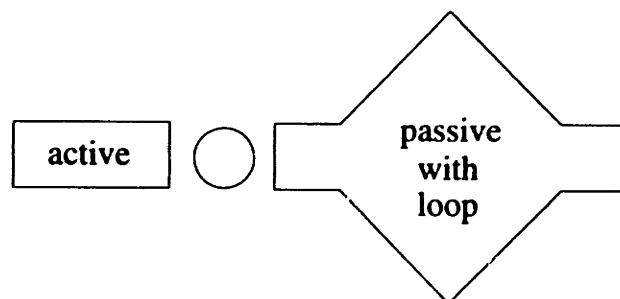
As mentioned before, a loop consists of a circular power line. Such a power line is a closed chain of bonds and 0- and 1-junctions. The chosen scheme of nodes can result in the following 3 cases :

1. No junction of the closed chain is a node. In this case, the loop will be an integral part of a subsystem and no information equation will contain more than one variable of the same subsystem. Therefore, this case yields a set of subsystems completely similar to those studied in chapters 3 and 4. The theory worked out there is applicable for this scheme of nodes. The computation of the initial state  $\mathbf{x}_0$  may be more difficult if it requires the state space description of the loop-containing subsystem. Figure 5.1 gives an example of such a node scheme and the resulting subsystems in part b).
2. Two or more junctions of the closed chain are nodes. Such a node scheme divides the loop over several subsystems with at least two boundary nodes. Therefore, this case only occurs in MIMO systems, or when different subsystems have the same eigenvalue. The methods derived in chapter 4 can solve this problem. Of course, the active subsystem should have enough degrees of freedom to solve all information equations. Figure 5.2 shows a loop “broken” by two nodes.
3. If exactly one junction of the closed chain is a node, a subsystem will result with two branches bounded by that same node. The information equation of that node, therefore, contains variables of the same subsystem. In some situations, that causes additional zeros and more complicated calculations. Figure 5.3 gives the subsystem structure for this case. If subsystem 1 in that figure is considered relaxed or in modal motion (both possible for active and passive subsystems, see table 4.1 on page 103), the corresponding variables in the information equation of the node either vanish or can be added together. For instance, if the node is a 1-junction, the equation will have the following form :

$$e_1 + e_2 = e_3. \quad (5.4)$$



a)



b)

Figure 5.1: Loop within a subsystem : the loop contains no node

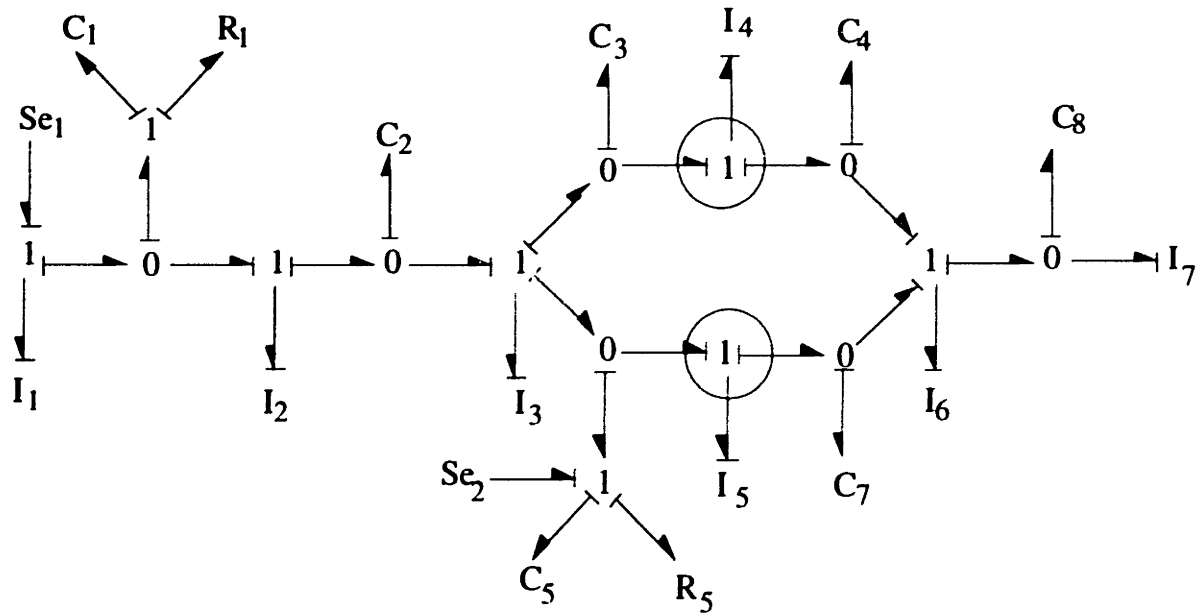
The modal motion results in following time evolutions of variables  $e_1$  and  $e_2$  :

$$e_1 = ae^{zt} \quad (5.5)$$

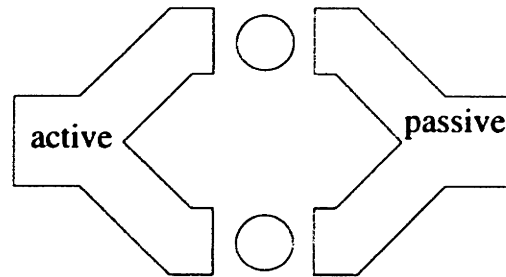
$$e_2 = be^{zt}, \quad (5.6)$$

with  $a$  and  $b$  constants depending on the modal eigenvector. Substituting these expressions in the information equation yields :

$$ae^{zt} + be^{zt} = (a + b)e^{zt} = e_3. \quad (5.7)$$



a)

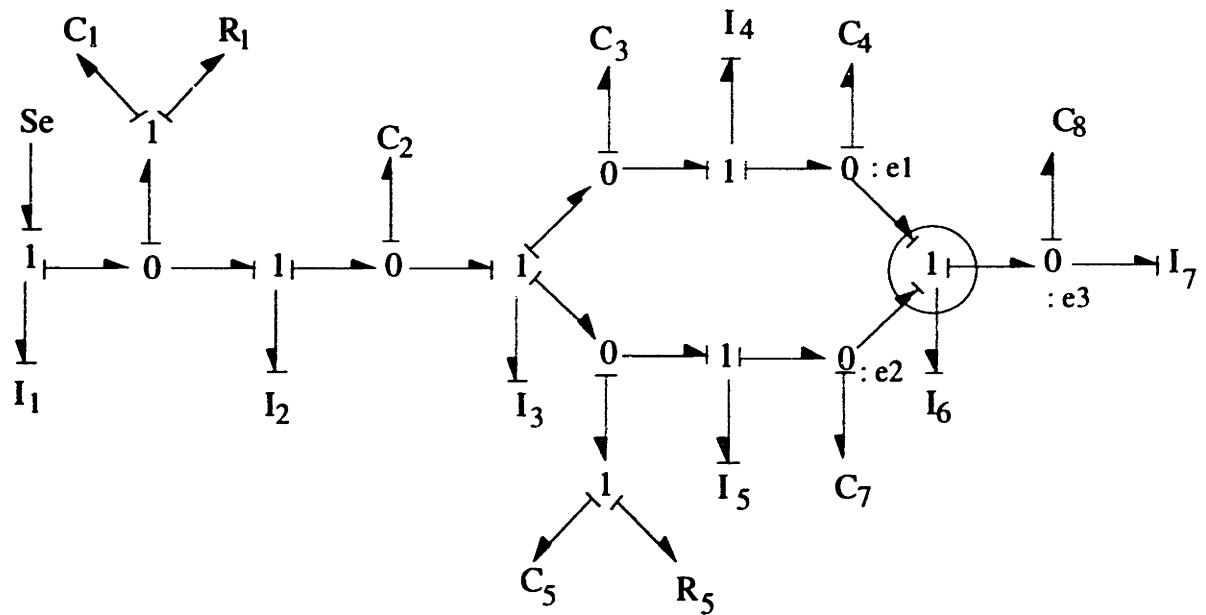


b)

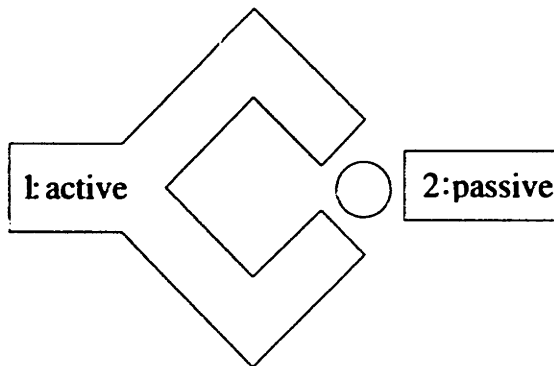
Figure 5.2: Loop broken by two nodes

When solving for  $e_3$ , the left hand side appears as one variable. Therefore, a loop at the end of a relaxed or modal motion subsystem forms no obstacle for the theory in previous chapters.

However, if subsystem 1 in figure 5.3 is an energized active subsystem, the appearance of two of its variables in the information equation has an effect on the solvability; techniques of previous chapters based on the eigenvalues of



a)



b)

Figure 5.3: Loop broken by one node

subsystems will not work.

In conclusion, loops do not always cause problems. For many node schemes the previously derived methods are sufficient. When the loop is at the end of an energized active subsystem, another calculation method should be used, which can yield unstable zeros, even when other conditions for unstable zeros, such as negative damping, are not met.

## 5.3 Loops Providing Additional Zeros

### 5.3.1 Loops at the End of Active Subsystems with one Source

In all previous methods, the variables belonging to active subsystems formed the “independent variables” of the set of information equations. They in their turn determine the input sources (the actual degrees of freedom of the whole system).

However, an equation featuring two such variables belonging to the same active subsystem, introduces a relationship between those two variables, for instance :

$$e_1 + e_2 \equiv 0. \quad (5.8)$$

If the active subsystem has only one input  $ue^{zt}$ , and the initial state of the subsystem is chosen to yield a rectilinear motion in state space, then the two variables can be written as :

$$e_1 = u.g(z)e^{zt} \quad (5.9)$$

$$e_2 = u.h(z)e^{zt}. \quad (5.10)$$

The relationship between the two variables forms a requirement on  $u$  and  $z$  :

$$u [g(z) + h(z)] = 0. \quad (5.11)$$

Since  $u$  cannot be 0 to energize the active subsystem, this is equivalent to :

$$g(z) + h(z) = 0. \quad (5.12)$$

Consequently, the active subsystem can only satisfy its constraint (5.8) when its time exponent  $z$  has a certain value. Clearly, this yields another zero for the whole system. However, at this time it is not identified as an eigenvalue, but as the time exponent needed to meet the constraint.



Notice that, if the right hand side of equations (5.8) and (5.11) is not zero, the constraint can be satisfied for every value of  $z$  by choosing an appropriate input  $u$ . This happens if, besides the loop containing active subsystem, also another non relaxed subsystem is adjacent to the same node. Often, that subsystem is in modal motion and will dictate the necessary value of  $z$ . If, on the other hand, all adjacent subsystems are relaxed, the before mentioned right hand side will be zero, and the constraint leads to an additional zero. The sequel of this chapter will call such zero a “loop zero”.

### 5.3.2 Calculation of Loop Zeros

Since only the active subsystem is not relaxed, it seems obvious that the value of  $z$ , satisfying the constraint of the outputs, depends on the whole subsystem. This is not true : only the two branches of the loop leading to the node influence  $z$ . Therefore, the calculation of the value  $z$  can concentrate on that part. Notice that these branches do not form a real subsystem in the sense of energetic isolation. Power can flow throughout the active subsystem without any barrier.

The reason why the  $z$  only depends on the branches of the loop is in close relationship to the existence of the constraint. Consider figure 5.4. The active subsystem actually exists of 3 branches, one containing the input source, the two others leading towards the 0-junctions of the variables  $e_1$  and  $e_2$ . The requirement  $e_1 + e_2 = 0$  can only be satisfied for a certain  $z$ , because there is only one input available (actuation number equal to 1 to determine 2 outputs). However, the 1-junction of variable  $f$  is the last junction in common of the paths between the input and both output variables. In fact, the variable of this last common junction, through which power flows from the source into the branches, forms a second constraint comparable to that of the node. The 1-junction can be seen as a fictitious flow source  $S_f = fe^{zt}$  (a last

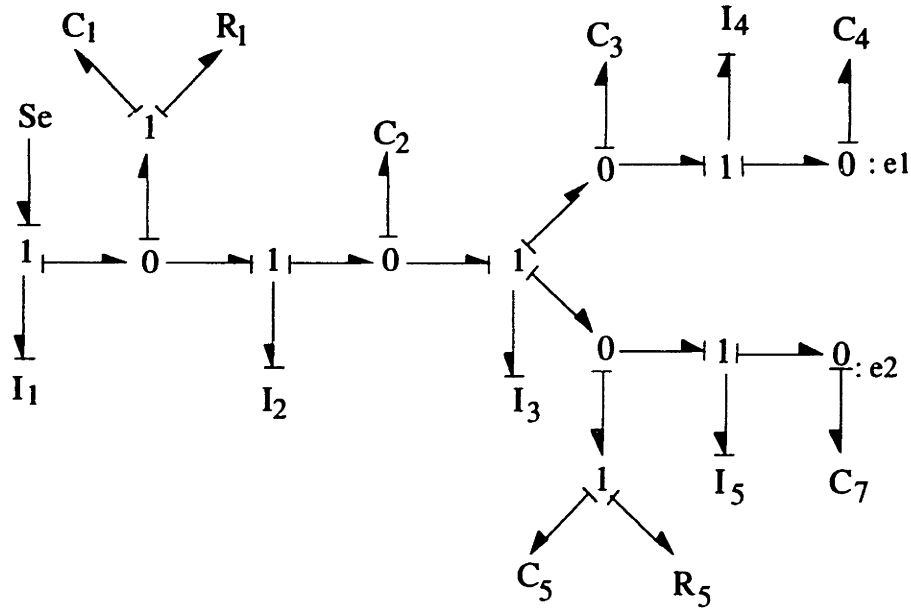


Figure 5.4: The active subsystem containing the loop branches

common 0-junction would introduce a fictitious effort source). The essence of the

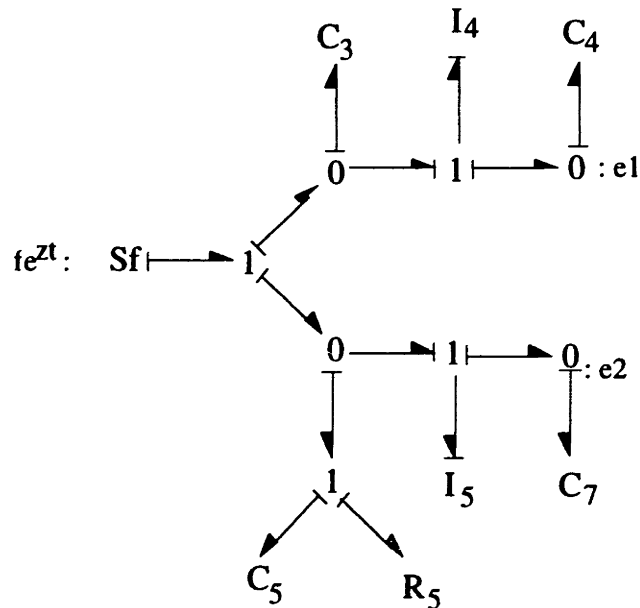


Figure 5.5: The fictitious source actuating the loop branches

loop zero clearly lies only in the branches of the loop : the outputs at their ends are related to each other, and a common junction connects them to the actuation.

To find the value of  $z$ , it is sufficient to find the  $z$  in the flow source of figure 5.5 such that the constraint involving  $e_1$  and  $e_2$  is satisfied. Notice that we know the time evolution of the fictitious flow source. Previous section gives expressions for its time derivatives (equation (5.2)). It is possible to find the state space equation matrices  $\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p, \mathbf{D}_p$  of the two branches actuated by that fictitious source. The output is the constraint ( $e_1 + e_2$ ) which should be identically zero. The inputs are the flow variable of the fictitious source and an additional input for all  $q$  necessary time derivatives. The equation for  $z$  expressing the zero output is :

$$\left[ \mathbf{C}_p (z\mathbf{I} - \mathbf{A}_p)^{-1} \mathbf{B}_p + \mathbf{D}_p \right] \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^q \end{pmatrix} = 0. \quad (5.13)$$

Though this equation seems very complicated, often the matrices  $\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p$  and  $\mathbf{D}_p$  are of much smaller dimension than the whole system. Moreover, this reasoning identifies those physical elements that influence the zero. If the resulting value of  $z$  is nonminimum phase, then only changes to these elements can bring that zero to the left half complex plane.

If several loops are next to each other, the same procedure can be used for each of them. Every loop will provide zeros. The number of zeros for each loop cannot be predicted and follows from the equation (5.13).

### 5.3.3 Calculation of the Initial State

Since power can flow throughout the active subsystem, the initial state must be calculated for the whole subsystem at the same time. The equation from chapter 4 can be used :

$$\mathbf{x}_{0s} = (z\mathbf{I} - \mathbf{A}_s)^{-1} \mathbf{B}_s \left( \mathbf{C}_s (z\mathbf{I} - \mathbf{A}_s)^{-1} \mathbf{B}_s + \mathbf{D}_s \right)^{-1} \mathbf{y}_s, \quad (5.14)$$

with  $\mathbf{A}_s, \mathbf{B}_s, \mathbf{C}_s, \mathbf{D}_s$  the state space matrices of the whole subsystem, and  $\mathbf{y}_s$  a vector of outputs that satisfies the constraint. For example, for the constraint  $e_1 + e_2 \equiv 0$ ,  $\mathbf{y}_s$  can be  $(1, -1)^T$ . Consequently, a change to an element not belonging to the branches does not affect the zero value, but does alter the corresponding initial state.

### 5.3.4 Active Subsystems in MIMO Systems

Loops at the end of active subsystems containing more than one source do not always introduce loop zeros. The basic reason for such zeros is a configuration of two branches with mutual constraints at both sides. This is only possible in active subsystem with multiple sources, if all paths from the sources to the loop enter the loop at the same junction. That junction will form the second mutual constraint for the branches. In all other cases, there are no loop zeros. Figure 5.6 presents the two cases : in case a) all paths enter the loop via the same junction, whereas in case b) both sources have effect on different junctions of the loop.

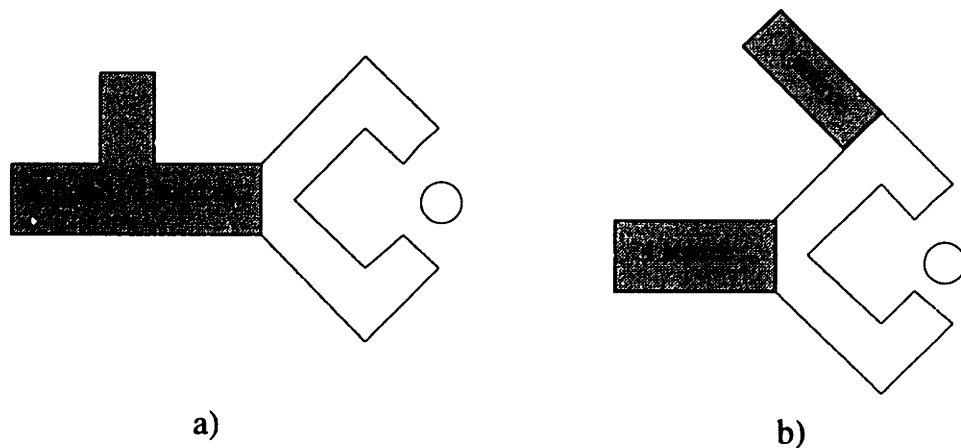


Figure 5.6: Different cases for loops in active subsystems with multiple sources

## 5.4 Examples

### 5.4.1 A Mechanical System with a Loop

The first example is already introduced in figure 2.13 on page 55. The mechanical structure is build in a loop; also the bond graph exhibits a loop as shown in figure 5.7. The position of the second mass ( $I_2$ ) is the output variable, so the 1-junction of  $f_2$  is

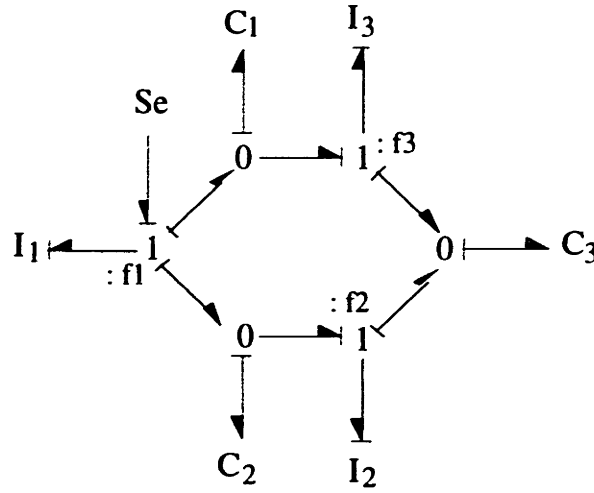


Figure 5.7: Bondgraph of a loop shaped mechanical system

a node. Since the system is actuated by only one source, there will be a loop zero. Unlike the procedure explained in section 2.5.2, this strategy does not introduce an artificial node or constraint at the input source. On the contrary, from that 1-junction the fictitious flow source will actuate the branches of the loop. Figure 5.8 presents the branches and that flow source. The only elements that determine the value of  $z$  are the elements of the branches :  $C_1, C_2, C_3$  and  $I_3$ . The state equations for this system are set up, using  $x_3, \dot{x}_3$  and  $x_1$  as states. The output is the difference between  $e_2$  and  $e_3$ . The equation (5.13) for the loop zero of this system is :

$$\begin{pmatrix} k_3 & 0 & k_2 \end{pmatrix} \cdot \begin{bmatrix} z & -1 & 0 \\ \frac{k_1+k_3}{m_3} & z & 0 \\ 0 & 0 & z \end{bmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ \frac{k_1}{m_3} \\ z \end{pmatrix} = 0. \quad (5.15)$$

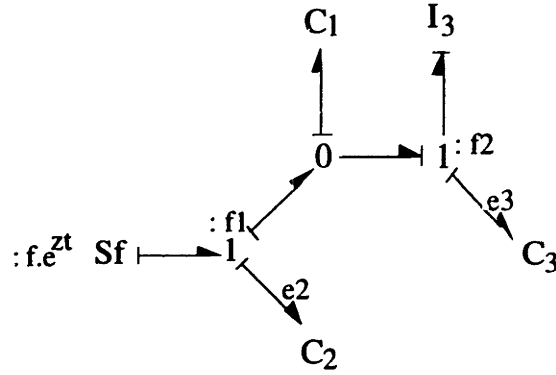


Figure 5.8: Bondgraph of the two loop branches actuated by the fictitious flow source

This equation is satisfied for

$$z_{1,2} = \pm j \sqrt{\frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{m_3 k_2}}, \quad (5.16)$$

which is the same result as that of the straightforward calculation of the zero, using the state space matrices of the whole system.

The reduction in computation effort is not so considerable (from a 6 state system to a 3 state system). Moreover, the calculation of a loop zero is often more difficult than the eigenvalue problem of a zero of an isolated subsystem. However, the most important result from this example is the identification of the structural parameters  $k_1$ ,  $k_2$ ,  $k_3$  and  $m_3$ , that influence the zero value. Only these parameters deserve attention, if the zero should be changed to obtain better performance in control.

### 5.4.2 An Unstable Zero

Following example of a non minimum phase system is also introduced on page 130 in [10]. Figure 5.9 depicts the schematic and bond graph representation of the system. The particularity is the lever, pivoted in the middle, between the second spring  $k_2$  and the third mass  $m_3$ . The loop in the bond graph is unreducible, because of the arrow directions in the loop. The output of the system is the position of the third

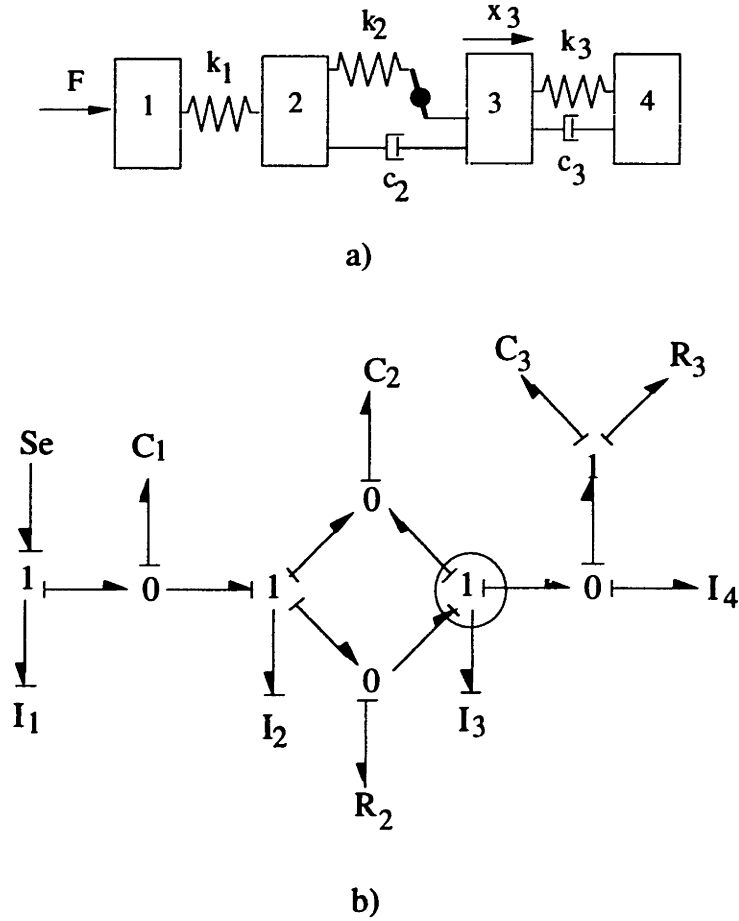


Figure 5.9: Schematic and bond graph representation of a non minimum phase system mass,  $x_3$ , the input is the force acting on the first mass. The node divides the system in an active subsystem at the left of the node, and a passive subsystem at the right. The node is on the loop and the branches belong to the active subsystem, so there will be a loop zero. The corresponding fictitious source and branches form a very simple system. The constraint is :

$$e_1 - e_2 \equiv 0 \quad (5.17)$$

If  $e_1$  is chosen as state, and the constraint is the output, the state space matrices for the subsystem of figure 5.10 are :

$$\mathbf{A} = 0 \quad (5.18)$$

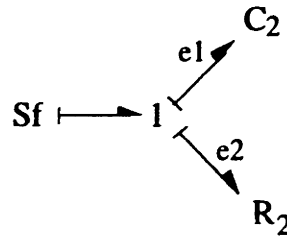


Figure 5.10: Fictitious source and branches for the non minimum phase system

$$\mathbf{B} = \frac{1}{k_2} \quad (5.19)$$

$$\mathbf{C} = 1 \quad (5.20)$$

$$\mathbf{D} = -R_2. \quad (5.21)$$

$$(5.22)$$

The resulting equation (5.13) for the loop zero is :

$$1.(z)^{-1} \cdot \frac{1}{k_2} + (-R_2) = 0. \quad (5.23)$$

Finally, the zero is  $+\frac{k_2}{R_2}$ , which is an unstable zero.



## Chapter 6

# Conclusion

---

This thesis presents a physical interpretation of zeros in SISO and transmission zeros in MIMO systems. It contributes to a better integration of the structural and control design of systems. Such integration is necessary to meet the ever increasing performance requirements for new systems. Zeros form an important aspect of the coupling between both designs : they have much effect on the achievable performance of the control algorithms, but are mainly determined by the structural design.

Therefore, this thesis consists of two parts. First, it studies in chapter 2 the impact of the system zeros on the control performance. Rules are derived to find the most appropriate zero locations with respect to as well the input as the output behaviour of the system. Once the control design has decided on the zero location using these rules, the structural design should be revised. That requires a good insight into the relationship between the zeros and the structural design parameters. The second part of the thesis investigates that relationship in chapters 3,4 and 5. It first shows that zeros are eigenvalues of subsystems. Then some techniques are presented to identify the corresponding subsystems for every zero. This identification utilizes the bond graph model of the system, as that clearly displays the energetical structure of the system, on which the theory is based. Chapter 4 answers the same questions for MIMO systems. The fifth chapter reveals the origin of unstable zeros : they are introduced by one-sided actuated loops in the bond graph. Finally, the structural parameters of the subsystem corresponding to a zero inappropriate for the control design can be changed.

This better insight into the physical meaning of zeros facilitates to large extent the revise process of the structural design : not only can it focus on a small part of the system, it also can use the well known techniques to find or modify the eigenvalues of subsystems. The procedure for loop zeros is more complicated, but here too, all involved elements are only located on the branches of the loop.

The physical interpretation of SISO zeros and MIMO transmission zeros show a striking similarity : both create power flow barriers in the system and subdivide it in subsystems. This supports the different theoretical definitions of transmission zeros discussed in chapter 2. Further research, however, can investigate a more general correspondence between the loop zeros in SISO and MIMO systems.

## Appendix A

# Numerical Results for the SISO Example

---

This appendix contains the numerical results for the second node scheme in the example of section 3.6 on page 79. The passive subsystems P3, P3 and P4 are considered relaxed, whereas P1 is in modal motion. The parameters of figure 3.8 have following numerical values :

- All masses  $m_1$  through  $m_8$  have have mass 1 kg.
- $k_1 = 1 \frac{N}{m}$ ,  $k_2 = 2 \frac{N}{m}$ ,  $k_3 = 3 \frac{N}{m}$ ,  $k_4 = 4 \frac{N}{m}$ ,  $k_5 = 5 \frac{N}{m}$ ,  $k_6 = 6 \frac{N}{m}$ ,  $k_7 = 7 \frac{N}{m}$
- $c_3 = 0.7 \frac{Ns}{m}$ ,  $c_4 = 0.8 \frac{Ns}{m}$ ,  $c_5 = 1.5 \frac{Ns}{m}$ ,  $c_7 = 2.8 \frac{Ns}{m}$

The passive subsystem P1 has following structure : The state space equations are :

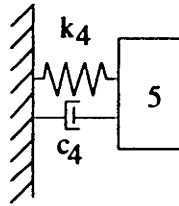


Figure A.1: Passive subsystem P1 in SISO example

$$\begin{pmatrix} \dot{x}_5 \\ \ddot{x}_5 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_4}{m_4} & -\frac{c_4}{m_4} \end{bmatrix} \begin{pmatrix} x_5 \\ \dot{x}_5 \end{pmatrix}. \quad (\text{A.1})$$

The eigenvalue of this subsystem and third zero  $z_3$  of the whole system is  $-0.4 + 1.9596j$  with corresponding eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 0.4382 - 0.0894j \\ 0.8944j \end{pmatrix}. \quad (\text{A.2})$$

The output of this passive system towards the node is :

$$e_4 = ce^{z_3 t} = \begin{bmatrix} -k_4 & -c_4 \end{bmatrix} \mathbf{v}_3 = (-1.7527 - 0.3578j)e^{z_3 t}. \quad (\text{A.3})$$

Figure A.2 shows the active subsystem consisting of 3 masses. Its state space equa-

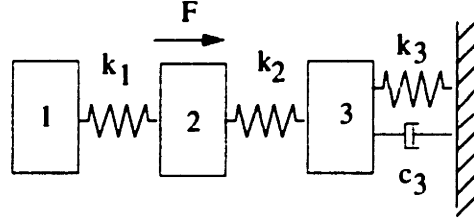


Figure A.2: Active subsystem in SISO example

tions with matrices  $\mathbf{A}_{a2}$ ,  $\mathbf{B}_{a2}$ ,  $\mathbf{C}_{a2}$  are :

$$\begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \\ \dot{x}_3 \\ \ddot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & 0 & \frac{k_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{m_2} & 0 & -\frac{k_1+k_2}{m_2} & 0 & \frac{k_2}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_2}{m_3} & 0 & -\frac{k_2+k_3}{m_3} & -\frac{c_3}{m_3} \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \\ 0 \\ 0 \end{pmatrix} F \quad (\text{A.4})$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & k_3 & c_3 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{pmatrix}. \quad (\text{A.5})$$

The necessary initial state  $\mathbf{x}_{01}$  for the active subsystem is :

$$\mathbf{x}_{01} = (z_3 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2} \cdot (\mathbf{C}_{a2} (z_3 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2})^{-1} c \quad (\text{A.6})$$

or,

$$\mathbf{x}_{01} = \begin{pmatrix} 0.0556 - 0.0832j \\ 0.1408 + 0.1423j \\ -0.2795 + 0.1357j \\ -0.1541 - 0.6020j \\ -0.5666 + 0.1542j \\ -0.0755 - 1.1720j \end{pmatrix} \quad (\text{A.7})$$

The total initial state  $\mathbf{x}_0$  is :

$$\begin{pmatrix} 0.0556 - 0.0832j \\ 0.1408 + 0.1423j \\ -0.2795 + 0.1357j \\ -0.1541 - 0.6020j \\ -0.5666 + 0.1542j \\ -0.0755 - 1.1720j \\ 0_{2 \times 1} \\ 0.4382 - 0.0894j \\ 0.8944j \\ 0_{6 \times 1} \end{pmatrix} \quad (\text{A.8})$$

## Appendix B

# Numerical Results for the MIMO Example

---

This appendix contains the numerical results for the first node scheme in the example of section 4.5 on page 103. The passive subsystems P3 and P3 are considered relaxed, whereas P1 is in modal motion. The active subsystems A1 and A2 are energized (see figure 4.7 on page 105. The parameters of figure 4.5 have the same numerical values as mentioned in appendix A. The passive subsystem P1 has following structure : The

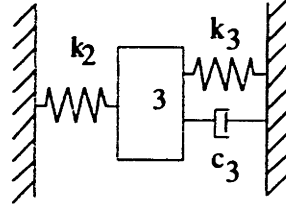


Figure B.1: Passive subsystem P1 in MIMO example

state space equations are :

$$\begin{pmatrix} \dot{x}_3 \\ \ddot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_2+k_3}{m_3} & -\frac{c_3}{m_3} \end{bmatrix} \begin{pmatrix} x_3 \\ \dot{x}_3 \end{pmatrix}. \quad (\text{B.1})$$

The first eigenvalue  $z_1$  is  $-0.35 + 2.2085j$  with corresponding eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 0.4032 - 0.0639j \\ 0.9129j \end{pmatrix}. \quad (\text{B.2})$$

The output of this passive system towards the nodes are :

$$e_2 = a e^{z_1 t} = \begin{bmatrix} -k_2 & 0 \end{bmatrix} \mathbf{v}_1 = (-0.8064 + 0.1278j) e^{z_1 t}. \quad (\text{B.3})$$

$$e_3 = b e^{z_1 t} = \begin{bmatrix} k_3 & c_3 \end{bmatrix} \mathbf{v}_1 = (1.2096 + 0.4473j) e^{z_1 t}. \quad (\text{B.4})$$

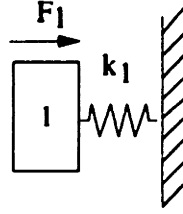


Figure B.2: Active subsystem 1 in MIMO example

Figure B.2 shows the first active subsystem. Its state space equations with matrices  $\mathbf{A}_{a1}$ ,  $\mathbf{B}_{a1}$ ,  $\mathbf{C}_{a1}$  are :

$$\begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m_1} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m_1} \end{pmatrix} F_1. \quad (\text{B.5})$$

$$e_1 = k_1 x_1 \quad (\text{B.6})$$

The necessary initial state  $\mathbf{x}_{01}$  for the first active subsystem is :

$$\mathbf{x}_{01} = (z_1 \mathbf{I} - \mathbf{A}_{a1})^{-1} \mathbf{B}_{a1} \cdot (\mathbf{C}_{a1} (z_1 \mathbf{I} - \mathbf{A}_{a1})^{-1} \mathbf{B}_{a1})^{-1} a \quad (\text{B.7})$$

or,

$$\mathbf{x}_{01} = \begin{pmatrix} -0.8064 + 0.1278j \\ -1.8257j \end{pmatrix} \quad (\text{B.8})$$

Figure B.3 shows the second active subsystem. Its state space equations with

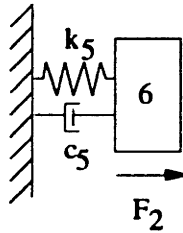


Figure B.3: Active subsystem 2 in MIMO example

matrices  $\mathbf{A}_{a2}$ ,  $\mathbf{B}_{a2}$ ,  $\mathbf{C}_{a2}$  are :

$$\begin{pmatrix} \dot{x}_6 \\ \ddot{x}_6 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_5}{m_6} & -\frac{c_5}{m_6} \end{bmatrix} \begin{pmatrix} x_6 \\ \dot{x}_6 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m_6} \end{pmatrix} F_2. \quad (\text{B.9})$$

$$e_5 = \begin{bmatrix} -k_5 & -c_5 \end{bmatrix} \begin{pmatrix} x_6 \\ \dot{x}_6 \end{pmatrix} \quad (\text{B.10})$$

The necessary initial state  $\mathbf{x}_{02}$  for the second active subsystem is :

$$\mathbf{x}_{02} = (z_1 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2} \cdot (\mathbf{C}_{a2}(z_1 \mathbf{I} - \mathbf{A}_{a2})^{-1} \mathbf{B}_{a2})^{-1} b \quad (\text{B.11})$$

or,

$$\mathbf{x}_{02} = \begin{pmatrix} -0.2224 + 0.0647j \\ -0.0650 - 0.5139j \end{pmatrix} \quad (\text{B.12})$$

The total initial state  $\mathbf{x}_0$  is :

$$\begin{pmatrix} -0.8064 + 0.1278j \\ -1.8257j \\ 0_{2 \times 1} \\ 0.4032 - 0.0639j \\ 0.9129j \\ 0_{4 \times 1} \\ -0.2224 + 0.0647j \\ -0.0650 - 0.5139j \\ 0_{4 \times 1} \end{pmatrix} \quad (\text{B.13})$$

Finally, the input vector  $\mathbf{u}$  is :

$$\begin{pmatrix} 3.2257 + 0.7668j \\ -0.0520 - 0.4111j \end{pmatrix}. \quad (\text{B.14})$$



## Bibliography

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- [1] Bode, H.W., "Network Analysis and Feedback Amplifier Design", Van Nostrand, Princeton, New Jersey, 1945.
- [2] Freudenberg, J.S. and Looze, D.P., "Right Half Plane Poles and Zeros and Design Tradeoffs in Feedback Systems", IEEE Transactions on Automatic Control, Vol. AC-30, No. 6, June 1985.
- [3] Sueur, C. and Dauphin-Tangy, G. "Bond Graph Approach for Structural Analysis of MIMO Linear Systems", J. Franklin Inst., Vol. 328, pp. 55-70, 1991.
- [4] Sueur, C. and Dauphin-Tangy, G., "Structural Controllability/Observability of Linear Systems Represented by Bond Graphs"
- [5] Margolis, D.L., "Bond Graphs for Distributed Systems Models Admitting Mixed Causal Inputs", Transactions of the ASME, Vol. 102, pp. 94-100, June 1980.
- [6] Margolis, D.L., "A Survey of Bond Graph Modelling for Interacting Lumped and Distributed Systems", Journal of the Franklin Institute, Vol. 319, No. 1/2, pp. 125-135, January/February 1985.
- [7] Rosenberg, R.C., "On Gyrobondgraphs and their Uses", Transactions of the ASME, Vol. 100, pp. 76-82, March 1978.
- [8] Zeid, A. and Rosenberg, R., "Estimating Eigenvalues for a Class of Dynamic Systems", Journal of the Franklin Institute, Vol. 320, No. 1, pp. 21-40, July 1985.

- [9] Zeid, A., "Some Bond Graph Structural Properties : Eigen Spectra and Stability", Proceedings of ASME Winter Annual Meeting, pp. 75-82, 1988.
- [10] Wu, Shang-Teh, "Input/Output Linearization of Uncertain Systems with Time Delay Control", PhD. Thesis, Department of Mechanical Engineering. M.I.T., 1993.
- [11] Miu, D.K., "Physical Interpretation of Transfer Function Zeros for Simple Control Systems with Mechanical Flexibilities", Journal of Dynamic Systems, Measurement and Control, Vol. 113, pp. 419-424, September 1991.
- [12] Pohjalainen, S., "Computation of Transmission Zeros for Distributed Parameter Systems", Internat. Journal of Control, Vol. 33, No. 2, 199-212, 1981.
- [13] Macfarlane, G.J. and Karcanias, N., "Poles and Zeros of Linear Multivariable Systems : a Survey of the Algebraic, Geometric and Complex Variable Theory", Internat. Journal of Control, Vol.245, No. 1, pp.33-74, 1976.
- [14] Desoer and Schulman, "Zeros and Poles of Matrix Transferfunctions and their Dynamical Interpretation", IEEE Transactions on Circuits and Systems, CAS 21, pp. 3-8, 1974.
- [15] Rosenbrock, H.H., "The Zeros of a System", Internat. Journal of Control, Vol. 18, No. 2, pp. 297-299, 1973.
- [16] Rosenbrock, H.H., "State Space and Multivariable Theory", New York, Wiley, 1970.
- [17] Davison, E.J. and Wang, S.H., "Properties and Calculation of Transmission Zeros of Linear Multivariable Systems", Automatica, Vol. 10, pp. 643-658, 1974.

- [18] Wolovich, W.A., "On Determining the Zeros of State Space Systems", IEEE Transactions on Automatic Control, pp. 542-544, Oct. 1973.
- [19] Davison, E.J., "Remark on Multiple Transmission Zeros of a System", Automatica, Vol. 12, p. 195, 1976.
- [20] Francis, B.A. and Wonham, W.M., "The Role of Transmission Zeros in Linear Multivariable Regulators", Internat. Journal of Control, Vol. 22, p. 657, 1975.
- [21] Macfarlane, A.G.J., "Relationships between Recent Developments in Linear Control Theory and Classical Design Techniques", Control System Design by Pole-Zero Assignment, pp. 51-122, Cambridge Academic Press, 1977.
- [22] Maciejowski, J.M., "Multivariable Feedback Design", Workingham G.B.; Reading, Mass : Addison-Wesley, pp. 386-389, 1989.
- [23] Enami, A. and Van Dooren, P., "Computation of Zeros of Multivariable Linear Systems", Automatica, Vol. 18, No. 4, pp.415-430, July 1982.
- [24] Youcef-Toumi, K., Lecture Notes for M.I.T. mechanical engineering course 2.151, Fall 1994.