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C^∞ SCALING ASYMPTOTICS FOR THE SPECTRAL PROJECTOR OF THE LAPLACIAN

YAIZA CANZANI AND BORIS HANIN

Abstract. This article concerns new off-diagonal estimates on the remainder and its derivatives in the pointwise Weyl law on a compact n-dimensional Riemannian manifold. As an application, we prove that near any non self-focal point, the scaling limit of the spectral projector of the Laplacian onto frequency windows of constant size is a normalized Bessel function depending only on n.

0. Introduction

Let (M, g) be a compact, smooth, Riemannian manifold without boundary. We assume throughout that the dimension of M is $n \geq 2$ and write Δ_g for the non-negative Laplace-Beltrami operator. Denote the spectrum of Δ_g by

$$0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots \nearrow \infty.$$

This article concerns the behavior of the Schwarz kernel of the projection operators

$$E_I : L^2(M) \to \bigoplus_{\lambda_j \in I} \ker(\Delta_g - \lambda_j^2),$$

where $I \subset [0, \infty)$. Given an orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ of $L^2(M, g)$ consisting of real-valued eigenfunctions,

$$\Delta_g \varphi_j = \lambda_j^2 \varphi_j \qquad \text{and} \qquad \|\varphi_j\|_{L^2} = 1, \tag{1}$$

the Schwarz kernel of E_I is

$$E_I(x, y) = \sum_{\lambda_j \in I} \varphi_j(x)\varphi_j(y). \tag{2}$$

The study of $E_{[0,\lambda]}(x, y)$ as $\lambda \to \infty$ has a long history, especially when $x = y$. For instance, it has been studied notably in [7, 8, 9, 10] for its close relation to the asymptotics of the spectral counting function

$$\#\{j : \lambda_j \leq \lambda\} = \int_M E_{[0,\lambda]}(x, x)\, dv_g(x), \tag{3}$$

where dv_g is the Riemannian volume form. An important result, going back to Hörmander [8, Thm 4.4], is the pointwise Weyl law (see also [4, 18]), which says that there exists $\varepsilon > 0$ so that if the Riemannian distance $d_g(x, y)$ between x and y is less than ε, then

$$E_{[0,\lambda]}(x, y) = \frac{1}{(2\pi)^n} \int_{|\xi|_{g_y} < \lambda} e^{i\langle \exp_y^{-1}(x), \xi\rangle} \frac{d\xi}{\sqrt{|g_y|}} + R(x, y, \lambda). \tag{4}$$

The integral in (4) is over the cotangent fiber T_y^*M and the integration measure is the quotient of the symplectic form $d\xi \wedge dy$ by the Riemannian volume form $dv_g = \sqrt{|g_y|}dy$. In Hörmander's original theorem, the phase function $\langle \exp_y^{-1}(x), \xi\rangle$ is replaced by any so-called adapted phase function and one still obtains that

$$\sup_{d_g(x,y)<\varepsilon} \left|\nabla_x^j \nabla_y^k R(x,y,\lambda)\right| = O(\lambda^{n-1+j+k}) \tag{5}$$

as $\lambda \to \infty$, where ∇ denotes covariant differentiation. The estimate (5) for $j = k = 0$ is already in [8, Thm 4.4], while the general case follows from the wave kernel method (e.g. as in §4 of [16] see also [3, Thm 3.1]).

Our main technical result, Theorem 2, shows that the remainder estimate (5) for $R(x, y, \lambda)$ can be improved from $O(\lambda^{n-1+j+k})$ to $o(\lambda^{n-1+j+k})$ under the assumption that x and y are near a non self-focal point (defined below). This paper is a continuation of [4] where the authors proved Theorem 2 for $j = k = 0$. An application of our improved remainder estimates is Theorem 1, which shows that we can compute the scaling limit of $E_{(\lambda,\lambda+1]}(x, y)$ and its derivatives near a non self-focal point as $\lambda \to \infty$.

Definition 1. A point $x \in M$ is *non self-focal* if the loopset

$$\mathcal{L}_x := \{\xi \in S_x^*M : \exists\, t > 0 \text{ with } \exp_x(t\xi) = x\}$$

has measure 0 with respect to the natural measure on T_x^*M induced by g. Note that $\mathcal{L}_x$ can be dense in S_x^*M while still having measure 0 (e.g. for points on a flat torus).

Theorem 1. *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, with no boundary. Suppose $x_0 \in M$ is a non self-focal point and consider a non-negative function r_λ satisfying $r_\lambda = o(\lambda)$ as $\lambda \to \infty$. Define the rescaled kernel*

$$E^{x_0}_{(\lambda,\lambda+1]}(u, v) := \lambda^{-(n-1)} E_{(\lambda,\lambda+1]}\left(\exp_{x_0}\left(\frac{u}{\lambda}\right),\ \exp_{x_0}\left(\frac{v}{\lambda}\right)\right).$$

Then, for all $k, j \geq 0$,

$$\sup_{|u|,|v|\leq r_\lambda} \left|\partial_u^j \partial_v^k \left(E^{x_0}_{(\lambda,\lambda+1]}(u, v) - \frac{1}{(2\pi)^n}\int_{S^*_{x_0}M} e^{i\langle u-v,\omega\rangle} d\omega\right)\right| = o(1)$$

as $\lambda \to \infty$. The inner product in the integral over the unit sphere $S^*_{x_0}M$ is with respect to the flat metric $g(x_0)$ and $d\omega$ is the hypersurface measure on $S^*_{x_0}M$ induced by $g(x_0)$.

Remark 1. Theorem 1 holds for $\Pi_{(\lambda,\lambda+\delta]}$ with arbitrary fixed $\delta > 0$. The difference is that the limiting kernel is multiplied by δ and the rate of convergence in the $o(1)$ term depends on δ.

Remark 2. One can replace the shrinking ball $B(x_0, r_\lambda)$ in Theorem 1 by a compact set $S \subset M$ in which for any $x, y \in S$ the measure of the set of geodesics joining x and y is zero (see Remark 3 after Theorem 2).

Theorem 1 follows from Theorem 2 by combining (9) with the relation $E_{(\lambda,\lambda+1]} = E_{[0,\lambda+1]} - E_{[0,\lambda]}$. In normal coordinates at x_0, Theorem 1 shows that the scaling limit of $E^{x_0}_{(\lambda,\lambda+1]}$ in the C^∞ topology is

$$E_1^{\mathbb{R}^n}(u, v) = \frac{1}{(2\pi)^n}\int_{S^{n-1}} e^{i\langle u-v,\omega\rangle} d\omega,$$

which is the kernel of the frequency 1 spectral projector for the flat Laplacian on $\mathbb{R}^n$. Theorem 1 can therefore be applied to studying the local behavior of random waves on (M,g). More precisely, a frequency λ monochromatic random wave φ_λ on (M,g) is a Gaussian random linear combination

$$\varphi_\lambda = \sum_{\lambda_j\in(\lambda,\lambda+1]} a_j\varphi_j \qquad a_j \sim N(0,1) \text{ i.i.d,}$$

of eigenfunctions with frequencies in $\lambda_j \in (\lambda, \lambda+1]$. In this context, random waves were first introduced by Zelditch in [20]. Since the Gaussian field φ_λ is centered, its law is determined by its covariance function, which is precisely $E_{(\lambda,\lambda+1]}(x,y)$. In the language of Nazarov-Sodin [11] (cf [6, 14]), the estimate (6) means that frenquency λ monochromatic random waves on (M,g) have frequeny 1 random waves on $\mathbb{R}^n$ as their translation invariant local limits at every non self-focal point. This point of view is taken up in the forthcoming article [5].

Theorem 2. *Let (M,g) be a compact, smooth, Riemannian manifold of dimension $n\geq 2$, with no boundary. Let $K\subseteq M$ be the set of all non self-focal points in M. Then for all $k,j\geq 0$ and all $\varepsilon>0$ there is a neighborhood $\mathcal{U}=\mathcal{U}(\varepsilon,k,j)$ of K and constants $\Lambda=\Lambda(\varepsilon,k,j)$ and $C=C(\varepsilon,k,j)$ for which*

$$\|R(x,y,\lambda)\|_{C^k_x(\mathcal{U})\times C^j_y(\mathcal{U})} \leq \varepsilon\lambda^{n-1+j+k} + C\lambda^{n-2+j+k} \tag{6}$$

for all $\lambda>\Lambda$. Hence, if $x_0\in K$ and $\mathcal{U}_\lambda$ is any sequence of sets containing x_0 with diameter tending to 0 as $\lambda\to\infty$, then

$$\|R(x,y,\lambda)\|_{C^k_x(\mathcal{U}_\lambda)\times C^j_y(\mathcal{U}_\lambda)} = o(\lambda^{n-1+j+k}). \tag{7}$$

Remark 3. One can consider more generally any compact $S\subseteq M$ such that all $x,y\in S$ are mutually non-focal, whic means

$$\mathcal{L}_{x,y} := \{\xi\in S^*_xM :\ \exists\, t>0 \text{ with } \exp_x(t\xi)=y\}$$

has measure zero. Then, combining [12, Thm 3.3] with Theorem 2, for every $\varepsilon>0$, there exists a neighborhood $\mathcal{U}=\mathcal{U}(\varepsilon,j)$ of S and constants $\Lambda=\Lambda(\varepsilon,j,S)$ and $C=C(\varepsilon,j,S)$ such that

$$\sup_{x,y\in S}\left|\nabla^j_x\nabla^j_y R(x,y,\lambda)\right| \leq \varepsilon\lambda^{n-1+2j} + C\lambda^{n-2+2j}.$$

We believe that this statement is true even when the number of derivatives in x,y is not the same but do no take this issue up here.

Our proof of Theorem 2 relies heavily on the argument for Theorem 1 in [4], which treated the case $j=k=0$. That result was in turn was based on the work of Sogge-Zelditch [18, 19], who studied $j=k=0$ and $x=y$. This last situation was also studied (independently and significantly before [4, 18, 19]) by Safarov in [12] (cf [13]) using a somewhat different method. The case $j=k=1$ and $x=y$ is essentially Proposition 2.3 in [20]. We refer the reader to the introduction of [4] for more bground on estimates like (6).

1. Proof of Theorem 2

Let x_0 be a non-self focal point. Let I, J be multi-indices and set

$$\Omega := |I| + |J| .$$

We abbreviate

$$E_\lambda \equiv E_{[0,\lambda]}.$$

Using that

$$\int_{S^{n-1}} e^{i\langle u,w\rangle} dw = (2\pi)^{n/2} J_{\frac{n-2}{2}}(|u|)|u|^{-\frac{n-2}{2}} \tag{8}$$

for all $u \in \mathbb{R}^n$, we have

$$\frac{1}{(2\pi)^n} \int_{|\xi|_{g_y}<\lambda} e^{i\langle \exp_y^{-1}(x),\, \xi\rangle} \frac{d\xi}{\sqrt{|g_y|}} = \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{\frac{n}{2}}} \left(\frac{J_{\frac{n-2}{2}}\left(\mu d_g(x,y)\right)}{\left(\mu d_g(x,y)\right)^{\frac{n-2}{2}}} \right) d\mu. \tag{9}$$

Choose coordinates around x_0. We seek to show that there exists a constant $c > 0$ so that for every $\varepsilon > 0$ there is an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and a constant c_ε so that we have

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left| \partial_x^I \partial_y^J E_\lambda(x,y) - \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{\frac{n}{2}}} \partial_x^I \partial_y^J \left(\frac{J_{\frac{n-2}{2}}\left(\mu d_g(x,y)\right)}{\left(\mu d_g(x,y)\right)^{\frac{n-2}{2}}} \right) d\mu \right| \leq c\,\varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \tag{10}$$

Let $\rho \in \mathcal{S}(\mathbb{R})$ satisfy $\operatorname{supp}(\hat\rho) \subseteq (-\operatorname{inj}(M,g), \operatorname{inj}(M,g))$ and

$$\hat\rho(t) = 1 \qquad \text{for all} \qquad |t| < \tfrac{1}{2}\operatorname{inj}(M,g). \tag{11}$$

We prove (10) by first showing that it holds for the convolved measure $\rho * \partial_x^I \partial_y^J E_\lambda(x,y)$ and then estimating the difference $\left|\rho * \partial_x^I \partial_y^J E_\lambda(x,y) - \partial_x^I \partial_y^J E_\lambda(x,y)\right|$ in the following two propositions.

Proposition 3. *Let x_0 be a non-self focal point. Let I, J be multi-indices and set $\Omega = |I| + |J|$. There exists a constant c so that for every $\varepsilon > 0$ there exist an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and a constant c_ε so that we have*

$$\left| \rho * \partial_x^I \partial_y^J E_\lambda(x,y) - \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{\frac{n}{2}}} \partial_x^I \partial_y^J \left(\frac{J_{\frac{n-2}{2}}\left(\mu d_g(x,y)\right)}{\left(\mu d_g(x,y)\right)^{\frac{n-2}{2}}} \right) d\mu \right| \leq c\,\varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega},$$

for all $x, y \in \mathcal{U}_\varepsilon$.

Proposition 4. *Let x_0 be a non-self focal point. There exists a constant c so that for every $\varepsilon > 0$ there exist an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and a constant c_ε so that for all multi-indices I, J we have*

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left|\rho * \partial_x^I \partial_y^J E_\lambda(x,y) - \partial_x^I \partial_y^J E_\lambda(x,y)\right| \leq c\,\varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}.$$

The proof of Proposition 4 hinges on the fact that x_0 is a non self-focal point. Indeed, for each $\varepsilon > 0$, Lemma 15 in [4] (which is a generalization of Lemma 3.1 in

[18]) yields the existence of a neighborhood $\mathcal{O}_\varepsilon$ of x_0, a function $\psi_\varepsilon \in C_c^\infty(M)$ and operators $B_\varepsilon, C_\varepsilon \in \Psi^0(M)$ supported in $\mathcal{O}_\varepsilon$ satisfying both:

$$\bullet\ \operatorname{supp}(\psi_\varepsilon) \subset \mathcal{O}_\varepsilon \ \text{ and } \ \psi_\varepsilon = 1 \ \text{ on a neighborhood of}\, x_0, \tag{12}$$

$$\bullet\ B_\varepsilon + C_\varepsilon = \psi_\varepsilon^2. \tag{13}$$

The operator B_ε is built so that it is microlocally supported on the set of cotangent directions that generate geodesic loops at x_0. Since x_0 is non self-focal, the construction can be carried so that the principal symbol $b_0(x,\xi)$ satisfies $\|b_0(x,\cdot)\|_{L^2(B_x^*M)} \leq \varepsilon$ for all $x \in M$. The operator C_ε is built so that $U(t)C_\varepsilon^*$ is a smoothing operator for $\frac{1}{2}\mathrm{inj}(M,g) < |t| < \frac{1}{\varepsilon}$. In addition, the principal symbols of B_ε and C_ε are real valued and their sub-principal symbols vanish in a neighborhood of x_0 (when regarded as operators acting on half-densities).

In what follows we use the construction above to decompose E_λ, up to an $O(\lambda^{-\infty})$ error, as

$$E_\lambda(x,y) = E_\lambda B_\varepsilon^*(x,y) + E_\lambda C_\varepsilon^*(x,y) \tag{14}$$

for all x, y sufficiently close to x_0. This decomposition is valid since $\psi_\varepsilon \equiv 1$ near x_0.

1.1. **Proof of Proposition 3.** The proof of Proposition 3 consists of writing

$$\rho * \partial_x^I \partial_y^J E_\lambda(x,y) = \int_0^\lambda \partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu(x,y))\, d\mu,$$

and on finding an estimate for $\partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu(x,y))$. Such an estimate is given in Lemma 5, which is stated for the more general case $\partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu Q^*(x,y))$ with $Q \in \{Id, B_\varepsilon, C_\varepsilon\}$ that is needed in the proof of Proposition 4.

Lemma 5. *Let (M,g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, with no boundary. Let $Q \in \{Id, B_\varepsilon, C_\varepsilon\}$ have principal symbol D_0^Q. Consider ρ as in* (11)*, and define*

$$\Omega = |I| + |J|\,.$$

Then, for all $x, y \in M$ with $d_g(x,y) \leq \frac{1}{2}\,\mathrm{inj}(M,g)$, all multi-indices I, J, and all $\mu \geq 1$, we have

$$\begin{aligned} &\partial_\mu(\rho * \partial_x^I \partial_y^J E_\mu Q^*)(x,y) \\ &= \frac{\mu^{n-1}}{(2\pi)^n} \partial_x^I \partial_y^J \left(\int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x), \omega\rangle_{g_y}} \left(D_0^Q(y,\omega) + \mu^{-1} D_{-1}^Q(y,\omega) \right) \frac{d\omega}{\sqrt{|g_y|}} \right) \\ &+ W_{I,J}(x,y,\mu). \end{aligned} \tag{15}$$

Here, $d\omega$ is the Euclidean surface measure on S_y^*M, and D_{-1}^Q is a homogeneous symbol of order -1. The latter satisfy

$$D_{-1}^{B_\varepsilon}(y,\cdot) + D_{-1}^{C_\varepsilon}(y,\cdot) = 0 \qquad \forall\, y \in \mathcal{O}_\varepsilon, \tag{16}$$

where $\mathcal{O}_\varepsilon$ is as in (12)*. Moreover, there exists $C > 0$ so that for every $\varepsilon > 0$*

$$\sup_{x,y \in \mathcal{O}_\varepsilon} \left| \int_{S_y^*M} e^{i\langle \exp_y^{-1}(x), \omega\rangle_{g_y}} D_{-1}^Q(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} \right| \leq C\,\varepsilon. \tag{17}$$

Finally, $W_{I,J}$ is a smooth function in (x,y) for which there exists $C>0$ such that for all x,y satisfying $d_g(x,y)\le \frac{1}{2}\operatorname{inj}(M,g)$ and all $\mu>0$

$$|W_{I,J}(x,y,\mu)|\le C\mu^{n-2+\Omega}\left(d_g(x,y)+(1+\mu)^{-1}\right). \tag{18}$$

Remark 4. Note that Lemma 5 does not assume that x,y are near an non self-focal point.

Remark 5. We note that Lemma 5 is valid for more general operators Q. Indeed, if $Q\in\Psi^k(M)$ has vanishing subprincipal symbol (when regarded as an operator acting on half-densities), then (15) holds with $D_0^Q(y,\omega)$ substituted by $\mu^k D_k^Q(y,\omega)$ and with $\mu^{-1}D_{-1}^Q(y,\omega)$ substituted by $\mu^{k-1}D_{k-1}^Q(y,\omega)$. Here, D_k^Q is the principal symbol of Q and D_{k-1}^Q is a homogeneous polynomial of degree $k-1$. In this setting, the error term satisfies $|W_{I,J}(x,y,\mu)|\le C\mu^{n+k-2+\Omega}\left(d_g(x,y)+(1+\mu)^{-1}\right)$.

Proof of Lemma 5. We use that

$$\partial_\mu(\rho * EQ^*)(x,y,\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\lambda}\hat\rho(t)\,U(t)Q^*(x,y)dt, \tag{19}$$

where $Q\in\Psi(M)$ is any pseudo-differential operator and $U(t)=e^{-it\sqrt{\Delta_g}}$ is the half-wave propagator. The argument from here is identical to that of [4, Proposition 12], which relies on a parametrix for the half-wave propagator for which the kernel can be controlled to high accuracy when x and y are close to the diagonal. The main corrections to the proof of [4, Proposition 12] are that $\partial_x^I\partial_y^J$ gives an $O(\mu^{n-3+\Omega})$ error in equations (54) and (60), and gives an $O(\mu^{n-1})$ error in (59). We must also take into account that $\partial_x\Theta(x,y)^{1/2}$ and $\partial_y\Theta(x,y)^{1/2}$ are both $O(d_g(x,y))$. □

Proof of Proposition 3. Following the technique for proving [4, Proposition 7], we obtain Proposition 3 by applying Lemma 5 to $Q=Id$ (this gives $D_0^{Id}=1$ and $D_{-1}^{Id}=0$) and integrating the expression in (15) from $\mu=0$ to $\mu=\lambda$. One needs to choose $\mathcal{U}_\varepsilon$ so that its diameter is smaller than ε, since this makes $\int_0^\lambda W_{I,J}(x,y,\mu)d\mu = O(\varepsilon\lambda^{n-1+\Omega}+\lambda^{n-2+\Omega})$ as needed. One also uses identity (9) to obtain the exact statement in Proposition 3. □

1.2. **Proof of Proposition 4.** As in (14),

$$E_\lambda(x,y)=E_\lambda B_\varepsilon^*(x,y)+E_\lambda C_\varepsilon^*(x,y)+O\left(\lambda^{-\infty}\right)$$

for all x,y sufficiently close to x_0. Proposition 4 therefore reduces to showing that there exist a constant c independent of ε, a constant $c_\varepsilon=c_\varepsilon(I,J,x_0)$, and a neighborhood $\mathcal{U}_\varepsilon$ of x_0 such that

$$\sup_{x,y\in\mathcal{U}_\varepsilon}\left|\partial_x^I\partial_y^J E_\lambda B_\varepsilon^*(x,y)-\rho*\partial_x^I\partial_y^J E_\lambda B_\varepsilon^*(x,y)\right|\le c\,\varepsilon\lambda^{n-1+\Omega}+c_\varepsilon\lambda^{n-2+\Omega}, \tag{20}$$

and

$$\sup_{x,y\in\mathcal{U}_\varepsilon}\left|\partial_x^I\partial_y^J E_\lambda C_\varepsilon^*(x,y)-\rho*\partial_x^I\partial_y^J E_\lambda C_\varepsilon^*(x,y)\right|\le c\,\varepsilon\lambda^{n-1+\Omega}+c_\varepsilon\lambda^{n-2+\Omega}. \tag{21}$$

Our proofs of (20) and (21) use that these estimates hold on diagonal when $|I|=|J|=0$ (i.e. no derivatives are involved). This is the content of the following result, which was proved in [18] for $Q = Id$. Its proof extends without modification to general $Q \in \Psi^0(M)$.

Lemma 6 (Theorem 1.2 and Proposition 2.2 in [18])**.** *Let $Q \in \Psi^0(M)$ have real-valued principal symbol q. Fix a non-self focal point $x_0 \in M$ and write $\sigma_{sub}(QQ^*)$ for the subprincipal symbol of QQ^* (acting on half-densities). Then, there exists $c > 0$ so that for every $\varepsilon > 0$ there exist a neighborhood $\mathcal{O}_\varepsilon$ and a constant C_ε making*

$$QE_\lambda Q^*(x,x) = (2\pi)^{-n} \int_{|\xi|_{g_x}<\lambda} \left(|q(x,\xi)|^2 + \sigma_{sub}(QQ^*)(x,\xi) \right) \frac{d\xi}{\sqrt{|g_x|}} + R_Q(x,\lambda),$$

with

$$\sup_{x\in\mathcal{U}} |R_Q(x,\lambda)| \leq c\,\varepsilon\lambda^{n-1} + C_\varepsilon \lambda^{n-2}$$

for all $\lambda \geq 1$.

We prove relation (20) in Section 1.2.1 and relation (21) in Section 1.2.2.

1.2.1. *Proof of relation* (20). Define

$$g_{I,J}(x,y,\lambda) := \partial_x^I \partial_y^J E_\lambda B_\varepsilon^*(x,y) - \rho * \partial_x^I \partial_y^J E_\lambda B_\varepsilon^*(x,y).$$

Note that $g_{I,J}(x,y,\cdot)$ is a piecewise continuous function. We aim to find c, c_ε and $\mathcal{U}_\varepsilon$ so that $x_0 \in \mathcal{U}_\varepsilon$ and

$$\sup_{x,y\in\mathcal{U}_\varepsilon} |g_{I,J}(x,y,\lambda)| \leq c\,\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon\lambda^{n-2+\Omega}. \tag{22}$$

By [4, Lemma 17], which is a Tauberian Theorem for non-monotone functions, relation (22) reduces to checking the following two conditions:

- $$\mathcal{F}_{\lambda\to t}(g_{I,J})(x,y,t) = 0 \quad \text{for all} \quad |t| < \frac{1}{2}\,\mathrm{inj}(M,g), \tag{23}$$
- $$\sup_{x,y\in\mathcal{U}_\varepsilon} \sup_{s\in[0,1]} |g_{I,J}(x,y,\lambda+s) - g_{I,J}(x,y,\lambda)| \leq c\,\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon\lambda^{n-2+\Omega}. \tag{24}$$

By construction, $\mathcal{F}_{\lambda\to t}(\partial_\lambda g_{I,J})(x,y,t) = (1-\hat{\rho}(t))\partial_x^I\partial_y^J U(t) B_\varepsilon^*(x,y) = 0$ for all $|t| < \frac{1}{2}\,\mathrm{inj}(M,g)$. Hence, since $\mathcal{F}_{\lambda\to t}(g_{I,J})$ is continuous at $t = 0$, we have (23). To prove (24) we write

$$\begin{aligned} \sup_{s\in[0,1]} &|g_{I,J}(x,y,\lambda+s) - g_{I,J}(x,y,\lambda)| \\ &\leq \sup_{s\in[0,1]} \left|\partial_x^I\partial_y^J E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)\right| + \sup_{s\in[0,1]} \left|\rho * \partial_x^I\partial_y^J E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)\right|. \end{aligned} \tag{25}$$

The second term in (25) is bounded above by the right hand side of (24) by Lemma 5. To bound the first term, use Cauchy-Schwartz to get

$$\begin{aligned} \sup_{s\in[0,1]} \left|\partial_x^I\partial_y^J [E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)]\right| &= \sup_{s\in[0,1]} \Big| \sum_{\lambda_j\in(\lambda,\lambda+1]} \partial_x^I \varphi_j(x)\cdot \partial_y^J B_\varepsilon \varphi_j(y) \Big| \\ &\leq \sum_{\lambda_j\in(\lambda,\lambda+1]} \left| \left[\left(B_\varepsilon \partial_y^J + [\partial_y^J, B_\varepsilon] \right) \varphi_j(y) \right] \right| \cdot \left| \partial_x^I \varphi_j(x) \right|. \end{aligned}$$

Write b_0 for the principal symbol of B_ε. By construction, for all y in a neighborhood of x_0, we have $\partial_y b_0(y,\xi) = 0$. Therefore, $\sigma_{|J|-1}\left([\partial_y^J, B_\varepsilon]\right) = i^{|J|}\{\xi^J, b_0(y,\xi)\} = 0$ and we conclude that $[\partial_y^J, B_\varepsilon] \in \Psi^{|J|-2}$. Thus, by the usual pointwise Weyl Law (e.g. [19, Equation (2.31)]),

$$\sup_{s\in[0,1]} \left|\partial_x^I \partial_y^J [E_{(\lambda,\lambda+s]} B_\varepsilon^*(x,y)]\right| \leq \sum_{\lambda_j \in (\lambda,\lambda+1]} \left|B_\varepsilon \partial_y^J \varphi_j(y)\right| \cdot \left|\partial_x^I \varphi_j(x)\right| + O(\lambda^{n-3+\Omega})$$

Next, define for each multi-index $K \in \mathbb{N}^n$ the order zero pseudo-differential operator

$$P_K := \partial^K \Delta_g^{-|K|/2}.$$

Using Cauchy-Schwarz and that $\partial^K \varphi_j = \lambda_j^{|K|} P_K \varphi_j$, we find

$$\sum_{\lambda_j \in (\lambda,\lambda+1]} \left|B_\varepsilon \partial_y^J \varphi_j(y)\right| \cdot \left|\partial_x^I \varphi_j(x)\right|$$
$$\leq (\lambda+1)^\Omega [(B_\varepsilon P_J) E_{(\lambda,\lambda+1]} (B_\varepsilon P_J)^*(y,y)]^{\frac{1}{2}} [P_I E_{(\lambda,\lambda+1]} P_I^*(x,x)]^{\frac{1}{2}}.$$

Again using the pointwise Weyl Law (see [19, Equation (2.31)]), we have $[P_I E_{(\lambda,\lambda+1]} P_I^*(x,x)]^{\frac{1}{2}}$ is $O(\lambda^{\frac{n-1}{2}})$. Next, since according to the construction of B_ε we have

$$\sup_{x\in\mathcal{U}_\varepsilon} \|b_0(x,\cdot)\|_{L^2(B_x^* M)} \leq \varepsilon$$

and $\partial_x b_0(x,\xi) = 0$ for x in a neighborhood $\mathcal{U}_\varepsilon$ of x_0, we conclude that

$$\sup_{x\in\mathcal{U}_\varepsilon} \|\sigma_{sub}(B_\varepsilon P_J (B_\varepsilon P_J)^*)(x,\cdot)\|_{L^2(B_x^* M)} \leq \varepsilon^2.$$

Proposition 6 therefore shows that there exists $c > 0$ making

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left|(B_\varepsilon P_J) E_{(\lambda,\lambda+1]} (B_\varepsilon P_J)^*(y,y)\right|^{\frac{1}{2}} \leq c\varepsilon \lambda^{\frac{n-1}{2}}. \tag{26}$$

This proves (24), which together with (23) allows us to conclude (22).

1.2.2. *Proof of relation* (21). Write

$$\partial_x^I \partial_y^J E_\lambda C_\varepsilon^*(x,y) = \sum_{\lambda_j \leq \lambda} \lambda_j^\Omega \left(P_I \varphi_j(x)\right) \cdot \left(C_\varepsilon P_J \varphi_j(y)\right) + \sum_{\lambda_j \leq \lambda} \lambda_j^{|I|} \left(P_I \varphi_j(x)\right) \cdot \left([\partial^J, C_\varepsilon] \varphi_j(y)\right). \tag{27}$$

As before, $[\partial^J, C_\varepsilon] \in \Psi^{|J|-2}$. Hence, by the usual pointwise Weyl law, the second term in (27) and its convolution with ρ are both $O(\lambda^{n-2+\Omega})$. Hence,

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left|\partial_x^I \partial_y^J E_\lambda C_\varepsilon^*(x,y) - \rho * \partial_x^I \partial_y^J E_\lambda C_\varepsilon^*(x,y)\right| = \sup_{x,y\in\mathcal{U}_\varepsilon} |V(x,y,\lambda) - \rho * V(x,y,\lambda)| + O\left(\lambda^{n+\Omega-2}\right),$$

where we have set

$$V(x,y,\lambda) := \partial^I E_\lambda (C_\varepsilon \partial^J)^*(x,y) = \sum_{\lambda_j \leq \lambda} \lambda_j^\Omega \left(P_I \varphi_j(x)\right) \cdot \left(C_\varepsilon P_J \varphi_j(y)\right).$$

Define

$$\alpha_{I,J}(x,y,\lambda) := V(x,y,\lambda) + \frac{1}{2}\sum_{\lambda_j\le\lambda} \lambda_j^{\Omega}\left(\left|P_I\varphi_{\lambda_j}(x)\right|^2 + \left|C_\varepsilon P_J\varphi_{\lambda_j}(y)\right|^2\right) \tag{28}$$

$$\beta_{I,J}(x,y,\lambda) := \rho * V(x,y,\lambda) + \frac{1}{2}\sum_{\lambda_j\le\lambda} \lambda_j^{\Omega}\left(\left|P_I\varphi_{\lambda_j}(x)\right|^2 + \left|C_\varepsilon P_J\varphi_{\lambda_j}(y)\right|^2\right). \tag{29}$$

By construction, $\alpha_{I,J}(x,y,\cdot)$ is a monotone function of λ for x,y fixed, and $\alpha_{I,J}(x,y,\lambda) - \beta_{I,J}(x,y,\lambda) = V(x,y,\lambda) - \rho * V(x,y,\lambda)$. So we aim to show that

$$\sup_{x,y\in\mathcal{U}_\varepsilon} |\alpha_{I,J}(x,y,\lambda) - \beta_{I,J}(x,y,\lambda)| \le c\,\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon\lambda^{n-2+\Omega}. \tag{30}$$

We control the difference in (30) applying a Tauberian theorem for monotone functions [4, Lemma 16]. To apply it we need to show the following:

- There exists $c > 0$ and $c_\varepsilon > 0$ making

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |\partial_\mu\beta_{I,J}(x,y,\mu)|\,d\mu \le c\varepsilon\lambda^{n-1+\Omega} + c_\varepsilon\lambda^{n-2+\Omega}. \tag{31}$$

- For all N there exists $M_{\varepsilon,N}$ so that for all $\lambda > 0$

$$|\partial_\lambda\left(\alpha_{I,J}(x,y,\cdot) - \beta_{I,J}(x,y,\cdot)\right) * \phi_\varepsilon(\mu)| \le M_{\varepsilon,N}\left(1+|\lambda|\right)^{-N}. \tag{32}$$

In equation (32) we have set $\phi_\varepsilon(\lambda) := \frac{1}{\varepsilon}\phi\left(\frac{\lambda}{\varepsilon}\right)$ for some $\phi \in \mathcal{S}(\mathbb{R})$ chosen so that $\operatorname{supp}\hat{\phi} \subseteq (-1,1)$ and $\hat{\phi}(0) = 1$.

Relation (31) follows after applying Lemma 6 to the piece of the integral corresponding to the second term in (29) and from applying Lemma 5 together with Remark 5 to $\rho * V = \rho * \partial^I E_\lambda Q^*$, where $Q := C_\varepsilon\partial^J$ has vanishing subprincipal symbol.

To verify (32) note that $\operatorname{supp}(1-\widehat{\rho}) \subseteq \{t : |t| \ge \operatorname{inj}(M,g)/2\}$ and $\operatorname{supp}(\widehat{\phi_\varepsilon}) \subseteq \{t : |t| \le \frac{1}{\varepsilon}\}$. Observe that

$$\partial_\lambda\Big(\alpha_{I,J}(x,y,\cdot) - \beta_{I,J}(x,y,\cdot)\Big) * \phi_\varepsilon\,(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\Big((1-\hat{\rho}(t))\,\hat{\phi}_\varepsilon(t)\partial^I U(t)(\partial^J C_\varepsilon)^*(x,y)\Big)(\lambda).$$

By construction $U(t)C_\varepsilon^*$ is a smoothing operator for $\frac{1}{2}\operatorname{inj}(M,g) < |t| < \frac{1}{\varepsilon}$. Thus, so is $\partial^I U(t)\left(\partial^J C_\varepsilon\right)^*$ which implies (32). This concludes the proof of relation (21). □

References

[1] P. Bérard. On the wave equation on a compact Riemannian manifold without conjugate points. *Math. Z.* 155 (1977), 249-276.

[2] M. Berger, P. Gauduchon, and E. Mazet. Le spectre d'une variété Riemannienne. *Lecture Notes in Math.* Springer (1971), 251p.

[3] B. Xu. Derivatives of the spectral function and Sobolev norms of eigenfunctions on a closed Riemannian manifold. *Anal. of Global. Anal. and Geom.*, (2004) 231–252.

[4] Y. Canzani, B. Hanin. Scaling Limit for the Kernel of the Spectral Projector and Remainder Estimates in the Pointwise Weyl Law. *Analysis and PDE*, 8 (2015) no. 7, 1707–1731.

[5] Y. Canzani, B. Hanin. Local Universality of Random Waves. In preparation.

[6] Y. Canzani, P. Sarnak. On the topology of the zero sets of monochromatic random waves. Preprint available: arXiv:1412.4437.

[7] J. Duistermaat and V. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Inventiones Mathematicae*, 29 (1975) no. 1, 39–79.

[8] L. Hörmander. The spectral function of an elliptic operator. *Acta mathematica*, 121.1 (1968), 193-218.
[9] V. Ivrii. Precise spectral asymptotics for elliptic operators. *Lecture Notes in Math.* 1100, Springer, (1984).
[10] V. Ivrii. The second term of the spectral asymptotics for a Laplace Beltrami operator on manifolds with boundary. *(Russian) Funksional. Anal. i Prolzhen.* (14) (1980), no. 2, 25–34.
[11] F. Nazarov and M. Sodin. Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. *Preprint: http://arxiv.org/abs/1507.02017.*
[12] Yu. Safarov. Asymptotics of the spectral function of a positive elliptic operator without a non-trapping condition. *Funktsional. Anal. i Prilozhen.* 22 (1988), no. 3, 53-65, 96 (Russian). English translation in *Funct. Anal. Appl.* 22 (1988), no. 3, 213-223
[13] Y. Safarov and D. Vassiliev. The asymptotic distribution of eigenvalues of partial differential operators. *American Mathematical Society,* 1996.
[14] P. Sarnak and I. Wigman Topologies of nodal sets of random band limited functions. *Preprint: http://arxiv.org/abs/1510.08500 (2015).*
[15] C. Sogge, J. Toth, and S. Zelditch. About the blowup of quasimodes on Riemannian manifolds. *J. Geom. Anal.* 21.1 (2011), 150-173.
[16] C. Sogge. Fourier integrals in classical analysis. *Cambridge Tracts in Mathematics* 105, Cambridge University Press, (1993).
[17] C. Sogge. Hangzhou lectures on eigenfunctions of the Laplacian Annals of Math Studies. 188, Princeton Univ. Press, 2014.
[18] C. Sogge and S. Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Mathematical Journal* 114.3 (2002), 387 – 437.
[19] C. Sogge and S. Zelditch. Focal points and sup-norms of eigenfunctions. *Preprint: http://arxiv.org/abs/1311.3999.*
[20] S. Zelditch. Real and complex zeros of Riemannian random waves. *Contemporary Mathematics* 14 (2009), 321.

(Y. Canzani) Department of Mathematics, Harvard University, Cambridge, United States.

E-mail address: canzani@math.harvard.edu

(B. Hanin) Department of Mathematics, MIT, Cambridge, United States.

E-mail address: bhanin@mit.edu