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# Classification of linearly compact simple Nambu-Poisson algebras

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## Abstract

We introduce the notion of universal odd generalized Poisson superalgebra associated to an associative algebra  $A$ , by generalizing a construction made in [5]. By making use of this notion we give a complete classification of simple linearly compact (generalized)  $n$ -Nambu-Poisson algebras over an algebraically closed field of characteristic zero.

## Introduction

In 1973 Y. Nambu proposed a generalization of Hamiltonian mechanics, based on the notion of  $n$ -ary bracket in place of the usual binary Poisson bracket [9]. Nambu dynamics is described by the flow, given by a system of ordinary differential equations which involves  $n - 1$  Hamiltonians:

$$(0.1) \quad \frac{du}{dt} = \{u, h_1, \dots, h_{n-1}\}.$$

The (only) example, proposed by Nambu is the following  $n$ -ary bracket on the space of functions in  $N \geq n$  variables:

$$(0.2) \quad \{f_1, \dots, f_n\} = \det \left( \frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^n.$$

He pointed out that this  $n$ -ary bracket satisfies the following axioms, similar to that of a Poisson bracket:

$$(\text{Leibniz rule}) \quad \{f_1, \dots, f_i \tilde{f}_i, \dots, f_n\} = f_i \{f_1, \dots, \tilde{f}_i, \dots, f_n\} + \tilde{f}_i \{f_1, \dots, f_i, \dots, f_n\};$$

$$(\text{skewsymmetry}) \quad \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\} = (\text{sign} \sigma) \{f_1, \dots, f_n\}.$$

Twelve years later this example was rediscovered by F. T. Filippov in his theory of  $n$ -Lie algebras which is a natural generalization of ordinary (binary) Lie algebras [7]. Namely, an  $n$ -Lie algebra is a vector space with  $n$ -ary bracket  $[a_1, \dots, a_n]$ , which is skewsymmetric (as above) and satisfies the following Filippov-Jacobi identity:

$$(0.3) \quad [a_1, \dots, a_{n-1}, [b_1, \dots, b_n]] = [[a_1, \dots, a_{n-1}, b_1], b_2, \dots, b_n] + [b_1, [a_1, \dots, a_{n-1}, b_2], b_3, \dots, b_n] + \dots \\ + [b_1, \dots, b_{n-1}, [a_1, \dots, a_{n-1}, b_n]].$$

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In particular, Filippov proved that the Nambu bracket (0.2) satisfies the Filippov-Jacobi identity.

Following Takhtajan [10], we call an  $n$ -Nambu-Poisson algebra a unital commutative associative algebra  $\mathcal{N}$ , endowed with an  $n$ -ary bracket, satisfying the Leibniz rule, skew-symmetry and Filippov-Jacobi identity. Of course for  $n = 2$  this is the definition of a Poisson algebra.

In [4] we classified simple linearly compact  $n$ -Lie algebras with  $n > 2$  over a field  $\mathbb{F}$  of characteristic 0. The classification is based on a bijective correspondence between  $n$ -Lie algebras and pairs  $(L, \mu)$ , where  $L$  is a  $\mathbb{Z}$ -graded Lie superalgebra of the form  $L = \bigoplus_{j=-1}^{n-1} L_j$  satisfying certain additional properties, and  $L_{n-1} = \mathbb{F}\mu$ , thereby reducing it to the known classification of simple linearly compact Lie superalgebras and their  $\mathbb{Z}$ -gradings [8], [1]. For this construction we used the universal  $\mathbb{Z}$ -graded Lie superalgebra, associated to a vector superspace.

In the present paper we use an analogous correspondence between linearly compact  $n$ -Nambu-Poisson algebras and certain "good" pairs  $(\mathcal{P}, \mu)$ , where  $\mathcal{P}$  is a  $\mathbb{Z}_+$ -graded odd Poisson superalgebra  $\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$  and  $\mu \in \mathcal{P}_{n-1}$  is an element of parity  $n \bmod 2$ . For this construction we use the universal  $\mathbb{Z}$ -graded odd Poisson superalgebra, associated to an associative algebra, considered in [5]. As a result, using the classification of simple linearly compact odd Poisson superalgebras [3], we obtain the following theorem.

**Theorem 0.1** *For  $n > 2$ , any simple linearly compact  $n$ -Nambu-Poisson algebra is isomorphic to the algebra  $\mathbb{F}[[x_1, \dots, x_n]]$  with the  $n$ -ary bracket (0.2).*

Note the sharp difference with the Poisson case, when each algebra  $\mathbb{F}[[p_1, \dots, p_n, q_1, \dots, q_n]]$  carries a Poisson bracket

$$(0.4) \quad \{f, g\}_P = \sum_{i=1}^n \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right),$$

making it a simple linearly compact Poisson algebra (and these are all, up to isomorphism [2]).

In the present paper we treat also the case of a generalized  $n$ -Nambu-Poisson bracket, which is an  $n$ -ary analogue of the generalized Poisson bracket, called also the Lagrange's bracket. For the latter bracket the Leibniz rule is modified by adding an extra term:

$$\{a, bc\} = \{a, b\}c + \{a, c\}b - \{a, 1\}bc.$$

In order to treat this case along similar lines, we construct the universal  $\mathbb{Z}$ -graded generalized odd Poisson superalgebra, associated to an associative algebra, which is a generalization of the construction in [5]. Our main result in this direction is the following theorem, which uses the classification of simple linearly compact odd generalized Poisson superalgebras [3].

**Theorem 0.2** *For  $n > 2$ , any simple linearly compact generalized  $n$ -Nambu-Poisson algebra is gauge equivalent (see Remark 1.4 for the definition) either to the Nambu  $n$ -algebra from Theorem 0.1 or to the Dzhumadil'daev  $n$ -algebra [6], which is  $\mathbb{F}[[x_1, \dots, x_{n-1}]]$  with the  $n$ -ary bracket*

$$(0.5) \quad \{f_1, \dots, f_n\} = \det \begin{pmatrix} f_1 & \dots & f_n \\ \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_1} \\ \dots & \dots & \dots \\ \frac{\partial f_1}{\partial x_{n-1}} & \dots & \frac{\partial f_n}{\partial x_{n-1}} \end{pmatrix}.$$

Note again the sharp difference with the generalized Poisson case, when each algebra  $\mathbb{F}[[p_1, \dots, p_n, q_1, \dots, q_n, t]]$  carries a Lagrange bracket

$$(0.6) \quad \{f, g\}_L = \{f, g\}_P + (2 - E)f \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)g,$$

where  $\{f, g\}_P$  is given by (0.4) and  $E = \sum_{i=1}^n (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i})$ , making it a simple linearly compact generalized Poisson algebra (and those, along with (0.4), are all, up to gauge equivalence).

Throughout the paper our base field  $\mathbb{F}$  has characteristic 0 and is algebraically closed.

## 1 Nambu-Poisson algebras

**Definition 1.1** A generalized  $n$ -Nambu-Poisson algebra is a triple  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  such that

- $(\mathcal{N}, \cdot)$  is a unital associative commutative algebra;
- $(\mathcal{N}, \{\cdot, \dots, \cdot\})$  is an  $n$ -Lie algebra;
- the following generalized Leibniz rule holds:

$$(1.1) \quad \{a_1, \dots, a_{n-1}, bc\} = \{a_1, \dots, a_{n-1}, b\}c + b\{a_1, \dots, a_{n-1}, c\} - \{a_1, \dots, a_{n-1}, 1\}bc.$$

If  $\{a_1, \dots, a_{n-1}, 1\} = 0$ , then (1.1) is the usual Leibniz rule and  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  is called simply  $n$ -Nambu-Poisson algebra.

For  $n = 2$  Definition 1.1 is the definition of a generalized Poisson algebra. Simple linearly compact generalized Poisson (super)algebras were classified in [2, Corollary 7.1].

**Example 1.2** Let  $\mathcal{N} = \mathbb{F}[[x_1, \dots, x_n]]$  with the usual commutative associative product and  $n$ -ary bracket defined, for  $f_1, \dots, f_n \in \mathcal{N}$ , by:

$$\{f_1, \dots, f_n\} = \det \begin{pmatrix} D_1(f_1) & \dots & D_1(f_n) \\ \dots & \dots & \dots \\ D_n(f_1) & \dots & D_n(f_n) \end{pmatrix}$$

where  $D_i = \frac{\partial}{\partial x_i}$ ,  $i = 1, \dots, n$ . Then  $\mathcal{N}$  is an  $n$ -Nambu-Poisson algebra, introduced by Nambu [9], that we will call the  $n$ -Nambu algebra (cf. [9], [7], [4]).

**Example 1.3** Let  $\mathcal{N} = \mathbb{F}[[x_1, \dots, x_{n-1}]]$  with the usual commutative associative product and  $n$ -ary bracket defined, for  $f_1, \dots, f_n \in \mathcal{N}$ , by

$$\{f_1, \dots, f_n\} = \det \begin{pmatrix} f_1 & \dots & f_n \\ D_1(f_1) & \dots & D_1(f_n) \\ \dots & \dots & \dots \\ D_{n-1}(f_1) & \dots & D_{n-1}(f_n) \end{pmatrix}$$

where  $D_i = \frac{\partial}{\partial x_i}$ ,  $i = 1, \dots, n-1$ . Then  $\mathcal{N}$  is a generalized Nambu-Poisson algebra that we will call the  $n$ -Dzhumadil'daev algebra (cf. [6], [4]).

**Remark 1.4** Let  $N = (\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  be a generalized  $n$ -Nambu-Poisson algebra. For any invertible element  $\varphi \in \mathcal{N}$  define the following bracket on  $\mathcal{N}$ :

$$(1.2) \quad \{f_1, \dots, f_n\}^\varphi = \varphi^{-1} \{\varphi f_1, \dots, \varphi f_n\}.$$

Then  $N^\varphi = (\mathcal{N}, \{\cdot, \dots, \cdot\}^\varphi, \cdot)$  is another generalized  $n$ -Nambu-Poisson algebra. Indeed, the skew-symmetry of the bracket is straightforward and the Filippov-Jacobi identity for the bracket  $\{\cdot, \dots, \cdot\}^\varphi$  easily follows from the Filippov-Jacobi identity for the bracket  $\{\cdot, \dots, \cdot\}$ . Let us check that  $\{\cdot, \dots, \cdot\}^\varphi$  satisfies the generalized Leibniz rule. We have:

$$\begin{aligned} \{f_1, \dots, f_{n-1}, gh\}^\varphi &= \varphi^{-1} \{\varphi f_1, \dots, \varphi f_{n-1}, \varphi gh\} = \varphi^{-1} (\{\varphi f_1, \dots, \varphi f_{n-1}, \varphi g\} h \\ &+ \varphi g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} - \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} \varphi gh) = \{f_1, \dots, f_{n-1}, g\}^\varphi h \\ &+ g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} - \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh = \{f_1, \dots, f_{n-1}, g\}^\varphi h \\ &+ g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} - \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh + g \{f_1, \dots, f_{n-1}, h\}^\varphi \\ &- \varphi^{-1} g \{\varphi f_1, \dots, \varphi f_{n-1}, \varphi h\} = \{f_1, \dots, f_{n-1}, g\}^\varphi h + g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} \\ &- \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh + g \{f_1, \dots, f_{n-1}, h\}^\varphi - g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} \\ &- \varphi^{-1} gh \{\varphi f_1, \dots, \varphi f_{n-1}, \varphi\} + \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh \\ &= \{f_1, \dots, f_{n-1}, g\}^\varphi h + g \{f_1, \dots, f_{n-1}, h\}^\varphi - \{f_1, \dots, f_{n-1}, 1\}^\varphi gh. \end{aligned}$$

We shall say that the generalized Nambu-Poisson algebras  $N$  and  $N^\varphi$  are *gauge equivalent*.

## 2 Odd generalized Poisson superalgebras

**Definition 2.1** An odd generalized Poisson superalgebra  $(\mathcal{P}, [\cdot, \cdot], \wedge)$  is a triple such that

- $(\mathcal{P}, \wedge)$  is a unital associative commutative superalgebra with parity  $p$ ;
- $(\Pi\mathcal{P}, [\cdot, \cdot])$  is a Lie superalgebra (here  $\Pi\mathcal{P}$  denotes the space  $\mathcal{P}$  with parity  $\bar{p} = p + \bar{1}$ );
- the following generalized odd Leibniz rule holds:

$$(2.1) \quad [a, b \wedge c] = [a, b] \wedge c + (-1)^{(p(a)+1)p(b)} b \wedge [a, c] + (-1)^{p(a)+1} D(a) \wedge b \wedge c,$$

where  $D(a) = [1, a]$ . If  $D = 0$ , then relation (2.1) becomes the odd Leibniz rule; in this case  $(\mathcal{P}, [\cdot, \cdot], \wedge)$  is called an odd Poisson superalgebra (or Gerstenhaber superalgebra). Note that  $D$  is an odd derivation of the associative product and of the Lie superalgebra bracket.

**Example 2.2** Consider the commutative associative superalgebra  $\mathcal{O}(m, n) = \Lambda(n)[[x_1, \dots, x_m]]$ , where  $\Lambda(n)$  denotes the Grassmann algebra over  $\mathbb{F}$  on  $n$  anti-commuting indeterminates  $\xi_1, \dots, \xi_n$ , and the superalgebra parity is defined by  $p(x_i) = \bar{0}$ ,  $p(\xi_j) = \bar{1}$ .

Set  $m = n$  and define the following bracket, known as the Buttin bracket, on  $\mathcal{O}(n, n)$  ( $f, g \in \mathcal{O}(n, n)$ ):

$$(2.2) \quad [f, g]_{HO} = \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} \right).$$

Then  $\mathcal{O}(n, n)$  with this bracket is an odd Poisson superalgebra, which we denote by  $PO(n, n)$ .

**Example 2.3** Consider the associative superalgebra  $\mathcal{O}(n, n+1)$  with even indeterminates  $x_1, \dots, x_n$  and odd indeterminates  $\xi_1, \dots, \xi_n, \xi_{n+1} = \tau$ . Define on  $\mathcal{O}(n, n+1)$  the following bracket ( $f, g \in \mathcal{O}(n, n+1)$ ):

$$(2.3) \quad [f, g]_{KO} = [f, g]_{HO} + (E - 2)(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} (E - 2)(g),$$

where  $[\cdot, \cdot]_{HO}$  is the Buttin bracket (2.2) and  $E = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$  is the Euler operator. Then  $\mathcal{O}(n, n+1)$  with bracket  $[\cdot, \cdot]_{KO}$  is an odd generalized Poisson superalgebra with  $D = -2 \frac{\partial}{\partial \tau}$  [1, Remark 4.1], which we denote by  $PO(n, n+1)$ .

**Remark 2.4** Let  $P = (\mathcal{P}, [\cdot, \cdot], \cdot)$  be an odd generalized Poisson superalgebra. For any invertible element  $\varphi \in \mathcal{P}$ , such that  $p(\varphi) = \bar{0}$  and  $[\varphi, \varphi] = 0$ , define the following bracket on  $P$ :

$$(2.4) \quad [a, b]^\varphi = \varphi^{-1} [\varphi a, \varphi b].$$

Then  $P^\varphi = (\mathcal{P}, [\cdot, \cdot]^\varphi, \cdot)$  is another odd generalized Poisson superalgebra, with derivation

$$D_\varphi(a) = [1, a]^\varphi = [\varphi, a] - D(\varphi)a.$$

The odd generalized Poisson superalgebras  $P$  and  $P^\varphi$  are called *gauge equivalent* (cf. [3, Example 3.4]). Note that the associative products in  $P$  and  $P^\varphi$  are the same.

**Theorem 2.5** [3, Corollary 9.2]

- a) Any simple linearly compact odd generalized Poisson superalgebra is gauge equivalent to  $PO(n, n)$  or  $PO(n, n+1)$ .
- b) Any simple linearly compact odd Poisson superalgebra is isomorphic to  $PO(n, n)$ .

**Definition 2.6** A  $\mathbb{Z}$ -graded (resp.  $\mathbb{Z}_+$ -graded) odd generalized Poisson superalgebra is an odd generalized Poisson superalgebra  $(\mathcal{P}, [\cdot, \cdot], \wedge)$  such that  $(\Pi\mathcal{P}, [\cdot, \cdot])$  is a  $\mathbb{Z}$ -graded Lie superalgebra:  $\Pi\mathcal{P} = \bigoplus_{j \in \mathbb{Z}} \Pi\mathcal{P}_j$  (resp. a  $\mathbb{Z}$ -graded Lie superalgebra of depth 1:  $\Pi\mathcal{P} = \bigoplus_{j \geq -1} \Pi\mathcal{P}_j$ ) and  $(\mathcal{P}, \wedge)$  is a  $\mathbb{Z}$ -graded commutative associative superalgebra:  $\mathcal{P} = \bigoplus_{k \in \mathbb{Z}} \mathcal{Q}_k$  (resp. a  $\mathbb{Z}_+$ -graded commutative associative superalgebra:  $\mathcal{P} = \bigoplus_{k \in \mathbb{Z}_+} \mathcal{Q}_k$ ) such that  $\mathcal{P}_j = \Pi\mathcal{Q}_{j+1}$ .

**Example 2.7** Let us consider the odd Poisson superalgebra  $PO(n, n)$  (resp.  $PO(n, n+1)$ ). Set  $\deg x_i = 0$  and  $\deg \xi_i = 1$  for every  $i = 1, \dots, n$  (resp.  $\deg x_i = 0$ ,  $\deg \xi_i = 1$  for every  $i = 1, \dots, n$  and  $\deg \tau = 1$ ). Then  $PO(n, n)$  (resp.  $PO(n, n+1)$ ) becomes a  $\mathbb{Z}_+$  graded odd (resp. generalized) Poisson superalgebra with

$$\mathcal{Q}_j = \{f \in \mathcal{O}(n, n) \mid \deg(f) = j\}$$

and

$$\mathcal{P}_j = \{f \in \mathcal{O}(n, n) \mid \deg(f) = j+1\}.$$

We will call this grading a grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ). We thus have, for  $\mathcal{P} = PO(n, n)$ :

$$\Pi\mathcal{P}_{-1} = \mathcal{Q}_0 = \mathbb{F}[[x_1, \dots, x_n]]$$

and, for  $j \geq 0$ ,

$$\Pi\mathcal{P}_j = \mathcal{Q}_{j+1} = \langle \xi_{i_1} \dots \xi_{i_{j+1}} \mid 1 \leq i_1 < \dots < i_{j+1} \leq n \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]].$$

Similarly, for  $\mathcal{P} = PO(n, n+1)$ , we have:

$$\mathcal{P}_{-1} = \mathcal{Q}_0 = \mathbb{F}[[x_1, \dots, x_n]]$$

$$\mathcal{P}_j = \mathcal{Q}_{j+1} = \langle \xi_{i_1} \dots \xi_{i_{j+1}} \mid 1 \leq i_1 < \dots < i_{j+1} \leq n+1 \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]].$$

**Remark 2.8** From the properties of the  $\mathbb{Z}$ -gradings of the Lie superalgebras  $HO(n, n)$  and  $KO(n, n+1)$  (see, for example, [8]), one can deduce that the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ) is, up to isomorphisms, the only  $\mathbb{Z}_+$ -grading of  $\mathcal{P} = PO(n, n)$  (resp.  $\mathcal{P} = PO(n, n+1)$ ) such that  $\mathcal{P}_{-1}$  is completely odd.

**Remark 2.9** Let  $\mathcal{P} = PO(n, n)$  or  $\mathcal{P} = PO(n, n+1)$  and let  $\mathcal{P}^\varphi$  be an odd generalized Poisson superalgebra gauge equivalent to  $\mathcal{P}$ . Then the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ) is, up to isomorphisms, the only  $\mathbb{Z}_+$ -grading of  $\mathcal{P}^\varphi$  such that  $\mathcal{P}_{-1}^\varphi$  is completely odd. Indeed, let  $\mathcal{P}^\varphi = \oplus_{k \in \mathbb{Z}_+} \mathcal{Q}_k^\varphi = \oplus_{j \geq -1} \mathcal{P}_j^\varphi$ , with  $\mathcal{P}_j^\varphi = \Pi\mathcal{Q}_{j+1}^\varphi$  a  $\mathbb{Z}_+$ -grading of  $\mathcal{P}^\varphi$ . Suppose that,  $x_i \in \mathcal{Q}_k^\varphi$  and  $\xi_i \in \mathcal{Q}_j^\varphi$  for some  $1 \leq i \leq n$  and some  $k, j \in \mathbb{Z}_+$ . Then

$$(2.5) \quad [x_i, \xi_i]^\varphi \in \Pi\mathcal{P}_{k+j-2}^\varphi = \mathcal{Q}_{k+j-1}^\varphi.$$

On the other hand, by (2.4), we have:

$$\begin{aligned} [x_i, \xi_i]^\varphi &= \varphi^{-1}[\varphi x_i, \varphi \xi_i] = \varphi^{-1}([\varphi x_i, \varphi] \xi_i + \varphi[\varphi x_i, \xi_i] - D(\varphi x_i) \varphi \xi_i) = \\ &= [x_i, \varphi] \xi_i - D(\varphi) x_i \xi_i + [\varphi x_i, \xi_i] - D(\varphi x_i) \xi_i = \frac{\partial \varphi}{\partial \xi_i} \xi_i + \frac{1}{2} x_i \xi_i D(\varphi) - D(\varphi) x_i \xi_i + \frac{\partial \varphi}{\partial x_i} x_i + \\ &\quad + \varphi + \frac{1}{2} D(\varphi) x_i \xi_i - D(\varphi) x_i \xi_i = \frac{\partial \varphi}{\partial \xi_i} \xi_i - D(\varphi) x_i \xi_i + \frac{\partial \varphi}{\partial x_i} x_i + \varphi, \end{aligned}$$

where  $D = 0$  if  $\mathcal{P} = PO(n, n)$  and  $D = -2\frac{\partial}{\partial \tau}$  if  $\mathcal{P} = PO(n, n+1)$ . Note that  $[x_i, \xi_i]^\varphi$  is invertible since  $\varphi$  is invertible and, by (2.5), it is homogeneous, hence  $k+j=1$ , i.e., either  $k=0$  and  $j=1$  or  $k=1$  and  $j=0$ . It follows that the only  $\mathbb{Z}_+$ -grading of  $\mathcal{P}^\varphi$  such that  $\mathcal{P}_{-1}^\varphi$  is completely odd is the grading of type  $(0, \dots, 0|1, \dots, 1)$ . We can thus simply denote the graded components of  $\mathcal{P}^\varphi$  with respect to this grading by  $\mathcal{P}_j = \Pi\mathcal{Q}_{j+1}$ .

Now let  $a \in \mathcal{Q}_i = \Pi\mathcal{P}_{i-1}$  and  $b \in \mathcal{Q}_k = \Pi\mathcal{P}_{k-1}$ . We have:  $[a, b]^\varphi = [a, \varphi]b + [\varphi a, b] + (-1)^{p(a)+1}(D(\varphi)ab + D(\varphi a)b)$ . Suppose that  $\varphi = \sum_{j \geq 0} \varphi_j$  with  $\varphi_j \in \mathcal{Q}_j$ . Then one can show, using the fact that  $[a, b]^\varphi \in \Pi\mathcal{P}_{i+k-2} = \mathcal{Q}_{i+k-1}$ , that  $[a, b]^\varphi = [a, b]^{\varphi^0}$ . It follows that when dealing with the  $\mathbb{Z}_+$ -graded odd generalized Poisson superalgebras  $\mathcal{P}^\varphi$  we can always assume  $\varphi \in \mathcal{Q}_0$ .

### 3 The universal odd generalized Poisson superalgebra

**Definition 3.1** Let  $A$  be a unital commutative associative superalgebra with parity  $p$ . A linear map  $X : A \rightarrow A$  is called a generalized derivation of  $A$  if it satisfies the generalized Leibniz rule:

$$(3.1) \quad X(bc) = X(b)c + (-1)^{p(b)p(c)} X(c)b - X(1)bc.$$

We denote by  $GDer(A)$  the set of generalized derivations of  $A$ . If  $X(1) = 0$ , relation (3.1) becomes the usual Leibniz rule and  $X$  is called a derivation. We denote by  $Der(A)$  the set of derivations of  $A$ .

**Proposition 3.2** The set  $GDer(A)$  is a subalgebra of the Lie superalgebra  $End(A)$ .

**Proof.** This follows by direct computations. □

Our construction of the universal odd generalized Poisson superalgebra is inspired by the one of the universal odd Poisson superalgebra explained in [5]. The universal odd Poisson superalgebra associated to  $A$  is the full prolongation of the subalgebra  $Der(A)$  of the Lie superalgebra  $End(A)$  (the definitions will be given below). In this section we generalize this construction when  $Der(A)$  is replaced by the subalgebra  $GDer(A)$ .

Consider the universal Lie superalgebra  $W(\Pi A)$  associated to the vector superspace  $\Pi A$ : this is the  $\mathbb{Z}_+$ -graded Lie superalgebra:

$$W(\Pi A) = \bigoplus_{k=-1}^{\infty} W_k(\Pi A)$$

where  $W_{-1} = \Pi A$  and for all  $k \geq 0$ ,  $W_k(V) = \text{Hom}(S^{k+1}(\Pi A), \Pi A)$  is the vector superspace of  $(k+1)$ -linear supersymmetric functions on  $\Pi A$  with values in  $\Pi A$ . The Lie superalgebra structure on  $W(\Pi A)$  is defined as follows: for  $X \in W_p(\Pi A)$  and  $Y \in W_q(\Pi A)$  with  $p, q \geq -1$ , we define  $X \square Y \in W_{p+q}(\Pi A)$  by:

$$(3.2) \quad X \square Y(a_0, \dots, a_{p+q}) = \sum_{\substack{i_0 < \dots < i_q \\ i_{q+1} < \dots < i_{q+p}}} \epsilon_a(i_0, \dots, i_{p+q}) X(Y(a_{i_0}, \dots, a_{i_q}), a_{i_{q+1}}, \dots, a_{i_{q+p}}).$$

Here  $\epsilon_a(i_0, \dots, i_{p+q}) = (-1)^N$  where  $N$  is the number of interchanges of indices of odd  $a_i$ 's in the permutation  $\sigma(s) = i_s$ ,  $s = 0, \dots, p+q$ . Then the bracket on  $W(\Pi A)$  is given by:

$$[X, Y] = X \square Y - (-1)^{\bar{p}(X)\bar{p}(Y)} Y \square X.$$

As  $GDer(A)$  is a subalgebra of the Lie superalgebra  $W_0(\Pi A) = \text{End}(\Pi A)$ , we can consider its full prolongation  $\mathcal{G}W^{as}(\Pi A)$ : this is the  $\mathbb{Z}_+$ -graded subalgebra  $\mathcal{G}W^{as}(\Pi A) = \bigoplus_{k=-1}^{\infty} \mathcal{G}W_k^{as}(\Pi A)$  of the Lie superalgebra  $W(\Pi A)$  defined by setting  $\mathcal{G}W_{-1}^{as}(\Pi A) = \Pi A$ ,  $\mathcal{G}W_0^{as}(\Pi A) = GDer(\Pi A)$ , and inductively for  $k \geq 1$ ,

$$\mathcal{G}W_k^{as}(\Pi A) = \{X \in W_k(\Pi A) | [X, W_{-1}(\Pi A)] \subset \mathcal{G}W_{k-1}^{as}(\Pi A)\}.$$



**Proposition 3.3** For  $k \geq 0$ , the superspace  $\mathcal{GW}_k^{as}(\Pi A)$  consists of linear maps  $X : S^{k+1}(\Pi A) \rightarrow \Pi A$  satisfying the following generalized Leibniz rule:

$$(3.3) \quad X(a_0, \dots, a_{k-1}, bc) = X(a_0, \dots, a_{k-1}, b)c + (-1)^{p(b)p(c)}X(a_0, \dots, a_{k-1}, c)b - X(a_0, \dots, a_{k-1}, 1)bc$$

for  $a_0, \dots, a_{k-1}, b, c \in \Pi A$ .

**Proof.** According to formula (3.2), for all  $X \in W_p(\Pi A)$  and  $Y \in W_{-1}(\Pi A) = \Pi A$ , we have:

$$(3.4) \quad [X, Y](a_1, \dots, a_p) = X(Y, a_1, \dots, a_p)$$

with  $a_1, \dots, a_p \in \Pi A$ . Now we proceed by induction on  $k \geq 0$ : for  $k = 0$ ,  $\mathcal{GW}_0^{as}(\Pi A) = GDer(A)$  and equality (3.3) holds by definition of generalized derivation. Assume property (3.3) for elements in  $\mathcal{GW}_{k-1}^{as}(\Pi A)$ , and let  $X$  in  $\mathcal{GW}_k^{as}(\Pi A)$ . For any  $a_0, a_1, \dots, a_{k-1}, b, c \in \Pi A$ , we have by (3.4):

$$X(a_0, a_1, \dots, a_{k-1}, bc) = [X, a_0](a_1, \dots, a_{k-1}, bc).$$

By definition of  $\mathcal{GW}^{as}(\Pi A)$ , we have  $[X, a_0] \in \mathcal{GW}_{k-1}^{as}(\Pi A)$ . Using the inductive hypothesis on  $[X, a_0]$ , we get:

$$\begin{aligned} [X, a_0](a_1, \dots, a_{k-1}, bc) &= [X, a_0](a_1, \dots, a_{k-1}, b)c + (-1)^{p(b)p(c)}[X, a_0](a_1, \dots, a_{k-1}, c)b \\ &\quad - [X, a_0](a_1, \dots, a_{k-1}, 1)bc \end{aligned}$$

which is exactly formula (3.3) for  $X$ . □

For  $X \in \Pi W_{h-1}(\Pi A)$  and  $Y \in \Pi W_{k-1}(\Pi A)$  with  $h, k \geq 0$ , we define their concatenation product  $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$  by

$$(3.5) \quad \begin{aligned} X \wedge Y(a_1, \dots, a_{h+k}) &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{h+k}}} \epsilon_a(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ &\quad \times X(a_{i_1}, \dots, a_{i_h}) Y(a_{i_{h+1}}, \dots, a_{i_{h+k}}) \end{aligned}$$

where  $\epsilon_a$  is defined as in (3.2) with  $a_1, \dots, a_{h+k} \in \Pi A$ .

**Proposition 3.4**  $(\Pi \mathcal{GW}^{as}(\Pi A), [\cdot, \cdot], \wedge)$  is a  $\mathbb{Z}_+$ -graded odd generalized Poisson superalgebra.

We will denote  $(\Pi \mathcal{GW}^{as}(\Pi A), [\cdot, \cdot], \wedge)$  by  $\mathcal{G}(A)$  and call it the universal odd generalized Poisson superalgebra associated to  $A$ . The rest of this section is devoted to the proof of Proposition 3.4.

**Lemma 3.5**  $(\Pi \mathcal{GW}^{as}(\Pi A), \wedge)$  is a unital  $\mathbb{Z}_+$ -graded associative commutative superalgebra with parity  $p$ .

**Proof.** It is already proved in [5] that  $(\Pi W(\Pi A), \wedge)$  is a unital  $\mathbb{Z}_+$ -graded associative commutative superalgebra with parity  $p$ , therefore we only need to prove that for  $X \in \Pi \mathcal{GW}_{h-1}^{as}(\Pi A)$  and

$Y \in \Pi\mathcal{GW}_{k-1}^{as}(\Pi A)$  with  $h, k \geq 0$ ,  $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$  satisfies the generalized Leibniz rule (3.3). We have:

$$\begin{aligned}
& X \wedge Y(a_1, \dots, a_{h+k-1}, bc) = \\
&= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{h+k} = h+k}} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\
& \quad \times X(a_{i_1}, \dots, a_{i_h}) Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, bc) \\
&+ \sum_{\substack{i_1 < \dots < i_h = h+k \\ i_{h+1} < \dots < i_{h+k}}} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{h-1}}) + \bar{p}(bc))} \\
& \quad \times X(a_{i_1}, \dots, a_{i_{h-1}}, bc) Y(a_{i_h}, \dots, a_{i_{h+k}})
\end{aligned} \tag{3.6}$$

For the first summand in the right hand side, since  $i_{h+k} = h+k$ , we have:

$$\begin{aligned}
\epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) &= \epsilon_{a_1, \dots, a_{h+k-1}, b}(i_1, \dots, i_{h+k}) \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, c}(i_1, \dots, i_{h+k}) \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, 1}(i_1, \dots, i_{h+k})
\end{aligned}$$

and

$$\begin{aligned}
Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, bc) &= Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, b)c + (-1)^{p(b)p(c)} Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, c)b \\
&\quad - Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, 1)bc.
\end{aligned}$$

In the second summand, since  $i_{h+k} = h$ , we have:

$$\begin{aligned}
\epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) &= \epsilon_{a_1, \dots, a_{h+k-1}, b}(i_1, \dots, i_{h+k}) (-1)^{p(c)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, c}(i_1, \dots, i_{h+k}) (-1)^{p(b)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, 1}(i_1, \dots, i_{h+k}) (-1)^{p(bc)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))}
\end{aligned}$$

and

$$\begin{aligned}
& X(a_{i_1}, \dots, a_{i_{h-1}}, bc) Y(a_{i_h}, \dots, a_{i_{h+k}}) = \\
& (-1)^{p(c)(p(Y) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} X(a_{i_1}, \dots, a_{i_{h-1}}, b) Y(a_{i_h}, \dots, a_{i_{h+k}}) c \\
& + (-1)^{p(b)(p(c) + p(Y) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} X(a_{i_1}, \dots, a_{i_{h-1}}, c) Y(a_{i_h}, \dots, a_{i_{h+k}}) b \\
& - (-1)^{p(bc)(p(Y) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} X(a_{i_1}, \dots, a_{i_{h-1}}, 1) Y(a_{i_h}, \dots, a_{i_{h+k}}) bc
\end{aligned}$$

The generalized Leibniz rule for  $X \wedge Y$  then follows by replacing these equalities in (3.6).  $\square$

It remains to prove that the Lie bracket on  $\Pi\mathcal{GW}^{as}(\Pi A)$  satisfies the generalized odd Leibniz rule (2.1). This follows from the following lemma.

**Lemma 3.6** *The following equalities hold for  $X, Y, Z \in \Pi\mathcal{GW}^{as}(\Pi A)$ :*

$$\begin{aligned} X \square (Y \wedge Z) &= (X \square Y) \wedge Z + (-1)^{\bar{p}(X)p(Y)} Y \wedge (X \square Z) - (X \square 1) \wedge Y \wedge Z, \\ (X \wedge Y) \square Z &= X \wedge (Y \square Z) + (-1)^{p(Y)\bar{p}(Z)} (X \square Z) \wedge Y. \end{aligned}$$

**Proof.** An analogue result is proved in [5, Lemma 3.5]. For  $X \in \Pi\mathcal{GW}_{l-k}^{as}(\Pi A)$ ,  $Y \in \Pi\mathcal{GW}_{h-1}^{as}(\Pi A)$  and  $Z \in \Pi\mathcal{GW}_{k-h-1}^{as}(\Pi A)$  with  $h, k-h, l-k+1 \geq 0$ , we have:

$$(3.7) \quad \begin{aligned} X \square (Y \wedge Z)(a_1, \dots, a_l) &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ &\quad \times X(Y(a_{i_1}, \dots, a_{i_h})Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) \end{aligned}$$

The generalized Leibniz rule for  $X$  can be rewritten in the following way:

$$\begin{aligned} X(bc, a_{k+1}, \dots, a_l) &= (-1)^{p(c)(\bar{p}(a_{k+1}) + \dots + \bar{p}(a_l))} X(b, a_{k+1}, \dots, a_l)c \\ &\quad + (-1)^{p(b)\bar{p}(X)} bX(c, a_{k+1}, \dots, a_l) \\ &\quad - (-1)^{p(bc)(\bar{p}(a_{k+1}) + \dots + \bar{p}(a_l))} X(1, a_{k+1}, \dots, a_l)bc. \end{aligned}$$

Using this equality in (3.7),  $X \square (Y \wedge Z)(a_1, \dots, a_l)$  is then of the form:

$$X \square (Y \wedge Z)(a_1, \dots, a_l) = A + B - C.$$

The first term  $A$  is equal to

$$\begin{aligned} &\sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Z) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\ &\quad \times X(Y(a_{i_1}, \dots, a_{i_h}), a_{i_{k+1}}, \dots, a_{i_l})Z(a_{i_{h+1}}, \dots, a_{i_k}) = \\ &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{l-k+h} \\ i_{l-k+h+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{l-k+h}}))(\bar{p}(a_{i_{l-k+h+1}}) + \dots + \bar{p}(a_{i_l}))} \\ &\quad \times (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Z) + \bar{p}(a_{i_{l-k+h+1}}) + \dots + \bar{p}(a_{i_l}))(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{l-k+h}}))} \\ &\quad \times X(Y(a_{i_1}, \dots, a_{i_h}), a_{i_{h+1}}, \dots, a_{i_{l-k+h}})Z(a_{i_{l-k+h+1}}, \dots, a_{i_l}) = \\ &= (X \square Y) \wedge Z(a_1, \dots, a_l). \end{aligned}$$

The second term  $B$  is equal to

$$\begin{aligned} &\sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Y) + \bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))\bar{p}(X)} \\ &\quad \times Y(a_{i_1}, \dots, a_{i_h})X(Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) \\ &= (-1)^{p(Y)\bar{p}(X)} \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(p(Z) + \bar{p}(X))(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ &\quad \times Y(a_{i_1}, \dots, a_{i_h})X(Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) = (-1)^{\bar{p}(X)p(Y)} Y \wedge (X \square Z)(a_1, \dots, a_l) \end{aligned}$$

since  $p(X \square Z) = \bar{p}(X) + p(Z)$ .

Finally, the third term  $C$  is equal to

$$\begin{aligned}
& \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Y) + p(Z) + \bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\
& \quad \times X(1, a_{i_{k+1}}, \dots, a_{i_l}) Y(a_{i_1}, \dots, a_{i_h}) Z(a_{i_{h+1}}, \dots, a_{i_k}) \\
& = \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\
& \quad \times (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}) + \bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} (-1)^{p(Y)(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\
& \quad \times X(1, a_{i_{k+1}}, \dots, a_{i_l}) Y(a_{i_1}, \dots, a_{i_h}) Z(a_{i_{h+1}}, \dots, a_{i_k}) = \\
& = \sum_{\substack{i_1 < \dots < i_{l-k} \\ i_{l-k+1} < \dots < i_{l-k+h} \\ i_{l-k+h+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{l-k+h}}))} (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{l-k}}))} \\
& \quad \times (X \square 1)(a_{i_1}, \dots, a_{i_{l-k}}) Y(a_{i_{l-k+1}}, \dots, a_{i_{l-k+h}}) Z(a_{i_{l-k+h+1}}, \dots, a_{i_l}) = (X \square 1) \wedge Y \wedge Z(a_1, \dots, a_l).
\end{aligned}$$

This proves the first equality. The second equality can be proved in the same way, using the definition of the box product (3.2) and the concatenation product (3.5).  $\square$

## 4 The main construction

Let  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$  be a generalized  $n$ -Nambu-Poisson algebra and denote by  $\Pi\mathcal{N}$  the space  $\mathcal{N}$  with reversed parity. Define

$$\begin{aligned}
(4.1) \quad & \mu : \Pi\mathcal{N} \otimes \dots \otimes \Pi\mathcal{N} \rightarrow \Pi\mathcal{N} \\
& \mu(f_1, \dots, f_n) = \{f_1, \dots, f_n\}.
\end{aligned}$$

Then  $\mu$  is a supersymmetric function on  $(\Pi\mathcal{N})^{\otimes n}$  [3, Lemma 1.2]. Furthermore  $\mu$  satisfies the generalized Leibniz rule

$$\mu(f_1, \dots, f_{n-1}, gh) = \mu(f_1, \dots, f_{n-1}, g)h + g\mu(f_1, \dots, f_{n-1}, h) - \mu(f_1, \dots, f_{n-1}, 1)gh,$$

hence  $\mu$  lies in  $\mathcal{GW}_{n-1}^{as}(\Pi\mathcal{N})$ .

Let  $OP(\mathcal{N})$  be the odd Poisson subalgebra of  $\mathcal{G}(\mathcal{N})$  generated by  $\Pi\mathcal{N}$  and  $\mu$ . Then, by construction,  $OP(\mathcal{N})$  is a transitive Lie subalgebra of  $\mathcal{GW}^{as}(\Pi\mathcal{N})$ , hence it is a transitive subalgebra of  $W(\Pi\mathcal{N})$ . Furthermore  $OP(\mathcal{N})$  is a  $\mathbb{Z}_+$ -graded odd Poisson subalgebra of  $\mathcal{G}(\mathcal{N})$ . Let us denote by  $OP(\mathcal{N}) = \oplus_{j \geq -1} \mathcal{P}_j(\mathcal{N})$  its depth 1  $\mathbb{Z}$ -grading as a Lie superalgebra.

**Proposition 4.1** *If  $\mathcal{N}$  is a simple generalized  $n$ -Nambu-Poisson algebra then  $OP(\mathcal{N})$  is a simple generalized odd Poisson superalgebra.*

**Proof.** Let  $I$  be a non-zero ideal of  $OP(\mathcal{N})$ . Then, by transitivity,  $I \cap \mathcal{P}_{-1}(\mathcal{N}) = I \cap \mathcal{N} \neq 0$ . Note that  $I \cap \mathcal{N}$  is a Nambu-Poisson ideal of  $\mathcal{N}$ . Indeed,  $(I \cap \mathcal{N}) \cdot \mathcal{N} = (I \cap \mathcal{N}) \wedge \mathcal{N} \subset I \cap \mathcal{N}$  and  $[I \cap \mathcal{N}, \mathcal{N}] \subset [\mathcal{N}, \mathcal{N}] = 0$ . Since  $\mathcal{N}$  is simple,  $I \cap \mathcal{N} = \mathcal{N}$ , hence  $1 \in I$ , hence  $I = OP(\mathcal{N})$ .  $\square$

**Remark 4.2** We recall that since  $(\mathcal{N}, \{\cdot, \dots, \cdot\})$  is an  $n$ -Lie algebra, the Filippov-Jacobi identity holds, i.e., for every  $a_1, \dots, a_{n-1} \in \mathcal{N}$ , the map  $D_{a_1, \dots, a_{n-1}} : \mathcal{N} \rightarrow \mathcal{N}$ ,  $D_{a_1, \dots, a_{n-1}}(a) = \{a_1, \dots, a_{n-1}, a\}$  is a derivation of  $(\mathcal{N}, \{\cdot, \dots, \cdot\})$ . By [4, Lemma 2.1(b)], this is equivalent to the condition  $[\mu, D_{a_1, \dots, a_{n-1}}] = 0$  in  $OP(\mathcal{N})$ . By (4.1), we have:  $D_{a_1, \dots, a_{n-1}} = [[\mu, a_1], \dots, a_{n-1}]$ , therefore  $\mu$  satisfies the following condition:

$$[\mu, [[\mu, a_1], \dots, a_{n-1}]] = 0 \quad \text{for every } a_1, \dots, a_{n-1} \in \mathcal{N}.$$

**Definition 4.3** We say that a pair  $(\mathcal{P}, \mu)$ , consisting of a  $\mathbb{Z}_+$ -graded generalized odd Poisson superalgebra  $\mathcal{P}$  and an element  $\mu \in \mathcal{P}_{n-1}$  of parity  $p(\mu) \equiv n \pmod{2}$ , is a good  $n$ -pair if it satisfies the following properties:

- G1)  $\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$  is a transitive  $\mathbb{Z}$ -graded Lie superalgebra of depth 1 such that  $\mathcal{P}_{-1}$  is completely odd;
- G2)  $\mu$  and  $\mathcal{P}_{-1}$  generate  $\mathcal{P}$  as a (generalized) odd Poisson superalgebra;
- G3)  $[\mu, [[\mu, a_1], \dots, a_{n-1}]] = 0$  for every  $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1}$ .

**Example 4.4** Let  $\mathcal{P} = PO(2h, 2h)$ ,  $h \geq 1$ , with the grading of type  $(0, \dots, 0|1, \dots, 1)$ , and let  $\mu = \sum_{i=1}^h \xi_i \xi_{i+h}$ . Then  $(\mathcal{P}, \mu)$  is a good 2-pair. Indeed, for  $1 \leq i \leq h$ ,  $[x_i, \mu]_{HO} = \xi_{h+i}$  and  $[x_{h+i}, \mu]_{HO} = -\xi_i$ , therefore  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Furthermore, for  $f \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$ , we have:  $[f, \mu]_{HO} = \sum_{i=1}^h (\frac{\partial f}{\partial x_i} \xi_{i+h} - \frac{\partial f}{\partial x_{i+h}} \xi_i)$ , hence

$$\begin{aligned} [\mu, [f, \mu]_{HO}]_{HO} &= \sum_{i,j=1}^h [\xi_j \xi_{j+h}, \frac{\partial f}{\partial x_i} \xi_{i+h} - \frac{\partial f}{\partial x_{i+h}} \xi_i]_{HO} = \\ &= \sum_{i,j=1}^h (\frac{\partial^2 f}{\partial x_i \partial x_j} \xi_{j+h} \xi_{i+h} - \frac{\partial^2 f}{\partial x_{j+h} \partial x_i} \xi_j \xi_{i+h} - \frac{\partial^2 f}{\partial x_j \partial x_{i+h}} \xi_{j+h} \xi_i + \frac{\partial^2 f}{\partial x_{i+h} \partial x_{j+h}} \xi_j \xi_i) = 0. \end{aligned}$$

Therefore  $(\mathcal{P}, \mu)$  satisfies property G3).

**Example 4.5** Let  $\mathcal{P} = PO(n, n)$  with the grading of type  $(0, \dots, 0|1, \dots, 1)$ , and let  $\mu = \xi_1 \dots \xi_n$ . Then  $(\mathcal{P}, \mu)$  is a good  $n$ -pair. Indeed,  $[x_{n-1}, [\dots, [x_2, [x_1, \mu]]]]_{HO} = \xi_n$ , and, similarly all the  $\xi_i$ 's can be obtained by commuting  $\mu$  with different  $x_j$ 's. Therefore  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Furthermore, let  $f = \sum_{i=1}^n f_i \xi_i \in \mathcal{P}_0$ , with  $f_i \in \mathbb{F}[[x_1, \dots, x_n]]$ , such that

$$(4.2) \quad \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = 0.$$

Then  $[f, \mu]_{HO} = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \xi_1 \dots \xi_n = 0$ . Notice that all elements of the form  $[[\mu, a_1], \dots, a_{n-1}]$  with  $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$  satisfy property (4.2), hence  $(\mathcal{P}, \mu)$  satisfies property G3).

**Example 4.6** Let  $\mathcal{P} = PO(2h+1, 2h+2)$ ,  $h \geq 1$ , with the grading of type  $(0, \dots, 0|1, \dots, 1, 1)$ , and let  $\mu = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$  (recall that  $\xi_{2h+2} = \tau$ ). Then  $(\mathcal{P}, \mu)$  is a good 2-pair. Indeed, we have:  $[1, \mu]_{KO} = 2\xi_{h+1}$  and  $[x_i, \mu]_{KO} = \xi_{i+h+1} - x_i \xi_{h+1}$  for  $1 \leq i \leq h+1$ ,  $[x_{i+h+1}, \mu]_{KO} = -\xi_i - x_{i+h+1} \xi_{h+1}$  for  $1 \leq i \leq h$ . Hence  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Furthermore, if  $f \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$ , we have:

$$[f, \mu]_{KO} = \sum_{i=1}^h \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E-2)(f) \xi_{h+1},$$

hence

$$\begin{aligned} [\mu, [f, \mu]_{KO}]_{KO} &= \left[ \sum_{j=1}^{h+1} \xi_j \xi_{h+1+j}, \sum_{i=1}^h \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E-2)(f) \xi_{h+1} \right]_{KO} \\ &= \sum_{i=1, \dots, h; j=1, \dots, h+1} \xi_{h+1+j} \left( \frac{\partial^2 f}{\partial x_j \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_j \partial x_{i+h+1}} \xi_i \right) + \sum_{j=1}^{h+1} \xi_{h+1+j} \left( \frac{\partial^2 f}{\partial x_j \partial x_{h+1}} \xi_{2h+2} \right. \\ &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_j} \xi_{h+1} \right) - \sum_{i,j=1}^h \xi_j \left( \frac{\partial^2 f}{\partial x_{h+1+j} \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_{h+1+j} \partial x_{i+h+1}} \xi_i \right) - \sum_{j=1}^h \xi_j \left( \frac{\partial^2 f}{\partial x_{h+1+j} \partial x_{h+1}} \xi_{2h+2} \right. \\ &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_{h+1} \right) - \xi_{h+1} (E-2)([f, \mu]_{KO}) = \xi_{2h+2} \sum_{i=1}^h \left( \frac{\partial^2 f}{\partial x_{h+1} \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_{h+1} \partial x_{i+h+1}} \xi_i \right) \\ &\quad + \sum_{j=1}^{h+1} \frac{\partial^2 f}{\partial x_j \partial x_{h+1}} \xi_{h+1+j} \xi_{2h+2} - \sum_{j=1}^{h+1} \xi_{h+1+j} \frac{\partial((E-2)(f))}{\partial x_j} \xi_{h+1} - \sum_{j=1}^h \left( \frac{\partial^2 f}{\partial x_{h+j+1} \partial x_{h+1}} \xi_j \xi_{2h+2} \right. \\ &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_j \xi_{h+1} \right) - \xi_{h+1} (E-2)([f, \mu]_{KO}) = \xi_{h+1} \left( \sum_{j=1}^{h+1} \xi_{h+1+j} \frac{\partial((E-2)(f))}{\partial x_j} \right. \\ &\quad \left. - \sum_{j=1}^h \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_j - (E-2) \left( \sum_{i=1}^h \left( \frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} \right) \right) = 0. \end{aligned}$$

**Example 4.7** Let  $\mathcal{P} = PO(n, n+1)$ , with the grading of type  $(0, \dots, 0|1, \dots, 1, 1)$ , and let  $\mu = \xi_1 \dots \xi_n \tau$  (recall that  $\tau = \xi_{n+1}$ ). Then  $(\mathcal{P}, \mu)$  is a good  $(n+1)$ -pair. Indeed, we have:  $[1, \mu]_{KO} = 2(-1)^{n+1} \xi_1 \dots \xi_n$ ,  $[x_{i_1}, [\dots, [x_{i_{n-1}}, \xi_1 \dots \xi_n]_{KO}]_{KO}]_{KO} = \pm \xi_{i_n}$  for  $i_1 \neq \dots \neq i_{n-1} \neq i_n$ ,  $[x_i, \xi_i \dots \xi_n \tau]_{KO} = \xi_{i+1} \dots \xi_n \tau + (-1)^{n-i} x_i \xi_i \dots \xi_n$  for  $1 \leq i \leq n$ . Hence  $\mathcal{P}_{-1}$  and  $\mu$  generate  $\mathcal{P}$ . Now let  $\text{div}_1 = \Delta + (E-n) \frac{\partial}{\partial \tau}$  where  $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial \xi_i}$  is the odd Laplacian, and let  $f = \sum_{i=1}^{n+1} f_i \xi_i \in \mathcal{P}_0$ ,  $f_i \in \mathbb{F}[[x_1, \dots, x_n]]$ , such that  $0 = \text{div}_1(f) = \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} + (E-n)(f_{n+1})$ . Then we have:

$$\begin{aligned} \left[ \sum_{i=1}^{n+1} f_i \xi_i, \mu \right]_{KO} &= \left[ \sum_{i=1}^{n+1} f_i \xi_i, \mu \right]_{HO} + \sum_{i=1}^{n+1} (E-2)(f_i \xi_i) (-1)^n \xi_1 \dots \xi_n - f_{n+1} (n-2) \xi_1 \dots \xi_n \tau \\ &= \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} \xi_1 \dots \xi_n \tau + (-1)^n (E-2)(f_{n+1} \xi_{n+1}) \xi_1 \dots \xi_n - (n-2) f_{n+1} \xi_1 \dots \xi_n \tau \end{aligned}$$

$$= \left( \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} + (E-2)(f_{n+1}) - (n-2)f_{n+1} \xi_1 \dots \xi_n \tau \right) = 0.$$

Notice that, since  $\text{div}_1(\mu) = 0$  and  $\text{div}_1(f) = 0$  for every  $f \in \mathcal{P}_{-1}$ , then  $\text{div}_1([\mu, a_1], \dots, a_{n-1}]) = 0$  for every  $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1}$ . Hence property *G3*) is satisfied.

**Remark 4.8** Let us consider  $\mathcal{P} = PO(k, k)$  (resp.  $\mathcal{P} = PO(k, k+1)$ ) with the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ). Let  $\varphi \in \mathcal{P}_{-1}$  be an invertible element. By Remark 2.9, the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ) defines a  $\mathbb{Z}_+$ -graded structure on the odd generalized Poisson superalgebra  $\mathcal{P}^\varphi$ , such that  $\mathcal{P}_j = \mathcal{P}_j^\varphi$ . Then, by (2.4),  $(\mathcal{P}, \mu)$  is a good  $n$ -pair with respect to this grading if and only if  $(\mathcal{P}^\varphi, \varphi^{-1}\mu)$  is.

The map  $\mathcal{N} \mapsto (OP(\mathcal{N}), \mu)$  establishes a correspondence between (simple) generalized  $n$ -Nambu-Poisson algebras  $\mathcal{N}$  and good  $n$ -pairs  $(OP(\mathcal{N}), \mu)$ . We now want to show that this correspondence is bijective.

**Lemma 4.9** *Let  $\mathcal{N}$  be a generalized  $n$ -Nambu-Poisson algebra. Then the 0-th graded component  $\mathcal{P}_0(\mathcal{N})$  of  $OP(\mathcal{N})$  is generated, as a Lie superalgebra, by elements of the form*

$$[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b$$

with  $a_i, b \in \Pi\mathcal{N}$ .

**Proof.** Let  $L_{-1} := \Pi\mathcal{N}$  and let  $L_0$  be the Lie subsuperalgebra of  $\mathcal{GW}_0^{as}(\Pi\mathcal{N}) = GDer(\Pi\mathcal{N})$  generated by the elements of the form  $[a_1, [a_2, \dots, [a_{n-1}, \mu b]]]$  with  $a_1, \dots, a_{n-1}, b \in \Pi\mathcal{N}$ . Note that, since  $\mathcal{GW}_0^{as}(\Pi\mathcal{N})$  is  $\mathbb{Z}$ -graded of depth 1, and  $1 \in \mathcal{N}$ , the restriction to  $\mathcal{N}$  of the derivation  $D$  of  $\mathcal{G}(\mathcal{N})$  is zero, hence

$$[a_1, [a_2, \dots, [a_{n-1}, \mu b]]] = [a_1, [a_2, \dots, [a_{n-1}, \mu]]]b.$$

An induction argument on the length of the commutators of the generating elements of  $L_0$  shows that  $L_0$  is stable with respect to the concatenation product by elements of  $\Pi\mathcal{N}$ .

Let  $L$  be the full prolongation of  $L_{-1} \oplus L_0$ , i.e.,  $L = L_{-1} \oplus L_0 \oplus (\oplus_{j \geq 1} L_j)$ , where  $L_j = \{\varphi \in \mathcal{GW}^{as}(\Pi\mathcal{N}) \mid [\varphi, L_{-1}] \subset L_{j-1}\}$ . Note that  $L_j$ , for  $j \geq 1$ , is stable with respect to the concatenation product by elements of  $\Pi\mathcal{N}$ . Indeed, if  $\varphi \in L_j$ , then

$$[\varphi \Pi\mathcal{N}, L_{-1}] = [\varphi, L_{-1}] \Pi\mathcal{N} \subset L_{j-1} \Pi\mathcal{N},$$

hence one can conclude by induction on  $j$  since  $L_0 \Pi\mathcal{N} \subset L_0$ . It follows that  $L$  is closed under the concatenation product, hence it is an odd generalized Poisson subsuperalgebra of  $\mathcal{GW}^{as}(\Pi\mathcal{N})$ . Indeed, using induction on  $i+j \geq 0$ , one shows that  $L_i L_j \subset L$  for every  $i, j \geq 0$ .

It follows that  $OP(\mathcal{N})$  is an odd generalized subsuperalgebra of  $L$ , since  $L$  is an odd generalized Poisson superalgebra containing  $\Pi\mathcal{N}$  and  $\mu$ . As a consequence, the 0-th graded component  $\mathcal{P}_0(\mathcal{N})$  of  $OP(\mathcal{N})$  is generated, as a Lie superalgebra, by elements of the form

$$[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b$$

with  $a_i, b \in \Pi\mathcal{N}$ . □

**Proposition 4.10** *Let  $(\mathcal{P}, \mu)$  be a good  $n$ -pair, and define on  $\mathcal{N} := \Pi\mathcal{P}_{-1}$  the following product:*

$$\{x_1, \dots, x_n\} = [\dots [[\mu, x_1], \dots, x_n]].$$

*Then:*

- (a)  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \wedge)$  is a generalized Nambu-Poisson algebra,  $\wedge$  being the restriction to  $\mathcal{N}$  of the commutative associative product  $\wedge$  defined on  $\mathcal{P}$ .
- (b) If  $\mathcal{P}$  is a simple odd generalized Poisson superalgebra, then  $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \wedge)$  is a simple generalized Nambu-Poisson algebra.

**Proof.** (a) By Definitions 2.1 and 2.6,  $\mathcal{N} = \mathcal{Q}_0$  is a commutative associative subalgebra of  $\mathcal{P}$ . Furthermore  $\{\cdot, \dots, \cdot\}$  is an  $n$ -Lie bracket due to [4, Prop. 2.4] and property G3). Finally, for  $f_1, \dots, f_{n-1}, g, h \in \Pi\mathcal{P}_{-1}$ , we have:

$$\begin{aligned} \{f_1, \dots, f_{n-1}, gh\} &= [[\dots [\mu, f_1], \dots, f_{n-1}], gh] = [[\dots [\mu, f_1], \dots, f_{n-1}], g]h + g[[\dots [\mu, f_1], \dots, f_{n-1}], h] \\ &+ (-1)^{p([\dots [\mu, f_1], \dots, f_{n-1}]) + 1} [1, [\dots [\mu, f_1], \dots, f_{n-1}]]gh = \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\} \\ &- (-1)^{p([\dots [\mu, f_1], \dots, f_{n-1}]) + 1} (-1)^{\bar{p}([\dots [\mu, f_1], \dots, f_{n-1}])} \{f_1, \dots, f_{n-1}, 1\}gh \\ &= \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\} - \{f_1, \dots, f_{n-1}, 1\}gh. \end{aligned}$$

(b) Now we want to show that if  $\mathcal{P}$  is simple, then  $\mathcal{N}$  is simple. Suppose that  $I$  is a non zero ideal of  $\mathcal{N}$ , and let  $\tilde{I}$  be the ideal of  $\mathcal{P}$  generated by  $\Pi I$  and  $\mu$ :  $\tilde{I} = \oplus_{j \geq -1} \tilde{I}_j$ , with  $\tilde{I}_j \subset \mathcal{P}_j$ . We want to show that  $\tilde{I}_{-1} = \tilde{I} \cap \mathcal{P}_{-1} = \Pi I$ . In fact, the concatenation product by elements in  $\oplus_{j \geq 1} \mathcal{Q}_j$  maps  $\mathcal{Q}_0$  to  $\oplus_{j \geq 1} \mathcal{Q}_j$  hence it does not produce any element in  $\mathcal{P}_{-1} = \mathcal{Q}_0$ . On the other hand,  $I \wedge \mathcal{Q}_0 = I \wedge \mathcal{N} \subset I$  since  $I$  is an ideal of  $\mathcal{N}$ . The bracket between elements in  $\oplus_{j \geq 0} \mathcal{P}_j$  lies in  $\oplus_{j \geq 0} \mathcal{P}_j$  and the bracket between  $I$  and elements in  $\oplus_{j \geq 1} \mathcal{P}_j$  lies in  $\oplus_{j \geq 0} \mathcal{P}_j$ . Therefore we just need to consider the brackets between elements in  $I$  and elements in  $\mathcal{P}_0$ . By hypothesis,  $\mathcal{P}$  is generated by  $\mathcal{P}_{-1}$  and  $\mu$ , hence, by the same argument as in Lemma 4.9,  $\mathcal{P}_0$  is generated by elements of the form  $[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b$  with  $a_i, b \in \Pi\mathcal{P}_{-1}$ . We have:

$$[I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]b] = [I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]]b$$

since  $[I, b] = 0$  and  $D|_I = 0$ . Since  $[I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]] = \{I, a_1, \dots, a_{n-1}\}$  and  $I$  is an ideal of  $\mathcal{N}$ ,  $[I, \mathcal{P}_0] \subset I$ .  $\square$

**Definition 4.11** *Two good  $n$ -pairs  $(\mathcal{P}, \mu)$  and  $(\mathcal{P}', \mu')$  are called isomorphic if there exists an odd Poisson superalgebras isomorphism  $\Phi : \mathcal{P} \rightarrow \mathcal{P}'$  such that  $\Phi(\mathcal{P}_j) = \mathcal{P}'_j$ ,  $\Phi(\mathcal{Q}_j) = \mathcal{Q}'_j$  for all  $j$  and  $\phi(\mu) \in \mathbb{F}^\times \mu'$ .*

**Theorem 4.12** *The map*

$$\mathcal{N} \rightarrow (OP(\mathcal{N}), \mu)$$

*with  $\mu$  defined as in (4.1), establishes a bijection between isomorphism classes of generalized  $n$ -Nambu-Poisson algebras and isomorphism classes of good  $n$ -pairs. Moreover:*

- (i)  $\mathcal{N}$  is simple (linearly compact) if and only if  $OP(\mathcal{N})$  is;
- (ii)  $\mathcal{N}$  is a Nambu-Poisson algebra if and only if  $OP(\mathcal{N})$  is an odd Poisson superalgebra.



**Proof.** The proof follows immediately from Propositions 4.1 and 4.10. The fact that the linear compactness of  $\mathcal{N}$  implies that of  $OP(\mathcal{N})$  can be proved in the same way as in [4, Proposition 2.4].  $\square$

**Remark 4.13** One can check (see also [4]) that if  $\mathcal{N}$  is the  $n$ -Nambu algebra, then  $(OP(\mathcal{N}), \mu) = (PO(n, n), \xi_1 \dots \xi_n)$  and if  $\mathcal{N}$  is the  $n$ -Dzhumaldidaev algebra, then  $(OP(\mathcal{N}), \mu) = (PO(n - 1, n), \xi_1 \dots \xi_{n-1} \tau)$ .

## 5 Classification of good pairs

In this section we will consider the odd Poisson (resp. generalized odd Poisson) superalgebra  $PO(n, n)$  (resp.  $PO(n, n+1)$ ) with the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ).

**Proposition 5.1** *Let  $\mathcal{P} = PO(n, n)$  or  $\mathcal{P} = PO(n, n+1)$  and  $(\mathcal{P}, \mu)$  be a good  $k$ -pair. Then the Lie subalgebra  $\mathcal{P}_0$  of  $\mathcal{P}$  is spanned by elements of the form:*

$$[[\mu, a_1], \dots, a_{k-1}]b$$

with  $a_1, \dots, a_{k-1}, b \in \mathcal{P}_{-1}$ .

**Proof.** By Theorem 4.12,  $\mathcal{P} = OP(\mathcal{N})$  for some  $k$ -Nambu-Poisson algebra  $\mathcal{N}$ . Hence, by Lemma 4.9,  $\mathcal{P}_0$  is generated as a Lie algebra by elements of the form

$$[[\mu, a_1], \dots, a_{k-1}]b$$

with  $a_1, \dots, a_{k-1}, b \in \mathcal{P}_{-1}$ . Let  $S = \langle [[\mu, a_1], \dots, a_{k-1}] \mid a_1, \dots, a_{k-1} \in \mathcal{P}_{-1} \rangle \subset \mathcal{P}_0$ .

Let  $\mathcal{P} = PO(n, n)$ . Then, for  $z_1, z_2 \in S$ ,  $b_1, b_2 \in \mathcal{P}_{-1}$ , we have:

$$\begin{aligned} [z_1 b_1, z_2 b_2] &= [z_1 b_1, z_2] b_2 + (-1)^{p(z_2)(p(z_1)+p(b_1)+1)} z_2 [z_1 b_1, b_2] = (-1)^{p(b_1)(p(z_2)+1)} [z_1, z_2] b_1 b_2 + \\ &\quad + z_1 [b_1, z_2] b_2 + (-1)^{p(b_1)(p(b_2)+1)+p(z_2)(p(z_1)+p(b_1)+1)} z_2 [z_1, b_2] b_1 \end{aligned}$$

since  $[b_1, b_2] = 0$ . We recall that  $[z_1, z_2]$  lies in  $S$  by [4, Theorem 0.2]. Finally, note that  $[z_1, b_2]$  and  $[b_1, z_2]$  lie in  $\mathcal{P}_{-1}$ . It follows that  $\mathcal{P}_0 \subseteq \langle [[\mu, a_1], \dots, a_{k-1}]b \mid a_i, b \in \mathcal{P}_{-1} \rangle \subseteq \mathcal{P}_0$ , hence the statement holds for  $\mathcal{P} = PO(n, n)$ .

If  $\mathcal{P} = PO(n, n+1)$ , one uses exactly the same argument and the fact that  $D|_{\mathcal{P}_{-1}} = 0$ ,  $D(S) \subseteq \mathcal{P}_{-1}$ .  $\square$

For any element  $f \in \mathcal{P}_{k-1} = \mathbb{F}[[x_1, \dots, x_n]] \otimes \wedge^k \mathbb{F}^n$ , we let  $f_0 = f|_{x_1=\dots=x_n=0} \in \wedge^k \mathbb{F}^n$ . We shall say that  $f$  has positive order if  $f_0 = 0$ .

**Corollary 5.2** *Let  $\mathcal{P} = PO(n, n)$  (resp.  $PO(n, n+1)$ ) with the grading of type  $(0, \dots, 0|1, \dots, 1)$  (resp.  $(0, \dots, 0|1, \dots, 1, 1)$ ). If  $\mu \in \mathcal{P}_{k-1}$  is such that  $\mu_0$  lies in the Grassmann subalgebra of  $\wedge^k(\mathbb{F}^n)$  (resp.  $\wedge^k(\mathbb{F}^{n+1})$ ) generated by some variables  $\xi_{i_1}, \dots, \xi_{i_h}$ , for some  $h < n$  (resp.  $h < n+1$ ), then  $\mu$  does not satisfy property G2). In particular, if  $\mu_0 = 0$ , then  $\mu$  does not satisfy property G2).*

**Proof.** Suppose, on the contrary, that some  $\xi_i$  does not appear in the expression of  $\mu_0$ . Then, by Proposition 5.1,  $\mathcal{P}_0$  does not contain  $\xi_i$  and this is a contradiction since if  $\mathcal{P} = PO(n, n)$  (resp.  $\mathcal{P} = PO(n, n+1)$ ),  $\mathcal{P}_0 = \langle \xi_1, \dots, \xi_n \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]]$  (resp.  $\mathcal{P}_0 = \langle \xi_1, \dots, \xi_{n+1} \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]]$ ).  $\square$

### 5.1 The case $PO(n, n)$

In this subsection we shall determine good  $k$ -pairs  $(\mathcal{P}, \mu)$  for  $\mathcal{P} = PO(n, n)$  with the  $\mathbb{Z}_+$ -grading of type  $(0, \dots, 0 | 1, \dots, 1)$ . We will denote the Lie superalgebra bracket in  $PO(n, n)$  simply by  $[\cdot, \cdot]$ . Recall the corresponding description of the  $\mathbb{Z}_+$ -grading given in Example 2.7. When writing a monomial in  $\xi_i$ 's we will assume that the indices increase; elements from  $\wedge^k \mathbb{F}^n$  will be written as linear combinations of such monomials.

**Lemma 5.3** *Let  $2 < k < n - 1$  and suppose that  $\mu \in PO(n, n)_{k-1}$  can be written in the following form:*

$$(5.1) \quad \mu = \xi_1 \dots \xi_k + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi + \psi,$$

where:

$$\mu_0 = \xi_1 \dots \xi_k + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi, \quad \varphi \in \wedge^k \mathbb{F}^n, \quad \psi_0 = 0,$$

$$h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and some } r, s > k\},$$

$$\frac{\partial^{k-1} \varphi}{\partial \xi_1 \dots \partial \xi_{k-1}} = 0, \quad \frac{\partial^k \varphi}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \xi_{k+2} \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-2}}} = 0.$$

Then  $\mu$  does not satisfy property G3).

**Proof.** Let us first suppose that  $h \geq 1$ . We have:

$$[x_{k+1}, \mu] = (-1)^h \xi_1 \dots \xi_h \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \frac{\partial(\varphi + \psi)}{\partial \xi_{k+1}};$$

$$[x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] = 2(-1)^{k-2} x_1 \xi_{k+2} + 2x_1 \frac{\partial^{k-1}(\varphi + \psi)}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}}.$$

Therefore  $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] =$

$$= 2(-1)^k (\xi_2 \dots \xi_k ((-1)^{k-2} \xi_{k+2} + \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}}) +$$

$$+ \xi_2 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}} +$$

$$\frac{\partial \varphi}{\partial \xi_1} ((-1)^{k-2} \xi_{k+2} + \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}})) + \omega,$$

for some  $\omega$  of positive order. Note that, the summand  $2\xi_2 \dots \xi_k \xi_{k+2}$  in the expression of  $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]]$  does not cancel out. Indeed, due to the hypotheses on  $\varphi$ , the only possibility to cancel the summand  $2\xi_2 \dots \xi_k \xi_{k+2}$  is that the expression of  $\varphi$  contains the sum  $a\xi_1 \dots \xi_h \xi_{k+1} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} \xi_t + b\xi_1 \dots \xi_{t-1} \xi_{t+1} \dots \xi_k \xi_{k+2}$ , for some  $t$ ,  $2 \leq t \leq k$ , and some suitable coefficients  $a, b \in \mathbb{F}^*$ . But this is impossible since it is in contradiction with the maximality of  $h$  if  $h = k - 2$ , and with the hypotheses on  $\varphi$  if  $h < k - 2$ . It follows that  $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] \neq 0$  and property G3) is not satisfied.

If  $h = 0$ , then one can use the same argument by showing that the commutator

$$[\mu, [x_1 x_{k+1}, [x_{i_1}, \dots, [x_{i_{k-2}}, \mu]]]]$$

□

**Theorem 5.4** *Let  $\mathcal{P} = PO(n, n)$ . Suppose that  $2 < k < n - 1$  and that  $\mu \in PO(n, n)_{k-1}$ . Then  $(\mathcal{P}, \mu)$  is not a good  $k$ -pair.*

**Proof.** By Corollary 5.2, if  $\mu_0 = 0$  then  $\mu$  does not satisfy property G2). Now suppose  $\mu_0 \neq 0$ . Since  $\mu_0$  lies in  $\wedge^k(\mathbb{F}^n)$ , we can assume, up to a linear change of indeterminates, that  $\mu_0 = \xi_1 \dots \xi_k + f$  for some  $f \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k f}{\partial \xi_1 \dots \partial \xi_k} = 0$ . Then, either  $\mu$  does not satisfy property G2) and  $(\mathcal{P}, \mu)$  is not a good  $k$ -pair, or, again by Corollary 5.2, all  $\xi_i$ 's appear in the expression of  $\mu_0$ . Let us thus assume to be in the latter case. Then, since  $k < n - 1$ , either there exist some  $r, s > k$  such that the indeterminates  $\xi_r$  and  $\xi_s$  both appear in the expression of  $\mu_0$  in at least one monomial (case A), or all the indeterminates  $\xi_r$  and  $\xi_s$  with  $r, s > k$  appear in distinct monomials (case B).

Suppose we are in case A), and let  $h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, i_1 < \dots < i_j \leq k; r, s > k\}$ . Then we can write

$$\mu_0 = \xi_1 \dots \xi_k + \xi_{i_1} \dots \xi_{i_h} \xi_r \xi_s \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi$$

for some  $r, s, i_{h+1}, \dots, i_{k-2} > k$ ,  $i_1, \dots, i_h \leq k$  and some  $\varphi \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k \varphi}{\partial \xi_1 \dots \partial \xi_k} = 0$  and  $\frac{\partial^k \varphi}{\partial \xi_{i_1} \dots \partial \xi_{i_h} \partial \xi_r \partial \xi_s \partial \xi_{i_1} \dots \partial \xi_{i_{k-2}}} = 0$ . Up to a permutation of indices we can assume  $r = k + 1$ ,  $s = k + 2$ ,  $\{i_1, \dots, i_h\} = \{1, \dots, h\}$  and up to a linear change of indeterminates we can assume  $\frac{\partial^{k-1} \varphi}{\partial \xi_1 \dots \partial \xi_{k-1}} = 0$ . Therefore  $\mu$  satisfies the hypotheses of Lemma 5.3, hence it does not satisfy property G3).

Now suppose we are in case B). Then

$$\mu_0 = \xi_1 \dots \xi_k + \xi_{i_1} \dots \xi_{i_{k-1}} \xi_{k+1} + \xi_{j_1} \dots \xi_{j_{k-1}} \xi_{k+2} + \psi$$

for some  $i_1 < \dots < i_{k-1} \leq k$ ,  $j_1 < \dots < j_{k-1} \leq k$  and  $\psi \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k \psi}{\partial \xi_1 \dots \partial \xi_k} = 0$ ,  $\frac{\partial^k \psi}{\partial \xi_{i_1} \dots \partial \xi_{i_{k-1}} \partial \xi_{k+1}} = 0$ ,  $\frac{\partial^k \psi}{\partial \xi_{j_1} \dots \partial \xi_{j_{k-1}} \partial \xi_{k+2}} = 0$ ,  $\frac{\partial^2 \psi}{\partial \xi_r \partial \xi_s} = 0$  for every  $r, s > k$ . Again by Corollary 5.2, we can assume that  $\{i_1, \dots, i_{k-1}\} \neq \{j_1, \dots, j_{k-1}\} \neq \{1, \dots, k - 1\}$ . Therefore there exists an index  $j_l \in \{1, \dots, k\} \cap \{j_1, \dots, j_{k-1}\}$  such that  $j_l \notin \{i_1, \dots, i_{k-1}\}$ .

Now consider the following change of indeterminates:

$$\xi'_{j_l} = \xi_{j_l} + \xi_{k+1}; \quad \xi'_j = \xi_j \quad \forall j \neq j_l.$$

Then

$$\mu_0 = \xi'_1 \dots \xi'_k + \xi'_{j_1} \dots \xi'_{j_{k-1}} \xi'_{k+2} + \xi'_{j_1} \dots \xi'_{j_{i-1}} \xi'_{j_{i+1}} \dots \xi'_{j_{k-1}} \xi'_{k+1} \xi'_{k+2} + \rho$$

for some  $\rho \in \wedge^k(\mathbb{F}^n)$  such that  $\frac{\partial^k \rho}{\partial \xi'_1 \dots \partial \xi'_k} = 0$ ,  $\frac{\partial^k \rho}{\partial \xi'_{j_1} \dots \partial \xi'_{j_{k-1}} \partial \xi'_{k+2}} = 0$ ,  $\frac{\partial^k \rho}{\partial \xi'_{j_1} \dots \partial \xi'_{j_{i-1}} \partial \xi'_{j_{i+1}} \dots \partial \xi'_{j_{k-1}} \partial \xi'_{k+1} \partial \xi'_{k+2}} = 0$ .

We are now again in case A) hence the proof is concluded.  $\square$

**Theorem 5.5** *Let  $\mathcal{P} = PO(n, n)$ . If  $(\mathcal{P}, \mu)$  is a good  $k$ -pair, then, up to isomorphisms, one of the following possibilities may occur:*

a) If  $n = 2h$ :

a1)  $k = 2$  and  $\mu_0 = \sum_{i=1}^h \xi_i \xi_{i+h}$ ;

a2)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_n$ .

b) If  $n = 2h + 1$ :

b1)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_n$ .

**Proof.** By Theorem 5.4, the only possibilities for  $k$  are  $k = 2$ ,  $k = n - 1$  or  $k = n$ .

By Corollary 5.2,  $\frac{\partial \mu_0}{\partial \xi_i} \neq 0$  for every  $i = 1, \dots, n$ . Using the classification of non-degenerate skew-symmetric bilinear forms, it thus follows that the case  $k = 2$  can occur only if  $n = 2h$  and, up to equivalence,  $\mu_0 = \sum_{i=1}^h \xi_i \xi_{i+h}$ , hence we get a1).

If  $k = n$  then, up to rescaling the odd indeterminates,  $\mu_0 = \xi_1 \dots \xi_n$  and we get cases a2) and b1).

Now assume  $k = n - 1$ . Assume that  $\frac{\partial^{n-2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_{n-2}}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n}$  for some  $i_1 < \dots < i_{n-2}$ ,  $i_{n-1} < i_n$ , and some  $\alpha, \beta \in \mathbb{F}^*$ . Consider the following change of indeterminates:

$$\xi'_{i_{n-1}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n} \quad \xi'_{i_j} = \xi_{i_j} \quad \forall j \neq n-1.$$

Then  $\frac{\partial^{n-2} \mu_0}{\partial \xi'_{i_1} \dots \partial \xi'_{i_{n-2}}} = \xi'_{i_{n-1}}$ . By using induction on the lexicographic order of the indices  $i_1 < \dots < i_{n-2}$ , one can thus show that, up to a linear change of indeterminates,  $\mu_0 = \xi_1 \dots \xi_{n-1}$ , hence  $(\mathcal{P}, \mu)$  is not a good  $k$ -pair due to Corollary 5.2.  $\square$

## 5.2 The case $PO(n, n+1)$

In this subsection we shall determine good pairs  $(\mathcal{P}, \mu)$  for  $\mathcal{P} = PO(n, n+1)$  with the  $\mathbb{Z}$ -grading of type  $(0, \dots, 0|1, \dots, 1, 1)$ . We shall adopt the same notation as in the previous subsection.

**Lemma 5.6** *Let  $2 \leq k < n - 1$ ,  $\mu \in PO(n, n+1)_k$  and suppose that  $\mu_0$  can be written in one of the following forms:*

1.

$$(5.2) \quad \mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi$$

where:

$$(a) \quad h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and } r, s > k\};$$

$$(b) \quad \varphi \in \wedge^{k+1} \mathbb{F}^{n+1} \text{ is such that } \frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0;$$

2.

$$(5.3) \quad \mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \tau \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi$$

where:

$$(a) \quad h = \max\{0 \leq j < k \mid \frac{\partial^{j+1} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k\};$$

$$(b) \quad \varphi \in \wedge^{k+1} \mathbb{F}^{n+1} \text{ is such that } \frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \tau \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-1}}} = 0 \text{ and } \frac{\partial^{k+1} \varphi}{\partial \xi_2 \dots \partial \xi_k \partial \tau} = 0.$$

Then  $\mu$  does not satisfy property G3).

**Proof.** Let us first suppose that  $\mu_0$  is of the form (5.2). Then, using the same arguments as in the proof of Lemma 5.3, one can show that  $[\mu, [x_{i_{k-1}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] \neq 0$ , since in its expression the summand  $\xi_2 \dots \xi_k \xi_{k+2} \tau$  does not cancel out.

Similarly, if  $\mu_0$  is of the form (5.3), then one can show that  $[\mu, [x_{i_{k-1}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [1, \mu]]]]]] \neq 0$ , since in its expression the summand  $\xi_2 \dots \xi_{k+1} \tau$  does not cancel out.  $\square$

**Theorem 5.7** Let  $\mathcal{P} = PO(n, n+1)$ . Suppose that  $2 \leq k < n-1$  and that  $\mu \in \mathcal{P}_k$ . Then  $(\mathcal{P}, \mu)$  is not a good  $(k+1)$ -pair.

**Proof.** Let us fix a set of odd indeterminates  $\xi_1, \dots, \xi_n, \xi_{n+1} = \tau$  and the corresponding basis of monomials of  $\wedge(\mathbb{F}^{n+1})$ . By Corollary 5.2, if  $\mu_0 = 0$  or  $\frac{\partial \mu_0}{\partial \tau} = 0$ , then  $\mu$  does not satisfy property G2). Hence suppose that  $\frac{\partial \mu_0}{\partial \tau} \neq 0$ . Then we may assume, up to a linear change of indeterminates, that  $\mu_0 = \xi_1 \dots \xi_k \tau + \varphi$  for some  $\varphi \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$ . Then, either  $\frac{\partial \varphi}{\partial \tau} = 0$  or  $\frac{\partial \varphi}{\partial \tau} \neq 0$ .

Suppose first  $\frac{\partial \varphi}{\partial \tau} = 0$ . Then, either for every  $r, s > k$  the indeterminates  $\xi_r, \xi_s$  appear in different monomials in the expression of  $\varphi$ , or there exist some  $r, s > k$  such that  $\xi_r, \xi_s$  appear in the same monomial.

In the first case  $\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_k (\xi_{k+1} + \xi_{k+2}) + \rho$  for some  $\rho \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \rho}{\partial \xi_1 \dots \partial \xi_k \partial \xi_{k+1}} = 0 = \frac{\partial^{k+1} \rho}{\partial \xi_1 \dots \partial \xi_k \partial \xi_{k+2}}$ . By Corollary 5.2 such an element does not satisfy property G2). Therefore we may assume that there exist some  $r, s > k$  such that  $\xi_r, \xi_s$  appear in the same monomial, i.e., that, up to a linear change of indeterminates,  $\mu_0$  is of the following form:

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi'$$

for some  $\varphi' \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \varphi'}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \xi_{k+2} \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-1}}} = 0$  and  $\frac{\partial^{k+1} \varphi'}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$ , where  $h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and } r, s > k\}$ . Therefore  $\mu$  satisfies hypothesis 1. of Lemma 5.6, hence it does not satisfy property G3).

Now suppose  $\frac{\partial \varphi}{\partial \tau} \neq 0$ . Then

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_{i_1} \dots \xi_{i_h} \tau \xi_{i_{h+1}} \dots \xi_{i_k} + \psi$$

for some  $i_1 < \dots < i_h \leq k < i_{h+1} < \dots < i_k$ , for some  $\psi \in \wedge^{k+1}(\mathbb{F}^{n+1})$  such that  $\frac{\partial^{k+1} \psi}{\partial \xi_{i_1} \dots \partial \xi_{i_k} \partial \tau} = 0$  and  $\frac{\partial^{k+1} \psi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$ , where  $h = \max\{0 \leq j < k \mid \frac{\partial^{j+1} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k\}$ . Now, up to a permutation of indices, we may assume that  $\{i_1, \dots, i_h\} = \{1, \dots, h\}$  and  $i_{h+1} = k+1$ . Then, either  $\mu$  does not satisfy property G2), or we may also assume that  $\frac{\partial^k \psi}{\partial \xi_2 \dots \partial \xi_k \partial \tau} = 0$ . Therefore  $\mu$  satisfies hypothesis 2. of Lemma 5.6, hence it does not satisfy property G3).  $\square$

**Theorem 5.8** Let  $\mathcal{P} = PO(n, n+1)$ . If  $(\mathcal{P}, \mu)$  is a good  $(k+1)$ -pair, then, up to isomorphisms, one of the following possibilities occur:

a) If  $n = 2h + 1$ :

a1)  $k = 1$  and  $\mu_0 = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$ ;

a2)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_{n+1}$ .

b) If  $n = 2h$ :

b1)  $k = n$  and  $\mu_0 = \xi_1 \dots \xi_{n+1}$ .

**Proof.** By Theorem 5.7, the only possibilities for  $k$  are  $k = 1$ ,  $k = n - 1$  or  $k = n$ .

By Corollary 5.2,  $\frac{\partial \mu_0}{\partial \xi_i} \neq 0$  for every  $i = 1, \dots, n + 1$ . It follows that, due to the classification of non-degenerate skew-symmetric bilinear forms, the case  $k = 2$  can occur only if  $n = 2h + 1$  and, up to equivalence,  $\mu_0 = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$ , hence we get a1).

If  $k = n$  then, up to rescaling the odd indeterminates,  $\mu_0 = \xi_1 \dots \xi_n \xi_{n+1}$  and we get cases a2) and b1).

Now assume  $k = n - 1$ . Then, using the same argument as in the proof of Theorem 5.5, one can show that, up to a linear change of indeterminates, we may assume  $\mu_0 = \xi_1 \dots \xi_{n-1} \xi_{n+1} + f$  for some  $f \in \wedge^n(\mathbb{F}^{n+1})$  such that  $\frac{\partial f}{\partial \xi_{n+1}} = 0$ . If  $f = 0$  then  $\mu$  does not satisfy property G2) by Corollary 5.2. If  $f \neq 0$ , then, up to a linear change of indeterminates,  $\mu_0 = \xi_1 \dots \xi_{n-1} \xi_{n+1} + \xi_1 \dots \xi_n = \xi_1 \dots \xi_{n-1}(\xi_{n+1} + \xi_n)$ . Then, by Proposition 5.1,  $\mu$  does not satisfy property G2).  $\square$

## 6 The classification theorem

**Remark 6.1** For every invertible element  $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$ , the following change of indeterminates preserves the odd symplectic form, i.e., the bracket in  $HO(n, n)$ , and maps  $\varphi \xi_1 \dots \xi_n$  to  $\xi'_1 \dots \xi'_n$ :

$$\begin{aligned} x'_1 &= \int_0^{x_1} \varphi^{-1}(t, x_2, \dots, x_n) dt =: \Phi, & \xi'_1 &= \varphi \xi_1, \\ x'_i &= x_i \quad \forall i \neq 1, & \xi'_i &= \xi_i - \varphi \frac{\partial \Phi}{\partial x_i} \xi_1 \quad \forall i \neq 1. \end{aligned}$$

Indeed one can check that  $\{x'_i, x'_j\}_{HO} = 0 = \{\xi'_i, \xi'_j\}_{HO}$  and  $\{x'_i, \xi'_j\}_{HO} = \delta_{ij}$  for every  $i, j = 1, \dots, n$ .

Note that the same change of variables, with the extra condition  $\tau' = \tau$ , preserves the bracket in the Lie superalgebra  $KO(n, n + 1)$ , and maps  $\varphi \xi_1 \dots \xi_n \tau$  to  $\xi'_1 \dots \xi'_n \tau'$ .

**Theorem 6.2** *A complete list, up to isomorphisms, of good  $k$ -pairs with  $k > 2$ , is the following:*

- i)  $(\mathcal{P}^\varphi, \varphi^{-1} \mu)$  with  $\mathcal{P} = PO(n, n)$ ,  $n > 2$ ,  $k = n$ ,  $\mu = \xi_1 \dots \xi_n$ ,  $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$ ;
- ii)  $(\mathcal{P}^\varphi, \varphi^{-1} \mu)$  with  $\mathcal{P} = PO(n, n + 1)$ ,  $n > 1$ ,  $k = n + 1$ ,  $\mu = \xi_1 \dots \xi_n \tau$ ,  $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$ .

**Proof.** Let  $\mathcal{P} = PO(n, n)$  with the grading of type  $(0, \dots, 0 | 1, \dots, 1)$ , and let  $(\mathcal{P}, \mu)$  be a good  $k$ -pair for  $k > 2$ . Then, by Theorem 5.5, we have necessarily  $n > 2$ ,  $k = n$ , and  $\mu_0 = \xi_1 \dots \xi_n$ . It follows that  $\mu = \xi_1 \dots \xi_n \psi$  for some invertible element  $\psi$  in  $\mathbb{F}[[x_1, \dots, x_n]]$ . By Remark 6.1, up to a change of variables, we may assume  $\psi = 1$ . In Example 4.5 we showed that the pair  $(\mathcal{P}, \xi_1 \dots \xi_n)$  is a good  $n$ -pair. Statement i) then follows from Theorem 2.5, Remark 2.9 and Remark 4.8.

Likewise, if  $\mathcal{P} = PO(n, n + 1)$  with the grading of type  $(0, \dots, 0 | 1, \dots, 1, 1)$  and  $(\mathcal{P}, \mu)$  is a good  $k$ -pair for  $k > 2$ , by Theorem 5.8 we have necessarily  $n > 1$ ,  $k = n + 1$  and  $\mu_0 = \xi_1 \dots \xi_n \tau$ . It follows that  $\mu = \xi_1 \dots \xi_n \tau \psi$  for some invertible element  $\psi$  in  $\mathbb{F}[[x_1, \dots, x_n]]$ . Again by Remark 6.1, we may assume  $\psi = 1$ . Furthermore in Example 4.7 we showed that  $(\mathcal{P}, \xi_1 \dots \xi_n \tau)$  is a good  $n$ -pair. Statement ii) then follows from Theorem 2.5, Remark 2.9 and Remark 4.8.  $\square$

**Theorem 6.3** *Let  $n > 2$ .*

- a) *Any simple linearly compact generalized  $n$ -Nambu-Poisson algebra is gauge equivalent either to the  $n$ -Nambu algebra or to the  $n$ -Dzhumadil'daev algebra.*
- b) *Any simple linearly compact  $n$ -Nambu-Poisson algebra is isomorphic to the  $n$ -Nambu algebra.*

**Proof.** By Theorems 4.12 and 2.5, we first need to consider good  $n$ -pairs  $(\mathcal{P}^\varphi, \mu)$  where  $\mathcal{P} = PO(k, k)$  or  $\mathcal{P} = PO(k, k+1)$  and  $n > 2$ . A complete list, up to isomorphisms, of such pairs is given in Theorem 6.2. The statement then follows from the construction described in Proposition 4.10. We point out that the pair  $(\mathcal{P}^\varphi, \varphi^{-1}\xi_1 \dots \xi_n)$ , with  $\mathcal{P} = PO(n, n)$ , corresponds to  $\mathcal{N}^\varphi$  where  $\mathcal{N}$  is the  $n$ -Nambu algebra; similarly, the pair  $(\mathcal{P}^\varphi, \varphi^{-1}\xi_1 \dots \xi_n \tau)$ , with  $\mathcal{P} = PO(n, n+1)$ , corresponds to  $\mathcal{N}^\varphi$ , where  $\mathcal{N}$  is the  $n$ -Dzhumadildaev algebra (see also Remark 4.13).  $\square$

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