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CURVES ON $K3$ SURFACES AND MODULAR FORMS

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WITH AN APPENDIX BY A. PIXTON

ABSTRACT. We study the virtual geometry of the moduli spaces of curves and sheaves on $K3$ surfaces in primitive classes. Equivalences relating the reduced Gromov-Witten invariants of $K3$ surfaces to characteristic numbers of stable pairs moduli spaces are proven. As a consequence, we prove the Katz-Klemm-Vafa conjecture evaluating λ_g integrals (in all genera) in terms of explicit modular forms. Indeed, all $K3$ invariants in primitive classes are shown to be governed by modular forms.

The method of proof is by degeneration to elliptically fibered rational surfaces. New formulas relating reduced virtual classes on $K3$ surfaces to standard virtual classes after degeneration are needed for both maps and sheaves. We also prove a Gromov-Witten/Pairs correspondence for toric 3-folds.

Our approach uses a result of Kiem and Li to produce reduced classes. In Appendix A, we answer a number of questions about the relationship between the Kiem-Li approach, traditional virtual cycles, and symmetric obstruction theories.

The interplay between the boundary geometry of the moduli spaces of curves, $K3$ surfaces, and modular forms is explored in Appendix B by A. Pixton.

CONTENTS

INTRODUCTION

0.1. **Stable maps and reduced classes.** Let S be a complex algebraic $K3$ surface, and let $\beta \in H_2(S, \mathbb{Z})$ be a nonzero effective curve class. The moduli space $\overline{M}_g(S, \beta)$ of stable maps from connected genus g curves to S representing β has expected dimension

$$\dim_{\mathbb{C}}^{\mathrm{vir}}(\overline{M}_g(S, \beta)) = \int_\beta c_1(S) + (1 - g)\big(\dim_{\mathbb{C}}(S) - 3\big) = g - 1.$$

However, via the holomorphic symplectic form on S, the standard obstruction theory for $\overline{M}_g(S, \beta)$ admits a trivial quotient. As a result,

$$[\overline{M}_g(S, \beta)]^{\mathrm{vir}} = 0\,.$$

The vanishing reflects the deformation invariance of Gromov-Witten theory: S admits deformations for which β is not of type $(1, 1)$ and thus not represented by holomorphic curves.

A *reduced* obstruction theory, obtained by removing the trivial factor, yields a reduced virtual class[1]

$$[\overline{M}_g(S, \beta)]^{\mathrm{red}} \in A_g(\overline{M}_g(S, \beta), \mathbb{Q})$$

of dimension g. A rich Gromov-Witten theory is obtained by integrating codimension g tautological classes on $\overline{M}_g(S, \beta)$ against $[\overline{M}_g(S, \beta)]^{\mathrm{red}}$. Such integrals are invariant with respect to deformations of S for which the class β remains of type $(1, 1)$.

The class $\beta \in H_2(S, \mathbb{Z})$ is *primitive* if β is not divisible.[2] While the reduced Gromov-Witten theory of S is defined for all β, here we primarily study the primitive case.

0.2. **Hodge classes.** The rank g Hodge bundle,

$$\mathbb{E} \to \overline{M}_g(S, \beta),$$

with fiber $H^0(C, \omega_C)$ over the point $[f : C \to S] \in \overline{M}_g(S, \beta)$, is well defined for all g. The Hodge bundle is pulled back from the moduli space of curves

$$\overline{M}_g(S, \beta) \to \overline{M}_g$$

when g is at least 2. The top Chern class λ_g of $\mathbb{E}$ is the most beautiful and well-behaved integrand in the reduced theory of S. Define

(1) $$R_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g\,.$$

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[1] See [6, 36] for foundational discussions.

[2] Primitive implies nonzero.

Let X be a polarized Calabi-Yau 3-fold which admits a $K3$-fibration,

$$\pi : X \to \mathbb{P}^1 \, .$$

Such a fibration determines a map of the base $\mathbb{P}^1$ to the moduli space of polarized $K3$ surfaces. The integrals $R_{g,\beta}$ precisely relate the Gromov-Witten invariants of X to the intersection numbers of $\mathbb{P}^1$ with Noether-Lefschetz divisors in the moduli of $K3$ surfaces [36].

0.3. **Katz-Klemm-Vafa conjecture.** Let $\beta \in H_2(S, \mathbb{Z})$ be a primitive effective curve class. The Gromov-Witten partition function[3] for β is

$$\mathsf{Z}_{GW,\beta} = \sum_{g=0}^{\infty} R_{g,\beta} \, u^{2g-2} \, .$$

The BPS counts $r_{g,\beta}$ are uniquely defined by

$$\mathsf{Z}_{GW,\beta} = \sum_{g=0}^{\infty} r_{g,\beta} \, u^{2g-2} \left(\frac{\sin(u/2)}{u/2} \right)^{2g-2} \, .$$

By deformation invariance[4], both $R_{g,\beta}$ and $r_{g,\beta}$ depend only upon the norm $\langle \beta, \beta \rangle$. We write $R_{g,h}$ and $r_{g,h}$ for $R_{g,\beta}$ and $r_{g,\beta}$ respectively when $\langle \beta, \beta \rangle = 2h - 2$.

The evaluation of $r_{g,h}$ in terms of modular forms was conjectured by S. Katz, A. Klemm, and C. Vafa [17]. The Fourier expansion of the discriminant modular form is

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \, .$$

Define the series

$$\Delta(y, q) = q \prod_{n=1}^{\infty} (1 - q^n)^{20} (1 - yq^n)^2 (1 - y^{-1}q^n)^2$$

where $\Delta(1, q) = \Delta(q)$. Our first result is the proof of the Katz-Klemm-Vafa conjecture.

Theorem 1. *The invariants $r_{g,h}$ for primitive curve classes are determined by*

$$\sum_{g=0}^{\infty} \sum_{h=0}^{\infty} (-1)^g r_{g,h} \left(\sqrt{y} - \frac{1}{\sqrt{y}} \right)^{2g} q^{h-1} = \frac{1}{\Delta(y, q)} \, .$$

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[3]The partition function only contains connected contributions. The reduced class suppresses contributions from stable maps with disconnected domains.

[4] The moduli space of quasi-polarized $K3$ surfaces with $\langle \beta, \beta \rangle = 2h - 2$ is connected (and, in fact, is a ball quotient), see [9].

By Theorem 1, the invariants $r_{g,h}$ are integers. The formula may also be directly written for the integrals $R_{g,h}$. For $n \geq 1$, let E_{2n} be the Eisenstein series

$$E_{2n}(q) = 1 - \frac{4n}{B_{2n}} \sum_{k \geq 1} \frac{k^{2n-1}q^k}{1 - q^k} \ ,$$

where B_{2n} is the corresponding Bernoulli number.

Corollary 2. *For primitive curve classes,*

$$\sum_{g=0}^{\infty} \sum_{h=0}^{\infty} R_{g,h} \ u^{2g-2}q^{h-1} = \frac{1}{u^2\Delta(q)} \cdot \exp\left(\sum_{g=1}^{\infty} u^{2g} \frac{|B_{2g}|}{g \cdot (2g)!} E_{2g}(q)\right) \ .$$

Theorem 1 specializes in genus 0 to the rational curve counts on $K3$ surfaces predicted by S.-T. Yau and E. Zaslow [53]. The Yau-Zaslow formula was proven for primitive classes in [1, 6].

Of course, the integrals $R_{g,\beta}$ may also be considered in the non-primitive case. A complete conjecture is explained in [36, 45] based on [17]. While the genus 0 integrals $R_{0,\beta}$ have been calculated for all classes β in [21], new methods appear to be required in higher genus.

0.4. **Descendents.** Let $\beta \in H_2(S, \mathbb{Z})$ be a primitive effective curve class. The moduli space of stable maps from connected genus g curves with r ordered marked points $\overline{M}_{g,r}(S, \beta)$ comes with r evaluation maps

$$\mathrm{ev}_i : \overline{M}_{g,r}(S, \beta) \to S \,.$$

Pulling back cohomology classes on S via ev_i gives *primary* classes on $\overline{M}_{g,r}(S, \beta)$. *Descendent* classes are obtained from the Chern classes of the cotangent lines

$$L_i \to \overline{M}_{g,r}(S, \beta)$$

at the marked points.

Let $\gamma_1, \ldots, \gamma_r \in H^*(S, \mathbb{Z})$, and let

$$\psi_i = c_1(L_i) \in H^2(\overline{M}_{g,r}(S, \beta), \mathbb{Q}).$$

The insertion $\tau_k(\gamma)$ corresponds to the class $\psi_i^k \cup \mathrm{ev}_i^*(\gamma)$ on the moduli space of maps. Let

$$(2) \qquad \left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \right\rangle_{g,\beta}^S = \int_{[\overline{M}_{g,r}(S,\beta)]^{\mathrm{red}}} \prod_{i=1}^{r} \psi_i^{k_i} \cup \mathrm{ev}_i^*(\gamma_i)$$

denote the reduced descendent Gromov-Witten invariants. By convention, the descendent vanishes if the degree of the integrand does not match the dimension $g + r$ of the reduced virtual class.

If only descendents of classes in $H^0(S,\mathbb{Z})$ and $H^4(S,\mathbb{Z})$ appear in (2), the bracket for primitive β depends only upon the norm

$$\langle \beta, \beta \rangle = 2h - 2$$

by deformation invariance. Since the classes in $H^2(S,\mathbb{Z})$ are *not* monodromy invariant, the bracket (2) may depend upon β if descendents of $H^2(S,\mathbb{Z})$ are present. When possible, we will replace the subscript β of the descendent bracket by h.

0.5. **Point insertions.** The evaluation of Theorem 1 extends naturally to the integrals

$$\left\langle (-1)^{g-k} \lambda_{g-k} \tau_0(\mathsf{p})^k \right\rangle^S_{g,h} = \int_{[\overline{M}_{g,k}(S,\beta)]^{\mathrm{red}}} (-1)^{g-k} \lambda_{g-k} \prod_{i=1}^{k} \mathrm{ev}_i^*(\mathsf{p}) \ ,$$

where λ_i is the ith Chern class of the Hodge bundle $\mathbb{E}$ and $\mathsf{p} \in H^4(S,\mathbb{Z})$ is the point class.

Theorem 3. *For primitive classes on $K3$ surfaces, we have*

$$\sum_{g=0}^{\infty} \sum_{h=0}^{\infty} \left\langle (-1)^{g-k} \lambda_{g-k} \tau_0(\mathsf{p})^k \right\rangle^S_{g,h} u^{2g-2} q^{h-1} =$$

$$\frac{1}{u^2 \Delta(q)} \cdot \exp \left(\sum_{g=1}^{\infty} u^{2g} \frac{|B_{2g}|}{g(2g)!} E_{2g}(q) \right)$$

$$\cdot \left(\sum_{m=1}^{\infty} q^m \sum_{d|m} \frac{m}{d} \big(2 \sin(du/2) \big)^2 \right)^k .$$

The last factor is related to the point insertions. In the $k = 0$ case, when no points are inserted, Theorem 3 specializes to Theorem 1 by Corollary 2.

0.6. **Quasimodular forms.** The ring of *quasimodular forms* with possible poles at $q = 0$ is the algebra generated by the Eisenstein series E_2 over the ring of $\mathbf{SL}(2,\mathbb{Z})$ modular forms with possible poles at $q = 0$. The ring of quasimodular forms is closed under $q \frac{d}{dq}$. See [5] for a basic treatment.

By deformation invariance, the full descendent theory of algebraic $K3$ surfaces is captured by elliptically fibered $K3$ surfaces. Let S be an elliptically fibered $K3$ surface with section. Let

$$\mathsf{s}, \mathsf{f} \in H_2(S,\mathbb{Z})$$

denote the section and fiber classes. A descendent potential function for the reduced theory of $K3$ surfaces in primitive classes is defined by

$$\mathsf{F}_g^S\big(\tau_{k_1}(\gamma_{l_1})\cdots\tau_{k_r}(\gamma_{l_r})\big) = \sum_{n=0}^{\infty}\Big\langle\tau_{k_1}(\gamma_{l_1})\cdots\tau_{k_r}(\gamma_{l_r})\Big\rangle_{g,\mathbf{s}+h\mathbf{f}}^S q^{h-1}$$

for $g \geq 0$. For arbitrary insertions, we prove the following result.

Theorem 4. $\mathsf{F}_g^S\big(\tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\big)$ *is the Fourier expansion in q of a quasimodular form with pole at $q = 0$ of order at most 1.*

The simplest of the $K3$ series is the count of genus g curves passing through g points,

$$(3) \qquad\qquad \mathsf{F}_g^S(\tau_0(p)^g) = \frac{1}{\Delta(q)}\cdot\left(-\frac{1}{24}\,q\frac{d}{dq}E_2\right)^g,$$

first calculated[5] by J. Bryan and C. Leung [6]. Formula (3) is also a specialization of Theorem 3.

In the non-primitive case, we conjecture the genus g reduced descendent potential to be a quasimodular form of higher level. A precise statement is made in Section 7.5.

0.7. **Stable pairs on $K3$ surfaces.** We will relate the reduced Gromov-Witten invariants of $K3$ surfaces to integrals over the moduli spaces of sheaves on $K3$ surfaces.

Let S be a $K3$ surface. A *pair* (F, s) consists of a sheaf F on S supported in dimension 1 together with a section $s \in H^0(S, F)$. A pair (F, s) is *stable* if

    (i) the sheaf F is pure,
    (ii) the section $\mathcal{O}_S \xrightarrow{s} F$ has 0-dimensional cokernel.

Purity here simply means every nonzero subsheaf of F has support of dimension 1. As a consequence, the scheme theoretic support $C \subset S$ of F is a curve. The discrete invariants of a stable pair are the holomorphic Euler characteristic $\chi(F) \in \mathbb{Z}$ and the class[6] $[F] \in H_2(S, \mathbb{Z})$.

Let $\beta \in H_2(S, \mathbb{Z})$ be a nonzero effective curve class. Let $P_n(S, \beta)$ be the moduli space of stable pairs satisfying

$$\chi(F) = n, \quad [F] = \beta.$$

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[5]Our indexing conventions differ slightly from those adopted in [6].

[6]$[F]$ is the sum of the classes of the irreducible 1-dimensional curves on which F is supported weighted by the generic length of F on the curve. Equivalently, $[F] = c_1(F)$.

After appropriate choices [46], pair stability coincides with stability arising from geometric invariant theory in Le Potier's study [22]. Hence, the moduli space $P_n(S, \beta)$ is a projective scheme.

The class β is *irreducible* if β is not a sum of two nonzero effective curve classes.[7] A basic result proven in [18, 48] is the following.

Proposition 5. *If β is irreducible, $P_n(S, \beta)$ is nonsingular of dimension $n + \langle \beta, \beta \rangle + 1$.*

When studying stable pairs, we will often assume β is irreducible. In the irreducible case, $P_n(S, \beta)$ depends, up to deformation equivalence, only upon the norm of β. We will use the notation $P_n(S, h)$ when $\langle \beta, \beta \rangle = 2h - 2$.

0.8. **Euler characteristic.** Let $\beta \in H_2(S, \mathbb{Z})$ be an irreducible effective curve class with norm $\langle \beta, \beta \rangle = 2h - 2$.

Let Ω_P be the cotangent bundle of the moduli space $P_n(S, h)$. Define the partition function

$$
\begin{aligned}
\mathsf{Z}_h^P(y) &= \sum_n \int_{P_n(S,h)} c_{n+2h-1}(\Omega_P) \, y^n \\
&= \sum_n (-1)^{n+2h-1} \mathsf{e}(P_n(S, h)) \, y^n.
\end{aligned}
$$

Here, e denotes the topological Euler characteristic. We have written the stable pairs partition function in the variable y instead of the traditional q since the latter will be reserved for the Fourier expansions of modular forms.[8] Since $P_n(S, h)$ is empty if $n < 1 - h$, we see Z_h^P is a Laurent series in y.

The topological Euler characteristics of $P_n(S, h)$ have been calculated by T. Kawai and K. Yoshioka. By Theorem 5.80 of [18],

$$
\sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} \mathsf{e}(P_n(S, h)) \, y^n q^{h-1} = \left(\sqrt{y} - \frac{1}{\sqrt{y}} \right)^{-2} \frac{1}{\Delta(y, q)}.
$$

We require the signed Euler characteristics,

$$
\sum_{h=0}^{\infty} \mathsf{Z}_h^P(y) \, q^{h-1} = \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} (-1)^{n+2h-1} \mathsf{e}(P_n(S, h)) \, y^n q^{h-1}.
$$

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[7]An irreducible class is primitive. By deforming S, every primitive curve class $\beta \in H_2(S, \mathbb{Z})$ can be made irreducible.

[8]The conflicting uses of q seem impossible to avoid. The possibilities for confusion are great.

Therefore, $\sum_{h=0}^{\infty} \mathsf{Z}_h^P(y)\, q^{h-1}$ equals

(4)
$$- \left(\sqrt{-y} - \frac{1}{\sqrt{-y}} \right)^{-2} \frac{1}{\Delta(-y, q)} \; .$$

0.9. **Correspondence.** To prove Theorem 1, we formulate and prove a Gromov-Witten/Pairs correspondence in the setting of reduced classes.

Let $\beta \in H_2(S, \mathbb{Z})$ be an irreducible effective curve class. We write the Gromov-Witten partition function as

$$\mathsf{Z}_h^{GW}(u) = \sum_{g=0}^{\infty} r_{g,h}\, u^{2g-2} \left(\frac{\sin(u/2)}{u/2} \right)^{2g-2} \; .$$

Our Gromov-Witten/Pairs correspondence for the reduced theories of the 3-fold $S \times \mathbb{C}$ implies[9]

(5)
$$\mathsf{Z}_h^{GW}(u) = \mathsf{Z}_h^P(y)$$

after the substitution $-e^{iu} = y$. Together with the Euler characteristic calculation (4), the correspondence (5) immediately yields Theorem 1.

To complete the proof of Theorem 1, we must establish the reduced Gromov-Witten/Pairs correspondence for $S \times \mathbb{C}$. There are two main ideas in the argument:

(i) Let R be the rational elliptic surface obtained by blowing-up the base locus of a pencil of cubics in $\mathbb{P}^2$. Let $E \subset R$ be a nonsingular member of the pencil. Using special degenerations of elliptically fibered $K3$ surfaces S to unions of rational elliptic surfaces $R \cup_E R$, we prove a new formula relating the reduced virtual classes of $S \times \mathbb{C}$ to the standard virtual classes of $R \times \mathbb{C}$. We prove the formula separately for stable maps and stable pairs.

(ii) Since R is isomorphic to $\mathbb{P}^2$ blown-up at 9 points, $R \times \mathbb{C}$ is deformation equivalent to a toric 3-fold. We prove a Gromov-Witten/Pairs correspondence for toric 3-folds following [34].

Together (i) and (ii) yield the correspondence (5) and complete the proof of Theorem 1.

We have no direct approach to the λ_g integrals $R_{g,\beta}$ on $\overline{M}_g(S, \beta)$. The moduli space of stable maps has contracted components and subtle virtual contributions. The nonsingularity of the corresponding moduli

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[9]The standard Gromov-Witten/Pairs conjecture of [46] applies to virtual classes for 3-fold theories. Our analogue is for reduced classes (in a $\mathbb{C}^*$-equivariant context). The $\mathbb{C}^*$-equivariant theories of $S \times \mathbb{C}$ are equivalent to the theories of S.
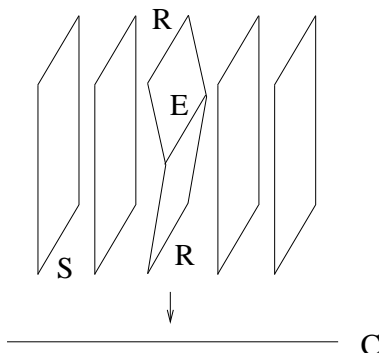
FIGURE 1. A degeneration of $K3$ surfaces

spaces of stable pairs is remarkable. Theorem 1 provides a model use of the Gromov-Witten/Pairs correspondence.

Part (i) constitutes the technical heart of the paper. The primitivity of $\beta \in H_2(S, \mathbb{Z})$ is crucial. In Section 4.6, we state a degeneration formula in the non-primitive case which leads to much more subtle invariants of R. Unfortunately, the toric correspondence (ii) is not sufficient to conclude a Gromov-Witten/Pairs correspondence for non-primitive classes $\beta \in H_2(S, \mathbb{Z})$. The non-primitive degeneration formula will be pursued in a sequel [37].

0.10. **Point insertions for stable pairs.** Let $\beta \in H_2(S, \mathbb{Z})$ be an irreducible effective curve class of norm $\langle \beta, \beta \rangle = 2h - 2$.

The linear system of curves of class β is h-dimensional. Let

$$(6) \qquad \rho : P_n(S, h) \to \mathbb{P}^h$$

be the canonical morphism obtained by sending (F, s) to the support of F. A point incidence condition for stable pairs corresponds to the ρ pullback of a hyperplane $H \subset \mathbb{P}^h$. The integral for stable pairs associated to k point conditions is defined by

$$\mathsf{C}_{n,h}^k = \int_{P_n(S,h)} c_{n+2h-1-k}(\Omega_P) \cup \rho^*(H^k) \, .$$

By Bertini, the subvariety

$$P_n^k(S, h) = \rho^{-1}(H_1) \cap \ldots \cap \rho^{-1}(H_k) \subset P_n(S, h)$$

is nonsingular of dimension $n + 2h - 1 - k$ for generic hyperplanes. Using Gauss-Bonnet, the Euler characteristics of the spaces $P_n^k(S, h)$

are expressible in terms of the integrals $\mathsf{C}_{n,h}^k$ by the formula

$$(7) \qquad \mathsf{e}\left(P_n^k(S,h)\right) = (-1)^{n+2h-1-k} \sum_{i=0}^{n+2h-1-k} (-1)^i \binom{i+k-1}{k-1} \mathsf{C}_{n,h}^{k+i}.$$

In fact, equation (7) may be easily inverted to express $\mathsf{C}_{n,h}^k$ in terms of the Euler characteristics.

Theorem 6. *The point conditions for irreducible classes on $K3$ surfaces are evaluated by*

$$\sum_n \sum_{h=0}^\infty \mathsf{C}_{n,h}^k(-y)^n q^{h-1} =$$

$$\frac{(-1)^{k+1}}{\Delta(y,q)} \cdot \frac{\left(\sum_{m=1}^\infty q^m \sum_{d|m} \frac{m}{d}\left(y^d - 2 + y^{-d}\right)\right)^k}{y - 2 + y^{-1}}.$$

Point conditions in the reduced Gromov-Witten theory of S are evaluated by Theorem 3. We derive Theorem 3 from Theorem 1 using degeneration and exact Gromov-Witten calculations for Hodge integrals. Theorem 3 then implies Theorem 6 by the equivariant Gromov-Witten/Pairs correspondence for $S \times \mathbb{C}$.

We do not know a direct approach along the lines of [18] for determining the integrals $\mathsf{C}_{n,h}^k$ or the Euler characteristics of $P_n^k(S,h)$.

0.11. **Plan of the paper.** We start, in Section 1, with a precise statement of the Gromov-Witten/Pairs correspondence for the reduced theory of $S \times \mathbb{C}$ with primary insertions, leaving many of the proofs for later Sections. Elliptically fibered $K3$ surfaces are reviewed in Section 2. The degeneration formulas in terms of the standard virtual classes of the rational elliptic surface are proven in Section 3 for stable pairs and in Section 4 for Gromov-Witten theory. We give full details for stable pairs and a briefer account for the more standard Gromov-Witten theory.

The Gromov-Witten/Pairs correspondence for toric 3-folds is established in Section 5, completing the proof of Theorem 1. Theorems 3 and 6 are proven in Section 6. The quasimodularity of Theorem 4 is obtained in Section 7 from a boundary induction in the tautological ring of the moduli space of curves using the strong form of Getzler-Ionel vanishing proven in [13].

Our approach uses a result of Kiem-Li [19] to construct reduced classes[10] . In Appendix A, we compare the Kiem-Li method to standard virtual cycle techniques. The inquiry leads naturally to a counterexample to a question of Behrend and Fantechi concerning symmetric obstruction theories that is explained in Section **??**.

In Appendix B, by A. Pixton [50], the interplay between Theorem 1 and boundary expressions for λ_g in low genus are explored.

1. Reduced Gromov-Witten/Pairs correspondence

1.1. **Stable maps.** Let S be a complex projective $K3$ surface, and let $\beta \in H_2(S, \mathbb{Z})$ be a primitive effective curve class. Consider the noncompact Calabi-Yau 3-fold

$$X = S \times \mathbb{C}$$

equipped with the $\mathbb{C}^*$-action defined by scaling the second factor. Let

$$\iota : S \to X$$

denote the inclusion given by the identification $S = S \times \{0\}$.

Let $\overline{M}_g(X, \iota_*\beta)$ be the moduli space of connected genus g stable maps to X representing the class $\iota_*\beta$. Since X is a Calabi-Yau 3-fold, the moduli space has expected dimension 0 with respect to the standard obstruction theory. Since S has a holomorphic symplectic form, $\overline{M}_g(X, \iota_*\beta)$ admits a reduced obstruction theory and reduced virtual class,

$$[\overline{M}_g(X, \iota_*\beta)]^{\mathrm{red}} \in A_1(\overline{M}_g(X, \iota_*\beta), \mathbb{Q}) \,.$$

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[10]A more detailed account of the deformation theory in [19] has appeared very recently [20].

The construction of the reduced theory exactly follows Section 2.2 of [36]. Although $\overline{M}_g(X, \iota_*\beta)$ is not compact, the $\mathbb{C}^*$-fixed locus

$$\overline{M}_g(X, \iota_*\beta)^{\mathbb{C}^*} \subset \overline{M}_g(X, \iota_*\beta)$$

is compact, so we can consider the reduced residue invariants[11]

$$N_{g,\beta} = \int_{[\overline{M}_g(X, \iota_*\beta)]^{\mathrm{red}}} 1 \in \mathbb{Q}(t) \,.$$

Here, t is the first Chern class of the standard representation of $\mathbb{C}^*$ and the generator of $H^*_{\mathbb{C}^*}(\bullet)$, the $\mathbb{C}^*$-equivariant cohomology of a point. The relationship between the residue invariants of $S \times \mathbb{C}$ and the invariants (1) of S is the following.

Lemma 7. $N_{g,\beta} = \frac{1}{t} R_{g,\beta}$.

Proof. The result is a direct consequence of the virtual localization formula of [14],

$$
\begin{aligned}
N_{g,\beta} &= \int_{[\overline{M}_g(X, \iota_*\beta)^{\mathbb{C}^*}]^{\mathrm{red}}} \frac{1}{e(\mathrm{Nor}^{vir})} \\
&= \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} \frac{t^g - \lambda_1 t^{g-1} + \lambda_2 t^{g-2} - \ldots + (-1)^g \lambda_g}{t} \\
&= \frac{1}{t} R_{g,\beta} \,.
\end{aligned}
$$

The first equality is by localization. The denominator on the right is the equivariant Euler class of the virtual normal bundle. Over a stable map $[f\colon C \to S]$, the virtual normal bundle has fiber

$$H^0(C, f^*N) - H^1(C, f^*N) \,,$$

where N is the normal bundle to S in X. Since $N \cong t$, we have

$$\mathrm{Nor}^{vir} \cong t - \mathbb{E}^\vee \otimes t \,,$$

from which the above formula follows. $\qquad\square$

If β is irreducible, then $\overline{M}_g(X, \iota_*(\beta)) = \overline{M}_g(S, \beta) \times \mathbb{C}$ and the reduced virtual class is pulled back from the projection to the first factor. In the irreducible case, Lemma 7 is immediate. An alternative proof of Lemma 7 for primitive β is obtained by deforming to the irreducible case.

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[11]See [7] for a discussion of residue Gromov-Witten theories.

1.2. **Stable pairs.** Let $P_n(X, \iota_*\beta)$ the moduli space of stable pairs (F, s) on $X = S \times \mathbb{C}$ with

$$\chi(F) = n, \quad [F] = \beta.$$

We will construct in Section 3.3 a reduced virtual class in dimension 1,

$$[P_n(X, \iota_*\beta)]^{\mathrm{red}} \in A_1(P_n(X, \iota_*\beta), \mathbb{Q}).$$

Again, we consider the reduced residue invariants

$$P_{n,\beta} = \int_{[P_n(X, \iota_*\beta)]^{\mathrm{red}}} 1 \in \mathbb{Q}(t).$$

By deformation invariance of the reduced theory, the invariant $P_{n,\beta}$ can be computed when β is irreducible.[12] By standard arguments[13]

$$P_n(X, \iota_*\beta) = P_n(S, \beta) \times \mathbb{C}$$

in the irreducible case. By Proposition 5, $P_n(X, \iota_*\beta)$ is nonsingular of dimension $n + \langle \beta, \beta \rangle + 2$. The obstruction bundle of the standard deformation theory [15, 46] of $P_n(X, \iota_*\beta)$ has fiber

$$\mathrm{Ext}^2(I^\bullet, I^\bullet)_0 \cong \mathrm{Ext}^1(I^\bullet, I^\bullet \otimes K_X)_0^*$$

over the moduli point of the pair

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}.$$

Here, K_X is the canonical bundle and the isomorphism is by Serre duality. Since $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ is the tangent space to $P_n(S, \beta) \times \mathbb{C}$, the moduli of stable pairs on X, and K_X^* is trivial with the standard representation, the obstruction bundle is

$$(\Omega_{P_n(S,\beta)} \oplus -t) \otimes t \cong (\Omega_P \otimes t) \oplus \mathbb{C}.$$

The reduced class is obtained by removing the trivial factor $\mathbb{C}$, as we show in Section 3.4.

Lemma 8. $P_{n,\beta} = \frac{1}{t}(-1)^{n+\langle\beta,\beta\rangle+1}\mathsf{e}(P_n(S, \beta))$.

---

[12]A primitive $(1,1)$-class β on a $K3$ surface S can always be deformed through curve classes to an irreducible $(1,1)$-class on another $K3$ surface S'.

[13]The only subtlety is to show the deformations of such pairs on X remain supported scheme-theoretically on the fibers of the projection $X \to \mathbb{C}$. The result follows from the tangent space analysis of Lemma C.7. of [48].

Proof. We calculate the residue of the top Chern class of the reduced obstruction bundle,

$$
\begin{aligned}
P_{n,\beta} &= \int_{P_n(X,\iota_*\beta)} e(\Omega_P \otimes t) \\
&= \int_{P_n(S,\beta)} \frac{e(\Omega_P)}{t} \\
&= \frac{1}{t}(-1)^{n+\langle\beta,\beta\rangle+1} e(P_n(S,\beta)) \,.
\end{aligned}
$$

The second equality comes from localisation. We have omitted all of the terms in $e(\Omega_P \otimes t)$ which do not contribute. $\qquad\square$

1.3. Point insertions.
For both theories of X, we can define reduced residue invariants with point insertions. For Gromov-Witten theory, define[14]

$$
\left\langle \tau_0(\mathsf{p})^k \right\rangle_{g,\beta}^{GW} = \int_{[\overline{M}_{g,k}(X,\iota_*\beta)]^{\mathrm{red}}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\mathsf{p}) \ \in \mathbb{Q}(t)
$$

where the evaluation maps are taken to S

$$
\mathrm{ev}_i : \overline{M}_{g,k}(X, \iota_*(\beta)) \to S
$$

and $\mathsf{p} \in H^4(S, \mathbb{Z})$ is the point class.

For stable pairs, the product $P_n(X, \iota_*\beta) \times X$ is equipped with a universal sheaf $\mathbb{F}$. Define operations

$$
\tau_0(\mathsf{p}) : A_*^{\mathbb{C}^*}(P_n(X, \iota_*\beta)) \to A_*^{\mathbb{C}^*}(P_n(X, \iota_*\beta))
$$

by the slant product

$$
\tau_0(\mathsf{p})(\bullet) = \pi_{P*}\big(\pi_S^*(\mathsf{p}) \cdot \mathrm{ch}_2(\mathbb{F}) \cap \pi_P^*(\bullet)\big) \,,
$$

where π_P and π_S are the projections of $P_n(X, \iota_*\beta) \times X$ to the first factor and to S (via the second factor). Notice that $\mathrm{ch}_2(\mathbb{F})$ is the pull-back via the map ρ of (6) of the universal curve in $S \times \mathbb{P}^h$. Define the residue invariants

$$
\left\langle \tau_0(\mathsf{p})^k \right\rangle_{n,\beta}^{P} = \int_{P_n(X,\iota_*\beta)} \tau_0(\mathsf{p})^k\big([P_n(X, \iota_*\beta)]^{\mathrm{red}}\big) \ \in \mathbb{Q}(t)
$$

following Section 6.1 of [47].

---

[14]We have dropped X from the bracket to simplify the notation.

The calculations of Lemmas 7 and 8 immediately extend to yield the following formulas,

$$
\begin{aligned}
\left\langle \tau_0(\mathsf{p})^k \right\rangle_{g,\beta}^{GW} &= t^{k-1} \left\langle (-1)^{g-k} \lambda_{g-k} \tau_0(\mathsf{p})^k \right\rangle_{g,\beta}^{S}, \\
\left\langle \tau_0(\mathsf{p})^k \right\rangle_{n,\beta}^{P} &= t^{k-1} \int_{P_n(S,\beta)} c_{n+2h-1-k}(\Omega_P) \cup \rho^*(H^k).
\end{aligned}
$$

1.4. **Correspondence.** The reduced Gromov-Witten/Pairs correspondence is stated in terms of the generating series

$$
\begin{aligned}
\mathsf{Z}_\beta^{GW}\left(\tau_0(\mathsf{p})^k, u\right) &= \sum_{g=0}^{\infty} \left\langle \tau_0(\mathsf{p})^k \right\rangle_{g,\beta}^{GW} u^{2g-2}, \\
\mathsf{Z}_\beta^{P}\left(\tau_0(\mathsf{p})^k, y\right) &= \sum_n \left\langle \tau_0(\mathsf{p})^k \right\rangle_{g,\beta}^{P} y^n.
\end{aligned}
$$

The stable pairs series is a Laurent function in y since $P_n(X, \iota_*\beta)$ is empty for sufficiently negative n. The above partition functions specialize to the partition function Z_β^P and Z_β^{GW} of Sections 0.8 and 0.9 when $k = 0$ and $t = 1$.

Theorem 9. *For primitive $\beta \in H_2(S, \mathbb{Z})$,*
  (i) $\mathsf{Z}_\beta^P\left(\tau_0(\mathsf{p})^k\right)$ *is a rational function of y.*
  (ii) *After the variable change $-e^{iu} = y$,*
$$
\mathsf{Z}_\beta^{GW}\left(\tau_0(\mathsf{p})^k\right) = \mathsf{Z}_\beta^P\left(\tau_0(\mathsf{p})^k\right).
$$

Theorem 9 is not a specialization of the Gromov-Witten/Pairs correspondence for 3-folds conjectured in [46]. The main difference is the occurrence of the reduced class. Since the reduced class suppresses contributions from stable maps with disconnected domains, the correspondence here may be viewed here as concerning only *connected* curves. Theorem 9 will be proven in Sections 2-5.

2. Elliptically fibered $K3$ surfaces

2.1. **Elliptic fibrations.** We fix here some notation which will be used throughout the paper. Let S be an elliptically fibered $K3$ surface

$$
(8) \qquad\qquad \pi : S \to \mathbb{P}^1
$$

with a section. We assume π is smooth except for 24 nodal rational fibers. Let

$$
\mathsf{s}, \mathsf{f} \in H_2(S, \mathbb{Z})
$$

denote the classes of the section and the elliptic fiber. The intersection pairings are

$$\langle \mathsf{s}, \mathsf{s} \rangle = -2, \quad \langle \mathsf{s}, \mathsf{f} \rangle = 1, \quad \langle \mathsf{f}, \mathsf{f} \rangle = 0 \,.$$

By deformation invariance, the reduced Gromov-Witten and stable pairs theories for primitive effective classes depend only on the norm $\langle \beta, \beta \rangle = 2h - 2$. By deformation invariance, we can fully capture the both theories for primitive classes on all algebraic $K3$ surfaces by studying

$$\beta = \mathsf{s} + h\mathsf{f}$$

on elliptically fibered $K3$ surfaces S.

2.2. Rational elliptic surface.
A rational elliptic surface R is obtained by blowing-up the 9 points of the base locus of a generic pencil of cubics. The pencil determines a map

$$\pi : R \to \mathbb{P}^1$$

with nonsingular elliptic fibers (except for 12 nodal rational fibers). Let $D \subset R$ be one of the 9 sections of π, and let $E \subset R$ be a fixed elliptic fiber with distinguished point

(9) $p = E \cap D.$

Let R_1 and R_2 be two copies of a rational elliptic surface R. Let $D_1 = D_2$, $E_1 = E_2$, and $p_1 = p_2$ be identical choices of the auxiliary data. A reducible surface

$$R_1 \cup_E R_2$$

is obtained by attaching R_1 and R_2 along the respective fibers E_i (with the corresponding distinguished points p_i identified). The singular surface is elliptically fibered over a broken rational curve,

(10) $R_1 \cup_E R_2 \to \mathbb{P}^1 \cup \mathbb{P}^1 \,.$

The fibration (10) has a distinguished section $D_1 \cup D_2$.

2.3. Degeneration.
The fibration (10) is a degeneration of (8). More precisely, there exists a family of fibrations

(11) $\mathcal{S} \xrightarrow{\pi} \mathcal{C} \to B$

over a pointed curve $(B, 0)$ with the following properties:

  (i) $\mathcal{S}$ is a nonsingular 3-fold, and $\mathcal{C}$ is a nonsingular surface.
 (ii) π has a section.
(iii) When specialized to nonzero $\xi \in B$, we obtain a nonsingular elliptically fibered $K3$ surface of the form (8).

(iv) When specialized to $0 \in B$, we obtain (10).

(v) The relative canonical bundle $\omega_{\mathcal{S}/B}$ is trivial.

Denote by ϵ the degenerating family of $K3$ surfaces obtained from composing (11),

$$\epsilon : \mathcal{S} \to B \,.$$

Since the section and fiber classes are globally defined by (ii), the sublattice of $H^2(\mathcal{S}_\xi, \mathbb{Z})$ spanned by s and f is fixed by the monodromy of ϵ around $0 \in B$.

The degenerating family (11) will play an essential role in our proof of Theorem 9.

3. Reduced stable pairs

3.1. **Definitions.** Let S be a complex algebraic $K3$ surface, and let $\beta \in H_2(S, \mathbb{Z})$ be an effective curve class. Let

$$X = S \times \mathbb{C}.$$

We include S as the fiber over $0 \in \mathbb{C}$,

$$\iota \colon S \hookrightarrow X \,.$$

Let $P_n(X, \iota_* \beta)$ be the quasi-projective moduli space of stable pairs (F, s) on X with holomorphic Euler characteristic and class

$$\chi(F) = n, \qquad [F] = \iota_* \beta \in H_2(X, \mathbb{Z}) \,.$$

Strictly speaking, to construct $P_n(X, \iota_*(\beta))$, we apply Le Potier's results [22] to the projective 3-fold

$$\overline{X} = S \times \mathbb{P}^1$$

to obtain a projective moduli space containing $P_n(X, \iota_* \beta)$ as an open subscheme.

We can also consider stable pairs on families of $K3$ surfaces. Let

$$\epsilon \colon \mathcal{S} \to B$$

be a smooth[15] family of $K3$ surfaces, and let

$$\mathcal{X} = \mathcal{S} \times \mathbb{C} \to B$$

be the corresponding family of 3-folds. We consider $\mathcal{S}$ as a subvariety via the inclusion

$$\iota \colon \mathcal{S} \times \{0\} \hookrightarrow \mathcal{S} \times \mathbb{C} = \mathcal{X}.$$

Let β be a section of the local system with fiber $H_2(\mathcal{S}_\xi, \mathbb{Z})$ over $\xi \in B$.

---

[15]We will later consider families degenerating to $R_1 \cup_E R_2$ as in Section 2.3.

By making B smaller if necessary, we can choose a holomorphic 2-form which is symplectic on every fiber of $\mathcal{S} \to B$,

$$\sigma \in H^0(\epsilon_* \Omega^2_{\mathcal{S}/B}). \tag{12}$$

In particular, $\omega_{\mathcal{X}/B}$ is trivial. By [22], there is a family of moduli spaces

$$\mathcal{P} \to B$$

representing the functor taking B-schemes $A \to B$ to the set of flat families of stable pairs in the class $\iota_* \beta$ on the fibers of

$$A \times_B \mathcal{X} \to A.$$

In addition, there is a universal sheaf $\mathbb{F}$ on $\mathcal{P} \times_B \mathcal{X}$, flat over $\mathcal{P}$, with a global section $\mathbb{S}$, such that the restriction of

$$\mathcal{O}_{\mathcal{P} \times_B \mathcal{X}} \xrightarrow{\mathbb{S}} \mathbb{F} \tag{13}$$

to the fiber over any closed point $(F, s) \in \mathcal{P}$ over $\xi \in B$ is the corresponding stable pair $\mathcal{O}_{\mathcal{X}_\xi} \xrightarrow{s} F$.

3.2. **Standard obstruction theory.** As in [46], given a stable pair (F, s) on X, we let $I^\bullet \in D^b(X)$ denote the complex of sheaves

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}$$

in degrees 0 and 1. When the section is onto, $I^\bullet$ is quasi-isomorphic to the kernel $\mathscr{I}_C$, the ideal sheaf of the Cohen-Macaulay curve C which is the scheme theoretical support of F. Similarly we let

$$\mathbb{I}^\bullet = \{\mathcal{O}_{\mathcal{P} \times_B \mathcal{X}} \xrightarrow{\mathbb{S}} \mathbb{F}\}$$

denote the universal complex.

From the perspective of [46], the trace-free Ext groups,

$$\mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \quad \text{and} \quad \mathrm{Ext}^2(I^\bullet, I^\bullet)_0$$

provide deformation and obstruction spaces for the stable pair (F, s). More generally, let $\mathbb{L}^\vee_{\mathcal{P}/B}$ denote the derived dual of the truncated relative cotangent complex of $\mathcal{P}$, and consider the map

$$\mathbb{L}^\vee_{\mathcal{P}/B} \longrightarrow R\pi_{\mathcal{P}*}(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0)[1], \tag{14}$$

given by the image of the relative Atiyah class of $\mathbb{I}^\bullet$ under the projection

$$\mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \mathbb{L}_{(\mathcal{P} \times_B \mathcal{X})/B}) \longrightarrow \mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \pi_{\mathcal{P}}^* \mathbb{L}_{\mathcal{P}/B})_0$$
$$= \mathrm{Hom}(\pi_{\mathcal{P}}^* \mathbb{L}^\vee_{\mathcal{P}/B}, R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1])$$
$$= \mathrm{Hom}(\mathbb{L}^\vee_{\mathcal{P}/B}, R\pi_{\mathcal{P}*}R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]).$$

Here $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{X}}$ are the projections from $\mathcal{P} \times_B \mathcal{X}$ to $\mathcal{P}$ and $\mathcal{X}$ respectively.

Proposition 10. *The map* (14) *is a perfect theory for the morphism* $\mathcal{P} \to B$.

Proof. The result is proved in [46, Section 2.3] and [15, Theorem 4.1] for *projective* morphisms $\pi_{\mathcal{P}}$. Since the fibers of our $\pi_{\mathcal{P}}$ are noncompact, we need a small modification to check that the complexes

$$(15) \qquad R\pi_{\mathcal{P}*}(R\mathscr{H}om(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})_0 \otimes \omega_{\pi_{\mathcal{P}}})[2] \ , \quad R\pi_{\mathcal{P}*}(R\mathscr{H}om(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})_0)[1]$$

are still naturally dual to each other. The relative canonical bundle

$$\omega_{\pi_{\mathcal{P}}} = \pi_{\mathcal{X}}^* \omega_{\mathcal{X}/B}$$

is trivial in our situation. The rest of the proofs in [15, 46] go through as before.

The proper way to deal with the noncompactness is to work with local cohomology in place of $R\pi_{\mathcal{P}*}$. However, we follow a simpler approach obtained by compactifying the fibers of $\mathcal{X} \to B$ by

$$\overline{\mathcal{X}} = \mathcal{S} \times \mathbb{P}^1 \to B \, .$$

Pairs extend trivially over $\overline{\mathcal{X}} \backslash \mathcal{X}$ by pushing forward the sheaf and section, allowing us to view $\mathcal{P}$ as the moduli space of stable pairs on the fibers of $\overline{\mathcal{X}} \to B$ whose underlying sheaf has support in $\mathcal{X} \subset \overline{\mathcal{X}}$. Suppressing the pushforward maps, we get a universal pair $(\mathbb{F}, \mathbb{S})$ on $\mathcal{P} \times_B \overline{\mathcal{X}}$ and a universal complex

$$\overline{\mathbb{I}}^{\bullet} = \{\mathcal{O}_{\mathcal{P} \times_B \overline{\mathcal{X}}} \xrightarrow{\mathbb{S}} \mathbb{F}\}$$

whose restriction to $\mathcal{P} \times_B \mathcal{X}$ is $\mathbb{I}^{\bullet}$.

Let $\overline{\pi}_{\mathcal{P}}$ denote the projection $\mathcal{P} \times_B \overline{\mathcal{X}} \to \mathcal{P}$. Since $R\mathscr{H}om(\overline{\mathbb{I}}^{\bullet}, \overline{\mathbb{I}}^{\bullet})_0$ is supported on $\mathcal{X} \subset \overline{\mathcal{X}}$, the two complexes (15) are

$$(16) \quad R\overline{\pi}_{\mathcal{P}*}(R\mathscr{H}om(\overline{\mathbb{I}}^{\bullet}, \overline{\mathbb{I}}^{\bullet})_0 \otimes \omega_{\overline{\pi}_{\mathcal{P}}})[2] \ , \quad R\overline{\pi}_{\mathcal{P}*}(R\mathscr{H}om(\overline{\mathbb{I}}^{\bullet}, \overline{\mathbb{I}}^{\bullet})_0)[1].$$

Therefore the usual relative Serre duality down the projective fibers of $\overline{\pi}_{\mathcal{P}}$ applies to give the duality (15). In particular, by [46, Lemma 2.10], the first complex is quasi-isomorphic to a complex

$$E^{\bullet} = \{E^{-1} \to E^0\} \simeq R\pi_{\mathcal{P}*}(R\mathscr{H}om(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})_0 \otimes \pi_{\mathcal{X}}^* \omega_{\mathcal{X}/B})[2]$$

of locally free sheaves on $\mathcal{P}$ in degrees -1 and 0. We denote the second, the dual of the first, by

$$E_{\bullet} = \{E_0 \to E_1\}$$

in degrees 0 and 1. Dualising (14), we obtain the more familiar form,

$$(17) \qquad\qquad E^\bullet \longrightarrow \mathbb{L}_{\mathcal{P}/B}.$$

As in [15, 46], the morphism (17) is surjective on h^{-1} and an isomorphism on h^0, verifying the axioms [4] of a perfect relative obstruction theory. $\qquad\square$

3.3. **Trivial quotient.** For the 3-fold $X = S \times \mathbb{C}$, the obstruction theory constructed in Section 3.2 has virtual cycle equal to 0 because of the existence of a trivial factor $\mathbb{C}$ obstructing extensions of $I^\bullet$ along deformations of S which take β out of the $(1,1)$ locus. To construct a nonzero virtual cycle, we must remove this trivial piece of the obstruction theory.

The obstruction sheaf of the deformation-obstruction theory (14) is the degree 1 cohomology sheaf[16]

$$(18) \qquad\qquad \mathrm{Ob} = \mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 .$$

As in (16), we also have

$$\mathrm{Ob} = \mathscr{E}xt^2_{\overline{\pi}_{\mathcal{P}}}(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet)_0 .$$

Consider the image of the relative Atiyah class of $\overline{\mathbb{I}}^\bullet$ under the map

$$\mathrm{Ext}^1(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet \otimes \mathbb{L}_{(\mathcal{P} \times_B \overline{\mathcal{X}})/B}) \longrightarrow \mathrm{Ext}^1(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet \otimes \pi^*_{\overline{\mathcal{X}}} \mathbb{L}_{\overline{\mathcal{X}}/B})_0$$
$$\longrightarrow H^0(\mathscr{E}xt^1_{\overline{\pi}_{\mathcal{P}}}(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet \otimes \pi^*_{\overline{\mathcal{X}}} \Omega_{\overline{\mathcal{X}}/B})).$$

Cup product with its image At defines the map

$$\mathscr{E}xt^2_{\overline{\pi}_{\mathcal{P}}}(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet)_0 \xrightarrow{\ \cup \mathrm{At}\ } \mathscr{E}xt^3_{\overline{\pi}_{\mathcal{P}}}(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet \otimes \pi^*_{\overline{\mathcal{X}}} \Omega_{\overline{\mathcal{X}}/B}) \xrightarrow{\ \mathrm{tr}\ } R^3 \overline{\pi}_{\mathcal{P}*}(\pi^*_{\overline{\mathcal{X}}} \Omega_{\overline{\mathcal{X}}/B}).$$

Pulling back the fiberwise symplectic form σ of (12) to $\mathcal{P} \times_B \overline{\mathcal{X}} \to B$ gives a section $\bar{\sigma}$ of $\pi_{\mathcal{P}*}(\pi^*_{\overline{\mathcal{X}}} \Omega^2_{\overline{\mathcal{X}}/B})$. Wedging with $\bar{\sigma}$, the upshot is a map from (18) to $\mathcal{O}_{\mathcal{P}}$:

$$(19) \qquad \mathrm{Ob} \longrightarrow R^3 \overline{\pi}_{\mathcal{P}*}(\pi^*_{\overline{\mathcal{X}}} \Omega^3_{\overline{\mathcal{X}}/B}) = R^3 \overline{\pi}_{\mathcal{P}*} \omega_{\overline{\pi}_{\mathcal{P}}} = \mathcal{O}_{\mathcal{P}}.$$

Proposition 11. *The map (19) is onto.*

Proof. Since the higher $\mathscr{E}xt^i_{\overline{\pi}_{\mathcal{P}}}(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet)_0$ sheaves on B vanish for $i \geq 3$, we can work at closed points. For a stable pair (F, s) on

$$\overline{\mathcal{X}}_\xi = \overline{X} = S \times \mathbb{P}^1 ,$$

---

[16]Here $\mathscr{E}xt^i_{\overline{\pi}_{\mathcal{P}}}$ denotes the ith cohomology sheaf of $R\pi_{\mathcal{P}*}R\mathscr{H}om$. We abbreviate the latter to $R\mathscr{H}om_{\pi_P}$, the derived functor of $\mathscr{H}om_{\pi_{\mathcal{P}}} = \pi_{\mathcal{P}*}\mathscr{H}om$.

we must show the composition

$$(20) \quad \mathrm{Ext}^2(\overline{I}^\bullet, \overline{I}^\bullet)_0 \xrightarrow{\cup \mathrm{At}(\overline{I}^\bullet)} \mathrm{Ext}^3(\overline{I}^\bullet, \overline{I}^\bullet \otimes \Omega_{\overline{X}}) \xrightarrow{\mathrm{tr}}$$

$$H^3(\Omega_{\overline{X}}) \xrightarrow{\cup \bar{\sigma}} H^{3,3}(\overline{X}) \xrightarrow{\sim} \mathbb{C}$$

is onto. Here, $\bar{\sigma}$ is the pull-back of the holomorphic symplectic form σ (12) from the $K3$ surface S to $\overline{X}$, and $\overline{I}^\bullet$ is the complex $\{\mathcal{O}_{\overline{X}} \to F\}$ on $\overline{X}$.

To show the map (20) is surjective, we exhibit a class in $\mathrm{Ext}^2(\overline{I}^\bullet, \overline{I}^\bullet)_0$ on which the composition is nonzero. Choose a first order deformation $\kappa_S \in H^1(T_S)$ of S which, via the holomorphic symplectic form

$$\sigma \colon T_S \cong \Omega_S \ ,$$

corresponds to a class $\kappa_S \lrcorner \sigma \in H^{1,1}(S)$ whose pairing with β is nonzero,

$$(21) \qquad \int_\beta \kappa_S \lrcorner \sigma \neq 0.$$

Let $\bar{\kappa} \in H^1(T_{\overline{X}})$ denote the pull-back of the Kodaira-Spencer class κ_S to $\overline{X}$. Let

$$(22) \qquad \bar{\kappa} \circ \mathrm{At}(\overline{I}^\bullet) \ \in \ \mathrm{Ext}^2(\overline{I}^\bullet, \overline{I}^\bullet)$$

be the cup product of $\bar{\kappa}$ with $\mathrm{At}(\overline{I}^\bullet) \in \mathrm{Ext}^1(\overline{I}^\bullet, \overline{I}^\bullet \otimes \Omega_{\overline{X}})$ followed by the contraction of $T_{\overline{X}}$ with $\Omega_{\overline{X}}$. By [15], the element (22) is the obstruction to deforming $\overline{I}^\bullet$ to first order with the deformation κ of $\overline{X}$, and in fact lies in $\mathrm{Ext}^2(\overline{I}^\bullet, \overline{I}^\bullet)_0 \subset \mathrm{Ext}^2(\overline{I}^\bullet, \overline{I}^\bullet)$ since the determinant $\mathcal{O}_{\overline{X}}$ of $\overline{I}^\bullet$ deforms trivially.

By [8, Proposition 4.2], $\mathrm{tr}\left(\bar{\kappa} \circ \mathrm{At}(\overline{I}^\bullet) \circ \mathrm{At}(\overline{I}^\bullet)\right) \in H^3(\Omega_{\overline{X}})$ equals $2\bar{\kappa} \lrcorner \mathrm{ch}_2(\overline{I}^\bullet)$. Therefore the image of $\bar{\kappa} \circ \mathrm{At}(\overline{I}^\bullet)$ under the map (20) is

$$(23) \qquad 2 \int_{\overline{X}} (\bar{\kappa} \lrcorner \mathrm{ch}_2(\overline{I}^\bullet)) \wedge \bar{\sigma} \ = \ -2 \int_{\overline{X}} (\bar{\kappa} \lrcorner \bar{\sigma}) \wedge \mathrm{ch}_2(\overline{I}^\bullet),$$

by the homotopy formula

$$(24) \qquad 0 \ = \ \bar{\kappa} \lrcorner (\mathrm{ch}_2 \wedge \bar{\sigma}) \ = \ (\bar{\kappa} \lrcorner \mathrm{ch}_2) \wedge \bar{\sigma} + (\bar{\kappa} \lrcorner \bar{\sigma}) \wedge \mathrm{ch}_2.$$

Since $\mathrm{ch}_2(\overline{I}^\bullet)$ is Poincaré dual to $-\iota_* \beta$, we conclude (23) equals

$$2 \int_\beta \kappa_S \lrcorner \sigma,$$

which by construction (21) is nonzero. $\qquad \square$

3.4. **Symmetric obstruction theories.** By Proposition 10, the two term complex of locally free sheaves $E^\bullet$ over $\mathcal{P}$, quasi-isomorphic to

$$R\pi_{\mathcal{P}*}(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_{\mathcal{X}}^*\omega_{\mathcal{X}/B})[2] \ ,$$

provides a perfect obstruction theory for $\mathcal{P} \to B$. Via the trivialisation of $\omega_{\mathcal{X}/B}$ and the equality between (15) and (16), the Serre duality of (16) shows that $E^\bullet$ is isomorphic to its own derived dual shifted by [1],

(25) $$(E^\bullet)^\vee[1] \cong E^\bullet.$$

Moreover the pairing between $E^\bullet$ and $E^\bullet[1]$ is given by trace, which satisfies

$$\mathrm{tr}(a \cup b) = \mathrm{tr}(b \cup a) \ .$$

Hence, the isomorphism (25) is also equal to its own dual [3, Lemma 1.23], and the deformation-obstruction theory of Proposition 10 is *symmetric* in the sense of [2, 3].

Since $E^\bullet$ is an obstruction theory, $h^0(E^\bullet) = \Omega_{\mathcal{P}/B}$. By definition, the obstruction sheaf Ob is h^1 of the dual complex $E_\bullet = (E^\bullet)^\vee$. Since $E^\bullet$ is symmetric, we have

$$\mathrm{Ob} = \Omega_{\mathcal{P}/B}.$$

A map $\mathrm{Ob} \to \mathcal{O}_{\mathcal{P}}$ is therefore equivalent to a section of the tangent sheaf

(26) $$T_{\mathcal{P}/B} = \mathscr{H}om(\Omega_{\mathcal{P}/B}, \mathcal{O}_{\mathcal{P}}) = h^{-1}(E^\bullet).$$

In our case, the product geometry provides a section of $T_{\mathcal{P}/B}$ by moving all stable pairs by the vector field ∂_t lifted from the second factor $\mathbb{C}$ of

$$\mathcal{X} = \mathcal{S} \times \mathbb{C} \,.$$

Explicitly, the section is

(27) $$\mathcal{O}_{\mathcal{P}} \xrightarrow{\ \partial_t \lrcorner \mathrm{At}(\mathbb{I}^\bullet)\ } \mathscr{E}xt^1_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$$

where the sheaf on the right is the 0th cohomology of $E^\bullet$. By the duality (15) and the vanishing of higher $\mathscr{E}xt$s,

$$\mathscr{E}xt^1_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \ = \ \mathscr{H}om(\mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0, \mathcal{O}_{\mathcal{P}}),$$

just as in (26). Therefore (27) gives

(28) $$\mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \to \mathcal{O}_{\mathcal{P}}.$$

Lemma 12. *The map* (28) *is the same as* (19).

Proof. The perfect pairing between the two complexes (15) is provided by composition of the derived Homs in (16) followed by the trace,

$$(29) \quad R\overline{\pi}_{\mathcal{P}*}(R\mathscr{H}om(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet) \otimes \omega_{\overline{\pi}_{\mathcal{P}}})[3] \xrightarrow{\text{tr}} R\overline{\pi}_{\mathcal{P}*}\omega_{\overline{\pi}_{\mathcal{P}}}[3] \longrightarrow R^3\overline{\pi}_{\mathcal{P}*}\omega_{\overline{\pi}_{\mathcal{P}}}.$$

The final map takes the highest nonvanishing cohomology sheaf of the complex. Since the fibers of $\overline{\pi}_{\mathcal{P}}$ are projective, last sheaf is $\mathcal{O}_{\mathcal{P}}$, as required.

To prove the Lemma, we must show sections f of $\mathscr{E}xt^2_{\overline{\pi}_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ satisfy

$$\text{tr}(f \cup \text{At}(\mathbb{I}^\bullet)) \wedge \sigma \; = \; \text{tr}(f \cup (\partial_t \lrcorner \text{At}(\mathbb{I}^\bullet))) \wedge (\sigma \wedge dt),$$

where $\sigma \wedge dt$ is the trivialisation of $\omega_{\mathcal{X}}$. The result follows from the homotopy formula $a \lrcorner (b \wedge c) = (a \lrcorner b) \wedge c \pm b \wedge (a \lrcorner c)$ used before. $\square$

In particular, since the map (27) is clearly pointwise injective for pairs with curve class β (supported in the $K3$ fibers of our threefolds), we recover Proposition 11.

3.5. **Reduced obstruction theory.** We now assume that our smooth family of $K3$ surfaces

$$\epsilon \colon \mathcal{S} \to B$$

has base a nonsingular curve B, and that β is of type $(1,1)$ on every fiber $\mathcal{S}_\xi$, $\xi \in B$.

Following the notation of Section 3.1, let

$$(30) \qquad\qquad \mathcal{X} = \mathcal{S} \times \mathbb{C} \to B$$

be a family of 3-folds, and let

$$\mathcal{P} \to B$$

be the associated family of moduli spaces of stable pairs in class $\iota_*\beta$ on the fibers of (30). We will construct and prove the deformation invariance of the reduced virtual class on the family $\mathcal{P} \to B$.

Since B is nonsingular, every perfect obstruction theory

$$E^\bullet \to \mathbb{L}_{\mathcal{P}/B}$$

for $\mathcal{P} \to B$ induces a perfect absolute obstruction theory for $\mathcal{P}$ by virtue of the exact triangle

$$(31) \qquad\qquad \Omega_B \to \mathbb{L}_{\mathcal{P}} \to \mathbb{L}_{\mathcal{P}/B},$$

where we have suppressed a pull-back map. Using the composition $E^\bullet \to \mathbb{L}_{\mathcal{P}/B} \to \Omega_B[1]$, we define

$$\mathcal{E}^\bullet = \text{Cone}\left(E^\bullet \to \Omega_B[1]\right)[-1].$$

We have diagram of exact triangles

(32)
$$
\begin{array}{ccccc}
\mathcal{E}^\bullet & \longrightarrow & E^\bullet & \longrightarrow & \Omega_B[1] \\
\downarrow & & \downarrow & & \| \\
\mathbb{L}_{\mathcal{P}} & \longrightarrow & \mathbb{L}_{\mathcal{P}/B} & \longrightarrow & \Omega_B[1] \, .
\end{array}
$$

By computation with the long exact sequence in cohomology sheaves of the above diagram, $\mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{P}}$ is an isomorphism on h^0 and surjective on h^{-1}. Since

$$ E^\bullet = \{E^{-1} \to E^0\} $$

is a complex of locally free sheaves, the induced map $E^\bullet \to \Omega_B[1]$ can be represented *locally* by a genuine map of complexes

$$
\begin{array}{ccc}
\{E^{-1} & \longrightarrow & E^0\} \\
\downarrow & & \\
\Omega_B \, . & &
\end{array}
$$

Hence, $\mathcal{E}^\bullet$ is locally represented by the 2-term complex of locally free sheaves

(33) $$ \mathcal{E}^\bullet \cong \{E^{-1} \to E^0 \oplus \Omega_B\}. $$

Since $\mathcal{P}$ is quasi-projective, $\mathcal{E}^\bullet$ is *globally* a 2-term complex of locally free sheaves. So $\mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{P}}$ is indeed a perfect obstruction theory.

From the perfect relative obstruction theory of Proposition 10,

$$ E^\bullet = \big(R\mathscr{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1] \big)^\vee, $$

we obtain a perfect absolute obstruction theory $\mathcal{E}^\bullet$ for $\mathcal{P}$. From the dual of the top row of (32) we have the long exact sequence of cohomology sheaves

$$ 0 \to \mathscr{E}xt^1_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \to T_{\mathcal{P}} \to T_B \to \mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \to \mathrm{Ob}_{\mathcal{P}} \to 0. $$

Here $\mathscr{E}xt^1_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 = T_{\mathcal{P}/B}$ is the relative tangent sheaf of P/B, and $T_{\mathcal{P}}$ is the absolute tangent sheaf. Similarly, $\mathrm{Ob}_{\mathcal{P}}$ is the obstruction sheaf $h^1((\mathcal{E}^\bullet)^\vee)$ of the absolute obstruction theory $\mathcal{E}^\bullet$, the quotient of the relative obstruction sheaf

$$ \mathrm{Ob} = \mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 $$

by the image of T_B.

By Proposition 11, the map (19) is a surjection $\mathrm{Ob} \to \mathcal{O}_{\mathcal{P}}$. To apply the construction of Kiem-Li to $\mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{P}}$, we must show the map (19)

annihilates T_B and so descends to a surjection $\mathrm{Ob}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$. To do so we need a description of the composition $E^\bullet \to \mathbb{L}_{\mathcal{P}/B} \to \Omega_B[1]$.

By the description (14) of $(\mathbb{L}_{\mathcal{P}/B})^\vee \to (E^\bullet)^\vee$, the dual of

$$E^\bullet \to \mathbb{L}_{\mathcal{P}/B} \to \Omega_B[1]$$

is the composition

$$(34) \qquad T_B[-1] \longrightarrow \mathbb{L}^\vee_{\mathcal{P}/B} \xrightarrow{\mathrm{At}_{\mathcal{P}/B}} R\mathscr{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)[1],$$

which actually factors through the trace-free part $R\mathscr{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]$ as in [15, Theorem 4.1] since all of the complexes $\mathbb{I}^\bullet$ have fixed trivial determinant. Here, the first map is the Kodaira-Spencer class of the fibers of $\mathcal{P} \to B$ obtained from the exact triangle (31). The second is cup product with the image $\mathrm{At}_{\mathcal{P}/B}$ of the relative Atiyah class $\mathrm{At}_{\mathcal{X}\times_B\mathcal{P}}$ of $\mathbb{I}^\bullet$ under the map $\mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \mathbb{L}_{\mathcal{X}\times_B\mathcal{P}}) \to \mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \pi_{\mathcal{P}}^*\mathbb{L}_{\mathcal{P}/B})$.

We can construct a similar composition

$$(35) \qquad T_B[-1] \longrightarrow R\pi_{\mathcal{P}*}\mathbb{L}^\vee_{\mathcal{X}/B} \xrightarrow{\mathrm{At}_{\mathcal{X}/B}} R\mathscr{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)[1]$$

from the projection $\mathrm{At}_{\mathcal{X}/B}$ of $\mathrm{At}_{\mathcal{X}\times_B\mathcal{P}}$ under $\mathbb{L}_{\mathcal{X}\times_B\mathcal{P}} \to \mathbb{L}_{\mathcal{X}/B}$. The first map of (35) is the Kodaira-Spencer class of the fibers of $\mathcal{X} \to B$ obtained from $(\pi_{\mathcal{P}*}\pi_{\mathcal{X}}^*$ applied to) the exact triangle

$$(36) \qquad \Omega_B \to \mathbb{L}_{\mathcal{X}} \to \mathbb{L}_{\mathcal{X}/B} \,.$$

Proposition 13. *The two composition* (34) *and* (35) *coincide.*

Proof. We relate both maps to a third, constructed in the same way using the full Atiyah class

$$(37) \qquad T_B[-1] \longrightarrow R\pi_{\mathcal{P}*}\mathbb{L}^\vee_{\mathcal{X}\times_B\mathcal{P}} \xrightarrow{\mathrm{At}_{\mathcal{X}\times_B\mathcal{P}}} R\mathscr{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)[1] \,.$$

Here, the first map is the Kodaira-Spencer class in $\mathrm{Ext}^1(\mathbb{L}_{\mathcal{X}\times_B\mathcal{P}}, \Omega_B)$ of the inclusion

$$i\colon \mathcal{X} \times_B \mathcal{P} \subset \mathcal{X} \times \mathcal{P}$$

coming from the exact triangle

$$(38) \qquad \Omega_B \to Li^*\mathbb{L}_{\mathcal{X}\times\mathcal{P}} \to \mathbb{L}_{\mathcal{X}\times_B\mathcal{P}} \,.$$

The composition (37) coincides with (34) by the following commutative diagram of exact triangles on $\mathcal{X} \times_B \mathcal{P}$ relating the Kodaira-Spencer

classes (38) and (31),

$$
\begin{array}{ccc}
\mathbb{L}_{\mathcal{X}} & == & \mathbb{L}_{\mathcal{X}} \\
\downarrow & & \downarrow \\
\Omega_B \longrightarrow Li^*\mathbb{L}_{\mathcal{X}\times\mathcal{P}} \longrightarrow \mathbb{L}_{\mathcal{X}\times_B\mathcal{P}} \\
\| & \downarrow & \downarrow \\
\Omega_B \longrightarrow \mathbb{L}_{\mathcal{P}} \longrightarrow \mathbb{L}_{\mathcal{P}/B}.
\end{array}
$$

We have suppressed several pull-back maps. The central row gives rise to (37) while the bottom row induces (34).

Similarly we have the following diagram relating the Kodaira-Spencer classes (38) and (36),

$$
\begin{array}{ccc}
\mathbb{L}_{\mathcal{P}} & == & \mathbb{L}_{\mathcal{P}} \\
\downarrow & & \downarrow \\
\Omega_B \longrightarrow Li^*\mathbb{L}_{\mathcal{X}\times\mathcal{P}} \longrightarrow \mathbb{L}_{\mathcal{X}\times_B\mathcal{P}} \\
\| & \downarrow & \downarrow \\
\Omega_B \longrightarrow \mathbb{L}_{\mathcal{X}} \longrightarrow \mathbb{L}_{\mathcal{X}/B}.
\end{array}
$$

The central row gives rise to (37) while the bottom row induces (35), so the two compositions coincide. $\square$

By Proposition 13, we may use the description (35) of the map

$$
T_B \to \mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0
$$

to compute the composition with

$$
\mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \to \mathcal{O}_{\mathcal{P}}.
$$

As in Proposition 11, we use the extension $\overline{\mathbb{I}}^\bullet$ of $\mathbb{I}^\bullet$ over $\overline{\mathcal{X}} \supset \mathcal{X}$. The result is

$$
T_B \longrightarrow R^1\overline{\pi}_{\mathcal{P}*}(\mathbb{L}^\vee_{\overline{\mathcal{X}}/B}) \xrightarrow{\mathrm{At}_{\overline{\mathcal{X}}/B}} \mathscr{E}xt^2_{\overline{\pi}_{\mathcal{P}}}(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet)_0 \xrightarrow{\mathrm{At}_{\overline{\mathcal{X}}/B}}
$$

$$
(39) \quad \mathscr{E}xt^3_{\overline{\pi}_{\mathcal{P}}}(\overline{\mathbb{I}}^\bullet, \overline{\mathbb{I}}^\bullet \otimes \pi_{\overline{\mathcal{X}}}^*\Omega_{\overline{\mathcal{X}}/B}) \xrightarrow{\mathrm{tr}} R^3\overline{\pi}_{\mathcal{P}*}(\pi_{\overline{\mathcal{X}}}^*\Omega_{\overline{\mathcal{X}}/B}) \xrightarrow{\wedge\bar{\sigma}} R^3\overline{\pi}_{\mathcal{P}*}(\omega_{\overline{\pi}_{\mathcal{P}}}).
$$

Working locally over $\mathcal{P}$, we will show the composition vanishes when applied to any section of T_B. Let

$$
\mathsf{KS} \in R^1\overline{\pi}_{\mathcal{P}*}(T_{\overline{\mathcal{X}}/B})
$$

denote the associated Kodaira-Spencer class. By [8, Proposition 4.2] (applied to $\mathcal{N} = \mathcal{O}_{\mathcal{P}}$), the image of our section under the first four maps above can be computed as

$$
(40) \quad\quad\quad \mathsf{KS} \lrcorner \, \mathrm{ch}_2(\overline{\mathbb{I}}^\bullet) \in R^3\overline{\pi}_{\mathcal{P}*}(\pi_{\overline{\mathcal{X}}}^*\Omega_{\overline{\mathcal{X}}/B}).
$$

The class (40) is the (1,3)-part of the derivative down the vector field along B of $\iota_*(\beta)$. By assumption the class $\iota_*\beta$ is of pure type (2,2) over all of B, so the class (40) is zero. We have proven the following result.

Proposition 14. *We have a surjection* $\mathrm{Ob}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$ *extending the surjection* $\mathrm{Ob} \to \mathcal{O}_{\mathcal{P}}$ *of* (19).

3.6. **Reduced classes.** From the perfect absolute obstruction theory $\mathcal{E}^\bullet$, the constructions of [4, 27] produce a normal cone C and an embedding

$$C \subset \mathcal{E}_1$$

into the total space of the vector bundle $\mathcal{E}_1$ dual to $\mathcal{E}^{-1}$. Without loss of generality we may assume that $\mathcal{E}_1 \cong E_1$ (as in (33)), so

$$C \subset E_1 \,.$$

Restricting (17) to a fiber

$$\iota_\xi \colon \mathcal{P}_\xi \hookrightarrow \mathcal{P}$$

over $\xi \in B$ yields the perfect obstruction theory

$$E^\bullet|_{\mathcal{P}_\xi} \longrightarrow \mathbb{L}_{\mathcal{P}_\xi},$$

a normal cone C_ξ, and an embedding $C_\xi \subset E_1|_{\mathcal{P}_\xi}$. By [4], the cone C specialises to the cone C_ξ,

(41) $$[C_\xi] = \iota_\xi^![C].$$

The cone C_ξ lies on C and relation (41) is valid on C. Intersecting C_ξ with the zero section of $E_1|_{\mathcal{P}_\xi}$ yields the usual virtual cycle $[\mathcal{P}_\xi]^{vir}$ employed in [46] (and which vanishes here).

A reduced virtual class is obtained by the following construction. Let

$$F_1 \subset \mathcal{E}_1$$

on $\mathcal{P}$ denote the locally free kernel of the surjective composition (19)

$$\mathcal{E}_1 \to \mathrm{Ob}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}.$$

By results of Kiem and Li [19], the normal cone $C \subset \mathcal{E}_1$ lies in $F_1 \subset \mathcal{E}_1$ *as a cycle* (rather than scheme theoretically[17]). Therefore we may view C as a cycle in F_1 and C_ξ as a cycle in $F_1|_{\mathcal{P}_\xi}$. We define

$$[\mathcal{P}]^{\mathrm{red}} = 0^![C] \in A_2(\mathcal{P}, \mathbb{Z})$$

---

[17]In Appendix A we explain why, for our particular moduli space $\mathcal{P}$ and obstruction theory $\mathcal{E}^\bullet$, replacing $\mathcal{E}_1$ by F_1 gives a genuine perfect obstruction theory. Therefore C lies in F_1 scheme theoretically. This is not the case for general obstruction theories, however, and is not necessary for what follows.

and by intersecting C with the zero section 0 of F_1 and

$$[\mathcal{P}_\xi]^{\mathrm{red}} = 0^!_\xi[C_\xi] \ \in A_1(\mathcal{P}_\xi, \mathbb{Z})$$

by intersecting C_ξ with the zero section 0_ξ of $F_1|_{\mathcal{P}_\xi} \subset E_1|_{\mathcal{P}_\xi}$. The deformation invariance of $[\mathcal{P}_\xi]^{\mathrm{red}}$ is a consequence of the equation

$$\iota^!_\xi[\mathcal{P}]^{\mathrm{red}} = [\mathcal{P}_\xi]^{\mathrm{red}},$$

obtained by the identity (41) on C.

3.7. Reduced invariants.

Since $X = S \times \mathbb{C}$ is not compact, neither is the moduli space $P_n(X, \iota_*\beta)$. However, the fixed point set

$$P_n(X, \beta)^{\mathbb{C}^*} \subset P_n(X, \beta)$$

of the $\mathbb{C}^*$-action induced by scaling the second factor of X is compact. We can therefore define invariants by residues.

The $\mathbb{C}^*$-action on $P_n(X, \beta)$ lifts to the perfect obstruction theory (17). Since the map (19) is easily seen to be $\mathbb{C}^*$-invariant, we obtain a $\mathbb{C}^*$-equivariant reduced virtual class. We define

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{red}}} 1 \,,$$

where the right side is the $\mathbb{C}^*$-equivariant residue.

By Lemma 8, the integral can be evaluated as

$$P_{n,\beta} = \frac{1}{t}(-1)^{n+\langle\beta,\beta\rangle+1}\mathsf{e}(P_n(S,\beta))$$

when $\beta \in H_2(S, \mathbb{Z})$ is irreducible.

3.8. Degenerating family of $K3$ surfaces.

We now consider the family of $K3$ surfaces

$$\epsilon : \mathcal{S} \to B$$

over a pointed curve $(B, 0)$ defined in Section 2.3. The family ϵ satisfies conditions (i-v) of Section 2.3 and has special fiber

$$\mathcal{S}_0 = R_1 \cup_E R_2 \,.$$

As before, let

(42) $\mathcal{X} = \mathcal{S} \times \mathbb{C} \to B \,.$

Denote the special fiber by $X[0] = \mathcal{S}_0 \times \mathbb{C}$, and let

$$X[0] = Y_1 \cup_{E \times \mathbb{C}} Y_2$$

denote the decomposition where $Y_i = R_i \times \mathbb{C}$.

Following the notation of [28], let $\mathscr{B} = \mathscr{B}(\beta, n)$ denote the Artin stack of (β, n)-decorated semistable models of $\mathcal{X}/B$, with the associated universal family

$$\text{(43)} \qquad\qquad \widetilde{\mathcal{X}} \to \mathscr{B} \,.$$

The stack $\mathscr{B}$ has a (non-representable) morphism to B with the fiber over $0 \in B$ denoted by $\mathscr{B}_0$. Away from $\mathscr{B}_0$, the universal family (43) is just the family of quasi-projective schemes

$$(\mathcal{X} \backslash X[0]) \to B \backslash \{0\} \,.$$

Replacing the special fiber $X[0]$ is the union over all k of the k-step semistable models

$$X[k] = \left(R_1 \cup_E (E \times \mathbb{P}^1) \cup_E \ldots \cup_E (E \times \mathbb{P}^1) \cup_E R_2 \right) \times \mathbb{C}$$

with automorphisms $(\mathbb{C}^*)^k$ covering the identity on $X[0]$ (k is the number of extra components $(E \times \mathbb{P}^1) \times \mathbb{C}$ in the semistable model). The decoration is an assignment of H_2 classes and integers for each component of the fibers of $\widetilde{\mathcal{X}}/\mathscr{B}$, satisfying standard gluing and continuity conditions described in [28]. In particular, on the nonsingular fibers, the decoration is simply (β, n).

A relative stable pair on the special fiber is

$$\text{(44)} \qquad\qquad \mathcal{O}_{X[k]} \xrightarrow{s} F$$

where F is a sheaf on $X[k]$ with holomorphic Euler characteristic $\chi(F) = n$ and class which pushes down to

$$\iota_* \beta \in H_2(X[0], \mathbb{Z}).$$

The stability conditions for the pair are

(i) F is pure with finite locally free resolution,
(ii) F is transverse to the singular loci E_i of $X[k]$,

$$\text{Tor}_j(F, \mathcal{O}_{E_i}) = 0 \quad \text{for all } i \text{ and } j > 1 \,,$$

(iii) the section s has 0-dimensional cokernel supported away from the singular loci E_i of $X[k]$, and
(iv) the pair (44) has only finitely many automorphisms covering the automorphisms $(\mathbb{C}^*)^k$ of $X[k]/X[0]$,

see [28, 46].

There is a Deligne-Mumford moduli stack

$$\mathcal{P} \to \mathscr{B}$$

of stable pairs on the fibers of $\widetilde{\mathcal{X}} \to \mathscr{B}$ whose restriction to each component of $X[k]$ has support and holomorphic Euler characteristic equal to the decoration. There is universal complex $\mathbb{I}^\bullet$ over

$$\widetilde{\mathcal{X}} \times_{\mathscr{B}} \mathcal{P}$$

by condition (iv). Composing with $\mathscr{B} \to B$ gives $\mathcal{P} \to B$ which, away from the special fiber, is the quasi-projective moduli space studied in Sections 3.1-3.4.

As before, $\mathcal{P}$ is an open subset of a *proper* Deligne-Mumford stack formed by considering relative stable pairs on the compactification $\overline{\mathcal{X}} \to \mathscr{B}$ given by replacing the $\mathbb{C}$ factor in (42) by $\mathbb{P}^1$ everywhere.

The universal complex $\mathbb{I}^\bullet$ is *perfect* due to condition (i) above. The deformation theory and Serre duality of [15, 46] go through exactly as before. Let $\pi_{\widetilde{\mathcal{X}}}, \pi_{\mathcal{P}}$ denote the projections from $\widetilde{\mathcal{X}} \times_{\mathscr{B}} \mathcal{P}$ to $\widetilde{\mathcal{X}}$ and $\mathcal{P}$ respectively. Just as in (14), the Atiyah class of $\mathbb{I}^\bullet$ gives a perfect obstruction theory

$$(45) \qquad E^\bullet = R\pi_{\mathcal{P}*}(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \pi_{\widetilde{\mathcal{X}}}^* \omega_{\widetilde{\mathcal{X}}/\mathscr{B}})[2] \to \mathbb{L}_{\mathcal{P}/\mathscr{B}}.$$

Moreover, Serre duality applies to give the duality (25):

$$(E^\bullet)^\vee[1] \cong E^\bullet.$$

Therefore the relative obstruction sheaf over $\mathcal{P}/\mathscr{B}$ is $\mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ with a map

$$(46) \qquad\qquad\qquad \mathscr{E}xt^2_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \to \mathcal{O}_{\mathcal{P}}$$

defined exactly as in (28). The map coincides, as before, with the map defined[18] in (19). While the proof of Proposition 11 is valid with the right notion of Chern classes for perfect complexes on $X[k]$, the dual description (28) is technically easier. Since the vector field ∂_t is nowhere zero on $\mathcal{P}$, the map (46) is again a surjection.

3.9. **Degeneration of the reduced class.** Let

$$(47) \qquad\qquad\qquad\qquad \mathcal{X} \to B$$

be the degenerating family of 3-folds considered in Section 3.8 above. Let $\beta = \mathsf{s} + h\mathsf{f}$ be a vertical curve class. Let

$$\mathcal{P} \to \mathscr{B}$$

---

[18]Since $\widetilde{\mathcal{X}}/\mathscr{B}$ is a reduced local complete intersection morphism, $\mathbb{L}_{\widetilde{\mathcal{X}}/\mathscr{B}} = \Omega_{\widetilde{\mathcal{X}}/\mathscr{B}}$.

be the moduli space of stable pairs on the fibers of (47) with holomorphic Euler characteristic n and class β. Let $\mathcal{P}_0$ be the special fiber of the composition

$$\mathcal{P} \to \mathscr{B} \to B$$

parameterizing stable pairs on semistable degenerations $X[k]$ of

$$X[0] = Y_1 \cup_{E \times \mathbb{C}} Y_2\,.$$

Given data $\eta = (n_1, n_2, h_1, h_2)$ defining a splitting

$$h = h_1 + h_2, \quad n + 1 = n_1 + n_2,$$

we can construct the moduli spaces $\mathcal{P}_{\eta_1}$ and $\mathcal{P}_{\eta_2}$ of *relative* stable pairs on Y_1 and Y_2 of classes $(\mathsf{s} + h_1\mathsf{f}, n_1)$ and $(\mathsf{s} + h_2\mathsf{f}, n_2)$ respectively. By restriction of relative stable pairs to the boundary divisor, $\mathcal{P}_{\eta_i}$ maps to $E \times \mathbb{C}$. We define

$$(48) \qquad \mathcal{P}_\eta = \mathcal{P}_{\eta_1} \times_{E \times \mathbb{C}} \mathcal{P}_{\eta_2}$$

which embeds into $\mathcal{P}_0$. In fact, $\mathcal{P}_0$ the union (not disjoint!) of the $\mathcal{P}_\eta$ over all possible splitting types η.

The perfect obstruction theory $E^\bullet \to \mathbb{L}_{\mathcal{P}/\mathscr{B}}$ (45) for $\mathcal{P} \to \mathscr{B}$ fits into the following commutative diagram of exact triangles:

$$(49) \qquad \begin{array}{ccccc} \mathcal{E}^\bullet & \longrightarrow & E^\bullet & \longrightarrow & \mathbb{L}_{\mathscr{B}}[1] \\ \downarrow & & \downarrow & & \| \\ \mathbb{L}_{\mathcal{P}} & \longrightarrow & \mathbb{L}_{\mathcal{P}/\mathscr{B}} & \longrightarrow & \mathbb{L}_{\mathscr{B}}[1]. \end{array}$$

The bottom row induces the map $E^\bullet \to \mathbb{L}_{\mathscr{B}}[1]$ whose cone we define to be $\mathcal{E}^\bullet[1]$.

Since $\mathscr{B}$ is nonsingular, $h^{-1}(\mathbb{L}_{\mathscr{B}}) = 0$. But $\mathscr{B}$ has nontrivial isotropy groups, so $h^1(\mathbb{L}_{\mathscr{B}})$ is nonzero. However, stable pairs have no continuous automorphisms (since $\mathcal{P}$ is a Deligne-Mumford stack with no continuous stabilizers by condition (iv)) so

$$h^0(\mathbb{L}_{\mathcal{P}/\mathscr{B}}) \to h^1(\mathbb{L}_{\mathscr{B}})$$

is onto. From the long exact sequences in cohomology of the above diagram, $\mathcal{E}^\bullet$ has cohomology only in degrees -1 and 0. Just as in (32), we conclude $\mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{P}}$ is a perfect absolute obstruction theory for $\mathcal{P}$.

Restriction to $\mathcal{P}_0 \subset \mathcal{P}$ yields $E^\bullet|_{\mathcal{P}_0} \to \mathbb{L}_{\mathcal{P}/\mathscr{B}}|_{\mathcal{P}_0}$ which we can compose with $\mathbb{L}_{\mathcal{P}/\mathscr{B}}|_{\mathcal{P}_0} \to \mathbb{L}_{\mathcal{P}_0/\mathscr{B}_0}$ to give a perfect obstruction theory

$$E^\bullet|_{\mathcal{P}_0} \longrightarrow \mathbb{L}_{\mathcal{P}_0/\mathscr{B}_0}$$

for $\mathcal{P}_0 \to \mathscr{B}_0$. Just as in (49), we can construct a perfect absolute obstruction theory for $\mathcal{P}_0$ via the diagram

(50)
$$
\begin{array}{ccccc}
\mathcal{E}_0^\bullet & \longrightarrow & E^\bullet|_{\mathcal{P}_0} & \longrightarrow & \mathbb{L}_{\mathscr{B}_0}[1] \\
\downarrow & & \downarrow & & \| \\
\mathbb{L}_{\mathcal{P}_0} & \longrightarrow & \mathbb{L}_{\mathcal{P}_0/\mathscr{B}_0} & \longrightarrow & \mathbb{L}_{\mathscr{B}_0}[1],
\end{array}
$$

which defines $\mathcal{E}_0^\bullet$ and the map to $\mathbb{L}_{\mathcal{P}_0}$.

The top row of (50) and the pull-back to $\mathcal{P}_0$ of (49) give the diagram of exact triangles

$$
\begin{array}{ccccccc}
& & & & \mathbb{L}_{\mathscr{B}_0/\mathscr{B}} & = & \mathbb{L}_{\mathscr{B}_0/\mathscr{B}} \\
& & & & \downarrow & & \downarrow \\
\mathcal{E}^\bullet|_{\mathcal{P}_0} & \longrightarrow & E^\bullet|_{\mathcal{P}_0} & \longrightarrow & \mathbb{L}_{\mathscr{B}}\big|_{\mathscr{B}_0}[1] & \longrightarrow & \mathcal{E}^\bullet|_{\mathcal{P}_0}[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\mathcal{E}_0^\bullet & \longrightarrow & E^\bullet|_{\mathcal{P}_0} & \longrightarrow & \mathbb{L}_{\mathscr{B}_0}[1] & \longrightarrow & \mathcal{E}_0^\bullet[1].
\end{array}
$$

Since $\mathscr{B}_0 \subset \mathscr{B}$ is the pull-back from B of the divisor $\{0\} \subset B$ with associated line bundle L_0, we have $\mathbb{L}_{\mathscr{B}_0/\mathscr{B}} \cong L_0^\vee[1]$. Therefore, the rightmost column of the above diagram gives the exact triangle

(51)
$$
\mathcal{E}^\bullet|_{\mathcal{P}_0} \longrightarrow \mathcal{E}_0^\bullet \longrightarrow L_0^\vee[1],
$$

relating the obstruction theories of $\mathcal{P}$ (pulled back to $\mathcal{P}_0$) and $\mathcal{P}_0$.

There is a divisor $\mathscr{B}_\eta \subset \mathscr{B}$ in the stack of decorated semistable models whose pull-back to $\mathcal{P}$ is $\mathcal{P}_\eta$. The associated line bundles L_η satisfy

$$
\bigotimes_\eta L_\eta = L_0.
$$

We can replace $\mathcal{P}_0 \subset \mathcal{P}$ over $\mathscr{B}_0 \subset \mathscr{B}$ by $\mathcal{P}_\eta \subset \mathcal{P}$ over $\mathscr{B}_\eta \subset \mathscr{B}$ and L_0 by L_η in the above diagrams. The result is a perfect obstruction theory $\mathcal{E}_\eta^\bullet \to \mathbb{L}_{\mathcal{P}_\eta}$ sitting in an exact triangle:

(52)
$$
\mathcal{E}^\bullet|_{\mathcal{P}_\eta} \longrightarrow \mathcal{E}_\eta^\bullet \longrightarrow L_\eta^\vee[1].
$$

The map $\mathcal{O}_\mathcal{P}[1] \to E^\bullet$ of (28), extended to singular $K3$s in (46), was shown in Section 3.5 to lift to the absolute obstruction theory

(53)
$$
\mathcal{O}_\mathcal{P}[1] \to \mathcal{E}^\bullet
$$

over the nonsingular locus $B \setminus \{0\}$. Since $\mathbb{I}^\bullet$ is a perfect complex the same proof extends to the singular $K3$s, as far as (39). To finish off we

must show that the composition (39) is zero. By the usual homotopy formula (24) we compute the composition as

$$-2(\mathsf{KS}\,\lrcorner\,\bar{\sigma})\cup\mathrm{ch}_2(\overline{\mathbb{I}}^{\bullet})\ \in\ R^3\overline{\pi}_{\mathcal{P}*}(\omega_{\overline{\pi}_{\mathcal{P}}})\ \cong\ \mathcal{O}_{\mathcal{P}}.$$

Even in the singular geometry, the above cup product equals the integral of $2\mathsf{KS}\,\lrcorner\,\bar{\sigma}$ over the class $\iota_*\beta$. The integral vanishes since β is always of type $(1,1)$ in the family B. Hence, the lift (53) extends over all of $\mathscr{B}$.

Let $\mathcal{E}^{\mathrm{red}}$ be the cone of (53). Restricting (53) to $\mathcal{P}_0$ and $\mathcal{P}_\eta$ and using (51, 52) yields the compositions

$$\mathcal{O}_{\mathcal{P}_0}[1]\to\mathcal{E}^{\bullet}|_{\mathcal{P}_0}\to\mathcal{E}_0^{\bullet},$$
$$\mathcal{O}_{\mathcal{P}_\eta}[1]\to\mathcal{E}^{\bullet}|_{\mathcal{P}_\eta}\to\mathcal{E}_\eta^{\bullet}.$$

Taking the cones defines the respective reduced theories $\mathcal{E}_0^{\mathrm{red}}$ and $\mathcal{E}_\eta^{\mathrm{red}}$. By (51) and (52), we obtain the exact triangles

$$(54) \qquad \mathcal{E}^{\mathrm{red}}|_{\mathcal{P}_0}\longrightarrow\mathcal{E}_0^{\mathrm{red}}\longrightarrow L_0^{\vee}[1]\ ,$$

$$(55) \qquad \mathcal{E}^{\mathrm{red}}|_{\mathcal{P}_\eta}\longrightarrow\mathcal{E}_\eta^{\mathrm{red}}\longrightarrow L_\eta^{\vee}[1]\,.$$

We have now worked out the compatibilities of the reduced obstruction theories for $\mathcal{P}$, $\mathcal{P}_\eta$, and $\mathcal{P}_0$. We now turn to the compatibility between the reduced obstruction theory of $\mathcal{P}_\eta$ and the usual obstruction theories of $\mathcal{P}_{\eta_1}$ and $\mathcal{P}_{\eta_2}$.

Consider a point $[I^{\bullet}]\in\mathcal{P}_\eta$ of the moduli space corresponding to a stable pair on $X[k]$. A decomposition of $X[k]$ as $X_1\cup_{E\times\mathbb{C}}X_2$ yields

$$(56) \qquad 0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_{X_1}\oplus\mathcal{O}_{X_2}\xrightarrow{(1,-1)}\mathcal{O}_{E\times\mathbb{C}}\longrightarrow 0.$$

The stable pair $I^{\bullet}$ restricts to stable pairs $I_1^{\bullet}$ and $I_2^{\bullet}$ over X_1 and X_2 respectively. On $E\times\mathbb{C}$, $I^{\bullet}$ restricts to the ideal sheaf $\mathscr{I}_p$ of a point in the intersection $(\mathsf{s}\cap E)\times\mathbb{C}$. Tensoring (56) with the perfect complex $R\mathscr{H}om(I^{\bullet},I^{\bullet})_0$ and taking sheaf cohomology gives the exact triangle on the bottom row of the following main diagram,

$$\begin{array}{ccccc}
\mathbb{L}_{\mathcal{P}_\eta/\mathscr{B}_\eta}^{\vee}[-1] & \longrightarrow & \mathbb{L}_{\mathcal{P}_{\eta_1}/\mathscr{B}_{\eta_1}\times\mathcal{P}_{\eta_2}/\mathscr{B}_{\eta_2}}^{\vee}[-1] & \longrightarrow & \mathbb{L}_{\mathcal{P}_\eta/(\mathcal{P}_{\eta_1}\times\mathcal{P}_{\eta_2})}^{\vee} \\
\downarrow & & \downarrow & & \downarrow \\
R\mathrm{Hom}_{X[k]}(I^{\bullet},I^{\bullet})_0 & \longrightarrow & \bigoplus_{i=1}^{2}R\mathrm{Hom}_{X_i}(I_i^{\bullet},I_i^{\bullet})_0 & \longrightarrow & R\mathrm{Hom}_{E\times\mathbb{C}}(\mathscr{I}_p,\mathscr{I}_p)_0.
\end{array}$$

Here, $\mathscr{B}_{\eta_i}$ denotes the stack of expanded degenerations of $(Y_i,E\times\mathbb{C})$ decorated by η_i, so

$$\mathscr{B}_\eta=\mathscr{B}_{\eta_1}\times\mathscr{B}_{\eta_2}\,.$$

The top row is the exact triangle of dual cotangent complexes for the fiber product structure (48) relative to $\mathscr{B}_\eta$ (all restricted to the point $I^\bullet \in \mathcal{P}$). The vertical arrows are the dual perfect obstruction theories provided by (14).

The last term of the top row has first cohomology sheaf $(J/J^2)^\vee$ at p, where J is the ideal of

$$\mathcal{P}_\eta \subset \mathcal{P}_{\eta_1} \times \mathcal{P}_{\eta_2} .$$

Since $\mathcal{P}_\eta$ is the basechange of $\mathcal{P}_{\eta_1} \times \mathcal{P}_{\eta_2}$ to the diagonal in $(E \times \mathbb{C})^2$, the conormal bundle to the diagonal in $(E \times \mathbb{C})^2$ surjects onto J/J^2. The right hand vertical arrow is the dual of this surjection (at p). The normal bundle to the diagonal,

$$T_p(E \times \mathbb{C}) \cong R\operatorname{Hom}_{E \times \mathbb{C}}(\mathscr{I}_p, \mathscr{I}_p)_0[1] \ ,$$

is identified with its image in $T_{(p,p)}\big((E \times \mathbb{C}) \times (E \times \mathbb{C})\big)$ by the map $(1, -1)$. Comparing with the map $(1, -1)$ in (56) shows the diagram is commutative.

After composing the coboundary map of the exact triangle occurring on the bottom row of the main diagram with the map (28)

$$\operatorname{Ext}^2(I^\bullet, I^\bullet)_0 \longrightarrow \mathbb{C},$$

we obtain the following morphism

(57) $T_p E \hookrightarrow T_p(E \times \mathbb{C}) \longrightarrow R\operatorname{Hom}(I^\bullet, I^\bullet)_0[2] \xrightarrow{h^2}$
$$\operatorname{Ext}^2(I^\bullet, I^\bullet)_0 \longrightarrow \mathbb{C}.$$

Proposition 15. *The composition* (57) *is an isomorphism.*

Proof. The result is straightforward using the dual description (27)

(58) $$\mathbb{C} \xrightarrow{\partial_t} \operatorname{Ext}^1(I^\bullet, I^\bullet)_0$$

of the map (28). The map is obtained from the deformation of $I^\bullet$ given by translation in the $\mathbb{C}$-direction.

The dual of the sequence (56) is

(59) $$0 \to \omega_{X_1} \oplus \omega_{X_2} \to \omega_X \to \mathcal{O}_{E \times \mathbb{C}} \to 0,$$

where of course $\omega_X \cong \mathcal{O}_X$.

Tensoring with the perfect complex $R\mathscr{H}om(I^\bullet, I^\bullet)_0$ and taking sheaf cohomology gives the exact triangle

$$\bigoplus_{i=1}^{2} R\operatorname{Hom}_{X_i}(I_i^\bullet, I_i^\bullet \otimes \omega_{X_i})_0 \to R\operatorname{Hom}_X(I^\bullet, I^\bullet)_0 \to R\operatorname{Hom}_{E \times \mathbb{C}}(\mathscr{I}_p, \mathscr{I}_p)_0.$$

Since the complex of sheaves $R\mathcal{H}om(I^\bullet, I^\bullet)_0$ is derived dual to itself (modulo a shift), this exact triangle is the Serre dual of the bottom row of the main diagram modulo a shift.

By the construction of the triangle using (59), we see the deformation ∂_t in (H^1 of) the second term maps to the corresponding deformation ∂_t in the third term. Hence, the composition of (58) and the second map in the exact triangle,

$$(60) \quad \mathbb{C} \xrightarrow{\partial_t} \mathrm{Ext}_X^1(I^\bullet, I^\bullet)_0 \to \mathrm{Ext}_{E\times\mathbb{C}}^1(\mathscr{I}_p, \mathscr{I}_p)_0 = T_p(E \times \mathbb{C}) \to T_p\mathbb{C},$$

is an isomorphism.

Under Serre duality, the splitting $T(E\times\mathbb{C}) \cong TE \oplus T\mathbb{C}$ is dual to the splitting $T(E \times \mathbb{C}) \cong T\mathbb{C} \oplus TE$ in the opposite order, since the pairing between the two spaces is by wedging (and the triviality of $\Lambda^2 T(E\times\mathbb{C})$). Therefore, (60) is precisely Serre dual to the composition (57). $\qquad\square$

After passing to absolute perfect obstruction theories, the dual of the main diagram yields

$$(61) \qquad \begin{array}{ccccc}
\mathcal{E}_1^\bullet \oplus \mathcal{E}_2^\bullet & \longrightarrow & \mathcal{E}_\eta^\bullet & \longrightarrow & \Omega_{E\times\mathbb{C}}[1] \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}_{\mathcal{P}_{\eta_1}\times\mathcal{P}_{\eta_2}} & \longrightarrow & \mathbb{L}_{\mathcal{P}_\eta} & \longrightarrow & \mathbb{L}_{\mathcal{P}_\eta/(\mathcal{P}_{\eta_1}\times\mathcal{P}_{\eta_2})} \,,
\end{array}$$

at the point $[I^\bullet] \in \mathcal{P}_\eta$. Here $\mathcal{E}_i^\bullet$ is the absolute obstruction theory for $\mathcal{P}_{\eta_i}$ derived in the usual way (49) from the relative obstruction theory $\mathbb{L}_{\mathcal{P}_{\eta_1}/\mathscr{B}_{\eta_1}}^\vee \to R\mathrm{Hom}_{X_i}(I_i^\bullet, I_i^\bullet)_0[1]$ of (14).

We have already shown the map (27) factors through the absolute obstruction theory,

$$(62) \qquad\qquad\qquad \mathbb{C}[1] \to \mathcal{E}_\eta^\bullet.$$

By Proposition 15, the composition of (62) and $\mathcal{E}_\eta^\bullet \to \Omega_{E\times\mathbb{C}}[1]$ is isomorphic to the inclusion $\Omega_E \hookrightarrow \Omega_{E\times\mathbb{C}}$ (all shifted by [1]). Since the latter is nontrivial, we can divide out the two top right hand terms of (61) by $\mathbb{C} \cong \Omega_E$ to give

$$(63) \qquad\qquad\qquad \mathcal{E}_1^\bullet \oplus \mathcal{E}_2^\bullet \to \mathcal{E}_\eta^{\mathrm{red}} \to N^\vee[1],$$

where $N^\vee$ is the conormal bundle of the diagonal $\mathbb{C} \hookrightarrow \mathbb{C} \times \mathbb{C}$.

Theorem 16. *The reduced virtual class of the moduli space of pairs $\mathcal{P}$ of the degeneration*

$$\epsilon : \mathcal{X} \to B$$

with holomorphic Euler characteristic n and primitive class

$$\beta = \mathsf{s} + h\mathsf{f}$$

satisfies three basic properties:

(i) *For all $\xi \in B$,*

$$\iota_\xi^![\mathcal{P}]^{\mathrm{red}} = [P_n(\mathcal{X}_\xi, \beta)]^{\mathrm{red}}.$$

(ii) *For the special fiber,*

$$[\mathcal{P}_0]^{\mathrm{red}} = \sum_\eta \iota_{\eta*}[P_\eta]^{\mathrm{red}}.$$

(iii) *The factorizations*

$$[P_n(\mathcal{X}_0, \eta)]^{\mathrm{red}} = [P_{n_1}((R_1/E) \times \mathbb{C}, \beta_1)]^{\mathrm{vir}} \times_\mathbb{C} [P_{n_2}((R_2/E) \times \mathbb{C}, \beta_2)]^{\mathrm{vir}}$$

hold for $\beta_i = \mathsf{s} + h_i\mathsf{f}$.

Remark. By the fiber product in (iii) we really mean

$$\Delta^! \left([P_{n_1}((R_1/E) \times \mathbb{C}, \beta_1)]^{\mathrm{vir}} \times [P_{n_2}((R_2/E) \times \mathbb{C}, \beta_2)]^{\mathrm{vir}}\right),$$

where $\Delta\colon \mathbb{C} \to \mathbb{C}^2$ is the diagonal and both virtual cycles have obvious maps to $\mathbb{C}$. However, since all of the cycles involved can be taken to be linear combinations of products of varieties with $\mathbb{C}$, this agrees with the resulting linear combination of fiber products over $\mathbb{C}$.

Proof. This follows the proof of the parallel statements in [28] for the degeneration formula for the standard obstruction theory. We use the compatibilities (54) and (55) of reduced obstruction theories in place of the standard compatibilities (51) and (52) of usual obstruction theories. The statement (iii) is immediate from the exact triangle (63). $\qquad\square$

Finally, we note all steps in the proof of Theorem 16 respect the $\mathbb{C}^*$-action on

$$\mathcal{X} = \mathcal{S} \times \mathbb{C}$$

given by scaling of the second factor. As a result, Theorem 16 holds for the reduced and ordinary virtual classes in $\mathbb{C}^*$-equivariant cycle theory.

4. Reduced Gromov-Witten

4.1. **Stable maps to the fibers of ϵ.** We again work with the family $\epsilon\colon \mathcal{S} \to B$ of Section 2.3. Denote the moduli space of connected stable maps to the fibers of ϵ, as constructed in [25, 26], by

(64) $$\overline{M}_g(\epsilon, \beta) \to B.$$

Over nonzero $\xi \in B$, the moduli space is simply $\overline{M}_g(\mathcal{S}_\xi, \beta)$. Over $0 \in B$, the moduli space parameterizes stable predeformable maps from a genus g curve to an expanded target degeneration of $\mathcal{S}_0 = R_1 \cup_E R_2$ of the form

$$R_1 \cup_E (E \times \mathbb{P}^1) \cup_E \cdots \cup_E (E \times \mathbb{P}^1) \cup_E R_2.$$

Here, we have inserted a non-negative number of copies of $E \times \mathbb{P}^1$ at the singular locus of $\mathcal{S}_0$, attached at $E \times \{0\}$ and $E \times \{\infty\}$. We will denote the standard inclusion of the fiber over $\xi \in B$ by

$$\iota_\xi : \overline{M}_g(\mathcal{S}_\xi, \beta) \hookrightarrow \overline{M}_g(\epsilon, \beta).$$

If $\beta = \mathsf{s} + h\mathsf{f}$, then the moduli space (64) has a simple structure. A stable map to any fiber $\mathcal{S}_\xi$ in class β has degree 1 over the base of the elliptic fibration

$$\pi : \mathcal{S}_\xi \to \mathbb{P}^1 \,.$$

Since the section s is rigid, the map consists of a fixed genus 0 curve mapping isomorphically to s attached to possibly higher genus curves mapping to the fibers of π. When $\xi = 0$, the intersection of the genus 0 curve with the singular locus of the expanded degeneration always has multiplicity 1 at the distinguished point $p \in E$, see (9).

4.2. **Relative maps.** Let $\beta = \mathsf{s} + h\mathsf{f}$, and let $\overline{M}_g(R/E, \beta)$ denote the moduli space [25] of stable relative maps to expanded degenerations of (R, E) with multiplicity 1 along the relative divisor E. There is an evaluation map

(65) $$\overline{M}_g(R/E, \beta) \to E$$

determined by the location of the relative point. In fact, since s is rigid, the evaluation map *always* has value $p \in E$.

A stable map to $\mathcal{S}_0$ can be split (non-uniquely) into relative stable maps to (R_1, E) and (R_2, E). Let

$$\eta = (g_1, g_2, h_1, h_2)$$

denote a quadruple of non-negative integers satisfying

$$g = g_1 + g_2, \quad h = h_1 + h_2 \,.$$

Define $\overline{M}_g(\mathcal{S}_0, \eta)$ to be the fibered product over the evaluation maps (65) on both sides,

$$\overline{M}_{g_1}(R_1/E_1, \mathsf{s} + h_1\mathsf{f}) \times_E \overline{M}_{g_2}(R_2/E_2, \mathsf{s} + h_2\mathsf{f}) \,.$$

Since the evaluations maps both factor through the point $p \in E$,

$$\overline{M}_g(\mathcal{S}_0, \eta) = \overline{M}_{g_1}(R_1/E_1, \mathsf{s} + h_1\mathsf{f}) \times \overline{M}_{g_2}(R_2/E_2, \mathsf{s} + h_2\mathsf{f}).$$

By standard results [25, 26], we have an embedding

$$\iota_\eta : \overline{M}_g(\mathcal{S}_0, \eta) \hookrightarrow \overline{M}_g(\epsilon, \beta),$$

and the full moduli space to $\mathcal{S}_0$ is the union

$$\overline{M}_g(\mathcal{S}_0, \beta) = \bigcup_\eta \overline{M}_g(\mathcal{S}_0, \eta).$$

In [26], the embedding ι_η is explicitly realized as given by a Cartier pseudo-divisor (in the sense of Fulton): there exists a line bundle L_η on $\overline{M}_g(\epsilon, \beta)$ with section $s_\eta \in \Gamma(L_\eta)$ whose zero locus is $\overline{M}_g(\mathcal{S}_0, \eta)$. If (L_0, s_0) denotes the pseudo-divisor given by pulling back the Cartier divisor $0 \in B$, we also have the identity

$$(66) \qquad\qquad (L_0, s_0) = \bigotimes_\eta (L_\eta, s_\eta).$$

4.3. **Obstruction theories.** We follow the construction of the perfect obstruction theory on $\overline{M}_g(\epsilon, \beta)$,

$$(67) \qquad\qquad E_\epsilon^\bullet \to \mathbb{L}_{\overline{M}_g(\epsilon, \beta)},$$

presented in [26]. We will always take $\beta = \mathsf{s} + h\mathsf{f}$. Since the multiplicity of the stable map at the singular locus of $\mathcal{S}_0$ is always 1, the obstruction theory of [26] simplifies substantially.

Let $\mathscr{B}$ denote the nonsingular Artin stack parameterizing expanded target degenerations of $\mathcal{S}$ over B. We first describe the relative obstruction theory for the morphism

$$\phi : \overline{M}_g(\epsilon, \beta) \to \mathscr{B}.$$

For convenience, we just describe the tangent and obstruction spaces at a closed point of the moduli space. For the general case see [26]. Fix $\xi \in B$, an expanded target degeneration[19]

$$\widetilde{S}_\xi \to \mathcal{S}_\xi,$$

and a stable map

$$f : C \to \widetilde{S}_\xi.$$

---

[19]Unless we are over the special point $\xi = 0$, we have $\widetilde{S}_\xi = \mathcal{S}_\xi$. Nontrivial degenerations occur only over $0 \in B$.

The tangent and obstruction spaces relative to the morphism ϕ are given by the cohomology groups in degrees 0 and 1 of the complex of vector spaces

(68) $$R \operatorname{Hom}_C(f^*\Omega_{\widetilde{S}_\xi} \to \Omega_C, \mathcal{O}_C).$$

Here, the complex $f^*\Omega_{\widetilde{S}_\xi} \to \Omega_C$ is obtained from the map f and is placed in degrees -1 and 0. Following the method explained in Section 3.9, the absolute obstruction theory $E_\epsilon^\bullet$ is then easily obtained from the relative obstruction theory since $\mathscr{B}$ is nonsingular and $\overline{M}_g(\epsilon, \beta)$ has no continuous automorphisms.

We can similarly construct perfect obstruction theories $E_0^\bullet$ and $E_\eta^\bullet$ for $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \eta)$ respectively. Just as in Section 3.9, both are related to $E_\epsilon^\bullet$ via exact triangles (cf. (51) and (52))

(69) $$L_0^\vee \to \iota_0^* E_\epsilon^\bullet \to E_0^\bullet \to L_0^\vee[1] \ ,$$

$$L_\eta^\vee \to \iota_\eta^* E_\epsilon^\bullet \to E_\eta^\bullet \to L_\eta^\vee[1] \ ,$$

where L_0 and L_η are the line bundles in (66).

In order to split the associated virtual class for $E_\eta^\bullet$ into contributions from R_1 and R_2, we will require an exact triangle relating $E_\eta^\bullet$ to the obstruction theories $E_1^\bullet, E_2^\bullet$ associated to each moduli space of relative stable maps $\overline{M}_{g_i}(R_i/E_i, \mathsf{s} + h_i\mathsf{f})$. This is the analogue of the exact triangle (61) for stable pairs:

(70) $$N_{\Delta/E\times E}^\vee \to E_1^\bullet \boxplus E_2^\bullet \to E_\eta^\bullet \to N_{\Delta/E\times E}^\vee[1].$$

Here, $N_{\Delta/E\times E}^\vee$ denotes the conormal bundle to the diagonal of $E \times E$, pulled back to $\overline{M}_g(\mathcal{S}_0, \eta)$ via the evaluation maps.

4.4. **Reduced classes.** The moduli space $\overline{M}_g(\epsilon, \beta)$ carries both an absolute obstruction theory (67) and an obstruction theory relative to $\mathscr{B}$ (68). To define a reduced class as in Section 3.6, we explain how to construct a 1-dimensional quotient of the relative obstruction space when $\beta = \mathsf{s} + h\mathsf{f}$. We present a uniform treatment over all $\xi \in B$. However, unless $\xi = 0$, all structures involved with the singularities of $\mathcal{S}_\xi$ are trivial. By an elementary analysis, the relative obstruction space over $\mathscr{B}$ equals the absolute obstruction space since there is *no obstruction* to deforming connected maps along with deformations of the fibers of ϵ. Hence, we obtain a 1-dimensional quotient of the absolute obstruction theory of $\overline{M}_g(\epsilon, \beta)$ also.

Let $\Omega^{\log}_{\widetilde{S}_\xi}$ denote the sheaf of differentials with logarithmic poles allowed along the singular locus (the residues along each branch are required to add to zero). The sheaf $\Omega^{\log}_{\widetilde{S}_\xi}$ is locally free of rank 2. Alternately, $\Omega^{\log}_{\widetilde{S}_\xi}$ is the sheaf of differentials for the log structure on $\widetilde{S}_\xi$ associated to the smoothing of $\widetilde{S}_\xi$. The surface $\widetilde{S}_\xi$ is log $K3$: we have an isomorphism

$$\wedge^2 \Omega^{\log}_{\widetilde{S}_\xi} \cong \mathcal{O}_{\widetilde{S}_\xi}.$$

Hence, there is a nondegenerate symplectic pairing on $\Omega^{\log}_{\widetilde{S}_\xi}$.

Similarly, let $\Omega^{\log}_C$ denote the sheaf of differentials on C with simple poles allowed only at the nodes mapping to the singular locus of $\widetilde{S}_\xi$ (again with the matching residue condition). In a neighborhood of such nodes, $\Omega^{\log}_C$ is locally free. We may view $\Omega^{\log}_C$ as the sheaf of differentials associated to the log structure on C pulled back from $\widetilde{S}_\xi$ via f.

Lemma 17. *The natural map of complexes*

$$[f^* \Omega_{\widetilde{S}_\xi} \to \Omega_C] \to [f^* \Omega^{\log}_{\widetilde{S}_\xi} \to \Omega^{\log}_C]$$

is a quasi-isomorphism.

Proof. The result follows from the diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & f^* \mathcal{O}_E & \longrightarrow & f^* \Omega_{\widetilde{S}_\xi} & \longrightarrow & f^* \Omega^{\log}_{\widetilde{S}_\xi} & \longrightarrow & f^* \mathcal{O}_E & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_p & \longrightarrow & \Omega_C & \longrightarrow & \Omega^{\log}_C & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0.
\end{array}
$$

The leftmost terms are the torsion subsheaves of the cotangent sheaves and the rightmost terms are the residues of the logarithmic forms. $\qquad\square$

The reduced obstruction space is given by the kernel of the following composition of morphisms:

$$
\begin{aligned}
\psi : \mathrm{Ext}^1([f^* \Omega_{\widetilde{S}_\xi} \to \Omega_C], \mathcal{O}_C) &= \mathrm{Ext}^1([f^* \Omega^{\log}_{\widetilde{S}_\xi} \to \Omega^{\log}_C], \mathcal{O}_C) \\
&\cong \mathrm{Ext}^1([(f^* \Omega^{\log}_{\widetilde{S}_\xi})^\vee \to \Omega^{\log}_C], \mathcal{O}_C) \\
&\to \mathrm{Ext}^1([\omega_C^\vee \to \Omega^{\log}_C], \mathcal{O}_C) \\
&\to H^1(C, \omega_C) = \mathbb{C}.
\end{aligned}
$$

Lemma 17 yields the first equality. The second equality is a consequence of the symplectic pairing. The third map is constructed via the pullback

$$(71) \qquad f^*(\Omega^{\log}_{\widetilde{S}_\xi}) \to \omega_C.$$

The last map is obtained from the vanishing of the composition

$$\omega_C^\vee \to (f^*\Omega^{\log}_{\widetilde{S}_\xi})^\vee \to f^*\Omega^{\log}_{\widetilde{S}_\xi} \to \Omega^{\log}_C .$$

Here we use the fact that the symplectic structure vanishes when pulled back to C.

Lemma 18. *The map ψ is surjective.*

Proof. For $\xi \neq 0$, the claim is part of the usual construction of the reduced class. We discuss only the singular case $\widetilde{S}_0$. The map ψ is induced by

$$(72) \qquad \mathrm{Ext}^1\big((f^*\Omega^{\log}_{\widetilde{S}_0})^\vee, \mathcal{O}_C\big) \cong H^1\big(C, f^*\Omega^{\log}_{\widetilde{S}_0}\big) \to H^1(C, \omega_C),$$

where the latter arrow is obtained from (71). We will prove (72) is surjective.

Pick a connected component of the singular locus of $\widetilde{S}_0$ and take the corresponding separating node p of C. Let C_1 and C_2 denote the connected components of the normalization of C at p. Consider the sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_p \to 0.$$

Tensoring with the bundles $f^*\Omega^{\log}_{\widetilde{S}_0}$ and ω_C, taking cohomology and using the map (71) gives the commutative diagram

$$
\begin{array}{ccc}
H^0\left(f^*\Omega^{\log}_{\widetilde{S}_0}\big|_p\right) & \longrightarrow & H^1\big(C, f^*\Omega^{\log}_{\widetilde{S}_0}\big) \\
\downarrow & & \downarrow \\
H^0(\omega_C|_p) & \longrightarrow & H^1(C, \omega_C) .
\end{array}
$$

The left hand arrow is surjective since C is an embedding in a neighborhood of p. The lower arrow is surjective by standard curve theory. Therefore the right hand arrow (72) is also surjective. $\qquad\square$

By taking duals and applying the above construction in families, we obtain a morphism

$$\gamma : \mathcal{O}_{\overline{M}_g(\epsilon,\beta)}[1] \to E_\epsilon^\bullet.$$

for which the induced map on obstruction sheaves

$$h^1((E_\epsilon^\bullet)^\vee) \to \mathcal{O}_{\overline{M}_g(\epsilon,\beta)}$$

is surjective. We refer to $E_\epsilon^{\bullet\mathrm{red}} = \mathrm{Cone}(\gamma)$ as the associated reduced obstruction theory, even though *a priori* not all the conditions of an obstruction theory may be satisfied.[20] Just as in Section 3.6 the results of Kiem-Li allow us to construct a reduced virtual class from $E_\epsilon^{\bullet\mathrm{red}}$.

4.5. **Degeneration of the reduced class.** We can define the reduced obstruction theory for $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \eta)$ via the compositions

$$\gamma_0 : \mathcal{O}[1] \to \iota_0^* E_\epsilon^\bullet \to E_0^\bullet \ ,$$

$$\gamma_\eta : \mathcal{O}[1] \to \iota_\eta^* E_\epsilon^\bullet \to E_\eta^\bullet.$$

From the exact triangles (69), we deduce the induced co-section of the obstruction space is surjective in both cases. We can take the cones of γ_0 and γ_η to obtain the reduced obstruction theories $E_0^{\bullet\mathrm{red}}$ and $E_\eta^{\bullet\mathrm{red}}$ respectively. The compatibility statements:

(73) $$L_0^\vee \to \iota_0^* E_\epsilon^{\bullet\mathrm{red}} \to E_0^{\bullet\mathrm{red}} \to L_0^\vee[1] \ ,$$

$$L_\eta^\vee \to \iota_\eta^* E_\epsilon^{\bullet\mathrm{red}} \to E_\eta^{\bullet\mathrm{red}} \to L_\eta^\vee[1] \ ,$$

continue to hold.

Lemma 19. *We have a quasi-isomorphism on* $\overline{M}_g(\mathcal{S}_0, \eta)$,

$$E_1^\bullet \boxplus E_2^\bullet \cong E_\eta^{\bullet\mathrm{red}},$$

compatible with the structure maps to the cotangent complex of $\overline{M}_g(\mathcal{S}_0, \eta)$.

Proof. By the second compatibility sequence (70) for $E_\eta^\bullet$, we have a natural map

(74) $$E_1^\bullet \boxplus E_2^\bullet \to E_\eta^{\bullet\mathrm{red}} \ .$$

If the induced map

(75) $$\mathcal{O}[1] \to N_{\Delta/E\times E}^\vee[1]$$

is a quasi-isomorphism, then (74) is also a quasi-isomorphism.

---

[20]With greater effort, standard reduced obstruction theories could surely be constructed here. For families of nonsingular $K3$ surfaces, the most difficult aspect of the construction of the reduced theory is the obstruction study of Ran [51] and Manetti [29]. In order to avoid such a study for the broken $K3$ surface over $0 \in B$, we restrict ourselves to a weak reduced theory which is technically simpler and sufficient for our purposes.

We prove the quasi-isomorphism statement (75) after taking duals and passing to closed points. After fixing a curve, expanded target, and map

$$f : C \to \widetilde{S}_0,$$

we will prove the composition

$$(76) \quad T_p E \to \mathrm{Ext}^1(f^* \Omega_{\widetilde{S}_0}^{\log}, \mathcal{O}_C) \cong \mathrm{Ext}^1((f^* \Omega_{\widetilde{S}_0}^{\log})^\vee, \mathcal{O}_C) \to H^1(C, \omega_C)$$

is an isomorphism.

As in the proof of Lemma 18, we will use the isomorphism

$$H^0(\omega_C|_p) \to H^1(C, \omega_C)$$

to factor the composition in terms of the fibers of the vector bundles $f^* \Omega_{\widetilde{S}}^{\log}$ and ω_C over the point p. In fact, the composition (76) is the same as the composition

$$(77) \qquad T_p E \to \left(\Omega_{\widetilde{S}_0}^{\log} \right)^\vee \Big|_p \cong \Omega_{\widetilde{S}_0}^{\log} \Big|_p \to \omega_C|_p \,.$$

From an explicit local description of the symplectic pairing on $\Omega_{\widetilde{S}_0}^{\log}$, the tangent direction along the elliptic fiber E is identified with the dual of the tangent direction along the section C. As a consequence, composition (77) is an isomorphism. $\square$

Theorem 20. *The reduced virtual class of maps to the degeneration*

$$\epsilon : \mathcal{S} \to B$$

for primitive $\beta = \mathsf{s} + h\mathsf{f}$ satisfies three basic properties:

(i) *For all $\xi \in B$,*

$$\iota_\xi^! [\overline{M}_g(\epsilon, \beta)]^{\mathrm{red}} = [\overline{M}_g(\mathcal{S}_\xi, \beta)]^{\mathrm{red}} \,.$$

(ii) *For the special fiber,*

$$[\overline{M}_g(\mathcal{S}_0, \beta)]^{\mathrm{red}} = \sum_\eta \iota_{\eta *} [\overline{M}_g(\mathcal{S}_0, \eta)]^{\mathrm{red}} \,.$$

(iii) *The factorizations*

$$[\overline{M}_g(\mathcal{S}_0, \eta)]^{\mathrm{red}} = [\overline{M}_{g_1}(R_1/E, \beta_1)]^{\mathrm{vir}} \times [\overline{M}_{g_2}(R_2/E, \beta_2)]^{\mathrm{vir}}$$

hold for $\beta_i = \mathsf{s} + h_i\mathsf{f}$.

Proof. The proof exactly follows the proof of the parallel statements in J. Li's papers [26] for the degeneration formula for the standard obstruction theory. We use the compatibility of reduced obstruction theories (73) instead of (69). The statement (iii) is immediate from Lemma 19. $\square$

4.6. **Non-primitive degeneration.** We announce here a degeneration formula for arbitrary (not necessarily primitive) classes

$$\beta = m\mathsf{s} + h\mathsf{f}$$

with respect to the family ϵ. Proofs and applications will appear in [37].

As before, a stable map to $\mathcal{S}_0$ can be split (non-uniquely) into relative stable maps to (R_1, E) and (R_2, E). The distribution of genera and fiber classes is given by the data

$$\eta = (g_1, g_2, h_1, h_2) \,.$$

Let $\beta_i = m\mathsf{s} + h_i\mathsf{f}$ as before. The additional data of an ordered partition γ of m

$$\gamma = (\gamma_1, \ldots, \gamma_\ell), \quad \sum_{i=1}^{\ell} \gamma_i = m$$

specifies the multiplicities with which l distinct points on the two sides map to the relative divisor.

The moduli spaces of relative maps[21]

$$\overline{M}^\bullet_{g_i}(R_i/E, \beta_i)_\gamma$$

are well-defined and admit boundary evaluation maps

(78) $$\mathrm{ev}_{\mathrm{rel}} : \overline{M}^\bullet_{g_i}(R_i/E, \beta_i)_\gamma \to E^\ell \,.$$

We define $\overline{M}^\bullet_g(\mathcal{S}_0, \eta)_\gamma$ by the fiber product of the boundary evaluations,

(79) $$\overline{M}^\bullet_g(\mathcal{S}_0, \eta)_\gamma = \overline{M}^\bullet_{g_1}(R_1/E, \beta_1)_\gamma \times_{E^\ell} \overline{M}^\bullet_{g_2}(R_2/E, \beta_2)_\gamma \,.$$

After a construction of the reduced virtual class for the family ϵ, Parts (i) and (ii) of Theorem 20 hold without change.

(i) For all $\xi \in B$,

$$\iota_\xi^! [\overline{M}^\bullet_g(\epsilon, \beta)]^{\mathrm{red}} = [\overline{M}^\bullet_g(\mathcal{S}_\xi, \beta)]^{\mathrm{red}} \,.$$

(ii) For the special fiber,

$$[\overline{M}^\bullet_g(\mathcal{S}_0, \beta)]^{\mathrm{red}} = \sum_\eta \sum_\gamma \iota_{\eta,\gamma,*} \left(\frac{1}{|\mathrm{Aut}(\gamma)|} [\overline{M}^\bullet_g(\mathcal{S}_0, \eta)_\gamma]^{\mathrm{red}} \right) \,.$$

$\mathrm{Aut}(\gamma)$ is the symmetry group permuting equal parts of γ.

---

[21]The superscript $\bullet$ denotes the moduli of maps with possibly disconnected domain curves. The map, however, is required to be nonconstant on every connected component of the domain.

However, the factorization rule (iii) is more interesting.

The class s determines, by intersection, a line bundle $\mathcal{L}$ of degree 1 on the relative $E \subset R_1$. Since the class f restricts to the trivial line bundle on E, the image of the boundary evaluation must lie in the subvariety $V \subset E^\ell$ defined by

$$V = \left\{ (p_1, \ldots, p_\ell) \mid \mathcal{O}_E\left(\sum_i^\ell \gamma_i p_i \right) \cong \mathcal{L}^m \right\}.$$

The subvariety V is nonsingular[22] of pure codimension 1.

The product of the boundary evaluation maps (78) yields

$$(80) \qquad \mathrm{ev}_{V \times V} : \overline{M}^\bullet_{g_1}(R_1/E, \beta_1)_\gamma \times \overline{M}^\bullet_{g_2}(R_2/E, \beta_2)_\gamma \to V \times V.$$

Let $\Delta \subset V \times V$ be the diagonal. We also denote the inclusion map by Δ. By definition (79),

$$\overline{M}^\bullet_g(\mathcal{S}_0, \eta)_\gamma \cong \mathrm{ev}^{-1}_{V \times V}(\Delta).$$

The following rule replaces part (iii) of Theorem 20.

(iii) The reduced virtual class of $\overline{M}^\bullet_g(\mathcal{S}_0, \eta)_\gamma$ is given by the Gysin pull-back of the standard virtual classes of $\overline{M}_{g_i}(R_i/E, \beta_i)_\gamma$,

$$[\overline{M}^\bullet_g(\mathcal{S}_0, \eta)_\gamma]^{\mathrm{red}} = \Delta^!\left([\overline{M}^\bullet_{g_1}(R_1/E, \beta_1)_\gamma]^{\mathrm{vir}} \times [\overline{M}^\bullet_{g_2}(R_2/E, \beta_2)_\gamma]^{\mathrm{vir}} \right).$$

A detailed discussion will be given in [37].

While the reduced Gromov-Witten invariants of S with disconnected domains vanish, the disconnected theory is more natural for the factorization (iii). Disconnected maps to (R_1, E) and (R_2, E) may glue to produce a connected map to $\mathcal{S}_0$.

5. TORIC GROMOV-WITTEN/PAIRS CORRESPONDENCE

5.1. **Toric 3-folds.** Let V be a nonsingular toric 3-fold acted upon by a 3-dimensional algebraic torus T. The conjectural Gromov-Witten/Pairs correspondence of [46, 47, 48] for toric varieties equates the T-equivariant Gromov-Witten theory of V to T-equivariant stable pairs theory of V. Since both sides can be defined by T-equivariant residues, V is not required to be compact. We refer the reader to Sections 3.1-3.2 of [46] for background.

Theorem 21. *The T-equivariant Gromov-Witten/Pairs correspondence with primary field insertions holds for V.*

---

[22]However, V need not be connected.

Proof. The T-equivariant Gromov-Witten/Donaldson-Thomas correspondence with primary field insertions has been established for toric V in [34]. The strategy of [34] is to match the capped Gromov-Witten and Donaldson-Thomas vertices. The 1-leg case was previously proven in [43] using [7, 42]. The Gromov-Witten and Donaldson-Thomas theory of A_n-resolutions was studied in [30, 32, 33]. Once the local theories of A_n-resolutions are matched, the method of [34] is completely formal, relying only on properties of localization and degeneration of the relevant moduli spaces.

To prove the Gromov-Witten/Pairs correspondence, we take precisely the same path. All the required results for the Gromov-Witten capped vertex have already been proven. Only the parallel results for the local stable pairs theory of A_n-resolutions over curves are required. For the latter, we follow, step by step, the arguments from [33, 43].

Stable pairs have the same formal properties (with respect to localization and degeneration) as the Donaldson-Thomas theory of ideal sheaves. Moreover, the relative moduli spaces in each theory both involve the Hilbert scheme of points on a surface. Thus, there is no difficulty in following the arguments. Indeed, stable pairs are simpler to treat since the support is of pure dimension 1.

Let us first review the basic steps in the proof of the Gromov-Witten/Donaldson-Thomas correspondence, afterwards indicating which aspects require modification.

- *Reduction to special geometries*

  In [34], a formal procedure is given to reduce invariants with primary insertions for arbitrary toric threefolds to relative invariants on the threefolds

  $$\mathbb{C}^2 \times \mathbb{P}^1 \quad \text{and} \quad A_n \times \mathbb{P}^1 \text{ for } n = 1, 2 \ .$$

  The arguments in [34] rely on the degeneration formalism, localization for relative moduli spaces, and dimension counts - all of which behave the same for stable maps, ideal sheaves, and stable pairs. In particular, the formal procedure can be applied to stable pairs to reduce the correspondence to the case of these specific 3-folds.

  We view $\mathbb{C}^2$ as A_0. For the three special geometries, the proofs in [43, 33] are based on the following principles.
- *Vanishing results*

Let σ be a holomorphic symplectic form on A_n fixed by a 1-dimensional torus action on A_n. Let

$$\mathsf{T}_0 \subset \mathsf{T}$$

be the 2-dimensional subtorus which acts on the toric 3-fold $A_n \times \mathbb{P}^1$ and fixes the pull-back of σ.

(i) The T_0-equivariant theory of $A_n \times \mathbb{P}^1$ vanishes unless the curve class β is contracted over A_n.

(ii) In case the curve class β is contracted over A_n, the T_0-equivariant theory of $A_n \times \mathbb{P}^1$ vanishes unless the holomorphic Euler characteristic χ is minimal.

• *Reduction to specific computations*

Using the vanishing results (i-ii), the full theory of $A_n \times \mathbb{P}^1$ is reduced to the calculation of specific 2-point invariants on the relative theory of $A_n \times \mathbb{P}^1$ modulo $(s_1 + s_2)^2$, where $s_1 + s_2$ is the equivariant weight of the holomorphic symplectic form σ. For $\mathbb{C}^2$, the argument is given in [43]. For A_n, the argument is given in [33, Propositions 4.3 and 4.4]. In each case, the proof consists of comparing the Nakajima and fixed-point bases of $\mathrm{Hilb}(A_n)$ and applying the vanishing statement along with certain dimension counts. Again the argument is completely formal since relative insertions behave in the same manner for stable pairs and ideal sheaves.

• *Reduction to Hilbert scheme geometry*

The last step is to prove that the specific calculations discussed above can be identified with 2-point invariants in the quantum cohomology of $\mathrm{Hilb}(A_n, d)$ via a basic matching result.

Consider the moduli space of genus 0, 2-pointed stable maps to the Hilbert scheme of d points of A_n of curve class β,

$$\overline{M}_{0,\{0,\infty\}}(\mathrm{Hilb}(A_n, d), \beta) \ .$$

There is an open set

$$U_{n,d,\beta} \subset \overline{M}_{0,\{0,\infty\}}(\mathrm{Hilb}(A_n, d), \beta)$$

corresponding to the locus where the domain is a simple chain of rational curves.

(iii) The canonical map from $U_{n,d,\beta}$ to the moduli space of ideal sheaves of $A_n \times \mathbb{P}^1$- rubber relative to $0, \infty \in \mathbb{P}^1$ is an isomorphism which respects the obstruction theory.

For any 2-point invariant, we study the contribution of virtual localization to any given fixed-point locus. If the locus does not lie in the open set of (iii), then the vanishing results imply the contribution is divisible by $(s_1 + s_2)^2$ and so is irrelevant for the computation.

To complete the proof of the Gromov-Witten/Pairs correspondence, we follow the same 4 steps with stable pairs instead of ideal sheaves. Since the endpoint of the argument is given by calculating 2-point invariants on $\mathrm{Hilb}(A_n, d)$, the correspondence is independent of which sheaf theory we use.

All arguments in the above outline are formal except for the vanishing statements and the matching with the Hilbert scheme geometry. In particular, it remains to establish properties (i-iii) for the stable pairs theory of the 3-fold $A_n \times \mathbb{P}^1$.

For (i), a direct approach is obtained by constructing a trivial quotient of the obstruction sheaf following Section 3. Property (ii) is equivalent to a vanishing for the 1-leg stable pairs vertex which can be checked explicitly from the localization formulas of [47]. The proof is presented in Section 5.2. Property (iii) is immediate since $U_{n,d,\beta}$ maps to the locus where the moduli of ideal sheaves is isomorphic to the moduli of stable pairs. $\qquad\qquad\square$

5.2. **1-leg stable pairs vertex.** We prove the necessary divisibility statement for stable pairs invariants for

$$X = \mathbb{C}^2 \times \mathbb{P}^1 \ .$$

The first two factors of the torus $\mathsf{T} = (\mathbb{C}^*)^3$ act on $\mathbb{C}^2$ by scaling the coordinates, and the last factor acts on $\mathbb{P}^1$ in the standard manner. The stable invariants take values in the equivariant cohomology ring $\mathbb{Q}(s_1, s_2, s_3)$. Via localization [47], the contribution of the virtual class can be decomposed into vertex and edge terms associated to the moment polytope of X. Given a partition μ, the vertex contribution is given by a Laurent series

$$\mathsf{W}_\mu^P(q; s_1, s_2, s_3) \in \mathbb{Q}(s_1, s_2, s_3)((q)).$$

Lemma 22. *The q^n coefficient of W_μ^P is divisible by $s_1 + s_2$ for $n > |\mu|$.*

Proof. The proof here is nearly identical to the calculation in [43] and [31] with a slightly different combinatorial formula. We again follow notation from [47].

By definition, $\mathsf{W}^P_\mu(q; s_1, s_2, s_3)$ is a weighted sum

$$\mathsf{W}^P_\mu(q; s_1, s_2, s_3) = \sum_{\mathcal{Q}} \mathsf{w}_{\mathcal{Q}}(s_1, s_2, s_3) q^{|\mathcal{Q}|}$$

over T-invariant stable pairs $\mathcal{Q}$ on $\mathbb{C}^3$ with limiting profile in the x_3- direction given by the subscheme of $\mathbb{C}^2$ defined by monomial ideal $\mu[x_1, x_2]$ associated to μ. We prove the divisibility statement for each summand with $|\mathcal{Q}| > |\mu|$.

Such a T-invariant stable pair $\mathcal{Q}$ has the following description. Let

$$M = \mathbb{C}[x_3, x_3^{-1}] \otimes \frac{\mathbb{C}[x_1, x_2]}{\mu[x_1, x_2]} .$$

We have a finitely generated T-invariant $\mathbb{C}[x_1, x_2, x_3]$-module

$$\mathbb{C}[x_3] \otimes \frac{\mathbb{C}[x_1, x_2]}{\mu[x_1, x_2]} \subset Q \subset M.$$

Let $F(t_1, t_2, t_3)$ denote the Laurent series indexing the torus weights of Q, and let $G(t_1, t_2)$ denote the same for $\frac{\mathbb{C}[x_1,x_2]}{\mu[x_1,x_2]}$. The equivariant vertex contribution for $\mathcal{Q}$ is obtained from the Laurent polynomial

$$H(t_1, t_2, t_3) = F - \frac{\overline{F}}{t_1 t_2 t_3} + F\overline{F} \prod_{i=1}^{3} \frac{(1 - t_i)}{t_i}$$
$$- \frac{1}{1 - t_3} \left(G + \frac{\overline{G}}{t_1 t_2} - G\overline{G} \frac{(1 - t_1)(1 - t_2)}{t_1 t_2} \right)$$

where $\overline{F} = F(1/t_1, 1/t_2, 1/t_3)$ and similarly for G. More precisely, $\mathsf{w}_{\mathcal{Q}}(s_1, s_2, s_3)$ is a product of linear factors associated to monomials in the above expression. The divisibility of the vertex contribution by $s_1 + s_2$ is equivalent to the negativity of the constant term in $H(t, t^{-1}, u)$.

The T-weights of Q define a labelled box configuration, which can be described by a sequence of skew Young diagrams of the form

$$\rho_k = \mu \backslash \nu_k$$

for Young diagrams $\nu_k \subset \mu$ satisfying inclusions

$$\emptyset = \rho_{-m} \subset \rho_{-m+1} \cdots \subset \rho_{-1} \subset \rho_0 = \rho_1 = \cdots = \mu .$$

Here, ρ_k contains a box $(a, b) \in \mathbb{Z}^2_{\geq 0}$ if (a, b, k) is a weight of Q. We have

$$|\mathcal{Q}| = \sum_{k=-m}^{-1} |\rho_k| .$$

If $|\mathcal{Q}| > |\mu|$, we must have $\rho_{-1} \neq 0$.

Let $c_r(\rho_k)$ denote the number of boxes $(a, b) \in \rho_k$ for which $a - b = r$, and let $d_r(\rho_k) = c_r(\rho_k) - c_{r+1}(\rho_k)$. The constant term of $H(t, t^{-1}, u)$ is given by

$$(81) \qquad -c_0(\rho_{-1}) + \frac{1}{2} \sum_r \left(d_r(\rho_0)^2 - \sum_{k \leq 0} (d_r(\rho_k) - d_r(\rho_{k-1}))^2 \right).$$

Since for each r, we have

$$\sum_{k \leq 0} d_r(\rho_k) - d_r(\rho_{k-1}) = d_r(\rho_0) \in \{-1, 0, 1\},$$

the second term of (81) is non-positive, and the entire expression is bounded above by $-c_0(\rho_{-1}) \leq 0$.

There are two cases. If $c_0(\rho_{-1}) > 0$, then we are done. If not, since $\rho_{-1} \neq \emptyset$, there must be a box of ρ_{-1} which minimizes $|r|$. If $a > b$ for such a box, then

$$c_{r-1}(\rho_0) = c_r(\rho_0)$$

and $d_{r-1}(\rho_0) = 0$. But,

$$c_{r-1}(\rho_{-1}) < c_r(\rho_{-1})$$

so $d_{r-1}(\rho_{-1}) < 0$ and the second term of (81) is strictly negative. A similar logic applies if the $|r|$-minimal box satisfies $a < b$. $\qquad \square$

5.3. **Proof of Theorems 1 and 9.** Let S be a $K3$ surface with primitive effective curve class $\beta \in H_2(S, \mathbb{Z})$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2.$$

Following the definitions of Section 1, to establish Theorem 9, we must prove:

    (i) $\mathsf{Z}_\beta^P \left(\tau_0(\mathsf{p})^k \right)$ is a rational function of y.
    (ii) After the variable change $-e^{iu} = y$,

$$\mathsf{Z}_\beta^{GW} \left(\tau_0(\mathsf{p})^k \right) = \mathsf{Z}_\beta^P \left(\tau_0(\mathsf{p})^k \right).$$

We consider both the reduced Gromov-Witten and the reduced stable pairs theory of the 3-fold

$$X = S \times \mathbb{C}$$

equivariant with respect to the scaling of $\mathbb{C}$. Properties (i) and (ii) above are reduced Gromov-Witten/Pairs correspondence for X with point insertions.

Let R be the rational elliptic surface considered in Section 2.2, and let

$$Y = R \times \mathbb{C}\,.$$

By Theorems 16 and 20, the reduced theory of X may be calculated on the broken 3-fold

$$Y \cup_{E \times \mathbb{C}} Y\,.$$

The original k point conditions are distributed to two sides (the precise distribution will not matter). In order to prove the reduced Gromov-Witten/Pairs correspondence for X, we need only establish the standard Gromov-Witten/Pairs correspondence for the relative geometry $(Y, E \times \mathbb{C})$ equivariant with respect to the scaling of $\mathbb{C}$.

Since R is deformation equivalent to a toric surface, the toric Gromov-Witten/Pairs correspondence implies the Gromov-Witten/Pairs correspondence for Y with primary field insertions equivariant with respect to the scaling of $\mathbb{C}$.

The final step is to deduce the Gromov-Witten/Pairs correspondence for $(Y, E \times \mathbb{C})$ from the established correspondence for Y. Let

$$Z = E \times \mathbb{P}^1 \times \mathbb{C}$$

and let $E \times \mathbb{C} \subset Z$ be a fiber over $\mathbb{P}^1$. The degeneration

(82) $$Y \rightsquigarrow Y \cup_{E \times \mathbb{C}} Z$$

is obtained from the deformation to the normal cone of $E \times \mathbb{C} \subset Y$. We will prove the correspondence for $(Y, E \times \mathbb{C})$ from the correspondences for Y and $(Z, E \times \mathbb{C})$ via the degeneration formula. In the degeneration (82), all point insertions are kept on Y.

The relative geometry $(Z, E \times \mathbb{C})$ with no point insertions is very simple. We are only interested in curve classes on Z which are degree 1 over the $\mathbb{P}^1$ factor. The dimension of the moduli spaces on both the Gromov-Witten and stable pairs sides is 2. Hence, a point condition in the relative divisor $E \times \mathbb{C}$ *must* be imposed to produce nonvanishing invariants (which are then nonequivariant constants). The theories of $(Z, E \times \mathbb{C})$ equivariant with respect to the scaling of $\mathbb{C}$ are equal to the corresponding nonequivariant theories of

(83) $$(E \times \mathbb{P}^1 \times E, E \times E)\,.$$

The curve classes we are considering are degree 0 over the last E factor. Since $E \times E$ is holomorphic symplectic, the reduced class constructions on both the Gromov-Witten and stable pairs sides lead to vanishing

unless the curve class is also degree 0 over the first E factor. Finally, the match between the Gromov-Witten and stable pairs theory for the relative geometry (83) is a consequence of the Gromov-Witten/Pairs correspondence for local curves.

We have now proven the Gromov-Witten correspondence in the relevant curve classes for two out of the three geometries in the degeneration (82). Moreover, the calculation for $(Z, E \times \mathbb{C})$ indicated above yields invertibility of the corresponding factor in the degeneration formula in both Gromov-Witten and stable pairs theory. We conclude the required Gromov-Witten/Pairs correspondence for $(Y, E \times \mathbb{C})$ holds. We have completed the proof of Theorem 9.

By Lemma 7, Lemma 8, and the Euler characteristic calculations of Kawai-Yoshioka, Theorem 1 is a consequence of Theorem 9 with no point insertions, see Section 0.9. $\qquad\square$

5.4. **Proof of Corollary 2.** Let $\mathcal{S}(u) = \frac{\sin(u/2)}{u/2}$. We have

$$
\begin{aligned}
\sum_{g,h \geq 0} R_{g,h} u^{2g-2} q^{h-1} &= \sum_{g,h \geq 0} r_{g,h} (u\mathcal{S}(u))^{2g-2} q^{h-1} \\
&= (u\mathcal{S}(u))^{-2} \sum_{g,h \geq 0} r_{g,h} (-1)^g (e^{iu/2} - e^{-iu/2})^{2g} q^{h-1} \\
&= (u\mathcal{S}(u))^{-2} \frac{1}{\Delta(e^{iu}, q)} \ .
\end{aligned}
$$

The first two equalities are by definition. The third is consequence of Theorem 1.

Corollary 2 is then a consequence of the following identity,

$$
\begin{aligned}
\log\left(\frac{1}{\mathcal{S}(u)^2} \cdot \frac{\Delta(q)}{\Delta(e^{iu}, q)} \right) &= -2\log\mathcal{S}(u) + \sum_{n \geq 1} 4\log(1 - q^n) \\
&\quad - 2\sum_{n \geq 1} \left(\log(1 - e^{iu}q^n) + \log(1 - e^{-iu}q^n) \right) \\
&= -2\log\mathcal{S}(u) + 4 \sum_{n,g,k \geq 1} \frac{(-1)^g k^{2g-1}}{(2g)!} u^{2g} q^{kn} \\
&= \sum_{g \geq 1} u^{2g} \frac{(-1)^{g+1} B_{2g}}{g \cdot (2g)!} E_{2g}(q) \ .
\end{aligned}
$$

For the last equality, we have used the expansion

$$\log \mathcal{S}(u) = \sum_{g \geq 1} \frac{(-1)^g B_{2g}}{2g \cdot (2g)!} u^{2g}.$$

Note also $(-1)^{g+1} B_{2g} = |B_{2g}|$ for $g \geq 1$. $\qquad\qquad\square$

6. Point insertions

6.1. **Overview.** Our strategy for proving Theorem 3 governing point insertions is by degeneration. We take the $K3$ surface, as before, to be fibered

$$\pi : S \to \mathbb{P}^1$$

with a section, and we take the primitive curve class to be of the form

$$\beta = \mathsf{s} + h \cdot \mathsf{f}.$$

Fix a nonsingular elliptic fiber E of π and consider the degeneration to the normal cone of E,

$$S_0 = S \cup_E (E \times \mathbb{P}^1).$$

We will use the degeneration to the normal cone to reduce the integrals of Theorem 3 to Theorem 1 and calculations in the relative Gromov-Witten theory of $E \times \mathbb{P}^1$.

6.2. **Degeneration formula.** We view $E \times \mathbb{P}^1$ as elliptically fibered

$$\pi : E \times \mathbb{P}^1 \to \mathbb{P}^1$$

with a section (contracted over E). Let $\mathsf{s}, \mathsf{f} \in H^2(E \times \mathbb{P}^1, \mathbb{Z})$ be the classes of the section and fiber respectively. Let

$$\overline{M}_{g,r}((E \times \mathbb{P}^1)/0, \mathsf{s} + h\mathsf{f}), \quad \overline{M}_{g,r}((E \times \mathbb{P}^1)/\{0, \infty\}, \mathsf{s} + h\mathsf{f})$$

denote the moduli spaces of stable relative maps to the targets

$$(E \times \mathbb{P}^1)/E_0, \quad (E \times \mathbb{P}^1)/(E_0 \cup E_\infty).$$

Here, $E_0, E_\infty \subset E \times \mathbb{P}^1$ are the fibers over $0, \infty \in \mathbb{P}^1$.

Since the class $\mathsf{s} + h\mathsf{f}$ has intersection number 1 with the fibers E_0 and E_∞, the relative conditions are given by cohomology classes on E. We will only require the identity and point classes

$$\mathbf{1}, \omega \in H^*(E, \mathbb{Z}).$$

We denote the relative conditions by subscripts after the relative moduli space. For example,

$$\overline{M}_{g,r}((E \times \mathbb{P}^1)/\{0, \infty\}, \mathsf{s} + h\mathsf{f})_{\omega, \mathbf{1}}$$

denotes the moduli space with the relative conditions ω and $\mathbf{1}$ imposed over 0 and ∞ respectively.

We will use bracket notation for the relative invariants of $E \times \mathbb{P}^1$. The insertion $\mathbb{E}^\vee(1)$ stands for the Chern polynomial of the dual of the Hodge bundle,

$$\left\langle \omega \mid \mathbb{E}^\vee(1)\, \tau_0(\mathsf{p}) \mid \mathbf{1} \right\rangle_{g,\mathsf{s}+h\mathsf{f}}^{(E\times\mathbb{P}^1)/\{0,\infty\}} =$$

$$\int_{[\overline{M}_{g,r}((E\times\mathbb{P}^1)/\{0,\infty\},\ \mathsf{s}+h\mathsf{f})_{\omega,1}]^{vir}} \left(1 - \lambda_1 + \lambda_2 - \ldots + (-1)^g\lambda_g \right) \cdot \mathrm{ev}_1^*(\mathsf{p}).$$

Denote the generating function of relative invariants by

$$\left\langle \omega \mid \mathbb{E}^\vee(1)\, \tau_0(\mathsf{p}) \mid \mathbf{1} \right\rangle^{(E\times\mathbb{P}^1)/\{0,\infty\}} =$$

$$\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} \left\langle \omega \mid \mathbb{E}^\vee(1)\, \tau_0(\mathsf{p}) \mid \mathbf{1} \right\rangle_{g,\mathsf{s}+h\cdot\mathsf{f}}^{(E\times\mathbb{P}^1)/\{0,\infty\}} u^{2g-2}q^{h-1}.$$

The integrals of Theorem 3 may be written as

$$\left\langle \mathbb{E}^\vee(1)\, \tau_0(\mathsf{p})^k \right\rangle^S = \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} \left\langle \mathbb{E}^\vee(1)\, \tau_0(\mathsf{p})^k \right\rangle_{g,h}^S u^{2g-2}q^{h-1}$$

$$= \sum_{g=0}^{\infty}\sum_{h=0}^{\infty} \left\langle (-1)^{g-k}\lambda_{g-k}\, \tau_0(\mathsf{p})^k \right\rangle_{g,h}^S u^{2g-2}q^{h-1}.$$

Proposition 23. *We have*

$$\left\langle \mathbb{E}^\vee(1)\, \tau_0(\mathsf{p})^k \right\rangle^S =$$

$$\left\langle \mathbb{E}^\vee(1) \right\rangle^S (u^2 q)^k \left(\left\langle \omega \mid \mathbb{E}^\vee(1)\, \tau_0(\mathsf{p}) \mid \mathbf{1} \right\rangle^{(E\times\mathbb{P}^1)/\{0,\infty\}} \right)^k.$$

Proof. For the degeneration of S to S_0, there already exists a degeneration formula for the reduced Gromov-Witten theory of S proven by Lee and Leung [23, 24].[23] The formula expresses the reduced invariants of S in terms of the reduced relative Gromov-Witten theory of the pair (S, E) and the standard relative Gromov-Witten theory of the pair $(E \times \mathbb{P}^1, E)$.

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[23]An algebraic proof can be obtained by the methods of Section 4.

We denote the generating series of reduced relative invariants of S/E by

$$\left\langle \mathbb{E}^\vee(1) \,\middle|\, \mathbf{1} \right\rangle^{S/E} = \sum_{g=0}^\infty \sum_{h=0}^\infty \left\langle \mathbb{E}_g^\vee(1) \,\middle|\, \mathbf{1} \right\rangle^{S/E}_{g,h} u^{2g-2} q^{h-1} .$$

The degeneration formula then gives the identity

(84) $\quad \left\langle \mathbb{E}^\vee(1) \, \tau_0(\mathsf{p})^k \right\rangle^S =$

$$\left\langle \mathbb{E}^\vee(1) \,\middle|\, \mathbf{1} \right\rangle^{S/E} u^2 q \left\langle \omega \,\middle|\, \mathbb{E}^\vee(1) \, \tau_0(\mathsf{p})^k \right\rangle^{(E\times\mathbb{P}^1)/0} .$$

For the first term on the right, we easily see

$$\left\langle \mathbb{E}^\vee(1) \,\middle|\, \mathbf{1} \right\rangle^{S/E} = \left\langle \mathbb{E}^\vee(1) \right\rangle^S .$$

Using the degeneration of $E \times \mathbb{P}^1$ to

$$(E \times \mathbb{P}^1) \cup_E (E \times \mathbb{P}^1) \cup \cdots \cup_E (E \times \mathbb{P}^1) ,$$

the second invariant on the right side of (84) can be calculated,

(85) $\quad \left\langle \omega \,\middle|\, \mathbb{E}^\vee(1) \, \tau_0(\mathsf{p})^k \right\rangle^{(E\times\mathbb{P}^1)/0} =$

$$\left(\left\langle \omega \,\middle|\, \mathbb{E}^\vee(1) \, \tau_0(\mathsf{p}) \,\middle|\, \mathbf{1} \right\rangle^{(E\times\mathbb{P}^1)/\{0,\infty\}} \right)^k (u^2 q)^k \left\langle \omega \,\middle|\, \mathbb{E}^\vee(1) \right\rangle^{(E\times\mathbb{P}^1)/0} .$$

The Proposition then follows from Lemma 24 below applied to equations (84)-(85) for $k \geq 0$. $\qquad\square$

Lemma 24. *The following two series are trivial:*

$$\left\langle \omega \,\middle|\, \mathbb{E}^\vee(1) \right\rangle^{(E\times\mathbb{P}^1)/0} = \frac{1}{u^2 q} , \qquad \left\langle \mathbf{1} \,\middle|\, \mathbb{E}^\vee(1) \, \tau_0(\mathsf{p}) \right\rangle^{(E\times\mathbb{P}^1)/0} = \frac{1}{u^2 q} .$$

Proof. We only calculate the first. The argument for the second is the same. For dimension reasons, the Hodge class insertion is $(-1)^g \lambda_g$. Consider the curve class $\mathsf{s} + h\mathsf{f}$.

- If $h = 0$, the invariant can be expressed in terms of the relative Gromov-Witten theory of $\mathbb{P}^1$ with an obstruction bundle term which is given by another $(-1)^g \lambda_g$ insertion. Since $\lambda_g^2 = 0$ for $g > 0$, only the $g = 0$ term survives.
- If $h > 0$, we can express the invariant in terms of the relative Gromov-Witten theory of $E \times E \times \mathbb{P}^1$, with degree $(0, h, 1)$, where again the obstruction bundle associated to the degree 0 direction contributes the original $(-1)^g \lambda_g$ insertion. However,

since the Gromov-Witten theory of abelian surfaces is trivial for noncontracted curves, all such terms vanish.

Only the $g = 0$ and $h = 0$ terms of the series $\left\langle \omega \mid \mathbb{E}^\vee(1) \right\rangle^{(E \times \mathbb{P}^1)/0}$ are nonvanishing. Direct calculation shows the nonvanishing term is 1. $\square$

The next result replaces the relative invariants of $E \times \mathbb{P}^1$ in Proposition 23 with absolute invariants.

Lemma 25. *We have*

$$\left\langle \mathbb{E}^\vee(1) \ \tau_0(\mathsf{p})^2 \right\rangle^{E \times \mathbb{P}^1} = 2 \left\langle \omega \mid \mathbb{E}^\vee(1) \ \tau_0(\mathsf{p}) \mid 1 \right\rangle^{(E \times \mathbb{P}^1)/\{0, \infty\}} .$$

Proof. We degenerate $E \times \mathbb{P}^1$ to two copies of $E \times \mathbb{P}^1$ attached along a fiber E with the two insertions $\tau_0(p)$ sent to different sides. Since we only consider curve classes intersecting the relative fiber once, the configurations in the degeneration formula are associated to the diagonal splittings of cohomology classes along the relative point. For parity reasons, we only need to consider even cohomology, so splittings must be of type $(1, \omega)$ or $(\omega, 1)$. The two configurations are clearly symmetric, and the second claim of Lemma 24 yields

$$\left\langle \mathbb{E}^\vee(1) \ \tau_0(\mathsf{p})^2 \right\rangle^{E \times \mathbb{P}^1} = 2 \left\langle \omega \mid \mathbb{E}^\vee(1) \ \tau_0(\mathsf{p}) \right\rangle^{(E \times \mathbb{P}^1)/0} .$$

The Lemma then follows from applying the first claim of Lemma 24 to the degeneration of $E \times \mathbb{P}^1$ along the divisor $E \times \infty$. $\square$

6.3. **Proof of Theorem 3.** Using Corollary 2, Proposition 23, and Lemma 25, the proof of Theorem 3 is reduced to the following two results.

Lemma 26. *We have*

$$\left\langle \mathbb{E}^\vee(1) \ \tau_0(\mathsf{p})^2 \right\rangle^{E \times \mathbb{P}^1} = \frac{2}{u^2 q} \ \sum_{m=1}^\infty q^m \sum_{d \mid m} \frac{m}{d} \left(2 \sin(du/2) \right)^2 .$$

Proof. Fix the genus g and the curve class $\mathsf{s} + h \cdot \mathsf{f}$. The Hodge class is determined by dimension constraints to be $(-1)^{g-1} \lambda_{g-1}$.

We degenerate E to two rational curves P_1 and P_2 intersecting each other at 0 and ∞ with the two $\tau_0(\mathsf{p})$ insertions sent to different components. The distribution of curve degree to $P_1 \times \mathbb{P}^1$ and $P_2 \times \mathbb{P}^1$ is either of the form

$$(h, 1) + (h, 0) \quad \text{or} \quad (h, 0) + (h, 1) .$$

We consider the first degree distribution, since the second case will yield the same answer.

In the degeneration formula, components of the domain map to $P_1 \times \mathbb{P}^1$ or $P_2 \times \mathbb{P}^1$. While the curve has a connected domain, the preimages C_1 and C_2 of the irreducible components of the target can each be disconnected. Let Γ denote the dual graph with vertices given by the connected components of C_1 and C_2 and edges determined by the intersections. Each vertex of Γ is forced to have valence at least two since, for degree reasons, the corresponding subcurve must intersect both relative divisors. The Hodge class λ_{g-1} vanishes on the boundary cycles of $\overline{M}_g$ for which the dual graph Γ has Betti number at least 2, see [12]. As a result, the graph Γ must be a single cycle with vertices alternating between $P_1 \times \mathbb{P}^1$ and $P_2 \times \mathbb{P}^1$.

There must exist a divisor d of h with the following property: each connected component of each C_i intersects the relative divisors at 0 and ∞ at single points of multiplicity d. Each C_i then consists of $\frac{h}{d}$ connected components. Let C_0 denote the connected component of C_1 which has degree $(d, 1)$. Every other component has degree $(d, 0)$. Dimension constraints force C_0 to contain one of the point insertions. There are h/d choices for which connected component of C_2 contains the other point insertion.

We first consider the connected components C' *other* than C_0. If such a component has genus g', the Hodge class λ_{g-1} will restrict to $\lambda_{g'}$ on C'. The contributions are given by the following evaluations:

$$\left\langle (d, \omega) \,\middle|\, \lambda_{g'} \,\middle|\, (d, \mathbf{1}) \right\rangle_{g', (d, 0)}^{(\mathbb{P}^1 \times \mathbb{P}^1)/\{0, \infty\}} = \frac{1}{d}\, \delta_{g', 0} \ ,$$

$$\left\langle (d, \mathbf{1}) \,\middle|\, \lambda_{g'}\, \tau_0(\mathsf{p}) \,\middle|\, (d, \mathbf{1}) \right\rangle_{g', (d, 0)}^{(\mathbb{P}^1 \times \mathbb{P}^1)/\{0, \infty\}} = \delta_{g', 0} \ .$$

Here, the relative divisors have coordinates 0 and ∞ on the first $\mathbb{P}^1$ factor. The classes $\mathbf{1}, \omega \in H^2(\mathbb{P}^1, \mathbb{Z})$ correspond to the identity and the point. The higher genus vanishings are obtained, as before, from $\lambda_{g'}^2 = 0$. In order to get a nontrivial invariant, the components C' must have genus 0. Hence, C_0 must have genus $g - 1$.

We are left with the evaluation of the connected component C_0 mapping to $\mathbb{P}^1 \times \mathbb{P}^1$ with degree $(d, 1)$. The result then follows from Lemma 27 below. $\qquad\square$

Lemma 27. *We have*

$$\sum_{g=0}^{\infty} \Big\langle\, (d,\omega) \,\Big|\, (-1)^g \lambda_g \tau_0(\mathsf{p}) \,\Big|\, (d,\omega) \,\Big\rangle_{g,(d,1)}^{(\mathbb{P}^1\times\mathbb{P}^1)/\{0,\infty\}} u^{2g-2} = \frac{1}{u^2}\mathcal{S}(du)^2,$$

where

$$\mathcal{S}(u) = \frac{\sin(u/2)}{u/2}\,.$$

Proof. We replace $(-1)^g\lambda_g$ with $\mathbb{E}_g^\vee(s)$ with a formal variable s. We can remove the $\tau_0(\mathsf{p})$ insertion by the identity

$$\Big\langle (d,\omega) \,\Big|\, \mathbb{E}^\vee(s)\, \tau_0(\mathsf{p}) \,\Big|\, (d,\omega) \,\Big\rangle_{(d,1)} = \Big\langle (d,\omega) \,\Big|\, \mathbb{E}^\vee(s) \,\Big|\, (d,\omega) \,\Big\rangle_{(d,1)}^{\sim}.$$

The tilde on the right side denotes the rubber moduli space of maps to $\mathbb{P}^1 \times \mathbb{P}^1$ relative to 0 and ∞ up to $\mathbb{C}^*$-scaling of the first $\mathbb{P}^1$ factor of the target. The identity follows by adding an insertion with the divisor equation and then using the marking to rigidify the $\mathbb{C}^*$-scaling.

Let us orient $\mathbb{P}^1 \times \mathbb{P}^1$ so the first factor is a horizontal coordinate and the second factor is vertical. Then the relative divisors are on the left and right and the $\mathbb{C}^*$-scaling acts horizontally. While there is no nontrivial horizontal torus action, there is a nontrivial $\mathbb{C}^*$-action in the vertical direction. We fix the vertical $\mathbb{C}^*$-action to have weight t for the normal direction along the upper edge and weight $-t$ for the normal direction along the lower edge. We choose equivariant lifts of the relative point insertions on each side corresponding to the upper fixed point on each vertical edge.
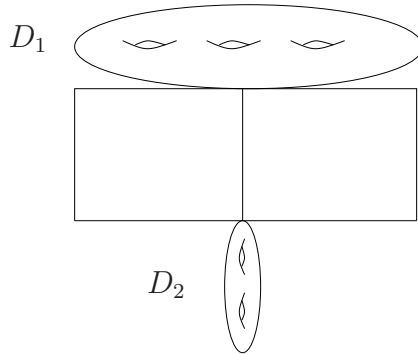

FIGURE 2. Rubber localization

If we localize, the $\mathbb{C}^*$-fixed stable maps have the following structure.[24] First, there is an arbitrary curve D_1 mapping with degree d to the

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[24]The analysis follows the localization calculation of [30].

upper horizontal $\mathbb{P}^1$, relative to both 0 and ∞, with a single relative point of multiplicity d. There is an arbitrary curve D_2 mapping with degree 0 to the lower horizontal $\mathbb{P}^1$ with no relative points. There is a single nonsingular rational curve mapping to $\mathbb{P}^1 \times \mathbb{P}^1$ with degree $(0,1)$, connecting D_1 and D_2. We may use the latter rational curve to rigidify the $\mathbb{C}^*$-scaling. The fixed-point contribution is

$$\sum_{g_1+g_2=g} t^2 \cdot \left\langle (d,\mathbf{1}) \; \middle| \; \mathbb{E}_{g_1}^\vee(s) \, \mathbb{E}_{g_1}^\vee(t) \, \frac{\mathrm{ev}_1^*(\mathsf{p})}{t-\psi_1} \; \middle| \; (d,\mathbf{1}) \right\rangle_{g_1,d}^{\mathbb{P}^1/\{0,\infty\}}$$

$$\cdot \frac{1}{t(-t)} \cdot \left\langle \emptyset \; \middle| \; \mathbb{E}_{g_2}^\vee(s) \, \mathbb{E}_{g_2}^\vee(-t) \, \frac{\mathrm{ev}_1^*(\mathsf{p})}{-t-\psi_1} \; \middle| \; \emptyset \right\rangle_{g_2,0}^{\mathbb{P}^1/\{0,\infty\}} .$$

Here, the first factor of t^2 comes from the relative insertions and the intermediate term $\frac{1}{t(-t)}$ comes from the $\mathbb{P}^1$ connecting D_1 and D_2. The point insertion in the first relative invariant comes from rigidifying the point where D_1 meets the connecting $\mathbb{P}^1$.

To simplify this expression, we choose the substitution $s = 1$ and $t = -1$ so we can apply the Mumford relation

$$\mathbb{E}_{g_1}^\vee(1)\mathbb{E}_{g_1}^\vee(-1) = (-1)^{g_1} .$$

The result is

$$\sum_{g_1+g_2=g} (-1)^g \left\langle (d,\mathbf{1}) \; \middle| \; \tau_{2g_1}(p) \; \middle| \; (d,\mathbf{1}) \right\rangle_{g_1,d}^{\mathbb{P}^1/\{0,\infty\}} \cdot \int_{\overline{M}_{g_2,1}} \frac{\mathbb{E}^\vee(1)\mathbb{E}^\vee(1)\mathbb{E}^\vee(0)}{1-\psi_1} .$$

The 1-pointed relative invariants of $\mathbb{P}^1$ are given in [40]

$$\sum_g (-1)^g u^{2g-2} \left\langle (d,\mathbf{1}) \; \middle| \; \tau_{2g}(\mathsf{p}) \; \middle| \; (d,\mathbf{1}) \right\rangle_{g,d}^{\mathbb{P}^1/\{0,\infty\}} = \frac{1}{u^2} \frac{\mathcal{S}(du)^2}{\mathcal{S}(u)} ,$$

and the triple Hodge integrals for $\overline{M}_{g,1}$ are given in [12] as

$$\sum_g (-1)^g u^{2g-2} \int_{\overline{M}_{g,1}} \frac{\mathbb{E}^\vee(1)\mathbb{E}^\vee(1)\mathbb{E}^\vee(0)}{1-\psi_1} = \frac{1}{u^2}\mathcal{S}(u) .$$

Putting everything together completes the proof. $\qquad\square$

7. Quasimodular forms

7.1. **Overview.** We follow the notation of Section 0.6. Let S be an elliptically fibered $K3$ surface, and let

$$\gamma_1, \ldots, \gamma_r \in H^*(S, \mathbb{Z})$$

be cohomology classes. Our main result here, Theorem 4, states the descendent series

$$\mathsf{F}_g^S(\tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r))$$

is a quasimodular form with simple pole at $q = 0$. The ring

$$\mathrm{QMod} = \mathbb{Q}[E_2(q), E_4(q), E_6(q)]$$

of holomorphic quasimodular forms (of level 1) is the $\mathbb{Q}$-algebra generated by Eisenstein series E_{2k}, see [5]. The ring QMod is naturally graded by weight (where E_{2k} has weight $2k$) and inherits an increasing filtration

$$\mathrm{QMod}_{\leq 2k} \subset \mathrm{QMod}$$

given by forms of weight $\leq 2k$. More precisely, we will prove the descendent series are of the form

$$\mathsf{F}_g^S(\tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)) \in \frac{1}{\Delta(q)}\mathrm{QMod}_{\leq 2g+2r}\,.$$

Our results follow from an explicit (though difficult) algorithm for calculating descendent series which reduces the claims to the case of $g = 0$ and the Gromov-Witten theory of elliptic curves.

7.2. **Elliptic curves.** We begin by explaining the quasimodularity statement for elliptic curves E. Given cohomology classes

$$\gamma_1,\ldots,\gamma_r \in H^*(E,\mathbb{Z}),$$

consider the following descendent series for connected Gromov-Witten invariants of E,

$$\mathsf{F}_g^E(\tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)) = \sum_{d\geq 0}\left\langle\tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\right\rangle_{g,d}^E q^d.$$

Proposition 28. *For $r > 0$, $\mathsf{F}_g^E(\tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r))$ is the Fourier expansion in q of a holomorphic quasimodular form of weight*

$$2g - 2 + 2\sum_{i=1}^r \deg(\gamma_i).$$

We use $\deg(\gamma_i)$ here to denote half the cohomological degree. For instance, point classes in $H^2(E,\mathbb{Z})$ are degree 1. The $r = 0$ case concerns the well-known counting of genus 1 covers of E. The resulting series is not in QMod.

Proof. When all the γ_i are point classes in $H^2(E, \mathbb{Z})$, the result is stated as a corollary of Theorem 5 in [39]. For the general case, we need only check that quasimodularity is preserved by the extended Virasoro relations for curves proved in [41] which provide rules for removing insertions $\tau_k(\gamma)$ for $\gamma \in H^{\leq 1}(E, \mathbb{Z})$. □

7.3. **Tautological classes.** We will rephrase Theorem 4 in terms of tautological classes. For $2g - 2 + r > 0$, let

$$R(\overline{M}_{g,r}) \subset H^*(\overline{M}_{g,r}, \mathbb{Q})$$

denote the subring of *tautological* cohomology classes. The tautological ring $R(\overline{M}_{g,r})$ is spanned by the push-forwards of products of ψ-classes on boundary strata, see [44] for an introduction.

Given $\alpha \in R(\overline{M}_{g,r})$ and $\gamma_1, \ldots, \gamma_r \in H^*(S, \mathbb{Z})$, we can use the forgetful morphism

$$\pi : \overline{M}_{g,r}(S, \beta) \to \overline{M}_{g,r}$$

to define the reduced invariant

$$\left\langle \pi^*(\alpha), \gamma_1, \cdots, \gamma_r \right\rangle_{g,\beta}^S = \int_{[\overline{M}_{g,r}(S,\beta)]^{\text{red}}} \pi^*(\alpha) \cup \prod_{i=1}^{r} \text{ev}_i^*(\gamma_i)$$

when $2g - 2 + r > 0$. Let

(86) $$\mathsf{F}_g^S(\alpha; \gamma_1, \ldots, \gamma_r) = \sum_{h=0}^{\infty} \left\langle \pi^*(\alpha), \gamma_{l_1}, \cdots, \gamma_{l_r} \right\rangle_{g,h}^S q^{h-1}$$

be the associated generating function. The index h of the sum stands for a primitive effective curve class $\beta \in H^2(S, \mathbb{Z})$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2.$$

Proposition 29. *We have*

$$\mathsf{F}_g^S(\alpha; \gamma_1, \ldots, \gamma_r) \in \frac{1}{\Delta(q)} \text{QMod}_{\leq 2g+2r}.$$

The splitting formula for standard Gromov-Witten theory takes a slightly modified form for the reduced Gromov-Witten theory of $K3$ surfaces. Given a reducible boundary divisor

$$\iota : \overline{M}_{g_1, r_1+1} \times \overline{M}_{g_2, r_2+1} \to \overline{M}_{g,r},$$

let Δ denote the pushforward of the fundamental class of the left side. The splitting formula for reduced classes is

$$\pi^*(\Delta) \cap [\overline{M}_{g,r}(S,\beta)]^{\mathrm{red}} =$$
$$\iota_*([\overline{M}_{g_1,r_1+1}(S,\beta)]^{\mathrm{red}} \times_S [\overline{M}_{g_2,r_2+1}(S,0)]^{\mathrm{vir}}$$
$$+ [\overline{M}_{g_1,r_1+1}(S,0)]^{\mathrm{vir}} \times_S [\overline{M}_{g_2,r_2+1}(S,\beta)]^{\mathrm{red}}) \ .$$

Similarly, for the irreducible boundary divisor given by

$$\iota : \overline{M}_{g-1,r+2} \to \overline{M}_{g,r} \ ,$$

we have

$$\pi^*(\Delta) \cap [\overline{M}_{g,r}(S,\beta)]^{\mathrm{red}} = [\overline{M}_{g-1,r+2}(S,\beta)]^{\mathrm{red}}.$$

See [23, 24] for proofs (again the methods of Section 4 could also be used).

Using the usual trading of cotangent line classes on $\overline{M}_{g,r}(S,\beta)$ and the above splitting formulas for the reduced class, we can easily express the descendent series

$$\mathsf{F}_g^S(\tau_{k_1}(\gamma_1) \cdots \tau_r(\gamma_{l_r}))$$

in terms of linear combinations of the series

$$\mathsf{F}_{g'}^S(\alpha'; \gamma_1', \ldots, \gamma_{r'}')$$

where $g' \leq g$ or $g' = g$ and $r' \leq r$. Hence, Proposition 29 implies Theorem 4.

7.4. Proof of Proposition 29.

We proceed by induction on the pair (g,r) where g is the genus of the domain curve and r is the number of insertions. We order such pairs (g,r) so $(g',r') < (g,r)$ if either

- $g' < g$ or
- $g' = g$ and $r' < r$.

We assume the Proposition is known for all series with $(g',r') < (g,r)$. We will give a procedure for reducing all invariants of type (g,r) to invariants of lower type in a manner preserving quasimodularity.

Base case: $(g,r) = (0, r \leq 2)$.

If $(g,r) = (0,0)$, then the Proposition follows from the Yau-Zaslow formula. If $g = 0$ and $r \leq 2$, we are in the unstable range $2g - 2 + r \leq 0$, so no α insertion appears. Then, for dimension reasons, there must be an insertion of the form

$$\tau_0(1), \tau_1(1) \text{ or } \tau_0(\gamma),$$

where $\gamma \in H^2(S, \mathbb{Z})$. Since the claim of the Proposition is preserved by applying either the string, dilaton or divisor equation, we can remove one insertion and strictly reduce r.

Case (i): $\deg \gamma_{l_i} \leq 1$ for all i.

We again use $\deg(\gamma)$ to denote half the cohomological degree of γ. In case (i), there are no point insertions. Since the reduced virtual dimension is $g + r$, the dimension constraint then implies

$$\deg \alpha \geq g.$$

By a strong form of Getzler-Ionel vanishing proved in Proposition 2 of [13], there exists a tautological class $\alpha' \in R(\partial \overline{M}_{g,r})$ satisfying

$$\iota_* \alpha' = \alpha,$$

where ι denote the inclusion of the boundary,

$$\iota : \partial \overline{M}_{g,r} \hookrightarrow \overline{M}_{g,r} .$$

Using α' and the splitting formula for the virtual class, we can express the series

$$\mathsf{F}_g(\alpha; \gamma_1, \ldots, \gamma_r)$$

as a linear combination of series of type $(g', r') < (g, r)$.

Case (ii): $\gamma_1 = \mathsf{p} \in H^4(S, \mathbb{Z})$.

We will degenerate S to the normal cone of an elliptic fiber E,

$$(87) \qquad\qquad S \rightsquigarrow S \times_E (E \times \mathbb{P}^1) ,$$

and use the degeneration formula for the reduced virtual class proven by Lee and Leung [23, 24]. Following the notation of Section 6, the required generating series are:

$$\mathsf{F}_g^{S/E}(\alpha; \gamma_1, \ldots, \gamma_r) = \sum_{h=0}^{\infty} \left\langle \pi^* \alpha \cup \prod \tau_0(\gamma_i) \,\Big|\, 1 \right\rangle_{g,h}^{S/E} q^{h-1} ,$$

$$\mathsf{F}_g^{(E \times \mathbb{P}^1)/E}(\alpha; \gamma_1, \ldots, \gamma_r) = \sum_{h=0}^{\infty} \left\langle \pi^* \alpha \cup \prod \tau_0(\gamma_i) \,\Big|\, \omega \right\rangle_{g, \mathsf{s} + h \cdot \mathsf{f}}^{(E \times \mathbb{P}^1)/E} q^h .$$

The Gromov-Witten invariants for S/E are reduced. The first step is to prove a quasimodularity result for the latter series.

Lemma 30. $\mathsf{F}_g^{(E \times \mathbb{P}^1)/E}(\alpha; \gamma_1, \ldots, \gamma_r) \in \mathrm{QMod}_{\leq 2g+2r}.$

Proof. The relative invariants of $(E \times \mathbb{P}^1)/E$ are algorithmically determined from the Gromov-Witten theory of E using localization (via the $\mathbb{C}^*$-action on $\mathbb{P}^1$) and the absolute/relative correspondence of [35]. The result is a combinatorial expression for the series

$$\mathsf{F}_g^{(E \times \mathbb{P}^1)/E}(\alpha; \gamma_1, \ldots, \gamma_r)$$

in terms of descendent series for E.

Each insertion in a descendent series for E contributes to the weight of the final answer by Proposition 28. The bound of $2g + 2r$ is obtained by an elementary analysis of the possible $\mathbb{C}^*$-fixed point loci of the moduli space $\overline{M}_{g,r}(E \times \mathbb{P}^1, \mathsf{s} + h\mathsf{f})$. $\qquad\square$

Lemma 31. *For $(g', r') < (g, r)$ and insertions*

$$\alpha' \in R(\overline{M}_{g',r'}), \quad \gamma_1', \ldots, \gamma_{r'}' \in H^*(S, \mathbb{Z}),$$

we have

$$\mathsf{F}_{g'}^{S/E}(\alpha'; \gamma_1', \ldots, \gamma_{r'}') \in \frac{1}{\Delta(q)} \mathrm{QMod}_{\leq 2g' + 2r'}.$$

Proof. The result follow from the degeneration formula and the inductive hypothesis. The degeneration formula of Lee and Leung applied to (87) yields

$$\begin{aligned}
\mathsf{F}_{g'}^{S}(\alpha'; \gamma_1', \ldots, \gamma_{r'}') = {}& \mathsf{F}_{g'}^{S/E}(\alpha'; \gamma_1', \ldots, \gamma_{r'}') \\
& + \sum_{(g'', r'') < (g', r')} \mathsf{F}_{g''}^{S/E}(\cdots) \cdot \mathsf{F}_{g'-g''}^{(E \times \mathbb{P}^1)/E}(\cdots).
\end{aligned}$$

The second sum is also over all distribution of insertions (both α' and the γ_i'). We conclude the relative series $\mathsf{F}^{S/E}$ can be expressed in terms of the absolute series F^S by a change of basis which is upper-triangular with respect to the ordering on pairs bounded by (g', r'). Moreover, the coefficients of the change of basis are given by the series $\mathsf{F}^{(E \times \mathbb{P}^1)/E}$. The Lemma then follows from Lemma 30 and the inductive hypothesis on F^S. $\qquad\square$

We now complete the analysis of Case (ii) by explaining a genus reduction procedure dependent upon the insertion $\tau_0(\mathsf{p})$. We use the formula of Lee and Leung applied to the degeneration (87). The insertion $\tau_0(\mathsf{p})$ is specialized to lie on the bubble $E \times \mathbb{P}^1$, and the remaining insertions are specialized arbitrarily. We obtain

$$(88) \quad \mathsf{F}_g^S(\alpha; \mathsf{p}, \gamma_2, \ldots, \gamma_r) = \sum_{(g', r') \leq (g, r)} \mathsf{F}_{g'}^{S/E}(\alpha', \cdots) \cdot \mathsf{F}_{g''}^{(E \times \mathbb{P}^1)/E}(\alpha''; \mathsf{p}, \cdots).$$

Again, we have suppressed the summation over the splitting of the tautological class α and distribution of the insertions.

Since the relative invariants on $E \times \mathbb{P}^1$ occurring in (88) have both relative and absolute point insertions, the domain genus must be positive. So $g'' > 0$ and $g' < g$. By Lemmas 30 and 31, every nonzero term on the right side of (88) is quasimodular. Therefore, the left side is quasimodular of the desired form. $\qquad\square$

7.5. Non-primitive classes.

Let S be an elliptically fibered $K3$ surface with section. Let
$$\mathsf{s}, \mathsf{f} \in H_2(S, \mathbb{Z})$$
denote the section and fiber classes as before. A natural descendent potential function for the reduced theory of $K3$ surfaces is defined by

$$\mathsf{F}^S_{g,m}\big(\tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r})\big) = \sum_{n=0}^{\infty} \Big\langle \tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r}) \Big\rangle^{\mathrm{red}}_{g,m\mathsf{s}+n\mathsf{f}} q^{m(n-m)}$$

for $g \geq 0$ and $m \geq 1$. The following conjecture[25] specializes to Theorem 4 in the primitive ($m = 1$) case.

Conjecture. $\mathsf{F}^S_{g,m}\big(\tau_{k_1}(\gamma_{l_1}) \cdots \tau_{k_r}(\gamma_{l_r})\big)$ *is the Fourier expansion in q of a quasimodular form of level m^2 with pole at $q = 0$ of order at most m^2.*

By the ring of quasimodular forms of level m^2 with possible poles at $q = 0$, we mean the algebra generated by the Eisenstein series E_2 over the ring of modular forms of level m^2.

APPENDIX A. REDUCED THEORIES REVISITED

A.1. Reduced obstruction theories.

Given a perfect obstruction theory for a scheme P, together with a surjection

$$(89) \qquad\qquad\qquad \mathrm{Ob} \to \mathcal{O}_P \ ,$$

we will attempt now to produce a reduced obstruction theory.

For our discussion, we assume the obstruction theory of P arises from the following standard model (which always holds locally). Let

$$P \subset A$$

---

[25]The conjecture was made earlier by two of us in [45].

be an embedding of P in a nonsingular ambient space A, cut out as the zero locus of a section s of a vector bundle $E \to A$. In other words,

$$s \colon E^{\vee} \to \mathcal{O}_A$$

generates the ideal $I \subset \mathcal{O}_A$ defining $P \subset A$. We obtain

$$
\begin{array}{ccc}
E^{\vee}|_P \longrightarrow \Omega_A|_P \longrightarrow \Omega_P \\
\downarrow{\scriptstyle s} \qquad\quad \| \qquad\qquad \| \\
I/I^2 \xrightarrow{\ d\ } \Omega_A|_P \longrightarrow \Omega_P,
\end{array}
$$

where the lower row is the exact sequence of Kähler differentials for $P \subset A$. The first horizontal arrow is the composition ds of the other arrows in the first square. We denote the resulting 2-term complex of locally free sheaves on P by

$$E^{\bullet} = \{E^{-1} \xrightarrow{\ ds\ } E^0\}.$$

Since $\{I/I^2 \xrightarrow{d} \Omega_A|_P\}$ is quasi-isomorphic to the truncated cotangent complex $\mathbb{L}_P$ of P, we obtain a morphism of complexes

$$
\begin{array}{ccccc}
E^{\bullet} & & & E^{-1} \xrightarrow{\ ds\ } E^0 \\
\downarrow & & = & \downarrow{\scriptstyle s} \qquad\qquad \| \\
\mathbb{L}_P & & & I/I^2 \xrightarrow{\ d\ } \Omega_A|_P,
\end{array}
$$

which is surjective on h^{-1} and an isomorphism on h^0. We have constructed a perfect obstruction[26] theory for P with obstruction sheaf Ob defined to be the cokernel of the *dual* map

$$E_0 \xrightarrow{\ ds\ } E_1 \ ,$$

where E_i denotes the dual of E^{-i}.

The surjection (89) yields a surjection $E_1 \to \mathcal{O}_P$ whose locally free kernel we denote by F_1,

(90) $$0 \to F_1 \to E_1 \to \mathcal{O}_P \to 0.$$

The map $E_0 \to E_1$ factors through F_1. After dualizing, we obtain the complex

(91) $$F^{\bullet} = \{F^{-1} \to E^0\}.$$

We would like to know when $F^{\bullet}$ gives a (smaller) obstruction theory for P.

---

[26]Of course, a different choice of (E, s) generating the same $I \subset \mathcal{O}_A$ gives a *different* obstruction theory for P, with a potentially different obstruction sheaf.

A simple example is given by considering the scheme

$$P = P_n = \operatorname{Spec} \mathbb{C}[x]/(x^n), \quad n \geq 2,$$

cut out of the ambient space $A = \mathbb{C}$ by the section $s = (x^n, 0)$ of the trivial rank 2 bundle E. Since

$$E_0 \xrightarrow{ds} E_1 \quad \text{is} \quad \mathcal{O}_P \xrightarrow{(nx^{n-1}, 0)} \mathcal{O}_P^{\oplus 2} \ ,$$

we find

(92) $$\mathrm{Ob} = \mathcal{O}_{P_{n-1}} \oplus \mathcal{O}_P$$

is the direct sum of the structure sheaf of $P_{n-1} \subset P$ and the structure sheaf of P. The second summand is locally free so we have a surjection

(93) $$\mathrm{Ob} \xrightarrow{(0,1)} \mathcal{O}_{\mathcal{P}} \ .$$

Here, replacing E_1 by F_1 amounts to removing the second summand of the original trivial rank 2 bundle E and working in the first summand. Since the section s lies in the first summand, the result is another obstruction theory for P.

A.2. **Questions.** The Kiem-Li construction of the reduced class, the powerful T^1-lifting result of [8, Proposition 6.13][27] and the simple example discussed in Section A.1 suggest the following very natural question.

(Q1) Given a perfect obstruction theory whose obstruction sheaf admits a locally free quotient, can the quotient be removed to leave another perfect obstruction theory ?

Since the problem is local we restrict ourselves to the standard model of Section A.1 for the obstruction theory of P, restricted to the case where the locally free quotient is trivial of rank one (89). We then ask if the obstruction theory $E^\bullet \to \mathbb{L}_P$ factors through the complex $F^\bullet$ of (91).

Dualizing the surjection $E_1 \to \mathcal{O}_P$ obtained from (89), yields a subsheaf

$$0 \to \mathcal{O}_P \to E^{-1} \ .$$

---

[27]Given a thickening $S_0 \subset S$ with square-zero ideal I, Buchweitz-Flenner show the obstructions to extending a map

$$f_0 \colon S_0 \to P \quad \text{to} \quad f \colon S \to P$$

lie in the kernel of any surjection $\mathrm{Ob} \to \mathcal{O}_P$ so long as the Kodaira-Spencer class of the thickening lies in $\mathrm{Ext}^1(\Omega_{S_0}, I)$. By [4, Theorem 4.5], Question 1 is equivalent to asking whether their result extends to thickenings in the larger group $\mathrm{Ext}^1(\mathbb{L}_{S_0}, I)$.

Then, question (Q1) specializes as follows.

(Q1') Does the composition $\mathcal{O}_P \to E^{-1} \to I/I^2$ always vanish?

In general, the answer to (Q1') is no. In the example of Section A.1, if we replace the surjection (93) by $(x, 1)$, then the resulting composition

$$\mathcal{O}_P \to E^{-1} \to I/I^2$$

sends 1 to x^{n+1}, nonzero in I/I^2 even for $n = 2$.

So we consider a weaker question. Given a splitting of the sequence (90), we write

$$E^{-1} = F^{-1} \oplus \mathcal{O}_P$$

after dualizing. We consider the diagram

(94)
$$
\begin{array}{ccc}
F^{-1} \oplus \mathcal{O}_P & \longrightarrow & E^0 \\
{\scriptstyle s=(s_1,s_2)}\big\downarrow & & \big\| \\
I/I^2 & \xrightarrow{\ d\ } & \Omega_A|_P
\end{array}
$$

and ask instead whether s_1 is surjective.

(Q2) Is the first component

$$s_1 \colon F^{-1} \to I/I^2$$

of the arrow s in (94) surjective ?

An affirmative answer to (Q1) implies an affirmative answer to (Q2), so (Q2) is weaker. However, an affirmative answer to (Q2) leads to a reduced obstruction theory (at least locally). We can also further weaken the questions.

(Q3) Is the answer to Questions 1' or 2 positive if the obstruction theory $E^\bullet$ is symmetric [3, 2] ?

While (Q2) does hold for the example of Section A.1, the following example shows the answer to both (Q2) and (Q3) is negative in general.

A.3. **Counterexample I.** Let f be a polynomial in x_1 and x_2 which is *not* homogeneous with respect to any grading of x_1 and x_2. For example,

$$f = x_1^2 + x_1 x_2 + x_2^3 .$$

We view f as polynomial in the ring $\mathbb{C}[x_1, x_2, t]$ which does not depend on t. Define $P \subset A = \mathrm{Spec}(\mathbb{C}[x_1, x_2, t])$ by the vanishing of the 1-form

$$\sigma = df + f\,dt = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + f\,dt\,.$$

In other words, P is defined by the ideal

$$(95) \qquad I = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f \right).$$

Since f is has no t dependence, $d\sigma = df \wedge dt$ vanishes on P (as df lies in the ideal I). Hence, σ is *almost closed* in the terminology of [2], and the resulting obstruction theory

$$E^\bullet = \{ TA|_P \xrightarrow{D\sigma} \Omega_A|_P \}$$

is symmetric.

The scheme P is cut out by the section[28] $s = (f_{x_1}, f_{x_2}, f)$ of the trivial rank 3 vector bundle on A. The induced map $E_0 \to E_1$ is

$$\begin{pmatrix} f_{x_1 x_1} & f_{x_2 x_1} & f_{t x_1} \\ f_{x_1 x_2} & f_{x_2 x_2} & f_{t x_2} \\ f_{x_1} & f_{x_2} & f_t \end{pmatrix} = \begin{pmatrix} f_{x_1 x_1} & f_{x_2 x_1} & 0 \\ f_{x_1 x_2} & f_{x_2 x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \ \mathcal{O}_P^{\oplus 3} \longrightarrow \mathcal{O}_P^{\oplus 3},$$

since all functions are t-independent and the partial derivatives of f vanish on P by definition. The symmetry of the matrix reflects the fact that the obstruction theory is symmetric.

The third summand of $E_1 = \mathcal{O}_P^{\oplus 3}$ is also a summand of the obstruction sheaf Ob, so Questions 2 and 3 ask whether s_1 is surjective. But

$$s_1 : \mathcal{O}_P^{\oplus 2} \to I/I^2 \quad \text{is} \quad s_1 = (f_{x_1}, f_{x_2}),$$

and the element $[f] \in I/I^2$ is not in the image of this map precisely because f is not quasihomogeneous [52].

A.4. **Critical locus condition.** An *a priori* stronger condition than having a symmetric obstruction theory is that $P \subset A$ should be the critical locus $\mathrm{Crit}(\phi)$ of a holomorphic function ϕ on A, with the induced obstruction theory

$$E^\bullet \ = \ \{ TA|_P \xrightarrow{D(d\phi)} \Omega_A|_P \}.$$

The symmetry of the Hessian of ϕ implies that $E^\bullet$ is indeed symmetric.

---

[28]We use the notation f_x for $\partial f / \partial x$.

(Q4) Is the answer to Questions $1'$ or 2 positive if there exists a holomorphic function ϕ on A for which

$$P = \mathrm{Crit}(\phi) \subset A$$

with the induced symmetric obstruction theory ?

In fact, the previous example (95) provides a counterexample also to (Q4) as it is easily seen[29] to be the critical locus of the holomorphic function

$$\Phi = e^t f$$

on $\mathbb{C}^3$. However, we will prove the answer to (Q4) is *yes* when the surjection $\mathrm{Ob} \to \mathcal{O}_P$ (coming from a vector field on P as in (26)) is the restriction of a vector field on A *along which the holomorphic function is constant.* In our application to stable pairs, A and P are products with $\mathbb{C}$, the vector field is ∂_t pulled back from $\mathbb{C}$, and the potential ϕ is $\mathbb{C}$-invariant.

Proposition 32. *Let ϕ be a holomorphic function on A, and let v be a vector field on A satisfying $v(\phi) = 0$. Suppose v does not vanish on*

$$P = \mathrm{Crit}(\phi) \subset A \ ,$$

and let $\mathrm{Ob} \to \mathcal{O}_P$ be the induced surjection (26). Then, the composition

$$\mathcal{O}_P \to E^{-1} \to I/I^2$$

is zero. Therefore $\{F^{-1} \to E^0\}$ defines a reduced perfect obstruction theory for P.

Proof. The map $\mathcal{O}_P \to E^{-1} = T_A|_P$ is defined by the section v of the tangent bundle. The map $E^{-1} \to I/I^2$ takes v to

$$[d\phi(v)] \in I/I^2 \ .$$

But, $d\phi(v) = v(\phi) = 0$. $\qquad\square$

The moduli space $\mathcal{P}/B$, obtained from families of nonsingular $K3$ surfaces, *is* the (relative) critical locus of a $\mathbb{C}$-invariant holomorphic function, so we can use Proposition 32 to produce a reduced obstruction theory.

---

[29]We thank Dominic Joyce for this observation. The published version of this paper erroneously claimed to prove that the scheme defined by the ideal (95) is not the critical locus of any holomorphic function on any smooth variety. A correct example of a zero scheme of an almost closed 1-form with this property is now constructed in [49].

Theorem 33. *The perfect obstruction theory $E^\bullet \to \mathbb{L}_{\mathcal{P}/B}$ of (17) factors through $F^\bullet$. The reduced class of Section 3.6 is really a virtual cycle for the obstruction theory $F^\bullet$.*

The only missing step in the proof of Theorem 33 is the expression of the moduli space $\mathcal{P} \to B$ locally as $\mathrm{Crit}(\phi)$ for a $\mathbb{C}$-invariant holomorphic function ϕ. There are two approaches to the question: via the methods of Joyce-Song [16] or via the announced work of Behrend-Getzler. Since we do not need Theorem 33, we omit the discussion here.

Appendix B. Boundary expressions *by A. Pixton*

Let β be a primitive effective class with self-intersection $2h - 2$ on a $K3$ surface S. We will compute here the Hodge integrals

$$R_{g,h} = R_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g$$

for $g \leq 3$ via the boundary geometry of the moduli space of curves. The results provide an alternate verification of the Katz-Klemm-Vafa conjecture in low genus.

For convenience, we scale the Eisenstein series to

$$C_{2g} = -\frac{B_{2g}}{2g \cdot (2g)!} E_{2g}.$$

By the proof of Corollary 2, we can rewrite the KKV conjecture as

$$\sum_{g \geq 0} \sum_{h \geq 0} R_{g,h} t^g q^{h-1} = \frac{1}{\Delta(q)} \exp\left(\sum_{g \geq 1} (-1)^g 2 C_{2g} t^g \right).$$

Our approach has two steps. First, we write λ_g as a linear combination of boundary strata of $\overline{M}_g$. This allows us to replace the Hodge integral with a linear combination of descendent integrals, which can be reduced for $g \leq 3$ to the purely stationary descendent integrals

$$\int_{[\overline{M}_{g,n}(S,\beta)]^{\mathrm{red}}} \tau_{k_1}(\mathsf{p}) \cdots \tau_{k_n}(\mathsf{p}).$$

Second, these integrals can be evaluated in terms of products of Gromov-Witten invariants of elliptic curves and the Yau-Zaslow formula.

Define quasimodular forms T_k in terms of the Gromov-Witten invariants of an elliptic curve E by

$$T_k = \sum_{\substack{i,j \geq 0 \\ 2i+j \leq k}} (-1)^{i+j} \frac{C_2^i}{i!} \sum_{d \geq 0} q^d \int_{[\overline{M}_{g,2}(E,d[E])]^{\mathrm{vir}}} \lambda_j \tau_k(\mathsf{p}) \tau_{k-2i-j}(\mathsf{p}) \,.$$

By the degeneration formula of [23], standard localization arguments and the Yau-Zaslow formula, we find

$$(96) \quad \sum_{h \geq 0} \left(\int_{[\overline{M}_{\sum_{i=1}^n k_i+n,n}(S,\beta)]^{\mathrm{red}}} \tau_{k_1}(\mathsf{p}) \cdots \tau_{k_n}(\mathsf{p}) \right) q^{h-1} =$$

$$\frac{1}{\Delta(q)} T_{k_1} \cdots T_{k_n} \,.$$

Formula (96) in the case $k_1 = \cdots = k_n = 0$ was proven by Bryan and Leung [6].

Faber and Pandharipande [12] describe how to replace Hodge insertions λ_j with descendent insertions, and the work of Okounkov and Pandharipande in [39], [40], [41] completely determines the descendent Gromov-Witten theory of target curves. Thus, the quasimodular forms T_k can be explicitly computed. We list the first two, expressed in terms of our scaled Eisenstein series C_{2m}:

$$T_0 = q\frac{d}{dq}C_2 = -2C_2^2 + 10C_4 \,,$$

$$T_1 = q\frac{d}{dq}\left(\frac{2}{3}C_2^2 - \frac{1}{3}C_4\right) = -\frac{8}{3}C_2^3 + 16C_2C_4 - 7C_6 \,.$$

We will also need the following formulas for the action of the differential operator $q\frac{d}{dq}$ on quasimodular forms:

$$q\frac{d}{dq}C_2 = -2C_2^2 + 10C_4 \,,$$

$$q\frac{d}{dq}C_4 = -8C_2C_4 + 21C_6 \,,$$

$$q\frac{d}{dq}C_6 = -12C_2C_6 + \frac{160}{7}C_4^2 \,,$$

$$q\frac{d}{dq}\left(\frac{1}{\Delta(q)}\right) = \frac{24C_2}{\Delta(q)} \,.$$

Since the $g = 0$ and $n = 0$ case of (96) is the Yau-Zaslow formula, we have

$$R_{0,h} = [q^{h-1}]\frac{1}{\Delta(q)} = [q^{h-1}][t^0]\frac{1}{\Delta(q)} \exp\left(\sum_{k\geq 0}(-1)^k 2C_{2k}t^k\right).$$

The case $g = 1$ is more involved. We want to rewrite λ_1 in terms of $\mathbb{Q}$-classes of boundary strata on a moduli space of curves. However, $\overline{M}_1$ is not stable, so we add a marked point. By the divisor equation,

$$R_{1,h} = \int_{[\overline{M}_{1,0}(S,\beta)]^{\text{red}}} (-1)^1\lambda_1 = \int_{[\overline{M}_{1,1}(S,\beta)]^{\text{red}}} (-1)^1\lambda_1 \operatorname{ev}_1^*(\beta^\vee),$$

where $\beta^\vee \cdot \beta = 1$.

Now, let $\delta_0 \in H^2(\overline{M}_{1,1})$ denote the $\mathbb{Q}$-class $[\Delta_0]$, where Δ_0 is the boundary locus of genus 0 curves with one node (and one marked point), which is just a single point. Since

$$\lambda_1 = \frac{1}{12}\delta_0 ,$$

we can remove the λ_1 insertion, restrict to maps from Δ_0, and resolve the node to obtain

$$\int_{[\overline{M}_{1,1}(S,\beta)]^{\text{red}}} (-1)^1\lambda_1 \operatorname{ev}_1^*(\beta^\vee) =$$

$$-\frac{1}{12}\cdot\frac{1}{2}\int_{[\overline{M}_{0,3}(S,\beta)]^{\text{red}}} \operatorname{ev}_1^*(\beta^\vee)(\operatorname{ev}_2 \times \operatorname{ev}_3)^*(\mathsf{D}),$$

where $\mathsf{D} \in H^4(S \times S)$ is the Poincaré dual of the diagonal embedding of S in $S \times S$. The extra factor of $\frac{1}{2}$ appears because there are two different ways of labeling the two new marked points. If we choose a basis

$$\gamma_0 = 1, \gamma_1, \ldots, \gamma_{22} \in H^2(S), \gamma_{23} = \mathsf{p}$$

for the cohomology of the $K3$ surface S, with dual basis $\{\gamma_i^\vee\}$, then

$$\mathsf{D} = \sum_{i=0}^{23} \gamma_i \times \gamma_i^\vee .$$

Now, the genus zero invariants involving pull-backs of $\gamma_{23} = \mathsf{p}$ all vanish for reasons of dimension. Therefore, we only have the terms

$$-\frac{1}{24}\sum_{i=1}^{22}\int_{[\overline{M}_{0,3}(S,\beta)]^{\text{red}}} \operatorname{ev}_1^*(\beta^\vee)\operatorname{ev}_2^*(\gamma_i)\operatorname{ev}_3^*(\gamma_i^\vee) .$$

Applying the divisor equation again yields

$$-\frac{1}{24}\sum_{i=1}^{22}(\beta\cdot\gamma_i)(\beta\cdot\gamma_i^\vee)\int_{[\overline{M}_{0,0}(S,\beta)]^{\mathrm{red}}}1\,.$$

The integral on the right is simply $R_{0,h}$, so

$$\begin{aligned}
R_{1,h} &= -\frac{1}{24}(\beta\cdot\beta)R_{0,h}\\
&= -\frac{h-1}{12}[q^{h-1}]\frac{1}{\Delta(q)}\\
&= [q^{h-1}]\left(-\frac{1}{12}q\frac{d}{dq}\left(\frac{1}{\Delta(q)}\right)\right)\\
&= [q^{h-1}]\left(-\frac{2C_2}{\Delta(q)}\right),
\end{aligned}$$

as predicted by the KKV conjecture.

In genus 2, we can write λ_2 in terms of boundary classes on $\overline{M}_2$. The relevant boundary strata are Δ_{00}, the generic element of which is a genus 0 curve with 2 nodes, and Δ_{01}, where the generic element is a genus 0 curve with 1 node intersecting a smooth genus 1 curve in a single point. The corresponding $\mathbb{Q}$-classes are $\delta_{00},\delta_{01}\in H^4(\overline{M}_2)$. The relation

$$\lambda_2=\frac{1}{120}(\delta_{00}+\delta_{01})$$

is well-known, see [38, Section 8].

Again, we can replace λ_2 by the classes δ_{00} and δ_{01} and then remove these classes by restricting to maps from curves in the corresponding boundary loci. After resolving the singularities of the source curves, we find

$$\begin{aligned}
R_{2,h} &= \int_{[\overline{M}_2(S,\beta)]^{\mathrm{red}}}(-1)^2\lambda_2\\
&= \frac{1}{120}\cdot\frac{1}{8}\int_{[\overline{M}_{0,4}(S,\beta)]^{\mathrm{red}}}(\mathrm{ev}_1\times\mathrm{ev}_2)^*(\mathsf{D})(\mathrm{ev}_3\times\mathrm{ev}_4)^*(\mathsf{D})\\
&\quad+\frac{1}{120}\cdot\frac{1}{2}\int_{[\overline{M}_{1,1}(S,\beta)]^{\mathrm{red}}\times[\overline{M}_{0,3}(S,0)]^{\mathrm{vir}}}(\mathrm{ev}_1\times\mathrm{ev}_2)^*(\mathsf{D})(\mathrm{ev}_3\times\mathrm{ev}_4)^*(\mathsf{D}).
\end{aligned}$$

In the second term, the curve class β cannot split nontrivially between the two irreducible components because β is primitive. One of the two components must be contracted to a point. In fact, only the rational component may be contracted because the moduli space $\overline{M}_{0,3}$ has dimension 0.

We now compute the two terms of $R_{2,h}$. The first term is completely analogous to the calculation in genus 1. We obtain

$$\frac{1}{120} \cdot \frac{1}{8} \int_{[\overline{M}_{0,4}(S,\beta)]^{\mathrm{red}}} (\mathrm{ev}_1 \times \mathrm{ev}_2)^*(\mathsf{D})(\mathrm{ev}_3 \times \mathrm{ev}_4)^*(\mathsf{D})$$

$$= \frac{1}{960}(2h-2)^2[q^{h-1}]\frac{1}{\Delta(q)}$$

$$= [q^{h-1}]\frac{1}{240}\left(q\frac{d}{dq}\right)^2\left(\frac{1}{\Delta(q)}\right)$$

$$= [q^{h-1}]\frac{\frac{11}{5}C_2^2 + C_4}{\Delta(q)}.$$

For the second term we have $\overline{M}_{0,3}(S,0) = S$ and

$$(\mathrm{ev}_3 \times \mathrm{ev}_4)^*(\mathsf{D}) = 24\mathsf{p} ,$$

so the integral reduces to

$$\frac{1}{10} \int_{[\overline{M}_{1,1}(S,\beta)]^{\mathrm{red}}} \mathrm{ev}^*(\mathsf{p}) .$$

Using (96), we find

$$\frac{1}{10}[q^{h-1}]\frac{T_0}{\Delta} = [q^{h-1}]\frac{-\frac{1}{5}C_2^2 + C_4}{\Delta(q)} .$$

Adding the two terms of $R_{2,h}$ yields

$$R_{2,h} = [q^{h-1}]\frac{2C_2^2 + 2C_4}{\Delta(q)} ,$$

which agrees with the KKV conjecture.

The genus 3 case is significantly more complicated. To start with, the tautological cohomology space containing λ_3 on $\overline{M}_3$ has rank 10. A basis has been found by Faber in [10]: 9 of the 10 generators can be chosen to be $\mathbb{Q}$-classes corresponding to boundary strata $(a),(b),\ldots,(i)$ depicted in Figure 6 of [10]. For the last generator, we let $[(j)]_{\mathbb{Q}}$ be the $\mathbb{Q}$-class corresponding to a genus 0 curve with 1 node intersecting a smooth genus 2 curve at a point, with a cotangent line class at the intersection point. Then, we can write

$$(-1)^3\lambda_3 = -\frac{1}{504}\left(\frac{1}{2}[(a)]_{\mathbb{Q}} + [(b)]_{\mathbb{Q}} + [(c)]_{\mathbb{Q}} + \frac{3}{10}[(d)]_{\mathbb{Q}} - \frac{2}{5}[(f)]_{\mathbb{Q}}\right.$$
$$\left. + 2[(g)]_{\mathbb{Q}} + 2[(j)]_{\mathbb{Q}}\right),$$

see [11].

Through arguments similar to those used in the genus 2 calculation, we can show that the integrals of all of these classes vanish except for those of $[(a)]_\mathbb{Q}, [(d)]_\mathbb{Q}, [(e)]_\mathbb{Q}$, and $[(j)]_\mathbb{Q}$. Since $[(e)]_\mathbb{Q}$ does not appear in the above formula for λ_3, we need to calculate only three integrals.

First, the class $[(a)]_\mathbb{Q}$ can be handled analogously to δ_{00} in the genus 2 case, since (a) is just the locus of genus 0 curves with 3 nodes. We calculate:

$$\int_{[\overline{M}_{3,0}(S,\beta)]^{\mathrm{red}}} [(a)]_\mathbb{Q} = \frac{1}{48}(\beta^2)^3[q^{h-1}]\frac{1}{\Delta(q)}$$

$$= [q^{h-1}]\frac{1}{6}\left(q\frac{d}{dq}\right)^3\left(\frac{1}{\Delta(q)}\right)$$

$$= [q^{h-1}]\frac{1760C_2^3 + 2400C_2C_4 + 840C_6}{\Delta(q)}.$$

The class $[(d)]_\mathbb{Q}$ is similarly obtained by adding a node to the genus 2 case δ_{01}, so we can compute

$$\int_{[\overline{M}_{3,0}(S,\beta)]^{\mathrm{red}}} [(d)]_\mathbb{Q}$$

$$= \frac{1}{4}\int_{[\overline{M}_{1,3}(S,\beta)]^{\mathrm{red}}\times S}(\mathrm{ev}_1\times\mathrm{ev}_2)^*(\mathsf{D})(\mathrm{ev}_3\times\mathrm{id})^*(\mathsf{D})(\mathrm{id}\times\mathrm{id})^*(\mathsf{D})$$

$$= \frac{1}{4}\cdot 24(2h-2)\int_{[\overline{M}_{1,1}(S,\beta)]^{\mathrm{red}}}\tau_0(\mathsf{p})$$

$$= [q^{h-1}]12q\frac{d}{dq}\left(\frac{q\frac{d}{dq}C_2}{\Delta(q)}\right)$$

$$= [q^{h-1}]\frac{-480C_2^3 + 1440C_2C_4 + 2520C_6}{\Delta(q)}.$$

Finally, we calculate the integral of the ψ-class $[(j)]_\mathbb{Q}$:

$$\int_{[\overline{M}_{3,0}(S,\beta)]^{\mathrm{red}}}[(j)]_\mathbb{Q} = \frac{1}{2}\int_{[\overline{M}_{2,1}(S,\beta)]^{\mathrm{red}}\times S}\psi_1(\mathrm{ev}_1\times\mathrm{id})^*(\mathsf{D})(\mathrm{id}\times\mathrm{id})^*(\mathsf{D})$$

$$= \frac{1}{2}\cdot 24\int_{[\overline{M}_{2,1}(S,\beta)]^{\mathrm{red}}}\tau_1(\mathsf{p})$$

$$= [q^{h-1}]\frac{12T_1}{\Delta(q)}$$

$$= [q^{h-1}]\frac{-32C_2^3 + 192C_2C_4 - 84C_6}{\Delta(q)}.$$

We now use the boundary formula for λ_3 and the above three calculations to obtain

$$
\begin{aligned}
R_{3,h} &= \int_{[\overline{M}_{3,0}(S,\beta)]^{\mathrm{red}}} (-1)^3 \lambda_3 \\
&= -\frac{1}{504}[q^{h-1}]\frac{1}{\Delta(q)}\Big(\frac{1}{2}(1760C_2^3 + 2400C_2C_4 + 840C_6) \\
&\qquad\qquad + \frac{3}{10}(-480C_2^3 + 1440C_2C_4 + 2520C_6) \\
&\qquad\qquad + 2(-32C_2^3 + 192C_2C_4 - 84C_6)\Big) \\
&= [q^{h-1}]\left(-\frac{\frac{4}{3}C_2^3 + 4C_2C_4 + 2C_6}{\Delta(q)}\right),
\end{aligned}
$$

as predicted by the KKV conjecture.

In higher genus, the boundary expressions for λ_g will likely lead to nonstationary descendent invariants of $K3$ surfaces. Using Theorem 4, the calculations can be, in principle, continued. An interesting question is how the KKV conjecture, proven in Theorem 1, constrains the possible boundary expressions for λ_g.

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