Eigenvalue distributions of beta-Wishart matrices

SIAM J. MATRIX ANAL. APPL.
Vol. XX, No. X, pp. XX–XX

EIGENVALUE DISTRIBUTIONS OF BETA-WISHART MATRICES*

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Abstract. We derive explicit expressions for the distributions of the extreme eigenvalues of the Beta-Wishart random matrices in terms of the hypergeometric function of a matrix argument. These results generalize the classical results for the real ($\beta = 1$), complex ($\beta = 2$), and quaternion ($\beta = 4$) Wishart matrices to any $\beta > 0$.

Key words. random matrix, Wishart distribution, eigenavalue, hypergeometric function of a matrix argument

AMS subject classifications. 15A52, 60E05, 62H10, 65F15

DOI. XX.XXXX/SXXXXXXXXXXXXXXXX

1. Introduction.

Recently, the classical real ($\beta = 1$), complex ($\beta = 2$), and quaternion ($\beta = 4$) Wishart random matrix ensembles were generalized to any $\beta > 0$ by what is now called the Beta–Wishart ensemble [2, 9]. In this paper we derive the explicit distributions for the extreme eigenvalues and the trace of this ensemble as series of Jack functions and, in particular, in terms of the hypergeometric function of matrix argument. These results generalize the classical expressions for the real and complex cases as series of zonal polynomials [17] and Schur functions [18], respectively.

This paper is motivated by the importance of the real and complex Wishart matrices (and in particular their eigenvalue distributions) in multivariate statistical analysis. We refer to the classical text by Muirhead [17], as well as [8, 10, 18]. Recently, it is becoming increasingly apparent that the classical matrix ensembles (and perhaps all of random matrix theory) generalizes from the Dyson's three-fold way ($\beta = 1, 2, 4$) [5] to any $\beta > 0$ [6]. Recent examples include the Beta-Hermite [3], Beta–Laguerre [3], Beta–Jacobi [4, 7, 12, 14], and Beta–Wishart ensembles [2, 9].

Our new results are particularly convenient for practical evaluation using our algorithms for the hypergeometric function of matrix argument [13], see section 5.

2. Definitions and background.

Since the eigenvalue distributions will be expressed as series of multivariate symmetric polynomials, we start with the relevant definitions.

For an integer $k \geq 0$ we say that $\kappa = (\kappa_1, \kappa_2, \ldots)$ is a *partition* of k (denoted $\kappa \vdash k$) if $\kappa_1 \geq \kappa_2 \geq \cdots \geq 0$ are integers such that $|\kappa| \equiv \kappa_1 + \kappa_2 + \cdots = k$. Partitions with t equal parts are denoted as $(a)^t = (a, a, \ldots, a)$. For two partitions λ and κ we write $\lambda \subseteq \kappa$ to indicate that $\lambda_i \leq \kappa_i$ for all i.

*Received by the editors DATE; accepted for publication (in revised form) by EDITOR DATE; published electronically DATE. This research was supported by the National Science Foundation Grant DMS-1016086 and by the Woodward Fund for Applied Mathematics at San Jose State University. The Woodward Fund is a gift from the estate of Mrs. Marie Woodward in memory of her son, Henry Teynham Woodward. He was an alumnus of the Mathematics Department at San Jose State University and worked with research groups at NASA Ames.

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For a partition $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ and $\beta > 0$, the *Generalized Pochhammer symbol* of parameter β is defined as

$$(a)_\kappa^{(\beta)} \equiv \prod_{i=1}^{m} \prod_{j=1}^{\kappa_i} \left(a - \frac{(i-1)\beta}{2} + j - 1 \right). \tag{2.1}$$

The *multivariate Gamma function* of parameter β is defined as

$$\Gamma_m^{(\beta)}(c) = \pi^{\frac{m(m-1)\beta}{4}} \prod_{i=1}^{m} \Gamma\left(c - \frac{(i-1)\beta}{2} \right) \quad \text{for} \quad \Re(c) > \frac{(m-1)\beta}{2}. \tag{2.2}$$

For an $m \times m$ matrix X, the *Jack functions* $C_\kappa^{(\beta)}(X)$ are symmetric homogeneous polynomials in the eigenvalues $x_1, x_2, \ldots, x_m$ of X. The $C_\kappa^{(\beta)}$'s are indexed by partitions κ and form an orthogonal basis of the space multivariate symmetric polynomials. We refer to Stanley [19], Macdonald [16], and Forrester [8] for details and assume the reader has some familiarity with these functions.

For integers $p \geq 0$ and $q \geq 0$, and an $m \times m$ complex symmetric matrices X, Y, the *hypergeometric function of a matrix argument* is defined as

$$_pF_q^{(\beta)}(a; b; X, Y) \equiv \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{1}{k!} \cdot \frac{(a_1)_\kappa^{(\beta)} \cdots (a_p)_\kappa^{(\beta)}}{(b_1)_\kappa^{(\beta)} \cdots (b_q)_\kappa^{(\beta)}} \cdot \frac{C_\kappa^{(\beta)}(X) C_\kappa^{(\beta)}(Y)}{C_\kappa^{(\beta)}(I)}, \tag{2.3}$$

where $(a) = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, b_2, \ldots, b_q)$. For one matrix argument X,

$$_pF_q^{(\beta)}(a; b; X) \equiv {}_pF_q^{(\beta)}(a; b; X, I)$$

Following [2], an $m \times m$ Beta–Wishart matrix with n ($n \geq m$) degrees of freedom and covariance matrix Σ (denoted $A \sim \mathcal{W}_m^{(\beta)}(n, \Sigma)$) has joint eigenvalue density

$$\frac{|\Sigma|^{-\frac{n\beta}{2}}}{\mathcal{K}_m^{(\beta)}(\frac{n\beta}{2})} \prod_{i=1}^{m} \lambda_i^{\frac{(n-m+1)\beta}{2}-1} \prod_{j<k} |\lambda_k - \lambda_j|^\beta \cdot {}_0F_0^{(\beta)}(-\tfrac{\beta}{2}\Lambda, \Sigma^{-1}), \tag{2.4}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ are the eigenvalues of A, $|\Sigma| \equiv \det \Sigma$, and

$$\mathcal{K}_m^{(\beta)}(a) \equiv \frac{m!}{\pi^{\frac{m(m-1)\beta}{2}}} \cdot \left(\frac{2}{\beta} \right)^{ma} \cdot \frac{\Gamma_m^{(\beta)}(a) \Gamma_m^{(\beta)}\left(\frac{m\beta}{2}\right)}{\left(\Gamma\left(\frac{\beta}{2}\right)\right)^m}. \tag{2.5}$$

3. Identities involving $_pF_q^{(\beta)}$. We present several identities, including a new one, which we need in the next section.

First, we incorporate the Vandermonde determinant into a new measure μ to prevent it from appearing in all the integrals that follow:

$$\mathrm{d}\mu(X) \equiv \prod_{i<j} |x_i - x_j|^\beta \mathrm{d}X.$$

Also, define

$$T_m^{(\beta)}(a, b) \equiv \frac{\Gamma_m^{(\beta)}(b)}{\Gamma_m^{(\beta)}(a) \Gamma_m^{(\beta)}(b-a)};$$

$$c_m^{(\beta)} \equiv \frac{m!}{\pi^{\frac{m(m-1)\beta}{4}}} \prod_{i=1}^{m} \frac{\Gamma(\frac{i\beta}{2})}{\Gamma(\frac{\beta}{2})};$$

$$p \equiv \frac{m-1}{2}\beta + 1. \tag{3.1}$$

Then, for a nonnegative integer r,

$$
\begin{aligned}
&(3.2) \quad {}_0F_0^{(\beta)}(X, I+Y) = e^{\operatorname{tr} X}\, {}_0F_0^{(\beta)}(X,Y);\\
&(3.3) \quad {}_1F_0^{(\beta)}(-r;X) = |I-X|^r;\\
&(3.4) \quad {}_2F_1^{(\beta)}(a,b;c;X) = {}_2F_1^{(\beta)}\left(c-a,b;c;-X(I-X)^{-1}\right)\cdot |I-X|^{-b};\\
&(3.5) \quad {}_2F_1^{(\beta)}(a,-r;c;X) = \frac{\Gamma_m^{(\beta)}(c)\Gamma_m^{(\beta)}(c-a+r)}{\Gamma_m^{(\beta)}(c-a)\Gamma_m^{(\beta)}(c+r)}\\
&\qquad\qquad \times {}_2F_1^{(\beta)}\left(a,-r;a-r+1+\tfrac{m-1}{2}\beta-c;I-X\right);\\
&(3.6) \quad {}_1F_1^{(\beta)}(a;b;Y) = \frac{T_m^{(\beta)}(a,b)}{c_m^{(\beta)}}\int_{[0,1]^m} {}_0F_0^{(\beta)}(X,Y)|X|^{a-p}|I-X|^{b-a-p}\mathrm{d}\mu(X);\\
&(3.7) \quad {}_1F_0^{(\beta)}(a;Y) = \frac{1}{c_m^{(\beta)}\Gamma_m^{(\beta)}(a)}\int_{\mathbb{R}_+^m} e^{-\operatorname{tr}(X)}\, {}_0F_0^{(\beta)}(X,Y)|X|^{a-p}\mathrm{d}\mu(X);\\
&(3.8) \quad C_\kappa^{(\beta)}(Y^{-1}) = \frac{|Y|^{a+p}}{c_m^{(\beta)}\Gamma_m^{(\beta)}(a+p)\cdot(a+p)_\kappa^{(\beta)}}\\
&\qquad\qquad \times \int_{\mathbb{R}_m^+} {}_0F_0^{(\beta)}(-X,Y)|X|^a C_\kappa^{(\beta)}(X)\mathrm{d}\mu(X);\\
&(3.9) \quad {}_2F_0^{(\beta)}(p,-r;X) = \frac{\Gamma_m^{(\beta)}(p+r)}{\Gamma_m^{(\beta)}(p)}|-X|^r \sum_{k=0}^{mr}\sum_{\kappa\vdash k,\,\kappa_1\le r}\frac{C_\kappa^{(\beta)}(-X^{-1})}{k!}.
\end{aligned}
$$

The identities (3.2) and (3.8) were conjectured by Macdonald [15] and proven by Forrester and Baker [1, section 6].

The identities (3.3), (3.4), and (3.5) are due to Forrester: see (13.4) and Propositions 13.1.6 and 13.1.7 in [8], respectively. Note that (3.3) was established for arguments X whose eigenvalues are in $[0,1]$. When $r \geq 0$ is an integer, however, (3.3) is true for any X—we have polynomials on both sides,[1] which must be identical everywhere.

The integrals (3.6) and (3.7) are due to Macdonald [15, (6.20), (6.21)]. For completeness, we repeat his argument here.

The integral (3.6) follows directly from Kadell's extension of Selberg's integral [11, Theorem 1]

$$
(3.10) \qquad \int_{[0,1]^m}\frac{C_\kappa^{(\beta)}(X)}{C_\kappa^{(\beta)}(I)}|X|^{a-p}|I-X|^{b-a-p}\mathrm{d}\mu(X) = \frac{c_m^{(\beta)}}{T_m^{(\beta)}(a,b)}\cdot\frac{(a)_\kappa^{(\beta)}}{(b)_\kappa^{(\beta)}}
$$

by multiplying both sides by $\frac{1}{|\kappa|!}C_\kappa^{(\beta)}(Y)$ and summing over all κ.

To deduce (3.7), we change variables $y_i = Nx_i, i = 1,2,\ldots,m$, in (3.10), where $N \equiv b-p$, to obtain

$$
\begin{aligned}
\int_{[0,N]^m}\frac{C_\kappa^{(\beta)}(Y)}{C_\kappa^{(\beta)}(I)}|Y|^{a-p}\prod_{i=1}^m\left|1-\tfrac{y_i}{N}\right|^N\mathrm{d}\mu(Y) &= N^{ma+|\kappa|}\cdot\frac{c_m^{(\beta)}}{T_m^{(\beta)}(a,N+p)}\cdot\frac{(a)_\kappa^{(\beta)}}{(N+p)_\kappa^{(\beta)}}\\
&= \Gamma_m^{(\beta)}(a)c_m^{(\beta)}(a)_\kappa^{(\beta)}\frac{\Gamma_m^{(\beta)}(N+p-a)N^{ma+|\kappa|}}{\Gamma_m^{(\beta)}(N+p)(N+p)_\kappa^{(\beta)}}.
\end{aligned}
$$

[1] In the series for ${}_1F_0^{(\beta)}$ on the left, if κ has any part greater than r or has more than m parts, then $(-r)_\kappa^{(\beta)} = 0$ or $C_\kappa^{(\beta)}(X) = 0$, respectively. Either way the κ term in the series is 0.

We take limits as $N \to \infty$. The identity $\Gamma(z+1) = z\Gamma(z)$ along with the definition (2.2) imply

$$\lim_{N\to\infty} \frac{\Gamma_m^{(\beta)}(N+p-a)}{\Gamma_m^{(\beta)}(N+p)} N^{ma} = 1. \tag{3.11}$$

From (2.1), $\lim_{N\to\infty} N^{-|\kappa|}(N+p)_\kappa^{(\beta)} = 1$. Also, $\lim_{N\to\infty} \left|1 - \frac{y_i}{N}\right|^N = e^{-y_i}$ for $i = 1, 2, \ldots, m$, and therefore

$$\int_{[0,\infty]^m} \frac{C_\kappa^{(\beta)}(Y)}{C_\kappa^{(\beta)}(I)} |Y|^{a-p} e^{-\operatorname{tr} Y} \mathrm{d}\mu(Y) = \Gamma_m^{(\beta)}(a) c_m^{(\beta)} \cdot (a)_\kappa^{(\beta)}.$$

Multiplying both sides by $\frac{1}{|\kappa|!} C_\kappa^{(\beta)}(X)$ and summing over all κ, we get (3.7).

Finally, (3.9) is a new result which we now prove.

THEOREM 3.1. *The identity* (3.9) *holds for any nonnegative integer* r.

Proof. We combine (3.4), ${}_2F_1^{(\beta)}(p, -r; c; I - X) = {}_2F_1^{(\beta)}(c-p, -r; c; I - X^{-1})|X|^r$ with (3.5) to get

$$\frac{\Gamma_m^{(\beta)}(c-p+r)}{\Gamma_m^{(\beta)}(c-p)} \cdot {}_2F_1^{(\beta)}(p, -r; p-r+1+\tfrac{(m-1)\beta}{2} - c; X)$$
$$= \frac{\Gamma_m^{(\beta)}(p+r)}{\Gamma_m^{(\beta)}(p)} \cdot {}_2F_1^{(\beta)}(c-p, -r; -p-r+1+\tfrac{(m-1)\beta}{2}; X^{-1}) \cdot |X|^r.$$

We replace X by $-cX$, use the fact that $p = \frac{(m-1)\beta}{2} + 1$, and move a c^{mr} factor to the left to get:

$$\frac{\Gamma_m^{(\beta)}(c-p+r)}{\Gamma_m^{(\beta)}(c-a) \cdot c^{mr}} \cdot {}_2F_1^{(\beta)}(p, -r; -r-c; -cX) \tag{3.12}$$
$$= \frac{\Gamma_m^{(\beta)}(p+r)}{\Gamma_m^{(\beta)}(p)} \cdot {}_2F_1^{(\beta)}(c-p, -r; -r; -\tfrac{1}{c}X^{-1}) \cdot |-X|^r.$$

We take limits on both sides as $c \to \infty$. From [8, (13.5)]

$$\lim_{c\to\infty} {}_2F_1^{(\beta)}(p, -r; -r-c; -cX) = {}_2F_0^{(\beta)}(p, -r; X);$$
$$\lim_{c\to\infty} {}_2F_1^{(\beta)}(c-p, -r; -r; -\tfrac{1}{c}X^{-1}) = \sum_{k=0}^{mr} \sum_{\kappa\vdash k,\, \kappa_1 \le r} \frac{C_\kappa^{(\beta)}(-X^{-1})}{k!}.$$

Combining this with (3.11) above, we obtain (3.9). ∎

4. The distributions of the extreme eigenvalues and trace. We start with the extreme eigenvalues.

THEOREM 4.1. *Let* $p = \frac{m-1}{2}\beta + 1$ *and let* $t \equiv \frac{n-m+1}{2}\beta - 1$ *be a nonnegative integer. For the extreme eigenvalues of a Wishart matrix* $A \sim \mathcal{W}_m^{(\beta)}(n, \Sigma)$ *we have:*

$$P(\lambda_{\max}(A) < x) = \frac{\Gamma_m^{(\beta)}(p)}{\Gamma_m^{(\beta)}\left(\frac{n}{2}\beta + p\right)} \left|\tfrac{x\beta}{2}\Sigma^{-1}\right|^{\frac{n\beta}{2}} {}_1F_1^{(\beta)}\left(\tfrac{n\beta}{2}; \tfrac{n\beta}{2} + p; -\tfrac{x\beta}{2}\Sigma^{-1}\right) \tag{4.1}$$

$$P(\lambda_{\min}(A) < x) = 1 - e^{\operatorname{tr}\left(-\frac{x\beta}{2}\Sigma^{-1}\right)} \sum_{k=0}^{mt} \sum_{\kappa\vdash k,\, \kappa_1 \le t} \frac{C_\kappa^{(\beta)}\left(\frac{x\beta}{2}\Sigma^{-1}\right)}{k!}. \tag{4.2}$$

Proof. We follow the ideas in Muirhead [17].

For the first part, the condition $\lambda_{\max} < x$ is equivalent to $\lambda_i \in [0,x]$, $i = 1, 2, \ldots, n$. To obtain the desired distribution, we integrate the joint density (2.4) over $[0,x]^n$, make a change of variables $L = xZ$, then use $\mathrm{d}L = x^m \mathrm{d}Z$ and (3.6) to obtain

$$\begin{aligned} P(\lambda_{\max} < x) &= \int_{[0,x]^m} \frac{|\Sigma|^{-\frac{n\beta}{2}}}{\mathcal{K}_m^{(\beta)}(\frac{n\beta}{2})} \prod_{i=1}^m \lambda_i^{\frac{n-m+1}{2}\beta - 1} {}_0F_0^{(\beta)}(-\tfrac{\beta}{2}L, \Sigma^{-1}) \mathrm{d}\mu(L) \\ &= x^{\frac{mn\beta}{2}} \frac{|\Sigma|^{-\frac{n\beta}{2}}}{\mathcal{K}_m^{(\beta)}(\frac{n\beta}{2})} \int_{[0,1]^m} \prod_{i=1}^m z_i^{\frac{n-m+1}{2}\beta - 1} \cdot {}_0F_0^{(\beta)}(Z, -\tfrac{x\beta}{2}\Sigma^{-1}) \mathrm{d}\mu(Z) \\ &= \frac{|x\Sigma^{-1}|^{\frac{n\beta}{2}}}{\mathcal{K}_m^{(\beta)}(\frac{n\beta}{2})} \cdot c_m^{(\beta)} \cdot \frac{\Gamma_m^{(\beta)}(\frac{n\beta}{2})\Gamma_m^{(\beta)}(p)}{\Gamma_m^{(\beta)}(\frac{n}{2}\beta + p)} {}_1F_1^{(\beta)}\big(\tfrac{n\beta}{2}; \tfrac{n+m-1}{2}\beta + 1; -\tfrac{x\beta}{2}\Sigma^{-1}\big), \end{aligned}$$

with $c_m^{(\beta)}$ defined in (3.1). We use (2.2) and (2.5) to simplify

$$\frac{1}{\mathcal{K}_m^{(\beta)}(\frac{n\beta}{2})} \cdot c_m^{(\beta)} \Gamma_m^{(\beta)}\big(\tfrac{n\beta}{2}\big) = \frac{\pi^{\frac{m(m-1)\beta}{4}}}{\left(\frac{2}{\beta}\right)^{\frac{mn\beta}{2}} \Gamma_m^{(\beta)}\big(\frac{m\beta}{2}\big)} \prod_{i=1}^m \Gamma\big(\tfrac{i\beta}{2}\big) = \left(\frac{\beta}{2}\right)^{\frac{mn\beta}{2}},$$

implying (4.1).

For the second part, the change of variables is $L = x(I+Z)$ with $\mathrm{d}L = x^m \mathrm{d}Z$:

$$\begin{aligned} &P(\lambda_{\min} > x) \\ &\quad = \frac{|\Sigma|^{-\frac{n\beta}{2}}}{\mathcal{K}_m^{(\beta)}(\frac{n\beta}{2})} \int_{[x,\infty]^m} \prod_{i=1}^m \lambda_i^t \cdot {}_0F_0^{(\beta)}(-\tfrac{\beta}{2}L, \Sigma^{-1}) \mathrm{d}\mu(L) \\ &\quad = x^{\frac{mn\beta}{2}} \frac{|\Sigma|^{-\frac{n\beta}{2}}}{\mathcal{K}_m^{(\beta)}(\frac{n\beta}{2})} \int_{\mathbb{R}_+^m} |I+Z|^t \cdot {}_0F_0^{(\beta)}(I+Z, -\tfrac{x\beta}{2}\Sigma^{-1}) \mathrm{d}\mu(Z) && (4.3) \\ &\quad = e^{\operatorname{tr}(-\frac{x\beta}{2}\Sigma^{-1})} \frac{|x\Sigma^{-1}|^{\frac{n}{2}\beta}}{\mathcal{K}_m^{(\beta)}(\frac{n}{2}\beta)} \int_{\mathbb{R}_+^m} \sum_{\kappa \subseteq (m^t)} \frac{(-t)_\kappa^{(\beta)} C_\kappa^{(\beta)}(-Z)}{|\kappa|!} {}_0F_0^{(\beta)}(Z, -\tfrac{x\beta}{2}\Sigma^{-1}) \mathrm{d}\mu(Z), && (4.4) \end{aligned}$$

where we used (3.2) and (3.3) to go from (4.3) to (4.4). Next, from (3.8),

$$\begin{aligned} &P(\lambda_{\min} > x) \\ &\quad = e^{\operatorname{tr}(-\frac{x}{2}\beta\Sigma^{-1})} \frac{|x\Sigma^{-1}|^{\frac{n}{2}\beta}}{\mathcal{K}_m^{(\beta)}(\frac{n}{2}\beta)} && (4.5) \\ &\qquad \times \sum_{\kappa \subseteq (m^t)} \frac{(-1)^{|\kappa|}(-t)_\kappa^{(\beta)}}{|\kappa|!} \int_{\mathbb{R}_+^m} C_\kappa^{(\beta)}(Z) \cdot {}_0F_0^{(\beta)}(-Z, \tfrac{x}{2}\beta\Sigma^{-1}) \mathrm{d}\mu(Z) \\ &\quad = e^{\operatorname{tr}(-\frac{x}{2}\beta\Sigma^{-1})} \frac{|x\Sigma^{-1}|^{\frac{n}{2}\beta}}{\mathcal{K}_m^{(\beta)}(\frac{n}{2}\beta)} \sum_{\kappa \subseteq (m^t)} \frac{(-t)_\kappa^{(\beta)}}{|\kappa|!} C_\kappa^{(\beta)}(-\tfrac{2}{x\beta}\Sigma) |\tfrac{x}{2}\beta\Sigma^{-1}|^{-p} c_m^{(\beta)} \Gamma_m^{(\beta)}(p)(p)_\kappa && (4.6) \\ &\quad = e^{\operatorname{tr}(-\frac{x}{2}\beta\Sigma^{-1})} \left|\tfrac{x\beta}{2}\Sigma^{-1}\right|^t \frac{\Gamma_m^{(\beta)}(p)}{\Gamma_m^{(\beta)}(\frac{n\beta}{2})} \sum_{\kappa \subseteq (m^t)} \frac{(-t)_\kappa^{(\beta)}(p)_\kappa}{|\kappa|!} C_\kappa^{(\beta)}(-\tfrac{2}{x\beta}\Sigma), && (4.7) \end{aligned}$$

which along with (3.9) implies (4.2). ∎

Note that the sum in (4.2) is ${}_0F_0^{(\beta)}(\frac{x}{a}\Sigma^{-1})$, truncated for partitions $\kappa \subseteq (m)^t$, and is thus particularly convenient to evaluate using our algorithms [13].

The following theorem generalizes Theorem 8.3.4 in Muirhead [17] to all $\beta > 0$.

THEOREM 4.2. *If $A \sim \mathcal{W}_m^{(\beta)}(n, \Sigma)$, then the density of the trace of A is*

$$\left|\tfrac{x}{2}\beta\Sigma^{-1}\right|^{\frac{n}{2}\beta} e^{-\frac{x\beta}{2\lambda}} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\frac{nm}{2}\beta + k)} \cdot \tfrac{\beta}{2\lambda}\left(\tfrac{x\beta}{2\lambda}\right)^{k-1} \sum_{\kappa \vdash k} \left(\tfrac{n}{2}\beta\right)_\kappa^{(\beta)} \cdot \frac{1}{k!} \cdot C_\kappa^{(\beta)}(I - \lambda\Sigma^{-1}), \tag{4.8}$$

where λ is arbitrary.

Proof. We follow, Muirhead's argument, which carries over to any $\beta > 0$.

From (2.4), the moment generating function of $\operatorname{tr} A$ is:

$$\begin{aligned}\phi(t) &= E[e^{\operatorname{tr}(tL)}] \\ &= \frac{|\Sigma|^{-\frac{n}{2}\beta}}{\mathcal{K}_m^{(\beta)}(\frac{n}{2}\beta)} \int_{\mathbb{R}_+^m} e^{-\operatorname{tr}(-tL)} \prod_{i=1}^m \lambda_i^{\frac{n-m+1}{2}\beta - 1} \prod_{j<k} |\lambda_k - \lambda_j|^\beta {}_0F_0^{(\beta)}(-tL, \tfrac{\beta}{2t}\Sigma^{-1}) dL\end{aligned}$$

We make a change of variables $L \to -tL$ and use (3.7) to obtain

$$\begin{aligned}\phi(t) &= |\Sigma|^{-\frac{n}{2}\beta} \cdot \frac{(-t)^{-\frac{mn}{2}\beta}}{c_m'(\beta)\Gamma_m^{(\beta)}(\frac{n}{2}\beta)\mathcal{K}_m^{(\beta)}(\frac{n}{2}\beta)} {}_1F_0^{(\beta)}(\tfrac{n}{2}\beta; \tfrac{\beta}{2t}\Sigma^{-1}) \\ &= |-t\tfrac{2}{\beta}\Sigma|^{-\frac{n}{2}\beta} \cdot |I - \tfrac{\beta}{2t}\Sigma^{-1}|^{-\frac{n}{2}\beta} \\ &= \left|I - \tfrac{2t}{\beta}\Sigma\right|^{-\frac{n}{2}\beta}.\end{aligned}$$

For $0 < \lambda < \infty$ write

$$\begin{aligned}\phi(t) &= \left|I - \tfrac{2t}{\beta}\Sigma\right|^{-\frac{n}{2}\beta} \\ &= \left(1 - \tfrac{2t}{\beta}\lambda\right)^{-\frac{mn}{2}\beta} |\lambda^{-1}\Sigma|^{-\frac{n}{2}\beta} \left| I - \frac{1}{1 - \frac{2t}{\beta}\lambda}(I - \lambda\Sigma^{-1})\right|^{-\frac{n}{2}\beta} \\ &= \left(1 - \tfrac{2t}{\beta}\lambda\right)^{-\frac{mn}{2}\beta} |\lambda^{-1}\Sigma|^{-\frac{n}{2}\beta} \cdot {}_1F_0^{(\beta)}\left(\frac{n}{2}\beta; \frac{1}{1 - \frac{2t}{\beta}\lambda}(I - \lambda\Sigma^{-1})\right) \\ &= \left(1 - \tfrac{2t}{\beta}\lambda\right)^{-\frac{mn}{2}\beta} |\lambda^{-1}\Sigma|^{-\frac{n}{2}\beta} \sum_{k=0}^{\infty} \left(1 - \tfrac{2t\lambda}{\beta}\right)^{-k} \sum_{\kappa \vdash k} \frac{1}{k!} \left(\tfrac{n}{2}\beta\right)_\kappa^{(\beta)} C_\kappa^{(\beta)}(I - \lambda\Sigma^{-1}),\end{aligned} \tag{4.9}$$

where t and λ are such that $\|I - \lambda\Sigma^{-1}\| < \left|1 - \frac{2t\lambda}{\beta}\right|$, so that the ${}_1F_0^{(\beta)}$ function above converges.

Since $\left(1 - \frac{2t\lambda}{\beta}\right)^{-r}$ is the moment-generating function of the gamma distribution with parameters r and $\frac{2\lambda}{\beta}$ and density function

$$g_{r,\frac{2\lambda}{\beta}}(u) = \frac{e^{-\frac{u\beta}{2\lambda}} u^{r-1}}{\left(\frac{2\lambda}{\beta}\right)^r \Gamma(r)} \qquad (u > 0),$$

the moment-generating function (4.9) can be inverted term by term to obtain (4.8)—the density function of the trace of A, where λ is arbitrary and, as in Muirhead [17, p. 341], can be chosen as $\lambda = \frac{2\lambda'\lambda''}{\lambda' + \lambda''}$, where λ' and λ'' are the largest and the smallest eigenvalues of Σ, respectively. ☐

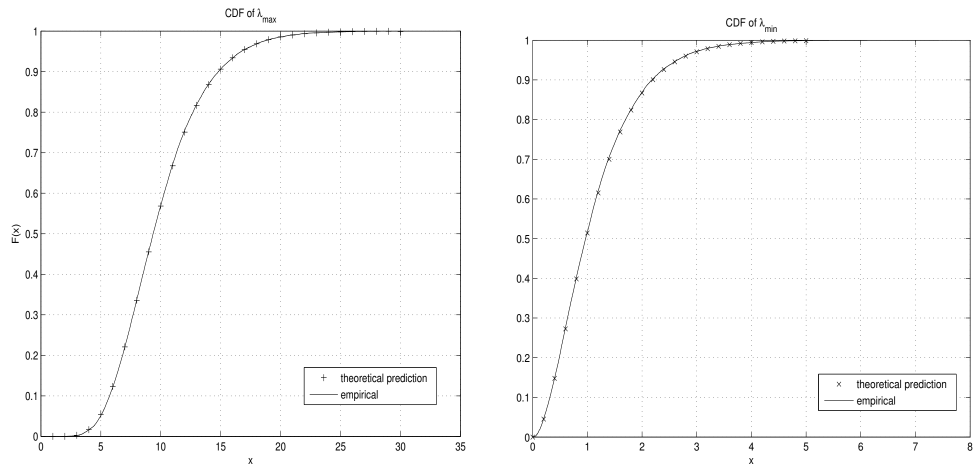

FIG. 5.1. *The empirical distributions from 20,000 replications of the largest and smallest eigenvalues, respectively, of a* 3×3 *Beta–Wishart matrix with 5 degrees of freedom,* $\beta = 4/3$, $\Sigma = \mathrm{diag}(1.3, 1, 0.7)$*, plotted against the theoretical predictions* (4.1) *and* (4.2).

5. Numerical experiments. The theoretical distributions of the extreme eigenvalues of the Beta–Wishart ensemble are particularly conveniently evaluated using our algorithm for computing the hypergeometric function of a matrix argument [13]. We performed extensive numerical tests of the theoretical eigenvalue distributions in this paper against the empirically predicted ones by the Beta–Wishart model from our recent paper [2]. The results were always a match. In Figure 5.1 we report one of our experiments in which we plotted the largest and the smallest eigenvalues of the *same* Beta–Wishart ensemble. More experiments are reported in [2].

Acknowledgments. We acknowledge the great insights that we have gathered into the subject of hypergeometric functions with matrix argument from Ian Macdonald's unpublished manuscript on the topic [15]. We also thank Iain Johnstone for many useful discussions in developing this work.

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