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Refined global Gan–Gross–Prasad conjecture for Bessel periods

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Abstract. We formulate a refined version of the global Gan–Gross–Prasad conjecture for general Bessel models, extending the work of Ichino–Ikeda and R. N. Harris in the co-rank 1 case. It is an explicit formula relating the automorphic period of Bessel type and the central value of certain L-function. To support such conjecture, we provide two examples for pairs $\mathrm{SO}_5 \times \mathrm{SO}_2$ and $\mathrm{SO}_6 \times \mathrm{SO}_3$ (both co-rank 3) in the endoscopic case via theta lifting.

Contents

1. Introduction

About twenty years ago, Gross and Prasad [14, 15] made a series of fascinating conjectures about restriction of representations of orthogonal groups, which are very important phenomena in the study of automorphic representations. Recently, such conjectures are completed to other cases of classical groups by the work of Gan, Gross and Prasad [8] in the framework of Bessel and Fourier–Jacobi models. Among other things, the conjecture predicts a deep relation between certain period integral of automorphic forms and special value of L-functions, the so-called *global Gan–Gross–Prasad conjecture*. Roughly speaking, in the cases we consider here, they conjecture that there exists a non-trivial period integral if and only if the central value of the standard L-function $L(1/2, \pi)$ is non-vanishing, where π is a cuspidal automorphic repre-

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sentation of the product group $G_2 \times G_0$. Here for $\alpha = 0, 2$, G_α is the special orthogonal (resp. unitary) group of a quadratic (hermitian) space V_α, where V_0 sits inside V_2 and its orthogonal complement is split of rank $2r + 1$ for $r \geq 0$.

Recently, the influential work of Ichino and Ikeda [23] has refined the above global Gan–Gross–Prasad conjecture to a conjectural formula which computes the norm of the period integral in terms of the central L-value, in the case of orthogonal groups of co-rank 1, that is, $r = 0$. For unitary groups and $r = 0$, the similar work was later accomplished by R. N. Harris [19]. In both cases, the domain of the period integral is simply the smaller group G_0. In this article, we propose a conjectural formula which computes the norm of the period integral, which will be introduced later, in terms of the central L-value, for both orthogonal and unitary groups and general r.

To simplify the introduction, we consider only orthogonal groups here. Let F be a number field with the ring of adèles $\mathbb{A}$. Let V_2 be a quadratic space over F which decomposes orthogonally as $V_2 = R \oplus V_0 \oplus \langle w \rangle \oplus R^*$ where $R \oplus R^*$ is the direct sum of r hyperbolic planes and w is an anisotropic vector. Put $G_\alpha = \mathrm{SO}(V_\alpha)$ for $\alpha = 0, 2$. Assume that $\dim V_2 \geq 3$ and if $\dim V_0 = 2$, G_0 is not split. We fix a complete filtration of R and let P be the parabolic subgroup of G_2 stabilizing such filtration, with the unipotent radical N. The group G_0, viewed as a subgroup of G_2, acts on N by conjugation. Let ψ be a generic automorphic character of N that is stable under the action of G_0. In particular, we may view ψ as an automorphic character of $\mathrm{N} \rtimes G_0$ which is usually called the Bessel subgroup of G_2. Let π_α ($\alpha = 0, 2$) be an irreducible tempered cuspidal automorphic representation of $G_\alpha(\mathbb{A})$. For $\varphi_\alpha \in \pi_\alpha$, define the period integral

$$\mathcal{P}(\varphi_2, \varphi_0) = \int_{\mathrm{N} \rtimes G_0(F) \backslash \mathrm{N} \rtimes G_0(\mathbb{A})} \varphi_2(u g_0) \varphi_0(g_0) \psi^{-1}(u) \, \mathrm{d}u \, \mathrm{d}g_0,$$

which is absolutely convergent. Then the global Gan–Gross–Prasad conjecture predicts that there exists some pair (φ_2, φ_0) in (the Vogan packet of) $\pi_2 \times \pi_0$ with $\mathcal{P}(\varphi_2, \varphi_0) \neq 0$ if and only if the central special L-value $L(1/2, \pi_2 \boxtimes \pi_0) \neq 0$.

Now suppose that π_α decomposes as $\otimes_{\mathrm{v}} \pi_{\alpha,\mathrm{v}}$ ($\alpha = 0, 2$) into admissible representations over local fields. Assume $\varphi_\alpha = \otimes \varphi_{\alpha,\mathrm{v}}$, then for each place v of F, we define a local term $\alpha_{\mathrm{v}}^{\natural}(\varphi_{2,\mathrm{v}}, \varphi_{0,\mathrm{v}})$ by a regularized integral of matrix coefficient. We show that it is always a non-negative real number, and equals 1 for almost all v. Therefore, the infinite product $\prod_{\mathrm{v}} \alpha_{\mathrm{v}}^{\natural}(\varphi_{2,\mathrm{v}}, \varphi_{0,\mathrm{v}})$ makes sense and is non-negative. Our refined Gan–Gross–Prasad conjecture is the following formula:

$$|\mathcal{P}(\varphi_2, \varphi_0)|^2 = \frac{1}{|\mathcal{S}_\Psi|} \frac{\Delta_{G_2} L(\frac{1}{2}, \pi_2 \boxtimes \pi_0)}{L(1, \pi_2, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})} \prod_{\mathrm{v}} \alpha_{\mathrm{v}}^{\natural}(\varphi_{2,\mathrm{v}}, \varphi_{0,\mathrm{v}}), \tag{1.1}$$

after normalizing the Haar measure and local unitary pairings on π_α. Here, Δ_{G_2} is a certain product of abelian L-values and $L(1, \pi_\alpha, \mathrm{Ad})$ is the adjoint L-value. See Section 2.1 for more details. In particular, when G_0 is the trivial group, the terms $L(\frac{1}{2}, \pi_2 \boxtimes \pi_0)$ and $L(1, \pi_0, \mathrm{Ad})$ are both 1, then the refined formula computes the (square norm of the) Whittaker–Fourier coefficient. In this case, the refined formula has also been conjectured by Lapid–Mao [33] for more general groups; see Section 2.2 for details. The extra constant term $1/|\mathcal{S}_\Psi|$ relates to the Arthur parameter Ψ of $\pi_2 \times \pi_0$, which was first observed by Ichino–Ikeda after summarizing many low-rank examples in the case $r = 0$. We conjecture that for general r, one should expect the same term.

As we have said, the conjecture for $r = 0$ has originally been formulated by Ichino–Ikeda and R. N. Harris. There are some new difficulties for $r > 0$. First, the matrix coefficient integral $\alpha_{\mathrm{v}}^{\natural}(\varphi_{2,\mathrm{v}}, \varphi_{0,\mathrm{v}})$ is absolutely convergent when $r = 0$, but *not* in general when $r > 0$ unless π_2 is square-integrable. Therefore, we need a regularization. For v non-archimedean, we use the notion of stable unipotent integrals introduced in [33]. Note that Waldspurger [46] also regularizes such integral via a way similar to ours, while our treatment seems to be more convenient for unramified calculation. For v archimedean, we use Fourier transform of tempered distribution to regularize $\alpha_{\mathrm{v}}^{\natural}$. Second, for the unramified computation of $\alpha_{\mathrm{v}}^{\natural}(\varphi_{2,\mathrm{v}}, \varphi_{0,\mathrm{v}})$, we encounter a new convergence issue, which does not exist in the case $r = 0$. This is overcome by using a variant construction of Whittaker–Shintani functions.

To support formula (1.1), we provide two examples for pairs $\mathrm{SO}_5 \times \mathrm{SO}_2$ and $\mathrm{SO}_6 \times \mathrm{SO}_3$ ($r = 1$ in both cases) in this article. Like many other examples, we make use of exceptional isomorphisms for low-rank orthogonal groups and exploit the machine of theta lifting. The following theorem is a special case of (1.1) for $\mathrm{SO}_5 \times \mathrm{SO}_2$, where we refer to Section 4 for the details.

Theorem (Theorem 4.3). *Let π be an irreducible cuspidal endoscopic automorphic representation of $\mathrm{SO}_5(\mathbb{A})$, where SO_5 is the split orthogonal group of rank 5; in other words π is a Yoshida lift. Let χ be an automorphic character of $\mathrm{SO}_2(\mathbb{A})$. For $\varphi = \otimes\varphi_{\mathrm{v}} \in \pi$,*

$$|\mathcal{P}(\varphi, \chi)|^2 = \frac{1}{8} \frac{\zeta_F(2)\zeta_F(4)L(\frac{1}{2}, \pi \boxtimes \chi)}{L(1, \pi, \mathrm{Ad})L(1, \chi_{K/F})} \prod_{\mathrm{v}} \alpha_{\mathrm{v}}^{\sharp}(\varphi_{\mathrm{v}}, \bar{\varphi}_{\mathrm{v}}; \chi_{\mathrm{v}}).$$

The non-refined version, that is, the global Gan–Gross–Prasad conjecture for the case in the above theorem has already been proved by Prasad and Takloo-Bighash [38]. In some special cases, the formula in the above theorem is known; see Remark 4.4 (2). Moreover, Qiu [39] has recently obtained a formula for Bessel periods of Saito–Kurokawa representations (which are not tempered).

There is a similar result for certain representations of $\mathrm{SO}_6 \times \mathrm{SO}_3$. Since the situation is a little bit technical, we will not state the actual theorem here. The reader may consult Theorem 5.10, Remark 5.11 and Corollary 5.13 for details.

To prove formula (1.1) for unitary groups, Jacquet and Rallis [24] proposed an approach using a relative trace formula in the case $r = 0$. Based on this relative trace formula, together with the associated fundamental lemma proved by Yun [50] and smooth matching proved by himself [53], W. Zhang [52] has recently made a significant progress toward the refined formula (1.1) for $\mathrm{U}_{n+1} \times \mathrm{U}_n$. On the other hand, the author [35] extended the relative trace formula approach for unitary groups for general r (and also the Fourier–Jacobi model which concerns the product group $\mathrm{U}_n \times \mathrm{U}_m$ with $n - m$ even). We hope that it will help to attack the refined conjecture proposed in this article as well. For orthogonal groups, there is generally no relative trace formula approach yet, except for $\mathrm{SO}_5 \times \mathrm{SO}_2$; see the work of Furusawa–Martin [7] and the references therein.

The article is organized as follows. We will formulate the refined conjecture in details in Section 2 and single out the particular case for Whittaker–Fourier coefficients in Section 2.2. In Section 3, we develop the local theory. Precisely, we introduce the notion of stable unipotent integrals in Section 3.1 and define the regularized Fourier coefficients for matrix coefficients at non-archimedean places. In Section 3.2, we prove the convergence of the regularized matrix coefficient integral and also the positivity. We dedicate Section 3.3 to the calculation in the unramified situation. In Section 3.4, we regularize the matrix coefficient integral at archimedean places and show the positivity. In Section 3.5, we calculate regularized Fourier coefficients for some local theta lifting, which will be used in our global examples later. Section 4 is devoted to the global example for $\mathrm{SO}_5 \times \mathrm{SO}_2$ where the representation on SO_5 is a Yoshida lift. We will reduce the refined conjecture in this case to the Waldspurger formula, recalled in Section 4.5. The final statement appears as Theorem 4.3. Section 5 is devoted to the global example for $\mathrm{SO}_6 \times \mathrm{SO}_3$ where the representation on SO_6 is induced from a (tempered) theta lifting from U_3. We will reduce the refined conjecture to the one for $\mathrm{U}_3 \times \mathrm{U}_2$, which is known for many cases by W. Zhang. The final statements appear as Theorem 5.10 and Corollary 5.13. Section 6 contains three appendices. The first two are both solely used in the unramified calculation; while the last appendix contains a list of explicit constructions of the exceptional isomorphism for quasi-split orthogonal groups up to rank 6, which is well known for many readers and hence simply for convenience.

Notation and convention. We fix a global field F of characteristic not 2 with $\mathbb{A}$ its ring of adèles. For any other global field F' containing F, put $\mathbb{A}_{F'} = \mathbb{A} \otimes_F F'$.

Let E/F be a field extension of degree ≤ 2 and c the unique automorphism of E such that $E^{c=1} = F$. Let $\chi_{E/F}\colon E^\times\backslash\mathbb{A}_E^\times \to \{\pm 1\}$ be the character associated to E/F via the class field theory, which is either trivial or quadratic. Put $\mathrm{N}E^\times = \{xx^c \mid x \in E^\times\}$ which is a subgroup of $F^\times$. Put $E^- = \{x \in E \mid x^c = -x\}$ and $E^{\times,1} = \{x \in E \mid xx^c = 1\}$.

For E/F as above, define the affine group

$$\mathrm{Herm}_n^E = \left\{ s \in \mathrm{Res}_F^E \mathrm{Mat}_n \mid {}^t s = s^c \right\}$$

of hermitian matrices of rank n, where Mat_n is the affine group of n-by-n square matrices. In particular, if $E = F$, then Herm_n^E is also denoted by Sym_n, the group of symmetric matrices of rank n. There is a determinant morphism $\det\colon \mathrm{Herm}_n^E \to \operatorname{Spec} F[X]$.

Let f be a function on an abstract group G. For $g \in G$, put $\mathrm{L}(g)f$ (resp. $\mathrm{R}(g)f$) to be the function sending g' to $f(g^{-1}g')$ (resp. $f(g'g)$). The identity element of any abstract group will be denoted by 1.

Introduce matrices

$$\mathbf{J}_n := \begin{bmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{bmatrix}; \quad \mathbf{w}_n := \begin{bmatrix} & \mathbf{w}_{n-1} \\ 1 & \end{bmatrix}, \quad \mathbf{w}_1 = 1.$$

Let v be a place of F. We write $|\ |_\mathrm{v}$ for the norm on F_v such that $\mathrm{d}(xy) = |x|_\mathrm{v}\,\mathrm{d}y$ for any Haar measure $\mathrm{d}y$ on F_v. In several cases where we focus on local situation, we suppress v from the notation and write $|\ |_F$ for $|\ |_\mathrm{v}$. If G is an algebraic group over F, we write $G_\mathrm{v} = G(F_\mathrm{v})$ as a Lie group. We denote by δ_{G_v} the modulus function and say G_v is unimodular if $\delta_{G_\mathrm{v}} = 1$. An admissible tempered representation of G_v where G is reductive is assumed to be unitary and of finite length. If G is reductive, we denote by $\mathcal{A}_0(G)$ the space of cusp forms on $G(\mathbb{A})$.

Let G be a linear algebraic group over F. We always take the adèlic measure $\mathrm{d}g$ on $G(\mathbb{A})$ to be the Tamagawa measure. Therefore, if G is unipotent, then $\mathrm{Vol}(G(F)\backslash G(\mathbb{A}), \mathrm{d}g) = 1$. Assume that G is reductive. We take positive local Haar measures $\mathrm{d}g_\mathrm{v}$ on G_v such that $\mathrm{Vol}(\mathcal{K}_\mathrm{v}, \mathrm{d}g_\mathrm{v}) = 1$ for almost all v, where $\mathcal{K}_\mathrm{v}$ is a maximal compact subgroup of G_v. Then we have $\mathrm{d}g = C_G \prod \mathrm{d}g_\mathrm{v}$ for some positive number C_G, called the *Haar measure constant* for G, following [23]. Let π be an irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$ realized on the space $\mathcal{V}_\pi \subset \mathcal{A}_0(G)$. Put $\mathcal{V}_{\bar{\pi}} = \{\bar{f} \mid f \in \mathcal{V}_\pi\}$ which is naturally an irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$, denoted by $\bar{\pi}$. Then we have a canonical bilinear pairing

$$\mathcal{B}_\pi : \mathcal{V}_\pi \otimes \mathcal{V}_{\bar{\pi}} \to \mathbb{C}$$

defined by the Petersson inner product

$$\mathcal{B}_\pi(\varphi_1, \varphi_2) = \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \varphi_1(g)\varphi_2(g)\, \mathrm{d}g,$$

where Z_G is the maximal split torus in the center of G and $\mathrm{d}g$ is the Tamagawa measure on $(Z_G\backslash G)(\mathbb{A})$. In particular, $\bar{\pi}$ is isomorphic to the contragredient representation $\check{\pi}$ of π.

Throughout the article, all quadratic, symplectic, hermitian, or skew-hermitian spaces are assumed to be non-degenerate.

For an even-dimensional quadratic space (V, q) over F, the discriminant quadratic algebra K_q is defined as $F(\sqrt{\operatorname{disc} q})$ (resp. $F \oplus F$) if $\operatorname{disc} q \notin (F^\times)^2$ (resp. $\operatorname{disc} q \in (F^\times)^2$), where $\operatorname{disc} q = (-1)^{\dim V/2} \det q$ is the discriminant.

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2. The refined formula for Bessel periods

2.1. Formulation of the formula. Let (V_1, q_1) be a hermitian space over E (with respect to the involution c), and

$$(V_2, q_2) = R \oplus (V_1, q_1) \oplus R^*,$$

where $R \oplus R^*$ is the direct sum of r hyperbolic planes with $r \geq 0$. In this notation, $q_1 : V_1 \times V_1 \to E$ is a (non-degenerate) hermitian form on V_1 such that the norm of an element $x \in V_1$ is $q_1(x, x)$, simply denoted as $q_1(x)$.

Fix an anisotropic element $w \in V_1$ and let (V_0, q_0) be the orthogonal complement of w in (V_1, q_1). Let n_α be the rank of V_α for $\alpha = 0, 1, 2$. Let $\operatorname{Ism}(V_\alpha, q_\alpha)$ be the isometry group of (V_α, q_α) and $G_\alpha = \operatorname{Ism}^0(V_\alpha, q_\alpha)$ its connected component, which are both reductive groups over F. We have successive embeddings $G_0 \hookrightarrow G_1 \hookrightarrow G_2$. When $E = F$, let K be the discriminant quadratic algebra of the pair (V_2, V_0), that is, $K = K_{q_0}$ (resp. K_{q_2}) if n_0 (resp. n_2) is even.

Assume that $n_2 + [E : F] \geq 4$ and if $n_0 = 2$, then G_0 is not split.

We fix maximal compact subgroups $\mathcal{K}_\alpha = \prod_{\mathrm{v}} \mathcal{K}_{\alpha,\mathrm{v}}$ of $G_\alpha(\mathbb{A})$ for $\alpha = 0, 1, 2$ such that $[\mathcal{K}_\alpha : \mathcal{K}_{\alpha+1} \cap \mathcal{K}_\alpha] < \infty$ for $\alpha = 0, 1$. Later in local situations, we will always assume that $\mathcal{K}_{\alpha,\mathrm{v}}$ is in good position with respect to a chosen minimal parabolic.

We define certain L-values attached to (dual) Gross motives [13] as follows:

$$\Delta_{G_2} = \begin{cases} \prod_{i=1}^{(n_2-1)/2} \zeta_F(2i) & \text{if } E = F \text{ and } n_2 \text{ is odd,} \\ \prod_{i=1}^{n_2/2-1} \zeta_F(2i) \cdot L(\frac{n_2}{2}, \chi_{K/F}) & \text{if } E = F \text{ and } n_2 \text{ is even,} \\ \prod_{i=1}^{n_2} L(i, \chi_{E/F}^i) & \text{if } E \neq F. \end{cases}$$

We fix a complete flag

$$\mathfrak{F}\colon 0 = R_0 \subset R_1 \subset \cdots \subset R_r = R$$

of the E-space R. Let $\mathrm{P}_{\mathfrak{F}}$ be the parabolic subgroup of G_2 stabilizing the flag

$$R_0 \subset R_1 \subset \cdots \subset R_r = R \subset R \oplus V_1,$$

and $\mathrm{N}_{\mathfrak{F}}$ its unipotent radical on which G_1 acts by conjugation. Fix a generic automorphic character

$$\psi_{\mathfrak{F},w} = \otimes_{\mathrm{v}} \psi_{\mathfrak{F},w,\mathrm{v}}\colon \mathrm{N}_{\mathfrak{F}}(F)\backslash \mathrm{N}_{\mathfrak{F}}(\mathbb{A}) \to \mathbb{C}^{\times}$$

of $\mathrm{N}_{\mathfrak{F}}$ that is invariant under the conjugation action of $G_0(\mathbb{A})$. For an automorphic form φ_2 of G_2, define

$$\mathcal{F}_{\psi_{\mathfrak{F},w}}(\varphi_2)(g_2) = \int_{\mathrm{N}_{\mathfrak{F}}(F)\backslash \mathrm{N}_{\mathfrak{F}}(\mathbb{A})} \varphi_2(ug_2)\psi_{\mathfrak{F},w}^{-1}(u)\,\mathrm{d}u.$$

It is clear that the restriction $\mathcal{F}_{\psi_{\mathfrak{F},w}}(\varphi_2)|_{G_0(\mathbb{A})}$ is invariant under the left translation by $G_0(F)$. For cusp forms φ_α of G_α for $\alpha = 0, 2$, we introduce the period integral

$$\mathcal{P}(\varphi_2, \varphi_0) = \int_{G_0(F)\backslash G_0(\mathbb{A})} \mathcal{F}_{\psi_{\mathfrak{F},w}}(\varphi_2)(g_0)\varphi_0(g_0)\,\mathrm{d}g_0,$$

which is absolutely convergent.

Now let us consider the local analogue of the above period integral. We fix a place v of F. Let $\pi_{\alpha,\mathrm{v}}$ be an irreducible admissible representation of $G_{\alpha,\mathrm{v}}$ for $\alpha = 0, 2$. For a (smooth) matrix coefficient $\Phi_{2,\mathrm{v}}$ of $\pi_{2,\mathrm{v}} \otimes \check{\pi}_{2,\mathrm{v}}$, the integral

$$\int_{\mathrm{N}_{\mathfrak{F},\mathrm{v}}} \Phi_{2,\mathrm{v}}(u_\mathrm{v})\psi_{\mathfrak{F},w,\mathrm{v}}^{-1}(u_\mathrm{v})\,\mathrm{d}u_\mathrm{v} \tag{2.1}$$

is absolutely convergent if $\pi_{2,\mathrm{v}}$ is square-integrable, but not in general. Here, $\mathrm{d}u_\mathrm{v}$ is the self-dual measure. Therefore, we need to regularize the above integral.

We briefly explain the idea on how to regularize the above integral for v non-archimedean. We prove that although not absolutely convergent, the integral (2.1) is *stable* in the sense that there exists a compact open subgroup N of $\mathrm{N}_{\mathfrak{F},\mathrm{v}}$ such that

$$\int_{N'} \Phi_{2,\mathrm{v}}(u_\mathrm{v})\psi_{\mathfrak{F},w,\mathrm{v}}^{-1}(u_\mathrm{v})\,\mathrm{d}u_\mathrm{v} = \int_{N} \Phi_{2,\mathrm{v}}(u_\mathrm{v})\psi_{\mathfrak{F},w,\mathrm{v}}^{-1}(u_\mathrm{v})\,\mathrm{d}u_\mathrm{v}$$

for any compact open subgroup N' of $\mathrm{N}_{\mathfrak{F},\mathrm{v}}$ containing N. We denote the above stable value of integral by

$$\int_{\mathrm{N}_{\mathfrak{F},\mathrm{v}}}^{\mathrm{st}} \Phi_{2,\mathrm{v}}(u_\mathrm{v})\psi_{\mathfrak{F},w,\mathrm{v}}^{-1}(u_\mathrm{v})\,\mathrm{d}u_\mathrm{v},$$

and define

$$\mathcal{F}_{\psi_{\mathfrak{F},w,\mathrm{v}}}(\Phi_{2,\mathrm{v}})(g_2) = \int_{\mathrm{N}_{\mathfrak{F},\mathrm{v}}}^{\mathrm{st}} \mathrm{L}(g_2^{-1})\Phi_{2,\mathrm{v}}(u_\mathrm{v})\psi_{\mathfrak{F},w,\mathrm{v}}^{-1}(u_\mathrm{v})\,\mathrm{d}u_\mathrm{v},$$

which is a locally constant function on $G_{2,\mathrm{v}}$.

For $\varphi_{\alpha,\mathrm{v}} \in \pi_{\alpha,\mathrm{v}}$ and $\check{\varphi}_{\alpha,\mathrm{v}} \in \check{\pi}_{\alpha,\mathrm{v}}$, we introduce the matrix coefficient

$$\Phi_{\varphi_{\alpha,\mathrm{v}}\otimes\check{\varphi}_{\alpha,\mathrm{v}}}(g_\alpha) = \mathcal{B}_{\pi_{\alpha,\mathrm{v}}}\big(\pi_{\alpha,\mathrm{v}}(g_\alpha)\varphi_{\alpha,\mathrm{v}}, \check{\varphi}_{\alpha,\mathrm{v}}\big),$$

where $\mathcal{B}_{\pi_{\alpha,\mathrm{v}}}\colon \pi_{\alpha,\mathrm{v}} \otimes \check{\pi}_{\alpha,\mathrm{v}} \to \mathbb{C}$ is the canonical G_α-invariant bilinear pairing for $\alpha = 0, 2$. Now we define

(2.2)

$$\alpha_\mathrm{v}(\varphi_{2,\mathrm{v}}, \check{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \check{\varphi}_{0,\mathrm{v}}) = \int_{G_{0,\mathrm{v}}} \mathcal{F}_{\psi_{\mathfrak{F},w,\mathrm{v}}}(\Phi_{\varphi_{2,\mathrm{v}}\otimes\check{\varphi}_{2,\mathrm{v}}})(g_{0,\mathrm{v}})\Phi_{\varphi_{0,\mathrm{v}}\otimes\check{\varphi}_{0,\mathrm{v}}}(g_{0,\mathrm{v}})\,\mathrm{d}g_{0,\mathrm{v}}$$

and

(2.3)

$$\alpha_\mathrm{v}^\natural(\varphi_{2,\mathrm{v}}, \check{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \check{\varphi}_{0,\mathrm{v}}) = \Big(\frac{\Delta_{G_{2,\mathrm{v}}} L(\frac{1}{2}, \pi_{2,\mathrm{v}} \boxtimes \pi_{0,\mathrm{v}})}{L(1, \pi_{2,\mathrm{v}}, \mathrm{Ad})L(1, \pi_{0,\mathrm{v}}, \mathrm{Ad})}\Big)^{-1} \alpha_\mathrm{v}(\varphi_{2,\mathrm{v}}, \check{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \check{\varphi}_{0,\mathrm{v}}).$$

If $\pi_{\alpha,\mathrm{v}}$ are unitary for $\alpha = 0, 2$, we may identify $\check{\pi}_{\alpha,\mathrm{v}}$ with $\bar{\pi}_{\alpha,\mathrm{v}}$ via a G_α-invariant pairing, denoted by $\mathcal{B}_{\pi_{\alpha,\mathrm{v}}}\colon \pi_{\alpha,\mathrm{v}} \otimes \bar{\pi}_{\alpha,\mathrm{v}} \to \mathbb{C}$ by abuse of notation. Put

$$\alpha_\mathrm{v}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}) = \alpha_\mathrm{v}(\varphi_{2,\mathrm{v}}, \bar{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \bar{\varphi}_{0,\mathrm{v}}), \tag{2.4}$$

$$\alpha_\mathrm{v}^\natural(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}) = \alpha_\mathrm{v}^\natural(\varphi_{2,\mathrm{v}}, \bar{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \bar{\varphi}_{0,\mathrm{v}}). \tag{2.5}$$

In Section 3.4, we define the above functionals for v archimedean as well, which will not be discussed here.

Theorem 2.1. *Assume that $\pi_{\alpha,\mathrm{v}}$ are tempered for $\alpha = 0, 2$.*

(1) *If* v *is non-archimedean, the integral* (2.2) *defining $\alpha_\mathrm{v}(\varphi_{2,\mathrm{v}}, \check{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \check{\varphi}_{0,\mathrm{v}})$ is absolutely convergent for all vectors $\varphi_{\alpha,\mathrm{v}} \in \pi_{\alpha,\mathrm{v}}$ and $\check{\varphi}_{\alpha,\mathrm{v}} \in \check{\pi}_{\alpha,\mathrm{v}}$ ($\alpha = 0, 2$).*

(2) *For all* v*, $\alpha_\mathrm{v}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}) \geq 0$ for all (smooth) vectors $\varphi_{\alpha,\mathrm{v}} \in \pi_{\alpha,\mathrm{v}}$ ($\alpha = 0, 2$).*

Let us consider the unramified situation. We say a place v of F is *good* (with respect to $\pi_{\alpha,\mathrm{v}}$ for $\alpha = 0, 2$) if

(1) v is non-archimedean;

(2) $\mathcal{K}_{\alpha,\mathrm{v}}$ is a hyperspecial maximal compact subgroup of $G_{\alpha,\mathrm{v}}$ (in particular, $G_{\alpha,\mathrm{v}}$ is unramified over F_v) for $\alpha = 0, 2$;

(3) $\mathcal{K}_{0,\mathrm{v}} \subset \mathcal{K}_{2,\mathrm{v}}$;

(4) $\mathcal{K}_{\alpha,\mathrm{v}}$ has volume 1 under the measure $\mathrm{d}g_{\alpha,\mathrm{v}}$ for $\alpha = 0, 2$;

(5) $\psi_{\mathfrak{F},w,\mathrm{v}}$ is unramified, that is, $\psi_{\mathfrak{F},w,\mathrm{v}}(uu') = 1$ for all $u' \in \mathrm{N}_{\mathfrak{F},\mathrm{v}} \cap \mathcal{K}_{2,\mathrm{v}}$ if and only if $u \in \mathrm{N}_{\mathfrak{F},\mathrm{v}} \cap \mathcal{K}_{2,\mathrm{v}}$;

(6) $\varphi_{\alpha,\mathrm{v}}$ and $\check{\varphi}_{\alpha,\mathrm{v}}$ are fixed by $\mathcal{K}_{\alpha,\mathrm{v}}$, and $\mathcal{B}_{\pi_{\alpha,\mathrm{v}}}(\varphi_{\alpha,\mathrm{v}}, \check{\varphi}_{\alpha,\mathrm{v}}) = 1$ (in particular, $\pi_{\alpha,\mathrm{v}}$ is an unramified representation) for $\alpha = 0, 2$.

Theorem 2.2. *If* v *is good and the integral defining* $\alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}},\check{\varphi}_{2,\mathrm{v}};\varphi_{0,\mathrm{v}},\check{\varphi}_{0,\mathrm{v}})$ *is absolutely convergent, then*

$$\alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}},\check{\varphi}_{2,\mathrm{v}};\varphi_{0,\mathrm{v}},\check{\varphi}_{0,\mathrm{v}}) = \frac{\Delta_{G_{2,\mathrm{v}}} L(\frac{1}{2},\pi_{2,\mathrm{v}}\boxtimes\pi_{0,\mathrm{v}})}{L(1,\pi_{2,\mathrm{v}},\mathrm{Ad})L(1,\pi_{0,\mathrm{v}},\mathrm{Ad})}.$$

In general, there is a meromorphic continuation of $\alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}},\check{\varphi}_{2,\mathrm{v}};\varphi_{0,\mathrm{v}},\check{\varphi}_{0,\mathrm{v}})$ *to the entire domain of the Satake parameters of* $\pi_{2,\mathrm{v}}\boxtimes\pi_{0,\mathrm{v}}$, *which is equal to the right-hand side of the above identity.*

Recall the multiplicity one theorem for general r proved in [25, Theorem A] and [8, Corollary 15.3], based on the assertion for $r=0$ (see [1, 42, 47]) that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,\mathrm{v}}\ltimes \mathrm{N}_{\mathfrak{F},\mathrm{v}}}(\pi_{2,\mathrm{v}}\boxtimes\pi_{0,\mathrm{v}}\otimes\psi^{-1}_{\mathfrak{F},w,\mathrm{v}},\mathbb{C}) \le 1.$$

We propose the following conjecture.

Conjecture 2.3. *Assume that* $\pi_{\alpha,\mathrm{v}}$ *are tempered for* $\alpha=0,2$. *Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,\mathrm{v}}\ltimes \mathrm{N}_{\mathfrak{F},\mathrm{v}}}(\pi_{2,\mathrm{v}}\boxtimes\pi_{0,\mathrm{v}}\otimes\psi^{-1}_{\mathfrak{F},w,\mathrm{v}},\mathbb{C}) = 1$$

if and only if $\alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}},\check{\varphi}_{2,\mathrm{v}};\varphi_{0,\mathrm{v}},\check{\varphi}_{0,\mathrm{v}}) \neq 0$ *for some* $\mathcal{K}_{\alpha,\mathrm{v}}$*-finite vectors* $\varphi_{\alpha,\mathrm{v}}\in\pi_{\alpha,\mathrm{v}}$ *and* $\check{\varphi}_{\alpha,\mathrm{v}}\in\check{\pi}_{\alpha,\mathrm{v}}$ $(\alpha=0,2)$.

Remark 2.4. In the case of orthogonal groups (where $E=F$), Waldspurger [46, Section 5.1] also defined the regularized integral $\alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}},\check{\varphi}_{2,\mathrm{v}};\varphi_{0,\mathrm{v}},\check{\varphi}_{0,\mathrm{v}})$ in a similar way which is equivalent to ours. Moreover, he confirmed the above conjecture in the case $E=F$ and v non-archimedean [46, Proposition 5.7]. Similar results in the unitary case (where E/F is a quadratic field extension) follow from [2]. When $r=0$ and v is non-archimedean, the above conjecture has also been proved by Sakellaridis and Venkatesh in [40, Proposition 6.4.1].

Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Following [23], we say that π is *almost locally generic* if for almost all places v of F, the local component π_{v} is generic with respect to the unramified generic character. As explained in [23, Section 2], such π should come from an elliptic Arthur parameter

$$\Psi(\pi)\colon \mathcal{L}_F \to {}^{L}G := \hat{G}\rtimes\mathcal{W}_F,$$

where $\mathcal{L}_F$ is the conjectural Langlands group of F and ${}^{L}G$ is the Langlands dual group. Define $\mathcal{S}_{\Psi(\pi)} := \mathrm{Cent}_{\hat{G}}(\mathrm{Im}\,\Psi(\pi))$ to be the centralizer of the image of $\Psi(\pi)$ in the complex dual group $\hat{G}$. It is a finite elementary 2-abelian group when $G=G_\alpha$ $(\alpha=0,2)$. Moreover, by the generalized Ramanujan conjecture, π_{v} should be tempered for all v.

Conjecture 2.5. *Let* $\pi_\alpha\simeq\otimes_{\mathrm{v}}\pi_{\alpha,\mathrm{v}}$ *be an irreducible cuspidal automorphic representation of* $G_\alpha(\mathbb{A})$ *that is almost locally generic. Identify* $\check{\pi}_\alpha$ *(resp.* $\check{\pi}_{\alpha,\mathrm{v}}$*) with* $\bar{\pi}_\alpha$ *(resp.* $\bar{\pi}_{\alpha,\mathrm{v}}$*) for* $\alpha=0,2$. *Let* S *be the set of places of* F *that are not good.*

(1) *The way we define* $\alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}};\varphi_{0,\mathrm{v}})$ *for* $\pi_{\alpha,\mathrm{v}}$ $(\alpha=0,2)$ *tempered should apply here as well.*

(2) *We have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,\mathrm{v}} \ltimes \mathrm{N}_{\mathfrak{F},\mathrm{v}}}(\pi_{2,\mathrm{v}} \boxtimes \pi_{0,\mathrm{v}} \otimes \psi^{-1}_{\mathfrak{F},w,\mathrm{v}}, \mathbb{C}) = 1$$

if and only if $\alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}) > 0$ *for some* $\mathcal{K}_{\alpha,\mathrm{v}}$*-finite vectors* $\varphi_{\alpha,\mathrm{v}} \in \pi_{\alpha,\mathrm{v}}$ *(*$\alpha = 0, 2$*).*

(3) *For every non-zero cusp forms* $\varphi_\alpha = \otimes_{\mathrm{v}} \varphi_{\alpha,\mathrm{v}}$ *in the space* $\mathcal{V}_{\pi_\alpha}$ *of* π_α *(*$\alpha = 0, 2$*),*

$$\frac{|\mathcal{P}(\varphi_2, \varphi_0)|^2}{\mathcal{B}_{\pi_2}(\varphi_2, \bar{\varphi}_2)\mathcal{B}_{\pi_0}(\varphi_0, \bar{\varphi}_0)} = \frac{C_{G_0}}{|\mathcal{S}_{\Psi(\pi_2)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{\Delta_{G_2} L(\frac{1}{2}, \pi_2 \boxtimes \pi_0)}{L(1, \pi_2, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})} \times \prod_{\mathrm{v}} \frac{\alpha^{\natural}_{\mathrm{v}}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}})}{\mathcal{B}_{\pi_{2,\mathrm{v}}}(\varphi_{2,\mathrm{v}}, \bar{\varphi}_{2,\mathrm{v}})\mathcal{B}_{\pi_{0,\mathrm{v}}}(\varphi_{0,\mathrm{v}}, \bar{\varphi}_{0,\mathrm{v}})},$$

where the product is in fact taken over the finite set S.

Remark 2.6. If we choose local Haar measures for G_0 such that $C_{G_0} = 1$, and $\mathcal{B}_{\pi_{\alpha,\mathrm{v}}}$ such that $\mathcal{B}_{\pi_\alpha} = \prod \mathcal{B}_{\pi_{\alpha,\mathrm{v}}}$ for $\alpha = 0, 2$, then the formula in Conjecture 2.5 (3) can be simplified as

$$(2.6) \quad |\mathcal{P}(\varphi_2, \varphi_0)|^2 = \frac{1}{|\mathcal{S}_{\Psi(\pi_2)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{\Delta_{G_2} L(\frac{1}{2}, \pi_2 \boxtimes \pi_0)}{L(1, \pi_2, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})} \prod_{\mathrm{v}} \alpha^{\natural}_{\mathrm{v}}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}).$$

By linearity, both $|\mathcal{P}(\varphi_2, \varphi_0)|^2$ and $\prod_{\mathrm{v}} \alpha^{\natural}_{\mathrm{v}}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}})$ define elements in the space

$$\otimes_{\mathrm{v}} \big(\mathrm{Hom}_{G_{0,\mathrm{v}} \ltimes \mathrm{N}_{\mathfrak{F},\mathrm{v}}}(\pi_{2,\mathrm{v}} \boxtimes \pi_{0,\mathrm{v}} \otimes \psi^{-1}_{\mathfrak{F},w,\mathrm{v}}, \mathbb{C}) \otimes \mathrm{Hom}_{G_{0,\mathrm{v}} \ltimes \mathrm{N}_{\mathfrak{F},\mathrm{v}}}(\bar{\pi}_{2,\mathrm{v}} \boxtimes \bar{\pi}_{0,\mathrm{v}} \otimes \psi_{\mathfrak{F},w,\mathrm{v}}, \mathbb{C})\big),$$

which has dimension ≤ 1. By Conjecture 2.5 (2), the functional $\prod_{\mathrm{v}} \alpha^{\natural}_{\mathrm{v}}$ is always a basis of the above space.

Therefore, $|\mathcal{P}(\varphi_2, \varphi_0)|^2$ must be proportional to $\prod_{\mathrm{v}} \alpha^{\natural}_{\mathrm{v}}$, with the ratio, according to (2.6), being

$$\frac{1}{|\mathcal{S}_{\Psi(\pi_2)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{\Delta_{G_2} L(\frac{1}{2}, \pi_2 \boxtimes \pi_0)}{L(1, \pi_2, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})}.$$

We may also formulate Conjecture 2.5 (3) or formula (2.6) using partial L-functions. Recall that away from a finite set S of places v, the representations $\pi_{\alpha,\mathrm{v}}$ are unramified for $\alpha = 0, 2$ and $\alpha^{\natural}_{\mathrm{v}}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}) = 1$. Therefore, (2.6) is equivalent to

$$|\mathcal{P}(\varphi_2, \varphi_0)|^2 = \frac{1}{|\mathcal{S}_{\Psi(\pi_2)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{\Delta^{\mathrm{S}}_{G_2} L^{\mathrm{S}}(\frac{1}{2}, \pi_2 \boxtimes \pi_0)}{L^{\mathrm{S}}(1, \pi_2, \mathrm{Ad}) L^{\mathrm{S}}(1, \pi_0, \mathrm{Ad})} \prod_{\mathrm{v} \in \mathrm{S}} \alpha_{\mathrm{v}}(\varphi_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}).$$

Finally, we remark that in the case $[E : F] = 2$, the adjoint L-function $L(s, \pi_\alpha, \mathrm{Ad})$ is simply the $\pm$-Asai L-function $L(s, \pi_\alpha, \mathrm{As}^{(-1)^{n_\alpha}})$.

Remark 2.7. As pointed out in [33, Section 5], in the formulation of Conjecture 2.5, one may avoid using Arthur parameters. The value $|\mathcal{S}_{\Psi(\pi_\alpha)}|$ may also be defined by functorial transfer to certain general linear groups. But the reader should be cautious that our size of $\mathcal{S}_{\Psi(\pi_\alpha)}$ is twice the one defined in [33, Section 5].

Remark 2.8. In Conjecture 2.5,

- when $r = 0$ and $E = F$, it is a conjecture of Ichino–Ikeda [23] (we refer to that paper for known examples in low-rank cases);
- when $r = 0$ and $[E : F] = 2$, it is a conjecture of R. N. Harris [19];
- when $r = 1$, $n_2 = 5$ and $E = F$ (that is, $\mathrm{SO}_5 \times \mathrm{SO}_2$), the identity (3) is also conjectured by Prasad and Takloo-Bighash [37, Conjecture], conditional on the regularization of α_{v} and unramified calculation.

2.2. Whittaker–Fourier coefficients. When $[E : F] + n_0 \leq 2$, the group G_2 is quasi-split; G_0 is trivial; and $\mathrm{N}_{\mathfrak{F}}$ is a maximal unipotent subgroup of G_2. Put $\psi_{\mathfrak{F}} = \psi_{\mathfrak{F},w}$. Then $\mathcal{F}_{\psi_{\mathfrak{F}}}(\varphi_2)$ is a Whittaker–Fourier coefficient of φ_2. In what follows, we will suppress the subscript 2 for simplicity.

Conjecture 2.9. *Let $\pi \simeq \otimes_{\mathrm{v}} \pi_{\mathrm{v}}$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ such that π_{v} is $\psi_{\mathfrak{F},\mathrm{v}}$-generic for every place v of F. We identify $\check{\pi}$ (resp. $\check{\pi}_{\mathrm{v}}$) with $\bar{\pi}$ (resp. $\bar{\pi}_{\mathrm{v}}$). Then for every non-zero cusp form $\varphi = \otimes_{\mathrm{v}} \varphi_{\mathrm{v}}$ in the space $\mathcal{V}_\pi$ of π,*

$$\frac{|\mathcal{F}_{\psi_{\mathfrak{F}}}(\varphi)(1)|^2}{\mathcal{B}_\pi(\varphi, \bar{\varphi})} = \frac{1}{|\mathcal{S}_{\Psi(\pi)}|} \frac{\Delta_G}{L(1, \pi, \mathrm{Ad})} \prod_{\mathrm{v}} \frac{\alpha_{\mathrm{v}}^{\natural}(\varphi_{\mathrm{v}})}{\mathcal{B}_{\pi,\mathrm{v}}(\varphi_{\mathrm{v}}, \bar{\varphi}_{\mathrm{v}})}.$$

Remark 2.10. (1) Recall that an irreducible admissible representation π_{v} of G_{v} is called $\psi_{\mathfrak{F},\mathrm{v}}$-*generic* if $\mathrm{Hom}_{\mathrm{N}_{\mathfrak{F},\mathrm{v}}}(\pi_{\mathrm{v}}, \psi_{\mathfrak{F},\mathrm{v}}) \neq 0$, which then must be of dimension 1.

(2) The above conjecture is also formulated by Lapid–Mao for general quasi-split reductive groups G, and even for metaplectic groups, in [33]. Moreover, substantial progress for the metaplectic case has been made in [32].

The following proposition is proved in [33, Proposition 2.10 and Section 2.5].

Proposition 2.11. *The representation π_{v} is $\psi_{\mathfrak{F},\mathrm{v}}$-generic if and only if $\alpha_{\mathrm{v}}(\varphi_{\mathrm{v}}; \check{\varphi}_{\mathrm{v}}) \neq 0$ for some $\mathcal{K}_{\mathrm{v}}$-finite vectors $\varphi_{\mathrm{v}} \in \pi_{\mathrm{v}}$ and $\check{\varphi}_{\mathrm{v}} \in \check{\pi}_{\mathrm{v}}$. Moreover, $\alpha_{\mathrm{v}}(\varphi_{\mathrm{v}}) \geq 0$ for all (smooth) vectors $\varphi_{\mathrm{v}} \in \pi_{\mathrm{v}}$.*

Corollary 2.12 (of Conjecture 2.9). *Let $\pi \simeq \otimes_{\mathrm{v}} \pi_{\mathrm{v}}$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Then π is globally $\psi_{\mathfrak{F}}$-generic if and only if it is locally $\psi_{\mathfrak{F}}$-generic, that is, π_{v} is $\psi_{\mathfrak{F},\mathrm{v}}$-generic for every place v of F.*

3. Local theory

In this section, we develop the local theory. Therefore, F is always a local field of characteristic not 2 and E/F is an étale algebra of degree at most 2. Moreover, from Section 3.1 to Section 3.3, F will be non-archimedean.

3.1. Stable unipotent integrals. We generalize [33, Sections 2.1–2.2] to any unipotent subgroup U of G ($= G_2$ in application). Lapid–Mao's results and arguments can be adopted here almost without change. Therefore, we follow closely the discussion in [33].

We recall [33, Definition 2.1, Remark 2.2]. Let U be a unipotent group and f a locally constant function on U.

(1) We say f has a *stable integral* if there exists a compact open subgroup N of U which *stabilizes* f in the following sense: for all other compact open subgroups N' of U containing N,
$$\int_{N'} f(u)\,\mathrm{d}u = \int_N f(u)\,\mathrm{d}u.$$
We denote the above stable value of integral by
$$\int_U^{\mathrm{st}} f(u)\,\mathrm{d}u.$$

(2) f is *compactly supported after averaging* if there exist compact open subgroups N_r and N_l of U such that $\mathrm{R}(\delta_{N_r})\mathrm{L}(\delta_{N_l})f$ is compactly supported, where δ denotes the Dirac measure, that is, the Haar measure with total volume 1.

(3) If f is compactly supported after averaging, then it has a stable integral and
$$\int_U^{\mathrm{st}} f(u)\,\mathrm{d}u = \int_U \big(\mathrm{R}(\delta_{N_r})\mathrm{L}(\delta_{N_l})f\big)(u)\,\mathrm{d}u.$$

Let G be an arbitrary reductive group over F with a fixed minimal parabolic subgroup $P_{\min}$. Let $P \supseteq P_{\min}$ be a parabolic subgroup with the Levi decomposition $P = M_P N_P$. We fix a Haar measure $\mathrm{d}u$ on N_P. Let π be an irreducible admissible representation of G. Let $\psi\colon N_P \to \mathbb{C}^\times$ be a *generic* character of N_P. We have the following proposition which, together with the proof, is essentially [33, Proposition 2.3].

Proposition 3.1. *For a matrix coefficient Φ of $\pi\otimes\check{\pi}$, the function $\Phi|_{N_P}\cdot\psi$ is compactly supported after averaging.*

It is clear that the function
$$\int_{N_P}^{\mathrm{st}} \mathrm{L}(g^{-1})\Phi(u)\psi(u)\,\mathrm{d}u$$
is locally constant on G, admitting the above proposition.

Let $Q \supseteq P_{\min}$ be another parabolic subgroup with the Levi decomposition $Q = M_Q N_Q$. Let W, W_P and W_Q be the Weil group of G, M_P and M_Q with the longest element w_0, w_P and w_Q, respectively. Denote by $\dot{\mathsf{w}} = \mathsf{w}_Q\mathsf{w}_0\mathsf{w}_P$ the longest element of $\mathsf{W}_Q\backslash\mathsf{W}/\mathsf{W}_P$. Then we have the Bruhat decomposition
$$G = \coprod_{\mathsf{w}\in\mathsf{W}_Q\backslash\mathsf{W}/\mathsf{W}_P} Q\mathsf{w}P.$$

Recall that the Bruhat order is defined in the following way: $\mathsf{w}_1 \le \mathsf{w}_2$ in $\mathsf{W}_Q\backslash\mathsf{W}/\mathsf{W}_P$ if $Q\mathsf{w}_1P$ is contained in the closure of $Q\mathsf{w}_2P$.

If σ is a representation of M_Q, then we write $\mathrm{Ind}\,\sigma := \mathrm{Ind}_Q^G\,\sigma$ for the normalized parabolic induction. Denote by $(\mathrm{Ind}\,\sigma)_P^\circ$ the P-invariant subspace of $\mathrm{Ind}\,\sigma$ of sections φ which are supported on the open cell $Q\dot{\mathsf{w}}P$.

Lemma 3.2. *Suppose that σ is a representation of M_Q and $\pi = \mathrm{Ind}\,\sigma$. Then for every $\varphi \in \pi$, there exists a compact open subgroup N of N_P such that $\varphi_{N,\psi} := \mathrm{R}(\delta_N)(\varphi \cdot \psi)$ is in $(\mathrm{Ind}\,\sigma)^\circ_P$.*

Proof. We prove by induction on the order of $\mathsf{w} \in \mathsf{W}_Q\backslash\mathsf{W}/\mathsf{W}_P$ that one can find a compact open subgroup N of N_P such that $\varphi_{N,\psi}$ vanishes on $\bigcup_{\mathsf{w}'<\mathsf{w}} Q\mathsf{w}'P$.

The statement is empty when w is minimal, that is, $Q\mathsf{w}P = QP$. Note that if $\varphi_{N,\psi}$ vanishes on $Q\mathsf{w}'P$, then $\varphi_{N',\psi}$ vanishes on $Q\mathsf{w}'P$ for any N' containing N. Therefore, we only need to show that for $\mathsf{w} \neq \dot{\mathsf{w}}$, if $\varphi|_{\bigcup_{\mathsf{w}'<\mathsf{w}} Q\mathsf{w}'P} = 0$ then there is a compact open subgroup $N \subset N_P$ such that $\varphi_{N,\psi}|_{Q\mathsf{w}P} = 0$.

Since $\mathsf{w} \neq \dot{\mathsf{w}}$, it holds that $\psi|_{N_P \cap \mathsf{w}^{-1}N_Q\mathsf{w}} \not\equiv 1$. For a sufficiently large N_1 satisfying $\psi|_{N_1 \cap \mathsf{w}^{-1}N_Q\mathsf{w}} \not\equiv 1$, we have

$$\begin{aligned}\varphi_{N_1,\psi}(\mathsf{w}) &= \int_{N_1} \varphi(\mathsf{w}u)\psi(u)\,\mathrm{d}u \\ &= \int_{N_1\cap\mathsf{w}^{-1}N_Q\mathsf{w}\backslash N_1} \int_{N_1\cap\mathsf{w}^{-1}N_Q\mathsf{w}} \varphi(\mathsf{w}u'u)\psi(u')\psi(u)\,\mathrm{d}u'\,\mathrm{d}u \\ &= \int_{N_1\cap\mathsf{w}^{-1}N_Q\mathsf{w}\backslash N_1} \varphi(\mathsf{w}u)\psi(u) \int_{N_1\cap\mathsf{w}^{-1}N_Q\mathsf{w}} \psi(u')\,\mathrm{d}u'\,\mathrm{d}u = 0.\end{aligned}$$

It follows that $\varphi_{N,\psi}(\mathsf{w}u) = (\mathrm{R}(u)\varphi)_{uNu^{-1},\psi}(\mathsf{w}) = 0$ for every compact open subset N of N_P and $u \in N_P$ such that

$$\psi|_{uNu^{-1}\cap\mathsf{w}^{-1}N_Q\mathsf{w}} \not\equiv 1. \tag{3.1}$$

The above condition is right N-invariant and also left P_w-invariant, where $P_\mathsf{w} = P \cap \mathsf{w}^{-1}Q\mathsf{w}$. By induction assumption, the support of $\mathrm{L}(\mathsf{w}^{-1})\varphi$ on P is compact modulo P_w. Choose a compact subset Ω of P such that the above support is contained in $P_\mathsf{w}\Omega$. Choose a compact open subset N of N_P containing $\bigcup_{u\in\Omega} u^{-1}N_1u$. Thus, (3.1) holds for all $u \in \Omega$, and hence for all $u \in P_\mathsf{w}\Omega N$. Therefore, $\varphi_{N,\psi}$ vanishes on $\mathsf{w}P$ and hence on $Q\mathsf{w}P$. □

Proof of Proposition 3.1. By Jacquet's subrepresentation theorem, we may assume that π is a subrepresentation of $\pi' := \mathrm{Ind}_Q^G\,\sigma$ for a supercuspidal irreducible representation σ of M_Q, and $\Phi = \Phi_{\varphi,\check\varphi}$ with $\varphi \in \mathrm{Ind}\,\sigma$, $\check\varphi \in \mathrm{Ind}\,\check\sigma$. For a compact open subgroup N of N_P, we have

$$\mathrm{R}(\delta_N)\mathrm{L}(\delta_N)(\Phi\cdot\psi) = \Phi_{\varphi_{N,\psi},\check\varphi_{N,\psi^{-1}}}.$$

Therefore, we may assume $\varphi \in (\mathrm{Ind}\,\sigma)^\circ_P$ and $\check\varphi \in (\mathrm{Ind}\,\check\sigma)^\circ_P$. We show that $\Phi_{\varphi,\check\varphi}|_{N_P}$ is compactly supported.

We may write

$$\Phi_{\varphi,\check\varphi}(u) = \mathscr{B}_{\pi'}(\pi'(u)\varphi,\check\varphi) = \int_{\dot{\mathsf{w}}^{-1}Q\dot{\mathsf{w}}\cap P\backslash P} \mathscr{B}_\sigma\big(\varphi(\dot{\mathsf{w}}pu),\check\varphi(\dot{\mathsf{w}}p)\big)\,\mathrm{d}p, \tag{3.2}$$

for some quotient measure $\mathrm{d}p$ on $\dot{\mathsf{w}}^{-1}Q\dot{\mathsf{w}} \cap P\backslash P$. Denote by Q' the parabolic subgroup containing $P_{\min}$ that is conjugate to the opposite parabolic subgroup of Q. Let $Q' = M_{Q'}N_{Q'}$ be the Levi decomposition with $M_{Q'} = \mathsf{w}_0^{-1}M_Q\mathsf{w}_0$. Let $N'_P = \mathsf{w}_P^{-1}M_{Q'}\mathsf{w}_P \cap N_P$ and $N''_P = \mathsf{w}_P^{-1}N_{Q'}\mathsf{w}_P \cap N_P$. Then $N_P = N'_PN''_P$ and $N'_P \subset \dot{\mathsf{w}}^{-1}Q\dot{\mathsf{w}}$. Since $\dot{\mathsf{w}}^{-1}Q\dot{\mathsf{w}} \cap M_P$

is a parabolic subgroup of M_P, we may choose a compact subgroup K of M_P such that $M_P = (\dot{\mathrm{w}}^{-1} Q \dot{\mathrm{w}} \cap M_P)K$. Then the integration (3.2) equals

$$C \int_K \int_{N''_P} \mathcal{B}_\sigma\big(\varphi(\dot{\mathrm{w}}u''ku), \check{\varphi}(\dot{\mathrm{w}}u''k)\big)\,\mathrm{d}u''\,\mathrm{d}k, \tag{3.3}$$

with some positive constant C and Haar measure $\mathrm{d}u''$ (resp. $\mathrm{d}k$) on N''_P (resp. K).

Introduce the following partial matrix coefficient:

$$\Phi^\flat_{\varphi,\check{\varphi}}(g) := C \int_{N''_P} \mathcal{B}_\sigma\big(\varphi(\dot{\mathrm{w}}u''g), \check{\varphi}(\dot{\mathrm{w}}u'')\big)\,\mathrm{d}u''.$$

Then

$$(3.3) = \int_K \Phi^\flat_{\pi(k)\varphi,\check{\pi}(k)\check{\varphi}}(kuk^{-1})\,\mathrm{d}k. \tag{3.4}$$

If we denote by Ω_k the support of the function $\Phi^\flat_{\pi(k)\varphi,\check{\pi}(k)\check{\varphi}}|_{N_P}$ which is locally constant in k, then the support of $\Phi_{\varphi,\check{\varphi}}|_{N_P}$ is contained in $\bigcup_{k\in K} k^{-1}\Omega_k k$. Therefore, we only need to show that $\Phi^\flat_{\varphi,\check{\varphi}}|_{N_P}$ has compact support for $\varphi \in (\mathrm{Ind}\,\sigma)^\circ_P$ and $\check{\varphi} \in (\mathrm{Ind}\,\check{\sigma})^\circ_P$ by noting that $\pi(k)\varphi \in (\mathrm{Ind}\,\sigma)^\circ_P$ and $\check{\pi}(k)\check{\varphi} \in (\mathrm{Ind}\,\check{\sigma})^\circ_P$ for every $k \in K$.

In the integral defining $\Phi^\flat_{\varphi,\check{\varphi}}$, u'' is confined to a compact open subset by the property of $\check{\varphi}$. Write $u''u = u_1u_2$ with $u_1 \in N'_P$ and $u_2 \in N''_P$. Then u_2 is confined to a compact open subset by the property of φ. Finally, we have

$$\mathcal{B}_\sigma\big(\varphi(\dot{\mathrm{w}}u_1u_2), \check{\varphi}(\dot{\mathrm{w}}u'')\big) = \mathcal{B}_\sigma\big(\sigma(\dot{\mathrm{w}}u_1\dot{\mathrm{w}}^{-1})\varphi(\dot{\mathrm{w}}u_2), \check{\varphi}(\dot{\mathrm{w}}u'')\big).$$

Since σ is supercuspidal, the support of its matrix coefficient is compact modulo center. Therefore, u_1 is confined to a compact open subset as well. Altogether, $\Phi^\flat_{\varphi,\check{\varphi}}|_{N_P}$ has compact support. □

3.2. Convergence and positivity. As in [23, Section 4], we choose minimal parabolic subgroups $P_{\alpha,\min}$ of G_α for $\alpha = 0, 1$. We also choose the minimal parabolic subgroup $P_{2,\min}$ of G_2 which stabilizes $\mathfrak{F}$ and such that $P_{2,\min} \cap G_1 = P_{1,\min}$. Denote by Ξ_α the Harish-Chandra's spherical function on G_α with respect to the pair $(P_{\alpha,\min}, \mathcal{K}_\alpha)$ for $\alpha = 0, 1, 2$. By definition, Ξ_α is the unique section in $\mathrm{Ind}^{G_\alpha}_{P_{\alpha,\min}} 1$ such that $\Xi_\alpha|_{\mathcal{K}_\alpha} \equiv 1$. Choose a height function $\|\cdot\|\colon G_2 \to \mathbb{R}_{\geq 1}$ and define $\varsigma(g) = \max\{1, \log\|g\|\}$ as in [46]. For $g, g' \in G_2$, we have

$$\varsigma(gg') \leq \varsigma(g) + \varsigma(g') \leq 2\varsigma(g)\varsigma(g').$$

Applying Proposition 3.1 to $G = G_2$, $P = \mathrm{P}_\mathfrak{F}$, $N_P = \mathrm{N}_\mathfrak{F}$ and $\psi = \psi^{-1}_{\mathfrak{F},w}$, we obtain a locally constant function $\mathcal{F}_{\psi_{\mathfrak{F},w}}(\Phi_{\varphi_2,\check{\varphi}_2})$. In this section, we show that the integral

$$\int_{G_0} \mathcal{F}_{\psi_{\mathfrak{F},w}}(\Phi_{\varphi_2,\check{\varphi}_2})(g_0)\Phi_{\varphi_0,\check{\varphi}_0}(g_0)\,\mathrm{d}g_0$$

is absolutely convergent for $\varphi_\alpha \in \pi_\alpha$ and $\check{\varphi}_\alpha \in \check{\pi}_\alpha$, assuming π_α are *tempered* for $\alpha = 0, 2$, which is the content of Theorem 2.1 (1).

Choose a compact open subgroup N of $N_P = \mathrm{N}_{\mathfrak{F}}$ as in Lemma 3.2 such that

$$\varphi_2' := (\varphi_2)_{N,\psi^{-1}_{\mathfrak{F},w}} \in (\mathrm{Ind}\,\sigma)^\circ_P \quad \text{and} \quad \check{\varphi}_2' = (\check{\varphi}_2)_{N,\psi_{\mathfrak{F},w}} \in (\mathrm{Ind}\,\check{\sigma})^\circ_P.$$

By definition,

$$\mathcal{F}_{\psi_{\mathfrak{F},w}}(\Phi_{\varphi_2,\check{\varphi}_2})(g_0) = \mathcal{F}_{\psi_{\mathfrak{F},w}}(\Phi_{\varphi_2',\check{\varphi}_2'})(g_0)$$

for $g_0 \in G_0$, since $\mathrm{N}_{\mathfrak{F}} \rtimes G_0$ is unimodular. Therefore, we may assume $\varphi_2 \in (\mathrm{Ind}\,\sigma)^\circ_P$ and $\check{\varphi}_2 \in (\mathrm{Ind}\,\check{\sigma})^\circ_P$.

Lemma 3.3. *Assume $\varphi_2 \in (\mathrm{Ind}\,\sigma)^\circ_P$ and $\check{\varphi}_2 \in (\mathrm{Ind}\,\check{\sigma})^\circ_P$. There exists a compact open subset Ω of $N_P = \mathrm{N}_{\mathfrak{F}}$ depending only on φ_2 and $\check{\varphi}_2$ such that, for every $g \in M_P$, the set $g \cdot \mathrm{supp}(\mathrm{L}(g^{-1})\Phi_{\varphi_2,\check{\varphi}_2}|_{N_P})$ is contained in $\Omega g \Omega$.*

Proof. For φ_2 and $\check{\varphi}_2$ as in the lemma, denote by $\Omega''_{\varphi_2,\check{\varphi}_2}$ the union of the support of functions $\varphi_2(\dot{\mathrm{w}}u)$ and $\check{\varphi}_2(\dot{\mathrm{w}}u)$ in $u \in N''_P$, which is a compact open subset of N''_P. Denote by $\Upsilon_{\varphi_2,\check{\varphi}_2}$ the union of the support of functions $\mathcal{B}_\sigma(\sigma(q)\varphi_2(\dot{\mathrm{w}}u), \check{\varphi}_2(\dot{\mathrm{w}}\check{u}))$ in $q \in M_Q \cap \dot{\mathrm{w}}P\dot{\mathrm{w}}^{-1}$ for all $u, \check{u} \in \Omega''_{\varphi_2,\check{\varphi}_2}$. The parabolic subgroup $M_Q \cap \dot{\mathrm{w}}P\dot{\mathrm{w}}^{-1}$ of M_Q has a Levi decomposition

$$(M_Q \cap \dot{\mathrm{w}}M_P\dot{\mathrm{w}}^{-1}) \cdot (M_Q \cap \dot{\mathrm{w}}N_P\dot{\mathrm{w}}^{-1}).$$

If we denote by

$$\alpha\colon M_Q \cap \dot{\mathrm{w}}P\dot{\mathrm{w}}^{-1} \to \dot{\mathrm{w}}^{-1}(M_Q \cap \dot{\mathrm{w}}N_P\dot{\mathrm{w}}^{-1})\dot{\mathrm{w}} = N'_P$$

the continuous map sending q to $\dot{\mathrm{w}}^{-1}u_q\dot{\mathrm{w}}$ where u_q is the unipotent part of q with respect to the above Levi decomposition, then

$$\Omega'_{\varphi_2,\check{\varphi}_2} := \alpha(\Upsilon_{\varphi_2,\check{\varphi}_2})$$

is a compact open subset of N'_P. We may choose a compact open subgroup Ω of N_P which contains

$$\Omega'_{\pi_2(k)\varphi_2,\check{\pi}_2(\check{k})\check{\varphi}_2}\Omega''_{\pi_2(k)\varphi_2,\check{\pi}_2(\check{k})\check{\varphi}_2}$$

for all $k, \check{k} \in K \subset M_P$. Moreover, we may assume Ω is normalized by K. We claim that Ω satisfies the requirement in the lemma.

Fixing $g \in M_P$, consider the function

$$\mathcal{B}_\sigma\big(\varphi_2(\dot{\mathrm{w}}u''kgu), \check{\varphi}_2(\dot{\mathrm{w}}u''k)\big) = \mathcal{B}_\sigma\big(\varphi_2(\dot{\mathrm{w}}u''kgk^{-1}kuk^{-1}k), \check{\varphi}_2(\dot{\mathrm{w}}u''k)\big) \tag{3.5}$$

in $u \in N_P$ where u'' varies in $\Omega \cap N''_P$ and k varies in K. Write $u_k = kuk^{-1}$, $kgk^{-1} = q_k\tilde{k}$ with $q_k \in \dot{\mathrm{w}}^{-1}Q\dot{\mathrm{w}} \cap M_P$ and $\tilde{k} \in K$. Then

$$\begin{aligned}(3.5) &= \mathcal{B}_\sigma\big(\pi_2(k)\varphi_2(\dot{\mathrm{w}}u''q_k\tilde{k}u_k), \check{\pi}_2(k)\check{\varphi}_2(\dot{\mathrm{w}}u'')\big)\\ &= \mathcal{B}_\sigma\big(\pi_2(\tilde{k}k)\varphi_2(\dot{\mathrm{w}}u''q_ku_{\tilde{k}k}), \check{\pi}_2(k)\check{\varphi}_2(\dot{\mathrm{w}}u'')\big).\end{aligned}$$

The above expression is non-zero only if $u''q_ku_{\tilde{k}k} \in q_k\Omega$. Then $u_{\tilde{k}k} = q_k^{-1}u_1q_ku_2$ for some $u_1, u_2 \in \Omega$, that is, $u = g^{-1}(k^{-1}u_1k)g(k^{-1}\tilde{k}^{-1}u_2\tilde{k}k)$. Since Ω is assumed to be normalized by K, we have $gu \in \Omega g\Omega$. The lemma follows by (3.4). □

For vectors $\varphi_\alpha \in \pi_\alpha$ and $\check{\varphi}_\alpha \in \check{\pi}_\alpha$ ($\alpha = 0, 2$), define

$$\alpha(\varphi_2, \check{\varphi}_2; \varphi_0, \check{\varphi}_0) = \int_{G_0} \mathcal{F}_{\psi_{\mathfrak{F}},w}(\Phi_{\varphi_2,\check{\varphi}_2})(g_0)\Phi_{\varphi_0,\check{\varphi}_0}(g_0)\,\mathrm{d}g_0,$$

and similarly for $\alpha^\natural(\varphi_2, \check{\varphi}_2; \varphi_0, \check{\varphi}_0)$, $\alpha(\varphi_2; \varphi_0)$ and $\alpha^\natural(\varphi_2; \varphi_0)$ as (2.3), (2.4) and (2.5).

Proof of Theorem 2.1 (1). We only need to prove the case in which $\Phi = \Phi_{\varphi_2,\check{\varphi}_2}$ for some $\varphi_2 \in (\operatorname{Ind}\sigma)^\circ_P$ and $\check{\varphi}_2 \in (\operatorname{Ind}\check{\sigma})^\circ_P$. To shorten notation, we write

$$N_g = g\,\mathrm{supp}\big(\mathrm{L}(g^{-1})\Phi_{\varphi_2,\check{\varphi}_2}|_{\mathrm{N}_{\mathfrak{F}}}\big)g^{-1}.$$

Let Ω be as in the above lemma and $C_\Omega = \max_\Omega \varsigma$. Recall that we have $\mathrm{P}_{\mathfrak{F}} = \mathrm{M}_{\mathfrak{F}}\mathrm{N}_{\mathfrak{F}}$ with $\mathrm{M}_{\mathfrak{F}} \simeq (\operatorname{Res}^E_F E^\times)^r \times G_1$. For $m \in G_1$, we have

$$\begin{aligned}
|\mathcal{F}_{\psi_{\mathfrak{F}},w}(\Phi)(m)| &\leq \int_{N_m} |\Phi(um)|\,\mathrm{d}u \\
&\leq \int_{N_m} A_0\,\Xi_2(um)\varsigma(um)^{B_0}\,\mathrm{d}u \\
&\leq A_0 \int_{N_m} \Xi_2(um)\big(2C_\Omega + \varsigma(m)\big)^{B_0}\,\mathrm{d}u \\
&\leq A_1\varsigma(m)^B \int_{N_m} \Xi_2(um)\,\mathrm{d}u
\end{aligned}$$

for some real constants $B_0, B \geq 0$, where the second inequality holds because π_2 is tempered. (In fact, we may take B_0 to be 0; see [45, Lemme VI.2.2].) By [45, Proposition II.4.5], there exist positive constants A_2 and B_1 such that

$$\int_{\mathrm{N}_{\mathfrak{F}}} \Xi_2(um)\varsigma(um)^{-B_1}\,\mathrm{d}u \leq A_2\,\Xi_1(m) \tag{3.6}$$

for all $m \in G_1$. In particular,

$$\begin{aligned}
\int_{N_m} \Xi_2(um)\,\mathrm{d}u &\leq \big(2C_\Omega + \varsigma(m)\big)^{B_1} \int_{N_m} \Xi_2(um)\varsigma(um)^{-B_1}\,\mathrm{d}u \\
&\leq \big(2C_\Omega + \varsigma(m)\big)^{B_1} \int_{\mathrm{N}_{\mathfrak{F}}} \Xi_2(um)\varsigma(um)^{-B_1}\,\mathrm{d}u \\
&\leq A_2\,\Xi_1(m)\big(2C_\Omega + \varsigma(m)\big)^{B_1} \\
&\leq A_3\,\Xi_1(m)\varsigma(m)^{B_2}.
\end{aligned}$$

Therefore,

$$|\mathcal{F}_{\psi_{\mathfrak{F}},w}(\Phi)(m)| \leq A\,\Xi_1(m)\varsigma(m)^B$$

for all $m \in G_1$. The rest follows from the proof on [23, p. 1388] when $E = F$ and from [19, Proposition 2.1] when $E \neq F$. □

Now we prove Theorem 2.1 (2) in the non-archimedean case, that is,

$$I := \alpha(\varphi_2; \varphi_0) = \int_{G_0} \mathcal{F}_{\psi_{\mathfrak{F}},w}(\Phi_{\varphi_2,\bar{\varphi}_2})(g_0)\Phi_{\varphi_0,\bar{\varphi}_0}(g_0)\,\mathrm{d}g_0 \geq 0$$

for every vector φ_2 (resp. φ_0) in the representation π_2 (resp. π_0). As before, we may assume that $\varphi_2 \in (\mathrm{Ind}\,\sigma)^{\circ}_{\mathrm{P}_{\mathfrak{F}}}$. Then

$$I = \int_{G_0} \int_{\mathrm{N}_{\mathfrak{F}}} \Phi_{\varphi_2,\bar{\varphi}_2}(g_0 u)\Phi_{\varphi_0,\bar{\varphi}_0}(g_0)\psi_{\mathfrak{F},w}(u)^{-1}\,\mathrm{d}u\,\mathrm{d}g_0.$$

Choose a sequence of compact open subgroups $N_1 \subset N_2 \subset \cdots$ of $\mathrm{N}_{\mathfrak{F}}$ such that $\bigcup N_i = \mathrm{N}_{\mathfrak{F}}$. For each $i \geq 1$, let

$$G^i := \{g_0 \in G_0 \mid \Phi_{\varphi_2,\bar{\varphi}_2}(g_0 u) = 0 \text{ for } u \notin N_i\},$$

which is an open subset of G_0. By the assumption on φ_2, we have $\bigcup G^i = G_0$. Put

$$I_i = \int_{G_0} \int_{N_i} \Phi_{\varphi_2,\bar{\varphi}_2}(g_0 u)\Phi_{\varphi_0,\bar{\varphi}_0}(g_0)\psi_{\mathfrak{F},w}(u)^{-1}\,\mathrm{d}u\,\mathrm{d}g_0.$$

Lemma 3.4. *The sequence $(I_i)_{i\geq 1}$ converges to I.*

Proof. In the above proof of Theorem 2.1 (1), we have actually shown that

$$\int_{G_0} \int_{\mathrm{N}_{\mathfrak{F}}} \left|\Phi_{\varphi_2,\bar{\varphi}_2}(g_0 u)\Phi_{\varphi_0,\bar{\varphi}_0}(g_0)\right|\mathrm{d}u\,\mathrm{d}g_0$$

is convergent. By definition,

$$\begin{aligned} |I - I_i| &= \left|\int_{G_0 - G^i} \int_{\mathrm{N}_{\mathfrak{F}}\backslash N_i} \Phi_{\varphi_2,\bar{\varphi}_2}(g_0 u)\Phi_{\varphi_0,\bar{\varphi}_0}(g_0)\psi_{\mathfrak{F}}(u)^{-1}\,\mathrm{d}u\,\mathrm{d}g_0\right| \\ &\leq \int_{G_0 - G^i} \int_{\mathrm{N}_{\mathfrak{F}}} \left|\Phi_{\varphi_2,\bar{\varphi}_2}(g_0 u)\Phi_{\varphi_0,\bar{\varphi}_0}(g_0)\right|\mathrm{d}u\,\mathrm{d}g_0, \end{aligned}$$

which converges to 0. □

Proof of Theorem 2.1 (2) *for* v *non-archimedean.* By the previous lemma, we only need to show that $I_i \geq 0$ for every $i \geq 1$. We closely follow the argument in [23, Section 4]. We have

$$\int_{N_i} \Phi_{\varphi_2,\bar{\varphi}_2}(g_0 u)\psi_{\mathfrak{F}}(u)^{-1}\,\mathrm{d}u = \mathrm{Vol}(N_i)\Phi_{(\varphi_2)_{N_i,\psi_{\mathfrak{F}}^{-1}},(\bar{\varphi}_2)_{N_i,\psi_{\mathfrak{F}}}}(g_0),$$

where the notation $(\varphi_2)_{N_i,\psi_{\mathfrak{F},w}^{-1}}$ and $(\bar{\varphi}_2)_{N_i,\psi_{\mathfrak{F},w}}$ are introduced in Lemma 3.2. Note that

$$(\bar{\varphi}_2)_{N_i,\psi_{\mathfrak{F},w}} = \overline{(\varphi_2)_{N_i,\psi_{\mathfrak{F},w}^{-1}}}.$$

Therefore, replacing φ_2 by $(\varphi_2)_{N_i,\psi_{\mathfrak{F},w}^{-1}}$, we only need to show that

$$\int_{G_0} \Phi_{\varphi_2,\bar{\varphi}_2}(g_0)\Phi_{\varphi_0,\bar{\varphi}_0}(g_0) \geq 0.$$

Let $\Phi_2 = \Phi_{\varphi_2,\bar{\varphi}_2}$ and $\Phi_0 = \Phi_{\varphi_0,\bar{\varphi}_0}$. Then $\Phi_2 \otimes \Phi_0$ is a positive definite function in $L^{2+\varepsilon}(G_2 \times G_0)$ since π_2 and π_0 are assumed to be tempered. To apply [20, Theorem 2.1] as argued in [23], we only need to show that

$$\int_{G_0} \Xi_2(g_0)\Xi_0(g_0)\,\mathrm{d}g_0 < \infty.$$

Since $\Xi_2(m)$ shares the same upper bound with $\Xi_1(m)$ for $m \in M_{1,\min}^+$, the above convergence follows from the proof of [23, Proposition 1.1, p. 1388]. Although only orthogonal groups $(E = F)$ are considered there, the argument on [23, p. 1388] works for unitary groups $(E \neq F)$ as well. □

When π_2 is square-integrable, the Fourier coefficient is already absolutely convergent. The following proposition shows the compatibility of the two different definitions.

Proposition 3.5. *Let π_2 be square-integrable and π_0 be tempered. Then for every matrix coefficient Φ_α of π_α $(\alpha = 0, 2)$, the integral*

$$\mathcal{F}_{\psi_{\mathfrak{F},w}}(\Phi_2)(g_0) := \int_{\mathrm{N}_{\mathfrak{F}}} \Phi_2(g_0 u)\psi_{\mathfrak{F},w}(u)^{-1}\,\mathrm{d}u$$

is absolutely convergent for every $g_0 \in G_0$; and the integral

$$\int_{G_0}\int_{\mathrm{N}_{\mathfrak{F}}} \Phi_2(g_0 u)\Phi_0(g_0)\psi_{\mathfrak{F},w}(u)^{-1}\,\mathrm{d}u\,\mathrm{d}g_0$$

is absolutely convergent as well. Moreover,

$$\int_{\mathrm{N}_{\mathfrak{F}}} \Phi_2(g_0 u)\psi_{\mathfrak{F},w}(u)^{-1}\,\mathrm{d}u = \int_{\mathrm{N}_{\mathfrak{F}}}^{\mathrm{st}} \Phi_2(g_0 u)\psi_{\mathfrak{F},w}(u)^{-1}\,\mathrm{d}u.$$

Proof. Since π_2 is square-integrable, the matrix coefficient Φ_2 is in the Harish-Chandra Schwartz space of the derived group of G_2 by [45, Corollaire III.1.2]. By (3.6), we have the estimate

$$\int_{\mathrm{N}_{\mathfrak{F}}} |\Phi_2(g_0 u)|\,\mathrm{d}u \le A\,\Xi_1(g_0)\varsigma(g_0)^B$$

for every $g_0 \in G_0$ and the rest of first statement follows from the proof on [23, p. 1388] when $E = F$ and from [19, Proposition 2.1] when $E \neq F$. The last statement follows from the fact that for a function $f \in \mathrm{L}^1(N_P)$ which also has a stable integral, we have

$$\int_{N_P}^{\mathrm{st}} f(u)\,\mathrm{d}u = \int_{N_P} f(u)\,\mathrm{d}u.$$ □

3.3. Unramified calculation. In this subsection, we prove Theorem 2.2. We assume the six conditions prior to that theorem. In particular, G_α are all quasi-split for $\alpha = 0, 1, 2$. Let $B_\alpha = T_\alpha N_\alpha$ be a Borel subgroup of G_α, where T_α is a maximal torus of G_α and N_α is the unipotent radical of B_α. We assume that $T_0 \subset T_1 \subset T_2$ and $N_0 \subset N_1 \subset N_2$.

If $n_0 + [E : F] \le 2$, that is, the group G_0 is trivial, Theorem 2.2 was proved by Lapid–Mao [33, Proposition 2.14]. If $r = 0$, that is, $G_1 = G_2$, Theorem 2.2 was proved by Ichino–Ikeda [23, Theorem 1.2] for $[E : F] = 1$ and by R. N. Harris [19, Theorem 2.12] for $[E : F] = 2$. In what follows, we will assume that G_0 is not trivial and $r > 0$.

Let $\pi_2 = \mathrm{I}(\Xi) = \mathrm{Ind}_{B_2}^{G_2}(\Xi)$ (resp. $\pi_0 = \mathrm{I}(\xi) = \mathrm{Ind}_{B_0}^{G_0}(\xi)$) be an unramified principal series of G_2 (resp. G_0). Here, Ξ (resp. ξ) is an unramified quasi-character of T_2 (resp. T_0). Let Φ_Ξ (resp. Φ_ξ) be the spherical matrix coefficient of π_2 (resp. π_0) such that $\Phi_\Xi(1) = 1$ (resp.

$\Phi_\xi(1) = 1$). We study the integral

$$I(g_2; \Xi, \xi) = \int_{G_0} \mathcal{F}_{\psi_{\mathfrak{F}}}(\Phi_\Xi)(g_2^{-1} g_0)\Phi_\xi(g_0)\,\mathrm{d}g_0,$$

where we write $\psi_{\mathfrak{F}}$ instead of $\psi_{\mathfrak{F},w}$ for short. We assume that both Ξ and ξ are sufficiently close to the unitary axis, and hence the above integral is absolutely convergent by (the proof of) Theorem 2.1 (1).

Let $f_\Xi \in \mathrm{I}(\Xi)$ (resp. $f_\xi \in \mathrm{I}(\xi)$) be the spherical vector such that $f_\Xi(1) = 1$ (resp. $f_\xi(1) = 1$). Then we have

$$\Phi_\Xi(g_2) = \int_{\mathcal{K}_2} f_\Xi(k_2 g_2)\,\mathrm{d}k_2, \quad \Phi_\xi(g_0) = \int_{\mathcal{K}_0} f_\xi(k_0 g_0)\,\mathrm{d}k_0.$$

Let $\mathcal{H}_\alpha = \mathcal{H}(\mathcal{K}_\alpha \backslash G_\alpha / \mathcal{K}_\alpha)$ be the spherical Hecke algebra of G_α. By the Satake isomorphism, we have homomorphisms

$$\omega_2 \colon \mathcal{H}_2 \to \mathbb{C}, \quad \omega_0 \colon \mathcal{H}_0 \to \mathbb{C}$$

of algebras associated to the unramified representations π_2 and π_0, respectively. Recall that a smooth function S on G_2 is called a *Whittaker–Shintani function* attached to (π_2, π_0) (see [29, Section 0.1]) if the following conditions are satisfied:

- $\mathrm{L}(uk_0)\mathrm{R}(k_2)S = \psi_{\mathfrak{F}}(u)S$ for any $k_\alpha \in \mathcal{K}_\alpha$ and $u \in \mathrm{N}_{\mathfrak{F}}$;
- $\mathrm{L}(\varphi_0)\mathrm{R}(\varphi_2)S = \omega_0(\varphi_0)\omega_2(\varphi_2)S$ for any $\varphi_\alpha \in \mathcal{H}_\alpha$.

Then $I(-; \Xi, \xi)$ is a Whittaker–Shintani function attached to $(\check{\pi}_2, \check{\pi}_0)$.

Applying Lemma 3.2 to $Q = B_2$, $\sigma = \Xi$ and $P = \mathrm{P}_{\mathfrak{F}}$, we know that for fixed elements $g_2, \check{g}_2 \in G_2$, there exists a compact open subgroup $N_{g_2,\check{g}_2}$ of $\mathrm{N}_{\mathfrak{F}}$ such that

$$\begin{aligned} f_\Xi^{g_2,\check{g}_2} &:= \mathrm{R}(\psi_{\mathfrak{F}}^{-1}\delta_{N_{g_2,\check{g}_2'}})(\pi_2(g_2) f_\Xi) \in (\mathrm{Ind}_{B_2}^{G_2} \Xi)^\circ_{\mathrm{P}_{\mathfrak{F}}}, \\ f_{\Xi^{-1}}^{g_2,\check{g}_2} &:= \mathrm{R}(\psi_{\mathfrak{F}}\delta_{N_{g_2,\check{g}_2}})(\check{\pi}_2(\check{g}_2) f_{\Xi^{-1}}) \in (\mathrm{Ind}_{B_2}^{G_2} \Xi^{-1})^\circ_{\mathrm{P}_{\mathfrak{F}}}. \end{aligned}$$

Lemma 3.6. *For $g_2 \in G_2$ and $g_0 \in G_0$, we have*

$$\mathcal{F}_{\psi_{\mathfrak{F}}}(\Phi_\Xi)(g_2^{-1} g_0) = w_{\mathfrak{F}}^{-1} \int_{\mathcal{K}_1} \int^{\mathrm{st}}_{\mathrm{N}_{\mathfrak{F}}} \mathcal{F}_{\psi_{\mathfrak{F}}}(f_\Xi)(k_1 \dot{\mathrm{w}} u g_0)(\check{\pi}_2(g_2) f_{\Xi^{-1}})(k_1 \dot{\mathrm{w}} u)\,\mathrm{d}u\,\mathrm{d}k_1,$$

where $\dot{\mathrm{w}}$ is the longest Weyl element in $B_2 \backslash G_2 / \mathrm{P}_{\mathfrak{F}}$, and

$$w_{\mathfrak{F}} = \int_{\mathrm{N}_{\mathfrak{F}}} f_\Xi(\dot{\mathrm{w}} u) f_{\Xi^{-1}}(\dot{\mathrm{w}} u)\,\mathrm{d}u = \frac{\Delta_{T_2}}{\Delta_{G_2}} \Big(\frac{\Delta_{T_1}}{\Delta_{G_1}}\Big)^{-1}.$$

Proof. We have, by definition,

$$\begin{aligned} \mathcal{F}_{\psi_{\mathfrak{F}}}(\Phi_\Xi)(g_2^{-1} g_0) &= \int^{\mathrm{st}}_{\mathrm{N}_{\mathfrak{F}}} \mathcal{B}_{\pi_2}\big(\pi_2(g_2^{-1} g_0 u') f_\Xi, f_{\Xi^{-1}}\big)\psi_{\mathfrak{F}}(u')^{-1}\,\mathrm{d}u' \\ &= \int^{\mathrm{st}}_{\mathrm{N}_{\mathfrak{F}}} \mathcal{B}_{\pi_2}\big(\pi_2(g_0 u') f_\Xi, \check{\pi}_2(g_2) f_{\Xi^{-1}}\big)\psi_{\mathfrak{F}}(u')^{-1}\,\mathrm{d}u' \end{aligned} \tag{3.7}$$

$$= \int_{\mathrm{N}_{\mathfrak{F}}}^{\mathrm{st}} \mathcal{B}_{\pi_2}\big(\pi_2(u')\pi_2(g_0) f_{\Xi}, \check{\pi}_2(g_2) f_{\Xi^{-1}}\big)\psi_{\mathfrak{F}}(u')^{-1}\,\mathrm{d}u'$$
$$= \int_{\mathrm{N}_{\mathfrak{F}}} \mathcal{B}_{\pi_2}\big(\pi_2(u') f_{\Xi}^{g_0,g_2}, f_{\Xi^{-1}}^{g_0,g_2}\big)\psi_{\mathfrak{F}}(u')^{-1}\,\mathrm{d}u'.$$

We use the following realization of $\mathcal{B}_{\pi_2}$:

$$\mathcal{B}_{\pi_2}(f, \check{f}) = w_{\mathfrak{F}}^{-1} \int_{\mathcal{K}_1} \int_{\mathrm{N}_{\mathfrak{F}}} f(k_1 \dot{\mathrm{w}} u)\check{f}(k_1 \dot{\mathrm{w}} u)\,\mathrm{d}u\,\mathrm{d}k_1.$$

Then

$$(3.7) = w_{\mathfrak{F}}^{-1} \int_{\mathrm{N}_{\mathfrak{F}}} \int_{\mathcal{K}_1} \int_{\mathrm{N}_{\mathfrak{F}}} f_{\Xi}^{g_0,g_2}(k_1 \dot{\mathrm{w}} u u') f_{\Xi^{-1}}^{g_0,g_2}(k_1 \dot{\mathrm{w}} u)\psi_{\mathfrak{F}}(u')^{-1}\,\mathrm{d}u\,\mathrm{d}k_1\,\mathrm{d}u',$$

where the integration is compactly supported. Then it equals

$$w_{\mathfrak{F}}^{-1} \int_{\mathcal{K}_1} \int_{\mathrm{N}_{\mathfrak{F}}} \mathcal{F}_{\psi_{\mathfrak{F}}}(\pi_2(g_0) f_{\Xi})(k_1 \dot{\mathrm{w}} u) f_{\Xi^{-1}}^{g_0,g_2}(k_1 \dot{\mathrm{w}} u)\,\mathrm{d}u\,\mathrm{d}k_1$$
$$= w_{\mathfrak{F}}^{-1} \int_{\mathcal{K}_1} \int_{\mathrm{N}_{\mathfrak{F}}}^{\mathrm{st}} \mathcal{F}_{\psi_{\mathfrak{F}}}(\pi_2(g_0) f_{\Xi})(k_1 \dot{\mathrm{w}} u)(\check{\pi}_2(g_2) f_{\Xi^{-1}})(k_1 \dot{\mathrm{w}} u)\,\mathrm{d}u\,\mathrm{d}k_1$$
$$= w_{\mathfrak{F}}^{-1} \int_{\mathcal{K}_1} \int_{\mathrm{N}_{\mathfrak{F}}}^{\mathrm{st}} \mathcal{F}_{\psi_{\mathfrak{F}}}(f_{\Xi})(k_1 \dot{\mathrm{w}} u g_0)(\check{\pi}_2(g_2) f_{\Xi^{-1}})(k_1 \dot{\mathrm{w}} u)\,\mathrm{d}u\,\mathrm{d}k_1.$$

The computation for $w_{\mathfrak{F}}$ is well known. □

Recall the Weyl element

$$\dot{\mathrm{w}} = \begin{bmatrix} & & \mathbf{w}_r \\ & \mathbf{1}_{n_2-2r} & \\ \mathbf{w}_r & & \end{bmatrix} \in G_2.$$

Put $\eta = \dot{\mathrm{w}}\tilde{\eta}$, where $\tilde{\eta} \in G_1 \subset G_2$ is the deliberately chosen representative of the dense orbit in $B_1 \backslash G_1 / B_0$ in the case $r = 0$, which is defined in [23, Lemma 5.3] when $E = F$ and similarly defined when $E \neq F$.

Let $Y_{\Xi,\xi}$ be the unique function on G_2 satisfying the following conditions:

(1) $Y_{\Xi,\xi}(b_2 g_2 b_0 u) = (\Xi^{-1}\delta_{B_2}^{1/2})(b_2)(\xi\delta_{B_0}^{-1/2})(b_0)\psi_{\mathfrak{F}}(u) Y_{\Xi,\xi}(g_2)$ for every $b_2 \in B_2$, $b_0 \in B_0$ and $u \in \mathrm{N}_{\mathfrak{F}}$;

(2) $Y_{\Xi,\xi}(\eta) = 1$;

(3) $Y_{\Xi,\xi}(g_2) = 0$ if $g_2 \notin B_2\eta(B_1 \ltimes \mathrm{N}_{\mathfrak{F}})$.

Put

$$T_{\Xi,\xi}(g_2) = \begin{cases} \int_{G_0} \mathcal{F}_{\psi_{\mathfrak{F}}}(f_{\Xi})(g_2 g_0) f_{\xi}(g_0)\,\mathrm{d}g_0 & \text{if } g_2 \in B_2\eta(B_1 \ltimes \mathrm{N}_{\mathfrak{F}}), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $T_{\Xi,\xi}(g_2) = T_{\Xi,\xi}(\eta) Y_{\Xi^{-1},\xi^{-1}}(g_2)$. Let us denote by $\tilde{\Xi}$ the restriction of Ξ to T_1.

Lemma 3.7. *The integral defining $T_{\Xi,\xi}(g_2)$ is absolutely convergent when $(\tilde{\Xi},\xi)$ are sufficiently close to the unitary axis.*

We will also write $\zeta(\Xi,\xi)$ for $T_{\Xi,\xi}(\eta)$.

Proof. We may assume $g_2=\eta$. By Lemma 3.3, we can choose a compact open subset N of $\mathrm{N}_{\mathfrak{F}}$ such that, for *every* $g_0\in G_0$,

$$\mathscr{F}_{\psi_{\mathfrak{F}}}(f_\Xi)(\eta g_0)=\int_N f_\Xi(\eta g_0 u)\psi_{\mathfrak{F}}(u)^{-1}\,\mathrm{d}u.$$

Let us write $f_{\Xi,N}=\mathrm{R}(\psi_{\mathfrak{F}}^{-1}N)f_\Xi$ which is a section in $\mathrm{I}(\Xi)$. There exists a positive constant C such that $|f_{\Xi,N}(g_2)|\le Cf_{|\Xi|}(g_2)$ for every $g_2\in G_2$. Therefore,

$$\begin{aligned}\int_{G_0}\big|\mathscr{F}_{\psi_{\mathfrak{F}}}(f_\Xi)(\eta g_0)f_\xi(g_0)\big|\,\mathrm{d}g_0&\le C\int_{G_0}f_{|\Xi|}(\eta g_0)f_{|\xi|}(g_0)\,\mathrm{d}g_0\\&=C\int_{G_0}f_{|\tilde{\Xi}|}(\tilde{\eta}g_0)f_{|\xi|}(g_0)\,\mathrm{d}g_0.\end{aligned}$$

Then the lemma follows from the case $r=0$; see [23, p. 1390]. □

By Lemma 3.6, we have

$$\begin{aligned}(3.8)\qquad I(g_2;\Xi,\xi)&=w_{\mathfrak{F}}^{-1}\int_{\mathcal{K}_0}\int_{\mathcal{K}_1}\int_{\mathrm{N}_{\mathfrak{F}}}^{\mathrm{st}}T_{\Xi^{-1},\xi^{-1}}(k_1\dot{\mathrm{w}}uk_0)(\check{\pi}_2(g_2)f_{\Xi^{-1}})\\&\qquad\times(k_1\dot{\mathrm{w}}u)\psi_{\mathfrak{F}}(u)\,\mathrm{d}u\,\mathrm{d}k_1\,\mathrm{d}k_0\\&=w_{\mathfrak{F}}^{-1}T_{\Xi,\xi}(\eta)\int_{\mathcal{K}_0}\int_{\mathcal{K}_1}\int_{\mathrm{N}_{\mathfrak{F}}}^{\mathrm{st}}Y_{\Xi^{-1},\xi^{-1}}(k_1\dot{\mathrm{w}}uk_0)(\check{\pi}_2(g_2)f_{\Xi^{-1}})\\&\qquad\times(k_1\dot{\mathrm{w}}u)\psi_{\mathfrak{F}}(u)\,\mathrm{d}u\,\mathrm{d}k_1\,\mathrm{d}k_0\\&=w_{\mathfrak{F}}^{-1}T_{\Xi,\xi}(\eta)\int_{\mathcal{K}_0}\int_{\mathcal{K}_1}Y_{\Xi^{-1},\xi^{-1}}(k_1\dot{\mathrm{w}}k_0)\mathscr{F}_{\psi_{\mathfrak{F}}^{-1}}(\check{\pi}_2(g_2)f_{\Xi^{-1}})\\&\qquad\times(k_1\dot{\mathrm{w}})\,\mathrm{d}k_1\,\mathrm{d}k_0,\end{aligned}$$

where the double integral is absolutely convergent when $(\tilde{\Xi},\xi)$ are sufficiently close to the unitary axis by the case $r=0$. Put

$$S'_{\Xi^{-1},\xi^{-1}}(g_2)=\int_{\mathcal{K}_0}\int_{\mathcal{K}_1}Y_{\Xi^{-1},\xi^{-1}}(k_1\dot{\mathrm{w}}k_0)\mathscr{F}_{\psi_{\mathfrak{F}}^{-1}}(\check{\pi}_2(g_2)f_{\Xi^{-1}})(k_1\dot{\mathrm{w}})\,\mathrm{d}k_1\,\mathrm{d}k_0.$$

The advantage of this Whittaker–Shintani function over the standard one is that it is easier to obtain a wider region of convergence for (Ξ,ξ), which is crucial for our computation of $I(1;\Xi,\xi)$.

The following lemma compares the Whittaker–Shintani function defined above with the standard one $S_{\Xi,\xi}$ in [29] which we recall as follows. Define a linear functional $l_{\Xi,\xi}$ in $\mathrm{Hom}_{G_0\ltimes\mathrm{N}_{\mathfrak{F}}}(\mathrm{I}(\Xi),\mathrm{I}(\xi)\otimes\psi_{\mathfrak{F}}^{-1})$ by

$$l_{\Xi,\xi}(f)(g_0)=\int_{G_1}f'(g_2g_0)Y_{\Xi,\xi}(g_2)\,\mathrm{d}g_2$$

for $f \in \mathrm{I}(\Xi)$, where f' is any function in $\mathrm{pr}_2^{-1}(f)$ and $\mathrm{pr}_2\colon \mathcal{C}_c^\infty(G_2) \to \mathrm{I}(\Xi)$ is defined similarly as pr_1. Put

$$S_{\Xi,\xi}(g_2) = \mathcal{B}_{\pi_0}\big(f_\xi, l_{\Xi,\xi}(\pi_2(g_2) f_\Xi)\big).$$

Lemma 3.8. *The function $S'_{\Xi,\xi}$ equals (the analytic continuation of) the standard Whittaker–Shintani function $S_{\Xi,\xi}$ multiplied by $w_{\mathfrak{F}}$.*

Proof. We only need to show the equality over the region where both are absolutely convergent, for example, when $Y_{\Xi,\xi}$ is continuous.

For $f \in \mathrm{I}(\Xi)$, consider the stable Fourier coefficient

$$\mathcal{F}'(f)(g_2) := \mathcal{F}_{\psi_{\mathfrak{F}}^{-1}}(f)(g_2\dot{\mathrm{w}}) = \int_{\mathrm{N}_{\mathfrak{F}}}^{\mathrm{st}} f(g_2\dot{\mathrm{w}}u)\psi_{\mathfrak{F}}(u)\,\mathrm{d}u,$$

which is a smooth function when restricted to G_1 and hence a vector in $\mathrm{I}(\tilde{\Xi})$. Let

$$\mathrm{pr}_1\colon \mathcal{C}_c^\infty(G_1) \to \mathrm{I}(\tilde{\Xi})$$

be the map given by

$$\mathrm{pr}_1(\tilde{f})(g_1) = \int_{B_1} (\tilde{\Xi}^{-1}\delta_{B_1}^{1/2})(b_1)\tilde{f}(b_1 g_1)\,\mathrm{d}b_1.$$

Define

$$l'_{\Xi,\xi}(f)(g_0) = \int_{G_1} \tilde{f}(g_1 g_0) Y_{\Xi,\xi}(g_1\dot{\mathrm{w}})\,\mathrm{d}g_1,$$

where $\tilde{f}$ is any function in $\mathrm{pr}_1^{-1}(\mathcal{W}'(f)|_{G_1})$. It is not hard to check that $l'_{\Xi,\xi}$ defines a functional in $\mathrm{Hom}_{G_0\ltimes \mathrm{N}_{\mathfrak{F}}}(\mathrm{I}(\Xi), \mathrm{I}(\xi)\otimes\psi_{\mathfrak{F}}^{-1})$. Put

$$S'_{\Xi,\xi}(g_2) = \mathcal{B}_{\pi_0}\big(f_\xi, l'_{\Xi,\xi}(\pi_2(g_2) f_\Xi)\big).$$

In particular, we may choose a function $\tilde{f}$ supported on $\mathcal{K}_1$ with $\mathrm{pr}_1(\tilde{f}) = \mathcal{F}'(\pi_2(g_2) f_\Xi)|_{G_1}$. Then

$$\begin{aligned}
S'_{\Xi,\xi}(g_2) &= \int_{\mathcal{K}_0} f_\xi(k_0)\int_{G_1} \tilde{f}(g_1 k_0) Y_{\Xi,\xi}(g_1\dot{\mathrm{w}})\,\mathrm{d}g_1\,\mathrm{d}k_0\\
&= \int_{\mathcal{K}_0}\int_{\mathcal{K}_1} \tilde{f}(k_1 k_0) Y_{\Xi,\xi}(k_1\dot{\mathrm{w}})\,\mathrm{d}k_1\,\mathrm{d}k_0\\
&= \int_{\mathcal{K}_0}\int_{\mathcal{K}_1} \mathcal{F}'(\pi_2(g_2) f_\Xi)(k_1 k_0) Y_{\Xi,\xi}(k_1\dot{\mathrm{w}})\,\mathrm{d}k_1\,\mathrm{d}k_0\\
&= \int_{\mathcal{K}_0}\int_{\mathcal{K}_1} \mathcal{F}_{\psi_{\mathfrak{F}}^{-1}}(\pi_2(g_2) f_\Xi)(k_1\dot{\mathrm{w}}) Y_{\Xi,\xi}(k_1\dot{\mathrm{w}}k_0)\,\mathrm{d}k_1\,\mathrm{d}k_0.
\end{aligned}$$

Therefore, we only need to show that $l'_{\Xi,\xi} = w_{\mathfrak{F}}\cdot l_{\Xi,\xi}$.

Let us first introduce more groups. For $\alpha = 0, 1, 2$, denote the opposite Borel subgroup by $\bar{B}_\alpha = T_\alpha \bar{N}_\alpha$. Put

$$T_\alpha^{(0)} = T_\alpha \cap \mathcal{K}_\alpha, \quad N_\alpha^{(0)} = N_\alpha \cap \mathcal{K}_\alpha, \quad \bar{N}_\alpha^{(0)} = \bar{N}_\alpha \cap \mathcal{K}_\alpha.$$

Choose a longest Weyl element $\mathsf{w}_{\alpha,\mathrm{long}} \in \mathcal{K}_\alpha$ of G_α such that $\mathsf{w}_{2,\mathrm{long}} = \dot{\mathsf{w}}\mathsf{w}_{1,\mathrm{long}}$. Choose an Iwahori subgroup $\mathcal{K}_\alpha^{(1)} \subset \mathcal{K}_\alpha$ such that $N_\alpha^{(0)} \subset \mathcal{K}_\alpha^{(1)}$. Put

$$\bar{N}_\alpha^{(1)} = \bar{N}_\alpha \cap \mathcal{K}_\alpha^{(1)}, \quad N_\alpha^{(1)} = \mathsf{w}_{\alpha,\mathrm{long}}^{-1} \bar{N}_\alpha^{(1)} \mathsf{w}_{\alpha,\mathrm{long}}.$$

Then we have an Iwahori decomposition $\mathcal{K}_\alpha^{(1)} = \bar{N}_\alpha^{(1)} T_\alpha^{(0)} N_\alpha^{(0)}$. Finally, let

$$\mathrm{N}_{\mathfrak{F}}^{(0)} = \mathrm{N}_{\mathfrak{F}} \cap \mathcal{K}_2, \quad \mathrm{N}_{\mathfrak{F}}^{(1)} = \mathrm{N}_{\mathfrak{F}} \cap N_2^{(1)}.$$

Since the dimension of $\mathrm{Hom}_{G_0 \ltimes \mathrm{N}_{\mathfrak{F}}}(\mathrm{I}(\Xi), \mathrm{I}(\xi) \otimes \psi_{\mathfrak{F}}^{-1})$ is 1, we only need to choose one function $f \in \mathrm{I}(\Xi)$ such that $l_{\Xi,\xi}(f)(1) \neq 0$ and compute the ratio $l'_{\Xi,\xi}(f)(1)/l_{\Xi,\xi}(f)(1)$. Take $f = \mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta})$. By the following lemma, we have

$$\frac{l'_{\Xi,\xi}(\mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta}))(1)}{l_{\Xi,\xi}(\mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta}))(1)} = \frac{\Delta_{T_2}\Delta_{G_1}}{\Delta_{T_1}\Delta_{G_2}} = w_{\mathfrak{F}}. \qquad \square$$

Lemma 3.9. *We have*

$$l'_{\Xi,\xi}\big(\mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta})\big)(1) = \mathrm{Vol}(\mathcal{K}_1^{(1)})\big[\mathrm{N}_{\mathfrak{F}}^{(0)} : \mathrm{N}_{\mathfrak{F}}^{(1)}\big]^{-1},$$
$$l_{\Xi,\xi}\big(\mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta})\big)(1) = \mathrm{Vol}(\mathcal{K}_2^{(1)}).$$

Proof. We first prove the second equation. By definition,

$$l_{\Xi,\xi}\big(\mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta})\big)(1) = \int_{\mathcal{K}_2^{(1)}} Y_{\Xi,\xi}(k_2\eta)\,\mathrm{d}k_2 = \mathrm{Vol}(\mathcal{K}_2^{(1)}),$$

since by our choice of η,

$$\mathcal{K}_2^{(1)}\eta\mathcal{K}_0^{(1)} \subset T_2^{(0)} N_2^{(0)} \eta T_0^{(0)} N_0^{(0)} \mathrm{N}_{\mathfrak{F}}^{(1)}.$$

By the same inclusion relation, we have

$$\mathcal{F}'\big(\mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta})\big)(g_2) = \mathrm{Vol}(\mathrm{N}_{\mathfrak{F}}^{(1)}) \int_{B_2} (\Xi^{-1}\delta_{B_2}^{1/2})(b_2)\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta}(b_2 g_2 \dot{\mathsf{w}})\,\mathrm{d}b_2. \tag{3.9}$$

Therefore,

$$\begin{aligned} l'_{\Xi,\xi}\big(\mathrm{pr}_2(\mathrm{ch}_{\mathcal{K}_2^{(1)}\eta})\big)(1) &= \mathrm{Vol}(\mathrm{N}_{\mathfrak{F}}^{(1)}) \int_{\mathcal{K}_1} \mathrm{ch}_{\mathcal{K}_2^{(1)}\eta}(k_1\dot{\mathsf{w}}) Y_{\Xi,\xi}(k_1\dot{\mathsf{w}})\,\mathrm{d}k_1 \\ &= \mathrm{Vol}(\mathrm{N}_{\mathfrak{F}}^{(1)}) \int_{\mathcal{K}_1^{(1)}} Y_{\tilde{\Xi},\xi}(k_1\tilde{\eta})\,\mathrm{d}k_1 \\ &= \mathrm{Vol}(\mathcal{K}_1^{(1)})\big[\mathrm{N}_{\mathfrak{F}}^{(0)} : \mathrm{N}_{\mathfrak{F}}^{(1)}\big]^{-1}, \end{aligned}$$

where the equality

$$\int_{\mathcal{K}_1^{(1)}} Y_{\tilde{\Xi},\xi}(k_1\tilde{\eta})\,\mathrm{d}k_1 = \mathrm{Vol}(\mathcal{K}_1^{(1)})$$

is obtained in a similar argument for the second equation of the lemma. □

Proof of Theorem 2.2. The assertion follows from (3.8), Lemma 3.8, Corollary 6.2, and Corollary 6.5. □

3.4. Regularization via Fourier transform. Let F be an archimedean local field. Choose a basis

$$\{v_1, \dots, v_m; v_{m+1}, \dots, v_{m+s}; v_m^*, \dots, v_1^*\}$$

of V_2 such that

- the hermitian form q_2 is given by the matrix
$$\begin{bmatrix} & & \mathbf{w}_m \\ & \nu \mathbf{1}_s & \\ \mathbf{w}_m & & \end{bmatrix}$$
under the above basis for some $\nu \in \{\pm 1\}$;
- $R_i = E\langle v_1, \dots, v_i\rangle$ for $i = 1, \dots, r$, and $R^* = E\langle v_r^*, \dots, v_1^*\rangle$;
- $\psi_{\mathfrak{F},w}(u) = \psi(\mathrm{Tr}_{E/F} \sum_{i=1}^r u_i)$ where $u_i = (uv_{i+1}, v_i^*)$ for $i = 1, \dots, r-1$ and $u_r = (uw, v_r^*)$ for $u \in \mathrm{N}_{\mathfrak{F}}$ for some non-trivial additive character $\psi\colon F \to \mathbb{C}^\times$.

Let $P_{2,\min}$ be the minimal parabolic subgroup that preserves subspaces $E\langle v_1, \dots, v_i\rangle$ for $1 \le i \le m$. Let $M_{2,\min} \subset P_{2,\min}$ be the subgroup that preserves $E\langle v_i\rangle$ for $1 \le i \le m$ and $A_2 \subset M_{2,\min}$ the maximal split torus. Put $P_{1,\min} = P_{2,\min} \cap G_1$ and $A_1 = A_2 \cap G_1$. Then $P_{1,\min}$ is a minimal parabolic subgroup of G_1 containing A_1 as a maximal split torus. We also choose a minimal parabolic subgroup $P_{0,\min}$ of G_0 which contains $A_0 := A_1 \cap G_0$ as a maximal split torus. We view G_2 as a subgroup of $\mathrm{GL}_{2m+s}(E)$ via the above chosen basis. Let $\mathcal{K}_\alpha = G_\alpha \cap \mathrm{U}_{2m+s}(E)$ which is a maximal compact subgroup of G_α for $\alpha = 1, 2$, where

$$\mathrm{U}_{2m+s}(E) = \{g \in \mathrm{GL}_{2m+s}(E) \mid {}^t g^c g = \mathbf{1}_{2m+s}\}$$

(resp. $\mathrm{U}_{2m+s}(E) = \mathrm{U}_{2m+s}(F) \times \mathrm{U}_{2m+s}(F)$) if E is a field (resp. $E = F \oplus F$). Then $G_\alpha = P_{\alpha,\min}\mathcal{K}_\alpha$ for $\alpha = 1, 2$. We also fix a maximal compact subgroup $\mathcal{K}_0$ of G_0 such that $[\mathcal{K}_0 : \mathcal{K}_1 \cap \mathcal{K}_0] < \infty$ and $G_0 = P_{0,\min}\mathcal{K}_0$.

For $0 \le j \le m$, let $P^j \subset G_2$ be the parabolic subgroup that preserves subspaces $E\langle v_1, \dots, v_i\rangle$ for $1 \le i \le j$, which has a Levi decomposition $P^j = M^j N^j$ where M^j is the subgroup that preserves subspaces $E\langle v_i\rangle$ for $1 \le i \le j$ and N^j is the unipotent radical. In particular, $P^0 = M^0 = G_2$, $P^m = P_{2,\min}$, $M^m = M_{2,\min}$, $P^r = \mathrm{P}_{\mathfrak{F}}$ and $N^r = \mathrm{N}_{\mathfrak{F}}$. Put $\mathcal{K}^j = \mathcal{K}_2 \cap M^j$. Then $\mathcal{K}^j$ is a maximal compact subgroup of M^j such that $M^j = (P^m \cap M^j)\mathcal{K}^j$, and in particular $\mathcal{K}^r = \mathcal{K}_1$. Denote by Ξ^j Harish-Chandra's spherical function on M^j with respect to the pair $(P^m \cap M^j, \mathcal{K}^j)$. In particular, $\Xi^0 = \Xi_2$ and $\Xi^r|_{G_1} = \Xi_1$. For $1 \le j \le r$ and $\gamma \ge -\infty$, put

$$N_\gamma^j = \{u \in N^j \mid |u_i| \le \mathrm{e}^\gamma \text{ for } 1 \le i \le j\}.$$

The unipotent subgroups N^j, $N_{-\infty}^j$ for $1 \le j \le r$ are all stable under the conjugation action of G_0. In what follow, for two non-negative real valued functions f and g, we write $f \prec g$ if there exists a real constant C such that $f \le C \cdot g$.

Proposition 3.10. *There exists some real constant $\beta \ge 0$ such that*

$$\int_{N_\gamma^r G_0} \Xi_2(ug_0)\Xi_0(g_0)\,\mathrm{d}u\,\mathrm{d}g_0 \prec \gamma^\beta$$

for $\gamma \ge 1$, where $\mathrm{d}u$ (resp. $\mathrm{d}g_0$) is a fixed Haar measure on N^r (resp. G_0).

For $0 \leq j \leq m$, let A^j denote the maximal split torus in M^j and $G^j \subset M^j$ the subgroup stabilizing every element in $E\langle v_1, \dots, v_j, v_j^*, \dots, v_1^* \rangle$. Denote by $\bar{P}^j \subset G_2$ the parabolic subgroup opposite to P^j with the unipotent radical $\bar{N}^j$. For an element $g \in G_2$, denote by $a_{\bar{P}^j}(g)$ the A^j-component in the Iwasawa decomposition $G_2 = \bar{P}^j \mathcal{K}^j$ and $a_{\bar{P}^j}(g)_i = a_{\bar{P}^j}(g) v_i / v_i$ for $1 \leq i \leq j$.

Lemma 3.11. *There exists a real constant $\alpha > 0$ such that, for $g, g' \in G_2$,*

$$\Xi_2(gg') \prec \Xi_2(g) \mathrm{e}^{\alpha \varsigma(g')}.$$

Proof. The proof is similarly to the proof of property (5) in [46, Section 3.3]. What we need is the following estimate in the archimedean case:

$$\sup_{k \in \mathcal{K}_2} \delta_{\bar{P}^m}(a_{\bar{P}^m}(kg')) \prec \mathrm{e}^{\alpha \varsigma(g')}$$

for some real constant $\alpha > 0$. One apparently can assume that $g' \in A_2$. Then it follows from (the much stronger result) [30, Theorem 4.1]. □

For a real number b, we introduce the function $\mathbf{1}_{\varsigma < b}$ as the characteristic function of the subset $\{g \in G_2 \mid \varsigma(g) < b\}$, and similarly for $\mathbf{1}_{\varsigma \geq b}$. Proposition 3.10 is essentially proved in [46, Section 4] where $E = F$ is non-archimedean. We provide the modification for F archimedean and possibly $E \neq F$.

Proof of Proposition 3.10. For a fixed $D \geq 0$, let

$$I_{r,D}(\gamma, g_0) = \int_{N_\gamma^r} \Xi_2(ug_0) \varsigma(ug_0)^D \, \mathrm{d}u.$$

We claim that

$$I_{r,D}(\gamma, g_0) \prec \gamma^\beta \varsigma(g_0)^{\beta'} \Xi_1(g_0) \tag{3.10}$$

for some real numbers $\beta, \beta' \geq 0$. Then the proposition follows from the case $r = 0$ which is proved in [23] and [19]. Now assume $r \geq 1$. For $x \in G^1$, put

$$I_{r,D}^1(\gamma, x) = \int_{N_\gamma^1} \Xi^0(ux) \varsigma(ux)^D \, \mathrm{d}u.$$

Assume that

$$I_{r,D}^1(\gamma, x) \prec \gamma^{D'} \delta_{P^1}(x)^{1/2} \varsigma(x)^{D'} \Xi^1(x) \tag{3.11}$$

for some $D' \geq 0$. Then

$$\begin{aligned} I_{r,D}(\gamma, g_0) &= \int_{G^1 \cap N_\gamma^r} \int_{N_\gamma^1} \Xi^0(uvg_0) \varsigma(uvg_0)^D \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{G^1 \cap N_\gamma^r} I_{r,D}^1(\gamma, vg_0) \, \mathrm{d}v \\ &\prec \gamma^{D'} \int_{G^1 \cap N_\gamma^r} \Xi^1(vg_0) \varsigma(vg_0)^{D'} \, \mathrm{d}v \\ &= \gamma^{D'} I_{r-1,D'}(\gamma, g_0). \end{aligned}$$

Therefore, (3.10) follows by induction on r.

To prove (3.11), we decompose $I^1_{r,D}(\gamma, x) = I^1_{r,D}(\gamma, x)_{<b} + I^1_{r,D}(\gamma, x)_{\geq b}$ for $b \geq 0$ where

$$I^1_{r,D}(\gamma, x)_{<b} = \int_{N^1_\gamma} \mathbf{1}_{\varsigma<b}(ux)\Xi^0(ux)\varsigma(ux)^D \,\mathrm{d}u,$$
$$I^1_{r,D}(\gamma, x)_{\geq b} = \int_{N^1_\gamma} \mathbf{1}_{\varsigma\geq b}(ux)\Xi^0(ux)\varsigma(ux)^D \,\mathrm{d}u.$$

By [17, Section 10, Lemma 2], there exists $D' > 0$ such that

$$\int_{N^1} \Xi^0(ux)\varsigma(ux)^{D-D'} \,\mathrm{d}u \prec \delta_{P^1}(x)^{1/2}\Xi^1(x).$$

Then

$$I^1_{r,D}(\gamma, x)_{<b} \prec b^{D'}\delta_{P^1}(x)^{1/2}\Xi^1(x).$$

Choose a splitting $N^1 = N^1_{-\infty}N^1_\dagger$ where $N^1_\dagger \subset N^1$ is a subgroup. Put $N^1_{\dagger,\gamma} = N^1_\gamma \cap N^1_\dagger$ which is compact. Then

$$I^1_{r,D}(\gamma, x)_{\geq b} = \int_{N^1_{\dagger,\gamma}} \int_{N^1_{-\infty}} \mathbf{1}_{\varsigma\geq b}(uvx)\Xi^0(uvx)\varsigma(uvx)^D \,\mathrm{d}u \,\mathrm{d}v.$$

Since $\varsigma(uvx) \leq \varsigma(u) + \varsigma(v) + \varsigma(x)$ and $\varsigma(v) \prec \gamma$, there exists $c_1 > 0$ such that

$$\varsigma(u) \geq c_1\varsigma(uvx) - \gamma - \varsigma(x) \geq bc_1 - \gamma - \varsigma(x).$$

By Lemma 3.11 and the fact that $\mathrm{Vol}(N^1_{\dagger,\gamma}) \prec \mathrm{e}^\gamma$, we have

$$I^1_{r,D}(\gamma, x)_{\geq b} \prec \mathrm{e}^{c_2\gamma}\mathrm{e}^{c_2\varsigma(x)} \int_{N^1_{-\infty}} \mathbf{1}_{\varsigma\geq bc_1-\gamma-\varsigma(x)}(u)\Xi^0(u)\varsigma(u)^D \,\mathrm{d}u$$

for some $c_2 > 0$. Again by Lemma 3.11, there exists $c_3 > 0$ such that

$$\mathrm{e}^{-c_3\varsigma(x)} \prec \delta_{P^1}(x)^{1/2}\Xi^1(x)$$

for all $x \in G^1$. By the lemma below, choose $b = \epsilon^{-1}c_1^{-1}(c_2+c_3+\epsilon)(\gamma+\varsigma(x))$ and we have

$$I^1_{r,D}(\gamma, x)_{\geq b} \prec \delta_{P^1}(x)^{1/2}\Xi^1(x).$$

Therefore, (3.11) holds. □

Lemma 3.12. *For $r \geq 1$, define*

$$J_{r,D}(b) = \int_{N^1_{-\infty}} \mathbf{1}_{\varsigma\geq b}(u)\Xi^0(u)\varsigma(u)^D \,\mathrm{d}u$$

for $b > 0$. Then there exists some constant $\epsilon > 0$ such that $J_{r,D}(b) \prec \mathrm{e}^{-\epsilon b}$.

If $E = F$ is non-archimedean, there are similar estimates in [46, Lemma 4.5] for $r \geq 2$ and in [46, Lemma 4.6] for $r = 1$. We verify the case where $r \geq 2$ and E is an (archimedean) field. The cases where $r = 1$ or $E = F \oplus F$ are much easier and will be left to the readers. The same argument can be applied to the archimedean case and possibly $E \neq F$ as well, except that in the second half we need a different trick.

Proof. We may assume that b is sufficiently large. In the proof, we take

$$\|g\| = \sum_{i,j=1}^{2m+s} |g_{ij}|_{\mathbb{C}} + |(g^{-1})_{ij}|_{\mathbb{C}}$$

for $g \in \mathrm{GL}_{2m+s}(E)$. We also put

$$\|v\| = \sum_{i=1}^{2m+s} |v^i|_{\mathbb{C}}$$

for an element $v = (v^1, \dots, v^{2m+s}) \in V_2 \simeq E^{2m+s}$.

Denote by $V_\flat$ the orthogonal complement of $E\langle v_1, v_2; v_2^*, v_1^*\rangle$ in V_2 whose dimension is at least 1. For $x \in V_\flat$, $y \in E$, $z \in E^-$, there is a unique element $u(x,z,y) \in N^1_{-\infty}$ such that

$$u(x,z,y)v_1^* = v_1^* + x + yv_2 - (q_2(x) + z)v_1.$$

The map

$$V_\flat \times E^- \times E \to N^1_{-\infty}, \quad (x,z,y) \mapsto u(x,z,y)$$

is an isomorphism. Put $u(x,z) = u(x,z,0)$ and $u(y) = u(0,0,y)$. Let $N^1_\sharp \subset N^1_{-\infty}$ be the subgroup consisting of $u(x,z,0)$ and $Y \subset N^1_{-\infty}$ be the subgroup consisting of $u(y)$. Let $V_\sharp$ be the orthogonal complement of $E\langle v_2, v_2^*\rangle$ in V_2. Put $G_\sharp = \mathrm{Ism}^0(V_\sharp)$ and $P^1_\sharp = P^1 \cap G_\sharp$. Then $N^1_\sharp$ is the unipotent radical of $P^1_\sharp$. Write $P^j_\sharp = P^j \cap G_\sharp$ and similarly for $\bar{P}^j_\sharp, A^j_\sharp, \bar{N}^j_\sharp$ for $3 \le j \le m$. Then $P^m_\sharp$ is a minimal parabolic subgroup of $G_\sharp$.

Write $u(x,z) = a_{\bar{P}^m}(u(x,z))u_{\bar{P}^m}(u(x,z))k(u(x,z))$ under the Iwasawa decomposition $G_2 = \bar{P}^m\mathcal{K}_2$. Then

$$k(u(x,z))^{-1}v_1^* = a_{\bar{P}^m}(u(x,z))_1^{-1}\big(v_1^* - x - (q_2(x) - z)v_1\big), \tag{3.12}$$

which implies that there exists a constant $\alpha' > 0$ such that

$$\big|a_{\bar{P}^m}(u(x,z))_1\big|_E \ge \mathrm{e}^{\alpha'\varsigma(u(x,z))}.$$

Since

$$a_{\bar{P}^m_\sharp}(u(x,z))_1 = a_{\bar{P}^m}(u(x,z))_1, \quad a_{\bar{P}^m}(u(x,z))_2 = 1,$$

and

$$a_{\bar{P}^m_\sharp}(u(x,z))_j = a_{\bar{P}^m}(u(x,z))_j$$

for $3 \le j \le m$, we have

$$\delta_{\bar{P}^m}\big(a_{\bar{P}^m}(u(x,z))\big)^{1/2} = \big|a_{\bar{P}^m}(u(x,z))_1\big|_E^{-1}\delta_{\bar{P}^m_\sharp}\big(a_{\bar{P}^m_\sharp}(u(x,z))\big)^{1/2}.$$

For a real number $\mu > 0$ which will be determined later, let

$$J^1_{r,D}(b) = \int_Y \int_{N^1_\sharp} \mathbf{1}_{\varsigma \ge b}(u(x,z)u(y))\mathbf{1}_{\varsigma \le \mu b}(u(y))\Xi^0(u(x,z)u(y)) \times \varsigma(u(x,z)u(y))^D \,\mathrm{d}u(x,z)u(y).$$

By Lemma 3.11 and [17, Section 10, Lemma 1 (2)], we have

$$J^1_{r,D}(b) \prec \int_Y \int_{N^1_\sharp} \mathbf{1}_{\varsigma \geq b}(u(x,z)u(y)) \mathbf{1}_{\varsigma \leq \mu b}(u(y)) \delta_{\bar{P}^m}(a_{\bar{P}^m}(u(x,z)))^{1/2} \\ \times \mathrm{e}^{\alpha\varsigma(u(y))} \varsigma(u(x,z)u(y))^{D'} \, \mathrm{d}u(x,z)u(y)$$

for some $D' \geq D$. Repeat the argument on [46, p. 212] and use [16, Lemma 89]. We may choose a sufficiently small constant $\mu > 0$ such that

$$J^1_{r,D}(b) \prec \mathrm{e}^{-\epsilon_1 b}$$

for some real constant $\epsilon_1 > 0$.

We need to estimate the difference

$$J^2_{r,D}(b) := J_{r,D}(b) - J^1_{r,D}(b).$$

Put $x' = x/\sqrt{|y|_{\mathbb{C}} + 1}$, $z' = z/(|y|_{\mathbb{C}} + 1)$ and $a(x',z') = a_{\bar{P}^m}(u(x',z'))$. By the Iwasawa decomposition

$$\begin{bmatrix} 1 & y \\ & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{|y|_{\mathbb{C}}+1} & \\ & \frac{1}{\sqrt{|y|_{\mathbb{C}}+1}} \end{bmatrix} \begin{bmatrix} 1 & \\ \frac{|y|_{\mathbb{C}}}{y} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{|y|_{\mathbb{C}}+1}} & \frac{y}{\sqrt{|y|_{\mathbb{C}}+1}} \\ -\frac{|y|_{\mathbb{C}}}{y\sqrt{|y|_{\mathbb{C}}+1}} & \frac{1}{\sqrt{|y|_{\mathbb{C}}+1}} \end{bmatrix}$$

in $\mathrm{SL}_2(E)$ and using the calculation on [46, p. 213], we have

$$u(x,z)u(y) \in \bar{N}^m a(y) a(x',z') a(x',z';y') \mathcal{K}_2,$$

where

$$a(y) = \begin{bmatrix} \sqrt{|y|_{\mathbb{C}}+1} & & & & \\ & \sqrt{|y|_{\mathbb{C}}+1} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{|y|_{\mathbb{C}}+1}} & \\ & & & & \frac{1}{\sqrt{|y|_{\mathbb{C}}+1}} \end{bmatrix},$$

and $a(x',z';y') \in A_2$ with

$$a(x',z';y')_1 = \sqrt{\left|a(x',z')_1^{-1} y q_V(x')\right|_{\mathbb{C}} + 1},$$

$a(x',z';y')_2 = a(x',z';y')_1^{-1}$ and all other components being 1. In particular,

$$\delta_{\bar{P}^m}(a(y)) = \left||y|_{\mathbb{C}} + 1\right|_E^{4-[E:F]+2m+s},$$
$$\delta_{\bar{P}^m}(a(x',z';y)) = \left|\left|a(x',z')_1^{-1} y(q_2(x') + z')\right|_{\mathbb{C}} + 1\right|_E^{-1}.$$

By [17, Section 10, Lemma 1 (2)] and changing variables $(x,z) \mapsto (x',z')$, we have

$$J^2_{r,D}(b) \prec \int_Y \int_{N^1_\sharp} \mathbf{1}_{\varsigma > \mu b}(u(y)) |a(x,z)_1|_E^{-1} \delta_{\bar{P}^m_\sharp}\big(a_{\bar{P}^m_\sharp}(u(x,z))\big)^{1/2} \\ \times \left|\left|a(x,z)_1^{-1} y(q_2(x)+z)\right|_{\mathbb{C}} + 1\right|_E^{-1/2} \\ \times \left||y|_{\mathbb{C}} + 1\right|_E^{-1/2} \varsigma(u(x,z))^D \varsigma(u(y))^D \, \mathrm{d}u(x,z) \, \mathrm{d}u(y). \tag{3.13}$$

Put $q(x,z) = q_2(x) + z \in \mathbb{C}$. Apparently,

$$
\begin{aligned}
(3.13) \prec & \int_Y \int_{N_\sharp^1} \mathbf{1}_{\varsigma > \mu b}(u(y)) |a(x,z)_1|_E^{-\beta} \delta_{\bar{P}_\sharp^m}\big(a_{\bar{P}_\sharp^m}(u(x,z))\big)^{1/2} \\
& \qquad \times \big||a(x,z)_1^{-1} y q(x,z)|_{\mathbb{C}} + 1\big|_E^{-\beta'} \big||y|_{\mathbb{C}} + 1\big|_E^{-1/2} \varsigma(u(y))^D \,\mathrm{d}u(x,z)\,\mathrm{d}u(y) \\
\prec & \int_Y \int_{N_\sharp^1} \mathbf{1}_{\varsigma > \mu b}(u(y)) |a(x,z)_1|_E^{2\beta'-\beta} \delta_{\bar{P}_\sharp^m}\big(a_{\bar{P}_\sharp^m}(u(x,z))\big)^{1/2} \\
& \qquad \times |q(x,z)|_E^{-2\beta'} |y|_E^{-1-2\beta'} \varsigma(u(y))^D \,\mathrm{d}u(x,z)\,\mathrm{d}u(y)
\end{aligned}
$$

for $0 < \beta < 1$ and $0 < \beta' < 1/2$ satisfying $\beta - 2\beta' > 0$ to be assigned later. We only need to show that the integral

$$
(3.14) \qquad \int_{N_\sharp^1} |a(x,z)_1|_E^{2\beta'-\beta} \delta_{\bar{P}_\sharp^m}\big(a_{\bar{P}_\sharp^m}(u(x,z))\big)^{1/2} |q(x,z)|_E^{-2\beta'} \,\mathrm{d}u(x,z)
$$

is convergent. Let $V_\flat^1$ be the subset of $V_\flat \oplus E^-$ consisting of (x,z) such that $|q(x,z)|_{\mathbb{C}} = 1$. The group $\{x \in E \mid |x|_{\mathbb{C}} = 1\} \times \mathrm{Ism}(V_\flat)$ acts transitively on $V_\flat^1$ by conjugation in $G_\sharp$. Choose a positive measure $\mathrm{d}u(x,z)$ on $V_\flat^1$ which is invariant under the above group action. Then

$$
\begin{aligned}
(3.15) \qquad (3.14) = \int_0^\infty \int_{V_\flat^1} & |a(tx,t^2z)_1|_E^{2\beta'-\beta} \delta_{\bar{P}_\sharp^m}\big(a_{\bar{P}_\sharp^m}(u(tx,t^2z))\big)^{1/2} \\
& \times |t|_E^{-4\beta'} |t|_E^{2m+s-5+[E:F]} \,\mathrm{d}u(x,z)\,\mathrm{d}^\times t,
\end{aligned}
$$

where $\mathrm{d}^\times t = t^{-1}\,\mathrm{d}t$. Decompose the integral (3.15) into two parts: $J_{\leq 1}$ for $0 < t \leq 1$ and $J_{>1}$ for $t > 1$. Then $J_{>1}$ is absolutely convergent by [16, Lemma 89]. The rest is to show that $J_{\leq 1}$ is convergent. If $\dim V_\flat = 0$, then it is trivial. If $\dim V_\flat = 1$ and $E = F$, one may choose β' small enough such that $J_{\leq 1}$ is convergent, since the set $\{x \in V_\sharp \mid |q(x)|_{\mathbb{C}} \leq 1\}$ is compact.

Now we assume that $\dim V_\flat + [E:F] \geq 3$, that is, $2m + s - 7 + [E:F] \geq 0$. Put

$$
J(t) = \int_{V_\flat^1} |a(tx,t^2z)_1|_E^{2\beta'-\beta} \delta_{\bar{P}_\sharp^m}\big(a_{\bar{P}_\sharp^m}(u(tx,t^2z))\big)^{1/2} \,\mathrm{d}u(x,z).
$$

Equation (3.12) implies that $|a(x,z)_1|_{\mathbb{C}} = 1 + \|x\| + |q(x,z)|_{\mathbb{C}}$. Hence for $0 < t \leq 1$ and $(x,z) \in V_\flat^1$, we have

$$
(3.16) \qquad |a(x,z)_1|_E \geq |a(tx,t^2z)_1|_E \geq |t|_E |a(x,z)_1|_E.
$$

We claim that for $(x,z) \in V_\flat^1$ and $0 < t \leq 1$,

$$
(3.17) \qquad 1 \leq \frac{\delta_{\bar{P}_\sharp^m}(a_{\bar{P}_\sharp^m}(u(tx,t^2z)))^{1/2}}{\delta_{\bar{P}_\sharp^m}(a_{\bar{P}_\sharp^m}(u(x,z)))^{1/2}} \leq |t|_E^{1-(2m+s-5+[E:F])}.
$$

Assume this inequality holds. Then, combining with (3.16), we get

$$
J(1) \leq J(t) \leq |t|_E^{1+2\beta'-\beta-(2m+s-5+[E:F])} J(1).
$$

Therefore, $J(1) < \infty$ and

$$
J_{\leq 1} = \int_0^1 J(t) |t|_E^{-4\beta'} |t|_E^{2m+s-5+[E:F]} \,\mathrm{d}^\times t \leq J(1) \int_0^1 |t|_E^{1-2\beta'-\beta} \,\mathrm{d}^\times t,
$$

which is convergent if we take, for example, $\beta = 1/2$ and $\beta' = 1/6$.

The remaining task is to show (3.17). Put

$$a_t = \begin{bmatrix} t & & \\ & \mathbf{1}_{2m+s} & \\ & & t^{-1} \end{bmatrix} \in A^m_\sharp \subset G_\sharp.$$

For an element $g \in G_\sharp$, let $g = \bar{n}(g)a(g)k(g)$ denote the components under the Iwasawa decomposition $G^\sharp = \bar{N}^m_\sharp A^m_\sharp \mathcal{K}_\sharp$ where $\mathcal{K}_\sharp = \mathcal{K}_2 \cap G_\sharp$. Then

$$\begin{aligned} u(tx, t^2 z) &= a_t u(x,z) a_t^{-1} = a_t \bar{n}(u(x,z)) a(u(x,z)) k(u(x,z)) a_t^{-1} \\ &= \bar{n}' a_t a(u(x,z)) \bar{n}\big(k(u(x,z))a_t^{-1}\big) a\big(k(u(x,z))a_t^{-1}\big) k\big(k(u(x,z))a_t^{-1}\big), \end{aligned}$$

where $\bar{n}' = a_t \bar{n}(u(x,z)) a_t^{-1} \in \bar{N}^m_\sharp$. Put

$$\bar{n}'' = a_t a(u(x,z)) \bar{n}\big(k(u(x,z))a_t^{-1}\big) a(u(x,z))^{-1} a_t^{-1} \in \bar{N}^m_\sharp,$$

$k = k(k(u(x,z))a_t^{-1}) \in \mathcal{K}_\sharp$. We also decompose $a(k(u(x,z))a_t^{-1}) = a'a''$, where a' acts trivially on the orthogonal complement of $E\langle v_1, v_1^* \rangle$ in $V^\sharp$ and $a''_1 = 1$. Recall that we have $a_1 = av_1/v_1$ for $a \in A^m_\sharp$. Therefore, $a(u(tx, t^2 z)) = a_t a(u(x,z)) a'a''$ and

$$(3.18) \qquad \frac{\delta_{\bar{P}^m_\sharp}(a_{\bar{P}^m_\sharp}(u(tx,t^2 z)))^{1/2}}{\delta_{\bar{P}^m_\sharp}(a_{\bar{P}^m_\sharp}(u(x,z)))^{1/2}} = \left| \frac{a(tx,t^2 z)_1}{a(x,z)_1} \right|_E^{-(2m+s-5+[E:F])/2} \delta_{P^m_\sharp}(a'')^{1/2}.$$

By [30, Theorem 4.1] and linear programming, we have

$$(3.19) \qquad |t|_E^{(2m+s-7+[E:F])/2} \le \delta_{P^m_\sharp}(a'')^{1/2} \le |t|_E^{-(2m+s-7+[E:F])/2}.$$

Then (3.17) follows from (3.16), (3.18) and (3.19). □

Corollary 3.13 (of Proposition 3.10). *Let π_α be an irreducible tempered Casselman–Wallach representation of G_α (see, for example,* [48, Chapter 11]*) and Φ_α a smooth matrix coefficient of π_α for $\alpha = 0, 2$. Then the integral*

$$\mathfrak{a}_{\Phi_2,\Phi_0}(u) := \int_{N^r_{-\infty} G_0} \Phi_2(uu'g_0)\Phi_0(g_0)\,\mathrm{d}u'\,\mathrm{d}g_0$$

is absolutely convergent. Moreover, the function $\mathfrak{a}_{\Phi_2,\Phi_0}$ defines a tempered distribution on $N^r/N^r_{-\infty} \simeq E^r$.

Proof. Choose a subgroup $N^r_\dagger$ of N^r such that the induced projection $N^r_\dagger \to N^r/N^r_{-\infty}$ is an isomorphism and G_0 acts trivially on $N^r_\dagger$. By the Dixmier–Malliavin theorem [5], we may assume $\Phi_2 = \mathrm{R}(f)\Phi'_2$ for some function $f \in \mathrm{C}^\infty_c(N^r_\dagger)$ and smooth matrix coefficient Φ'_2. Then

$$\mathfrak{a}_{\Phi_2,\Phi_0}(u) = \int_{N^r_\dagger} \int_{N^r_{-\infty} G_0} f(u'')\Phi'_2(uu'u''g_0)\Phi_0(g_0)\,\mathrm{d}u'\,\mathrm{d}g_0\,\mathrm{d}u'',$$

which is absolutely convergent by Proposition 3.10 and [41, Theorem 1.2]. The second part also follows from Proposition 3.10. □

Let N be an abelian Lie group over E. Denote by $\mathcal{D}(N)$ (resp. $\mathcal{S}(N)$) the space of tempered distributions (resp. Schwartz functions) on N. By definition, we have a bilinear pairing $(\ ,\ ): \mathcal{D}(N) \times \mathcal{S}(N) \to \mathbb{C}$. Define the Fourier transform $\hat{\ } : \mathcal{D}(N) \to \mathcal{D}(N)$ by the formula $(\hat{\mathfrak{a}}, \phi) = (\mathfrak{a}, \hat{\phi})$ for all $\phi \in \mathcal{S}(N)$.

Proposition 3.14. *The Fourier transform* $\widehat{\mathfrak{a}_{\Phi_2,\Phi_0}}$ *is smooth on the regular locus* $(\widehat{N^r/N^r_{-\infty}})^{\mathrm{reg}}$*, which is an open dense subset of* $\widehat{N^r/N^r_{-\infty}}$*.*

Proof. Denote by T the subgroup of M^r which acts trivially on V_1. It is isomorphic to $(E^\times)^r$. We may write an element $t \in T$ as $(t_1, \dots, t_r)$ where $t_i = tv_i/v_i$ for $1 \le i \le r$. We also write an element $u \in N^r/N^r_{-\infty}$ as $(u_1, \dots, u_r)$ where $u_i = q_2(uv_{i+1}, v_i^*)$ for $1 \le i \le r-1$ and $u_r = q_2(uw, v_r^*)$. The group T acts on $N^r/N^r_{-\infty}$ by conjugation. Precisely, $tut^{-1} = (t_1t_2^{-1}u_1, \dots, t_{r-1}t_r^{-1}u_{r-1}, t_ru_r)$ for $t \in T$ and $u \in N^r/N^r_{-\infty}$. Therefore, T acts on $\mathcal{D}(N^r/N^r_{-\infty})$, denoted by C. Fix a real number $\varepsilon > 0$. Then T acts on $\mathrm{L}^{2+\varepsilon}(G_2)$ by conjugation, which we also denote by C.

For an element $t \in T$, define ψ_t by $\psi_{\mathfrak{F},w}(tut^{-1})$. Then the assignment $t \mapsto \psi_t$ is an isomorphism from T to $(\widehat{N^r/N^r_{-\infty}})^{\mathrm{reg}}$ under which the measure $|t_1|\,\mathrm{d}t$ is the restriction of the dual measure of $\mathrm{d}u$ on $N^r/N^r_{-\infty}$ to the regular part, up to a positive constant. We may assume that the constant is 1. Pick up a Schwartz function $\phi \in \mathcal{S}(\widehat{N^r/N^r_{-\infty}})$ which is compactly supported on the regular part. We identify ϕ as an element in $\mathrm{C}^\infty_{\mathrm{c}}(T)$. Conversely, we may identify $f \in \mathrm{C}^\infty_{\mathrm{c}}(T)$ as an element in $\mathcal{S}(\widehat{N^r/N^r_{-\infty}})$. Then

$$
\begin{aligned}
\big(\mathrm{C}(f)\mathfrak{a}_{\Phi_2,\Phi_0}, \hat{\phi}\big) &= \int_{N^r/N^r_{-\infty}} \int_T f(t)\mathfrak{a}_{\Phi_2,\Phi_0}(t^{-1}ut)\hat{\phi}(u)\,\mathrm{d}t\,\mathrm{d}u \\
&= \int_T f(t)|t_1| \int_{N^r/N^r_{-\infty}} \mathfrak{a}_{\Phi_2,\Phi_0}(u)\hat{\phi}(tut^{-1})\,\mathrm{d}u\,\mathrm{d}t \\
&= \int_{N^r/N^r_{-\infty}} \int_T \mathfrak{a}_{\Phi_2,\Phi_0}(u) f(t')\hat{\phi}(t'ut'^{-1})|t_1'|\,\mathrm{d}t'\,\mathrm{d}u \\
&= \int_{N^r/N^r_{-\infty}} \int_T \mathfrak{a}_{\Phi_2,\Phi_0}(u) f(t') \int_T \phi(t)\psi_t^{-1}(t'ut'^{-1})|t_1|\,\mathrm{d}t\,|t_1'|\,\mathrm{d}t'\,\mathrm{d}u \\
&= \int_{N^r/N^r_{-\infty}} \int_T \mathfrak{a}_{\Phi_2,\Phi_0}(u)\phi(t) \int_T f(t')\psi_{t'}^{-1}(tut^{-1})|t_1'|\,\mathrm{d}t'|t_1|\,\mathrm{d}t\,\mathrm{d}u \\
&= \int_T \int_{N^r/N^r_{-\infty}} \mathfrak{a}_{\Phi_2,\Phi_0}(u)\hat{f}(tut^{-1})\,\mathrm{d}u\phi(t)|t_1|\,\mathrm{d}t.
\end{aligned}
$$

Thus, the restriction of $\widehat{\mathrm{C}(f)\mathfrak{a}_{\Phi_2,\Phi_0}}$ to $(\widehat{N^r/N^r_{-\infty}})^{\mathrm{reg}}$ is the function $t \mapsto (\mathfrak{a}_{\Phi_2,\Phi_0}, \mathrm{C}(t^{-1})\hat{f})$ that is smooth.

On the other hand, for $f \in \mathrm{C}^\infty_{\mathrm{c}}(T)$, we have

$$
\begin{aligned}
\mathrm{C}(f\delta_{N^r_{-\infty}\rtimes T})\mathfrak{a}_{\Phi_2,\Phi_0}(u) &= \int_T f(t)\delta_{N^r_{-\infty}\rtimes T}(t)\mathfrak{a}_{\Phi_2,\Phi_0}(t^{-1}ut)\,\mathrm{d}t \\
&= \int_T f(t)\delta_{N^r_{-\infty}\rtimes T}(t) \int_{N^r_{-\infty}G_0} \Phi_2(t^{-1}utu'g_0)\Phi_0(g_0)\,\mathrm{d}u'\,\mathrm{d}g_0\,\mathrm{d}t \\
&= \int_T \int_{N^r_{-\infty}G_0} f(t)\Phi_2(t^{-1}uu'g_0t)\Phi_0(g_0)\,\mathrm{d}u'\,\mathrm{d}g_0\,\mathrm{d}t
\end{aligned}
$$

$$= \int_T \int_{N^r_{-\infty} G_0} f(t)\Phi_2(t^{-1}uu'g_0t)\Phi_0(g_0)\,\mathrm{d}u'\,\mathrm{d}g_0\,\mathrm{d}t$$
$$= \int_{N^r_{-\infty} G_0} f(t)\mathrm{C}(f)\Phi_2(uu'g_0)\Phi_0(g_0)\,\mathrm{d}u'\,\mathrm{d}g_0$$
$$= \mathfrak{a}_{\mathrm{C}(f)\Phi_2,\Phi_0}.$$

Therefore, $\widehat{\mathfrak{a}_{\mathrm{C}(f)\Phi_2,\Phi_0}}$ is smooth on $(\widehat{N^r/N^r_{-\infty}})^{\mathrm{reg}}$. Finally, by the Dixmier–Malliavin theorem [5], every smooth matrix coefficient Φ_2 is a finite linear combination of $\mathrm{C}(f)\Phi_2'$. □

For smooth vectors $\varphi_\alpha \in \pi_\alpha$ and $\check{\varphi}_\alpha \in \check{\pi}_\alpha$ $(\alpha = 0, 2)$, define

$$\alpha(\varphi_2, \check{\varphi}_2; \varphi_0, \check{\varphi}_0) = \widehat{\mathfrak{a}_{\Phi_{\varphi_2\otimes\check{\varphi}_2},\Phi_{\varphi_0\otimes\check{\varphi}_0}}}(\psi_{\mathfrak{F},w}),$$

and similarly for $\alpha^\natural(\varphi_2, \check{\varphi}_2; \varphi_0, \check{\varphi}_0)$, $\alpha(\varphi_2; \varphi_0)$ and $\alpha^\natural(\varphi_2; \varphi_0)$ as (2.3), (2.4) and (2.5). In fact, for our purpose of regularizing the matrix coefficient integral, it is enough to show that $\widehat{\mathfrak{a}_{\Phi_2,\Phi_0}}$ is continuous on $(\widehat{N^r/N^r_{-\infty}})^{\mathrm{reg}}$ in the above proposition.

Proof of Theorem 2.1 (2) *for* v *archimedean.* For a smooth vector $\varphi_\alpha \in \pi_\alpha$ $(\alpha = 0, 2)$, put $\Phi_\alpha(g_\alpha) = (\pi_\alpha(g_\alpha)\varphi_\alpha, \bar{\varphi}_\alpha)$. We show that

$$\widehat{\mathfrak{a}_{\Phi_2,\Phi_0}}\big|_{(\widehat{N^r/N^r_{-\infty}})^{\mathrm{reg}}} \geq 0.$$

Equivalently, we show that

$$(\widehat{\mathfrak{a}_{\Phi_2,\Phi_0}}, |\phi|_{\mathbb{C}}) \geq 0$$

for every function $\phi \in \mathcal{S}(\widehat{N^r/N^r_{-\infty}})$ that is compactly supported in $(\widehat{N^r/N^r_{-\infty}})^{\mathrm{reg}}$. By the cross-correlation theorem, we have

$$\widehat{|\phi|_{\mathbb{C}}}(u) = \int_{N^r/N^r_{-\infty}} \hat{\phi}(uu')\overline{\hat{\phi}(u')}\,\mathrm{d}u'.$$

Therefore,

$$\begin{aligned}\big(\widehat{\mathfrak{a}_{\Phi_2,\Phi_0}}, |\phi|_{\mathbb{C}}\big) &= \big(\mathfrak{a}_{\Phi_2,\Phi_0}, \widehat{|\phi|_{\mathbb{C}}}\big)\\ &= \int_{N^r/N^r_{-\infty}} \int_{N^r/N^r_{-\infty}} \mathfrak{a}_{\Phi_2,\Phi_0}(u)\hat{\phi}(uu')\overline{\hat{\phi}(u')}\,\mathrm{d}u\,\mathrm{d}u'\\ &= \int_{N^r/N^r_{-\infty}} \int_{N^r/N^r_{-\infty}} \mathfrak{a}_{\Phi_2,\Phi_0}(u'^{-1}u)\hat{\phi}(u)\overline{\hat{\phi}(u')}\,\mathrm{d}u\,\mathrm{d}u'.\end{aligned}$$

View $\hat{\phi}$ as a function on $N^r_\dagger$, the group chosen in the proof of Corollary 3.13. The above expression is equal to

$$\int_{N^r_{-\infty} G_0} \mathcal{B}_{\pi_2}\big(\pi_2(u'g_0)\mathrm{R}(\hat{\phi})\varphi_2, \mathrm{R}(\hat{\phi})\varphi_2\big)\mathcal{B}_{\pi_0}(\pi_0(g_0)\varphi_0, \varphi_0)\,\mathrm{d}g_0\,\mathrm{d}u'.$$

The positivity follows by applying [20, Theorem 2.1] to the group $G = G_2 \times G_0$ and the unimodular subgroup $H = N^r_{-\infty} \rtimes G_0$ and using Proposition 3.10. □

Proposition 3.15. *Let π_2 be square-integrable and π_0 be tempered.*

(1) *For every smooth matrix coefficient Φ_α of π_α ($\alpha = 0, 2$), the integral*

$$\int_{G_0} \int_{\mathrm{N}_{\mathfrak{F}}} \Phi_2(ug_0)\Phi_0(g_0)\psi_{\mathfrak{F},w}(u)^{-1}\,\mathrm{d}u\,\mathrm{d}g_0$$

is absolutely convergent.

(2) *The above integral coincides with $\widehat{\alpha_{\Phi_2,\Phi_0}}(\psi_{\mathfrak{F},w})$.*

Proof. The proof of (1) is same as the proof of Proposition 3.10 (1), except that we use [48, Theorem 15.2.4] instead of [45, Corollaire III.1.2], and also use [17, Section 10, Lemma 2]. Part (2) is obvious. □

Remark 3.16. The way of regularization described above works for non-archimedean places as well, where the estimate in Proposition 3.10 has been obtained by Waldspurger [46, Assertion 4.3 (3)]. It is easy to see that such regularization coincides with the one described in Section 3.2. In fact in the cited article, Waldspurger regularizes the matrix coefficient integral α via a third (but equivalent) way that is more or less a combination of the two methods provided here.

The idea of using Fourier transform of tempered distribution to regularize the matrix coefficient integral appears in [40, Section 6.3] for the Whittaker case at non-archimedean places. Such idea is also used in [34].

3.5. Regularized unipotent integral for theta liftings. Let F be any local field of characteristic not 2 and E/F an étale algebra of degree ≤ 2. Let (V, q_V) be a hermitian space over E of dimension m. For $1 \leq n \leq m - 2 + [E : F]$ and an element $T \in \mathrm{Herm}_n^E(F)$, we write $\Omega_T = \{x \in V^n \mid q_V(x) = T\}$ where $q_V(x)$ is the moment matrix of x. If T is regular, that is, $\det T \neq 0$, Ω_T is a single orbit, if non-empty, under the action of $H = \mathrm{Ism}(V, q_V)$. In that case, fix a representative $\mu_T \in \Omega_T$ and let H_{μ_T} be the stabilizer of μ_T in H. Then $\Omega_T \simeq H_{\mu_T}\backslash H$. Fix a non-trivial additive character $\psi\colon F \to \mathbb{C}^\times$.

The abelian unipotent group $\mathrm{Herm}_n^E(F)$ is selfdual. In fact, all characters of $\mathrm{Herm}_n^E(F)$ are of the form ψ_u, defined by $\psi_u(T) = \psi(\mathrm{tr}\, uT)$, for some $u \in \mathrm{Herm}_n^E(F)$. Denote by $\mathrm{Herm}_n^E(F)^{\mathrm{reg}} \subset \mathrm{Herm}_n^E(F)$ the open dense subset of non-singular hermitian matrices and by $\mathrm{Herm}_n^E(F)^{\mathrm{deg}}$ its complement. We recall the following well-known lemma.

Lemma 3.17. *Let (V, q_V) be a hermitian space over E of dimension m. There is a unique H-invariant positive measure $\underline{\mathrm{d}}\,\underline{h}$ on $\Omega_T \simeq H_{\mu_T}\backslash H$ for every regular $T \in \mathrm{Herm}_n^E(F)$ such that, for every $\phi_2 \in \mathcal{S}(V^n)$,*

$$\int_{V^n} \phi(x)\,\mathrm{d}x = \int_{\mathrm{Herm}_n^E(F)^{\mathrm{reg}}} \int_{H_{\mu_T}\backslash H} \phi(\underline{h}^{-1}\mu_T)\,\underline{\mathrm{d}}\,\underline{h}\,\mathrm{d}T,$$

where $\mathrm{d}x$ (resp. $\mathrm{d}T$) is the selfdual measure on V^n (resp. $\mathrm{Herm}_n^E(F)$) with respect to ψ.

Remark 3.18. We call the measure $\underline{\mathrm{d}}\,\underline{h}$ in the above lemma the *Siegel–Weil measure*. If F is a global field, $\psi\colon F\backslash\mathbb{A} \to \mathbb{C}^\times$ is automorphic, and μ_T is in $V^n(E)$, then the product of Siegel–Weil measures (possibly regularized by a convergence factor) at all places is the quotient measure of Tamagawa measures on $H(\mathbb{A})$ and $H_{\mu_T}(\mathbb{A})$.

Choose an orthogonal decomposition $V = X \oplus V^{\mathrm{an}} \oplus X^*$, where V^{an} is an anisotropic kernel of V; X and X^* are totally isotropic subspaces. Choose a basis $\{v_1, \dots, v_r\}$ (resp. $\{v_1^*, \dots, v_r^*\}$) of X (resp. X^*) such that $(v_i, v_j^*) = \delta_{ij}$. Assume that $\dim V^{\mathrm{an}} = s$ and choose a basis $\{v_{r+1}, \dots, v_{r+s}\}$ if $s > 0$. We identity V with E^m under the basis

$$\{v_1, \dots, v_{r+s}, v_r^*, \dots, v_1^*\} \quad (2r + s = m),$$

and hence H with a subgroup of $\mathrm{GL}_m(E)$. Through the basis of X, we also get a minimal parabolic subgroup $P_{\min} = M_{\min} N_{\min}$ of H, which is the one preserving the flag

$$0 \subset \{v_1\} \subset \cdots \subset \{v_1, \dots, v_r\}.$$

Let $A \subset M_{\min}$ be the maximal split torus and fix a maximal compact subgroup $\mathcal{K}_H$ of H that is in good position with respect to $P_{\min}$.

Lemma 3.19. *Let σ be an irreducible unitary admissible representation of H satisfying the following estimate on matrix coefficients: There exists a positive constant λ with $\lambda > 1 - n/(m - 3 + [E : F])$ such that for every smooth matrix coefficient Φ, there exists a positive constant C_Φ such that $|\Phi(a)| \le C_\Phi \delta_{P_{\min}}(a)^\lambda$ for all $a \in A^+$. Then for any such Φ and any two Schwartz functions $\phi_1, \phi_2 \in \mathcal{S}(V^n)$, the following integral is absolutely convergent:*

$$\int_H \int_{V^n} \phi_1(h^{-1}x)\phi_2(x)\Phi(h)\,\mathrm{d}x\,\mathrm{d}h.$$

The above estimate is satisfied, for example, when

(1) $n \ge m - 3 + [E : F]$, or

(2) $2n > m - 3 + [E : F]$ and σ is tempered.

Proof. We have the minimal parabolic subgroup $P_{\min} = M_{\min} N_{\min}$ of H, where the Levi factor $M_{\min}$ is

$$\prod_{i=1}^{n} \mathrm{Res}_F^E \mathrm{GL}(\mathrm{Span}_E\{x_i\}) \times \mathrm{Ism}(V^{\mathrm{an}}) \simeq (E^\times)^r \times \mathrm{Ism}(V^{\mathrm{an}}),$$

and

$$A \simeq \begin{cases} (F^\times)^m & \text{if } E = F \oplus F, \\ (F^\times)^r & \text{otherwise.} \end{cases}$$

We write elements in A as $a = a(a_1, \dots, a_m)$ in the first case and $a = a(a_1, \dots, a_r)$ in the second case.

Recall the formula

$$\int_H f(h)\,\mathrm{d}h = \int_{A^+} \mu(a) \int_{\mathcal{K}\times\mathcal{K}} f(k_1 a k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}a$$

for every integrable smooth function f on H, where $\mu(a) = \mathrm{Vol}(\mathcal{K}a\mathcal{K})/\mathrm{Vol}(\mathcal{K})$. It suffices to show that

$$\int_{A^+} \int_{V^n} \phi_1(t^{-1}x)\phi_2(x)\Phi(a)\,\mathrm{d}x\,\mathrm{d}a$$

is absolutely convergent. For this, as we will see, it suffices to assume that $|\phi_1|$ and $|\phi_2|$ are bounded by Schwartz functions.

Case 1. E is a field and $[E:F]+s>1$. Then the simple roots of $(P_{\min}, A)$ are given by

$$\alpha_i(a) = a_i a_{i+1}^{-1}, \quad 1 \le i \le r-1; \quad \alpha_r(a) = a_r \text{ or } a_r^2.$$

Therefore $A^+ = \{a = a(a_1,\dots,a_r) \mid |a_1| \le \cdots \le |a_r| \le 1\}$. There exists a constant $C_1 > 0$ such that

$$\int_{V^n} \left|\phi_1(a^{-1}x)\phi_2(x)\right| \mathrm{d}x \le C_1 \prod_{i=1}^{r} |a_i|^n$$

for $a \in A^+$. Assume the bound $|\Phi(a)| \le C_2 \delta_{P_{\min}}(a)^{\lambda}$ where

$$\delta_{P_{\min}}(a) = \prod_{i=1}^{r} |a_i|^{m-1+[E:F]-2i}$$

and $0 \le \lambda \le \frac{1}{2}$. Then we have

$$\begin{aligned}&\int_{A^+}\int_{V^n} \left|\phi_1(a^{-1}x)\phi_2(x)\Phi(a)\right| \mathrm{d}x\, \mathrm{d}a \\ &\qquad \le C \int_{|a_1|\le\cdots\le|a_r|\le 1} \delta_{P_{\min}}(a)^{\lambda-1} \prod_{i=1}^{r}|a_i|^n\, \mathrm{d}^{\times}a_1 \cdots \mathrm{d}^{\times}a_r,\end{aligned}$$

which converges absolutely if and only if $\lambda > 1 - n/(m-3+[E:F])$.

Case 2. $s=0$ and $E=F$. Then the simple roots of $(P_{\min}, A)$ are given by

$$\alpha_i(a) = a_i a_{i+1}^{-1}, \quad 1 \le i \le r-1; \quad \alpha_r(a) = a_{r-1}a_r.$$

Therefore $A^+ = \{a = a(a_1,\dots,a_r) \mid |a_1| \le \cdots \le |a_r|; |a_{r-1}a_r| \le 1\}$, and

$$\int_{A^+}\int_{V^n} \left|\phi_1(a^{-1}x)\phi_2(x)\Phi(a)\right| \mathrm{d}x\, \mathrm{d}a \le C_1 I_1 + C_2 I_2,$$

where

$$\begin{aligned} I_1 &= \int_{|a_1|\le\cdots\le|a_r|\le 1} \delta_{P_{\min}}(a)^{\lambda-1} \prod_{i=1}^{r}|a_i|^n\, \mathrm{d}^{\times}a_1 \cdots \mathrm{d}^{\times}a_r,\\ I_2 &= \int_{|a_r|>1}\int_{|a_1|\le\cdots\le|a_{r-1}|\le|a_r|^{-1}} \delta_{P_{\min}}(a)^{\lambda-1}|a_r|^{-n} \prod_{i=1}^{r-1}|a_i|^n\, \mathrm{d}^{\times}a_1 \cdots \mathrm{d}^{\times}a_r \le I_1.\end{aligned}$$

Therefore, the integral is absolutely convergent if and only if $\lambda > 1 - n/(m-2)$.

Case 3. $E = F \oplus F$. Then the simple roots of $(P_{\min}, A)$ are given by $\alpha_i(a) = a_i a_{i+1}^{-1}$ for $1 \le i \le m-1$. Therefore $A^+ = \{a = a(a_1,\dots,a_m) \mid |a_1| \le \cdots \le |a_m|\}$, and

$$\int_{A^+}\int_{V^n} \left|\phi_1(a^{-1}x)\phi_2(x)\Phi(a)\right| \mathrm{d}x\, \mathrm{d}a \le \sum_{j=0}^{m} C_j I_j,$$

where

$$I_j = \int_{1<|a_{j+1}|\le\cdots\le|a_m|}\int_{|a_1|\le\cdots\le|a_j|\le 1} \delta_{P_{\min}}(a)^{\lambda-1} \prod_{i=1}^{j}|a_i|^n \prod_{i=j+1}^{m}|a_i|^{-n}\, \mathrm{d}^{\times}a_1 \cdots \mathrm{d}^{\times}a_m,$$

which are all absolutely convergent if and only if $\lambda > 1 - n/(m-1)$. □

From now on, we assume that $1 \le n \le m - 2 + [E : F]$ and σ satisfies the estimate in Lemma 3.19. We first consider the case where F is non-archimedean.

Lemma 3.20. *Let ϕ_1, ϕ_2, Φ be as before. Then the integral*

$$g_{\phi_1,\phi_2,\Phi}(T) = \int_{H_{\mu_T}\backslash H} \int_H \phi_1(h^{-1}\underline{h}^{-1}\mu_T)\phi_2(\underline{h}^{-1}\mu_T)\Phi(h)\,\mathrm{d}h\,\underline{\mathrm{d}}\,\underline{h} \tag{3.20}$$

is absolutely convergent for all $T \in \mathrm{Herm}_n^E(F)^{\mathrm{reg}}$ and is locally constant.

Proof. For $\gamma \in \mathrm{GL}_n(E)$, we have

$$g_{\omega_\psi(\gamma)\phi_1,\omega_\psi(\gamma)\phi_2,\Phi}(T) = g_{\phi_1,\phi_2,\Phi}({}^t\gamma^c T\gamma).$$

There is a compact open subgroup Γ of $\mathrm{GL}_n(E)$ which acts trivially on ϕ_1 and ϕ_2. Therefore, $g_{\phi_1,\phi_2,\Phi}({}^t\gamma^c T\gamma) = g_{\phi_1,\phi_2,\Phi}(T)$ for all $\gamma \in \Gamma$. In particular, $g_{\phi_1,\phi_2,\Phi}(T)$ is a locally constant function on $\mathrm{Herm}_n^E(F) \cap \mathrm{GL}_n(E)$. By Lemma 3.19, $g_{\phi_1,\phi_2,\Phi}$ is integrable on $\mathrm{Herm}_n^E(F)$, which implies that (3.20) is absolutely convergent for all $T \in \mathrm{Herm}_n^E(F)^{\mathrm{reg}}$. □

Proposition 3.21. *For $\phi_1, \phi_2 \in \mathcal{S}(V^n)$ and Φ a matrix coefficient of σ, define a function $f_{\phi_1,\phi_2,\Phi}$ on $\mathrm{Herm}_n^E(F)$ by the formula*

$$f_{\phi_1,\phi_2,\Phi}(u) := \int_H \int_{V^n} \phi_1(h^{-1}x)\phi_2(x)\psi_{q_V(x)}(u)\Phi(h)\,\mathrm{d}x\,\mathrm{d}h,$$

which is locally constant. Then $f_{\phi_1,\phi_2,\Phi} \cdot \psi_T^{-1}$ has a stable integral for $T \in \mathrm{Herm}_n^E(F)^{\mathrm{reg}}$. More precisely,

$$\begin{aligned}&\int_{\mathrm{Herm}_n^E(F)}^{\mathrm{st}} f_{\phi_1,\phi_2,\Phi}(u)\psi_T(u)^{-1}\,\mathrm{d}u \\ &\quad = \begin{cases} 0 & \text{if } \Omega_T = \emptyset, \\ \displaystyle\int_H \int_{H_\mu\backslash H} \phi_1(h^{-1}\underline{h}^{-1}\mu_T)\phi_2(\underline{h}^{-1}\mu_T)\Phi(h)\,\underline{\mathrm{d}}\,\underline{h}\,\mathrm{d}h & \text{otherwise,}\end{cases}\end{aligned}$$

if we take $\mathrm{d}x$ (resp. $\mathrm{d}u$) to be the selfdual measure on V^n (resp. $\mathrm{Herm}_n^E(F)$) with respect to ψ, and $\underline{\mathrm{d}}\,\underline{h}$ the Siegel–Weil measure.

We say a Schwartz function $\phi \in \mathcal{S}(V^n)$ is *regularly supported* if $(\mathrm{supp}\,\phi) \cap \Omega_T = \emptyset$ for all $T \in \mathrm{Herm}_n^E(F)^{\mathrm{deg}}$. In particular, for every $x \in \mathrm{supp}\,\phi$, the components of x are linearly independent.

Proof. First assume that $\mathrm{supp}\,\phi_2 \cap \Omega_T = \emptyset$. Choose a compact open subset U of $\mathrm{Herm}_n^E(F)$ such that $q_T(\mathrm{supp}\,\phi_2) \subset U$ and $T \notin U$. Then there exists a compact open subgroup $N \subset \mathrm{Herm}_n^E(F)$ such that $\psi_{T'-T}$ is non-trivial on N for all $T' \in U$. We have

$$\int_{N'} f_{\phi_1,\phi_2,\Phi}(u)\psi_T(u)^{-1}\,\mathrm{d}u = 0$$

for every compact open subgroup N' of $\mathrm{Herm}_n^E(F)$ containing N. In general, we may write $\phi_2 = \phi_2' + \phi_2''$ as a sum of two Schwartz functions such that $T \notin q_T(\mathrm{supp}\,\phi_2')$ and ϕ_2'' is regularly supported. Then

$$\int_{N'} f_{\phi_1,\phi_2,\Phi}(u)\psi_T(u)^{-1}\,\mathrm{d}u = \int_{N'} f_{\phi_1,\phi_2'',\Phi}(u)\psi_T(u)^{-1}\,\mathrm{d}u.$$

Therefore, we may assume ϕ_2 is regularly supported at the beginning. Then by Lemma 3.20,

$$f_{\phi_1,\phi_2,\Phi}(u) = \int_{\mathrm{Herm}_n^E(F)} g_{\phi_1,\phi_2,\Phi}(T)\psi_T(u)^{-1}\,\mathrm{d}T,$$

and the proposition follows from the Fourier inversion formula. □

Now we consider the case where F is archimedean.

Proposition 3.22. *Let ϕ_1, ϕ_2, Φ be as before and assume that Φ is smooth. Then the integral*

$$g_{\phi_1,\phi_2,\Phi}(T) = \int_{H_{\mu_T}\backslash H}\int_H \phi_1(h^{-1}\underline{h}^{-1}\mu_T)\phi_2(\underline{h}^{-1}\mu_T)\Phi(h)\,\mathrm{d}h\,\underline{\mathrm{d}}\,\underline{h} \tag{3.21}$$

is absolutely convergent for all $T \in \mathrm{Herm}_n^E(F)^{\mathrm{reg}}$. Moreover, $g_{\phi_1,\phi_2,\Phi}$ is a function in $\mathrm{L}^1(\mathrm{Herm}_n^E(F))$ (by Lemma 3.19) which is continuous on $\mathrm{Herm}_n^E(F)^{\mathrm{reg}}$.

Proof. The group $\mathrm{GL}_n(E)$ acts on $\mathcal{S}(V^n)$ by right multiplication. We may choose functions $\phi_0 \in \mathcal{S}(\mathrm{GL}_n(E))$ and $\phi_1', \phi_2' \in \mathcal{S}(V^n)$ such that

- $|\phi_1(x)| \le \mathrm{R}(\phi_0)\phi_1'(x)$ for all $x \in V^n$;
- $|\phi_2(x\gamma)| \le \phi_2'(x)$ for all $x \in V^n$ and $\gamma \in \operatorname{supp}\phi_0$.

Then

$$\begin{aligned}|g_{\phi_1,\phi_2,\Phi}(T)| &\le \int_{H_{\mu_T}\backslash H}\int_H\int_{\mathrm{GL}_n(E)} \phi_0(\gamma)\phi_1'(h^{-1}\underline{h}^{-1}\mu_T\gamma)\phi_2'(\underline{h}^{-1}\mu_T\gamma)|\Phi(h)|\,\mathrm{d}h\,\underline{\mathrm{d}}\,\underline{h}\,\mathrm{d}\gamma\\ &\le \int_{\mathrm{GL}_n(E)} \phi_0(\gamma)g_{\phi_1',\phi_2',|\Phi|}({}^t\gamma^cT\gamma)\,\mathrm{d}\gamma,\end{aligned}$$

which is absolutely convergent by Lemma 3.19.

For continuity, we only need to show that for a sequence $\gamma_i \in \mathrm{GL}_n(E)$ converging to 1, $g_{\phi_1,\phi_2,\Phi}({}^t\gamma_i^cT\gamma_i)$ converges to $g_{\phi_1,\phi_2,\Phi}(T)$. We can choose $\phi_1'', \phi_2'' \in \mathcal{S}(V^n)$ and a sequence c_i of positive real numbers converging to 0, such that

$$|\phi_\alpha(x\gamma_i) - \phi_\alpha(x)| \le c_i\phi_\alpha''(x)$$

for all $x \in V^n$ and $\alpha = 1, 2$. Then

$$\begin{aligned}&\left|g_{\phi_1,\phi_2,\Phi}({}^t\gamma_i^cT\gamma_i) - g_{\phi_1,\phi_2,\Phi}(T)\right|\\ &\quad\le c_i^2\int_{H_{\mu_T}\backslash H}\int_H \phi_1''(h^{-1}\underline{h}^{-1}\mu_T)\phi_2''(\underline{h}^{-1}\mu_T)|\Phi(x)|\,\mathrm{d}h\,\underline{\mathrm{d}}\,\underline{h}\\ &\qquad+ c_i\int_{H_{\mu_T}\backslash H}\int_H |\phi_1(h^{-1}\underline{h}^{-1}\mu_T)|\phi_2''(\underline{h}^{-1}\mu_T)|\Phi(x)|\,\mathrm{d}h\,\underline{\mathrm{d}}\,\underline{h}\\ &\qquad+ c_i\int_{H_{\mu_T}\backslash H}\int_H \phi_1''(h^{-1}\underline{h}^{-1}\mu_T)|\phi_2(\underline{h}^{-1}\mu_T)||\Phi(x)|\,\mathrm{d}h\,\underline{\mathrm{d}}\,\underline{h},\end{aligned}$$

which converges to 0. □

Corollary 3.23. *For $\phi_1, \phi_2 \in \mathcal{S}(V^n)$ and Φ a smooth matrix coefficient of σ, define a function $f_{\phi_1,\phi_2,\Phi}$ on $\mathrm{Herm}_n^E(F)$ by the formula*

$$f_{\phi_1,\phi_2,\Phi}(u) := \int_H \int_{V^n} \phi_1(h^{-1}x)\phi_2(x)\psi_{q_V(x)}(u)\Phi(h)\,\mathrm{d}x\,\mathrm{d}h.$$

Then $f_{\phi_1,\phi_2,\Phi}$ is a tempered distribution on $\mathrm{Herm}_n^E(F)$ and its Fourier transform is continuous on $\mathrm{Herm}_n^E(F)^{\mathrm{reg}}$. More precisely, for $T \in \mathrm{Herm}_n^E(F)^{\mathrm{reg}}$,

$$\widehat{f_{\phi_1,\phi_2,\Phi}}(\psi_T) = \begin{cases} 0 & \text{if } \Omega_T = \emptyset, \\ \displaystyle\int_H \int_{H_\mu \backslash H} \phi_1(h^{-1}\underline{h}^{-1}\mu_T)\phi_2(\underline{h}^{-1}\mu_T)\Phi(h)\,\mathrm{d}\,\underline{h}\,\mathrm{d}h & \text{otherwise,} \end{cases}$$

if we take $\mathrm{d}x$ (resp. $\mathrm{d}u$) to be the selfdual measure on V^n (resp. $\mathrm{Herm}_n^E(F)$) with respect to ψ, and $\underline{\mathrm{d}}\,\underline{h}$ the Siegel–Weil measure.

4. Global examples for $\mathrm{SO}_5 \times \mathrm{SO}_2$

In this section, F will be a number field and we fix a non-trivial additive character $\psi: F\backslash\mathbb{A} \to \mathbb{C}^\times$.

4.1. Theta integrals. Let $W, \langle\ ,\ \rangle$ be the standard symplectic space over F of dimension $2n$. Let $V, (\ ,\ )$ be a quadratic space over F of dimension $2m$ and $\mathbb{W} = V \otimes W$ equipped with the symplectic form $\langle\!\langle\ ,\ \rangle\!\rangle = (\ ,\ ) \otimes \langle\ ,\ \rangle$. Put $G = \mathrm{GSp}(W)$ and $H = \mathrm{GO}(V)$ which form a reductive dual pair in $\mathrm{GSp}(\mathbb{W})$. Denote by $H^0 = \mathrm{GSO}(V)$ the identity component of H. Write ν for the similitude character in various groups and put

$$R = \mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V)) := \{(g, h) \in G \times H \mid \nu(g) = \nu(h)\}.$$

Then $\mathrm{Sp}(W) \times \mathrm{O}(V) \subset R$. Put

$$G(\mathbb{A})^+ = \{g \in G(\mathbb{A}) \mid \text{there exists } h \in H(\mathbb{A}) \text{ such that } \nu(g) = \nu(h)\}$$

and $G(F)^+ = G(F) \cap G(\mathbb{A})^+$, $G_{\mathrm{v}}^+ = G(F_{\mathrm{v}}) \cap G(\mathbb{A})^+$ for a place v of F. We note that in the above definition of $G(\mathbb{A})^+$, we may replace $H(\mathbb{A})$ by $H^0(\mathbb{A})$.

Let $W_1 = \mathrm{Span}_F\{e_1, \dots, e_n\}$ and $W_1^* = \mathrm{Span}_F\{e_1^*, \dots, e_n^*\}$ which form a complete polarization of W such that $\langle e_i, e_j^* \rangle = \delta_{ij}$. Denote by $\mathrm{P} \subset G$ the Siegel parabolic subgroup stabilizing W_1, which has the standard Levi decomposition $\mathrm{P} = \mathrm{MN}$, where M is the subgroup stabilizing both W_1 and W_1^*, and N is the unipotent radical. Then $\mathrm{M} \simeq \mathrm{GL}_n \times F^\times$ and N is isomorphic to Sym_n. For an element $T \in \mathrm{Sym}_n(F)$, we define an automorphic character ψ_T of $\mathrm{N} \simeq \mathrm{Sym}_n$ by the following formula:

$$\psi_T(u(s)) = \psi(\mathrm{tr}\, Ts): \mathrm{N}(F)\backslash\mathrm{N}(\mathbb{A}) \to \mathbb{C}^\times.$$

In fact, every automorphic character of N is obtained in this way.

Let ω_ψ be the Weil representation of $R(\mathbb{A})$ which is realized on $\mathcal{S}(V^n(\mathbb{A}))$, the space of Schwartz functions on $V^n(\mathbb{A})$, where we identify $V \otimes W_1^*$ with V^n through the previously chosen basis. Precisely, for $(g,h) \in \mathrm{Sp}(W)(\mathbb{A}) \times \mathrm{O}(V)(\mathbb{A})$,

$$\begin{aligned}\omega_\psi(1,h)\phi(x) &= \phi(h^{-1}x),\\ \omega_\psi(m(a),1)\phi(x) &= \chi_V(\det a)|\det a|_{\mathbb{A}}^{\frac{m}{2}}\phi(xa),\\ \omega_\psi(u(s),1)\phi(x) &= \psi(\operatorname{tr} q_V(x)s)\phi(x),\\ \omega_\psi(\mathsf{w}_n,1)\phi(x) &= \hat{\phi}(x).\end{aligned}$$

For $(g,h) \in R(\mathbb{A})$, we have

$$\omega_\psi(g,h)\phi(x) = |\nu(h)|^{-\frac{n}{2}}\omega_\psi(g_1,1)\phi(h^{-1}x),$$

where

$$g_1 = \begin{bmatrix} \mathbf{1}_n & \\ & \nu(g)^{-1}\cdot \mathbf{1}_n \end{bmatrix} g \in \mathrm{Sp}(W)(\mathbb{A}).$$

For $\phi \in \mathcal{S}(V^n(\mathbb{A}))$, we attach the theta series

$$\theta(g,h;\phi) = \sum_{x \in V^n(F)} \omega_\psi(g,h)\phi(x),$$

which is a smooth function on $R(F)\backslash R(\mathbb{A})$ of moderate growth. For a cusp form $f \in \mathcal{A}_0(H)$, define the theta integral

$$\theta(f;\phi)(g) = \int_{\mathrm{O}(V)(F)\backslash \mathrm{O}(V)(\mathbb{A})} \theta(g,hh_g;\phi)f(hh_g)\,\mathrm{d}h$$

and its variant

$$\theta^0(f;\phi)(g) = \int_{\mathrm{SO}(V)(F)\backslash \mathrm{SO}(V)(\mathbb{A})} \theta(g,hh_g;\phi)f(hh_g)\,\mathrm{d}h$$

as functions on $G(\mathbb{A})^+$, where h_g is some element in $H^0(\mathbb{A})$ with $\nu(h_g) = \nu(g)$. They are left invariant under $G(F)^+$, and hence define functions on $G(F)^+\backslash G(\mathbb{A})^+$. We view them as functions on $G(F)\backslash G(\mathbb{A})$ through extension by zero with respect to the natural embedding $G(F)^+\backslash G(\mathbb{A})^+ \hookrightarrow G(F)\backslash G(\mathbb{A})$.

4.2. Global Bessel periods of theta lifting of GSp_4. Assume $m \geq n = 2$, that is, $G \simeq \mathrm{GSp}_4$. For $T \in \mathrm{Sym}_2(F)$, define

$$\mathrm{M}_T = \{(g, \det g) \in \mathrm{M} \mid {}^t gTg = \det g \cdot T\},$$

which contains the center of G that is isomorphic to $F^\times$. Assume that $\det T \neq 0$. Then T induces a (non-degenerate) quadratic form q_T on W_1. We have

$$\mathrm{M}_T = \mathrm{GSO}(W_1) \simeq \mathrm{Res}_F^K K^\times,$$

where $K = K_{q_T}$ is the discriminant quadratic algebra. According to our general convention, we will assume that K is a field. Put

$$\mathrm{M}_T(\mathbb{A})^+ = \{g \in \mathrm{M}_T(\mathbb{A}) \mid \text{there exists } h \in H^0(\mathbb{A}) \text{ such that } \nu(g) = \nu(h)\}$$

and

$$\mathrm{M}_T(F)^+ = \mathrm{M}_T(F) \cap \mathrm{M}_T(\mathbb{A})^+, \quad \mathrm{M}_{T,\mathrm{v}}^+ = \mathrm{M}_T(F_\mathrm{v}) \cap \mathrm{M}_T(\mathbb{A})^+.$$

Fix an automorphic character $\chi\colon \mathrm{M}_T(F)\backslash \mathrm{M}_T(\mathbb{A}) \to \mathbb{C}^\times$. For a cusp form $\varphi \in \mathcal{A}_0(G)$ with central character $\chi^{-1}|_{\mathbb{A}^\times}$, we define the integral

$$\mathcal{P}(\varphi,\chi) = \int_{\mathbb{A}^\times \mathrm{M}_T(F)\backslash \mathrm{M}_T(\mathbb{A})} \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \varphi(ug)\psi_T(u)^{-1}\chi(g)\,\mathrm{d}u\,\mathrm{d}g,$$

which is considered in [38].

Let σ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ realized on the space $\mathcal{V}_\sigma \subset \mathcal{A}_0(H)$, with central character $\chi_\sigma = \chi^{-1}|_{\mathbb{A}^\times}$. Let $\pi = \theta(\sigma)$ be the global theta lifting which is realized on the space

$$\mathcal{V}_\pi = \{\mathrm{R}(g)\theta(f;\phi) \mid f \in \mathcal{V}_\sigma,\ \phi \in \mathcal{S}(V^2(\mathbb{A})),\ g \in G(\mathbb{A})\}.$$

We will assume the conditions on σ specified in [9, Section 7.2] such that π is cuspidal (and hence unitary). Then for $g \in \mathrm{M}_T(\mathbb{A})^+$, we have

$$\begin{aligned}(4.1)\quad &\int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \theta^0(f;\phi)(ug)\psi_T(u)^{-1}\,\mathrm{d}u\\
&= \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \Big(\int_{\mathrm{SO}(V)(F)\backslash \mathrm{SO}(V)(\mathbb{A})} \theta(ug,hh_g;\phi) f(hh_g)\,\mathrm{d}h\Big)\psi_T(u)^{-1}\,\mathrm{d}u\\
&= \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \int_{\mathrm{SO}(V)(F)\backslash \mathrm{SO}(V)(\mathbb{A})} \sum_{x\in V^2(F)} \omega_\psi(ug,hh_g)\phi(x) f(hh_g)\psi_T(u)^{-1}\,\mathrm{d}h\,\mathrm{d}u\\
&= \int_{\mathrm{SO}(V)(F)\backslash \mathrm{SO}(V)(\mathbb{A})} \sum_{x\in V^2(F)} \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \omega_\psi(ug,hh_g)\phi(x)\psi_T(u)^{-1}\,\mathrm{d}u f(hh_g)\,\mathrm{d}h,\end{aligned}$$

where $h_g \in H^0(\mathbb{A})$ is any element such that $\nu(h_g) = \nu(g)$.

By definition,

$$\omega_\psi(ug,hh_g)\phi(x) = \chi_V(\nu(g))\phi(h_g^{-1}h^{-1}xg)\psi_{q_V(x)}(u).$$

Therefore,

$$\int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \omega_\psi(ug,hh_g)\phi(x)\psi_T(u)^{-1}\,\mathrm{d}u = 0,$$

unless there exists $x \in V^2(F)$ with the moment matrix $q_V(x) = T$. Denote by $V^2(F)_T$ the set of all such x, on which $\mathrm{SO}(V)(F)$ acts transitively. If $V^2(F)_T \neq \emptyset$, we pick up a base point μ in $V^2(F)_T$. The quadratic space (W_1, q_T) is isomorphic to the subspace generated by the components of μ. Therefore, we may identify W_1 as a quadratic subspace of V. The stabilizer of μ in $\mathrm{SO}(V)$ is $\mathrm{SO}(W_1^\perp)$. We have

$$(4.1) = \int_{\mathrm{SO}(W_1^\perp)(F)\backslash \mathrm{SO}(V)(\mathbb{A})} \phi(h^{-1}\mu) f(h_g h)\,\mathrm{d}h,$$

where $h_g \in H^0(\mathbb{A})$ is some element which preserves $W_1(\mathbb{A})$ and restricts to g on $W_1(\mathbb{A})$. Therefore,

$$(4.2)\qquad \mathcal{P}(\theta^0(f;\phi),\chi) = \int_{\mathrm{SO}(W_1^\perp)(\mathbb{A})\backslash \mathrm{SO}(V)(\mathbb{A})} \Lambda_\mu(\mathrm{R}(h)f,\chi)\phi(h^{-1}\mu)\,\mathrm{d}h,$$

in which

$$\begin{aligned}\Lambda_\mu(f,\chi) &= \int_{\mathbb{A}^\times \mathrm{GSO}(W_1)(F)\backslash \mathrm{GSO}(W_1)(\mathbb{A})} \Big(\int_{\mathrm{SO}(W_1^\perp)(F)\backslash \mathrm{SO}(W_1^\perp)(\mathbb{A})} f(g'h_g)\,\mathrm{d}g'\Big)\chi(g)\,\mathrm{d}g \\ &= \int_{\mathbb{A}^\times \mathrm{G}[\mathrm{SO}(W_1^\perp)\times\mathrm{SO}(W_1)](F)\backslash \mathrm{G}[\mathrm{SO}(W_1^\perp)\times\mathrm{SO}(W_1)](\mathbb{A})} f(g'h_g)\chi(g)\,\mathrm{d}g'\,\mathrm{d}g,\end{aligned}$$

where $(g',g) \in \mathrm{G}[\mathrm{SO}(W_1^\perp)\times\mathrm{SO}(W_1)](\mathbb{A}) \subset \mathrm{GSO}(W_1^\perp)(\mathbb{A}) \times \mathrm{GSO}(W_1)(\mathbb{A})$.

Recall that we have a short exact sequence

$$1 \to \mathrm{SO}(V) \to \mathrm{O}(V) \xrightarrow{\det} \mu_2 \to 1.$$

Fix a splitting $\mu_2 \to \mathrm{O}(V)$ and view μ_2 as a subgroup of $\mathrm{O}(V)$. For any integrable function Φ on $\mathrm{O}(V)(F)\backslash \mathrm{O}(V)(\mathbb{A})$, we have

$$\int_{\mathrm{O}(V)(F)\backslash \mathrm{O}(V)(\mathbb{A})} \Phi(h)\,\mathrm{d}h = \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})}\int_{\mathrm{SO}(V)(F)\backslash \mathrm{SO}(V)(\mathbb{A})} \mathrm{L}(\epsilon)\Phi(h_0)\,\mathrm{d}h_0\,\mathrm{d}\epsilon,$$

where $\mathrm{d}\epsilon$ is the Tamagawa measure on $\mu_2(\mathbb{A})$. In particular,

$$\theta(f;\phi)(g) = \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} \theta^0(f^\epsilon;\phi^\epsilon)(g)\,\mathrm{d}\epsilon,$$

where $f^\epsilon = \mathrm{L}(\epsilon)f$ and $\phi^\epsilon = \omega_\psi(\epsilon)\phi$.

We have

$$\begin{aligned}(4.3)\quad \big|\mathcal{P}(\theta(f;\phi),\chi)\big|^2 &= \iint_{[\mu_2(F)\backslash\mu_2(\mathbb{A})]^2} \mathcal{P}\big(\theta^0(f^\epsilon;\phi^\epsilon),\chi\big)\,\overline{\mathcal{P}(\theta^0(f^\varepsilon;\phi^\varepsilon),\chi)}\,\mathrm{d}\epsilon\,\mathrm{d}\varepsilon \\ &= \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} \mathbb{P}(f^\varepsilon;\phi^\varepsilon)\,\mathrm{d}\varepsilon,\end{aligned}$$

where

$$\mathbb{P}(f;\phi) = \int_{\mu_2(F)\backslash\mu_2(\mathbb{A})} \mathcal{P}\big(\theta^0(f^\epsilon;\phi^\epsilon),\chi\big)\,\overline{\mathcal{P}(\theta^0(f;\phi),\chi)}\,\mathrm{d}\epsilon.$$

4.3. The group GO_4. Now assume that V has dimension 4 and discriminant 1. Then there exists a unique quaternion algebra B over F with an F-algebra embedding $K \hookrightarrow B$ such that there exists an isometry $(V,q_V) \xrightarrow{\sim} (B,\nu_B)$ under which $W_1^\perp$ is mapped to K, where ν_B is the reduced norm on B. The group $B^\times \times B^\times$ acts on the vector space $V \simeq B$ by the formula $(b_1,b_2).x = b_1 x b_2^\iota$ where $\iota\colon B \to B$ is the canonical involution. In particular, we have

$$H = \mathrm{GO}(V) \simeq (B^\times \times B^\times) \rtimes \langle \mathbf{t}\rangle/\Delta F^\times,$$

where $\Delta F^\times = \{(x,x^{-1}) \mid x \in F^\times\} \subset B^\times \times B^\times$ and $\mathbf{t} \in \mathrm{O}(V)$ is the involution on $B^\times \times B^\times$ by changing two factors. We identify μ_2 with $\langle\mathbf{t}\rangle$. The following lemma is straightforward.

Lemma 4.1. *The subgroup* $\mathrm{G}[\mathrm{SO}(W_1^\perp)\times\mathrm{SO}(W_1)] \subset \mathrm{GSO}(V) \simeq B^\times \times B^\times/\Delta F^\times$ *consists of elements (x,y) with $x,y \in K^\times$, and the projection*

$$\mathrm{G}[\mathrm{SO}(W_1^\perp)\times\mathrm{SO}(W_1)] \to \mathrm{GSO}(W_1) \simeq \mathrm{Res}_F^K K^\times$$

is given by $(x,y) \mapsto xy$.

Put

$$(B^\times \times B^\times)^1 = \{(b_1, b_2) \in B^\times \times B^\times \mid \nu_B(b_1 b_2) = 1\}.$$

Then $\mathrm{SO}(V) = (B^\times \times B^\times)^1/\Delta F^\times$ and $\mathrm{SO}(W_1^\perp)\backslash \mathrm{SO}(V) = \Delta K^\times \backslash (B^\times \times B^\times)^1$ where $\Delta K^\times = \{(x, x^{-1}) \mid x \in K^\times\} \subset (B^\times \times B^\times)^1$.

4.4. Local Bessel periods of Yoshida lifts. We fix a place v of F which will be suppressed from notation in this section.

Let σ be an irreducible unitary admissible representation of H such that its big theta lifting $\Theta^+(\sigma) := \mathrm{Hom}_H(\text{c-ind}_R^{G^+\times H}\omega_\psi, \bar\sigma)$, which is a smooth representation of G^+, is non-zero. Then $\Theta^+(\sigma)$ has a unique irreducible quotient, denoted by $\theta^+(\sigma)$ (see [9, Theorem A.1]). We have a unique (up to scalar) R-equivariant map

$$\theta : \omega_\psi \otimes \sigma \to \theta^+(\sigma).$$

Let $\pi = \theta(\sigma) = \mathrm{Ind}_{G^+}^{G}\theta^+(\sigma)$, which is an irreducible admissible representation of G by [9, Lemma 5.2]. We assume that π is unitary (and non-zero), which is the case if σ is the local component of an irreducible unitary cuspidal automorphic representation that has non-zero cuspidal global theta lifting.

Let $\mathcal{B}_\omega\colon \omega_\psi \otimes \bar\omega_\psi \to \mathbb{C}$ be the canonical bilinear pairing defined by

$$\mathcal{B}_\omega(\phi, \tilde\phi) = \int_{V^2} \phi(x)\tilde\phi(x)\,\mathrm{d}x.$$

By [9, Lemma 5.6], the pairing $\mathcal{Z}^\natural\colon (\sigma\otimes\bar\sigma)\otimes(\omega_\psi\otimes\bar\omega_\psi) \to \mathbb{C}$, defined as

$$\mathcal{Z}^\natural(f, \tilde f; \phi, \tilde\phi) = \frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma,\mathrm{std})}\int_{\mathrm{O}(V)} \mathcal{B}_\omega\big(\omega_\psi(h)\phi, \tilde\phi\big)\mathcal{B}_\sigma\big(\sigma(h)f, \tilde f\big)\,\mathrm{d}h,$$

where the integral is absolutely convergent by Lemma 3.19, descends to a bilinear pairing $\mathcal{B}_\pi\colon \pi\otimes\bar\pi \to \mathbb{C}$. In other words, we have

$$\mathcal{B}_\pi\big(\theta(f,\phi), \theta(\tilde f, \tilde\phi)\big) = \mathcal{Z}^\natural(f, \tilde f; \phi, \tilde\phi).$$

Lemma 4.2. *Let $g \in \mathrm{M}_T^+$.*

(1) *If F is non-archimedean, the function $\mathcal{B}_\pi(\pi(\bullet g)\theta(f,\phi), \theta(\tilde f, \tilde\phi))\psi_T^{-1}$ has a stable integral. More precisely,*

$$\int_{\mathrm{N}}^{\mathrm{st}} \mathcal{B}_\pi\big(\pi(ug)\theta(f,\phi), \theta(\tilde f,\tilde\phi)\big)\psi_T(u)^{-1}\,\mathrm{d}u = \frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma,\mathrm{std})}$$
$$\times \int_{\mathrm{O}(V)}\int_{\mathrm{SO}(W_1^\perp)\backslash \mathrm{SO}(V)} \phi(h_g^{-1}h^{-1}\underline{h}^{-1}\mu g)\tilde\phi(\underline{h}^{-1}\mu)\mathcal{B}_\sigma\big(\sigma(hh_g)f, \tilde f\big)\,\underline{\mathrm{d}}\,\underline{h}\,\mathrm{d}h,$$

where $h_g \in H^0$ is some element which preserves W_1 and restricts to g on W_1; $\mathrm{d}u$ is the selfdual measure on N with respect to ψ; and $\underline{\mathrm{d}}\,\underline{h}$ is the Siegel–Weil measure in Lemma 3.17.

(2) *If F is archimedean, the function $\mathcal{B}_\pi(\pi(\bullet g)\theta(f,\phi),\theta(\tilde{f},\tilde{\phi}))$ is a tempered distribution on* N *whose Fourier transform is continuous on* $\mathrm{N}^{\mathrm{reg}} \simeq \mathrm{Sym}_2(F)^{\mathrm{reg}}$. *More precisely,*

$$\widehat{\mathcal{B}_\pi\big(\pi(\bullet g)\theta(f,\phi),\theta(\tilde{f},\tilde{\phi})\big)}(\psi_T) = \frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma,\mathrm{std})} \times \int_{\mathrm{O}(V)}\int_{\mathrm{SO}(W_1^\perp)\backslash\mathrm{SO}(V)} \phi(h_g^{-1}h^{-1}\underline{h}^{-1}\mu g)\tilde{\phi}(\underline{h}^{-1}\mu)\mathcal{B}_\sigma\big(\sigma(hh_g)f,\tilde{f}\big)\,\underline{\mathrm{d}}\,\underline{h}\,\mathrm{d}h,$$

where $h_g \in H^0$, du *and* $\underline{\mathrm{d}}\,\underline{h}$ *are as in case* (1).

In both cases, we use the same Haar measure dh *on* O(V) *on the two sides of the identities.*

Proof. We apply Proposition 3.21 for (1) and Corollary 3.23 for (2). Notice that $n = 2$, $m = 4$ and $E = F$. □

For $\varphi \in \pi$ and $\tilde{\varphi} \in \bar{\pi}$, we put

$$\mathfrak{b}_\pi^{\varphi,\tilde{\varphi}}(g) = \begin{cases} \int_{\mathrm{N}}^{\mathrm{st}} \mathcal{B}_\pi\big(\pi(ug)\varphi,\tilde{\varphi}\big)\psi_T(u)^{-1}\,\mathrm{d}u & \text{if } F \text{ is non-archimedean,} \\ \widehat{\mathcal{B}_\pi\big(\pi(\bullet g)\theta(f,\phi),\theta(\tilde{f},\tilde{\phi})\big)}(\psi_T) & \text{if } F \text{ is archimedean.} \end{cases}$$

Define

$$\alpha^\natural(\varphi,\tilde{\varphi};\chi) = \frac{L(1,\sigma,\mathrm{Ad})L(1,\chi_{K/F})}{\zeta_F(2)\zeta_F(4)L(\frac{1}{2},\sigma\boxtimes\chi)} \int_{F^\times\backslash\mathrm{M}_T} \mathfrak{b}_\pi^{\varphi,\tilde{\varphi}}(g)\chi(g)\,\mathrm{d}g. \tag{4.4}$$

Take the measure $\mathrm{d}h_0 = 2\,\mathrm{d}h|_{\mathrm{SO}(V)}$, which is twice the restriction of the measure dh to SO(V). Then

$$\left(\frac{\zeta_F(2)\zeta_F(4)}{L(1,\sigma,\mathrm{std})}\right)^{-1}\mathfrak{b}_\pi^{\varphi,\tilde{\varphi}}(g) = \sum_{\epsilon=\pm1}\frac{1}{2}\int_{\mathrm{SO}(V)}\int_{\mathrm{SO}(W_1^\perp)\backslash\mathrm{SO}(V)} \phi^\epsilon(h_g^{-1}h_0^{-1}\underline{h}^{-1}\mu g) \times \tilde{\phi}(\underline{h}^{-1}\mu)\mathcal{B}_\sigma\big(\sigma(h_0h_g)f^\epsilon,\tilde{f}\big)\,\underline{\mathrm{d}}\,\underline{h}\,\mathrm{d}h_0. \tag{4.5}$$

Simplify and substitute $h = h_g^{-1}\underline{h}h_0h_g$, $\tilde{h} = \underline{h}$. We have

$$(4.5) = \sum_{\epsilon=\pm1}\frac{1}{2}\int_{\mathrm{SO}(V)}\int_{\mathrm{SO}(W_1^\perp)\backslash\mathrm{SO}(V)} \phi^\epsilon(h^{-1}\mu)\tilde{\phi}(\tilde{h}^{-1}\mu)\mathcal{B}_\sigma\big(\sigma(\tilde{h}^{-1}h_gh)f^\epsilon,\tilde{f}\big)\,\underline{\mathrm{d}}\tilde{h}\,\mathrm{d}h,$$

which is equal to

$$\sum_{\epsilon=\pm1}\frac{1}{2}\int_{\mathrm{SO}(W_1^\perp)\backslash\mathrm{SO}(V)}\int_{\mathrm{SO}(W_1^\perp)\backslash\mathrm{SO}(V)}\int_{\mathrm{SO}(W_1^\perp)} \phi^\epsilon(h^{-1}\mu) \times \tilde{\phi}(\tilde{h}^{-1}\mu)\mathcal{B}_\sigma\big(\sigma(g'h_gh)f^\epsilon,\bar{\sigma}(\tilde{h})\tilde{f}\big)\,\mathrm{d}g'\,\underline{\mathrm{d}}\tilde{h}\,\mathrm{d}h.$$

Introduce a new measure

$$\mathrm{d}'\tilde{h} = \frac{\zeta_F(2)^2}{L(1,\chi_{K/F})}\,\underline{\mathrm{d}}\tilde{h}$$

on the (inside) quotient group $\mathrm{SO}(W_1^{\perp})\backslash \mathrm{SO}(V)$ to make the volume of the orbit of the maximal compact open subgroup to be 1 at unramified places. We have

$$(4.4) = \sum_{\epsilon=\pm 1} \frac{1}{2} \iint_{[\mathrm{SO}(W_1^{\perp})\backslash \mathrm{SO}(V)]^2} \phi^{\epsilon}(h^{-1}\mu)\tilde{\phi}(\tilde{h}^{-1}\mu)\mathbb{L}_{\mu}(\sigma(h)f^{\epsilon}, \bar{\sigma}(\tilde{h})\tilde{f}; \chi)\,\mathrm{d}'\tilde{h}\,\mathrm{d}h, \tag{4.6}$$

where

$$\mathbb{L}_{\mu}(f, \tilde{f}; \chi) = \frac{L(1, \sigma_1, \mathrm{Ad})L(1, \sigma_2, \mathrm{Ad})L(1, \chi_{K/F})^2}{\zeta_F(2)^2 L(\frac{1}{2}, \sigma_1, \chi)L(\frac{1}{2}, \sigma_2, \chi)} \times \int_{F^{\times}\backslash \mathrm{GSO}(W_1)} \Big(\int_{\mathrm{SO}(W_1^{\perp})} \mathcal{B}_{\sigma}\big(\sigma(g'h_g)f, \tilde{f}\big)\,\mathrm{d}g'\Big)\chi(g)\,\mathrm{d}g.$$

4.5. Waldspurger formula. Let K be a quadratic field extension of F. Let B be a quaternion algebra over F with a fixed embedding $K \hookrightarrow B$. Let $\sigma \simeq \otimes_{\mathrm{v}}\sigma_{\mathrm{v}}$ be an irreducible unitary cuspidal automorphic representation of $B^{\times}(\mathbb{A})$ realized on $\mathcal{V}_{\sigma} \subset \mathcal{A}_0(B^{\times})$ such that $\chi_{\sigma} \cdot \chi|_{\mathbb{A}^{\times}} = 1$. For $f \in \mathcal{V}_{\sigma}$, define the following period integral:

$$\mathcal{Q}_{\chi}(f) = \int_{\mathbb{A}^{\times}K^{\times}\backslash \mathbb{A}_K^{\times}} f(t)\chi(t)\,\mathrm{d}t.$$

Let v be a place of F. For $f_{\mathrm{v}} \in \sigma_{\mathrm{v}}$ and $\tilde{f}_{\mathrm{v}} \in \bar{\sigma}_{\mathrm{v}}$, the integral

$$\beta_{\mathrm{v}}(f_{\mathrm{v}}, \tilde{f}_{\mathrm{v}}; \chi_{\mathrm{v}}) = \int_{F_{\mathrm{v}}^{\times}\backslash K_{\mathrm{v}}^{\times}} \mathcal{B}_{\sigma_{\mathrm{v}}}\big(\sigma_{\mathrm{v}}(t_{\mathrm{v}})f_{\mathrm{v}}, \tilde{f}_{\mathrm{v}}\big)\chi_{\mathrm{v}}(t_{\mathrm{v}})\,\mathrm{d}t_{\mathrm{v}}$$

is absolutely convergent. Put

$$\beta_{\mathrm{v}}^{\natural}(f_{\mathrm{v}}, \tilde{f}_{\mathrm{v}}; \chi_{\mathrm{v}}) = \frac{L(1, \sigma_{\mathrm{v}}, \mathrm{Ad})L(1, \chi_{K_{\mathrm{v}}/F_{\mathrm{v}}})}{\zeta_{F_{\mathrm{v}}}(2)L(\frac{1}{2}, \sigma_{\mathrm{v}}, \chi_{\mathrm{v}})}\beta_{\mathrm{v}}(f_{\mathrm{v}}, \tilde{f}_{\mathrm{v}}; \chi_{\mathrm{v}}),$$

where the L-factors assure that $\beta_{\mathrm{v}}^{\natural}(f_{\mathrm{v}}, \tilde{f}_{\mathrm{v}}; \chi_{\mathrm{v}}) = 1$ for almost all places v.

Assume that $C_{B^{\times}} = 1$ and $\mathcal{B}_{\sigma} = \prod_{\mathrm{v}} \mathcal{B}_{\sigma_{\mathrm{v}}}$. We have the following formula due to Waldspurger (see [43] and [23, Section 6]):

$$\mathcal{Q}_{\chi}(f)\mathcal{Q}_{\bar{\chi}}(\tilde{f}) = \frac{1}{2}\frac{\zeta_F(2)L(\frac{1}{2}, \sigma, \chi)}{L(1, \sigma, \mathrm{Ad})L(1, \chi_{K/F})}\prod \beta_{\mathrm{v}}^{\natural}(f_{\mathrm{v}}, \tilde{f}_{\mathrm{v}}; \chi_{\mathrm{v}}). \tag{4.7}$$

4.6. Period formula of Yoshida lifts. Assume that σ is induced by the representation $\sigma_1 \boxtimes \sigma_2$ of $B^{\times}(\mathbb{A}) \times B^{\times}(\mathbb{A})$. For $f = f_1 \otimes f_2 \in \mathcal{V}_{\sigma_1} \otimes \mathcal{V}_{\sigma_2}$, we have

$$\Lambda_{\mu}(f, \chi)(h) = \int_{\mathbb{A}^{\times}K^{\times}\backslash \mathbb{A}_K^{\times}} f_1(x_1h_1)\chi(x_1)\,\mathrm{d}x_1 \int_{\mathbb{A}^{\times}K^{\times}\backslash \mathbb{A}_K^{\times}} f_2(x_2h_2)\chi(x_2)\,\mathrm{d}x_2$$

for $h = (h_1, h_2) \in \mathrm{SO}(V)(\mathbb{A})$. Locally by Lemma 4.1,

$$\mathbb{L}_{\mu}(f, \tilde{f}; \chi) = \beta^{\natural}(f_1, \tilde{f}_1; \chi)\beta^{\natural}(f_2, \tilde{f}_2; \chi).$$

Assume that $C_{\mathrm{SO}(V)} = C_{\mathrm{SO}(W_1)} = C_{\mathrm{SO}(W_1^{\perp})} = 1$. By (4.2) and the Waldspurger formula (4.7),

$$\begin{aligned}
\mathbb{P}(f;\phi) &= \frac{1}{4}\frac{\zeta_F(2)^2 L(\frac{1}{2},\sigma_1,\chi)L(\frac{1}{2},\sigma_2,\chi)}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})L(1,\chi_{K/F})^2}\int_{\mu_2(F)\backslash\mu_2(\mathbb{A})}\prod\iint_{[\mathrm{SO}(W_1^{\perp})_{\mathrm{v}}\backslash\,\mathrm{SO}(V)_{\mathrm{v}}]^2} \\
&\quad \Big(\prod_{\alpha=1,2}\beta_{\mathrm{v}}^{\natural}((\sigma(h_{\mathrm{v}})f_{\mathrm{v}}^{\epsilon})_\alpha,(\bar{\sigma}(\tilde{h}_{\mathrm{v}})\bar{f}_{\mathrm{v}})_\alpha;\chi_{\mathrm{v}})\Big)\phi_{\mathrm{v}}^{\epsilon}(h_{\mathrm{v}}^{-1}\mu)\bar{\phi}_{\mathrm{v}}(\tilde{h}_{\mathrm{v}}^{-1}\mu)\,\mathrm{d}h_{\mathrm{v}}\,\mathrm{d}\tilde{h}_{\mathrm{v}}\,\mathrm{d}\epsilon \\
&= \frac{1}{4}\frac{\zeta_F(2)^2 L(\frac{1}{2},\sigma_1,\chi)L(\frac{1}{2},\sigma_2,\chi)}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})L(1,\chi_{K/F})^2}\int_{\mu_2(F)\backslash\mu_2(\mathbb{A})}\prod\iint_{[\mathrm{SO}(W_1^{\perp})_{\mathrm{v}}\backslash\,\mathrm{SO}(V)_{\mathrm{v}}]^2} \\
&\quad \mathbb{L}_\mu\big(\sigma(h_{\mathrm{v}})f_{\mathrm{v}}^{\epsilon},\bar{\sigma}(\tilde{h}_{\mathrm{v}})\bar{f}_{\mathrm{v}};\chi_{\mathrm{v}}\big)\phi_{\mathrm{v}}^{\epsilon}(h_{\mathrm{v}}^{-1}\mu)\bar{\phi}_{\mathrm{v}}(\tilde{h}_{\mathrm{v}}^{-1}\mu)\,\mathrm{d}h_{\mathrm{v}}\,\mathrm{d}\tilde{h}_{\mathrm{v}}\,\mathrm{d}\epsilon \\
&= \frac{1}{8}\frac{L(\frac{1}{2},\sigma_1,\chi)L(\frac{1}{2},\sigma_2,\chi)}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})L(1,\chi_{K/F})}\int_{\mu_2(\mathbb{A})}\prod\iint_{[\mathrm{SO}(W_1^{\perp})_{\mathrm{v}}\backslash\,\mathrm{SO}(V)_{\mathrm{v}}]^2} \\
&\quad \mathbb{L}_\mu\big(\sigma(h_{\mathrm{v}})f_{\mathrm{v}}^{\epsilon},\bar{\sigma}(\tilde{h}_{\mathrm{v}})\bar{f}_{\mathrm{v}};\chi_{\mathrm{v}}\big)\phi_{\mathrm{v}}^{\epsilon}(h_{\mathrm{v}}^{-1}\mu)\bar{\phi}_{\mathrm{v}}(\tilde{h}_{\mathrm{v}}^{-1}\mu)\,\mathrm{d}h_{\mathrm{v}}\,\mathrm{d}'\tilde{h}_{\mathrm{v}}\,\mathrm{d}\epsilon.
\end{aligned}$$

By (4.6), the above expression equals

$$\frac{1}{8}\frac{L(\frac{1}{2},\sigma_1,\chi)L(\frac{1}{2},\sigma_2,\chi)}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})L(1,\chi_{K/F})}\prod\alpha_{\mathrm{v}}^{\natural}\big(\theta(f_{\mathrm{v}},\phi_{\mathrm{v}}),\theta(\bar{f}_{\mathrm{v}},\bar{\phi}_{\mathrm{v}});\chi_{\mathrm{v}}\big),$$

since

$$\alpha_{\mathrm{v}}^{\natural}\big(\theta(f_{\mathrm{v}},\phi_{\mathrm{v}}),\theta(\bar{f}_{\mathrm{v}},\bar{\phi}_{\mathrm{v}});\chi_{\mathrm{v}}\big) = \alpha_{\mathrm{v}}^{\natural}\big(\theta(f_{\mathrm{v}}^{\varepsilon},\phi_{\mathrm{v}}),\theta(\bar{f}_{\mathrm{v}}^{\varepsilon},\bar{\phi}_{\mathrm{v}});\chi_{\mathrm{v}}\big)$$

for any $\varepsilon\in\mu_2(\mathbb{A})$. We have

$$(4.3) = \frac{1}{16}\frac{L(\frac{1}{2},\sigma_1,\chi)L(\frac{1}{2},\sigma_2,\chi)}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})L(1,\chi_{K/F})}\prod\alpha_{\mathrm{v}}^{\natural}\big(\theta(f_{\mathrm{v}},\phi_{\mathrm{v}}),\theta(\bar{f}_{\mathrm{v}},\bar{\phi}_{\mathrm{v}});\chi_{\mathrm{v}}\big).$$

We fix a decomposition $\mathcal{B}_{\sigma_\alpha} = \prod\mathcal{B}_{\sigma_{\alpha,\mathrm{v}}}$ for $\alpha = 1,2$. The local bilinear pairing $\mathcal{B}_{\sigma_{1,\mathrm{v}}}\otimes\mathcal{B}_{\sigma_{2,\mathrm{v}}}$ induces a local bilinear pairing $\mathcal{B}_{\sigma_{\mathrm{v}}^0}$ for the representation σ_{v}^0 of H_{v}^0, and then induces a local bilinear pairing $\mathcal{B}_{\sigma_{\mathrm{v}}}$ for the representation σ_{v} of H_{v}. We have $\mathcal{B}_{\sigma^0} = \frac{1}{2}\prod\mathcal{B}_{\sigma_{\mathrm{v}}^0}$. By [9, Lemma 2.1], we have $\mathcal{B}_{\sigma^0} = 2\mathcal{B}_\sigma$. Therefore, $\mathcal{B}_\sigma = \frac{1}{4}\prod\mathcal{B}_{\sigma_{\mathrm{v}}^0}$. Combining with [9, Proposition 7.13], we have

$$\mathcal{B}_\pi = \frac{1}{4}\frac{L(1,\sigma,\mathrm{std})}{\zeta_F(2)\zeta_F(4)}\prod\mathcal{B}_{\pi_{\mathrm{v}}}.$$

Moreover, since the Tamagawa number of $\mathrm{M}_T/F^\times\simeq K^\times/F^\times$ is 2, we have

$$\mathcal{B}_\chi = 2\prod\mathcal{B}_{\chi_{\mathrm{v}}}.$$

Altogether, we have proved the following theorem.

Theorem 4.3. *Let $(\pi,\mathcal{V}_\pi)$ be an irreducible unitary cuspidal (endoscopic) automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, that is, a theta lifting from $\mathrm{GO}(V)$ for some quadratic space V of dimension 4 and discriminant 1. For a non-degenerate symmetric matrix $T\in\mathrm{Sym}_2(F)$, denote $\mathrm{M}_T = \mathrm{GSO}(W_1,q_T)\simeq\mathrm{GSO}_2$ as above. Let χ be an automorphic character of $\mathrm{M}_T(\mathbb{A})$ such that $\chi_\pi\cdot\chi|_{\mathbb{A}^\times} = 1$. Assume that $\mathcal{B}_\pi = \prod\mathcal{B}_{\pi_{\mathrm{v}}}$, $\mathcal{B}_\chi = \prod\mathcal{B}_{\chi_{\mathrm{v}}}$, and the Haar measure constant $C_{M_T} = 1$. Then for each vector $\varphi = \otimes\varphi_{\mathrm{v}}\in\mathcal{V}_\pi$,*

$$|\mathcal{P}(\varphi,\chi)|^2 = \frac{1}{8}\frac{\zeta_F(2)\zeta_F(4)L(\frac{1}{2},\pi\boxtimes\chi)}{L(1,\pi,\mathrm{Ad})L(1,\chi_{K/F})}\prod_{\mathrm{v}}\alpha_{\mathrm{v}}^{\sharp}(\varphi_{\mathrm{v}},\bar{\varphi}_{\mathrm{v}};\chi_{\mathrm{v}}).$$

Remark 4.4. (1) When π has the trivial central character, we may view π (resp. χ) as an irreducible cuspidal automorphic representation of $\mathbb{A}^\times\backslash \mathrm{GSp}_4(\mathbb{A}) \simeq \mathrm{SO}_5(\mathbb{A})$ (resp. $\mathbb{A}^\times\backslash \mathrm{M}_T(\mathbb{A}) \simeq \mathrm{SO}_2(\mathbb{A})$). Then the above theorem confirms Conjecture 2.5 for such (π, χ) since $|\mathcal{S}_{\Psi(\pi)}| = 4$ and $|\mathcal{S}_{\Psi(\chi)}| = 2$.

(2) When φ is the Yoshida lift of two classical normalized newforms of weight 2 (of $\mathrm{PGL}_{2,\mathbb{Q}}$) and χ is trivial, a similar formula in Theorem 4.3 can be deduced from the work of Böcherer and Schulze-Pillot [3, Corollary 2.4] and Furusawa [6, (4.3.4)], where they use classical language.

(3) By the Gan–Gross–Prasad conjecture [8], we should also consider pure inner forms of the pair (G_2, G_0). In this case, we should consider not only GSp_4, but also GSp_4^D, whose adjoint group is isomorphic to $\mathrm{SO}(V_2)$ with non-split quadratic space V_2 of dimension 5. More precisely, let D be a quaternion division algebra over F whose local Hasse invariants coincide with those of V_2. Define

$$G = \mathrm{GSp}_4^D := \{g \in \mathrm{GL}_2(D) \mid g\mathbf{w}_2\, {}^t g^\iota = \lambda \cdot \mathbf{w}_2,\ \lambda \in F^\times\},$$

where ι is the standard involution on D. Let $V, \langle\ ,\ \rangle$ be a skew-hermitian left D-module of rank 2 and $H = \mathrm{GU}(V)$. Then there is a theta lifting from H to G, analogous to Yoshida lifts discussed above. In fact, a cuspidal automorphic representation of $H(\mathbb{A})$ is given by two cuspidal automorphic representations, one of $B_1^\times(\mathbb{A})$ and the other of $B_2^\times(\mathbb{A})$, with the same central character, where B_1 and B_2 are two (different) quaternion algebras over F. The previous calculation works in this case as well, except that we need to use a regularized Siegel–Weil formula for quaternionic hermitian spaces (see [49, p. 1386, Theorem] for $\epsilon = 1$ and $m = n = 2$), or rather its corollary, the Rallis inner product formula. We then also have Theorem 4.3 for inner forms.

(4) The unrefined version of Theorem 4.3, that is, the equivalence between non-vanishing of L-values and non-triviality of period integrals, has been obtained in [38] (including inner forms). In fact, our explicit calculation in this section is inspired by that article.

5. Global examples for $\mathrm{SO}_6 \times \mathrm{SO}_3$

In this section, F will be a number field; E/F will be a quadratic field extension; and we fix a non-trivial additive character $\psi\colon F\backslash\mathbb{A} \to \mathbb{C}^\times$.

5.1. Theta integrals and regularized Rallis inner product formula. Let $W, \langle\ ,\ \rangle$ be the standard skew-hermitian space over E of dimension $2n$. Let $V, (\ ,\ )$ be a hermitian space over E of dimension m. Put $\mathbb{W} = \mathrm{Res}^E_F V \otimes_E W$ which is naturally a symplectic space over F. Put $G = \mathrm{GU}(W) \simeq \mathrm{GU}_{n,n}$ and $H = \mathrm{GU}(V)$ which form a reductive dual pair in $\mathrm{GSp}(\mathbb{W})$. Denote by ν the similitude character in various groups and put

$$R = \mathrm{G}(\mathrm{U}(W) \times \mathrm{U}(V)) := \{(g,h) \in G \times H \mid \nu(g) = \nu(h)\}.$$

Then $\mathrm{U}(W) \times \mathrm{U}(V) \subset R$. Put

$$G(\mathbb{A})^+ = \{g \in G(\mathbb{A}) \mid \text{there exists } h \in H(\mathbb{A}) \text{ such that } \nu(g) = \nu(h)\}$$

and $G(F)^+ = G(F) \cap G(\mathbb{A})^+$, $G_{\mathrm{v}}^+ = G(F_{\mathrm{v}}) \cap G(\mathbb{A})^+$ for a place v of F. The following lemma is elementary.

Lemma 5.1. *Assume that m is odd. Then $G(\mathbb{A})^+$ (resp. G_{v}^+) consists of all g such that $\nu(g) \in \mathrm{N}\,\mathbb{A}_E^\times$ (resp. $\nu(g) \in \mathrm{N}\,E_{\mathrm{v}}$). In particular, the natural map*

$$\mathbb{A}_E^{\times,1}\,\mathrm{U}(W)(F)\backslash \mathrm{U}(W)(\mathbb{A}) \to \mathbb{A}_E^\times G(F)^+\backslash G(\mathbb{A})^+$$

is an isomorphism.

Let $W_1 = \mathrm{Span}_E\{e_1, \dots, e_n\}$ and $W_1^* = \mathrm{Span}_E\{e_1^*, \dots, e_n^*\}$ which form a complete polarization of W such that $\langle e_i, e_j^*\rangle = \delta_{ij}$. Denote by $\mathrm{P} \subset G$ the hermitian Siegel parabolic subgroup stabilizing W_1, which has the standard Levi decomposition $\mathrm{P} = \mathrm{MN}$, where M is the subgroup stabilizing both W_1 and W_1^*, and N is the unipotent radical. Then N is isomorphic to Herm_n^E. For an element $T \in \mathrm{Herm}_n^E(F)$, we define an automorphic character ψ_T of $\mathrm{N} \simeq \mathrm{Herm}_n^E$ by the following formula:

$$\psi_T(u(s)) = \psi(\mathrm{tr}\, Ts)\colon \mathrm{N}(F)\backslash\mathrm{N}(\mathbb{A}) \to \mathbb{C}^\times.$$

In fact, every automorphic additive character of N is obtained in this way.

Fix an automorphic character $\chi\colon E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ such that $\chi|_{\mathbb{A}^\times} = \chi_{E/F}^m$. Let $\omega_{\psi,\chi}$ be the Weil representation of $R(\mathbb{A})$ which is realized on $\mathcal{S}(V^n(\mathbb{A}_E))$, the space of Schwartz functions on $V^n(\mathbb{A}_E)$, where we have identified $V \otimes_E W_1^*$ with V^n through the previously chosen basis. In particular,

$$\omega_{\psi,\chi}(m(a), 1)\phi(x) = \chi(\det a)|\det a|_{\mathbb{A}}^{\frac{m}{2}}\phi(xa)$$

for $m(a) \in \mathrm{M}(\mathbb{A})$. For $\phi \in \mathcal{S}(V^n(\mathbb{A}_E))$, we define the theta series to be

$$\theta_\chi(g, h; \phi) = \sum_{x \in V^n(E)} \omega_{\psi,\chi}(g, h)\phi(x),$$

which is a smooth function on $R(F)\backslash R(\mathbb{A})$ of moderate growth. For a cusp form $f \in \mathcal{A}_0(H)$, define the theta integral

$$\theta_\chi(f;\phi)(g) = \int_{\mathrm{U}(V)(F)\backslash\mathrm{U}(V)(\mathbb{A})} \theta_\chi(g, hh_g;\phi) f(hh_g)\,\mathrm{d}h$$

as a function on $G(\mathbb{A})^+$, where h_g is some element in $H(\mathbb{A})$ with $\nu(h) = \nu(g)$. The theta integral is left invariant under $G(F)^+$, and hence defines a function on $G(F)^+\backslash G(\mathbb{A})^+$. We view $\theta_\chi(f;\phi)$ as a function on $G(F)\backslash G(\mathbb{A})$ through extension by zero with respect to the natural embedding $G(F)^+\backslash G(\mathbb{A})^+ \hookrightarrow G(F)\backslash G(\mathbb{A})$.

We have the following special case of regularized Rallis inner product formula, in view of Lemma 5.1.

Theorem 5.2 ([10, Theorem 1.3]). *Assume that $n = 2$ and $m = 3$. Let σ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ realized on $\mathcal{V}_\sigma \subset \mathcal{A}(H)$. Then for $f = \otimes_{\mathrm{v}} f_{\mathrm{v}} \in \mathcal{V}_\sigma$ and $\phi = \otimes_{\mathrm{v}}\phi_{\mathrm{v}} \in \mathcal{S}(V^2(\mathbb{A}_E))$,*

$$\int_{\mathbb{A}^\times G(F)\backslash G(\mathbb{A})} \theta_\chi(f,\phi)(g)\,\overline{\theta_\chi(f,\phi)(g)}\,\mathrm{d}g = \frac{2L(1,\sigma,\mathrm{std})}{\prod_{i=1}^4 L(i, \chi_{E/F}^i)} \prod \mathcal{Z}_{\mathrm{v}}^\natural(f_{\mathrm{v}}, \bar{f}_{\mathrm{v}}; \phi_{\mathrm{v}}, \bar{\phi}_{\mathrm{v}}),$$

where the normalized zeta integral $\mathcal{Z}_{\mathrm{v}}^\natural$ is defined in (5.2). *Here, we normalize the local Haar measures such that $C_{\mathrm{U}(V)} = 1$ and the bilinear pairing $\mathcal{B}_{\sigma_{\mathrm{v}}}$ such that $\mathcal{B}_\sigma = \prod_{\mathrm{v}} \mathcal{B}_{\sigma_{\mathrm{v}}}$.*

5.2. Global Bessel periods of theta lifting of $\mathrm{GU}_{2,2}$. Assume that $n = 2$ and $m \geq n$. In particular, $G \simeq \mathrm{GU}_{2,2}$ and $\mathrm{M} \simeq \mathrm{Res}^E_F \mathrm{GL}_{2,E} \times F^\times$. For $T \in \mathrm{Herm}^E_2(F)$, define

$$\mathrm{M}_T = \{(g, \nu) \in \mathrm{M} \mid {}^t g^c T g = \det g \cdot T\}.$$

Assume that $\det T \neq 0$. Then T induces a (non-degenerate) hermitian form q_T on W_1. In particular, M_T is identified with $\mathrm{GU}(W_1, q_T)$ or simply $\mathrm{GU}(W_1)$. Put

$$\mathrm{M}_T(\mathbb{A})^+ = \{g \in \mathrm{M}_T(\mathbb{A}) \mid \text{there exists } h \in H(\mathbb{A}) \text{ such that } \nu(g) = \nu(h)\}$$

and

$$\mathrm{M}_T(F)^+ = \mathrm{M}_T(F) \cap \mathrm{M}_T(\mathbb{A})^+, \quad \mathrm{M}^+_{T,\mathrm{v}} = \mathrm{M}_T(F_\mathrm{v}) \cap \mathrm{M}_T(\mathbb{A})^+.$$

Let π (resp. π_0) be an irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$ (resp. $\mathrm{M}_T(\mathbb{A})$) realized on the space $\mathcal{V}_\pi$ (resp. $\mathcal{V}_{\pi_0}$) with the central character χ_π (resp. χ_{π_0}). Assume that $\chi_\pi \cdot \chi_{\pi_0} = 1$. For $\varphi \in \mathcal{V}_\pi$ and $\varphi_0 \in \mathcal{V}_{\pi_0}$, define the integral

$$\mathcal{P}(\varphi, \varphi_0) = \int_{\mathbb{A}^\times_E \mathrm{M}_T(F) \backslash \mathrm{M}_T(\mathbb{A})} \int_{\mathrm{N}(F) \backslash \mathrm{N}(\mathbb{A})} \varphi(ug) \psi_T(u)^{-1} \varphi_0(g) \, \mathrm{d}u \, \mathrm{d}g.$$

Now assume that $\pi \subset \theta_\chi(\sigma)$ for an irreducible unitary cuspidal automorphic representation $(\sigma, \mathcal{V}_\sigma)$ of $H(\mathbb{A})$ whose central character χ_σ satisfies $\chi^2 \cdot \chi_\sigma \cdot \chi_{\pi_0} = 1$. Here,

$$\theta_\chi(\sigma) = \{\mathrm{R}(g)\theta_\chi(f; \phi) \mid f \in \mathcal{V}_\sigma, \ \phi \in \mathcal{S}(V^2(\mathbb{A}_E)), \ g \in G(\mathbb{A})\}.$$

As in Section 4.2, $\mathcal{P}(\theta_\chi(f, \phi), \varphi_0) = 0$ if W_1 cannot be embedded into V as hermitian spaces. Otherwise, we fix an embedding $\mu\colon W_1 \to V$, that is, an element $\mu \in V^2(E)$ such that $q_V(\mu) = T$. Then

$$\mathcal{P}(\theta_\chi(f, \phi), \varphi_0) = \int_{\mathrm{U}(W_1^\perp)(\mathbb{A}) \backslash H(\mathbb{A})} \Lambda_\mu(\mathrm{R}(h) f, \varphi_0) \phi(h^{-1}\mu) \, \mathrm{d}h, \tag{5.1}$$

where

$$\Lambda_\mu(f, \varphi_0) = \int_{\mathbb{A}^\times_E \mathrm{M}_T(F)^+ \backslash \mathrm{M}_T(\mathbb{A})^+} \int_{\mathrm{U}(W_1^\perp)(F) \backslash \mathrm{U}(W_1^\perp)(\mathbb{A})} \chi(\det g) f(g' h_g) \varphi_0(g) \, \mathrm{d}g' \, \mathrm{d}g.$$

5.3. Local Bessel periods of theta lifting. We fix a place v of F which will be suppressed from notation in this section. We temporarily allow arbitrary m, n with $1 \leq m \leq 2n$.

Lemma 5.3. *Let σ and π^+ be irreducible admissible representations of H and G^+, respectively. If* $\mathrm{Hom}_R(\omega_{\psi,\chi}, \sigma \boxtimes \pi^+) \neq 0$, *then* $\pi^+ \circ \mathrm{Ad}(g_0) \not\simeq \pi^+$ *for any* $g_0 \in G \setminus G^+$.

Proof. Assume that G properly contains G^+; otherwise the lemma is vacuous. Choose a similitude $\tau\colon V \to V'$ with the similitude factor ν_0. Since $g_0 \notin G^+$, V' is not isometric to V. Let $H' = \mathrm{GU}(V')$ and $R' \subset G \times H'$ be the group defined similarly as R. For simplicity, we denote by ω (resp. ω') the corresponding Weil representation for V (resp. V') with respect to ψ and χ. Then, for every $\phi' \in \mathcal{S}(V'^n)$, $(g, h) \in R$,

$$\omega(g, h)(\phi' \circ \tau) = (\omega'(g^{\nu_0}, tht^{-1})) \circ \tau,$$

where

$$g^{\nu_0} = \begin{bmatrix} 1 & 0 \\ 0 & \nu_0 \end{bmatrix} g \begin{bmatrix} 1 & 0 \\ 0 & \nu_0 \end{bmatrix}^{c,-1}.$$

Suppose the lemma were false. We have both

$$\mathrm{Hom}_R(\omega, \sigma \boxtimes \pi^+) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{R'}(\omega', \sigma^t \boxtimes \pi^+) \neq 0,$$

where σ^t is the t-twist of σ. But this contradicts the theta dichotomy (predicted in [18]; see [36, Theorem 2.9] for the archimedean case, and the proof of [12, Theorem 2.10] in the non-archimedean case), since $m < 2n$. □

Let σ be an irreducible unitary admissible representation of H such that its big theta lifting $\Theta^+_\chi(\sigma) := \mathrm{Hom}_H(\text{c-ind}_R^{G^+ \times H} \omega_{\psi,\chi}, \bar{\sigma})$, which is a smooth representation of G^+, is non-zero.

Assumption 5.4. We say σ satisfies the *strong Howe duality* (SHD) (for the group G) if $\Theta^+_\chi(\sigma)$ has a unique irreducible quotient, denoted by $\theta^+_\chi(\sigma)$.

Lemma 5.5. *The SHD is satisfied if*

(1) *F is archimedean or has odd residue characteristic; or*

(2) *σ is supercuspidal.*

Proof. For the isometry case, (1) is due to [22] and [44], and (2) is due to [31]. The similitude case follows from [51]. □

If σ satisfies SHD, we have unique (up to scalar) R-equivariant maps

$$\theta_\chi \colon \omega_{\psi,\chi} \otimes \sigma \to \theta^+_\chi(\sigma), \quad \theta_{\bar{\chi}} \colon \bar{\omega}_{\psi,\chi} \otimes \bar{\sigma} \to \theta^+_{\bar{\chi}}(\bar{\sigma}).$$

Lemma 5.3 implies the following lemma.

Lemma 5.6. *Assume that $\theta^+_\chi(\sigma)$ is unitary. Then $\pi = \theta_\chi(\sigma) := \mathrm{Ind}_{G^+}^G \theta^+_\chi(\sigma)$ is irreducible and moreover, we have*

$$\mathcal{B}_\pi(\pi(g)\varphi_1, \varphi_2) = 0$$

for $g \in G \setminus G^+$ and $\varphi_1, \varphi_2 \in \theta^+_\chi(\sigma)$. Here, $\mathcal{B}_\pi \colon \pi \otimes \bar{\pi} \to \mathbb{C}$ is a G-invariant bilinear pairing and we regard $\theta^+_\chi(\sigma)$ as a subrepresentation of $\pi|_{G^+}$.

Remark 5.7. The above lemma implies that globally, given $\sigma = \otimes_{\mathrm{v}} \sigma_{\mathrm{v}}$ an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ satisfying SHD (that is, σ_{v} satisfies SHD for all v), if $\theta_\chi(\sigma)$ is cuspidal, it is irreducible. Moreover, $\theta_\chi(\sigma) \simeq \otimes_{\mathrm{v}} \theta_{\chi_{\mathrm{v}}}(\sigma_{\mathrm{v}})$.

From now on, we assume that $n = 2$, $m = 3$, and also that σ satisfies SHD. Let $\mathcal{B}_\omega \colon \omega_{\psi,\chi} \otimes \bar{\omega}_{\psi,\chi} \to \mathbb{C}$ be the canonical bilinear pairing defined by

$$\mathcal{B}_\omega(\phi, \tilde{\phi}) = \int_{V^2} \phi(x)\tilde{\phi}(x)\, \mathrm{d}x.$$

Define the pairing $\mathcal{Z}^\natural\colon (\sigma \otimes \bar{\sigma}) \otimes (\omega_{\psi,\chi} \otimes \bar{\omega}_{\psi,\chi}) \to \mathbb{C}$ by the formula

$$\mathcal{Z}^\natural(f, \tilde{f}; \phi, \tilde{\phi}) = \frac{\prod_{i=1}^4 L(i, \chi_{E/F}^i)}{L(1, \sigma, \mathrm{std})} \int_{\mathrm{U}(V)} \mathcal{B}_\omega\big(\omega_{\psi,\chi}(h)\phi, \tilde{\phi}\big) \mathcal{B}_\sigma\big(\sigma(h) f, \tilde{f}\big)\, \mathrm{d}h, \tag{5.2}$$

where the integral is absolutely convergent by Lemma 3.19, since $n = 2$, $m = 3$, and we have $[E : F] = 2$. By the assumption on σ, Lemma 5.6 and (the proof of) [9, Lemmas 5.6, 5.7], the pairing $\mathcal{Z}^\natural$ descents to a bilinear pairing $\mathcal{B}_\pi\colon \pi \otimes \bar{\pi} \to \mathbb{C}$.

Lemma 5.8. *Let $g \in \mathrm{M}_T^+$.*

(1) *If F is non-archimedean, the function $\mathcal{B}_\pi(\pi(\bullet g)\theta_\chi(f, \phi), \theta_{\bar{\chi}}(\tilde{f}, \tilde{\phi}))\psi_T^{-1}$ has a stable integral. More precisely,*

$$\int_{\mathrm{N}}^{\mathrm{st}} \mathcal{B}_\pi\big(\pi(ug)\theta_\chi(f, \phi), \theta_{\bar{\chi}}(\tilde{f}, \tilde{\phi})\big)\psi_T(u)^{-1}\, \mathrm{d}u = \frac{\prod_{i=1}^4 L(i, \chi_{E/F}^i)\chi(\det g)}{L(1, \sigma, \mathrm{std})} \times \int_{\mathrm{U}(V)} \int_{\mathrm{U}(W_1^\perp)\backslash \mathrm{U}(V)} \phi(h_g^{-1} h^{-1} \underline{h}^{-1} \mu g)\tilde{\phi}(\underline{h}^{-1}\mu) \mathcal{B}_\sigma\big(\sigma(h h_g) f, \tilde{f}\big)\, \underline{\mathrm{d}}\,\underline{h}\, \mathrm{d}h,$$

where $h_g \in H$ is some element which preserves W_1 and restricts to g on W_1; $\mathrm{d}u$ is the selfdual measure on N with respect to ψ; and $\underline{\mathrm{d}}\,\underline{h}$ is the Siegel–Weil measure in Lemma 3.17.

(2) *If F is archimedean, the function $\mathcal{B}_\pi(\pi(\bullet g)\theta_\chi(f, \phi), \theta_{\bar{\chi}}(\tilde{f}, \tilde{\phi}))$ is a tempered distribution on N whose Fourier transform is continuous on $\mathrm{N}^{\mathrm{reg}} \simeq \mathrm{Herm}_2^E(F)^{\mathrm{reg}}$. More precisely,*

$$\widehat{\mathcal{B}_\pi\big(\pi(\bullet g)\theta(f, \phi), \theta(\tilde{f}, \tilde{\phi})\big)}(\psi_T) = \frac{\prod_{i=1}^4 L(i, \chi_{E/F}^i)\chi(\det g)}{L(1, \sigma, \mathrm{std})} \times \int_{\mathrm{U}(V)} \int_{\mathrm{U}(W_1^\perp)\backslash \mathrm{U}(V)} \phi(h_g^{-1} h^{-1} \underline{h}^{-1} \mu g)\tilde{\phi}(\underline{h}^{-1}\mu) \mathcal{B}_\sigma\big(\sigma(h h_g) f, \tilde{f}\big)\, \underline{\mathrm{d}}\,\underline{h}\, \mathrm{d}h,$$

where $h_g \in H$, $\mathrm{d}u$ and $\underline{\mathrm{d}}\,\underline{h}$ are as in case (1).

In both cases, we use the same Haar measure $\mathrm{d}h$ on $\mathrm{U}(V)$ on the two sides of the identities.

Proof. We apply Proposition 3.21 for (1) and Corollary 3.23 for (2). Notice that $n = 2$, $m = 3$ and $[E : F] = 2$. □

For $\varphi \in \pi$, $\varphi_0 \in \pi_0$, $\tilde{\varphi} \in \bar{\pi}$, and $\tilde{\varphi}_0 \in \bar{\pi}_0$, we put

$$\mathfrak{b}_\pi^{\varphi,\tilde{\varphi}}(g) = \begin{cases} \displaystyle\int_{\mathrm{N}}^{\mathrm{st}} \mathcal{B}_\pi\big(\pi(ug)\theta_\chi(f, \phi), \theta_{\bar{\chi}}(\tilde{f}, \tilde{\phi})\big)\psi_T(u)^{-1}\, \mathrm{d}u & \text{if } F \text{ is non-archimedean,} \\ \widehat{\mathcal{B}_\pi\big(\pi(\bullet g)\theta(f, \phi), \theta(\tilde{f}, \tilde{\phi})\big)}(\psi_T) & \text{if } F \text{ is archimedean.} \end{cases}$$

Define

$$\alpha^\sharp(\varphi, \tilde{\varphi}; \varphi_0, \tilde{\varphi}_0) = \frac{L(1, \pi, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})}{L(1, \chi_{E/F})^3 \prod_{i=2}^4 L(i, \chi_{E/F}^i) L(\frac{1}{2}, \pi \boxtimes \pi_0)} \times \int_{E^\times\backslash \mathrm{M}_T^+} \chi(\det g)\mathfrak{b}_{\varphi,\tilde{\varphi}}(g)\mathcal{B}_{\pi_0}\big(\pi_0(g)\varphi_0, \tilde{\varphi}_0\big)\, \mathrm{d}g. \tag{5.3}$$

Similar to the calculation in Section 4.4, we have

$$\Big(\frac{\prod_{i=1}^{4} L(i,\chi_{E/F}^{i})}{L(1,\sigma,\mathrm{std})}\Big)^{-1} \mathfrak{b}_{\pi}^{\varphi,\tilde{\varphi}}(g) = \int_{\mathrm{U}(W_1^{\perp})\backslash \mathrm{U}(V)} \int_{\mathrm{U}(W_1^{\perp})\backslash \mathrm{U}(V)} \int_{\mathrm{U}(W_1^{\perp})} \phi(h^{-1}\mu) \\ \times \tilde{\phi}(\tilde{h}^{-1}\mu)\mathcal{B}_{\sigma}\big(\sigma(g'h_g h)f, \bar{\sigma}(\tilde{h})\tilde{f}\big)\,\mathrm{d}g'\,\underline{\mathrm{d}}\tilde{h}\,\mathrm{d}h.$$

Since $m = 3$, $W_1^{\perp}$ has dimension 1. We introduce a new measure

$$\mathrm{d}'\tilde{h} = \zeta_F(2)L(3,\chi_{E/F})\,\underline{\mathrm{d}}\tilde{h}$$

on the (inside) quotient group $\mathrm{U}(W_1^{\perp})\backslash \mathrm{U}(V)$ to make the volume of the orbit of the maximal compact open subgroup to be 1 at unramified places. Since

$$L(s,\pi,\mathrm{Ad}) = L(1,\chi_{E/F})L(s,\sigma,\mathrm{Ad})L(s,\sigma,\mathrm{std}),$$

we have

$$(5.4)\quad (5.3) = \iint_{[\mathrm{U}(W_1^{\perp})\backslash \mathrm{U}(V)]^2} \phi(h^{-1}\mu)\tilde{\phi}(\tilde{h}^{-1}\mu)\mathbb{L}_{\mu}\big(\sigma(h)f, \bar{\sigma}(\tilde{h})\tilde{f}; \varphi_0, \tilde{\varphi}_0\big)\,\mathrm{d}'\tilde{h}\,\mathrm{d}h,$$

where

$$\mathbb{L}_{\mu}(f,\tilde{f};\varphi_0,\tilde{\varphi}_0) = \frac{L(1,\sigma,\mathrm{Ad})L(1,\pi_0,\mathrm{Ad})}{\prod_{i=1}^{3} L(i,\chi_{E/F}^{i})L(\frac{1}{2},(\sigma\otimes\chi)\boxtimes\pi_0)} \int_{E^{\times}\backslash \mathrm{M}_T^{+}} \chi(\det g) \\ \times \Big(\int_{\mathrm{U}(W_1^{\perp})} \mathcal{B}_{\sigma}\big(\sigma(g'h_g)f,\tilde{f}\big)\,\mathrm{d}g'\Big)\mathcal{B}_{\pi_0}\big(\pi_0(g)\varphi_0,\tilde{\varphi}_0\big)\,\mathrm{d}g \\ = \frac{L(1,\sigma,\mathrm{Ad})L(1,\pi_0,\mathrm{Ad})}{\prod_{i=1}^{3} L(i,\chi_{E/F}^{i})L(\frac{1}{2},(\sigma\otimes\chi)\boxtimes\pi_0)} \\ \times \int_{\mathrm{U}(W_1)} \chi(\det g)\mathcal{B}_{\sigma}\big(\sigma(g)f,\tilde{f}\big)\mathcal{B}_{\pi_0}\big(\pi_0(g)\varphi_0,\tilde{\varphi}_0\big)\,\mathrm{d}g.$$

5.4. Period formula of tempered theta lifting. For $f \in \mathcal{V}_{\sigma}$ and $\varphi_0 \in \mathcal{V}_{\pi_0}$,

$$\Lambda_{\mu}(f,\varphi_0) = 2\int_{\mathbb{A}_E^{\times,1}\,\mathrm{U}(W_1)(F)\backslash \mathrm{U}(W_1)(\mathbb{A})} \chi(\det g)f(g)\varphi_0(g)\,\mathrm{d}g = \mathcal{Q}(f,\varphi_0\cdot\chi),$$

where

$$\mathcal{Q}(f,\varphi_0\cdot\chi) = \int_{\mathrm{U}(W_1)(F)\backslash \mathrm{U}(W_1)(\mathbb{A})} \chi(\det g)f(g)\varphi_0(g)\,\mathrm{d}g.$$

By (5.1),

$$\big|\mathcal{P}(\theta_{\chi}(f,\phi),\varphi_0)\big|^2 = \int_{[\mathrm{U}(W_1^{\perp})(\mathbb{A})\backslash H(\mathbb{A})]^2} \mathcal{Q}\big(\mathrm{R}(h)f,\varphi_0\cdot\chi\big)\,\overline{\mathcal{Q}\big(\mathrm{R}(\tilde{h})f,\varphi_0\cdot\chi\big)} \\ \times \phi(h^{-1}\mu)\bar{\phi}(\tilde{h}^{-1}\mu)\,\mathrm{d}h\,\mathrm{d}\tilde{h}.$$

Conjecture 5.9 (Conjecture 2.5 for $\mathrm{U}_3 \times \mathrm{U}_2$). *Let the notation be as above. Assume that $(\sigma, \mathcal{V}_\sigma)$ (resp. $(\pi_0, \mathcal{V}_{\pi_0})$) is a tempered cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A})$ (resp. $\mathrm{U}(W_1)(\mathbb{A})$), and $\mathcal{B}_\sigma = \prod \mathcal{B}_{\sigma_v}$, $\mathcal{B}_{\pi_0} = \prod \mathcal{B}_{\pi_{0,v}}$, $C_{\mathrm{U}(V)} = 1$. For $f = \otimes_v f_v$ and $f' = \otimes_v f'_v$, (resp. $\varphi_0 = \otimes_v \varphi_{0,v}$) in the space $\mathcal{V}_\sigma$ (resp. $\mathcal{V}_{\pi_0}$), we have*

$$\begin{aligned}&\mathcal{Q}(f, \varphi_0 \cdot \chi)\, \overline{\mathcal{Q}(f', \varphi_0 \cdot \chi)} \\ &\quad = \frac{1}{|\mathcal{S}_{\Psi(\sigma)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{\prod_{i=1}^3 L(i, \chi_{E/F}^i) L(\frac{1}{2}, (\sigma \otimes \chi) \boxtimes \pi_0)}{L(1, \sigma, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})} \prod_{v \in \Sigma} \mathbb{L}_\mu\big(f_v, \bar{f}'_v; \varphi_{0,v}, \bar{\varphi}_{0,v}\big).\end{aligned}$$

If σ and π_0 are representations of the similitude groups as in the previous discussion, then we say Conjecture 5.9 holds for (σ, π_0) if it holds for every pair (σ', π'_0) of irreducible sub-representations of $(\sigma|_{\mathrm{U}(V)(\mathbb{A})}, \pi_0|_{\mathrm{U}(W_1)(\mathbb{A})})$. Here, we are restricting functions, not representations.

Since m is odd, we have $|\mathcal{S}_{\Psi(\sigma)}| = |\mathcal{S}_{\Psi(\sigma')}|$ and $|\mathcal{S}_{\Psi(\pi)}| = |\mathcal{S}_{\Psi(\pi'_0)}|$. Assume that Conjecture 5.9 holds for (σ, π_0). Then we have

$$\begin{aligned}\big|\mathcal{P}(\theta_\chi(f, \phi), \varphi_0)\big|^2 &= \frac{1}{|\mathcal{S}_{\Psi(\sigma)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{L(1, \chi_{E/F}) L(\frac{1}{2}, (\sigma \otimes \chi) \boxtimes \pi_0)}{L(1, \sigma, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})} \\ &\quad \times \prod_{v \in \Sigma} \int_{[\mathrm{U}(W_1^\perp)_v \backslash H_v]^2} \mathbb{L}_\mu\big(\sigma(h_v) f_v, \bar{\sigma}(\tilde{h}_v) \bar{f}_v; \varphi_{0,v}, \bar{\varphi}_{0,v}\big) \phi_v(h_v^{-1}\mu) \bar{\phi}_v(\tilde{h}_v^{-1}\mu)\, \mathrm{d}h_v\, \mathrm{d}'\tilde{h}_v\end{aligned}$$

for $\phi = \otimes_v \phi_v$. By (5.4),

$$\begin{aligned}\big|\mathcal{P}(\theta_\chi(f, \phi), \varphi_0)\big|^2 &= \frac{1}{|\mathcal{S}_{\Psi(\sigma)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{L(1, \chi_{E/F}) L(\frac{1}{2}, (\sigma \otimes \chi) \boxtimes \pi_0)}{L(1, \sigma, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})} \\ &\quad \times \prod_{v \in \Sigma} \alpha_v^\sharp\big(\theta_{\chi_v}(f_v, \phi_v), \theta_{\bar{\chi}_v}(\bar{f}_v, \bar{\phi}_v); \varphi_{0,v}, \bar{\varphi}_{0,v}\big).\end{aligned}$$

By Theorem 5.2, we have

$$\mathcal{B}_{\pi_2} = \frac{2L(1, \sigma, \mathrm{std})}{\prod_{i=1}^4 L(i, \chi_{E/F}^i)} \prod \mathcal{B}_{\pi_{2,v}}.$$

Together, we prove the following theorem by noticing that $|\mathcal{S}_{\Psi(\pi)}| = 2|\mathcal{S}_{\Psi(\sigma)}|$.

Theorem 5.10. *Let $\pi = \theta_\chi(\sigma)$ be an irreducible unitary cuspidal (endoscopic) automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A})$ which is the global theta lifting of an irreducible tempered unitary cuspidal automorphic representation $(\sigma, \mathcal{V}_\sigma)$ of $\mathrm{GU}(V)(\mathbb{A})$ with $\dim V = 3$ satisfying SHD. For a non-degenerate hermitian matrix $T \in \mathrm{Herm}_2^E(F)$, let $(\pi_0, \mathcal{V}_{\pi_0})$ be an irreducible tempered unitary cuspidal automorphic representation of $\mathrm{M}_T(\mathbb{A})$, where $\mathrm{M}_T \simeq \mathrm{GU}(W_1, q_T)$ as above, such that $\chi^2 \cdot \chi_\sigma \cdot \chi_{\pi_0} = 1$. Assume that $\mathcal{B}_{\pi_\alpha} = \prod \mathcal{B}_{\pi_{\alpha,v}}$ for $\alpha = 0, 2$ and $C_{\mathrm{M}_T} = 1$. Suppose that Conjecture 5.9 holds for (σ, π_0). Then for vectors $\varphi = \otimes \varphi_v \in \mathcal{V}_\pi$ and $\varphi_0 = \otimes \varphi_{0,v} \in \mathcal{V}_{\pi_0}$, we have*

$$\begin{aligned}|\mathcal{P}(\varphi, \varphi_0)|^2 &= \frac{L_F(2) L_F(4) L(3, \chi_{E/F})}{|\mathcal{S}_{\Psi(\pi)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{L(1, \chi_{E/F})^2 L(\frac{1}{2}, \pi \boxtimes \pi_0)}{L(1, \pi, \mathrm{Ad}) L(1, \pi_0, \mathrm{Ad})} \\ &\quad \times \prod_v \alpha_v^\sharp(\varphi_v, \bar{\varphi}_v; \varphi_{0,v}, \bar{\varphi}_{0,v}).\end{aligned}$$

Remark 5.11. By [52], Conjecture 5.9 holds if the following conditions are satisfied:

(1) E/F splits at all archimedean places.

(2) There exists a split place v such that σ_{v} and $\pi_{0,\mathrm{v}}$ are supercuspidal.

(3) For a non-split place v, if σ_{v} and $\pi_{0,\mathrm{v}}$ are not both unramified, then they are supercuspidal.

In particular, condition (2) implies that $|\mathcal{S}_{\Psi(\pi)}| = 4$ and $|\mathcal{S}_{\Psi(\pi_0)}| = 2$. Moreover, we assume that σ_{v} is supercuspidal if v has residue characteristic 2 to meet SHD.

5.5. From $\mathrm{GU}_{2,2}$ to SO_6. So far, we have not touched orthogonal groups yet. In fact, they are related via the subgroup

$$G^\dagger = \mathrm{GU}^\dagger(W) := \{g \in G = \mathrm{GU}(W) \mid \det g = \nu(g)^n\}$$

of G. Notice that $\mathrm{SU}(W)$ is the intersection of $\mathrm{GU}^\dagger(W)$ and $\mathrm{U}(W)$. Let $n = 2$ be as above and let V_2 be the unique up to isometry quasi-split quadratic space over F of dimension 6 and discriminant algebra E. Then we have the short exact sequence

$$1 \to F^\times \to G^\dagger \to \mathrm{SO}(V_2) \to 1,$$

which identifies $G^\dagger$ with $\mathrm{GSpin}(V_2)$; see Section 6.3. Let $\mathrm{P}^\dagger = \mathrm{P} \cap G^\dagger$ whose unipotent radical is again N, and $\mathrm{M}^\dagger = \mathrm{P}^\dagger \cap \mathrm{M}$. We have the Levi decomposition $\mathrm{P}^\dagger = \mathrm{M}^\dagger\mathrm{N}$. Put $\mathrm{M}_T^\dagger = \mathrm{M}_T \cap \mathrm{M}^\dagger$. Then we have the short exact sequence

$$1 \to F^\times \to \mathrm{M}_T^\dagger \to \mathrm{SO}(V_0) \to 1,$$

which identifies $\mathrm{M}_T^\dagger$ with $\mathrm{GSpin}(V_0)$, where V_0 is the quadratic space over F of dimension 3 determined by T. Note that the natural inclusion $F^\times\backslash\mathrm{M}_T^\dagger \to E^\times\backslash\mathrm{M}_T$ is an isomorphism.

Let $(\sigma, \mathcal{V}_\sigma)$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$. For $f \in \mathcal{V}_\sigma$ and $\phi \in \mathcal{S}(V^n(\mathbb{A}_E))$, we put $\theta_\chi^\dagger(f;\phi) = \theta_\chi(f;\phi)|_{G^\dagger(\mathbb{A})}$. Define

$$\theta_\chi^\dagger(\sigma) = \{\varphi|_{G^\dagger(\mathbb{A})} \mid \varphi \in \theta_\chi(\sigma)\},$$

which is an automorphic representation of $G^\dagger(\mathbb{A})$, not necessarily irreducible.

From now on, we assume that $m = 3$, σ satisfies SHD, and $\pi = \theta_\chi(\sigma)$ is cuspidal. Define

$$\mathfrak{X}_\pi = \{\omega\colon \mathbb{A}_E^\times G^\dagger(\mathbb{A})G(F)\backslash G(\mathbb{A}) \to \mathbb{C}^\times \mid \pi \otimes \omega \simeq \pi\},$$

which is a finite group by [21, Lemma 4.11]. For $\varphi \in \pi = \theta_\chi(\sigma)$ being stabilized under all twists of $\mathfrak{X}_\pi$, we have

$$\int_{\mathbb{A}^\times G^\dagger(F)\backslash G^\dagger(\mathbb{A})} \varphi(g)\overline{\varphi(g)}\,\mathrm{d}g = |\mathfrak{X}_\pi|\mathcal{B}_\pi(\varphi,\tilde{\varphi}) \tag{5.5}$$

by [21, Remark 4.20]. Let $(\pi_2, \mathcal{V}_{\pi_2})$ be an irreducible summand of $\theta_\chi^\dagger(\sigma)$. The following lemma is a special case of [33, Lemma 3.6].

Lemma 5.12. *We have that* $|\mathcal{S}_{\Psi(\pi_2)}| = |\mathfrak{X}_\pi| \cdot |\mathcal{S}_{\Psi(\pi)}|$.

For $\varphi_\alpha \in \mathcal{V}_{\pi_\alpha}$ ($\alpha = 0, 2$), recall that

$$\mathcal{P}(\varphi_2, \varphi_0) = \int_{\mathbb{A}^\times \mathrm{M}_T^\dagger(F)\backslash \mathrm{M}_T^\dagger(\mathbb{A})} \int_{\mathrm{N}(F)\backslash \mathrm{N}(\mathbb{A})} \varphi_2(ug)\psi_T(u)^{-1}\varphi_0(g)\,\mathrm{d}u\,\mathrm{d}g.$$

Locally, for $\varphi_\alpha \in \pi_\alpha$ and $\tilde{\varphi}_\alpha \in \bar{\pi}_\alpha$ ($\alpha = 0, 2$), we define

$$\alpha^\natural(\varphi_2, \tilde{\varphi}_2; \varphi_0, \tilde{\varphi}_0) = \frac{L(1, \pi, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})}{L(1, \chi_{E/F})^3 \prod_{i=2}^4 L(i, \chi_{E/F}^i) L(\frac{1}{2}, \pi \boxtimes \pi_0)} \times \int_{F^\times\backslash \mathrm{M}_T^\dagger} \chi(\det g)\mathfrak{b}_{\pi_2}^{\varphi,\tilde{\varphi}}(g)\mathcal{B}_{\pi_0}\big(\pi_0(g)\varphi_0, \tilde{\varphi}_0\big)\,\mathrm{d}u\,\mathrm{d}g.$$

By (5.5), [21, Lemma 4.12] and Lemma 5.12, Theorem 5.10 implies

$$|\mathcal{P}(\varphi_2, \varphi_0)|^2 = \frac{\Delta_{G_{\mathrm{ad}}^\dagger}}{|\mathcal{S}_{\Psi(\pi_2)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{L(\frac{1}{2}, \pi_2 \boxtimes \pi_0)}{L(1, \pi_2, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \prod_{\mathrm{v}} \alpha_{\mathrm{v}}^\natural(\varphi_{2,\mathrm{v}}, \bar{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \bar{\varphi}_{0,\mathrm{v}})$$

for vectors $\varphi_\alpha = \otimes\varphi_{\alpha,\mathrm{v}} \in \mathcal{V}_{\pi_\alpha}$ ($\alpha = 0, 2$). Here, we view π_0 as a representation of $\mathrm{M}_T^\dagger(\mathbb{A})$ and hence the factor $L(1, \chi_{E/F})$ is removed from its adjoint L-function. We also have $L(1, \pi, \mathrm{Ad}) = L(1, \chi_{E/F})L(1, \pi_2, \mathrm{Ad})$. To summarize, we have the following corollary.

Corollary 5.13. *Let the situation be as above. When π_2 (resp. π_0) has the trivial central character, we may view π_2 (resp. π_0) as an irreducible cuspidal automorphic representation of $\mathbb{A}^\times\backslash \mathrm{GU}_{2,2}^\dagger(\mathbb{A}) \simeq \mathrm{SO}(V_2)(\mathbb{A})$ (resp. $\mathbb{A}^\times\backslash\mathrm{M}_T^\dagger(\mathbb{A}) \simeq \mathrm{SO}(V_0)(\mathbb{A})$). Then we have*

$$|\mathcal{P}(\varphi_2, \varphi_0)|^2 = \frac{1}{|\mathcal{S}_{\Psi(\pi_2)}||\mathcal{S}_{\Psi(\pi_0)}|} \frac{\Delta_{G_2} L(\frac{1}{2}, \pi_2 \boxtimes \pi_0)}{L(1, \pi_2, \mathrm{Ad})L(1, \pi_0, \mathrm{Ad})} \prod_{\mathrm{v}} \alpha_{\mathrm{v}}^\natural(\varphi_{2,\mathrm{v}}, \bar{\varphi}_{2,\mathrm{v}}; \varphi_{0,\mathrm{v}}, \bar{\varphi}_{0,\mathrm{v}}),$$

which confirms Conjecture 2.5 for such (π_2, π_0), where $G_2 = \mathrm{SO}(V_2)$ as always.

6. Appendices

6.1. Formulas for $\zeta(\Xi, \xi)$. We keep the notation and conventions from Section 3.3. In particular, G_0 is not trivial and $r > 0$. Choose a decomposition $V_2 = S \oplus V_{\mathrm{an}} \oplus S^*$ where V_{an} is an anisotropic kernel of V_2 and S, S^* are totally isotropic. Denote by m the dimension of S, which is the Witt index of V_2. Choose a basis $\{e_1, \dots, e_m\}$ of S and a dual basis $\{e_1^*, \dots, e_m^*\}$ of S^*. Assume that in the filtration $\mathfrak{F}$, $R_i = \mathrm{Span}_E\{e_1, \dots, e_i\}$ for $0 \le i \le r$ and $R^* = \mathrm{Span}_E\{e_1^*, \dots, e_r^*\}$. Since the group G_2 is quasi-split, we have that $\dim_E V_{\mathrm{an}} \le 2$. Precisely, we fix a basis of V_{an} as follows: $V_{\mathrm{an}} = \mathrm{Span}_E\{e_0\}$ when n_2 is odd; $V_{\mathrm{an}} = \mathrm{Span}_E\{e_0, e_0^*\}$ (e_0^* is not supposed to be the dual of e_0) when n_2 is even, $[E : F] = 2$ and K is a field; and $V_{\mathrm{an}} = 0$ otherwise. Note that $m > r$, since G_0 is not trivial. Therefore, we may also assume that the anisotropic vector w belongs to $\mathrm{Span}_E\{e_{r+1}, e_{r+1}^*\}$.

We identify T_2 with $\prod_{i=1}^m \mathrm{GL}_E(e_i) \times \mathrm{Ism}^0(V_0)$. Consider the maximal parabolic subgroup $P_{r+1} \subset G_2$ stabilizing the subspace $\mathrm{Span}_E\{e_1, \dots, e_{r+1}\}$ with the standard Levi subgroup $\mathrm{Res}_F^E \mathrm{GL}_{r+1} \times G_1'$, where $G_1' \subset G_0$. We separate the parameter Ξ into two parts: $\Xi' = (\Xi_1, \dots, \Xi_{r+1})$ which produces an unramified representation $\tau = \mathrm{Ind}(\Xi')$ of $\mathrm{Res}_F^E \mathrm{GL}_{r+1}$, and $\tilde{\Xi}' = (\Xi_{r+2}, \dots)$ which produces an unramified representation $\sigma = \mathrm{Ind}(\tilde{\Xi}')$ of G_1'. Then $\pi_2 \simeq \mathrm{Ind}_{P_{r+1}}^{G_2} \tau \otimes \sigma$. We choose the isomorphism such that f_Ξ maps to the unramified section which takes value $f_\tau \otimes f_\sigma$ at 1, where f_τ (resp. f_σ) is the unramified section of τ (resp. σ) such that $f_\tau(1) = 1$ (resp. $f_\sigma(1) = 1$).

Proposition 6.1. *We have the following relation:*

$$\zeta(\Xi, \xi) = \frac{L(\frac{1}{2}, \tau \boxtimes \pi_0)}{L(1, \tau \boxtimes \sigma) L(1, \tau, \rho)} \prod_{1 \le i < j \le r+1} \frac{1}{L_E(1, \Xi_i \Xi_j^{-1})} \zeta(\xi, \tilde{\Xi}'),$$

where

- $\rho = \wedge^2$, *if* $[E : F] = 1$ *and* n_2 *is even;*
- $\rho = \mathrm{Sym}^2$, *if* $[E : F] = 1$ *and* n_2 *is odd;*
- ρ *is the Asai representation* As^+, *if* $[E : F] = 2$ *and* n_2 *is even;*
- ρ *is the Asai representation* As^-, *if* $[E : F] = 2$ *and* n_2 *is odd.*

Proof. The proposition follows from [11, Theorem 4.5] for $[E : F] = 1$ and from [26, Theorem 3.9] for $[E : F] = 2$. Note that the extra factor $\prod_{1 \le i < j \le r+1} L_E(1, \Xi_i \Xi_j^{-1})^{-1}$ comes from the normalization of the unramified Whittaker function, which follows from the Casselman–Shalika formula [4]. □

Corollary 6.2. *We have the following formulas for* $\zeta(\Xi, \xi)$*:*

(1) *If* $[E : F] = 1$, $K = F \oplus F$ *and* n_2 *odd, then* $m = (n_2 - 1)/2$. *Let* $\Xi = (\Xi_1, \dots, \Xi_m)$ *and* $\xi = (\xi_{r+1}, \dots, \xi_m)$. *We have*

$$\begin{aligned}
\zeta(\Xi, \xi) = \prod_{i=1}^{m} L(1, \Xi_i^2)^{-1} & \prod_{1 \le i < j \le m} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\
& \times \prod_{r+1 \le i < j \le m} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\
& \times \prod_{1 \le i \le j \le m} L\Big(\frac{1}{2}, \Xi_i \xi_j\Big) L\Big(\frac{1}{2}, \Xi_i \xi_j^{-1}\Big) \\
& \times \prod_{r+1 \le j < i \le m} L\Big(\frac{1}{2}, \Xi_i \xi_j\Big) L\Big(\frac{1}{2}, \Xi_i^{-1} \xi_j\Big).
\end{aligned}$$

(2) *If* $[E : F] = 1$, $K = F \oplus F$ *and* n_2 *even, then* $m = n_2/2$. *Let* $\Xi = (\Xi_1, \dots, \Xi_m)$ *and* $\xi = (\xi_{r+1}, \dots, \xi_{m-1})$. *We have*

$$\begin{aligned}
\zeta(\Xi, \xi) = & \prod_{1 \le i < j \le m} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\
& \times \prod_{i=r+1}^{m-1} L(1, \xi_i^2)^{-1} \prod_{r+1 \le i < j \le m-1} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\
& \times \prod_{1 \le i \le j \le m-1} L\Big(\frac{1}{2}, \Xi_i \xi_j\Big) L\Big(\frac{1}{2}, \Xi_i \xi_j^{-1}\Big) \\
& \times \prod_{r+1 \le j < i \le m} L\Big(\frac{1}{2}, \Xi_i \xi_j\Big) L\Big(\frac{1}{2}, \Xi_i^{-1} \xi_j\Big).
\end{aligned}$$

(3) *If* $[E:F]=1$, K *is a field and* n_2 *odd, then* $m=(n_2-1)/2$. *Let* $\Xi=(\Xi_1,\dots,\Xi_m)$ *and* $\xi=(\xi_{r+1},\dots,\xi_{m-1})$. *We have*

$$\begin{aligned}\zeta(\Xi,\xi)=&\prod_{i=1}^{m}L(1,\Xi_i^2)^{-1}\prod_{1\le i<j\le m}L(1,\Xi_i\Xi_j)^{-1}L(1,\Xi_i\Xi_j^{-1})^{-1}\\&\times\prod_{i=r+1}^{m-1}L(1,\xi_i)^{-1}L(1,\chi\xi_i)^{-1}\prod_{r+1\le i<j\le m-1}L(1,\xi_i\xi_j)^{-1}L(1,\xi_i\xi_j^{-1})^{-1}\\&\times\prod_{1\le i\le j\le m-1}L\Big(\frac{1}{2},\Xi_i\xi_j\Big)L\Big(\frac{1}{2},\Xi_i\xi_j^{-1}\Big)\\&\times\prod_{r+1\le j<i\le m}L\Big(\frac{1}{2},\Xi_i\xi_j\Big)L\Big(\frac{1}{2},\Xi_i^{-1}\xi_j\Big)\prod_{i=1}^{m}L\Big(\frac{1}{2},\Xi_i\Big)L\Big(\frac{1}{2},\chi\Xi_i\Big).\end{aligned}$$

(4) *If* $[E:F]=1$, K *is a field and* n_2 *even, then* $m=n_2/2-1$. *Let* $\Xi=(\Xi_1,\dots,\Xi_m)$ *and* $\xi=(\xi_{r+1},\dots,\xi_m)$. *We have*

$$\begin{aligned}\zeta(\Xi,\xi)=&\prod_{i=1}^{m}L(1,\Xi_i)^{-1}L(1,\chi\Xi_i)^{-1}\prod_{1\le i<j\le m}L(1,\Xi_i\Xi_j)^{-1}L(1,\Xi_i\Xi_j^{-1})^{-1}\\&\times\prod_{i=r+1}^{m}L(1,\xi_i^2)^{-1}\prod_{r+1\le i<j\le m}L(1,\xi_i\xi_j)^{-1}L(1,\xi_i\xi_j^{-1})^{-1}\\&\times\prod_{1\le i\le j\le m}L\Big(\frac{1}{2},\Xi_i\xi_j\Big)L\Big(\frac{1}{2},\Xi_i\xi_j^{-1}\Big)\\&\times\prod_{r+1\le j<i\le m}L\Big(\frac{1}{2},\Xi_i\xi_j\Big)L\Big(\frac{1}{2},\Xi_i^{-1}\xi_j\Big)\prod_{i=r+1}^{m}L\Big(\frac{1}{2},\xi_i\Big)L\Big(\frac{1}{2},\chi\xi_i\Big).\end{aligned}$$

(5) *If* $[E:F]=2$, E *is a field and* n_2 *odd, then* $m=(n_2-1)/2$. *Let* $\Xi=(\Xi_1,\dots,\Xi_m)$ *and* $\xi=(\xi_{r+1},\dots,\xi_m)$. *We have*

$$\begin{aligned}\zeta(\Xi,\xi)=&\prod_{i=1}^{m}L_F(1,\chi\Xi_i)^{-1}L_E(1,\Xi_i)^{-1}\prod_{1\le i<j\le m}L_E(1,\Xi_i\Xi_j)^{-1}L_E(1,\Xi_i\Xi_j^{-1})^{-1}\\&\times\prod_{r+1\le i<j\le m}L_E(1,\xi_i\xi_j)^{-1}L_E(1,\xi_i\xi_j^{-1})^{-1}\\&\times\prod_{1\le i\le j\le m}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i\xi_j^{-1}\Big)\\&\times\prod_{r+1\le j<i\le m}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i^{-1}\xi_j\Big).\end{aligned}$$

(6) *If $[E:F]=2$, E is a field and n_2 even, then $m=n_2/2$. Let $\Xi=(\Xi_1,\dots,\Xi_m)$ and $\xi=(\xi_{r+1},\dots,\xi_{m-1})$. We have*

$$\begin{aligned}\zeta(\Xi,\xi)=&\prod_{1\le i<j\le m}L_E(1,\Xi_i\Xi_j)^{-1}L_E(1,\Xi_i\Xi_j^{-1})^{-1}\\&\times\prod_{i=r+1}^{m-1}L_F(1,\chi\xi_i)^{-1}L_E(1,\xi_i)^{-1}\\&\times\prod_{r+1\le i<j\le m-1}L_E(1,\xi_i\xi_j)^{-1}L_E(1,\xi_i\xi_j^{-1})^{-1}\\&\times\prod_{1\le i\le j\le m-1}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i\xi_j^{-1}\Big)\\&\times\prod_{r+1\le j<i\le m}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i^{-1}\xi_j\Big).\end{aligned}$$

(7) *If $E=F\oplus F$ and n_2 odd, then $m=(n_2-1)/2$. Let $\Xi=(\Xi_1,\dots,\Xi_m;\Xi_0)$ and $\xi=(\xi_{r+1},\dots,\xi_m)$. We have*

$$\begin{aligned}\zeta(\Xi,\xi)=&\prod_{i=1}^{m}L_F(1,\Xi_0\Xi_i)^{-1}\prod_{1\le i<j\le m}L_F(1,\Xi_i\Xi_j)^{-1}L_F(1,\Xi_i\Xi_j^{-1})^{-1}\\&\times\prod_{r+1\le i<j\le m}L_F(1,\xi_i\xi_j)^{-1}L_F(1,\xi_i\xi_j^{-1})^{-1}\\&\times\prod_{1\le i\le j\le m}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i\xi_j^{-1}\Big)\\&\times\prod_{r+1\le j<i\le m}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i^{-1}\xi_j\Big)\prod_{j=r+1}^{m}L_F\Big(\frac{1}{2},\Xi_0\xi_j\Big).\end{aligned}$$

(8) *If $E=F\oplus F$ and n_2 even, then $m=n_2/2$. Let $\Xi=(\Xi_1,\dots,\Xi_m)$ and $\xi=(\xi_{r+1},\dots,\xi_{m-1};\xi_0)$. We have*

$$\begin{aligned}\zeta(\Xi,\xi)=&\prod_{1\le i<j\le m}L_F(1,\Xi_i\Xi_j)^{-1}L_F(1,\Xi_i\Xi_j^{-1})^{-1}\\&\times\prod_{i=r+1}^{m-1}L_F(1,\xi_0\xi_i)^{-1}\prod_{r+1\le i<j\le m-1}L_F(1,\xi_i\xi_j)^{-1}L_F(1,\xi_i\xi_j^{-1})^{-1}\\&\times\prod_{1\le i\le j\le m-1}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i\xi_j^{-1}\Big)\\&\times\prod_{r+1\le j<i\le m}L_E\Big(\frac{1}{2},\Xi_i\xi_j\Big)L_E\Big(\frac{1}{2},\Xi_i^{-1}\xi_j\Big)\prod_{i=1}^{m}L_F\Big(\frac{1}{2},\Xi_i\xi_0\Big).\end{aligned}$$

In (3) *and* (4), *χ equals $\chi_{K/F}$. In* (5) *and* (6), *χ equals $\chi_{E/F}$.*

Proof. By Proposition 6.1, we only need to prove the corresponding formula for $\zeta(\xi, \tilde{\Xi}')$. Let q be the cardinality of the residue field of F. When q is odd, cases (1)–(4) follow from [23, Section 5] and cases (5)–(8) follow from [19, Section 2.2.1]. When q is even, it suffices to show that $\zeta(\xi, \tilde{\Xi}')$ is an element in $\mathbb{Q}(q^{\frac{1}{2}}, \xi, \tilde{\Xi}')$. By an argument at the end of [23, p. 1393], the function $I(1; \xi, \tilde{\Xi}')$ (defined in Section 3.3) is an element in $\mathbb{Q}(q^{\frac{1}{2}}, \xi, \tilde{\Xi}')$. From the decomposition $I(1; \xi, \tilde{\Xi}') = \zeta(\xi, \tilde{\Xi}') S_{\xi^{-1}, \tilde{\Xi}'^{-1}}(1)$, we only need to show that the Shintani function $S_{\xi^{-1}, \tilde{\Xi}'^{-1}}(1)$ is in $\mathbb{Q}(q^{\frac{1}{2}}, \xi, \tilde{\Xi}')$, which is the case due to the explicit formula for Shintani functions available for all q. Here, we only use the explicit formula for Shintani functions when $r = 0$, whose references will be listed in the next section. □

6.2. Whittaker–Shintani functions of Bessel type. We follow the notation and setting used in Section 3.3. By Lemma 3.8 and its preceding discussion, $S_{\Xi,\xi}(g_2)$, as a function of (Ξ, ξ), has a meromorphic continuation to a region Ω where (Ξ, ξ) are sufficiently closed to the unitary axis. We may assume that Ω is invariant under the action of the Weyl group $W_2 \times W_0$. Our goal in this subsection is to provide an explicit formula of $S_{\Xi,\xi}(1)$ for $(\Xi, \xi) \in \Omega$. For this purpose, we further assume that the residue characteristic of F is not 2. Such a formula already exists in literature in the following cases:

- $E = F$ and $K = F \oplus F$, due to [29];
- $E = F$ and $r = 0$, due to [23];
- E is a quadratic field extension of F and $r = 0$, due to [19, 27];
- $E = F \oplus F$ and $r = 0$, due to [28].

All remaining cases have $r > 0$, in which the one when $E = F$ and $K = F \oplus F$ (that is, G_2 and G_0 are both split orthogonal groups) has been treated in [29]. For the rest of them, we follow the method in [29, Sections 10, 11] and also [23, Section 5].

Lemma 6.3. *The quality $S_{\Xi,\xi}(g_2)\zeta(\Xi, \xi)^{-1}$ is $W_2 \times W_0$-invariant as a function of* (Ξ, ξ).

Proof. The proof is similar to [23, Lemma 5.2]. Note that $I(g_2, \Phi_\Xi, \Phi_\xi)$ is $W_2 \times W_0$-invariant, since Φ_Ξ, Φ_ξ and $\zeta(\Xi, \xi)\zeta(\Xi^{-1}, \xi^{-1})$ are also $W_2 \times W_0$-invariant by the explicit formulas in Corollary 6.2. The $W_2 \times W_0$-invariance of $S_{\Xi,\xi}(g_2)\zeta(\Xi, \xi)^{-1}$ follows from the equality

$$\frac{I(g_2, \Phi_\Xi, \Phi_\xi)}{\zeta(\Xi, \xi)\zeta(\Xi^{-1}, \xi^{-1})} = \frac{S_{\Xi^{-1},\xi^{-1}}(g_2)}{\zeta(\Xi^{-1}, \xi^{-1})}$$

by (3.8) and Lemma 3.8. □

Denote by $A_\alpha \subset T_\alpha$ the maximal split torus. Put

$$A_2^+ = \{t \in A_2 \mid |\alpha(t)| \leq 1 \text{ for any positive root } \alpha \text{ of } (G_2, A_2)\},$$
$$A_0^+ = \{t \in A_0 \mid |\alpha(t)| \leq 1 \text{ for any positive root } \alpha \text{ of } (G_0, A_0)\}.$$

See the proof of Lemma 3.19 for the explicit description of A_2^+ and A_0^+. Then we have the Cartan decompositions $G_2 = \mathcal{K}_2 A_2^+ \mathcal{K}_2$ and $G_0 = \mathcal{K}_0 A_0^+ \mathcal{K}_0$. For each positive root α of G_2

(resp. G_0), we denote the c-function of Harish-Chandra by $c_\alpha(\Xi)$ (resp. $c_\alpha(\xi)$) and put

$$\overline{c}_{w_2}(\Xi) = \prod_{\alpha>0, w_2\alpha>0} c_\alpha(\Xi), \qquad \overline{c}_{w_0}(\xi) = \prod_{\alpha>0, w_0\alpha>0} c_\alpha(\xi).$$

When w_2 (resp. w_0) is the identity element, we put $\mathbf{c}_2(\Xi) = \overline{c}_{w_2}(\Xi)$ (resp. $\mathbf{c}_0(\xi) = \overline{c}_{w_0}(\xi)$).

We fix a basis $\{g_{2,w_2}\}_{w_2\in W_2}$ (resp. $\{g_{0,w_0}\}_{w_0\in W_0}$) of $I(\Xi)^{\mathcal{B}_2}$ (resp. $I(\xi)^{\mathcal{B}_0}$) as in [29, Proposition 1.10] (whose proof works for any unramified group G). In view of the relation (3.9), we have the following formula similar to [29, Theorem 10.7]:

$$\begin{aligned}\frac{S_{\Xi,\xi}(t_0\eta^{-1}t_2)}{\zeta(\Xi,\xi)} = {} & \frac{\Delta_{G_2}\Delta_{G_0}}{\Delta_{T_2}\Delta_{T_0}} \\ & \times \sum_{w_2\in W_2, w_0\in W_0} \mathbf{c}_{\mathrm{WS}}(w_2\Xi, w_0\xi)\cdot(w_2\Xi)^{-1}\delta_2^{1/2}(t_2)\cdot(w_0\xi)^{-1}\delta_0^{1/2}(t_0),\end{aligned}$$

where $\mathbf{c}_{\mathrm{WS}}(\Xi,\xi) = \mathbf{c}_2(\Xi)\mathbf{c}_0(\xi)\zeta(\Xi,\xi)^{-1}$.

Proposition 6.4. *We have*

$$\sum_{w_2\in W_2, w_0\in W_0} \mathbf{c}_{\mathrm{WS}}(w_2\Xi, w_0\xi) = \Delta_{G_0}^{-1}.$$

Proof. We closely follow [29, Section 11]. As in [29, Section 10.2], let $\mathbf{e}_2(\Xi)$ (resp. $\mathbf{e}_0(\xi)$) be the numerator of $\mathbf{c}_2(\Xi)$ (resp. $\mathbf{c}_0(\xi)$) and $\mathbf{d}_2(\Xi)$ (resp. $\mathbf{d}_0(\xi)$) be the denominator of $\mathbf{c}_2(\Xi)$ (resp. $\mathbf{c}_0(\xi)$). Denote by $\mathbf{b}(\Xi,\xi)^{-1}$ the product of all terms involving $L(\frac{1}{2},-)$ in the formula of $\zeta(\Xi,\xi)$ in Corollary 6.2. Then we have

$$\zeta(\Xi,\xi) = \frac{\mathbf{e}_2(\Xi)\mathbf{e}_0(\xi)}{\mathbf{b}(\Xi,\xi)}, \qquad \mathbf{c}_{\mathrm{WS}}(\Xi,\xi) = \frac{\mathbf{b}(\Xi,\xi)}{\mathbf{d}_2(\Xi)\mathbf{d}_0(\xi)}.$$

Put

$$A_{r,n_0} = A_{r,n_0}(\Xi,\xi) = \sum_{w_2\in W_2, w_0\in W_0} \frac{\mathbf{b}(w_2\Xi, w_0\xi)}{\mathbf{d}_2(w_2\Xi)\mathbf{d}_0(w_0\xi)}, \qquad A^\dagger_{r,n_0} = A_{r,n_0}(\Xi^{-1},\xi^{-1}).$$

We regard Ξ and ξ as indeterminates (with suitably many components) and hence $A^\dagger_{r,n_0}$ is in the ring $\mathbb{C}[\Xi^{\pm1},\xi^{\pm1}]$. Similarly, we put

$$\mathbf{b}^\dagger_{r,n_0}(\Xi,\xi) = \mathbf{b}(\Xi^{-1},\xi^{-1}), \quad \mathbf{d}_2^\dagger(\Xi) = \mathbf{d}_2(\Xi^{-1}), \quad \mathbf{d}_0^\dagger(\xi) = \mathbf{d}_0(\xi^{-1}).$$

Let ρ_α be the half-sum of positive roots of (G_α, A_α) for $\alpha = 0, 2$. Put

$$\mathbf{D}_2(\Xi) = \Xi^{\rho_2}\mathbf{d}_2^\dagger(\Xi), \quad \mathbf{D}_0(\xi) = \xi^{\rho_0}\mathbf{d}_0^\dagger(\xi).$$

Then

$$A^\dagger_{r,n_0} = \mathbf{D}_2(\Xi)^{-1}\mathbf{D}_0(\xi)^{-1} \sum_{w_2\in W_2, w_0\in W_0} \operatorname{sgn}(w_2)\operatorname{sgn}(w_0)w_2w_0\big(\Xi^{\rho_2}\xi^{\rho_0}\mathbf{b}^\dagger_{r,n_0}(\Xi,\xi)\big).$$

We say a monomial $\Xi^{\lambda}\xi^{\mu}$ is *regular* if $\Xi^{w_2\lambda}\xi^{w_0\mu} = \Xi^{\lambda}\xi^{\mu}$ implies that both w_2 and w_0 are identities. Set $\mathbf{B}_{r,n_0} = \Xi^{\rho_2}\xi^{\rho_0}\mathbf{b}^{\dagger}_{r,n_0}(\Xi,\xi)$ and expand it as $\sum_{c_{\lambda,\mu}} \Xi^{\lambda}\xi^{\mu}$. We have

$$A^{\dagger}_{r,n_0} = \mathbf{D}_2(\Xi)^{-1}\mathbf{D}_0(\xi)^{-1} \sum_{\Xi^{\lambda}\xi^{\mu}\ \text{regular}} c_{\lambda,\mu} \sum_{w_2\in W_2, w_0\in W_0} \operatorname{sgn}(w_2)\operatorname{sgn}(w_0) w_2 w_0(\Xi^{\lambda}\xi^{\mu}).$$

Repeat the proof of [29, Lemma 11.5] for the eight cases in Corollary 6.2, where the original proof is for case (1). We conclude that $A^{\dagger}_{r,n_0}$ is constant in Ξ and independent of r. Therefore, the proposition reduces to the case $r = 0$ which is known. □

Corollary 6.5. *We have*

$$S_{\Xi,\xi}(1) = \frac{\Delta_{G_2}}{\Delta_{T_2}\Delta_{T_0}}\zeta(\Xi,\xi).$$

6.3. Realization of low-rank orthogonal groups. Let V be the unique up to isomorphism quasi-split quadratic space over F of dimension m and discriminant algebra E when m is even.

- If $m = 2$, then $\mathrm{SO}(V) \simeq E^{\times,1} \simeq \mathrm{Res}^E_F\, E^{\times}/F^{\times}$.
- If $m = 3$, then $\mathrm{SO}(V) \simeq \mathrm{PGL}_2$, which acts on the space

$$V = \left\{ v = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \,\middle|\, a, b, c \in F \right\}$$

with the norm $q(v) = a^2 + bc$ by $g.v = gvg^{-1}$.

- If $m = 4$, then $\mathrm{SO}(V) \simeq \mathrm{GL}_2^{E,\dagger}/F^{\times}$. We have

$$\mathrm{GL}_2^{E,\dagger} = \{ g \in \mathrm{Res}^E_F\, \mathrm{GL}_2 \mid \det g \in F^{\times} \},$$

which acts on the space

$$V = \left\{ v = \begin{bmatrix} e & b \\ c & -e^c \end{bmatrix} \,\middle|\, b, c \in F,\ e \in E \right\}$$

with the norm $q(v) = ee^c + bc$ by $g.v = gvg^{c,-1}$.

- If $m = 5$, then $\mathrm{SO}(V) \simeq \mathrm{PGSp}_4$. We have

$$\mathrm{GSp}_4 = \{ g \in \mathrm{GL}_4 \mid {}^t g \mathbf{J}_2 g = \nu(g)\mathbf{J}_2 \},$$

which acts on the space

$$V = \left\{ v = \begin{bmatrix} a & b & & x \\ c & -a & -x & \\ & -y & a & c \\ y & & b & -a \end{bmatrix} \,\middle|\, a, b, c, x, y \in F \right\}$$

with the norm $q(v) = a^2 + bc + xy$ by $g.v = gvg^{-1}$.

- If $m = 6$, then $\mathrm{SO}(V) \simeq \mathrm{GU}^{\dagger}_{2,2}/F^{\times}$ where

$$\mathrm{GU}^{\dagger}_{2,2} = \{g \in \mathrm{Res}^{E}_{F}\,\mathrm{GL}_4 \mid {}^t g^c \mathbf{J}_2 g = \nu(g)\mathbf{J}_2,\ \det g = \nu(g)^2\},$$

which acts on the space

$$V = \left\{ v = \begin{bmatrix} e & b & & x \\ c & -e^c & -x & \\ & -y & e & c \\ y & & b & -e^c \end{bmatrix} \,\middle|\, b, c, x, y \in F,\ e \in E \right\}$$

with the norm $q(v) = ee^c + bc + xy$ by $g.v = gvg^{c,-1}$.

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