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Graph Balancing for Distributed Subgradient Methods over Directed Graphs

Ali Makhdoumi and Asuman Ozdaglar MIT, Cambridge, MA 02139
Emails: makhdoum@mit.edu, asuman@mit.edu

Abstract—We consider a multi agent optimization problem where a set of agents collectively solves a global optimization problem with the objective function given by the sum of locally known convex functions. We focus on the case when information exchange among agents takes place over a directed network and propose a distributed subgradient algorithm in which each agent performs local processing based on information obtained from his incoming neighbors. Our algorithm uses weight balancing to overcome the asymmetries caused by the directed communication network, i.e., agents scale their outgoing information with dynamically updated weights that converge to balancing weights of the graph. We show that both the objective function values and the consensus violation, at the ergodic average of the estimates generated by the algorithm, converge with rate $O(\frac{\log T}{\sqrt{T}})$, where T is the number of iterations. A special case of our algorithm provides a new distributed method to compute average consensus over directed graphs.

I. INTRODUCTION

A. Motivation

Many of today's optimization problems in data science (including statistics, machine learning, and data mining) use distributed computation. Modern processors have access to thousands of data points much more than they can process. Therefore, there is a need to distribute data among different data centers and then process it in a decentralized way based on the information that is provided to them. Distributed computation can lead to gains in computational efficiency in statistical learning, as shown by a number of authors ([4], [37], [1], [9]). The applications in statistical learning along with other applications in distributed control (see e.g. distributed sensor networks [6], coordination [13], and flow control [21]) have motivated a flurry of research on distributed approaches for solving optimization problems where the objective function is the sum of local objective functions of agents (nodes) that are connected through a network (see [7], [45], [44], [27], [26], [20], [19], [36], [18], [34]).

Most of the existing algorithms assume information exchange over undirected networks where the connection between nodes are bidirectional, meaning that if node i can send information to node j , then node j can also send information to node i .¹ However, in many applications, the underlying graph is directed because nodes typically broadcast at different power levels and have varying interference and noise patterns, implying communication capability in one direction, but not the other. In this paper, we consider a multi-agent optimization problem where a set of agents collectively minimize the sum of locally known convex functions $f_i(x)$, using information exchange over a directed network.

¹We use the terms graph and network interchangeably.

B. Related Works and Contribution

Our paper is related to a large recent literature on distributed methods for solving multi agent optimization problems over networks. Much of this literature builds on the seminal works [42], [43], which proposed gradient methods that can parallelize computations across multiple processors. We summarize subgradient based distributed methods for solving multiagent optimization problems over undirected and directed graphs.

Undirected Graphs: A number of recent papers proposed subgradient type distributed methods that use consensus or averaging mechanisms for aggregating information among the agents over an undirected network (see [26], [27], [20], [35], [15], [41], [22], [23], [48], [2], [8]). For convex local objective functions, these methods achieve $O(\frac{\log T}{\sqrt{T}})$ convergence rate, where T is the number of iterations. The recent paper [47] adopts stronger assumptions on the local objective functions, i.e., they are convex with Lipschitz continuous gradients, and shows that a distributed gradient method with averaging converges at rate $O(\frac{1}{T})$ to an error neighborhood (with the further assumption of strongly convex local objective functions, linear rate is achieved to an error neighborhood). Another contribution [14] assumes local objective functions have continuous and bounded gradients and uses Nesterov's acceleration to design a distributed algorithm with rate $O(\frac{\log T}{T})$. The recent paper [40], proposes a gradient-based distributed algorithm for convex objective function with Lipschitz continuous gradients and shows its convergence with rate $O(\frac{1}{T})$.

Directed Graphs: A few recent papers proposed and studied distributed subgradient methods over directed graphs [41], [24], [25]. The key idea used in these papers is the incorporation of the *push-sum algorithm*, which is a distributed algorithm presented in [16] to obtain the average of initial values of nodes over a directed graph. Push-sum, while being an effective approach in obtaining averages over a directed graph, involves updates that include a nonlinear operation (division by weight estimates), which makes the analysis within a subgradient optimization method involved (see e.g. [24]). In this paper, we use an alternative approach based on weight-balancing to design a distributed subgradient algorithm over directed networks. The notion of weights that balance a directed graph was introduced in [12] and studied recently in a number of papers with the goal of designing algorithms that enable their computation in a distributed manner (see [10], [11], [38], [32]). We combine such an algorithm for updating weights together with a distributed subgradient algorithm and show that these updates implemented simultaneously in the

same time scale solves multiagent optimization problem over directed graphs. The update step for agent's estimates involves operators linear in each estimate, allowing the analysis to use techniques from time-varying non-homogeneous non-negative matrix theory [39]. Although we do not pursue this here, our algorithm can be generalized to work over time-varying directed graphs (see e.g. [26], [24] for the analysis of distributed subgradient methods over time-varying graphs). Note that our algorithm does not require the local functions to have Lipschitz continuous gradient nor being strongly convex (the only requirement of our algorithm is that the functions are convex and not necessarily smooth). Indeed with these strong assumptions the rate of convergence can be improved as shown in [47] and [25] for algorithms over undirected graphs.

Our work also contributes to the vast literature on the consensus problem, where agents have a more specific goal of aligning their estimates (see [13], [30], [46], [31] for consensus over undirected graphs and [16], [5], [33] for consensus over directed graphs). In particular, a special case of our algorithm provides a new distributed method for computing average of initial values over a directed graph. Our contribution here is most closely related to [33] and [5], which proposed distributed algorithms for average consensus over directed graphs using balancing weights. Reference [33] builds on earlier work [32], which presented a distributed algorithm for computing balancing node weights based on approximating the left eigenvector associated with the zero eigenvalue of the Laplacian matrix of the underlying directed network. In [33], the authors used this algorithm for updating weights in the same time scale as the update for estimates and showed convergence to average of initial values. Reference [5] provides a similar algorithm for average consensus based on an earlier work [11], which proposes an update rule for computing balancing edge weights.

C. Outline

The organization of paper is as follows. In Section II, we give the problem formulation and present the distributed algorithm. In Section III, we show the convergence of weights to balancing weights of the graph. In Section IV, we consider the sequence of ergodic (time) averages of the estimates generated by our algorithm. We then show optimality convergence of our algorithm and establish $O(\frac{\log T}{\sqrt{T}})$ rate of convergence. Finally, in Section V, we provide numerical results that illustrate the performance of our algorithm, which leads to concluding remarks in Section VI.

D. Basic Notations

A vector $\mathbf{x}$ is viewed as a column vector. For a matrix A , we write $[A]_i$ to denote the i th column of matrix A , $[A]^i$ to denote the i th row of matrix A , and A_{ij} to denote the entry at i th row and j th column. For a vector $\mathbf{x}$, x_i denotes the i th component of the vector. We show the L_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_1 \triangleq \sum_{i=1}^n |x_i|$. We also let $\|\mathbf{x}\|_\infty = \max_i |x_i|$. For a set S , $|S|$ denotes the number of elements of S .

II. PROBLEM SETUP AND ALGORITHM

A. Formulation

We consider a set of agents (nodes) $V = \{1, \dots, n\}$ connected through a directed graph $G = (V, E)$ where $E \subset V \times V$

is the set of directed edges, i.e., $(i, j) \in E$ represents a directed edge from agent i to agent j . We denote the in-neighbors and out-neighbors of an agent i by $N^{\text{in}}(i) = \{j \in V \mid (j, i) \in E\}$ and $N^{\text{out}}(i) = \{j \in V \mid (i, j) \in E\}$, respectively. We also use $d_i^{\text{in}} = |N^{\text{in}}(i)|$ and $d_i^{\text{out}} = |N^{\text{out}}(i)|$ to denote the number of in-neighbors and out-neighbors.

We consider the following optimization problem

$$\min_{x \in \mathbb{R}} \sum_{i=1}^n f_i(x), \quad (1)$$

where $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function (possibly non-smooth), known to agent i only. The goal is to solve (1) using an algorithm that involves each agent performing computations based on his *local objective function* f_i and exchanging information over the directed graph (i.e., receiving information from his in-neighbors and sending the outcome of his computation to his out-neighbors).

We adopt the following standard assumption on the underlying graph G .

Assumption 1 (Strongly connected graph). The graph $G = (V, E)$ is strongly connected, i.e., for all nodes $i, j \in V$, there exists a directed path from i to j .

This assumption ensures that every node i receives information from every other node j in the graph.

B. Algorithm

Our algorithm generalizes the distributed subgradient algorithm presented in [7] to allow its implementation over directed graphs. The algorithm in [7] requires some form of symmetry in information exchange between pairs of nodes (essentially that node j 's information in node i 's update is scaled with the same weight used in scaling node i 's information in node j 's update). This kind of symmetry does not exist in directed graphs with directed communication over edges. Hence, a direct application of the algorithm given in [7] will lead to information from high out-degree nodes to be disproportionately represented in the estimate formed by agents. To alleviate this, our proposed algorithm scales outgoing information from each node with time-varying weights which in the limit ensures the incoming and the outgoing information of a node to be balanced. The following definition introduces node weights that satisfy this balancing requirement. The notion of node weights that balance a directed graph was proposed in [12] and used recently for deriving a Lyapunov function for convergence analysis of average-consensus [29], [30], consensus on general functions [3], design of stable flocking algorithms [17], and traffic-flow problems [12].

Definition 1 (Balancing weights). The node weights w_i for $i \in V$ balance a directed graph G if for any i , we have

$$w_i d_i^{\text{out}} = \sum_{j \in N^{\text{in}}(i)} w_j.$$

This definition ensures that the total weight outgoing from node i (measured by $w_i d_i^{\text{out}}$) is equal to the total weight incoming to node i (measured by $\sum_{j \in N^{\text{in}}(i)} w_j$), hence the term balancing weights.

We next describe our algorithm. Let $x_i(t) \in \mathbb{R}$ denote the estimate of agent i at time t for the optimal solution of (1). Each agent i starts from arbitrary value $x_i(0) \in \mathbb{R}$ and weight $w_i(0) \in \mathbb{R}$. At time t , agent i updates its estimate $x_i(t)$ as

$$x_i(t+1) = x_i(t) \left(1 - w_i(t)d_i^{\text{out}}\right) + \left(\sum_{j \in N^{\text{in}}(i)} w_j(t)x_j(t)\right) - \alpha(t)g_i(t), \quad (2)$$

where $g_i(t)$ is a subgradient of f_i at $x_i(t)$, i.e., $g_i(t) \in \partial f_i(x_i(t))$, $\alpha(t)$ is a step size sequence and $w_i(t)$ is a scalar weight. Each agent i linearly combines the estimates of his incoming neighbors and his own estimate and takes a step along the negative subgradient of his local objective function. This is followed by the following weight update at node i

$$w_i(t+1) = \frac{1}{2}w_i(t) + \frac{1}{d_i^{\text{out}}} \sum_{j \in N^{\text{in}}(i)} \frac{1}{2}w_j(t). \quad (3)$$

In order to understand this update rule, note that if the sequence of weights $\{w_i(t)\}_t$, $i = 1, \dots, n$ converges, it follows from (3) that the limiting weights balance the graph. With these balancing weights, update (2) ensures that in the limit the incoming information and the outgoing information of node i is properly scaled.

In principle, update (2) is similar to the algorithm considered in [7] for updating estimates, i.e., each agent updates his estimate by linearly combining estimates of neighbors together with a local optimal step. However, note that the weight matrix, i.e., the matrix that contains the scalars that multiply the estimates of the agents, is not doubly stochastic. We will show in Sections III-IV that it is column stochastic and becomes doubly stochastic only in the limit and our analysis shows this property suffices to guarantee that the estimates (obtained by (2)) converge to the optimal solution of problem (1).

III. CONVERGENCE OF WEIGHTS

In this section, we show that the sequence of weights generated by the update (3) converges to balancing weights for G . Using the notation $\mathbf{w}(t) = [w_1(t), \dots, w_n(t)]'$, we can write the weight updates of our algorithm more compactly as

$$\mathbf{w}(t+1) = \frac{1}{2}(I + D^{-1}A)\mathbf{w}(t), \quad (4)$$

where $D = \text{diag}(d_1^{\text{out}}, \dots, d_n^{\text{out}})$ and A is the adjacency matrix of the directed graph defined as $A_{ij} = 1$ for all $i, j \in N^{\text{in}}(i)$. We let $P = \frac{1}{2}(I + D^{-1}A)$ and rewrite the weight updates as

$$\mathbf{w}(t+1) = P\mathbf{w}(t).$$

Next, we show the weight sequence generated by the update (3) converges to balancing weights.

Lemma 1. *Under Assumption 1, the sequence $\mathbf{w}(t)$ converges to $\mathbf{w}$ and $\mathbf{w} = [w_1, \dots, w_n]'$ balances the graph.*

Proof: We first show that P is primitive.² Note that all entries of P are non-negative, the diagonal entries of P are

²A matrix P is primitive if there exists $n \in \mathbb{N}$ such that all entries of P^n are positive.

positive, and the underlying graph is strongly connected. The result follows from these facts. We next show the spectral radius of P , $\rho(P)$, is one. We define an auxiliary matrix $\bar{P} = \frac{1}{2}(I + AD^{-1})$ and show it is column stochastic. This holds because the summation of the entries on the j th column of $\bar{P}$ is $\frac{1}{2} + \sum_i : j \in N^{\text{in}}(i) \frac{1}{2} \frac{1}{d_j^{\text{out}}} = 1$. Since $\bar{P}$ is column stochastic, we have $\rho(\bar{P}) = 1$. Therefore, we have

$$\begin{aligned} 1 &= \rho(\bar{P}) = \rho(\bar{P}DD^{-1}) = \rho(D^{-1}\bar{P}D) \\ &= \rho\left(\frac{1}{2}(I + D^{-1}AD^{-1}D)\right) = \rho(P), \end{aligned}$$

where the equality $\rho(\bar{P}DD^{-1}) = \rho(D^{-1}\bar{P}D)$ holds because the set of eigenvalues of AB is the same as the set of eigenvalues of BA for two arbitrary matrices A and B . Finally, since P is a primitive matrix with $\rho(P) = 1$, using Perron-Frobenius Theorem, the limit $\lim_{t \rightarrow \infty} P^t$ exists and as a result $\mathbf{w}(t)$ converges to some $\mathbf{w}$ that satisfies $\mathbf{w} = P\mathbf{w}$. Using the definition of P , this yields

$$w_i = \frac{1}{2}w_i + \sum_{j: j \in N^{\text{in}}(i)} \frac{1}{2} \frac{1}{d_i^{\text{out}}} w_j, \text{ for any } i,$$

which results in

$$d_i^{\text{out}} \frac{1}{2} w_i = \sum_{j \in N^{\text{in}}(i)} \frac{1}{2} w_j.$$

Therefore, $\{\mathbf{w}(t)\}_{t=0}^{\infty}$ converges to weights that balance the graph. ■

The generated weight sequence $\mathbf{w}(t)$ depends on the initial value of $\mathbf{w}(0)$. We consider the sequence update (2) and note that in order to guarantee the contribution of agent i 's estimate at time t in his estimate at time $t+1$ is positive, we need the coefficient of $x_i(t)$ which is $1 - w_i(t)d_i^{\text{out}}$ to be positive at each iteration. In the next lemma we show that by choosing a small $\mathbf{w}(0)$, we can guarantee this. Let D denote the diameter and $d^* = \max_i d_i^{\text{out}}$ denote the maximum out-degree of the graph G .

Lemma 2. *If for all $i = 1, \dots, n$, $w_i(0) \leq (1/d^*)^{2D+1}$, then $w_i(t)d_i^{\text{out}} < 1$ for all i and $t \geq 0$.*

Proof: Let $\mathbf{u}$ and $\mathbf{v}$ be left and right eigenvectors of P corresponding to eigenvalue one; i.e. $\mathbf{u}'P = \mathbf{u}'$ and $P\mathbf{v} = \mathbf{v}$. From Perron-Frobenius Theorem, we have $\lim_{t \rightarrow \infty} P^t = \mathbf{v}\mathbf{u}'$, where $\mathbf{u} > 0$, $\mathbf{v} > 0$, $\sum_{i=1}^n v_i = 1$, and $\sum_{i=1}^n v_i u_i = 1$. Next, we will bound the entries of $\mathbf{v}$. Let $v_M = \max_i v_i$ and $v_m = \min_i v_i$ denote the maximum and minimum entries among v_i 's. The i -th equation of $P\mathbf{v} = \mathbf{v}$ can be written as

$$v_i = \sum_{k \in N^{\text{in}}(i)} \frac{1}{d_i^{\text{out}} - 1} v_k \quad \text{for all } k \in N^{\text{in}}(i). \quad (5)$$

Using (5), for any i we can bound v_i by a factor of v_k for $k \in N^{\text{in}}(i)$ as follows

$$v_i \geq \frac{1}{d^*} v_k \quad \text{for all } k \in N^{\text{in}}(i), \quad (6)$$

where $d^* = \max_i d_i^{\text{out}}$. Using the above inequality, for any k , we obtain that $v_i \geq \left(\frac{1}{d^*}\right)^D v_M$, for all i , where D is the

the diameter of the graph. Using this inequality along with $\sum_{i=1}^n v_i = 1$, we obtain

$$1 = \sum_{i=1}^n v_i \geq v_M n \left(\frac{1}{d^*} \right)^D,$$

which yields to $v_M \leq \frac{1}{n} (d^*)^D$.

Next, we find a lower bound on the value of v_m . Again, using (6), we obtain that $v_m \geq \left(\frac{1}{d^*} \right)^D v_i$, for all i . Using this inequality along with $\sum_{i=1}^n v_i = 1$, we obtain

$$1 = \sum_{i=1}^n v_i \leq v_m n d^{*D},$$

which yields to $v_m \geq \frac{1}{n} \left(\frac{1}{d^*} \right)^D$.

Next, we will use the derived bounds on v_m and v_M in order to bound the summation of entries of each row of P^t . Since $P\mathbf{v} = \mathbf{v}$, for any t , we have $P^t \mathbf{v} = \mathbf{v}$. Because the entries of P^t are non-negative, for any i we have

$$v_M = \max_{1 \leq i \leq n} v_i \geq v_i = [P^t \mathbf{v}]_i = \sum_{j=1}^n [P^t]_{ij} v_j \geq v_m \sum_{j=1}^n [P^t]_{ij}.$$

Plugging in the bounds on v_m and v_M , we obtain

$$\sum_{j=1}^n [P^t]_{ij} \leq \frac{v_M}{v_m} \leq \frac{\frac{1}{n} d^{*D}}{\frac{1}{n} \left(\frac{1}{d^*} \right)^D} = d^{*2D}.$$

Therefore, if we let $w_i(0) < \frac{1}{d^{*(2D+1)}}$ for all $i \in V$, then

$$d_i^{\text{out}} w_i(t) = d_i^{\text{out}} \sum_{j=1}^n [P^t]_{ij} w_j(0) \leq d_i^{\text{out}} d^{*2D} \|\mathbf{w}(0)\|_\infty < 1,$$

and we obtain $1 - w_i(t) d_i^{\text{out}} > 0$ for all $t \geq 0$, and $i \in V$. ■

IV. CONVERGENCE OF ALGORITHM

A. Preliminary Results

We can write the updates of our algorithm (2) in a compact form as

$$\mathbf{x}(t+1) = Q(t) \mathbf{x}(t) - \alpha(t) \mathbf{g}(t), \quad (7)$$

where

$$\mathbf{g}(t) = [g_1(t), \dots, g_n(t)]',$$

$$\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]',$$

and $Q(t)$ is a matrix such that $[Q(t)]_{ii} = 1 - w_i(t) d_i^{\text{out}}$ and $[Q(t)]_{ij} = -w_j(t)$ for any i and $j \in N^{\text{in}}(i)$.

Using (7) recursively, we can relate the estimate at time $t+1$ to initial estimates and the intermediate matrices and subgradients as follows

$$\begin{aligned} \mathbf{x}(t+1) &= [Q(t)Q(t-1) \dots Q(0)] \mathbf{x}(0) \\ &\quad - \sum_{s=0}^{t-1} \alpha(s) [Q(t) \dots Q(s+1)] \mathbf{g}(s) - \alpha(t) \mathbf{g}(t). \end{aligned} \quad (8)$$

This motivates the following definition

$$\Phi(t : s) = Q(t) \dots Q(s), \quad \text{for } t \geq s,$$

with the convention that $\Phi(t : t+1) = I$. Using this definition (8) can be written as

$$\mathbf{x}(t+1) = \Phi(t : 0) \mathbf{x}(0) - \sum_{s=0}^t \Phi(t : s+1) \mathbf{g}(s) \alpha(s). \quad (9)$$

Next, we show that for any t the matrix $Q(t)$ is column stochastic and becomes doubly stochastic only in the limit (i.e., Q is doubly stochastic, where $\lim_{t \rightarrow \infty} Q(t) = Q$) and our analysis shows this suffices to guarantee the entries of $\Phi(t : s)$ converge to $\frac{1}{n}$ exponentially fast as $t-s$ grows large. In the analysis that follows, we will use Theorem 4.14 and Theorem 4.19 of [39].

Proposition 1 ([39]). 1) Let $Q(t)$ be a sequence of column stochastic matrices such that $\lim_{t \rightarrow \infty} Q(t) = Q$ (entry-wise), where Q is a primitive matrix. Then, for any s , we have that $\lim_{t \rightarrow \infty} [\Phi(t : s)]_{ij} = p_j$, where $\mathbf{p}$ is the unique probability vector (non-negative with summation equal to one) such that $Q\mathbf{p} = \mathbf{p}$.

2) Let $Q(t)$ be a sequence of column stochastic primitive matrices such that $\min_{ij}^+ [Q(t)]_{ij} \geq \gamma$, where $\min^+$ denotes the minimum among positive entries. Then there exist a probability vector $\mathbf{p}(s)$ such that

$$|[\Phi(t : s)]_{ij} - p_j(s)| \leq \frac{1}{1-\gamma} (1-\gamma)^{t-s+1}.$$

Lemma 3. There exist constants C and λ such that for any $t \geq s$ and $i, j \in V$, we have

$$|[\Phi(t : s)]_{ij} - \frac{1}{n}| \leq C \lambda^{(t-s+1)}. \quad (10)$$

Proof: Using Lemma 2, since for any i , $d_i^{\text{out}} w_i(t) < 1$, all the entries of matrices $Q(t)$ are non-negative for $t \geq 0$. Furthermore, for any t the matrix $Q(t)$ is column stochastic because for any i ,

$$\sum_{i=1}^n Q_{ij}(t) = 1 - w_j(t) d_j^{\text{out}} + \sum_{i: j \in N^{\text{in}}(i)} w_j(t) = 1.$$

Using Lemma 1, $\lim_{t \rightarrow \infty} Q(t) = Q$, where $Q_{ii} = 1 - w_i d_i^{\text{out}}$ and $Q_{ij} = w_i$ for $j \in N^{\text{in}}(i)$. The matrix Q is row stochastic as well because for any j ,

$$\sum_{i=1}^n Q_{ij} = 1 - d_j^{\text{out}} w_j + \sum_{i: j \in N^{\text{in}}(i)} w_j = 1.$$

Since Q is both column and row stochastic the unique probability vector $\mathbf{p}$ for which $Q\mathbf{p} = \mathbf{p}$ holds, is a vector with all entries equal to $\frac{1}{n}$. Also note that since all entries of Q are non-negative, the diagonal entries of Q are positive, and the underlying graph is strongly connected, we know that the matrix Q is primitive (with the same argument $Q(t)$ is primitive for any t).

We now have all the conditions to use the first part of Proposition 1, to obtain $\lim_{t \rightarrow \infty} [\Phi(s : t)]_{ij} = \frac{1}{n}$, for all $i, j \in V$.

Next, we use the second part of Proposition 1 in order to establish the exponential rate of convergence. Let $\delta = \min_{ij}^+ Q_{ij}$. Since $\lim_{t \rightarrow \infty} Q(t) = Q$, there exists t_0 such that for any $t \geq t_0$, we have that $\|Q(t) - Q\|_\infty \leq \frac{\delta}{2}$. Therefore,

for $t \geq t_0$, we have that $\min^+[Q(t)]_{ij} \geq \frac{\delta}{2}$. We let $\gamma = \min\{\frac{\delta}{2}, \min_{t \leq t_0, i, j}^+[Q(t)]_{ij}\}$ to obtain $\min_{ij}^+[Q(t)]_{ij} \geq \gamma$, for any t . Now we can use the second part of Proposition 1 to obtain that for any s ,

$$|[\Phi(t : s)]_{ij} - \frac{1}{n}| \leq \frac{1}{1-\gamma}(1-\gamma)^{t-s+1}.$$

This completes the proof. $\blacksquare$

Remark 1. When $f_i(x) = 0$ for all $i \in V$ and all $x \in \mathbb{R}$ (implying $g_i(t) = 0$, for all $i \in V$ and all $t \geq 0$), the update step (2) simplifies to

$$x_i(t+1) = x_i(t) (1 - w_i(t)d_i^{\text{out}}) + \sum_{j \in N^{\text{in}}(i)} w_j(t)x_j(t). \quad (11)$$

For this special case combining (9) with Lemma 3 shows that the estimates $x_i(t)$ generated by algorithm (2)-(3) converges to the average of the initial values, i.e.,

$$\lim_{t \rightarrow \infty} x_i(t) = \frac{1}{n} \sum_{i=1}^n x_i(0),$$

and the rate of convergence is exponential. Hence, our algorithm provides a new distributed method for computing average of initial values over a directed graph (see [33], [5] for similar algorithms).

B. Consensus in Estimates

We will first show that under some mild assumptions on the step size sequence, the disagreement between estimates of agents goes to zero, i.e., $\lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| = 0$, for all $i, j \in V$. To that end, define an auxiliary sequence as

$$y(t+1) = y(t) - \frac{\alpha(t)}{n} \sum_{i=1}^n g_i(t). \quad (12)$$

We let $y(0) = \frac{1}{n} \sum_{i=1}^n x_i(0)$ to obtain

$$y(t) = \frac{1}{n} \sum_{i=1}^n x_i(0) - \frac{1}{n} \sum_{s=0}^{t-1} \alpha(s) \sum_{i=1}^n g_i(s). \quad (13)$$

Using the compact form (9), the update of node i at time $t+1$ can be written as

$$x_i(t+1) = [\Phi(t : 0)\mathbf{x}(0)]_i - \sum_{s=0}^t [\Phi(t : s+1)\mathbf{g}(s)]_i \alpha(s). \quad (14)$$

We establish the convergence rate of our algorithm using the ergodic average of the sequence $\{x_i(t)\}$ generated by algorithm (2)-(3), defined as

$$\hat{x}_i(T) = \frac{1}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) x_i(t), \quad (15)$$

and

$$\hat{y}(T) = \frac{1}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) y(t). \quad (16)$$

Assumption 2. Suppose that the functions f_i have bounded subgradient, i.e., there exists L such that for any $i \in V$ and x

$$|g_i| \leq L, \text{ for any } g_i \in \partial f_i(x) \text{ and any } x.$$

Lemma 4. Let the sequence $\{\alpha(t)\}_{t=0}^\infty$ be non-negative. For all $i \in V$, we have

$$|x_i(t) - y(t)| \leq C\lambda^t \|\mathbf{x}(0)\|_1 + nL \sum_{s=0}^{t-1} C\lambda^{t-s-1} \alpha(s). \quad (17)$$

Moreover, if the sequence $\{\alpha(t)\}_{t=0}^\infty$ is non-increasing and converges to zero, then for any i and j ,

$$\lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| = 0.$$

Proof:

Using (14), (13), and Hölder's inequality, we have

$$\begin{aligned} |x_i(t) - y(t)| &= |[\Phi(t-1 : 0)\mathbf{x}(0)]_i \\ &\quad - \sum_{s=0}^{t-1} [\Phi(t-1 : s+1)\mathbf{g}(s)]_i \alpha(s) \\ &\quad - \frac{1}{n} \sum_{i=1}^n x_i(0) + \frac{1}{n} \sum_{s=0}^{t-1} \alpha(s) \sum_{i=1}^n g_i(s)| \\ &\leq \max_j |[\Phi(t-1 : 0)]_{ij} - \frac{1}{n}| \times \|\mathbf{x}(0)\|_1 \\ &\quad + n \sum_{s=0}^{t-1} \alpha(s) L \max_j |[\Phi(t-1 : s+1)]_{ij} - \frac{1}{n}| \\ &\leq C\lambda^t \|\mathbf{x}(0)\|_1 + nL \sum_{s=0}^{t-1} C\lambda^{t-1-s} \alpha(s), \end{aligned}$$

where we used Lemma 3 to obtain the last inequality. For a given ϵ , let t_0 be such that for $t \geq t_0$, $\alpha(t) \leq \frac{\epsilon(1-\lambda)}{2}$ to obtain

$$\begin{aligned} \sum_{s=0}^{t-1} \alpha(s) \lambda^{t-1-s} &= \sum_{s=0}^{t_0} \alpha(s) \lambda^{t-1-s} + \sum_{s=t_0+1}^{t-1} \alpha(s) \lambda^{t-1-s} \\ &\leq \left(\max_{0 \leq s \leq t_0} \alpha(s) \right) \lambda^{-t_0} \lambda^{t-1} \sum_{s=0}^{t_0} \lambda^s + \frac{\epsilon(1-\lambda)}{2} \sum_{s=t_0+1}^{t-1} \lambda^{t-1-s} \\ &\leq \alpha(0) \lambda^{-t_0} \frac{1}{1-\lambda} \lambda^{t-1} + \frac{\epsilon}{2}. \end{aligned}$$

Using this relation, we have $\lim_{t \rightarrow \infty} \sum_{s=0}^{t-1} \alpha(s) \lambda^{t-1-s} \leq \frac{\epsilon}{2}$. Therefore, for any i, j , we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| &\leq \left(\lim_{t \rightarrow \infty} 2\lambda^t \|\mathbf{x}(0)\|_1 \right) + nLC\epsilon = nLC\epsilon. \\ \text{Since } \epsilon \text{ is arbitrary, by taking } \epsilon \rightarrow 0, \text{ we conclude} \\ \lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| &= 0. \quad \blacksquare \end{aligned}$$

Lemma 4 establishes the convergence of all the sequences $\{x_i(t) - x_j(t)\}_{t=0}^\infty$ to zero. Next, we bound the difference between the ergodic average sequences $\{\hat{x}_i(t)\}_{t=0}^\infty$ and $\{\hat{y}(t)\}_{t=0}^\infty$.

Theorem 1. The ergodic average of the sequence generated by algorithm (2)-(3), satisfies

$$\begin{aligned} |\hat{x}_i(T) - \hat{y}(T)| &\leq \frac{2}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) \left(C\lambda^t \|\mathbf{x}(0)\|_1 + Ln \sum_{s=0}^{t-1} C\lambda^{t-1-s} \alpha(s) \right) \quad (18) \end{aligned}$$

Proof: Using convexity of norm, for any i , we have

$$|\hat{x}_i(T) - \hat{y}(T)| \leq \frac{1}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) |x_i(t) - y(t)|.$$

We next use Lemma 4 to upper bound each term on the right hand side of the previous relation, which yields to

$$\begin{aligned} |\hat{x}_i(T) - \hat{y}(T)| &\leq \frac{1}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) |x_i(t) - y(t)| \\ &\leq \frac{1}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) \\ &\quad \left(C\lambda^t \|\mathbf{x}(0)\|_1 + Ln \sum_{s=0}^{t-1} C\lambda^{t-1-s} \alpha(s) \right). \end{aligned}$$

C. Optimality Convergence

Let $F(\mathbf{x}(t)) = \sum_{i=1}^n f_i(x_i(t))$ and $F(\mathbf{x}^*) = \sum_{i=1}^n f_i(x_i^*)$, where $\mathbf{x}^* = (x^*, \dots, x^*) \in \mathbb{R}^n$ is an optimal solution of (1). Also let $\hat{\mathbf{x}}(T) = (\hat{x}_1(T), \dots, \hat{x}_n(T))$.

Lemma 5. *The ergodic average of the sequence generated by algorithm (2)-(3), satisfies*

$$\begin{aligned} F(\hat{\mathbf{x}}(T)) - F(\hat{\mathbf{y}}(T)) &\leq L \frac{1}{\sum_{t=0}^T \alpha(t)} \\ &\quad \left(\sum_{t=0}^T \alpha(t) \left(C\lambda^t \|\mathbf{x}(0)\|_1 + Ln \sum_{s=0}^{t-1} C\lambda^{t-1-s} \alpha(s) \right) \right) \end{aligned} \quad (19)$$

Proof: Using bounded subgradient assumption and convexity of norm function, we obtain that

$$\begin{aligned} F(\hat{\mathbf{x}}(T)) - F(\hat{\mathbf{y}}(T)) &\leq \sum_{j=1}^n g_j(\hat{x}_j(T)) (\hat{x}_j(T) - \hat{y}(T)) \\ &\leq \sum_{j=1}^n L |\hat{x}_j(T) - \hat{y}(T)| \leq L \frac{1}{\sum_{t=0}^T \alpha(t)} \\ &\quad \left(\sum_{t=0}^T \alpha(t) \left(C\lambda^t \|\mathbf{x}(0)\|_1 + Ln \sum_{s=0}^{t-1} C\lambda^{t-1-s} \alpha(s) \right) \right), \end{aligned}$$

where we used Theorem 1 to obtain the last inequality. ■

Lemma 6. *The ergodic average of the sequence generated by algorithm (2)-(3), satisfies*

$$\begin{aligned} F(\hat{\mathbf{y}}(T)) - F(\mathbf{x}^*) &\leq \frac{n}{2 \sum_{t=0}^T \alpha(t)} (y(0) - x^*)^2 \\ &\quad + \frac{nL^2}{2 \sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t)^2 \\ &\quad + \frac{2L}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) \sum_{i=1}^n |y(t) - x_i(t)|. \end{aligned} \quad (20)$$

Proof: Using (12), we have that

$$\begin{aligned} (y(t+1) - x^*)^2 &= (y(t) - x^*)^2 + \frac{\alpha(t)^2}{n^2} \left(\sum_{i=1}^n g_i(t) \right)^2 \\ &\quad - \frac{2\alpha(t)}{n} \sum_{i=1}^n g_i(t) (y(t) - x^*). \end{aligned} \quad (21)$$

Since $|g_i(t)| \leq L$, we also have that

$$\begin{aligned} g_i(t)(y(t) - x^*) &= g_i(t)(y(t) - x_i(t)) + g_i(t)(x_i(t) - x^*) \\ &\geq g_i(t)(y(t) - x_i(t)) + f_i(x_i(t)) - f_i(x^*) \\ &\geq -L|y(t) - x_i(t)| \\ &\quad + f_i(x_i(t)) - f_i(y(t)) + f_i(y(t)) - f_i(x^*). \end{aligned}$$

Using $f_i(x_i(t)) - f_i(y(t)) \geq -L|x_i(t) - y(t)|$ in the previous relation, we obtain

$$g_i(t)(y(t) - x^*) \geq -2L|y(t) - x_i(t)| + f_i(y(t)) - f_i(x^*).$$

Using the previous relation and $|g_i(t)| \leq L$ in (21), we have

$$\begin{aligned} (y(t+1) - x^*)^2 &\leq (y(t) - x^*)^2 + L^2 \alpha(t)^2 \\ &\quad - \frac{2\alpha(t)}{n} \sum_{i=1}^n (f_i(y(t)) - f_i(x^*)) + \frac{4L\alpha(t)}{n} \sum_{i=1}^n |y(t) - x_i(t)|. \end{aligned}$$

Rearranging the terms of the previous relation yields that

$$\begin{aligned} &\frac{2\alpha(t)}{n} \sum_{i=1}^n (f_i(y(t)) - f_i(x^*)) \\ &\leq - (y(t+1) - x^*)^2 + (y(t) - x^*)^2 + L^2 \alpha(t)^2 \\ &\quad + \frac{4L\alpha(t)}{n} \sum_{i=1}^n |y(t) - x_i(t)|. \end{aligned}$$

Taking summation of the previous relation from $t = 0$ to T and noting the telescopic cancellation of terms, we obtain

$$\begin{aligned} \frac{2}{n} \sum_{t=0}^T \alpha(t) (F(\mathbf{y}(t)) - F(\mathbf{x}^*)) &= \sum_{t=0}^T \frac{2\alpha(t)}{n} \sum_{i=1}^n (f_i(y(t)) - f_i(x^*)) \\ &\leq - (y(T+1) - x^*)^2 + (y(0) - x^*)^2 + L^2 \sum_{t=0}^T \alpha(t)^2 \\ &\quad + \sum_{t=0}^T \frac{4L\alpha(t)}{n} \sum_{i=1}^n |y(t) - x_i(t)| \\ &\leq (y(0) - x^*)^2 + L^2 \sum_{t=0}^T \alpha(t)^2 + \sum_{t=0}^T \frac{4L\alpha(t)}{n} \sum_{i=1}^n |y(t) - x_i(t)|. \end{aligned}$$

Since $\hat{y}(T) = \frac{1}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) y(t)$. Using convexity of the functions along with the previous relation, we obtain (20), which completes the proof. ■

Next, we will bound the difference between the objective function value at $\{\hat{\mathbf{x}}(t)\}_{t=0}^\infty$ and the optimal point.

Theorem 2. *The ergodic average of the sequence generated by algorithm (2)-(3), satisfies*

$$\begin{aligned} F(\hat{\mathbf{x}}(T)) - F(\mathbf{x}^*) &\leq \frac{n}{2 \sum_{t=0}^T \alpha(t)} (y(0) - x^*)^2 \\ &\quad + \frac{nL^2}{2 \sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t)^2 \\ &\quad + \frac{2L}{\sum_{t=0}^T \alpha(t)} \sum_{t=0}^T \alpha(t) \sum_{i=1}^n |y(t) - x_i(t)| + L \frac{1}{\sum_{t=0}^T \alpha(t)} \\ &\quad \left(\sum_{t=0}^T \alpha(t) \left(C\lambda^t \|\mathbf{x}(0)\|_1 + Ln \sum_{s=0}^{t-1} C\lambda^{t-1-s} \alpha(s) \right) \right) \end{aligned} \quad (22)$$

Proof: The proof follows by combining Lemma 5 and Lemma 6. ■

D. Convergence Rate: Choice of Step size

In this section we characterize the convergence rate of the algorithm for $\alpha(t) = \frac{1}{\sqrt{t+1}}$, $t \geq 0$, as the step size sequence. We will show this choice of step size yields $O(\frac{\log T}{\sqrt{T}})$ convergence iteration complexity, which is the optimal convergence rate of any first-order optimization method over the class of convex functions as shown in [28] (see Section 3 of [28] for general lower complexity bounds).

Theorem 3. *Using the ergodic average of the sequence generated by the proposed algorithm with $\alpha(t) = \frac{1}{\sqrt{t+1}}$, both rate of convergence to the optimal solution and rate of convergence of estimates to a consensus point are $O(\frac{\log T}{\sqrt{T}})$. In particular, with $\mathbf{x}(0) = 0$, for all $i, j \in V$ we have*

$$|\hat{x}_i(T) - \hat{x}_j(T)| \leq 2LnC \frac{4}{1-\lambda} \frac{\log T}{\sqrt{T}}, \quad (23)$$

and

$$\begin{aligned} F(\hat{\mathbf{x}}(T)) - F(\mathbf{x}^*) &\leq n(x^*)^2 \frac{1}{2\sqrt{T}} + \frac{nL^2}{2} \frac{2\log T}{\sqrt{T}} \\ &+ 2L^2nC \frac{4}{1-\lambda} \frac{\log T}{\sqrt{T}} + nL^2C \frac{4}{1-\lambda} \frac{\log T}{\sqrt{T}}. \end{aligned} \quad (24)$$

Proof: With $\alpha(t) = \frac{1}{\sqrt{t+1}}$, we have

$$\sum_{t=0}^T \alpha(t) \geq \int_1^{T+2} \frac{1}{\sqrt{x}} dx = 2(\sqrt{T+2} - 1) \geq \sqrt{T}.$$

Since $\alpha(t) \leq 1$, we have $\sum_{t=1}^T \alpha(t)\lambda^t \leq \sum_{t=1}^\infty \lambda^t = \frac{1}{1-\lambda}$, and

$$\begin{aligned} \sum_{t=0}^T \sum_{s=0}^{t-1} \lambda^{t-s-1} \alpha(s) \alpha(t) &= 1 + \sum_{i=1}^T \lambda^{i-1} \left(\sum_{j=1}^{T-i+1} \frac{1}{\sqrt{j}\sqrt{i+j}} \right) \\ &\leq 1 + \sum_{i=1}^T \lambda^{i-1} \int_{x=0}^{T-i+1} \frac{1}{\sqrt{x}\sqrt{x+i}} \\ &= 1 + \sum_{i=1}^T \lambda^{i-1} 2 \left(\log(\sqrt{T-i+1} + \sqrt{T-1}) \right) \\ &\leq 1 + 2 \frac{1}{1-\lambda} \left(\log 2 + \frac{1}{2} \log T \right) \leq \frac{4}{1-\lambda} \log T, \end{aligned}$$

for $T \geq 2$. Using the previous three relations in Theorem 1, for any i and T we obtain

$$|\hat{x}_i(T) - \hat{y}(T)| \leq C \frac{1}{1-\lambda} \|\mathbf{x}(0)\|_1 \frac{1}{\sqrt{T}} + LnC \frac{4}{1-\lambda} \frac{\log T}{\sqrt{T}}. \quad (25)$$

This shows that rate of convergence to a consensus point is $O(\frac{\log T}{\sqrt{T}})$. In particular, using this relation for i and j and setting $\mathbf{x}(0) = 0$, we obtain (23). Using the three relations along with Theorem 2 and $\sum_{t=0}^T \alpha(t)^2 \leq 1 + \log(T+1) \leq$

Fig. 1: Sample Network with $n = 20$ nodes.

Fig. 2: Performance of algorithm over sample network: $\hat{x}_i(T) - x^*$ for $i = 1$ and $i = 20$ as a function of T .

$2\log T$ (for $T \geq 4$), yields

$$\begin{aligned} F(\hat{\mathbf{x}}(T)) - F(\mathbf{x}^*) &\leq n(y(0) - x^*)^2 \frac{1}{2\sqrt{T}} + \frac{nL^2}{2} \frac{2\log T}{\sqrt{T}} \\ &+ 2LC \frac{1}{1-\lambda} \|\mathbf{x}(0)\|_1 \frac{1}{\sqrt{T}} + 2L^2nC \frac{4}{1-\lambda} \frac{\log T}{\sqrt{T}} \\ &+ LC \frac{1}{1-\lambda} \|\mathbf{x}(0)\|_1 \frac{1}{\sqrt{T}} + nL^2C \frac{4}{1-\lambda} \frac{\log T}{\sqrt{T}}, \end{aligned} \quad (26)$$

This shows that the rate of convergence to the optimal solution is $O(\frac{\log T}{\sqrt{T}})$. In particular, setting $\mathbf{x}(0) = 0$ in this relation, we obtain (24). ■

V. NUMERICAL RESULTS

In this section, we show numerical results based on the proposed distributed subgradient algorithm to demonstrate the performance of the algorithm. We consider minimizing the function $F(x) = \frac{1}{2} \sum_{i=1}^n (x - a_i)^2$ where a_i is a scalar that is known only to node i . This problem appears in distributed estimation where the goal is to estimate the parameter x^* , using local measurements $a_i = x^* + N_i$ at each node $i = 1, \dots, n$. Here N_i represents measurements noise, which we assume to be jointly Gaussian with mean zero and variance one. The maximum likelihood estimate is the minimizer x^* of $F(x)$. We let $n = 20$, $a_i = i$, and consider a directed graph shown in Figure 1. We plot the differences $\hat{x}_1(T) - x^*$ and $\hat{x}_n(T) - x^*$ as a function of T in Figure 2.

VI. CONCLUSION

We considered a multi agent optimization problem where a directed network of agents collectively solves a global optimization problem with the objective function given by the sum of locally known convex functions. We propose a distributed subgradient algorithm in which each agent performs local

processing based on his local information and information obtained from his incoming neighbors. In our algorithm, agents scale the information they send to others with dynamically updated weights that converge to balancing weights of the graph and we show this technique overcomes the asymmetries caused by the directed communication network. We show that both the rate of convergence of the objective function values to optimal function value, and the rate of convergence of the estimates to a consensus point is $O(\frac{\log T}{\sqrt{T}})$, where T is the number of iterations.

REFERENCES

- [1] A. Agarwal and J. C. Duchi. Distributed delayed stochastic optimization. In *Advances in Neural Information Processing Systems*, 2011.
- [2] J. Chen and A. H. Sayed. Diffusion adaptation strategies for distributed optimization and learning over networks. *IEEE Transactions on Signal Processing*, 2012.
- [3] J. Cortés. Distributed algorithms for reaching consensus on general functions. *Automatica*, 44(3):726–737, 2008.
- [4] O. Dekel, R. Gilad-Bachrach, O. Shamir, and L. Xiao. Optimal distributed online prediction using mini-batches. *The Journal of Machine Learning Research*, 2012.
- [5] A. D. Dominguez-Garcia and C. N. Hadjicostis. Distributed matrix scaling and application to average consensus in directed graphs. *IEEE Transactions on Automatic Control*, 2013.
- [6] M. F. Duarte and Y. H. Hu. Vehicle classification in distributed sensor networks. *Journal of Parallel and Distributed Computing*, 2014.
- [7] M. F. Duarte and Y. H. Hu. Vehicle classification in distributed sensor networks. *Journal of Parallel and Distributed Computing*, 2004.
- [8] J. C. Duchi, A. Agarwal, and M. J. Wainwright. Dual averaging for distributed optimization: convergence analysis and network scaling. *IEEE Transactions on Automatic Control*, 2012.
- [9] J. C. Duchi, M. I. Jordan, M. J. Wainwright, and Y. Zhang. Optimality guarantees for distributed statistical estimation. *arXiv preprint arXiv:1405.0782*, 2014.
- [10] B. Gharesifard and J. Cortés. Distributed continuous-time convex optimization on weight-balanced digraphs. *IEEE Transactions on Automatic Control*, 2014.
- [11] C. N. Hadjicostis and A. Rikos. Distributed strategies for balancing a weighted digraph. In *IEEE Mediterranean Conference on Control & Automation (MED)*, 2012.
- [12] L. Hooi-Tong. On a class of directed graphs with an application to traffic-flow problems. *Operations Research*, 18(1):87–94, 1970.
- [13] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 2003.
- [14] D. Jakovetic, J. Xavier, and J. M. Moura. Fast distributed gradient methods. *IEEE Transactions on Automatic Control*, 2014.
- [15] B. Johansson, T. Keviczky, M. Johansson, and K. H. Johansson. Subgradient methods and consensus algorithms for solving convex optimization problems. In *IEEE Conference on Decision and Control (CDC)*, 2008.
- [16] D. Kempe, A. Dobra, and J. Gehrke. Gossip-based computation of aggregate information. In *IEEE Symposium on Foundations of Computer Science*, 2003.
- [17] D. Lee and M. W. Spong. Stable flocking of multiple inertial agents on balanced graphs. *IEEE Transactions on Automatic Control*, 2007.
- [18] H. Li and Z. Han. Competitive spectrum access in cognitive radio networks: graphical game and learning. In *Wireless Communications and Networking Conference (WCNC)*, 2010.
- [19] I. Lobel and A. Ozdaglar. Distributed subgradient methods for convex optimization over random networks. *IEEE Transactions on Automatic Control*, 2011.
- [20] I. Lobel, A. Ozdaglar, and D. Feijer. Distributed multi-agent optimization with state-dependent communication. *Mathematical Programming*, 2011.
- [21] S. H. Low and D. E. Lapsley. Optimization flow control: basic algorithm and convergence. *IEEE/ACM Transactions on Networking (TON)*, 1999.
- [22] I. Matei and J. S. Baras. Performance evaluation of the consensus-based distributed subgradient method under random communication topologies. *IEEE Journal of Selected Topics in Signal Processing*, 2011.
- [23] A. Nedic. Asynchronous broadcast-based convex optimization over a network. *IEEE Transactions on Automatic Control*, 2011.
- [24] A. Nedic and A. Olshevsky. Distributed optimization over time-varying directed graphs. In *IEEE Conference on Decision and Control (CDC)*, 2013.
- [25] A. Nedic and A. Olshevsky. Distributed optimization of strongly convex functions over time-varying graphs. In *IEEE Global Conference on Signal and Information Processing*, 2013.
- [26] A. Nedic and A. Ozdaglar. On the rate of convergence of distributed subgradient methods for multi-agent optimization. In *IEEE Conference on Decision and Control (CDC)*, 2007.
- [27] A. Nedic, A. Ozdaglar, and P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 2010.
- [28] Y. Nesterov. *Introductory lectures on convex optimization*, volume 87. Springer Science & Business Media, 2004.
- [29] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 2007.
- [30] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 2004.
- [31] A. Olshevsky and J. N. Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM Journal on Control and Optimization*, 2009.
- [32] A. Priolo, A. Gasparri, E. Montijano, and C. Sagues. A decentralized algorithm for balancing a strongly connected weighted digraph. In *American Control Conference (ACC)*, 2013.
- [33] A. Priolo, A. Gasparri, E. Montijano, and C. Sagues. A distributed algorithm for average consensus on strongly connected weighted digraphs. *Automatica*, 2014.
- [34] M. Rabbat and R. Nowak. Distributed optimization in sensor networks. In *International symposium on Information processing in sensor networks*, 2004.
- [35] S. S. Ram, A. Nedić, and V. V. Veeravalli. Distributed stochastic subgradient projection algorithms for convex optimization. *Journal of optimization theory and applications*, 2010.
- [36] S. S. Ram, V. V. Veeravalli, and A. Nedic. Distributed non-autonomous power control through distributed convex optimization. In *IEEE INFOCOM*, 2009.
- [37] B. Recht, C. Re, S. Wright, and F. Niu. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In *Advances in Neural Information Processing Systems*, 2011.
- [38] A. I. Rikos, T. Charalambous, and C. N. Hadjicostis. Distributed weight balancing over digraphs. *IEEE Transactions on Control of Network Systems*, 2014.
- [39] E. Seneta. *Non-negative matrices and Markov chains*. Springer, 2006.
- [40] W. Shi, Q. Ling, G. Wu, and W. Yin. Extra: An exact first-order algorithm for decentralized consensus optimization. *arXiv preprint arXiv:1404.6264*, 2014.
- [41] K. I. Tsianos, S. Lawlor, and M. G. Rabbat. Push-sum distributed dual averaging for convex optimization. In *IEEE Conference on Decision and Control (CDC)*, 2012.
- [42] J. N. Tsitsiklis. Problems in decentralized decision making and computation. Technical report, DTIC Document, 1984.
- [43] J. N. Tsitsiklis, D. P. Bertsekas, M. Athans, et al. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 1986.
- [44] E. Wei and A. Ozdaglar. On the $O(1/k)$ convergence of asynchronous distributed alternating direction method of multipliers. *arXiv preprint arXiv:1307.8254*, 2013.
- [45] E. Wei and A. E. Ozdaglar. Distributed alternating direction method of multipliers. In *IEEE Conference on Decision and Control (CDC)*, 2012.
- [46] L. Xiao, S. Boyd, and S.-J. Kim. Distributed average consensus with least-mean-square deviation. *Journal of Parallel and Distributed Computing*, 2007.
- [47] K. Yuan, Q. Ling, and W. Yin. On the convergence of decentralized gradient descent. *arXiv preprint arXiv:1310.7063*, 2013.
- [48] M. Zhu and S. Martínez. On distributed convex optimization under inequality and equality constraints. *IEEE Transactions on Automatic Control*, 2012.