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## *Modeling human ad hoc coordination*

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# Modeling Human Ad Hoc Coordination

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## Abstract

Whether in groups of humans or groups of computer agents, collaboration is most effective between individuals who have the ability to coordinate on a joint strategy for collective action. However, in general a rational actor will only intend to coordinate if that actor believes the other group members have the same intention. This circular dependence makes rational coordination difficult in uncertain environments if communication between actors is unreliable and no prior agreements have been made. An important normative question with regard to coordination in these ad hoc settings is therefore how one can come to believe that other actors will coordinate, and with regard to systems involving humans, an important empirical question is how humans arrive at these expectations. We introduce an exact algorithm for computing the infinitely recursive hierarchy of graded beliefs required for rational coordination in uncertain environments, and we introduce a novel mechanism for multiagent coordination that uses it. Our algorithm is valid in any environment with a finite state space, and extensions to certain countably infinite state spaces are likely possible. We test our mechanism for multiagent coordination as a model for human decisions in a simple coordination game using existing experimental data. We then explore via simulations whether modeling humans in this way may improve human-agent collaboration.

Forming shared plans that support mutually beneficial behavior within a group is central to collaborative social interaction and collective intelligence (Grosz and Kraus 1996). Indeed, many common organizational practices are designed to facilitate shared knowledge of the structure and goals of organizations, as well as mutual recognition of the roles that individuals in the organizations play. Once teams become physically separated and responsiveness or frequency of communication declines, the challenge of forming shared plans increases. Part of this difficulty is fundamentally computational. In theory, coming to a fully mutually recognized agreement on even a simple action plan among two choices can be literally impossible if communication is even mildly unreliable, even if an arbitrary amount of communication is allowed (Halpern and Moses 1990; Lynch 1996).

This problem is well-studied within the AI literature (e.g., (Gmytrasiewicz and Durfee 1992)), though the core difficulties still manifest in contemporary research on "ad hoc coordination"—collaborative multiagent planning with previously unknown teammates (Stone et al. 2010). However, surprisingly little is known about the strategies that *humans* use to overcome the difficulties of coordination (Thomas et al. 2014). Understanding how and when people try to coordinate is critical to furthering our understanding of human group behavior, as well as to the design of agents for human-agent collectives (Jennings et al. 2014). Existing attempts at modeling human coordination have focused either on unstructured predictive models (e.g., (Frieder, Lin, and Kraus 2012)) or bounded depth socially recursive reasoning models (e.g., (Gal and Pfeffer 2008; Yoshida, Dolan, and Friston 2008)), but there is reason to believe that these accounts miss important aspects of human coordination.

One concept that appears repeatedly in formal treatments of coordination but has not appeared meaningfully in empirical modeling is common knowledge. Two agents have common knowledge if both agents have infinitely nested knowledge of the other agent's knowledge of a proposition, i.e. the first agent knows the second agent knows, the first agent knows the second agent knows the first agent knows, etc. Common knowledge has been shown to be necessary for exact coordination (Halpern and Moses 1990), and a probabilistic generalization of common knowledge, called common p-belief, has been been shown to be necessary for approximate coordination (Monderer and Samet 1989). While these notions are clearly important normatively, it is not entirely clear how important they are empirically in human coordination. Indeed, supposing that humans are able to mentally represent an infinitely recursive belief state seems a priori implausible, and the need to represent and infer this infinite recursive belief state has also been a barrier to empirically testing models involving common knowledge.

Nevertheless, building on the existing normative results, a group of researchers recently designed a set of experiments to test whether people are able to recognize situations in which common knowledge might obtain (Thomas et al. 2014) (hereafter referred to as the "Thomas experiments"). These researchers argued that people do possess a distinct mental representation of common knowledge by showing that people will attempt to coordinate more often in situa-

tions where common knowledge can be inferred. However, this previous work did not formalize this claim in a model or rigorously test it against plausible alternative computational models of coordination. This existing empirical work therefore leaves open several important scientific questions that a modeling effort can help address. In particular: How might people mentally represent common p-belief? Do people reason about graded levels of common p-belief, or just "sufficiently high" common p-belief? Finally, what computational processes could people use to infer common p-belief?

In this work we use a previously established fixed point characterization of common p-belief (Monderer and Samet 1989) to formulate a novel model of human coordination. In finite state spaces this characterization yields an exact finite representation of common p-belief, which we use to develop an efficient algorithm for computing common p-belief. This algorithm allows us to simulate models that rely on common p-belief. Because of the normative importance of common p-belief in coordination problems, our algorithm may also be independently useful for coordination in artificial multiagent systems. We show using data from the Thomas experiments that this model provides a better account of human decisions than three alternative models in a simple coordination task. Finally, we show via simulations based on the data from the Thomas experiments that modeling humans in this way may improve human-agent coordination.

# 1 Background

We first provide a description of the coordination task we will study in this paper: the well-known coordinated attack problem. We then provide an overview of the formal definitions of common knowledge and common p-belief, and their relationship to the coordinated attack problem.

## 1.1 Coordinated Attack Problem

The coordination task that we study in this paper is alternatively called the coordinated attack problem, the two generals problem, or the email game. The original formulation of this task was posed in the literature on distributed computer systems to illustrate the impossibility of achieving consensus among distributed computer processors that use an unreliable message-passing system (Halpern and Moses 1990), and the problem was later adapted by economists to a game theoretic context (Rubinstein 1989). Here we focus on the game theoretic adaptation, as this formulation is more amenable to decision-theoretic modeling and thus more relevant for modeling human behavior.[1]

In this task the world can be in one of two states, $x = 1$ or $x = 0$. The state of the world determines which of two games two players will play together. The payoff matrices for these two games are as follows ($a > c > \max(b, d)$):

| $x = 1$ | $A$ | $B$ | | $x = 0$ | $A$ | $B$ |
|---|---|---|---|---|---|---|
| $A$ | a,a | b,c | | $A$ | d,d | b,c |
| $B$ | c,b | c,c | | $B$ | c,b | c,c |

The players receive the optimal payoff if they coordinate on both playing $A$ when $x = 1$, but playing $A$ is risky. Playing $A$ is inferior to playing $B$ if $x = 0$ or if there is a mismatch between the players' actions. Playing $B$ is safe with a sure payoff of $c$. Thus in order for it to be rational to play $A$, a player must believe with sufficient confidence both that the world state is $x = 1$ and that the other player will play $A$.

## 1.2 Common p-Belief

In order for a player to believe the other player will play $A$ in this game, it is not enough for that player to believe that the other player knows $x = 1$. If the second player does not believe that the first player knows $x = 1$, then the second player will not try to coordinate. Therefore the first player must also at least believe that the second player believes the first player knows $x = 1$. However, it turns out even this amount of knowledge does not suffice. In fact, an infinite hierarchy of recursive belief is needed (Morris and Shin 1997). This infinite hierarchy of beliefs has been formalized using a construct called common p-belief, which we now define.

Using standard definitions from game theory, we define a two-player finite Bayesian game to be a tuple $((\Omega, \mu, (\Pi_0, \Pi_1)), (\mathcal{A}_0, \mathcal{A}_1), (u_0, u_1))$ consisting of a finite state space $\Omega = \{\omega_1, \ldots, \omega_{|\Omega|}\}$, a probability measure $\mu$ defined over that state space, the information partition $\Pi_i$ of each player $i$, the action set $\mathcal{A}_i$ of each player, and the utility function $u_i$ of each player. Elements of $\Omega$ are called states, and subsets of $\Omega$ are called events. For a given world state $\omega$ and player $i$ the partition of $\Omega$, $\Pi_i$, uniquely specifies the beliefs of player $i$ in the form of posterior probabilities. $\Pi_i(\omega)$, which indicates the unique element of $\Pi_i$ containing $\omega$, can be thought of as the observation that player $i$ receives when the true state $\omega$ occurs. Specifically for any event $E \subseteq \Omega$, $\mu(E \,|\, \Pi_i(\omega))$ is the probability that player $i$ assigns to $E$ having occurred given $\omega$ has occurred. As a shorthand, we write $P_i(E \,|\, \omega) = \mu(E \,|\, \Pi_i(\omega))$. Using another common shorthand, we will treat propositions and events satisfying those propositions interchangeably. For example, in the coordinated attack problem we will be interested in whether there is "common p-belief" that $x = 1$, which will refer to common p-belief in the event $C = \{\omega \in \Omega : x(\omega) = 1\}$, where $x$ formally is a random variable mapping $\Omega \to \{0, 1\}$.

Following (Dalkiran et al. 2012), we say that player $i$ p-believes[2] an event $E$ at $\omega$ if $P_i(E \,|\, \omega) \geq p$. An event $E$ is said to be p-evident if for all $\omega \in E$ and for all players $i$, player $i$ p-believes $E$ at $\omega$. In a slight divergence from the standard terminology of this literature, we say an event $E$ is super-p-evident if for all $\omega \in E$ and for all players $i$, $P_i(E \,|\, \omega) > p$ (the only difference being strict inequality). We say there is common p-belief in an event $C$ at state $\omega$ if there exists a p-evident event $E$ with $\omega \in E$, and for all $\omega' \in E$ and all players $i$, player $i$ p-believes $C$ at $\omega'$. Common knowledge is defined as common p-belief for $p = 1$.

A critically important result of (Monderer and Samet 1989) states that this definition of common p-belief is equiv-

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[1]Our exposition largely assumes familiarity with rudimentary game theory, and familiarity with measure-theoretic probability as it appears in incomplete information games.

[2]We use an italicized "p" when referring to specific values of p-belief and a non-italicized "p" when referring to the terms in general.

alent to a more intuitive infinitely recursive formulation. The importance of this definition of common p-belief is therefore that it provides a fixed point characterization of common p-belief strictly in terms of beliefs about events rather than directly in terms of beliefs about other players. When $\Omega$ is finite, common p-belief can thus be represented in terms of a finite set of states, rather than an infinite hierarchy of beliefs.

## 2 Models

We now describe four strategies for coordination in the coordinated attack game we study. Two of these strategies involve the computation of p-evident events and common p-belief, which we will use to test whether human coordination behavior could be explained in terms of reasoning about p-evident events. The other two strategies serve as baselines.

### 2.1 Rational p-Belief

The first strategy we consider represents an agent who maximizes expected utility at an equilibrium point of the coordinated attack problem we study. The strategy is implemented as follows: player $i$ plays action $A$ if and only if the player believes with probability at least $p^* = \frac{c-b}{a-b}$ that both players have common $p^*$-belief that $x = 1$. This strategy forms an equilibrium of the coordinated attack problem if $p^* > P_i(x = 1)$ and if evidence that $x = 1$ always leads to certain belief that $x = 1$, i.e. $P_i(x = 1 \,|\, \omega) > P_i(x = 1) \Rightarrow P_i(x = 1 \,|\, \omega) = 1$ for all $\omega$. These conditions will be satisfied by the specific state spaces and payoffs we use to represent the Thomas experiments. We call this model the rational p-belief strategy. The proof that this strategy forms an equilibrium, including a derivation for the specific form of $p^*$, is included in our supplementary materials.[3]

### 2.2 Matched p-Belief

The second strategy we consider is a novel probabilistic relaxation of the rational p-belief strategy. Humans have been shown to exhibit a behavior called probability matching in many decision-making settings (Herrnstein 1961; Vulkan 2000). Probability matching consists of taking the probability $p$ that a decision is the best decision available, and choosing to make that decision with probability $p$. While probability matching is not utility maximizing, it can be viewed as rational if players are performing sample-based Bayesian inference and if taking samples is costly (Vul et al. 2014). Motivated by this frequently observed behavior, we propose a model we call the matched p-belief strategy. A player $i$ using this strategy chooses action $A$ at $\omega$ with probability $p$ equal to the maximal common p-belief that the player perceives at $\omega$, i.e. the largest value such that $i$ p-believes at $\omega$ that there is common $p$-belief that $x = 1$.

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[3] A version of our paper that includes the supplementary materials (as well as any post-publication corrections to the main text) is available in multiple locations online, including at http://people.csail.mit.edu/pkrafft/papers/krafft-et-al-2016-modeling-human-ad-hoc-coordination.pdf.

### 2.3 Iterated Maximization

Next we consider a well-known model of boundedly rational behavior sometimes called a "level-$k$" depth of reasoning model. This family of models has been shown to be consistent with human behavior in a diversity of settings, including some coordination games (e.g., (Yoshida, Dolan, and Friston 2008)), and hence is a strong baseline. Since the term "level-$k$" is used for many slightly different models, we call our instantiation of this model the iterated maximization strategy. This strategy assumes that players have a certain fixed level of recursive social reasoning $k$. A player using the level-$k$ iterated maximization strategy chooses the action that maximizes that player's expected utility when playing with a player using the level-$(k - 1)$ strategy. The level-0 player takes action $A$ at $\omega$ if $P_i(x = 1 \,|\, \omega) > \frac{c-b}{a-b}$. This level-0 strategy corresponds to the player maximizing expected utility assuming the player can control the actions of both players, or equivalently that the optimal joint action is taken according to that player's beliefs. While in general the predictions of level-$k$ models depend strongly on the specification of the level-0 strategy, in informal exploration we found that the qualitative conclusions of our work are robust to whether we instead specify the level-0 strategy as always playing $A$ or choosing between $A$ and $B$ uniformly randomly.

### 2.4 Iterated Matching

Finally, we also consider a less common depth of reasoning model that combines the iterated maximization strategy with probability matching behavior, which we call iterated matching. Like the iterated maximization strategy, this strategy assumes that players have a certain fixed level of recursive social reasoning $k$. However, instead of choosing the action that maximizes expected utility, a level-$k$ player using the iterated matching strategy chooses to take action $A$ with probability equal to that player's belief that $x = 1$, times the expected probability that a level-$(k-1)$ companion player would play $A$. The level-0 player probability matches on $P_i(x = 1 \,|\, \omega)$.

## 3 Algorithms

In this section we present the algorithms we use to implement each of the models we consider. To the best of our knowledge the existing literature on common p-belief has yet to offer algorithms for computing common p-belief (or in our case the perceived maximal common p-belief) for a given world state and observation model. This computation is central to the rational and matched p-belief strategies. Hence we offer the first fully computational account of coordination via reasoning about p-evident events. Algorithms for iterated reasoning are straightforward and well-known.

The challenge in developing an algorithm for computing a player's perception of the maximal common p-belief is avoiding enumeration over all exponentially many possible subsets of $\Omega$. While it is straightforward to evaluate whether a particular given event is $p$-evident, the definition of common $p$-belief requires only the existence of some such event. Computing perceived maximal common $p$-belief therefore

**Algorithm 1** common_p_belief($C, i, \omega$)

$\quad E := \Omega;\ F := \Omega$
$\quad$**while** $P_i(F \mid \omega) > 0$ **do**
$\quad\quad p := $ evidence_level($F, C$)
$\quad\quad E := F$
$\quad\quad F := $ super_p_evident($E, C, p$)
$\quad p := $ evidence_level($E, C$)
$\quad$**return** $p$

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**Algorithm 2** evidence_level($E, C$)

$\quad$**return** $\min_{\omega \in E}$ min_belief($E, C, \omega$)

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**Algorithm 3** min_belief($E, C, \omega$)

$\quad$**return** $\min_{i \in \{0,1\}} \min(P_i(E \mid \omega), P_i(C \mid \omega))$

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**Algorithm 4** super_p_evident($E, C, p$)

$\quad$**while** $E$ has changed **do**
$\quad\quad$**if** $\exists\, \omega \in E :$ min_belief($E, C, \omega$) $\le p$ **then**
$\quad\quad\quad E := E \setminus \{\omega\}$
$\quad$**return** $E$

---

requires jointly searching over values of $p$ and over subsets of $\Omega$. We leverage the generic mathematical structure of p-evident events in finite state spaces in order to develop an exact algorithm that avoids enumerating all subsets. Our algorithm only requires a search that is polynomial in the size of the state space. Of course, the state spaces in many problems are often themselves exponential in some underlying parameter variables, and hence future improvements on this algorithm would be desirable. Extensions to at least certain countably infinite or continuous state spaces are likely possible as well, such as perhaps by refining the search to only consider events that have non-zero probability given the player's observations.

### 3.1 Computing Information Partitions

Our algorithms require access to the information partitions of each player. However, directly specifying the information partitions that people have in a naturalistic setting, such as in the data we use from the Thomas experiments, is difficult. Instead, we take the approach of specifying a plausible generative probabilistic world model, and we then construct the information partitions from this factored representation via a straightforward algorithm. The generative world model specifies the pieces of information, or "observations", that each player receives. The algorithm for generating information partitions, which is specified precisely in our supplementary materials, consists of iterating over all combinations of random values of variables in the generative world model, treating each such combination as a state in $\Omega$, and for each player grouping together the states that yield identical observations.

### 3.2 Computing Common p-Belief

Algorithms 1-4 present the functions needed to compute perceived maximal common p-belief. Formally, given a player $i$, a particular state $\omega$, and an event $C$, Algorithm 1 computes the largest value $p$ for which player $i$ p-believes that there is common $p$-belief in $C$. Note that it is insufficient to compute the largest value of $p$ for which there is common $p$-belief in $C$ at $\omega$, since in general at state $\omega$ player $i$ only knows that the event $\Pi_i(\omega)$ has occurred, not that $\omega$ specifically has occurred. Relatedly, note that while Algorithm 1 takes $\omega$ as input for convenience, the algorithm only depends on $\Pi_i(\omega)$, and hence could be executed by a player.

Formal proofs of the correctness of these algorithms are included in our supplementary materials. The basic logic of the algorithms is to maintain a candidate p-evident event $E$, and to gradually remove elements from $E$ to make it more p-evident until a point where player $i$ believes the event to be impossible because the elements of the event are no longer consistent with that player's observations. By only removing elements that either cause $E$ to be unlikely or cause $C$ to be unlikely, we are guaranteed to arrive at a more p-evident event at each iteration, and one that preserves belief in $C$. By starting with $E$ as the entire state space, the final candidate event must be the largest, most p-evident event that player $i$ p-believes at state $\omega$ in which $C$ is common $p$-belief. This algorithm can also be viewed as traversing a unique nested sequence of maximally evident events (independent of $i$ and $\omega$) induced by $C$ and the structure of $(\Omega, \mu, (\Pi_0, \Pi_1))$, halting at the first event in this sequence that does not include any elements of $\Pi_i(\omega)$.

The rational p-belief strategy consists of player $i$ taking action $A$ if common_p_belief($x = 1,\ i,\ \omega$) $> \frac{c-b}{a-b}$. The matched p-belief strategy consists of player $i$ choosing $A$ with probability common_p_belief($x = 1, i, \omega$).

### 3.3 Iterated Reasoning

We now present our algorithms for the iterated reasoning strategies. For level $k > 0$, given a player $i$ and a state $\omega$, the iterated maximization strategy computes

$$f_i^k(\omega) = \mathbb{1}\Big( \sum_{\omega' \in \Pi_i(\omega)} P_i(\omega' \mid \omega)\Big( P_i(x=1 \mid \omega') f_{1-i}^{k-1}(\omega')a$$

$$+ P_i(x=0 \mid \omega') f_{1-i}^{k-1}(\omega')d + \big(1 - f_{1-i}^{k-1}(\omega')\big)b \Big) > c \Big),$$

where $\mathbb{1}()$ is the indicator function, and $f_i^0(\omega) = \mathbb{1}(P_i(x = 1 \mid \omega) > \frac{c-b}{a-b})$. If $f_i^k(\omega) = 1$, then player $i$ plays $A$, and otherwise player $i$ plays $B$. For the iterated matching strategy,

$$q_i^k(\omega) = P_i(x = 1 \mid \omega) \cdot \sum_{\omega' \in \Pi_i(\omega)} P_i(\omega' \mid \omega) q_{1-i}^{k-1}(\omega').$$

$A$ is then played with probability $q_i^k$, $q_i^0 = P_i(x = 1 \mid \omega)$.

## 4 Data

We now present the data we use for our empirical results. The dataset comes from the Thomas experiments (Thomas et al. 2014). These experiments presented participants with a stylized coordinated attack problem couched in a story about
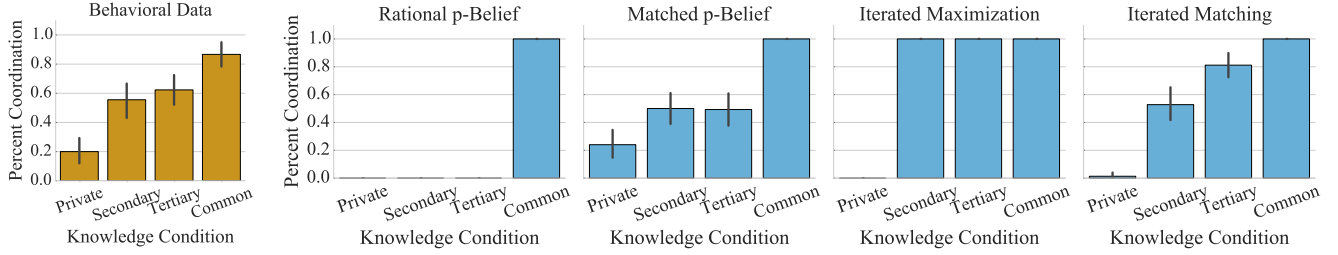
Figure 1: Data from the Thomas experiments and the predictions of each of the models we consider.

the butcher and the baker of a town. In their story, these merchants can either work together to produce hot dogs, or they can work separately to produce chicken wings and dinner rolls, respectively. The merchants can sell chicken wings and dinner rolls separately for a constant profit of $c$ each on any day, but the profit of hot dogs varies from day-to-day. The merchants make a profit of $a$ each if $x = 1$ on a particular day or $d$ if $x = 0$. There is also a loudspeaker that sometimes publicly announces the prices of hot dogs, and a messenger who runs around the town delivering messages. The experiments had four different knowledge conditions that specified the information that participants received:

1. **Private Knowledge:** "The Messenger Boy [sic] has not seen the Butcher today, so he cannot tell you anything about what the Butcher knows."

2. **Secondary Knowledge:** "The Messenger Boy says he stopped by the butcher shop before coming to your bakery. He tells you that the Butcher knows what today's hot dog price is. However, he says that he forgot to mention to the Butcher that he was coming to see you, so the Butcher is not aware that you know today's hot dog price."

3. **Tertiary Knowledge:** "The Messenger Boy mentions that he is heading over to the butcher shop, and will let the Butcher know today's price as well. The Messenger Boy will also tell the Butcher that he just came from your bakery and told you the price. However, the Messenger Boy will not inform the Butcher that he told you he would be heading over there. So, while the Butcher is aware that you know today's price, he is not aware that you know that he knows that."

4. **Common Knowledge:** "The loudspeaker broadcast the market price . . . The messenger boy did not come by. Because the market price was broadcast on the loudspeaker, the Butcher knows today's price, and he knows that you know this information as well."

After being shown this information as well as additional information indicating that $x = 1$, the participants were asked whether they would like to try to make hot dogs or not. The dataset from this experiment is visualized in Figure 1. Since the researchers provided evidence that the behavior of participants in their two-player experiments was invariant to payoffs, here we focus on their first payoff condition, in which $a = 1.1$, $b = 0$, $c = 1$, and $d = 0.4$.

We use this dataset to test whether the coordination strategies we have described are good models of human coordination in this setting. In order to be able to generate predictions for these models, we must determine a state space that represents the story in the Thomas experiments. We designed the following two probabilistic generative world models (one for the messenger, and one for the loudspeaker) to be consistent with a reading of the knowledge conditions from those experiments. The observe($i$,$o$) function indicates that player $i$ observes $o$.

**Messenger:**
$$x \sim \text{Bernoulli}(\delta)$$
$$\text{visit}_0 \sim \text{Bernoulli}(0.5)$$
$$\text{visit}_1 \sim \text{Bernoulli}(0.5)$$
$$\text{tell\_plan}_0 \sim \text{visit}_0 \wedge \text{Bernoulli}(0.5)$$
$$\text{tell\_plan}_1 \sim \text{visit}_1 \wedge \text{Bernoulli}(0.5)$$
**if** $\text{visit}_0$:
    $\text{observe}(0, x)$
    **if** $\text{tell\_plan}_0$:
        $\text{observe}(0, (\text{visit}_1, \text{tell\_plan}_1))$
**if** $\text{visit}_1$:
    $\text{observe}(1, (x, \text{visit}_0))$
    **if** $\text{tell\_plan}_1$:
        $\text{observe}(1, \text{tell\_plan}_0)$

**Loudspeaker:**
$$x \sim \text{Bernoulli}(\delta)$$
$$\text{broadcast} \sim \text{Bernoulli}(0.5)$$
**if** broadcast:
    $\text{observe}(0, x), \text{observe}(1, x)$

These models share a free parameter $\delta$. We take $\delta = 0.25$. This setting provides a closer fit to the empirical data than the maximum entropy setting of $\delta = 0.5$. We interpret statements that one player is "not aware" as meaning that the player could have been made aware, and assign a maximum entropy probability of 0.5 to these events.

The state spaces corresponding to these world models consist of the sets of all possible combinations of variables in the models' generative processes: ($x$, $\text{visit}_0$, $\text{visit}_1$, $\text{tell\_plan}_0$, $\text{tell\_plan}_0$) for $\Omega_{\text{messenger}}$ and ($x$, broadcast) for $\Omega_{\text{loudspeaker}}$. The generative processes also uniquely spec-
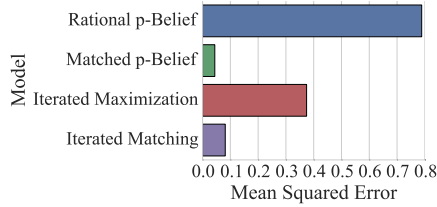
Figure 2: The mean-squared error of each model's predictions on the Thomas experiments' data.



Figure 3: Performance of agents in our simulated human-agent coordination experiments. A strategy's marginal value is the expected sum of payoffs the strategy obtained in each of the four knowledge conditions, minus the payoffs that could have been obtained by always playing $B$.

ify probability measures over each state space. The knowledge conditions correspond to the following states. Private: $(1, 1, 0, 1, 0) \in \Omega_{\text{messenger}}$, Secondary: $(1, 1, 1, 0, 1) \in \Omega_{\text{messenger}}$, Tertiary: $(1, 1, 1, 1, 0) \in \Omega_{\text{messenger}}$, and Common Knowledge: $(1, 1) \in \Omega_{\text{loudspeaker}}$. The participants act as player 0 in all but the secondary condition. Due to high ambiguity in the wording of the private knowledge condition, we considered two plausible readings. Either the messenger is communicating an intention to not visit the other player, or the messenger is being unhelpful in not offering any information about the messenger's plan. By using the state $(1, 1, 0, 1, 0)$ we assume the first interpretation. This interpretation results in a better empirical fit.

## 5 Results

We now present our empirical results. We first examine the predictions of each of the coordination strategies we consider given the generative processes representing the Thomas experiments. We then examine the extent to which a computer agent equipped with the best fitting model of human coordination is able to achieve higher payoffs in a simulated human-agent coordination problem. All of our code is available online at https://github.com/pkrafft/modeling-human-ad-hoc-coordination.

### 5.1 Model Comparison

To perform model comparison we compute the probability of choosing $A$ that each model predicts given our formal representations of each of the four knowledge conditions. We then compare these predicted probabilities to the actual probabilities observed in the Thomas experiments. For the two iterated reasoning models we use a grid search over $[0, 1, 2, 3, 4, 5]$ to find the best fitting $k$ for each model (ultimately $k = 1$ in iterated maximization and $k = 3$ in iterated matching). The specific predictions of each model are shown in Figure 1. As shown in Figure 2, the matched p-belief model achieves the lowest mean-squared error. Qualitatively, the most striking aspect of the data that the matched p-belief model successfully captures is the similarity in the probability of coordination between the secondary and tertiary knowledge conditions. The two models that involve maximizing agents (rational p-belief and iterated maximization) both make predictions that are too extreme. The iterated matching model offers a competitive second place fit to the data, but it fails to capture the similarity between the middle two knowledge conditions.
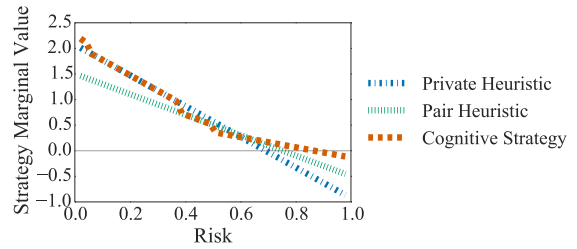
The reason that the matched p-belief model makes good predictions for the two middle conditions is that the player in both of those conditions has the same amount of uncertainty appearing at some level of that player's infinite recursive hierarchy of interpersonal beliefs. Common p-belief essentially represents a minimum taken over all these levels, and thus the common p-belief in each of those two conditions is the same. The rational p-belief model is aware of the uncertainty at higher levels of recursive belief, but its predictions are too coarse due to the assumption of utility maximization. An interesting avenue for future work is to examine whether the rational p-belief model can be relaxed in any other way to allow for intermediate predictions, such as by allowing for heterogeneity in interpretations of the world model across agents. It is possible that the matched p-belief model is approximating such a case.

### 5.2 Human-Agent Coordination

Besides testing the fit of the models of human coordination that we have proposed, we are also interested in whether the best fitting model helps us improve outcomes in human-agent coordination. We use the data from the Thomas experiments to evaluate this possibility. For this task our computer agents implement what we call a "cognitive strategy". An agent using the cognitive strategy chooses the action that maximizes expected utility under an assumption that the agent's companion is using the matched p-belief model. These agents play the humans' companion player in each of the four knowledge conditions of the Thomas experiments (player 1 in all but the Secondary condition). We evaluate the payoffs from the agents' actions using the human data from the Thomas experiments. In this simulation we vary the payoffs $(a, b, c, d)$, and we assume that the humans would remain payoff invariant across the range of payoffs that we use. This assumption is reasonable given that participants in the Thomas experiments displayed payoff invariance in the two-agent case. We vary the payoffs according to risk, taking the payoffs to be $(1, 0, p^*, 0)$ for a particular risk level $p^* = \frac{c-b}{a-b}$. As shown in Figure 3, we find that having the matched p-belief model of human coordination may help in human-agent coordination. We compared to two baseline strategies: an agent using a "private heuristic" who always

coordinates if the agent knows $x = 1$, and an agent using a "pair heuristic" who always coordinates if the agent knows that both the agent and the human know $x = 1$. The private heuristic achieves good performance for low risk levels, and the pair heuristic achieves good performance for high risk levels. The cognitive strategy achieves good performance at both low and high levels of risk, and only has negative marginal value over always playing the safe action B at very high levels of risk.

# 6 Discussion

In the present paper we focused on laying the groundwork for using common p-belief in AI and cognitive modeling. In particular, we developed an efficient algorithm for the inference of maximal perceived common p-belief, we showed that the coordination strategy of probability matching on common p-belief explains certain surprising qualitative features of existing data from a previous human experiment, and we showed that this model may also help improve agent outcomes in human-agent coordination. This work has three main limitations. Due to the small amount of human data we had and the lack of a held-out test set, our empirical results are necessarily only suggestive. While the data are inconsistent with the rational p-belief model and the iterated maximization model, the predictions of the iterated matching model and the matched p-belief model are both reasonably good. The strongest evidence we have favoring the matched p-belief model is this model's ability to produce equal amounts of coordination in the secondary and tertiary knowledge conditions as well as a low amount with private knowledge and a high amount with common knowledge. No iterated reasoning model under any formulation we could find of the Thomas experiments was able to capture the equality between the two middle conditions while maintaining good predictions at the extremes. Two other important limitations of our work are that the coordination task we consider did not involve intentional communication, and that the state and action spaces of the task were simple. While these features allowed us to easily test the predictions of each of our alternative models, it would be interesting to see how the models we considered would compare in more complex environments. A related interesting direction for future work is the application of inference of common p-belief through reasoning about p-evident events to artificial distributed systems, such as for developing or analyzing bitcoin/blockchain-like protocols, synchronizing remote servers, or distributed planning in ad hoc multi-robot teams.

# 7 Acknowledgments

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# Supplemental Materials

These supplementary sections present the proofs of the correctness of our algorithms. Our key proofs are self-contained, but they rely on some preliminary results involving the structure of p-evident events in finite probability spaces. We first offer proofs of these preliminary results in the "Initial Results" section. We then prove that the rational p-belief strategy is an equilibrium in the "Models" section. Finally, we show the correctness of our algorithm for computing perceived maximal common p-belief in the "Algorithms" section. This final section also includes our algorithm for computing information partitions from a given probabilistic generative world model.

All of the proofs are original. Some of the less trivial lemmas and propositions may be novel contributions to the literature on p-evident events. In particular, we know of no work that explores the structure of what we will call "maximally evident $C$-indicating events", i.e. maximally p-evident events in which $C$ is common knowledge at a given state $\omega$. Our main result along these lines will be that any finite state space has a unique representation as a nested sequence of maximally evident $C$-indicating events. This result is central to understanding our algorithm for computing common p-belief.

## S1 Definitions

We first restate the most relevant definitions from our main text in a clearer format, and we give several additional definitions that will be helpful in our proofs. As described in the main text, we assume a Bayesian game with a finite state space $\Omega = \{\omega_1, \ldots, \omega_{|\Omega|}\}$ and information partitions $\Pi_0$ and $\Pi_1$, which are simply partitions of the state space $\Omega$. The notation $\Pi_i(\omega)$ indicates the unique element of $\Pi_i$ that includes the state $\omega$. These information partitions and a measure $\mu$ defined over $\Omega$ induce the conditional distribution $P_i(E \,|\, \omega) = \mu(E \,|\, \Pi_i(\omega)) = \frac{\sum_{\omega' \in E \cap \Pi_i(\omega)} \mu(\omega')}{\sum_{\omega' \in \Pi_i(\omega)} \mu(\omega')}$, which specifies the belief about an event $E$ that the player $i$ holds at state $\omega$.

**Definition 1** ((Monderer and Samet 1989))**.** *Player $i$ is said to p-**believe** an event $E \subseteq \Omega$ at $\omega$ if $P_i(E \,|\, \omega) \geq p$.*

**Definition 2** ((Monderer and Samet 1989))**.** *An event $E$ is p-**evident** if for each $\omega \in E$, all players p-believe $E$ at $\omega$.*

**Definition 3.** *An event $E$ is **super**-p-**evident** if for all $\omega \in E$ and for all players $i$, $P_i(E \,|\, \omega) > p$.*

**Definition 4.** *An event $E$ is said to be a p-**evident** $C$-**indicating event** if $E$ is a p-evident event and if $P_i(C \,|\, \omega) \geq p$ for all $i$ and for all $\omega \in E$.*

**Definition 5** ((Monderer and Samet 1989))**.** *The players have **common** p-**belief** in an event $C$ at state $\omega$ if there exists a p-evident $C$-indicating event $E$ that includes $\omega$.*

**Definition 6.** *An event $E$ is the **largest** p-**evident** $C$-**indicating event** (for a particular $p$) if $E$ is a p-evident $C$-indicating event and for all events $F$, either $F$ is not a p-evident $C$-indicating event, or $F \subseteq E$.*

**Definition 7.** *The $C$-**evidence level** of an event $E$ is the maximum value of $p$ for which $E$ is a p-evident $C$-indicating event.*

**Definition 8.** *An event $E$ is the **maximally evident** $C$-**indicating superset** of $F$ if there exists a $p$ such that $p$ is the $C$-evidence level of $E$, $E$ is the largest p-evident $C$-indicating event, and for all other events $G$, either $G$ does not contain $F$ or $G$ has a lower $C$-evidence level than $E$.*

**Definition 9.** *An event $E$ is the **largest super-evident** $C$-**indicating subset** of $F$ if there exists a $p$ such that $p$ is the $C$-evidence level of $F$, $E$ is a super-p-evident $C$-indicating event, and for all other events $G$, either $G$ is not a subset of $F$, $G$ is a subset of $E$, or the $C$-evidence level of $G$ is less than or equal to $p$.*

## S2 Initial Results

Our first result is a simple lemma showing that the family of $p$-evident $C$-indicating events is closed under unions.

**Lemma 1.** *For a given event $C$, if $E$ and $F$ are p-evident $C$-indicating events, then $E \cup F$ is also a p-evident $C$-indicating event.*

*Proof.* Consider $\omega \in E \cup F$. For any player $i$, $P_i(E \cup F \,|\, \omega) \geq \min(P_i(E \,|\, \omega), P_i(F \,|\, \omega)) \geq p$ (since $E$ and $F$ are p-evident). Therefore $E \cup F$ is p-evident. Now suppose $\omega \in E$. In this case $P_i(C \,|\, \omega) \geq p$ since $E$ is $C$-indicating. But also, if $\omega \in F$, then $P_i(C \,|\, \omega) \geq p$ since $F$ is $C$-indicating. Therefore $E \cup F$ is $C$-indicating. $\square$

Our next result is an immediate corollary and establishes the existence and uniqueness of largest $p$-evident $C$-indicating events for any $p$.

**Corollary 1.** *For a given event $C$, if there exists a p-evident $C$-indicating event $E$, then there exists a unique largest p-evident $C$-indicating event.*

*Proof.* Assume there exists a $p$-evident $C$-indicating event. Take $\mathcal{F}$ to be the set of all $p$-evident $C$-indicating events. By lemma 1, the set $G = \cup_{F \in \mathcal{F}} F$ consisting of the union of all of these events is also a $p$-evident $C$-indicating event. Therefore $G$ is a $p$-evident $C$-indicating event containing all other $p$-evident $C$-indicating events. Moreover, $E$ must be unique since it contains all other $p$-evident events that indicate $C$. $\square$

The next lemma establishes a containment relationship between largest $p$-evident $C$-indicating events associated with different values of $p$.

**Lemma 2.** *For a given event $C$, if $E$ is the largest p-evident $C$-indicating event, and $F$ is the largest $p'$-evident $C$-indicating event, with $p \geq p'$, then $E \subseteq F$.*

*Proof.* Suppose $E$ is the largest $p$-evident $C$-indicating event. Since $p' \leq p$, $E$ is also a $p'$-evident $C$-indicating event. Thus $E$ must be contained by the largest $p'$-evident $C$-indicating event. $\square$

We now show that a unique maximally evident $C$-indicating event exists around any subset of $\Omega$.

**Lemma 3.** *For a given event $C$, and for any event $E$, there exists a unique maximally evident $C$-indicating superset of $E$.*

*Proof.* Consider an event $E$. The $C$-evidence level of $E$ is given by $p = \min_{i\in[0,1],\omega\in E}\min(P_i(E\,|\,\omega), P_i(C\,|\,\omega))$, and therefore exists. Moreover, since $E$ is a $p$-evident $C$-indicating event, $E$ must be contained by the unique largest $p$-evident $C$-indicating event $F$ (which is guaranteed to exist by corollary 1). Further, by definition $F$ must also contain all other $p$-evident supersets of $E$, and therefore $F$ is the maximally evident $C$-indicating superset of $E$. $\qquad\square$

The following lemma, which will be useful in proving our main theoretical result, shows that the family of maximally evident $C$-indicating events is highly constrained.

**Lemma 4.** *For a given event $C$, for any event $E$, there exists some $\omega \in E$ such that the maximally evident $C$-indicating superset of $E$ is equal to the maximally evident $C$-indicating superset of $\{\omega\}$.*

*Proof.* Let $F$ be the maximally evident $C$-indicating superset of $E$, guaranteed to exist by lemma 3. Let $p_F$ be the $C$-evidence level of $F$. Consider an $\omega \in E$. Let $G_\omega$ be the maximally evident $C$-indicating superset of $\{\omega\}$, and let $p_{G_\omega}$ be the $C$-evidence level of $G_\omega$. We must have that $p_{G_\omega} \geq p_F$ for all $\omega$ (Since $F$ contains $\omega$, $p_F > p_{G_\omega}$ would violate the fact that $G_\omega$ is maximally evident.) Suppose $p_{G_\omega} > p_F$ for all $\omega$. Then, by lemma 1, $H = \cup_{\omega\in E}G_\omega$ would be a $p_H$-evident $C$-indicating event with $p_H \geq \min_\omega p_{G_\omega} > p_F$. However, $H$ clearly contains $E$, and thus $p_H > p_F$ violates the fact that $F$ is the maximally evident $C$-indicating superset of $E$. Therefore, there must exist some $\omega$ such that $p_{G_\omega} = p_F$. Since largest p-evident $C$-indicating events are unique by corollary 1, we must have $H = G_\omega$. $\qquad\square$

The following theorem is our main theoretical result, and drives the correctness of our algorithms. This theorem, which is also illustrated in Figure S1, states that any finite $\Omega$ has a unique representation as a nested sequence of maximally evident $C$-indicating events. The efficiency of our algorithm stems from only searching through this sequence of subsets, rather than all possible subsets, in order to compute common p-belief at any state $\omega$.

**Theorem 1.** *Given an information structure $(\Omega, \mu, (\Pi_0, \Pi_1))$ with $|\Omega| < \infty$ and any event $C \subseteq \Omega$, let $\mathcal{F} = \{E_1', \ldots, E_n'\}$ be the set of maximally evident $C$-indicating events of $\Omega$, i.e. the set of events $E_i'$ for which there exists some $F \subseteq \Omega$ such that $E_i'$ is the maximally evident $C$-indicating superset of $F$. $\mathcal{F}$ can be ordered into a nested sequence of subsets, $E_1 \supset E_2 \supset \ldots \supset E_n$ such that $E_i$ is the largest super-evident $C$-indicating subset event of $E_{i-1}$ for any $i > 1$.*

*Proof.* We prove this theorem by construction. We know from lemma 3 that for each $\omega_j \in \Omega$, there exists a maximally evident $C$-indicating superset of $\{\omega_j\}$. We label this event $E_{\omega_j}'$. Lemma 4 implies that collection of events $\{E_{\omega_j}' : \omega_j \in \Omega\}$ is equal to $\mathcal{F}$, the entire set of maximally evident $C$-indicating events. We also know from lemma 3 that each

$E_{\omega_j}'$ is associated with a particular $C$-evidence level, which we will label $p_{\omega_j}$. Without loss of generality, we can assume that the index $j$ sorts these $C$-evidence levels by their magnitudes. This produces a finite non-decreasing sequence of values in $[0, 1]$: $p_{\omega_1} \leq p_{\omega_2} \leq \ldots \leq p_{\omega_{|\Omega|}}$. By lemma 2 the events $E_{\omega_j}'$ thus form a nested sequence of subsets: $E_{\omega_1}' \supseteq E_{\omega_2}' \supseteq \ldots \supseteq E_{\omega_{|\Omega|}}'$. We can collapse the events that are equal to each other to arrive at a sequence of strict subsets: $E_1 \supset E_2 \supset \ldots \supset E_n$.

Now consider $E_i$ for some $i > 1$. By construction $E_i$ is a super-$p_{\omega_{i-1}}$-evident $C$-indicating event, and $E_i \subset E_{i-1}$. Now take $F$ to be another super-$p_{\omega_{i-1}}$-evident $C$-indicating event, and let $p_F$ be the $C$-evidence level of $F$. If $p_F \geq p_{\omega_i}$, then we must have $E_i \supseteq F$ by lemma 2 since $E_i$ is the largest $p_{\omega_i}$-evident event. Now suppose $p_F < p_{\omega_i}$, so $p_{\omega_{i-1}} < p_F < p_{\omega_i}$. Let $G$ be the maximally evident $C$-indicating superset of $F$, and let $p_G$ be the $C$-evidence level of $G$. Clearly $p_G \geq p_F$, and by construction we must also have $p_G = p_{\omega_j}$ since $\mathcal{F}$ contains all maximally evident $C$-indicating supersets, and hence $G = E_j$, for some $j$. But since $p_{\omega_j} = p_G \geq p_F > p_{\omega_{i-1}}$, we must have $j \geq i$, and so $F \subseteq G = E_j \subseteq E_i$. Therefore again $F \subseteq E_i$. Hence $E_i$ must be the largest $C$-indicating subset of $E_{i-1}$. $\qquad\square$

The following corollary to this theorem provides the basis for our iterative algorithm.

**Corollary 2.** *Given a maximally evident $C$-indicating event $E$, either there exists a unique largest super-evident $C$-indicating subset of $E$, or no super-$p$-evident $C$-indicating subsets exist.*

*Proof.* Since $E$ is a maximally evident $C$-indicating event, $E$ is equal to some $E_i$ in the sequence $\mathcal{F}$ given by theorem 1. Therefore, if $i < n$, $E_{i+1}$ is the unique largest super-evident $C$-indicating subset of $E$ or no such subsets exist, as indicated in theorem 1. $\qquad\square$

Finally, we have four technical lemmas that will be useful for analyzing our specific algorithms and player strategies. The first lemma is a simple constraint on belief about p-evident events that follows from how information partitions work.

**Lemma 5.** *For any $p$-evident event $E$, either $P_i(E\,|\,\omega) = 0$ or $P_i(E\,|\,\omega) \geq p$ for all $\omega$ and for all $i$.*

*Proof.* If $P_i(E\,|\,\omega) > 0$, then there must exist an $\omega' \in \Pi(\omega)$ such that $\omega' \in E$. Since $E$ is $p$-evident, then $P_i(E\,|\,\omega') \geq p$. But then since $\Pi_i(\omega) = \Pi_i(\omega')$ by the definition of an information partition, $P_i(E\,|\,\omega) = \mu(E\,|\,\Pi_i(\omega)) = \mu(E\,|\,\Pi_i(\omega')) = P_i(E\,|\,\omega')$, and hence $P_i(E\,|\,\omega) \geq p$. $\qquad\square$

The second lemma connects the containment relationships between p-evident events to the relationship between beliefs about those events.

**Lemma 6.** *If player $i$ $p$-believes a $p$-evident $C$-indicating event, then player $i$ $p$-believes the largest $p$-evident $C$-indicating event.*
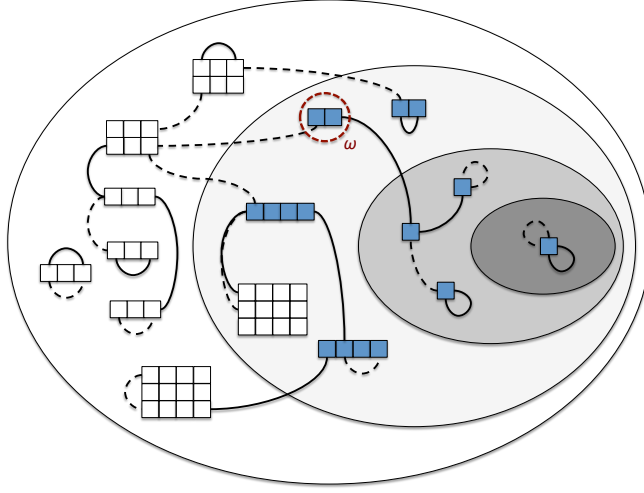
Figure S1: Any finite state space can be uniquely represented as a nested sequence of maximally evident $C$-indicating events. The diagram in this figure represents the generative process of the messenger described in the main text of our paper (with $\delta = 0.25$). Each contiguous rectangle of blocks represents a state in $\Omega$, and the measure of the state is given by the area of the rectangle. States that are shaded are members of $C = \{\omega \in \Omega : x(\omega) = 1\}$. The solid lines between states represent the information partition of player 0, while the the dotted lines represent that of player 1. Two states belong to the same element of a player's information partition if they are connected by some path in the graph induced by that player's edges. Self-loops indicate that a player has no uncertainty about the state when the state obtains. The four nested ovals are the four maximally evident $C$-indicating events in this state space ($\mathcal{F}$ in theorem 1), and the grey shading in the ovals represents the $C$-evidence levels of those events (0, 0.25, 0.5, and 1.0). Our algorithm iterates over these maximally evident events rather than over all possible events, and at $\omega$ for player $i$ returns the $C$-evidence level associated with the last such event containing some element of $\Pi_i(\omega)$. For instance, at the circled state the algorithm will find the third nested event for player 0 and the second for player 1.

*Proof.* Let $F$ be a $p$-evident $C$-indicating event and let $E$ be the largest $p$-evident $C$-indicating event. Since $E$ is largest, $F \subseteq E$, so we must have $P_i(E \,|\, \omega) = P_i(E \cup F \,|\, \omega) \geq P_i(F \,|\, \omega)$. Hence if $i$ $p$-believes $F$, $i$ must also $p$-believe $E$. $\square$

The next lemma states that p-belief is transitive.

**Lemma 7.** *If player $i$ $p$-believes $F$ at all elements of an event $E$, and player $i$ $p$-believes $G$ at all elements of $F$, then player $i$ $p$-believes $G$ at all elements of $E$.*

*Proof.* Let $\omega$ be an element of E. Since $i$ $p$-believes $F$ at $\omega$, $P_i(F \,|\, w) \geq p$. Therefore we must have some $\omega' \in \Pi_i(\omega) \cap F$. Since player $i$ $p$-believes $G$ at all states in $F$, we must have $P_i(G \,|\, \omega') \geq p$. By the definition of an information partition, we must then also have $P_i(G \,|\, \omega) = P_i(G \,|\, \omega') \geq p$. Hence player $i$ $p$-believes $G$ at all $\omega \in E$. $\square$

The final lemma states, roughly, that with regard to p-evident events a player's beliefs about another player cannot be too inconsistent with the first player's own beliefs.

**Lemma 8.** *If player $i$ $p$-believes that player $1-i$ $p$-believes the largest $p$-evident $C$-indicating event $E$, and player $i$ $p$-believes $C$, then player $i$ must $p$-believe $E$.*

*Proof.* Consider an arbitrary $\omega \in \Omega$ and an arbitrary $p$. Take the largest $p$-evident $C$-indicating event $E$. Suppose at $\omega$ player $i$ $p$-believes that player $1-i$ $p$-believes $E$, and

player $i$ $p$-believes $C$. To generate the first belief, there must be an element of the information partition of player $1 - i$, $T \in \Pi_{1-i}$, such that $P_{1-i}(E \,|\, T) \geq p$, and another set of states $S \subseteq \Pi_i(\omega) \cap T$ such that $P_i(S \,|\, \omega) \geq p$. But then, since states within a player's information partition are indistinguishable and since $S \subseteq \Pi_i(\omega) \cap T$, we must have for all $\omega' \in S$, $P_{1-i}(E \,|\, \omega') = P_{1-i}(E \,|\, T) \geq p$ and $P_i(S \,|\, \omega') = P_i(S \,|\, \omega) \geq p$. Therefore, since $E$ is $p$-evident, $S \cup E$ is also a $p$-evident event (if $\omega'' \in S$, then player $1 - i$ $p$-believes $E$ and player $i$ $p$-believes $S$, while if $\omega'' \in E$, then both players $p$-believe $E$). Moreover, since player $i$ $p$-believes $C$ at $\omega$ (and therefore at $\Pi(\omega)$ and $S$) and since $1-i$ $p$-believes $C$ at $E$ (and hence at $S$ by lemma 7), then both players $p$-believe $C$ at all states in $S \cup E$. Therefore $S \cup E$ is a $p$-evident $C$-indicating event (and, moreover, player $i$ $p$-believes $S \cup E$ at $\omega$ since player $i$ $p$-believes $S$ at $\omega$). But $E$ was assumed to be the largest $p$-evident $C$-indicating event, so we must have $S \cup E = E$. Hence player $i$ $p$-believes $E$. $\square$

## S3   Models

### Strategic Coordination

In this section we show the rational p-belief strategy forms an equilibrium in the coordination game we study. Recall the rational p-belief strategy is that player $i$ plays action $A$ if and only if player $i$ believes with at least probability $p^* = \frac{c-b}{a-b}$ that both players have common $p^*$-belief that $x = 1$.

**Proposition 1.** *Assuming noiseless messages and assuming $p^* > \mu(x = 1)$, the rational p-belief strategy maximizes the expected return of player $i$ at every $\omega \in \Omega$, given player $1-i$ also uses the same strategy.*

*Proof.* Suppose that messages are noiseless, i.e. $P_i(x = 1 \mid \omega) > P_i(x = 1) \Rightarrow P_i(x = 1 \mid \omega) = 1$ for all $\omega$. Take $\omega \in \Omega$.

Take $F$ to be some $p^*$-evident $(x = 1)$-indicating event. Take $E$ to be the largest such event. Suppose player $i$ $p^*$-believes $F$. By lemma 6, player $i$ also $p^*$-believes $E$. By the definition of p-evident events, for any $\omega' \in E$, we must then have that player $1 - i$ also $p^*$-believes $E$ at $\omega'$. Therefore by lemma 7 player $i$ also $p^*$-believes that player $1 - i$ $p^*$-believes $E$. Since player $1 - i$ is assumed to be using the rational p-belief strategy, player $i$ therefore $p^*$-believes player $1 - i$ will play $A$. Since we have assumed $p^* > P_i(x = 1)$ and messages are noiseless, we must have $P_i(x = 1 \mid \Pi_i(\omega)) = 1$. Then the expected return for $i$ of playing $A$ must be at least $p^* \cdot a + (1-p^*)b = \frac{c-b}{a-b}a + \frac{a-c}{a-b}b = c$. Therefore, playing $A$ maximizes player $i$'s expected return (and if player $i$ $p'$-believes a $p'$-evident $(x = 1)$-indicating event, $p' > p^*$, then the expected return from $A$ is strictly greater than for $B$).

Now suppose player $i$ $p^*$-believes $x = 1$ but that there is no $p^*$-evident $(x = 1)$-indicating event $F$ such that $P_i(F \mid \omega) \geq p^*$. Take $E$ to be the largest $p^*$-evident $(x = 1)$-indicating event. Then player $i$ cannot $p^*$-believe that player $1 - i$ $p^*$-believes $E$ (since otherwise player $i$ would $p^*$-believe $E$ by lemma 8). Thus, since player $1 - i$ is assumed to be using the rational p-belief strategy, player $i$ believes with probability at most $p' < p^*$ that player $1 - i$ will play $A$. Then since $p' \cdot a + (1-p')b < p^* \cdot a + (1-p^*)b = c$, the expected utility from playing $A$ to player $i$ is less than from playing $B$. (And if $i$ does not $p^*$-believe $x = 1$, the same arithmetic holds.) □

## S4    Algorithms
### Computing Information Partitions
In this section we state the simple algorithm for converting the generative process descriptions of the probabilistic models in our main text to information partitions. The "run($i$, $\omega$)" function takes as input a player $i$ and a state $\omega$ expressed as a tuple, uses the variables in the tuple for each random draw within the generative process, and returns a composition of the *observe* calls for player $i$ (For example, run(0, (1,1,0,1,0)) would return [(1),(0,0)] in the messenger model). Our algorithm groups together states with the same observations into elements of the information partitions.

### Computing Common p-Belief
We now prove the correctness of the common_p_belief algorithm given in the main text. We first prove a lemma stating the correctness of the super_p_evident algorithm.

**Lemma 9.** *Assuming $E$ is a maximally evident $C$-indicating event, and assuming $p$ is the $C$-evidence level of $E$, then the set returned by evaluating "super_p_evident($E, p$)" is the*

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**Algorithm 5** information_partition($i$)

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partition := dict()
for ω ∈ Ω do
    obs := run(i, ω)
    if obs ∈ partition then
        partitions[obs].append(ω)
    else
        partitions[obs] := [ω]
return partitions
```

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*largest super-evident $C$-indicating subset of $E$, or the empty set if and only if such an event does not exist.*

*Proof.* First note that from the condition in the "if" statement of the function, all states that remain in $E$ when the function returns will have the properties $P_i(E \mid \omega) > p$ and $P_i(C \mid \omega) > p$. Therefore, the function only returns super-$p$-evident $C$-indicating subsets of $E$ (and since $p$ is the $C$-evidence level of $E$, strict subsets must be returned), or the empty set if no super-$p$-evident subsets exists. Next, note that the "while" loop only removes elements of $E$ that cannot belong to the largest super-evident $C$-indicating subset of $E$, if such an event exists. We can see this by induction on the items removed. Suppose that the first $\omega$ removed belonged to $F$, the largest super-evident $C$-indicating subset of $E$. Then we would have $P_i(F \mid \omega) > p$ since $F$ must be super-$p$-evident. But since $F \subseteq E$, by lemma 2 we would then also have $P_i(E \mid \omega) > p$, which contradicts the fact that $\omega$ was removed. Now assume that the first $k$ elements removed do not belong to $F$, and let $E'$ be the set $E$ minus those elements. Then suppose the $(k + 1)$st element removed, if it exists, belonged to $F$. Since all the previous elements removed did not belong to $F$ by the inductive assumption, $F \subseteq E'$. But then analogous to the base case, the $(k + 1)$st element, $\omega_{k+1}$, belonging to $F$ implies $P_i(E' \mid \omega_{k+1}) > p$, which contradicts the fact that this element was removed. Therefore this function returns exactly the largest super-evident $C$-indicating subset of $E$. Finally, since $\Omega$ is assumed to be finite, and at least one element of $E$ is removed in each iteration of the while loop, the function must terminate. □

Lastly, we show the correctness of our main algorithm.

**Proposition 2.** *The value returned by evaluating "common_p_belief($C, i, \omega$)" is the $C$-evidence level of the maximally evident $C$-indicating event $E$ containing some element of $\Pi_i(\omega)$.*

*Proof.* First note "evidence_level(F, C)" computes the $C$-evidence level of $F$ (which must exist by lemma 3). Also note $\Omega$ is the maximally evident $C$-indicating superset of $\Omega$. Thus by lemma 9, the calls to "super_p_evident()" iteratively return the nested sequence of subsets of maximally evident subsets described in theorem 1.

If there does not exist a $p$-evident $C$-indicating event for $p > 0$, then by lemma 9 the first call to super_p_evident will return the empty set. In this case the $C$-evidence level of $\Omega$ will be 0, and hence the function will return 0.

Since the "while" loop will continue iterating until either $F$ is empty or $F$ is an event that player $i$ believes to be impossible (i.e., that does not contain any elements in $\Pi_i(\omega)$), the last $E$ before either of these cases occurred must be the maximally evident $C$-indicating event containing some element of $\Pi_i(\omega)$. The call to evidence_level then computes the $C$-evidence level of $E$. $\qquad\square$

Our final result interprets the last result in terms of common-p-belief.

**Corollary 3.** *The value returned by evaluating "common_p_belief$(C, i, \omega)$" is the maximum value of $p$ such that player $i$ $p$-believes there is common-$p$-belief in $C$ at $\omega$.*

*Proof.* By proposition 2, common_p_belief$(C, i, \omega)$ returns the $C$-evidence level of some event $E$ that player $i$ believes with probability greater than 0. Let $p$ equal the $C$-evidence level of $E$. By lemma 5, player $i$ must therefore $p$-believe $E$. Player $i$ thus $p$-believes there is common-$p$-belief in $C$ at $\omega$ since $E$ is a $p$-evident $C$-indicating event. Now suppose there was some $p' > p$ such that player $i$ $p'$-believes that there is common-$p'$-belief in $C$. Then by definition of common-p-belief there must exist a $p'$-evident event $F$ that player $i$ $p'$-believes. But then player $i$ must also $p'$-believe the maximally evident $C$-indicating superset of $F$. However, this contradicts the fact that $p$ was returned by common_p_belief$(C, i, \omega)$, since if player $i$ $p'$-believes $F$, then $F$ is a maximally evident event containing some element of $\Pi_i(\omega)$. $\qquad\square$

# References

Monderer, D., and Samet, D. 1989. Approximating common knowledge with common beliefs. *Games and Economic Behavior* 1(2):170–190.