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*A Tannakian Interpretation of the
Elliptic Infinitesimal Braid Lie Algebras*

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A TANNAKIAN INTERPRETATION OF THE ELLIPTIC INFINITESIMAL BRAID LIE ALGEBRAS

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To Aleksandr Aleksandrovich Kirillov on his 3²²-th birthday, with admiration

ABSTRACT. Let $n \geq 1$. The pro-unipotent completion of the pure braid group of n points on a genus 1 surface has been shown to be isomorphic to an explicit pro-unipotent group with graded Lie algebra using two types of tools: (a) minimal models (Bezrukavnikov), (b) the choice of a complex structure on the genus 1 surface, making it into an elliptic curve E , and an appropriate flat connection on the configuration space of n points in E (joint work of the authors with D. Calaque). Following a suggestion by P. Deligne, we give an interpretation of this isomorphism in the framework of the Riemann-Hilbert correspondence, using the total space $E^\#$ of an affine line bundle over E , which identifies with the moduli space of line bundles over E equipped with a flat connection.

Introduction. Let T be a topological 2-torus, i.e., a closed, compact topological surface of genus 1. For $n \geq 1$, let $C_n(T)$ be its configuration space, defined as the complement of the union of all the diagonals of T^n . A base point $x \in C_n(T)$ being fixed, we denote by $\mathrm{PB}_{1,n}^x$ the fundamental group of $C_n(T)$ relative to x ; it is called the pure braid group of genus 1. One attaches to this group its pronipotent completion over \mathbb{Q} , which is a pronipotent \mathbb{Q} -group, and the Lie algebra of this \mathbb{Q} -group, which is a pronilpotent \mathbb{Q} -Lie algebra, which we will denote $\mathrm{LiePB}_{1,n}^x$. This Lie algebra is equipped with the descending filtration associated with the lower central series. The associated graded Lie algebra $\mathrm{grLiePB}_{1,n}^x$ is then a positively graded \mathbb{Q} -Lie algebra.

We denote by $\mathrm{LiePB}_{1,n}^x \hat{\otimes} \mathbb{C}$ the completed tensor product of $\mathrm{LiePB}_{1,n}^x$ with \mathbb{C} (inverse limit of the tensor products with \mathbb{C} of the filtered quotients of $\mathrm{LiePB}_{1,n}^x$). Then $\mathrm{LiePB}_{1,n}^x \hat{\otimes} \mathbb{C}$ is a complete, filtered complex Lie algebra, and is associated graded is isomorphic to $\mathrm{grLiePB}_{1,n}^x \otimes \mathbb{C}$.

Let $\mathfrak{t}_{1,n}$ be the Lie algebra with generators x_i, y_i ($i \in [1, n]$), t_{ij} ($i \neq j \in [1, n]$), and relations

$$(1) \quad \forall i, j \in [1, n], \quad [x_i, x_j] = [y_i, y_j] = 0,$$

$$(2) \quad \forall i \neq j \in [1, n], \quad [x_i, y_j] = t_{ij} = t_{ji},$$

$$(3) \quad \forall i \in [1, n], \quad [x_i, y_i] = - \sum_{j|j \neq i} t_{ij},$$

$$(4) \quad \forall i, j, k \in [1, n] \quad \text{with} \quad \#\{i, j, k\} = 3, \quad [x_k, t_{ij}] = [y_k, t_{ij}] = 0,$$

$$(5) \quad \forall i \neq j \in [1, n], \quad [x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0.$$

We set $\deg(x_i) = 1$, $\deg(y_i) = 0$ for $i \in [n]$, which induces a grading on $\mathfrak{t}_{1,n}$.

We set $\mathfrak{t}_{1,n}^{\mathbb{C}} := \mathfrak{t}_{1,n} \otimes \mathbb{C}$. We denote by $\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}}$ the degree completion of $\mathfrak{t}_{1,n}^{\mathbb{C}}$.

Theorem 0.1 ([Bez]). • *The graded Lie algebras $\mathrm{grLiePB}_{1,n}^x \otimes \mathbb{C}$ and $\mathfrak{t}_{1,n}^{\mathbb{C}}$ are isomorphic.*

• *$\mathrm{LiePB}_{1,n}^x \hat{\otimes} \mathbb{C}$ is isomorphic, as a completed filtered Lie algebra, to $\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}}$.*

The proof from [Bez] is based on the minimal model theory. In [CEE], we gave another proof of this theorem by choosing a complex structure on T , making it into an elliptic curve E , and by constructing a suitable connection on a principal bundle over $C_n(E)$ with structure group $\exp(\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}})$ (for a survey of the construction of this connection see [H]). The monodromy of this connection defines a morphism

$$(6) \quad \text{PB}_{1,n}^x \rightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}}),$$

which gives rise to isomorphisms $\text{LiePB}_{1,n}^x \hat{\otimes} \mathbb{C} \simeq \hat{\mathfrak{t}}_{1,n}^{\mathbb{C}}$ and $\text{grLiePB}_{1,n}^x \otimes \mathbb{C} \simeq \mathfrak{t}_{1,n}^{\mathbb{C}}$ enabling one to prove the announced statements. The whole construction arises as an elliptic analogue of the similar genus zero construction, and may also be viewed as a universal version of the KZB connection.

One of the main features of Theorem 0.1 is that it says that $\text{LiePB}_{1,n}^x \hat{\otimes} \mathbb{C}$ is isomorphic to the completion of a graded Lie algebra.

The purpose of this paper is, following a suggestion of P. Deligne, to give an interpretation of this isomorphism in the framework of the Riemann-Hilbert (RH) correspondence, thus providing a categorical approach to the results of [CEE]. Let us recall the framework of this correspondence.

Let X be a smooth complex algebraic variety. According to [Del], there is an equivalence of tensor categories (the RH correspondence) between:

- (i) the category $\text{VBFC}(X)$ of vector bundles with a flat connection on X with regular singularities;
- (ii) the category $\text{LS}(X)$ of topological local systems on X ;

One attaches to each tensor category its unipotent part (see §1.1.2). The RH correspondence then induces an equivalence between the unipotent parts of its two sides, namely

$$(7) \quad \text{RH}_{unip} : \text{VBFC}(X)_{unip} \xrightarrow{\sim} \text{LS}(X)_{unip};$$

it attaches, to each object of $\text{VBFC}(X)$, the local system of its horizontal sections.

Any point $x \in X$ gives rise to a fiber functors $F_x^{ls} : \text{LS}(X) \rightarrow \text{Vec}_{\mathbb{C}}$ and $F_x^{vb} : \text{VBFC}(X) \rightarrow \text{Vec}_{\mathbb{C}}$, equipped with a canonical isomorphism $F_x^{ls} \circ \text{RH} \simeq F_x^{vb}$.

The Tannakian group corresponding to F_x^{ls} is $\text{Aut}^{\otimes}(F_x^{ls}) \simeq \pi_1^B(X, x)$ (this is the Betti fundamental group of X with base point x).

Set $X := C_n(E)$. Then $\pi_1^B(X, x) = \text{PB}_{1,n}^{x,unip}(\mathbb{C})$, where the exponent *unip* means the prounipotent completion of a discrete group and $-(\mathbb{C})$ denotes the group of \mathbb{C} -points, so that one of the sides of the isomorphism of Theorem 0.1 relates to the left-hand side of the RH equivalence (7).

We prove:

Theorem 0.2. 1) *There exists:*

- a) *an explicit tensor functor*

$$F : \text{VBFC}(C_n(E))_{unip} \rightarrow \text{Vec}_{\mathbb{C}}$$

- b) *a natural isomorphism*

$$(8) \quad \text{VBFC}(C_n(E))_{unip} \ni (\mathcal{E}, \nabla) \mapsto i_{(\mathcal{E}, \nabla)} \in \text{Iso}_{\text{Vec}_{\mathbb{C}}}(F(\mathcal{E}, \nabla), F_x^{vb}(\mathcal{E}, \nabla))$$

between the functors F and F_x^{vb} ,

- c) *a canonical isomorphism $\text{Aut}^{\otimes}(F) \simeq \exp(\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}})$.*

- 2) *The composed isomorphism*

$$\exp(\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}}) \xrightarrow{\sim} \text{Aut}^{\otimes}(F) \xrightarrow{\sim} \text{Aut}^{\otimes}(F_x^{vb}) \xrightarrow[\text{RH}]{\sim} \text{Aut}^{\otimes}(F_x^{ls}) \xrightarrow{\sim} \text{PB}_{1,n}^{x,unip}(\mathbb{C})$$

coincides with the inverse of the completion of (6).

The group $\text{Aut}^\otimes(F_x^{vb})$ is the de Rham fundamental group of $C_n(E)$ with base point x , denoted $\pi_1^{DR}(C_n(E), x)$, and the isomorphism $\pi_1^{DR}(C_n(E), x) \simeq \text{Aut}^\otimes(F_x^{vb}) \xrightarrow{\text{RH}} \text{Aut}^\otimes(F_x^{ls}) \simeq \pi_1^B(C_n(E), x)$ is the 'comparison isomorphism'.

The construction of the functor F depends on some geometric background. We fix a resolution of singularities $\pi_0 : \tilde{X}_0 \rightarrow E^n$, such that if $D_0 \subset E^n$ is the union of all diagonals, then $\tilde{D} := \pi_0^{-1}(D_0)$ is a normal crossing divisor. Let $E^\#$ be the universal additive extension of E . This is a 2-dimensional commutative algebraic group, fitting in an exact sequence $0 \rightarrow H^0(E, \mathcal{O})^\vee \rightarrow E^\# \rightarrow E \rightarrow 0$ (see §5). It gives rise to a morphism $p : (E^\#)^n \rightarrow E^n$. Let $\tilde{X} := (E^\#)^n \times_{E^n} \tilde{X}_0$ and let $D := D_0 \times_{E^n} (E^\#)^n$, $\tilde{D} := D_0 \times_{E^n} \tilde{X}$. Then $\tilde{D} \subset \tilde{X}$ is a normal crossing divisor and there is a commutative diagram (see §2.1)

$$\begin{array}{ccccc} \tilde{D} & \longrightarrow & D & \longrightarrow & D_0 \\ \downarrow \subset & & \downarrow \subset & & \downarrow \subset \\ \tilde{X} & \longrightarrow & (E^\#)^n & \longrightarrow & E^n \end{array}$$

Let us now explain the construction of the functor F , which was proposed by P. Deligne in [Del1, Del2]. Let (\mathcal{E}, ∇) be a unipotent vector bundle with flat connection on $E^n - D_0$. One lifts (\mathcal{E}, ∇) to a unipotent vector bundle with flat connection on $\tilde{X} - \tilde{D}$. It canonically gives rise to a vector bundle on \tilde{X} with flat connection on $\tilde{X} - \tilde{D}$ admitting simple poles at \tilde{D} (see §3.1, isomorphism (c)). This object is the lift of a vector bundle on $(E^\#)^n$, equipped with a flat connection on $(E^\#)^n - D$ and simple poles at D . Since this is a unipotent object, and a unipotent vector bundle on $(E^\#)^n$ is trivial by the homological properties of $(E^\#)^n$, the obtained vector bundle $\mathcal{E}^\#$ on $(E^\#)^n$ is trivial. We have then $\mathcal{E}^\# \simeq V \otimes \mathcal{O}$, where $V := H^0((E^\#)^n, \mathcal{E}^\#)$. We then set $F(\mathcal{E}, \nabla) := V$.

So we see that the homological properties of $E^\#$ enable one to apply to the elliptic situation the framework from [Del3], §12.

The equivalence (8) is then given by the specialization map

$$(\mathcal{E}, \nabla) \mapsto [F(\mathcal{E}, \nabla) = V \simeq (V \otimes \mathcal{O})_x \simeq \mathcal{E}_x = F_x^{vb}(\mathcal{E}, \nabla)].$$

In order to prove the isomorphism $\text{Aut}^\otimes(F) \simeq \exp(\hat{\mathfrak{t}}_{1,n}^\mathbb{C})$, we construct a category equivalence between $\text{VBFC}(C_n(E))_{\text{unip}}$ and the category $\text{Vec}((E^\#)^n, D)$ of flat connections on trivial vector bundles over $(E^\#)^n$, unipotent and with simple poles at D . The computation of the latter category relies on the study of the algebra of differential forms over $(E^\#)^n$. Computation then shows that the Lie algebra $\text{Der}^\otimes(F)$ is graded; this fact originates from the graded structure of differential forms over $(E^\#)^n$ with poles at D , which comes from homogeneity properties of the Fay relations.

The organization of the text is described in the following table of contents.

CONTENTS

Introduction	1
Acknowledgements	4
1. Basic material and constructions	4
1.1. Categories	4
1.2. Divisors and residues	5
1.3. Vector spaces and maps attached to special divisors	6
1.4. Tensor categories of geometric origin	9
2. The geometric setup	9
2.1. Geometric data	9

2.2.	A class of examples	10
3.	The main results	10
3.1.	The main result: construction of a tensor equivalence	10
3.2.	Tensor categories and affine fibrations (equiv. (a))	11
3.3.	Deligne extension (equiv. (c))	12
3.4.	Category equivalences induced by desingularization (equiv. (d))	13
3.5.	A tensor category $\text{Vec}(Y, D)$ (equiv. (e))	13
3.6.	A Lie coalgebra attached to a pair (Y, D) (equiv. (f))	14
3.7.	Computation of \mathfrak{G} in the context of §2.2 (equiv. (g))	16
3.8.	Relation with the universal KZB connection	16
4.	Category equivalences induced by desingularization (equiv. (d))	16
4.1.	A geometric result	16
4.2.	Proof of Lemma 3.9	17
5.	Elliptic material	17
5.1.	Elliptic curves in characteristic zero	17
5.2.	The functor $E \mapsto E^\#$	20
5.3.	Algebraic Fay identity on $(E^\#)^n$	22
6.	Cohomological computations related to $E^\#$	27
6.1.	Spaces of differentials on $E^\#$	27
6.2.	Spaces of differentials on $(E^\#)^n$	29
6.3.	Computation of the kernel of $\odot : \Lambda^2(\Omega^1) \rightarrow \Omega^2$	32
7.	Presentation and computation of the Lie algebra \mathfrak{G} (equivs. (f) and (g))	40
7.1.	Grading on Ω^1	40
7.2.	A graded space \mathbf{I}	40
7.3.	Computation of the map $d : \Omega^1 \rightarrow \Omega^2$	41
7.4.	Grading on the coalgebra \mathbf{C} and the Lie coalgebra \mathfrak{C}	41
7.5.	Computation of the Lie algebra \mathfrak{G}	41
7.6.	An isomorphism $\mathfrak{t}_{1,n}^{\mathbb{C}} \simeq \mathfrak{G}$	44
8.	Elements of a description of $\text{VBFC}(X, D)_{\text{unip}}$ (equiv. (e))	47
8.1.	Reduction of a space of forms to Σ_{\log}	47
8.2.	Equality $\Omega^1 = \Sigma_{\log}$	49
9.	Relation with the universal KZB connection	50
9.1.	A flat connection on $(E^\#)^n$	50
9.2.	The universal KZB system	50
9.3.	Relation between the two systems	51
	References	51

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1. BASIC MATERIAL AND CONSTRUCTIONS

In this paper, we work over an algebraically closed field \mathbf{k} of characteristic 0.

1.1. Categories. In this §, we attach, to each tensor functor $\mathcal{C} \rightarrow \mathcal{D}$, a tensor functor $\mathcal{C}_{\text{unip}} \rightarrow \mathcal{D}_{\text{unip}}$. We will denote by Vec the tensor category of finite dimensional \mathbf{k} -vector spaces.

1.1.1. *Tensor categories.* Let \mathcal{C} be a locally finite, \mathbf{k} -linear, abelian, rigid monoidal category, such that the endomorphism ring of its unit object $\mathbf{1}_{\mathcal{C}}$ is isomorphic to \mathbf{k} and such that its tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is bilinear on morphisms. Such a category is called *tensor* in [EGNO]. According to *loc. cit.*, Prop. 4.21, the tensor product bifunctor \otimes is then biexact.

1.1.2. *Unipotent parts of tensor categories.* Define $\text{Ob}(\mathcal{C}_{unip})$ to be the subclass of $\text{Ob}(\mathcal{C})$ consisting of all objects O that admit a filtration $0 = O_0 \subset O_1 \subset \cdots \subset O_n = O$, such that each quotient O_i/O_{i-1} is isomorphic to $\mathbf{1}_{\mathcal{C}}$. One checks that $\text{Ob}(\mathcal{C}_{unip})$ is stable under the tensor product of \mathcal{C} .

Define \mathcal{C}_{unip} to be the full subcategory of \mathcal{C} whose class of objects is $\text{Ob}(\mathcal{C}_{unip})$. Then \mathcal{C}_{unip} is again a tensor category.

Let \mathcal{D} be another tensor category. A tensor functor from \mathcal{C} to \mathcal{D} is a pair (F, J) of an exact and faithful \mathbf{k} -linear functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a functorial isomorphism $J : F(-) \otimes F(-) \rightarrow F(- \otimes -)$, satisfying diagram (2.23) in [EGNO]. One checks that such a tensor functor induces a tensor functor (F_{unip}, J_{unip}) from \mathcal{C}_{unip} to \mathcal{D}_{unip} . One then has a diagram of tensor functors

$$\begin{array}{ccc} \mathcal{C}_{unip} & \xrightarrow{F_{unip}} & \mathcal{D}_{unip} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

where the vertical functors are fully faithful.

1.2. **Divisors and residues.** Let X be a smooth irreducible \mathbf{k} -variety.

1.2.1. *Sheaves.* We denote by \mathcal{O}_X the structure sheaf of X and by \mathcal{K}_X its sheaf of rational functions; this is a constant sheaf. If δ is a divisor of X (union of codimension 1 subvarieties), we denote by $\mathcal{O}_{X,\delta}$ the subsheaf of \mathcal{K}_X , such that for each open subset U of X , the space $\Gamma(U, \mathcal{O}_{X,\delta})$ is the space of rational functions over U (or X), that are regular on a dense open set in $U \cap \delta$. Restriction to δ induces a \mathcal{O}_X -sheaf morphism $\mathcal{O}_{X,\delta} \rightarrow i_*(\mathcal{K}_\delta)$, where $i : \delta \rightarrow X$ is the canonical inclusion and \mathcal{K}_δ is the direct sum $\oplus_i \mathcal{K}_{\delta_i}$, where $(\delta_i)_i$ are the irreducible components of δ .

If \mathcal{E} is a quasi-coherent \mathcal{O}_X -sheaf, we define \mathcal{E}^{rat} to be the sheaf $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ and \mathcal{E}_δ to be the sheaf $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\delta}$. Then $(\mathcal{O}_X)_\delta = \mathcal{O}_{X,\delta}$.

We also denote by $\Gamma_{rat}(X, \mathcal{E})$ the space of rational sections of \mathcal{E} . So $\Gamma_{rat}(X, \mathcal{E}) = \Gamma(X, \mathcal{E}^{rat})$.

1.2.2. *Divisors.* A *special divisor* (SD) in X is a divisor whose components are non-singular and such that any pair of components intersects transversally.

A *reduced normal crossing divisor* (RNCD) is a divisor $D = \cup_{i \in I} D_i$, whose components are non-singular and satisfying the following condition. For each point p of X , let $J(p)$ be the set of indices j such that p lies in D_j . Then p should have a neighborhood $U(p)$, in which each D_j , $j \in J(p)$ may be defined by an equation $f_j = 0$, and the collection of differentials $(df_j(p))_{j \in J(p)}$ is a linearly independent family in $T_p^*(X)$.

1.2.3. *Logarithmic differential forms for RNCDs.* Let D be a divisor. For k an integer ≥ 0 , the sheaf $\Omega_X^k(\log D)$ is the subsheaf of $\Omega_X^k(*D)$ whose local sections are differentials α such that both α and $d\alpha$ are regular except for a possible simple pole along D . The definition shows that if D is a RNCD, then $\Omega_X^k(\log D)$ can also be defined as follows (see [EV]). For p as above, the space of sections of this sheaf over $U(p)$ is given by the linear span, over all subsets $S \subset J(p)$, of all the differentials $(\bigwedge_{s \in S} (df_s/f_s)) \wedge a_S$, where a_S lies in $\Gamma(U(p), \Omega_X^{k-|S|})$. This alternative definition shows that the collection of sheaves $\Omega_X^\bullet(\log D)$ is stable under the differential and the wedge product.

1.2.4. *Poincaré residue: a sheaf morphism* $\Omega^k(X, \log \delta)^{rat} \rightarrow (\Omega_\delta^{k-1})^{rat}$. Let δ be a smooth irreducible codimension 1 subvariety of X .

Let $k \geq 0$. According to §1.2.1, $\Omega_X^k(\log \delta)_\delta$ is the subsheaf of $(\Omega_X^k)^{rat}$ defined as follows. For U an open subset of X , in which δ is defined by an equation $f = 0$, where $f \in \Gamma(U, \mathcal{O}_X)$, the space $\Gamma(U, \Omega_X^k(\log \delta)_\delta)$ is the set of differentials in $\Gamma_{rat}(X, \Omega_X^k)$ of the form $\omega = (df/f) \wedge a + b$, where $a \in \Gamma(U, (\Omega_X^{k-1})_\delta)$ and $b \in \Gamma(U, (\Omega_X^k)_\delta)$. In other terms, this space is the space of rational forms α on U , such that both α and $d\alpha$ have at most a simple pole at $\delta \cap U$. The map taking ω to the restriction of a to δ is well-defined and induces a sheaf morphism to the sheaf of rational differentials on δ

$$\text{Res}_\delta^{(k)} : \Omega_X^k(\log \delta)_\delta \rightarrow (\Omega_\delta^{k-1})^{rat}.$$

Taking global sections over X of this sheaf morphism, we obtain a linear map

$$\Gamma_{rat}(X, \Omega_X^k) \hookrightarrow \Gamma(X, \Omega_X^k(\log \delta)_\delta) \xrightarrow{\text{Res}_\delta^{(k)}} \Gamma(\delta, (\Omega_\delta^{k-1})^{rat}) = \Gamma_{rat}(\delta, \Omega_\delta^{k-1}).$$

1.3. **Vector spaces and maps attached to special divisors.** Let X be an irreducible, smooth \mathbf{k} -variety. We fix a SD $D = \cup_{i \in I} D_i$ in X (for convenience, we assume I to be ordered).

1.3.1. *Vector spaces attached to special divisors.* Recall that for each $i \in I$, the space of global sections $\Gamma(X, \Omega_X^1(\log D_i))$ is a subspace of $\Gamma_{rat}(X, \Omega_X^1)$. The sum of these spaces is a subspace

$$(9) \quad \Omega^1 := \sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i)) \subset \Gamma_{rat}(X, \Omega_X^1).$$

Similarly, for each pair $\{i, j\} \in \mathcal{P}_2(I)$ (the set of parts of I with cardinality 2), the space of global sections $\Gamma(X, \Omega_X^2(\log D_i \cup D_j))$ is a subspace of $\Gamma_{rat}(X, \Omega_X^2)$. The sum of these spaces is a subspace

$$(10) \quad \Omega^2 := \sum_{\{i, j\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_i \cup D_j)) \subset \Gamma_{rat}(X, \Omega_X^2).$$

1.3.2. *Maps attached to special divisors.* Let $i \in I$. As $\Omega_X^\bullet(\log D_i)$ is stable under the wedge product, there is a commutative diagram

$$\begin{array}{ccc} \Lambda^2(\Gamma(X, \Omega_X^1(\log D_i))) & \hookrightarrow & \Lambda^2(\Gamma_{rat}(X, \Omega_X^1)) \\ \downarrow & & \downarrow \\ \Gamma(X, \Omega_X^2(\log D_i)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

Let $\{i, j\} \in \mathcal{P}_2(I)$. As there are injective morphisms $\Omega_X^\bullet(\log D_i) \rightarrow \Omega_X^\bullet(\log D_i \cup D_j)$, $\Omega_X^\bullet(\log D_j) \rightarrow \Omega_X^\bullet(\log D_i \cup D_j)$, and as $\Omega_X^\bullet(\log D_i \cup D_j)$ is stable under the wedge product, there is a commutative diagram

$$\begin{array}{ccc} \Gamma(X, \Omega_X^1(\log D_i)) \otimes \Gamma(X, \Omega_X^1(\log D_j)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^1)^{\otimes 2} \\ \downarrow & & \parallel \\ \Gamma(X, \Omega_X^1(\log D_i \cup D_j))^{\otimes 2} & & \Gamma_{rat}(X, \Omega_X^1)^{\otimes 2} \\ \downarrow & & \downarrow \\ \Gamma(X, \Omega_X^2(\log D_i \cup D_j)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

The direct sum of these diagrams, together with the decomposition

$$\begin{aligned} & \Lambda^2(\oplus_{i \in I} \Gamma(X, \Omega_X^1(\log D_i))) \\ & \simeq \oplus_{i \in I} \Lambda^2(\Gamma(X, \Omega_X^1(\log D_i))) \oplus \left(\oplus_{\{i,j\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^1(\log D_i)) \otimes \Gamma(X, \Omega_X^1(\log D_j)) \right) \end{aligned}$$

gives a commutative diagram

$$\begin{array}{ccc} \Lambda^2(\oplus_{i \in I} \Gamma(X, \Omega_X^1(\log D_i))) & \longrightarrow & \Lambda^2(\Gamma_{rat}(X, \Omega_X^1)) \\ \downarrow & & \downarrow \\ \oplus_{i \in I} \Gamma(X, \Omega_X^2(\log D_i)) \oplus \left(\oplus_{\{i,j\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_i \cup D_j)) \right) & \longrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

As the images of the horizontal maps are respectively $\Lambda^2(\sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i)))$ and

$$\sum_{\{i,j\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_i \cup D_j)),$$

this gives:

Lemma 1.1. *There is a commutative diagram*

$$(11) \quad \begin{array}{ccc} \Lambda^2(\Omega^1) = \Lambda^2(\sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i))) & \hookrightarrow & \Lambda^2(\Gamma_{rat}(X, \Omega_X^1)) \\ \downarrow & & \downarrow \\ \Omega^2 = \sum_{\{i,j\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_i \cup D_j)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

We denote the resulting map by $\otimes : \Lambda^2(\Omega^1) \rightarrow \Omega^2$. The product in $\Lambda^\bullet(\Omega^1)$ will be denoted $\underline{\Delta}$, and we use the notation $a \otimes b := \otimes(a \underline{\Delta} b)$ for $a, b \in \Omega^1$.

As $\Omega_X^\bullet(\log D_i)$ is stable under the differential, there is a commutative diagram

$$\begin{array}{ccc} \Gamma(X, \Omega_X^1(\log D_i)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^1) \\ d \downarrow & & \downarrow d \\ \Gamma(X, \Omega_X^2(\log D_i)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

The direct sum of these diagrams yields a commutative diagram

$$\begin{array}{ccc} \oplus_{i \in I} \Gamma(X, \Omega_X^1(\log D_i)) & \longrightarrow & \Gamma_{rat}(X, \Omega_X^1) \\ d \downarrow & & \downarrow d \\ \oplus_{i \in I} \Gamma(X, \Omega_X^2(\log D_i)) & \longrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

Taking images of the horizontal maps, we obtain:

Lemma 1.2. *There is a commutative diagram*

$$(12) \quad \begin{array}{ccc} \Omega^1 = \sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^1) \\ d \downarrow & & \downarrow d \\ \Omega^2 = \sum_{i \in I} \Gamma(X, \Omega_X^2(\log D_i)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

The resulting map is denoted $d : \Omega^1 \rightarrow \Omega^2$.

Let $i \in I$. For any $\{k, l\} \in \mathcal{P}_2(I)$, both sheaves $\Omega_X^2(\log D_k \cup D_l)$ and $\Omega_X^2(\log D_i)_{D_i}$ are subsheaves of $(\Omega_X^2)^{rat}$. Inspection shows that there exists a natural sheaf morphism

$$\Omega_X^2(\log D_k \cup D_l) \rightarrow \Omega_X^2(\log D_i)_{D_i}$$

making the diagram

$$\begin{array}{ccc} \Omega_X^2(\log D_k \cup D_l) & \longrightarrow & (\Omega_X^2)^{rat} \\ \downarrow & \nearrow & \\ \Omega_X^2(\log D_i)_{D_i} & & \end{array}$$

commute. Taking global sections over X , one obtains a linear map

$$\Gamma(X, \Omega_X^2(\log D_k \cup D_l)) \rightarrow \Gamma(X, \Omega_X^2(\log D_i)_{D_i}).$$

Taking sums over $\{k, l\} \in \mathcal{P}_2(I)$, one obtains the upper morphism of the following diagram

$$\begin{array}{ccc} \oplus_{\{k, l\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_k \cup D_l)) & \longrightarrow & \Gamma(X, \Omega_X^2(\log D_i)_{D_i}) \\ \downarrow & & \downarrow \\ \sum_{\{k, l\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_k \cup D_l)) & \hookrightarrow & \Gamma_{rat}(X, \Omega_X^2) \end{array}$$

The composition of the left and bottom arrows of this diagram is the decomposition of the map $\oplus_{\{k, l\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_k \cup D_l)) \rightarrow \Gamma_{rat}(X, \Omega_X^2)$ induced by the sum, and the composition of the top and right arrows is also shown to coincide with the same map, so that the diagram commutes. Inspecting the diagram, one obtains the inclusion

$$\sum_{\{k, l\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_k \cup D_l)) \subset \Gamma(X, \Omega_X^2(\log D_i)_{D_i}).$$

One may compose this map with $\text{Res}_{D_i}^{(2)} : \Gamma(X, \Omega_X^2(\log D_i)_{D_i}) \rightarrow \Gamma_{rat}(D_i, \Omega_{D_i}^1)$. Examining the form of the images of elements of $\Gamma(X, \Omega_X^2(\log D_k \cup D_l))$ through the composed map, one sees that these images are 0 if $k, l \neq i$ and lie in $\Gamma(D_i, \Omega_{D_i}^1(\log D_i \cap D_l))$ (resp., in $\Gamma(D_i, \Omega_{D_i}^1(\log D_i \cap D_k))$) if $k = i$ (resp., if $l = i$), one obtains:

Lemma 1.3. *There is a unique map*

$$(13) \quad \text{Res}_{D_i}^{(2)} : \Omega^2 = \sum_{\{k, l\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_k \cup D_l)) \rightarrow \sum_{j \in I | j \neq i} \Gamma(D_i, \Omega_{D_i}^1(\log D_i \cap D_j)) =: \mathbf{D}_i^1$$

making the following diagram commute

$$\begin{array}{ccc} \Gamma(X, \Omega_X^2(\log D_i)_{D_i}) & \xrightarrow{\text{Res}_{D_i}^{(2)}} & \Gamma_{rat}(D_i, \Omega_{D_i}^1) \\ \uparrow & & \uparrow \\ \sum_{\{k, l\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_k \cup D_l)) & \xrightarrow{\text{Res}_{D_i}^{(2)}} & \sum_{j \in I | j \neq i} \Gamma(D_i, \Omega_{D_i}^1(\log D_i \cap D_j)) \end{array}$$

Dropping in this Subsection the normal crossing condition from the hypotheses on \overline{D} and arguing as in the construction leading to Lemma 1.3, one obtains:

Lemma 1.4. *There is a unique linear map*

$$(14) \quad \text{Res}_{D_i}^{(1)} : \sum_{k \in I} \Gamma(X, \Omega_X^1(\log D_k)) \rightarrow \Gamma(D_i, \mathcal{O}_{D_i})$$

making the following diagram

$$\begin{array}{ccc}
\Gamma(X, \Omega_X^1(\log D_i)_{D_i}) & \xrightarrow{\text{Res}_{D_i}^{(1)}} & \Gamma_{\text{rat}}(D_i, \mathcal{O}_{D_i}) \\
\uparrow & & \uparrow \\
\sum_{k \in I} \Gamma(X, \Omega_X^1(\log D_k)) & \xrightarrow{\text{Res}_{D_i}^{(1)}} & \Gamma(D_i, \mathcal{O}_{D_i})
\end{array}$$

commute. The restriction of $\text{Res}_{D_i}^{(1)}$ to each $\Gamma(X, \Omega_X^1(\log D_k))$, $k \neq i$, is zero.

Pick $i \in I$. Replacing the collection $(X, I, (D_i)_{i \in I}, D, i)$ by $(D_i, I - \{i\}, (D_j \cap D_i)_{j \in I - \{i\}}, D \cap D_i, j)$, the map (14) yields a linear map

$$(15) \quad \text{Res}_{D_i \cap D_j}^{(1)} : \mathbf{D}_i^1 = \sum_{j \in I - \{i\}} \Gamma(D_i, \Omega_{D_i}^1(\log D_i \cap D_j)) \rightarrow \Gamma(D_i \cap D_j, \mathcal{O}_{D_i \cap D_j}) =: \mathbf{D}_{ij}^0.$$

The sum of residue maps is a map

$$\Gamma_{\text{rat}}(X, \Omega_X^1) \hookleftarrow \sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i)) \rightarrow \oplus_{i \in I} \Gamma(D_i, \mathcal{O}_{D_i}).$$

The elements of the middle space are rational differentials on X , which are regular except for simple poles at D_i . The elements of the kernel of the map $\sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i)) \rightarrow \oplus_{i \in I} \Gamma(D_i, \mathcal{O}_{D_i})$ are therefore regular everywhere, that is, elements of $\Gamma(X, \Omega_X^1)$. We derive from there the exact sequence

$$(16) \quad 0 \rightarrow \Gamma(X, \Omega_X^1) \rightarrow \sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i)) \rightarrow \oplus_{i \in I} \Gamma(D_i, \mathcal{O}_{D_i}).$$

Summarizing the results of this §, we obtain:

Lemma 1.5. *To X and its SD $D = \cup_{i \in I} D_i$ are attached:*

- vector spaces Ω^1 and Ω^2 given by (9) and (10);
- linear maps $\odot : \Lambda^2(\Omega^1) \rightarrow \Omega^2$ and $d : \Omega^1 \rightarrow \Omega^2$ (see (11) and (12));
- vector spaces $(\mathbf{D}_i^1)_{i \in I}$ and $(\mathbf{D}_{ij}^0)_{\{i,j\} \in \mathcal{P}_2(I)}$ (see (13) and (15));
- maps $\text{Res}_{D_i}^{(2)} : \Omega^2 \rightarrow \mathbf{D}_i^1$ and $\text{Res}_{D_i \cap D_j}^{(1)} : \mathbf{D}_i^1 \rightarrow \mathbf{D}_{ij}^0$;
- an exact sequence $0 \rightarrow \Gamma(X, \Omega_X^1) \rightarrow \Omega^1 \xrightarrow{\oplus_{i \in I} \text{Res}_{D_i}^{(1)}} \oplus_{i \in I} \Gamma(D_i, \mathcal{O}_{D_i})$ (see (16)).

1.4. Tensor categories of geometric origin. For X a smooth, irreducible quasiprojective variety over \mathbf{k} , we define $\text{VBFC}(X)$ to be the category of pairs $(\mathcal{F}, \nabla_{\mathcal{F}})$, where \mathcal{F} is a vector bundle over X and $\nabla_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1$ is a flat connection on \mathcal{F} . When equipped with the tensor product of vector bundles with connection, it is a symmetric tensor category with unit object the pair $(\mathcal{O}_X, d : \mathcal{O}_X \rightarrow \Omega_X^1)$.

If $D \subset X$ is a divisor, we define $\text{VBFC}(X, D)$ to be the tensor category of vector bundles over X with flat connection over $X - D$ and simple poles at D .

2. THE GEOMETRIC SETUP

2.1. Geometric data. We give ourselves the following data:

- a smooth projective variety X_0 ;
- an affine fibration $p : X \rightarrow X_0$;
- a special divisor $D_0 \subset X_0$ (see §1.2.2); we set $D_0 := \cup_{\alpha \in I} D_{0\alpha}$, so that the $D_{0\alpha}$ are smooth and pairwise normal crossing;
- a resolution of singularities $\pi_0 : \tilde{X}_0 \rightarrow X_0$ such that $\tilde{D}_0 := \pi_0^{-1}(D_0)$ is a normal crossing divisor (NCD).

We define $D \subset X$ by $D := p^{-1}(D_0)$. We then set $\tilde{X} := \tilde{X}_0 \times_{X_0} X$ and $\tilde{D} \subset \tilde{X}$ by $\tilde{D} := \tilde{D}_0 \times_{X_0} X$. Then there is a commutative diagram

$$(17) \quad \begin{array}{ccccc} & & \tilde{D} & \xrightarrow{\quad} & D \\ & \swarrow & & \searrow & \\ & & \tilde{X} & \xrightarrow{\pi} & X \\ & \downarrow \tilde{p} & & \downarrow p & \\ & & \tilde{X}_0 & \xrightarrow{\pi_0} & X_0 \\ & \swarrow & & \searrow & \\ \tilde{D}_0 & \xrightarrow{\quad} & D_0 & & \end{array}$$

where the squares are Cartesian, and the divisors $\tilde{D}_0 \subset \tilde{X}_0$ and $\tilde{D} \subset \tilde{X}$ are NCDs.

2.2. A class of examples. We will work with the following example, based on the datum of an elliptic curve E over \mathbf{k} and an integer $n \geq 1$:

- $X_0 := E^n$;
- the affine fibration $p : X \rightarrow X_0$ is $(E^\sharp)^n \rightarrow E^n$ (see §5);
- the divisor $D_0 \subset X_0$ is $\cup_{i < j \in [n]} D_{ij}$;
- the resolution of singularities $\pi_0 : \tilde{X}_0 \rightarrow X_0$ is obtained by the Hironaka desingularization theorem (explicit examples may be derived from [FM] or [U]).

3. THE MAIN RESULTS

In §3.1, we explain the main result of this paper, which consists of the construction of a sequence of equivalences of tensor categories. The construction of these equivalences is explained in the following subsections of the present §. In the last subsection (§3.8), we explain the relation between this construction and the KZB connection from [CEE].

3.1. The main result: construction of a tensor equivalence. Let X_0, D_0, \dots be as in §2.2.

We will construct a sequence of equivalences of tensor categories

$$(18) \quad \begin{aligned} \text{VBFC}(X_0 - D_0)_{unip} &\stackrel{(a)}{\simeq} \text{VBFC}(X - D)_{unip} \stackrel{(b)}{\simeq} \text{VBFC}(\tilde{X} - \tilde{D})_{unip} \stackrel{(c)}{\simeq} \text{VBFC}(\tilde{X}, \tilde{D})_{unip} \\ &\stackrel{(d)}{\simeq} \text{VBFC}(X, D)_{unip} \stackrel{(e)}{\simeq} \text{Vec}(X, D) \stackrel{(f)}{\simeq} \text{Mod}(\mathfrak{G})_{unip} \stackrel{(g)}{\simeq} \text{Mod}(\mathfrak{t}_{1,n})_{unip}, \end{aligned}$$

where $\text{Vec}(X, D)$ is the category defined in §3.5 and $\text{Mod}(\mathfrak{g})$ denote the category of finite dimensional modules over a Lie algebra \mathfrak{g} . In this diagram, (a) comes from §3.2, (b) comes from $X - D = \tilde{X} - \tilde{D}$, (c) comes from taking the unipotent part of the isomorphism from §3.3, (d) is the result from §4, (e) comes from §3.5, (f) comes from §3.6, (g) comes from §3.7. All this induces an equivalence of tensor categories $\text{VBFC}(C(E, n))_{unip} \simeq \text{Mod}(\mathfrak{t}_{1,n}^{\mathbb{C}})_{unip}$.

The fiber functor on $\text{VBFC}(X_0 - D_0)_{unip}$ by this equivalence is the functor

$$F : \text{VBFC}(X_0 - D_0)_{unip} \rightarrow \text{Vec}$$

constructed as follows. To a unipotent bundle with flat connection (\mathcal{E}, ∇) on $X_0 - D_0$, we attach its lift $(\tilde{\mathcal{E}}, \tilde{\nabla})$ to $X - D$. One then extends this bundle with flat connection to a pair $(\tilde{\mathcal{E}}, \tilde{\nabla})$ of a

bundle $\bar{\mathcal{E}}$ over X with a flat connection $\bar{\nabla}$ with regular singularities on D . Unipotent bundles over X are trivial, so $\bar{\mathcal{E}}$ is canonically isomorphic to $\Gamma(X, \bar{\mathcal{E}}) \otimes \mathcal{O}_X$. One then sets

$$(19) \quad F(\mathcal{E}, \nabla) := \Gamma(X, \bar{\mathcal{E}}).$$

It follows from (18) and from the computation (19):

Theorem 3.1. *The functor $F : \text{VBFC}(X_0 - D_0)_{\text{unip}} \rightarrow \text{Vec}$ defined by (19) is a fiber functor. There is a tensor equivalence*

$$\text{VBFC}(C(E, n))_{\text{unip}} \simeq \text{Mod}(\mathfrak{t}_{1,n}^{\mathbb{C}})_{\text{unip}},$$

compatible with the fiber functors on both sides.

Remark 3.2. This result could as well be obtained as the following sequence of equivalences of tensor categories

$$\begin{aligned} \text{VBFC}(X_0 - D_0)_{\text{unip}} &\stackrel{(b')}{\simeq} \text{VBFC}(\tilde{X}_0 - \tilde{D}_0)_{\text{unip}} \stackrel{(c')}{\simeq} \text{VBFC}(\tilde{X}_0, \tilde{D}_0)_{\text{unip}} \stackrel{(d')}{\simeq} \text{VBFC}(X_0, D_0)_{\text{unip}} \\ &\stackrel{(a')}{\simeq} \text{VBFC}(X, D)_{\text{unip}} \stackrel{(e)}{\simeq} \text{Vec}(X, D) \stackrel{(f)}{\simeq} \text{Mod}(\mathfrak{G})_{\text{unip}} \stackrel{(g)}{\simeq} \text{Mod}(\mathfrak{t}_{1,n}^{\mathbb{C}})_{\text{unip}}, \end{aligned}$$

based on the "left-bottom" part of diagram (17) (as opposed to the "top-right" part).

3.2. Tensor categories and affine fibrations (equiv. (a)). If Y is an affine variety over \mathbf{k} , then $H^1(Y, \mathcal{O}_Y) = \text{Ext}_Y^1(\mathcal{O}_Y, \mathcal{O}_Y) = 0$. Applying this for $Y = \mathbb{A}^m$, we obtain that if (\mathcal{E}, ∇) is an object of $\text{VBFC}(\mathbb{A}^m)_{\text{unip}}$, and if $0 = (\mathcal{E}_0, \nabla_0) \subset \cdots \subset (\mathcal{E}_N, \nabla_N) = (\mathcal{E}, \nabla)$ is a filtration whose subquotients are isomorphic to the trivial object, then the diagram $0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_N = \mathcal{E}$ is isomorphic to the diagram $0 = \mathcal{O}_{\mathbb{A}^m}^{\oplus 0} \subset \cdots \subset \mathcal{O}_{\mathbb{A}^m}^{\oplus N}$. The image of the connection ∇ is then necessarily of the form $d + A$, where A is a strictly upper-triangular $N \times N$ matrix with coefficients in $\Omega_{\mathbb{A}^m}^1$, such that $(d + A)^2 = 0$. Using the acyclicity of $\Omega_{\mathbb{A}^m}^0 \rightarrow \Omega_{\mathbb{A}^m}^1 \rightarrow \Omega_{\mathbb{A}^m}^2$ (vanishing of the first De Rham cohomology group of \mathbb{A}^m), one constructs a unipotent $N \times N$ matrix n such that $d + A = ndn^{-1}$. Therefore $(\mathcal{E}, \nabla) \simeq (\mathcal{O}_{\mathbb{A}^m}^{\oplus N}, d)$: any object of $\text{VBFC}(\mathbb{A}^m)$ is isomorphic to $(\mathcal{O}_{\mathbb{A}^m}, d)^{\oplus N}$ for some $N \geq 0$. All this implies:

Lemma 3.3. *The functor $\text{VBFC}(\mathbb{A}^m) \rightarrow \text{Vec}$ of global sections defines an equivalence of categories*

$$\text{VBFC}(\mathbb{A}^m)_{\text{unip}} \xrightarrow{\sim} \text{Vec}.$$

A quasi-inverse is given by the operation $- \otimes (\mathcal{O}_{\mathbb{A}^m}, d)$ of taking the tensor product of a vector space with the unit object $(\mathcal{O}_{\mathbb{A}^m}, d)$.

Let Y be a \mathbf{k} -variety and \mathcal{E} be a vector bundle over Y . According to [Del], §I.2.2.4, p. 6, a connection ∇ over \mathcal{E} is equivalent to the data, for any \mathbf{k} -scheme S , any pair of morphisms $x, y : S \rightarrow Y$, infinitely close in the sense that the resulting morphism $S \rightarrow Y \times Y$ factorizes through the first infinitesimal neighborhood of the diagonal, of an isomorphism $\gamma_{x,y} : x^* \mathcal{E} \simeq y^* \mathcal{E}$, functorial with respect to base change and such that $\gamma_{x,x}$ is the identity. Moreover, ∇ is flat iff one has $\gamma_{y,z} \circ \gamma_{x,y} = \gamma_{x,z}$.

Let $p : Y^\# \rightarrow Y$ be a morphism of smooth, quasiprojective varieties over \mathbf{k} , locally isomorphic to a projection $\mathbb{A}^m \times U \rightarrow U$. Let $(\mathcal{E}^\#, \nabla^\#) \in \text{VBFC}(Y^\#)_{\text{unip}}$. For $x : S \rightarrow Y$, set $\mathcal{E}_x := \Gamma_{\text{hor}}(\mathcal{E}_{Y_x^\#}, \nabla_{|Y_x^\#}^\#)$, where $Y_x^\# := Y^\# \times_x S$ and $(\mathcal{E}_{Y_x^\#}^\#, \nabla_{|Y_x^\#}^\#)$ is the pull-back of $(\mathcal{E}^\#, \nabla^\#)$ to $Y_x^\#$. Using a trivialization of the projection, one obtains an isomorphism $\gamma_{x,y} : \mathcal{E}_x \simeq \mathcal{E}_y$, satisfying the base change and flatness conditions. In this way, and using Lemma 3.3, we define a tensor functor

$$p_* : \text{VBFC}(Y^\#)_{\text{unip}} \rightarrow \text{VBFC}(Y)_{\text{unip}}.$$

On the other hand, pull-back defines a tensor functor $\mathrm{VBFC}(Y) \rightarrow \mathrm{VBFC}(Y^\#)$, which restricts to a tensor functor

$$p^* : \mathrm{VBFC}(Y)_{\mathrm{unip}} \rightarrow \mathrm{VBFC}(Y^\#)_{\mathrm{unip}}.$$

Arguing locally as in the proof of Lemma 3.3, one obtains:

Lemma 3.4. *Let $p : Y^\# \rightarrow Y$ be a morphism of smooth, quasiprojective varieties over \mathbf{k} , locally isomorphic to a projection $\mathbb{A}^m \times U \rightarrow U$. The functors p_*, p^* define quasi-inverse category equivalences between $\mathrm{VBFC}(Y)_{\mathrm{unip}}$ and $\mathrm{VBFC}(Y^\#)_{\mathrm{unip}}$.*

Remark 3.5. The present functor p_* is related to the direct image functor p_* for D -modules as follows

$$\begin{array}{ccc} \mathrm{VBFC}(Y^\#)_{\mathrm{unip}} & \xrightarrow{\mathrm{can}} & \mathcal{D}_{Y^\#}\text{-mod} \\ p_* \downarrow & & \downarrow p_* \\ \mathrm{VBFC}(Y)_{\mathrm{unip}} & \xrightarrow{\mathrm{can}} \mathcal{D}_Y\text{-mod} \xrightarrow{-[-d]} & D^b(\mathcal{D}_Y\text{-mod}) \end{array}$$

where $-[-d]$ is the shift by degree $-d$.

3.3. Deligne extension (equiv. (c)). In this §, we assume that Y is a smooth quasiprojective \mathbf{k} -variety and that $D \subset Y$ is a NCD.

Let $\tau : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}$ be a lift of the canonical projection $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$.

Let $\mathrm{VBFC}(Y-D)_{\mathrm{unip.mon.}}$ be the full subcategory of $\mathrm{VBFC}(Y-D)$ corresponding to the vector bundles with flat connection over $Y-D$ for which the monodromy around each component of D is unipotent.

Let $\mathrm{VBFC}(Y, D)_{\mathrm{nilp.res.}}$ (resp., $\mathrm{VBFC}(Y, D)_\tau$) be the full subcategory of $\mathrm{VBFC}(Y, D)$ corresponding to the vector bundles over Y with flat connection on $Y-D$ with at most simple poles at D , whose residue at each component of D is nilpotent (resp., has spectrum contained in the image of τ). Contrary to $\mathrm{VBFC}(Y, D)_\tau$, the categories $\mathrm{VBFC}(Y, D)_{\mathrm{nilp.res.}}$ and $\mathrm{VBFC}(Y-D)_{\mathrm{unip.mon.}}$ are tensor subcategories of their ambient categories.

We decorate by the superscript *an* the analytic analogues of these categories.

Restriction induces a tensor functor

$$(20) \quad \widetilde{res} : \mathrm{VBFC}(Y, D)^{\mathrm{an}} \rightarrow \mathrm{VBFC}(Y-D)^{\mathrm{an}},$$

which further restricts to a functor

$$res : \mathrm{VBFC}(Y, D)_{\mathrm{nilp.res.}}^{\mathrm{an}} \rightarrow \mathrm{VBFC}(Y-D)_{\mathrm{unip.mon.}}^{\mathrm{an}}.$$

In [Del], Prop. 5.2, p. 91, Deligne constructed a tensor functor

$$(21) \quad \mathrm{can.ext.} : \mathrm{VBFC}(Y-D)_{\mathrm{unip.mon.}}^{\mathrm{an}} \rightarrow \mathrm{VBFC}(Y, D)_{\mathrm{nilp.res.}}^{\mathrm{an}}.$$

(canonical extension), such that $res \circ \mathrm{can.ext.} = id$. This functor is extended in [Del], Prop. 5.4, to a functor

$$\widetilde{\mathrm{can.ext.}} : \mathrm{VBFC}(Y-D)^{\mathrm{an}} \rightarrow \mathrm{VBFC}(Y, D)_\tau^{\mathrm{an}}$$

such that $\widetilde{res} \circ \widetilde{\mathrm{can.ext.}} = id$. In [Bri], Cor. 2, it is proved that \widetilde{res} is a category equivalence. It follows that res is a (tensor) category equivalence. Taking unipotent parts, one obtains:

Lemma 3.6. *If Y is a quasiprojective \mathbf{k} -variety and if $D \subset Y$ is a NCD, then there is an equivalence of tensor categories*

$$\mathrm{VBFC}(Y-D)_{\mathrm{unip}}^{\mathrm{an}} \simeq \mathrm{VBFC}(Y, D)_{\mathrm{unip}}^{\mathrm{an}}$$

Remark 3.7. The category $\text{VBFC}(Y, D)_{\text{unip}}$ is "much smaller" than $\text{VBFC}(Y, D)_{\text{unip.mon.}}$. For example, if $Y = \mathbb{P}^1$ and D is a set of points in \mathbb{A}^1 , if $N \geq 1$, if $(A_p)_{p \in D}$ is a collection of nilpotent matrices in $M_N(\mathbf{k})$ with $\sum_{p \in D} A_p = 0$, and if $A := \sum_{p \in D} A_p \cdot dz/(z - p)$, then the pair $(\mathcal{O}_Y^{\oplus N}, d + A)$ lies in $\text{Ob}(\text{VBFC}(Y, D)_{\text{unip.mon.}})$, but it lies in $\text{Ob}(\text{VBFC}(Y, D)_{\text{unip}})$ only if the matrices $(A_p)_{p \in D}$ can be simultaneously be upper-triangularized.

Remark 3.8. The canonical extension map from [Del, Bri] makes sense in the analytic category. To make sense of it in the algebraic category (in the unipotent case), one proceeds as follows: (1) start with (\mathcal{E}, ∇) over $Y - D$; (2) as (\mathcal{E}, ∇) has regular singularities, it admits an extension $(\bar{\mathcal{E}}, \bar{\nabla})$ with simple poles; (3) because of unipotence of monodromy, residues should have integer eigenvalues; (4) this induces a splitting of the restriction of $\bar{\mathcal{E}}$ to D according to generalized eigenvalues of the residues; (5) one modifies $\bar{\mathcal{E}}$ in order that the eigenvalues of the residues become 0; (6) the uniqueness of the extension is proved as follows: an isomorphism (over $Y - D$) between two extensions satisfies a differential equation, which implies that it has logarithmic growth. On the other hand, it is rational. Hence it is regular.

3.4. Category equivalences induced by desingularization (equiv. (d)). One proves in §4:

Lemma 3.9. *Let Y be a smooth quasiprojective \mathbf{k} -variety and $D \subset Y$ be a divisor. Let us fix a desingularization of (Y, D) , that is a morphism $\pi_Y : \tilde{Y} \rightarrow Y$ of smooth quasiprojective \mathbf{k} -varieties, such that the preimage $\tilde{D} := \pi_Y^{-1}(D)$ of D under π_Y is a NCD in \tilde{Y} and such that π_Y restricts to an isomorphism $\tilde{Y} - \tilde{D} \simeq Y - D$. There is a tensor equivalence*

$$\text{VBFC}(\tilde{Y}, \tilde{D})_{\text{unip}} \simeq \text{VBFC}(Y, D)_{\text{unip}}.$$

3.5. A tensor category $\text{Vec}(Y, D)$ (equiv. (e)).

3.5.1. A tensor category $\text{Vec}(Y, D)$. Let Y be a quasiprojective \mathbf{k} -variety such that $H^0(Y, \mathcal{O}_Y) = \mathbf{k}$ and $H^1(Y, \mathcal{O}_Y) = 0$, and let $D \subset Y$ be a divisor. Let $\text{Vec}(Y, D)$ be the following category:

- objects are the pairs (V, ω) , where V is a finite dimensional vector space, and ω is an element of $\Gamma(Y, \Omega_Y^1(D)) \otimes \text{End}(V)$, such that ω is strictly compatible with some filtration of V (i.e., satisfies $\omega(V^i) \subset \Gamma(Y, \Omega_Y^1(D)) \otimes V^{i+1}$ where $V = V^0 \supset V^1 \supset \dots \supset V^N = 0$ is the filtration of V) and satisfies the Maurer-Cartan equation $d\omega + [\omega, \omega] = 0$ (equality in $\Gamma(Y, \Omega_X^2(2D)) \otimes \text{End}(V)$);
- the set of morphisms from (V, ω) to (V', ω') is the set of linear maps $f : V \rightarrow V'$, such that $f\omega = \omega'f$ (equality in $\Gamma(Y, \Omega_Y^1(D)) \otimes \text{Hom}_{\mathbf{k}}(V, V')$).

If (V, ω) and (V', ω') are two objects, then:

- their direct sum is defined as $(V, \omega) \oplus (V', \omega') := (V \oplus V', \omega + \omega')$, where ω is induced by the canonical map $\text{End}(V) \rightarrow \text{End}(V \oplus V')$, and ω' is defined in a similar way;
- their tensor product is defined as $(V, \omega) \otimes (V', \omega') := (V \otimes V', \omega \otimes 1 + 1 \otimes \omega')$, where $\omega \otimes 1$ is induced by the map $\Gamma(Y, \Omega_Y^1(D)) \otimes \text{End}(V) \rightarrow \Gamma(Y, \Omega_Y^1(D)) \otimes \text{End}(V) \otimes \text{End}(V') \simeq \Gamma(Y, \Omega_Y^1(D)) \otimes \text{End}(V \oplus V')$ given by tensor product with $\text{id}_{V'}$ and $1 \otimes \omega'$ is defined similarly.

Lemma 3.10. *Let Y be a projective \mathbf{k} -variety and let $D \subset Y$ be a divisor. Then $\text{Vec}(Y, D)$ is a tensor category with unit object $(\mathbf{k}, 0)$.*

3.5.2. Isomorphism $\text{VBFC}(X, D)_{\text{unip}} \simeq \text{Vec}(X, D)$. Assume that (X, D) are as in §2.2.

It is known that $H^1(E^\#, \mathcal{O}_{E^\#}) = \mathbf{k}$. Together with the Künneth formula and $X = (E^\#)^n$, this implies that $H^1(X, \mathcal{O}_X) = 0$. It follows that any diagram $0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_N = \mathcal{E}$ of vector bundles over X with subquotients $\simeq \mathcal{O}_X$ is isomorphic to a diagram $0 = \mathcal{O}_X \otimes V_0 \subset$

$\cdots \subset \mathcal{O}_X \otimes V_N = \mathcal{O}_X \otimes V$, where $0 = V_0 \subset \cdots \subset V_N = V$ is a maximal filtration of a finite dimensional vector space V .

If then (\mathcal{E}, ∇) is an object of $\text{VBFC}(X, D)_{\text{unip}}$, the above isomorphism necessarily takes ∇ to a connection on $\mathcal{O}_X \otimes V$ of the form $d + \omega$, where ω is as described in the definition of the category $\text{Vec}(X, D)$. We obtain in this way:

Lemma 3.11. *If (X, D) are as in §2.2, then there is an equivalence of tensor categories*

$$\text{VBFC}(X, D)_{\text{unip}} \simeq \text{Vec}(X, D).$$

3.5.3. Full subcategories of $\text{Vec}(X, D)$. Let us come back to the framework of §3.5.1, so Y is a projective \mathbf{k} -variety and $D \subset Y$ is a divisor.

Let $\Sigma \subset \Gamma(Y, \Omega_Y^1(D))$ be a vector subspace. Define $\text{Vec}_\Sigma(Y, D)$ as the full subcategory of $\text{Vec}(Y, D)$, where the objects are the pairs (V, ω) as in the definition of $\text{Vec}(Y, D)$, such that $\omega \in \Sigma \otimes \text{End}(V)$. Then:

Lemma 3.12. *Let Y be a projective \mathbf{k} -variety, let $D \subset Y$ be a divisor, and let $\Sigma \subset \Gamma(Y, \Omega_Y^1(D))$ be a vector subspace. Then $\text{Vec}_\Sigma(Y, D)$ is a tensor subcategory of $\text{Vec}(Y, D)$.*

3.5.4. Equality $\text{Vec}(Y, D) = \text{Vec}_{\Sigma_{\log}}(Y, D)$. Let $(D_i)_{i \in I}$ be the components of D , so $D = \cup_{i \in I} D_i$; we assume that each divisor D_i is smooth and that these divisors intersect pairwise transversally. Let $\Sigma_{\log} \subset \Gamma(Y, \Omega_Y^1(D))$ be the subspace of differentials α defined by the following conditions:

- a) α is a logarithmic form with poles only at D (see §1.2.3);
- b) for each $i \in I$, $\text{res}_{D_i}(\alpha)$ is regular of the whole of D_i .

Lemmas 8.1 and 8.2 from §8 imply:

Lemma 3.13. *Let Y be a projective \mathbf{k} -variety, let $D \subset Y$ be a divisor, then $\text{Vec}(Y, D) = \text{Vec}_{\Sigma_{\log}}(Y, D)$.*

Remark 3.14. There are bundle injections $\Omega_Y^1(\log D_i) \hookrightarrow \Omega_Y^1(D_i) \hookrightarrow \Omega_Y^1(\sum_{i \in I} D_i)$, inducing an injection $\sum_{i \in I} \Omega_Y^1(\log D_i) \hookrightarrow \Omega_Y^1(\sum_{i \in I} D_i)$. Then $\Sigma_{\log} = \Gamma(Y, \sum_{i \in I} \Omega_Y^1(\log D_i))$.

3.5.5. Computation of Σ_{\log} . Let (X, D) be as in §2.2. Recall that for each pair (i, j) with $i < j \in [n]$, the space of global sections $\Gamma(X, \Omega_X^1(\log D_{ij}))$ is a subspace of $\Gamma_{\text{rat}}(X, \Omega_X^1)$. Recall from §1.3.2 that the sum of these spaces is a subspace

$$\Omega^1 = \sum_{i < j \in [n]} \Gamma(X, \Omega_X^1(\log D_{ij})) \subset \Gamma(X, \Omega_X^1(D)).$$

In §8.2, we prove:

Lemma 3.15. *Assuming that (X, D) are as in §2.2, one has $\Sigma_{\log} = \Omega^1$.*

Combining Lemmas 3.11, 3.13 and 3.15, one gets:

Lemma 3.16. *If (X, D) are as in §2.2, then one has $\text{Vec}(X, D) = \text{Vec}_{\Omega^1}(X, D)$.*

3.6. A Lie coalgebra attached to a pair (Y, D) (equiv. (f)). Let again (Y, D) be a pair of a quasiprojective \mathbf{k} -variety and a divisor $D \subset Y$, and let $\Sigma \subset \Gamma(Y, \Omega_Y^1(D))$ be a vector subspace.

3.6.1. Relation between $\text{Vec}_\Sigma(Y, D)$ and a category of comodules over a Lie coalgebra. The composed map $\Sigma \subset \Gamma(Y, \Omega_X^1(D)) \xrightarrow{d} \Gamma(Y, \Omega_X^2(2D))$ induced by the differential will be denoted

$$\Sigma \xrightarrow{d} \Gamma(Y, \Omega_X^2(2D));$$

the wedge product of forms will be denoted $\otimes : \Lambda^2(\Gamma(Y, \Omega_X^1(D))) \rightarrow \Gamma(Y, \Omega_X^2(2D))$, and its composed map with the inclusion $\Lambda^2(\Sigma) \subset \Lambda^2(\Gamma(Y, \Omega_X^1(D)))$ will be denoted

$$\Lambda^2(\Sigma) \xrightarrow{\otimes} \Gamma(Y, \Omega_X^2(2D)).$$

The tensor algebra $T(\Sigma)$ is a commutative bialgebra, when equipped with the shuffle product \mathfrak{m} and the deconcatenation coproduct Δ_{conc} .

Let

$$\mu : T(\Sigma) \rightarrow T(\Sigma) \otimes \Gamma(Y, \Omega_X^2(2D)) \otimes T(\Sigma)$$

be the map defined by

$$\mu([h_1 | \dots | h_n]) := \sum_{i=1}^n [h_1 | \dots | h_{i-1}] \otimes dh_i \otimes [h_{i+1} | \dots | h_n] + \frac{1}{2} \sum_{i=1}^{n-1} [h_1 | \dots | h_{i-1}] \otimes (h_i \otimes h_{i+1}) \otimes [h_{i+2} | \dots | h_n],$$

for $n \geq 0$, $h_1, \dots, h_n \in \Sigma$ (in particular, $\mu(1) = 0$). Then the following diagram commutes

$$\begin{array}{ccc} T(\Sigma) & \xrightarrow{\mu} & T(\Sigma) \otimes \Gamma(Y, \Omega_X^2(2D)) \otimes T(\Sigma) \\ \Delta_{conc} \downarrow & & \downarrow \text{id} \otimes \text{id} \otimes \Delta_{conc} \oplus \Delta_{conc} \otimes \text{id} \otimes \text{id} \\ T(\Sigma)^{\otimes 2} & \xrightarrow{\mu \otimes \text{id} \oplus \text{id} \otimes \mu} & (T(\Sigma) \otimes \Gamma(Y, \Omega_X^2(2D))) \otimes T(\Sigma) \oplus T(\Sigma) \otimes (T(\Sigma) \otimes \Gamma(Y, \Omega_X^2(2D))) \otimes T(\Sigma) \end{array}$$

This implies that $\mathbf{C}_\Sigma := \text{Ker}(\mu)$ is a subbialgebra of the shuffle bialgebra $T(\Sigma)$. One associates to it the Lie coalgebra $\mathfrak{C}_\Sigma := \text{Coprime}(\mathbf{C}_\Sigma) = \text{Coker}(\mathbf{C}_\Sigma^{\otimes 2} \rightarrow \mathbf{C}_\Sigma, a \otimes b \mapsto a \mathfrak{m} b - a \epsilon(b) - \epsilon(a)b)$, where \mathfrak{m} is the product of \mathbf{C}_Σ and $\epsilon : \mathbf{C}_\Sigma \rightarrow \mathbf{k}$ is its counit map.

Finite dimensional comodules over \mathbf{C}_Σ bijectively correspond to finite dimensional comodules over the Lie coalgebra \mathfrak{C}_Σ . These form a tensor category, denoted $\text{Comod}(\mathfrak{C}_\Sigma)$. One then has a tautological equivalence of tensor categories

$$(22) \quad \text{Vec}_\Sigma(Y, D_Y) \simeq \text{Comod}(\mathfrak{C}_\Sigma)_{unip}.$$

3.6.2. Gradedness in a particular situation. Assume that (X, D) is as in §2.1, so $D = \cup_{i \in I} D_i$ is a special divisor in X . Then for each pair $\{i, j\} \in \mathcal{P}_2(I)$, the divisors D_i and D_j intersect transversally. The space of global sections $\Gamma(X, \Omega_X^2(\log D_i \cap D_j))$ is then a vector subspace of $\Gamma(X, \Omega_X^2(2D))$. Recall from §1.3.2 the space

$$\Omega^2 = \sum_{\{i, j\} \in \mathcal{P}_2(I)} \Gamma(X, \Omega_X^2(\log D_i \cup D_j)) \subset \Gamma(X, \Omega_X^2(2D))$$

and the maps

$$d : \Omega^1 \rightarrow \Omega^2, \quad \otimes : \Lambda^2(\Omega^1) \rightarrow \Omega^2.$$

In §§7.1 and 7.4, we prove:

Lemma 3.17. *If (X, D) is as in §2.2, then the maps d, \otimes corestrict to maps $\Omega^1 \xrightarrow{d} \mathbf{I}$ and $\Lambda^2(\Omega^1) \xrightarrow{\otimes} \mathbf{I}$, where Ω^1 and \mathbf{I} are graded by \mathbb{N} (the integers ≥ 0) and have finite dimensional components. These maps have degree 0.*

It follows that the Lie coalgebra \mathfrak{C}_{Ω^1} is graded and has finite dimensional components.

Define $\mathfrak{G} := \mathfrak{C}_{\Omega^1}^\vee$, the graded dual of \mathfrak{C}_{Ω^1} . This is a graded Lie algebra with finite dimensional components. Dualization sets up a tensor equivalence

$$(23) \quad \text{Comod}(\mathfrak{C}_{\Omega^1})_{unip} \simeq \text{Mod}(\mathfrak{G})_{unip}.$$

Combining (22) for $\Sigma = \Omega^1$ and (23), we get:

Lemma 3.18. *If (X, D) is as in §2.2, then there is a tensor equivalence*

$$\mathrm{Vec}_{\Omega^1}(X, D) \simeq \mathrm{Mod}(\mathfrak{G})_{\mathrm{unip}}.$$

This tensor equivalence identifies the obvious (forgetful) fiber functors of both sides with each other.

Let $\omega \in (\mathfrak{G} \otimes \Omega^1)^\wedge$ be the canonical element (where $(-)^\wedge$ is the degree completion). Then $d\omega + [\omega, \omega] = 0$ (equality in the degree completion of $\mathfrak{G} \otimes \Omega^2$). Therefore:

Lemma 3.19. *$d + \omega$ is a flat connection on the trivial principal $\exp(\mathfrak{G})$ -bundle over $X - D$.*

3.7. Computation of \mathfrak{G} in the context of §2.2 (equiv. (g)). Recall the Lie algebra $\mathfrak{t}_{1,n}^\mathbb{C}$ from the Introduction. In §7 (Prop. 7.1), we prove:

Lemma 3.20. *There is an isomorphism of graded Lie algebras $\mathfrak{G} \simeq \mathfrak{t}_{1,n}^\mathbb{C}$.*

3.8. Relation with the universal KZB connection. For $\tau \in \mathfrak{H}$, set $E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. This is an analytic elliptic curve. There is a commutative diagram of analytic varieties

$$\begin{array}{ccc} \mathbb{C}^{2n} & \longrightarrow & (E_\tau^\#)^n \\ \downarrow & & \downarrow \\ \mathbb{C}^n & \longrightarrow & E_\tau^n \end{array}$$

By [CEE], E_τ^n is equipped with a principal $\exp(\hat{\mathfrak{t}}_{1,n}^\mathbb{C})$ -bundle $\mathcal{P}_{\mathrm{KZB}}$ with flat connection ∇_{KZB} ; the lift of $(\mathcal{P}_{\mathrm{KZB}}, \nabla_{\mathrm{KZB}})$ to \mathbb{C}^n identifies with a principal bundle with flat connection over \mathbb{C}^n

(trivial $\exp(\hat{\mathfrak{t}}_{1,n}^\mathbb{C})$ -bundle, $d + A_{\mathrm{KZB}}$).

On the other hand, Lemma 3.19 gives rise to a principal bundle with flat connection over $(E_\tau^\#)^n$,
(trivial $\exp(\mathfrak{G})$ -bundle, $d + \omega$).

In §9, we prove:

Theorem 3.1. *There is an isomorphism between the following principal bundles with flat connections over \mathbb{C}^{2n} :*

- the pull-back under $\mathbb{C}^{2n} \rightarrow \mathbb{C}^n \rightarrow (E_\tau)^n$ of $(\mathcal{P}_{\mathrm{KZB}}, \nabla_{\mathrm{KZB}})$;
- the pull-back under $\mathbb{C}^{2n} \rightarrow (E_\tau^\#)^n$ of $((E_\tau^\#)^n \times \exp(\mathfrak{G}), d + \omega)$.

4. CATEGORY EQUIVALENCES INDUCED BY DESINGULARIZATION (EQUIV. (d))

4.1. A geometric result. We work over \mathbb{C} . Let X be a smooth variety.

We define a unipotent vector bundle on X to be an iterated extension of copies of \mathcal{O}_X , i.e., a vector bundle which admits a filtration with associated graded $\mathcal{O}_X^{\oplus N}$ for some $N \geq 0$.

Similarly, a unipotent connection is a pair (bundle, connection) which admits a filtration with associated graded $(\mathcal{O}_X, d)^{\oplus N}$ for some $N \geq 0$.

Let X be a smooth variety and D a divisor in X . Let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities, such that $\tilde{D} := \pi^{-1}(D)$ is a normal crossing divisor and $\pi : \tilde{X} - \tilde{D} \rightarrow X - D$ is an isomorphism. Let $V \in \mathrm{VBFC}(\tilde{X}, \tilde{D})_{\mathrm{unip}}$, i.e., V is a unipotent vector bundle on \tilde{X} with a flat unipotent connection ∇ outside \tilde{D} , which has first order poles and nilpotent residues at \tilde{D} .

The following proposition was suggested to us by P. Deligne.

Proposition 4.1. *The bundle V descends to a unipotent vector bundle \bar{V} on X with respect to which the connection ∇ on $\tilde{X} - \tilde{D} \cong X - D$ has first order poles at D with nilpotent residues.*

Proof. Let $U \subset X$ be an affine open set. We claim that the restriction of the bundle V to $\pi^{-1}(U)$ is trivial. This follows immediately from the following lemma.

Lemma 4.2. *One has $H^i(\pi^{-1}(U), \mathcal{O}) = 0$ for $i > 0$.*

Proof. It is well known that $\pi_* \mathcal{O} = \mathcal{O}$, i.e., there is no higher direct images; see e.g. the beginning and Theorem 1 in [CR]. Thus, by adjunction $H^i(\pi^{-1}(U), \mathcal{O}) = H^i(U, \mathcal{O}) = 0$ for $i > 0$ since U is affine. \square

Indeed, by Lemma 4.2, we have $\text{Ext}_{\pi^{-1}(U)}^1(\mathcal{O}, \mathcal{O}) = 0$, so $V|_{\pi^{-1}(U)}$ is trivial, as it is unipotent.

Thus, we see that $V = \pi^* \bar{V}$, where $\bar{V} := \pi_* V$. In other words, the fibers of V at all points of $\pi^{-1}(p)$ for any $p \in X$ are canonically isomorphic to each other, and thus give rise to a well defined vector space, which is the fiber of \bar{V} at p , and it varies algebraically in p . It is clear that the connection ∇ on $X - D$ has simple poles and nilpotent residues at D with respect to V . This proves the proposition. \square

4.2. Proof of Lemma 3.9. Let $V \in \text{VBFC}(\tilde{X}, \tilde{D})_{\text{unip}}$. By Proposition 4.1, we can canonically attach to it a bundle $\bar{V} \in \text{VBFC}(X, D)_{\text{unip}}$. This gives rise to a functor

$$\pi_* : V \in \text{VBFC}(\tilde{X}, \tilde{D})_{\text{unip}} \mapsto \bar{V} \in \text{VBFC}(X, D)_{\text{unip}}.$$

It is clear that this functor is fully faithful, so it remains to show that it is essentially surjective. To this end, it suffices to note that for any $W \in \text{VBFC}(X, D)_{\text{unip}}$, we have $W \cong \pi_* \pi^* W$, where $\pi^* W \in \text{VBFC}(\tilde{X}, \tilde{D})_{\text{unip}}$ is the ordinary pullback of W . Indeed, this is definitely so outside of a set of codimension 2 on X (since there the map π is an isomorphism). But any isomorphism of vector bundles on X defined outside of a set of codimension 2 extends to the whole X . Thus, the functor π_* is an equivalence whose inverse is π^* .

Remark 4.3. This argument automatically yields that the flat connection on $\pi^* W$ has simple poles and nilpotent residues on each component of the exceptional divisor D' of π . Let us prove this fact independently. Let C be a smooth algebraic curve on \tilde{X} passing transversally through a generic point P of some component D'_j of D' and having no other intersection points with \tilde{D} . Then $\pi^* W|_C$ is a vector bundle on C with a unipotent connection outside of P , and our job is show that this connection has a simple pole with respect to $\pi^* W|_C$, and moreover the corresponding residue is nilpotent. To this end, consider the point $\pi(P) \in D$. Let D_{i_1}, \dots, D_{i_r} be the components of D containing $\pi(P)$. Let D_{i_m} be described near P by the equation $z_m = 0$, where z_m is a rational function on X regular at P with $dz_m(P) \neq 0$. Then, trivializing W near P , we get that the connection form takes the form

$$\omega = \sum_m \frac{A_{i_m}}{z_m} dz_m + \dots$$

where \dots is the regular part and A_{i_m} is the residue of the connection at D_{i_m} . Let d_m be the degree of intersection of D_{i_m} with C , i.e. the order of vanishing of $z_m|_C$ at P . Then it is easy to see that the connection on C has a simple pole at P with residue $A = \sum_m d_m A_{i_m}$. (In fact, if C is generic, then $d_m = 1$.) Since the connection on W is unipotent, A_{i_m} are strictly upper triangular in the same basis. Hence, so is A . Thus A is nilpotent, as desired.

5. ELLIPTIC MATERIAL

This § presents the material related to elliptic curves alluded to in §2.2: (a) elliptic curves in char. 0 (§5.1); (b) the construction of the functor $E \mapsto E^\#$, where for each elliptic curve E , $E^\#$ is a surface equipped with an affine fibration $E^\# \rightarrow E$ (§5.2). In §5.3, we prove an identity between functions on $E^\#$ derived from the Fay identity, which will be used in the sequel of the paper.

5.1. Elliptic curves in characteristic zero.

5.1.1. *Universal elliptic and theta functions.* Let $\underline{g}_2, \underline{g}_3$ be formal commutative variables; they generate the polynomial ring $\mathbb{Q}[\underline{g}_2, \underline{g}_3]$.

Lemma 5.1. *There exists a unique family $(a_n(\underline{g}_2, \underline{g}_3))_{n \geq 0}$ of elements of $\mathbb{Q}[\underline{g}_2, \underline{g}_3]$, such that the element*

$$(24) \quad \wp_{univ} := \frac{1}{p^2} + \sum_{n \geq 0} a_n(\underline{g}_2, \underline{g}_3) p^n \in \mathbb{Q}[\underline{g}_2, \underline{g}_3]((p))$$

satisfies

$$(25) \quad (\wp'_{univ})^2 = 4\wp_{univ}^3 - \underline{g}_2\wp_{univ} - \underline{g}_3$$

(identity in $\mathbb{Q}[\underline{g}_2, \underline{g}_3]((p))$), where \wp'_{univ} is the derivative of \wp_{univ} with respect to p .

This statement is well-known. We give a proof for completeness.

Proof. Explicit computation imposes $a_n = 0$ for n odd, and

$$a_0 = 0, \quad a_2 = \frac{1}{20}\underline{g}_2, \quad a_4 = \frac{1}{28}\underline{g}_4.$$

The equation

$$(26) \quad \wp''_{univ} = 6\wp_{univ}^2 - \frac{1}{2}\underline{g}_2,$$

consequence of the identity (25), then implies for $n \geq 6$

$$(n^2 - n - 12)a_n = 6 \sum_{p, q \geq 2, p+q=n-2} a_p a_q,$$

which determine $(a_n)_{n \geq 0}$ uniquely. Substituting these values in (24), one then obtains a solution of (26), which by multiplying by \wp'_{univ} and integrating is such that $(\wp'_{univ})^2 - 4\wp_{univ}^3 + \underline{g}_2\wp_{univ}$ is constant with respect to p ; as the constant term in this expression is $-\underline{g}_3$, this solution of (26) satisfies (25). \square

Remark 5.2. If the variables $\underline{g}_2, \underline{g}_3$ are given weights 4, 6, then a_n is a polynomial of weight $n + 2$.

Define

$$\zeta_{univ} := \frac{1}{p} - \sum_{n \geq 0} a_n(\underline{g}_2, \underline{g}_3) \frac{p^{n+1}}{n+1} \in \mathbb{Q}[\underline{g}_2, \underline{g}_3]((p)),$$

and

$$\tilde{\theta}_{univ} := p \cdot \exp\left(- \sum_{n \geq 0} a_n(\underline{g}_2, \underline{g}_3) \frac{p^{n+2}}{(n+1)(n+2)}\right) \in \mathbb{Q}[\underline{g}_2, \underline{g}_3][[p]],$$

then

$$\wp_{univ} = -\partial_p^2 \log \tilde{\theta}_{univ}.$$

5.1.2. *Specializations.* Let \mathbf{k} be a field of characteristic 0, let $(E, 0, \omega)$ be a triple of: an elliptic curve E over \mathbf{k} , an element $0 \in E(\mathbf{k})$, and a nonzero regular differential ω on E . Assume that E has a model $y^2 = 4x^3 - g_2x - g_3$, where $g_2, g_3 \in \mathbf{k}$, such that 0 corresponds to the point at infinity and $\omega = dx/y$. Then the ring of regular functions on $E - \{0\}$ is

$$(27) \quad A := \Gamma(E, \mathcal{O}_E(*0)) = \mathbf{k}[x, y]/(y^2 - 4x^3 + g_2x + g_3).$$

The elements of $\mathbf{k}((p))$ obtained from $\wp_{univ}(p), \wp'_{univ}(p)$ by the specialization $\mathbb{Q}[\underline{g}_2, \underline{g}_3] \rightarrow \mathbf{k}$, induced by $\underline{g}_2 \mapsto g_2, \underline{g}_3 \mapsto g_3$ depend only on the pair (E, ω) , and will henceforth be denoted $\wp_{E, \omega}(p), \wp'_{E, \omega}(p)$.

If $(E, 0, \omega)$ corresponds to (g_2, g_3) and $\alpha \in \mathbf{k}^\times$, then $(E, 0, \alpha\omega)$ corresponds to $(\alpha^{-4}g_2, \alpha^{-6}g_3)$. Remark 5.2 in §5.1.1 then implies the identity $\tilde{\theta}_{E, \alpha\omega}(p) = \alpha^2 \tilde{\theta}_{E, \omega}(\alpha^{-2}p)$.

There is a unique ring morphism

$$\begin{aligned} A &\rightarrow \mathbf{k}((p)), \\ x &\mapsto \wp_{E, \omega}(p), \quad y \mapsto \wp'_{E, \omega}(p). \end{aligned}$$

This morphism is injective.

Moreover, there is a derivation ∂ of A , uniquely determined by

$$\partial : x \mapsto y, \quad y \mapsto 6y^2 - \frac{1}{2}g_2.$$

This derivation is compatible with the derivation d/dp of $\mathbf{k}((p))$, so that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{k}((p)) \\ \partial \downarrow & & \downarrow d/dp \\ A & \longrightarrow & \mathbf{k}((p)) \end{array}$$

The derivation ∂ (resp., d/dp) corresponds to the regular differential dx/y of E (resp., dp of the formal disc); the differentials dx/y and dp correspond to each other under the parametrization of the formal disc around 0 induced by $A \rightarrow \mathbf{k}((p))$.

5.1.3. Relation with uniformization. For τ a complex number with positive imaginary part, let $\theta(-|\tau) : \mathbb{C} \rightarrow \mathbb{C}$ be the holomorphic function defined by

$$(28) \quad \theta(z|\tau) := \frac{1}{2\pi i} (\mathbf{e}(\frac{z}{2}) - \mathbf{e}(-\frac{z}{2})) \prod_{j \geq 1} (1 - \mathbf{e}(z + j\tau)) \prod_{j \geq 1} (1 - \mathbf{e}(-z + j\tau)),$$

where $i := \sqrt{-1}$ and $\mathbf{e}(z) := \exp(2\pi iz)$. It is such that

$$\theta(z + 1|\tau) = -\theta(z|\tau), \quad \theta(z + \tau|\tau) = -e^{-i\pi\tau} e^{-2\pi iz} \theta(z|\tau), \quad \theta'(0|\tau) = 1,$$

where $'$ means the partial derivative with respect to z .

Assume that $\mathbf{k} = \mathbb{C}$ and that $(E, 0, \omega)$ is such that $E_{an} \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and that ω corresponds to dp . Then

$$(29) \quad \tilde{\theta}_{\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), dp}(p) = \theta(p|\tau) e^{\frac{1}{2}G_2(\tau)p^2}$$

for p formal near 0 and $G_2(\tau)$ is the quasimodular Eisenstein series given by

$$\frac{1}{2}G_2(\tau) = \frac{\pi^2}{6} - (2\pi)^2 \sum_{n \geq 1} \sigma_1(n) q^n,$$

where $q = \mathbf{e}(\tau)$.

5.1.4. Analytic Fay identity. Let \mathfrak{H} be the set of complex numbers with positive imaginary part. Let $\tau \in \mathfrak{H}$. Fay's identity (see [Fay]) is the identity

$$(30) \quad \begin{aligned} &\theta(p + z|\tau)\theta(p + p' + z'|\tau)\theta(p'|\tau)\theta(z + z'|\tau) - \theta(p' + z'|\tau)\theta(p + z + z'|\tau)\theta(z|\tau)\theta(p + p'|\tau) \\ &+ \theta(p + p' + z + z'|\tau)\theta(p' - z|\tau)\theta(p|\tau)\theta(z'|\tau) = 0 \end{aligned}$$

in $\text{Hol}(\mathbb{C}^4)$, where (z, z', p, p') is the current variable in \mathbb{C}^4 .

The function

$$\mathbb{C}^2 \times \mathfrak{H} \rightarrow \mathbb{C}, \quad (p, z, \tau) \mapsto F(p, z|\tau) := \frac{\theta(p + z|\tau)}{\theta(p|\tau)\theta(z|\tau)}$$

is meromorphic. It expands for $z \rightarrow 0$ as a series $\sum_{k \geq -1} F_k(p|\tau) z^k$, where $F_k(p|\tau) \in \frac{1}{p} \text{Hol}(\mathfrak{H})[[p]]$. It follows that $F(p, z|\tau)$ may be viewed as an element of $\frac{1}{pz} \text{Hol}(\mathfrak{H})[[p, z]]$.

View now z, z', p, p' in (29) as formal variables. Dividing this equation by the product $\theta(p|\tau)\theta(p'|\tau)\theta(z|\tau)\theta(z'|\tau)\theta(p+p'|\tau)\theta(z+z'|\tau)$, we obtain the identity

$$(31) \quad F(p, z|\tau)F(p+p', z'|\tau) - F(p', z'|\tau)F(p, z+z'|\tau) + F(p+p', z+z'|\tau)F(p', -z|\tau) = 0$$

in $\frac{1}{pp'zz'(p+p')(z+z')}\text{Hol}(\mathfrak{H})[[p, p', z, z']]$.

5.1.5. *Formal series Fay identity.* Let $(g_2, g_3) \in \mathbb{C}^2$. Let $(E, 0, \omega)$ be the triple of the elliptic curve $y^3 = 4x^3 - g_2x - g_3$, the point at infinity, and the differential $\omega = dx/y$. Choose a uniformization $i : E_{an} \xrightarrow{\sim} \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, taking 0 to the origin. Then there exists a unique $\alpha \in \mathbb{C}^\times$, such that $\omega = \alpha \cdot i^*(dp)$. Computation then yields

$$\tilde{\theta}_{E, \omega}(p) = \alpha^2 \theta(\alpha^{-2}p|\tau) e^{\frac{1}{2}G_2(\tau)(\alpha^{-2}p)^2}$$

(identity in $\mathbb{C}[[p]]$).

View now p, z as formal variables and set

$$(32) \quad \tilde{F}_{E, \omega}(p, z) := \frac{\tilde{\theta}_{E, \omega}(p+z)}{\tilde{\theta}_{E, \omega}(p)\tilde{\theta}_{E, \omega}(z)}$$

(element of $\frac{1}{pz}\mathbb{C}[[p, z]]$). Then

$$\tilde{F}_{E, \omega}(p, z) = \alpha^{-2} F(\alpha^{-2}p, \alpha^{-2}z|\tau) e^{G_2(\tau)\alpha^{-4}pz}$$

(identity in $\frac{1}{pz}\mathbb{C}[[p, z]]$). Combining this identity with (31), one gets

$$(33) \quad \tilde{F}_{E, \omega}(p, z)\tilde{F}_{E, \omega}(p+p', z') - \tilde{F}_{E, \omega}(p', z')\tilde{F}_{E, \omega}(p, z+z') + \tilde{F}_{E, \omega}(p+p', z+z')\tilde{F}_{E, \omega}(p', -z) = 0$$

(identity in $\frac{1}{pp'zz'(p+p')(z+z')}\mathbb{C}[[p, p', z, z']]$).

The coefficients of this identity are polynomials in (g_2, g_3) and therefore arise from specialization of the analogous identity, where $\tilde{\theta}_{E, \omega}$ is replaced by $\tilde{\theta}_{univ}$, under specialization $\mathbb{Q}[g_2, g_3] \rightarrow \mathbb{C}$. It follows that the identity analogous to (33) holds with $\tilde{\theta}_{E, \omega}$ replaced by $\tilde{\theta}_{univ}$, and then upon further specialization that identity (33) also holds when $(E, 0, \omega)$ is defined over a field \mathbf{k} of characteristic 0.

5.2. **The functor** $E \mapsto E^\#$. In this § and in the next one, we choose an elliptic curve E as in §5.1.2. It is therefore defined over a field \mathbf{k} of characteristic 0, has fixed origin 0, and a nonzero regular differential ω .

5.2.1. *A section* $\sigma \in \Gamma_{rat}(E \times E, \Omega_E^1 \boxtimes \mathcal{O}_E)$.

Lemma 5.3. *There exists a unique rational section σ of the bundle $\Omega_E^1 \boxtimes \mathcal{O}_E$ over $E \times E$, regular except for:*

- a simple pole at E_{diag} with residue 1,
- a simple pole at $\{0\} \times E$ with residue -1 ,
- a simple pole at $E \times \{0\}$,

and such that the ratio $\sigma/(\omega \otimes 1)$ (a rational function on $E \times E$) is antisymmetric w.r.t. the exchange of variables.

Proof. If σ_1, σ_2 are two such rational functions, then their difference σ_3 is regular on E^2 and is therefore of the form $c \cdot (\omega \otimes 1)$, where $c \in \mathbf{k}$. The antisymmetry condition then implies that it is zero.

If E has a model $y^2 = 4x^3 - g_2x - g_3$ and 0 is the point at infinity in this model, then σ is given by

$$\sigma(P, Q) := \frac{y_P + y_Q}{x_P - x_Q} \cdot \frac{dx_P}{2y_P},$$

where $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$. \square

Remark 5.4. Assume that $\mathbf{k} = \mathbb{C}$ and that a uniformization $E_{an} \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is fixed. Then the image of σ under $(E_{an})^2 \simeq (\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}))^2$ is the differential

$$\left(\frac{\theta'}{\theta}(p_1 - p_2|\tau) - \frac{\theta'}{\theta}(p_1|\tau) + \frac{\theta'}{\theta}(p_2|\tau) \right) dp_1,$$

where (p_1, p_2) are the coordinates on \mathbb{C}^2 .

5.2.2. *The functor $E \mapsto E^\#$.* Define $E^\#$ to be the moduli space of line bundles of degree zero over E , equipped with a flat connection. The tensor product of bundles with connections makes $E^\#$ into a commutative algebraic group over \mathbf{k} , fitting in an exact sequence

$$(34) \quad 0 \rightarrow \Gamma(E, \Omega_E^1) \xrightarrow{\text{can}} E^\# \xrightarrow{\pi} E \rightarrow 0,$$

where $\text{can} : \Gamma(E, \Omega_E^1) \rightarrow E^\#$ is the canonical group morphism. According to [Me, MaMe] (based on [Ro]; for a recent account see [BoK]), $E^\#$ is the universal extension of the algebraic group E by vector spaces.

Given a pair (\mathcal{L}, σ) of a line bundle of degree zero on E equipped with a nonzero rational section, one may use σ as a rational trivialization in order to express connections on \mathcal{L} ; this leads to a bijection

$$\underline{\text{iso}} : \{\text{connections } \nabla \text{ on } \mathcal{L}\} \leftrightarrow \{\text{rational differentials } \psi \text{ on } E \text{ regular except for simple poles at the zeroes/poles of } \sigma, \text{ and with } \sum_{P \in E} \text{res}_P(\psi)P = (\sigma)\},$$

$$\nabla \mapsto \nabla(\sigma)\sigma^{-1}.$$

A morphism

$$s : E - \{0\} \rightarrow E^\#,$$

which is a rational section of the morphism $\pi : E^\# \rightarrow E$ may be constructed as follows. To a point P in $E - \{0\}$, it associates the pair of the bundle $\mathcal{O}(P - 0)$ and of the connection ∇_P corresponding via $\underline{\text{iso}}$ to the rational differential $\sigma(-, P)$ (this makes sense as the divisor of this differential is $P - 0$). The morphism s gives rise to an isomorphism of schemes

$$(35) \quad \tilde{s} : (E - \{0\}) \times \Gamma(E, \Omega_E^1) \rightarrow E^\# - \pi^{-1}(0),$$

given by $(P, \tilde{c}) \mapsto (\mathcal{O}(P - 0), \nabla_P + \tilde{c})$. The behavior of \tilde{s} with respect to the group structures on both sides is described by

$$\tilde{s}(P + P', \tilde{c} + \tilde{c}') = \tilde{s}(P, \tilde{c}) + \tilde{s}(P', \tilde{c}') + f(P, -P') \cdot \text{can}(\omega) \quad (\text{identity in } E^\#),$$

for any $P, P' \in E$ such that P, P' and $P + P'$ are $\neq 0$; here $+$ denotes the addition both in E , in $\Gamma(E, \Omega_E^1)$ and in $E^\#$, can is as in (34), and $f(P, P') \in \mathbf{k}$ is defined by $\sigma(P, P') = f(P, -P')\omega(P)$.

5.2.3. *Formal neighborhood of $\pi^{-1}(0)$ in $E^\#$.* Let E be as in §5.1.2. The element $\omega = dx/y = dp \in \Gamma(E, \Omega_E^1)$ is a basis of this vector space and sets up an isomorphism $\Gamma(E, \Omega_E^1) \simeq \mathbb{A}^1$. Combining the isomorphism (35) with this isomorphism, we obtain an isomorphism

$$(36) \quad \Gamma(E^\#, \mathcal{O}_{E^\#}(*\pi^{-1}(0))) \simeq \mathbf{k}[(E - \{0\}) \times \mathbb{A}^1] = \mathbf{k}[x, y, \tilde{c}]/(y^2 = 4x^3 - g_2x - g_3).$$

(see §5.1.2). We set

$$(37) \quad \mathcal{A} := \Gamma(E^\#, \mathcal{O}_{E^\#}(*\pi^{-1}(0))).$$

There is an isomorphism $\pi^{-1}(0) \simeq \Gamma(E, \Omega_E^1)$, and an isomorphism of the formal neighborhood of $\pi^{-1}(0)$ in $E^\#$ with $\Gamma(E, \Omega_E^1) \times \mathrm{Spf}(\mathbf{k}[[p]]) \simeq \mathbb{A}^1 \times \mathrm{Spf}(\mathbf{k}[[p]]) = \mathrm{Spf}(\mathbf{k}[t][[p]])$, corresponding to the injective algebra morphism

$$(38) \quad \mathrm{can} : \mathcal{A} \rightarrow \mathbf{k}[t]((p))$$

given by

$$(39) \quad x \mapsto \wp_{E,\omega}(p), \quad y \mapsto \wp'_{E,\omega}(p), \quad \tilde{c} \mapsto t - \frac{d \log \tilde{\theta}_{E,\omega}(p)}{dp}.$$

5.2.4. *Uniformization of $E^\#$.* Assume that $\mathbf{k} = \mathbb{C}$ and that a uniformization

$$(40) \quad E_{an} \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$

is fixed, such that dx/y corresponds to dp , p being the canonical coordinate on \mathbb{C} .

There is a morphism $\mathbb{Z}^2 \rightarrow \mathbb{C}^2$, given by $(n, m) \mapsto (n + m\tau, -2\pi im)$. Then there is an analytic isomorphism

$$E_{an}^\# \simeq \mathrm{Coker}(\mathbb{Z}^2 \rightarrow \mathbb{C}^2),$$

whose inverse takes the class of $(p, c) \in \mathbb{C}^2$ to the bundle $\mathcal{O}([p] - [0])$ equipped with the connection

$$(41) \quad d + \left(\frac{\theta'}{\theta}(z - p|\tau) - \frac{\theta'}{\theta}(z|\tau) + c \right) dz,$$

z being the standard coordinate on \mathbb{C} . Here $[p]$ is the class of p in E_{an} , $[p] - [0]$ is a degree zero divisor, $\mathcal{O}([p] - [0])$ is the associated line bundle, and $\theta(-|\tau)$ is defined by (28).

Taking into account the role of the section σ in the definition of the isomorphism \tilde{s} , we derive that the composite map

$$E_{an}^\# - \pi^{-1}(0) \xrightarrow{\sim} E^\#(\mathbb{C}) - \pi^{-1}(0) \xrightarrow{\tilde{s}} (E(\mathbb{C}) - \{0\}) \times \mathbb{C} \rightarrow \mathbb{C},$$

where the last map is the projection on the last factor, is the map $\tilde{c} : E_{an}^\# - \pi^{-1}(0) \rightarrow \mathbb{C}$ taking the class of (p, c) to

$$\tilde{c}(p, c) := c - (\theta'/\theta)(p|\tau).$$

The map \tilde{c} is a rational function on $E^\#$, regular except at $\pi^{-1}(0)$ where it has a simple pole. Taking into account (39) and (29), the local coordinate systems (p, c) and (p, t) at the neighborhood of $\pi^{-1}(0)$ are related by

$$t = c + G_2(\tau)p.$$

The relation between the algebraic and analytic coordinates on $E^\#$ is then:

$$[(p, c)] \leftrightarrow (x, y, \tilde{c}) = (\wp(p|\tau), \wp'(p|\tau), c - \frac{\theta'}{\theta}(p|\tau)).$$

Remark 5.5. There is, up to isomorphism, a unique 2-dimensional bundle \mathcal{E} over E , which is a nontrivial extension of \mathcal{O} by itself. One then has an exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \xrightarrow{\varphi} \mathcal{O} \rightarrow 0$. Let $\mathrm{Tot}(\mathcal{F})$ be the total space of a bundle \mathcal{F} over E . Then φ induces a morphism $\tilde{\varphi} : \mathrm{Tot}(\mathcal{E}) \rightarrow \mathrm{Tot}(\mathcal{O}) \simeq E \times \mathbb{A}^1$, and $E^\#$ identifies with the preimage $\tilde{\varphi}^{-1}(E \times \{1\})$. When $\mathbf{k} = \mathbb{C}$, this can be checked using a uniformization (40) of E_{an} . The bundle \mathcal{E} then identifies with the bundle $(\mathbb{C} \times \mathbb{C}^2)/\mathbb{Z}^2$, where the action of \mathbb{Z}^2 on $\mathbb{C} \times \mathbb{C}^2$ is $(n, m) \cdot (z, \vec{v}) := (z + n + m\tau, \begin{pmatrix} 1 & -2\pi im \\ 0 & 1 \end{pmatrix} \cdot \vec{v})$.

5.3. Algebraic Fay identity on $(E^\#)^n$.

5.3.1. *Elements in $\mathcal{A} = \Gamma(E^\#, \mathcal{O}_{E^\#}(*\pi^{-1}(0)))$.* Recall from §5.1.2 that ∂ is a derivation of $A = \mathbf{k}[E - \{0\}]$ (see (27)). It may be uniquely extended to a derivation of $\mathcal{A} \simeq A[\tilde{c}]$, also denoted ∂ , by $\partial(\tilde{c}) = x$. This extension is compatible with the morphism $\text{can} : \mathcal{A} \rightarrow \mathbf{k}[t]((p))$ and with the derivation $\partial/\partial p$ of the latter ring.

Lemma 5.6. *Let z be a formal variable. One has*

$$\frac{1}{\tilde{\theta}_{E,\omega}(z)} \exp(-\tilde{c}z - \sum_{k \geq 2} \frac{1}{k!} \partial^{k-2}(x) z^k) - \frac{1}{z} \in \mathcal{A}[[z]].$$

Proof. The element $\tilde{\theta}_{E,\omega}(z)$ belongs to $z + z^2 \mathbf{k}[[z]]$, therefore

$$\frac{1}{\tilde{\theta}_{E,\omega}(z)} - \frac{1}{z} \in \mathbf{k}[[z]],$$

and moreover $-\tilde{c}z - \sum_{k \geq 2} \frac{1}{k!} \partial^{k-2}(x) z^k$ belongs to $z\mathcal{A}[[z]]$, therefore

$$\frac{1}{\tilde{\theta}_{E,\omega}(z)} \left(\exp(-\tilde{c}z - \sum_{k \geq 2} \frac{1}{k!} \partial^{k-2}(x) z^k) - 1 \right) \in \mathcal{A}[[z]].$$

All this implies the result. □

Definition 5.7. The elements $(f_\alpha)_{\alpha \geq -1}$ of \mathcal{A} are defined by

$$(42) \quad f_\alpha := \left[\frac{1}{\tilde{\theta}_{E,\omega}(z)} \exp(-\tilde{c}z - \sum_{k \geq 2} \frac{1}{k!} \partial^{k-2}(x) z^k) \Big| z^\alpha \right],$$

where $[-|z^\alpha]$ means the coefficient of z^α in the formal series expansion (so $f_{-1} = 1$).

Lemma 5.8. *The following identity*

$$(43) \quad \tilde{\theta}_{E,\omega}(z) \left(\frac{1}{z} + \sum_{\alpha \geq 0} \text{can}(f_\alpha) z^\alpha \right) = e^{-tz} \frac{\tilde{\theta}_{E,\omega}(p+z)}{\tilde{\theta}_{E,\omega}(p)}.$$

holds in $\mathbf{k}[t]((p))[[z]]$.

Proof. By Def. 42, we have

$$(44) \quad \tilde{\theta}_{E,\omega}(z) \left(\frac{1}{z} + \sum_{\alpha \geq 0} f_\alpha z^\alpha \right) = \exp(-\tilde{c}z - \sum_{k \geq 2} \frac{1}{k!} \partial^{k-2}(x) z^k)$$

(identity in $\mathcal{A}((z))$).

The image of this identity under the morphism $\text{can} : \mathcal{A} \rightarrow \mathbf{k}[t]((p))$ (see (38)) is the first equality in the following computation

$$\begin{aligned} \tilde{\theta}_{E,\omega}(z) \left(\frac{1}{z} + \sum_{\alpha \geq 0} \text{can}(f_\alpha) z^\alpha \right) &= \exp \left(- \left(t - \frac{\tilde{\theta}'_{E,\omega}(p)}{\tilde{\theta}_{E,\omega}(p)} \right) z - \sum_{k \geq 2} \partial_p^{k-2} (-\partial_p^2 \log \tilde{\theta}_{E,\omega}(p)) \frac{z^k}{k!} \right) \\ &= e^{-tz} \exp((e^{z\partial_p} - 1) \log \tilde{\theta}_{E,\omega}(p)) = e^{-tz} \frac{\tilde{\theta}_{E,\omega}(p+z)}{\tilde{\theta}_{E,\omega}(p)} \end{aligned}$$

(equality in $\mathbf{k}[t]((p))[[z]]$), which implies the result. □

5.3.2. *Operations on A and \mathcal{A} arising from the group laws of E and $E^\#$.* Let $(E, 0)$ be given by the model $y^2 = 4x^3 - g_2x - g_3$ and the point at infinity, so that $A = \mathbf{k}[E - \{0\}]$ is as in (27).

Lemma 5.9. *The addition law on $(E, 0)$ gives rise to a coproduct morphism*

$$\begin{aligned}\Delta_A : A &\rightarrow (A \otimes A) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right], & x &\mapsto -x^{(1)} - x^{(2)} + \frac{1}{4} \left(\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right)^2, \\ y &\mapsto -\frac{1}{2}(y^{(1)} + y^{(2)}) + \frac{3}{2}(x^{(1)} + x^{(2)}) \frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} - \frac{1}{4} \left(\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right)^3,\end{aligned}$$

where $a^{(1)} := a \otimes 1$, $a^{(2)} := 1 \otimes a$ for $a = x, y$.

Proof. The addition $E \times E \rightarrow E$ is given by $((x_P, y_P), (x_Q, y_Q)) \mapsto (x_R, y_R)$, where

$$\begin{aligned}x_R &:= -x_P - x_Q + \frac{1}{4} \left(\frac{y_P - y_Q}{x_P - x_Q} \right)^2, \\ y_R &:= -\frac{1}{2}(y_P + y_Q) + \frac{3}{2}(x_P + x_Q) \frac{y_P - y_Q}{x_P - x_Q} - \frac{1}{4} \left(\frac{y_P - y_Q}{x_P - x_Q} \right)^3.\end{aligned}$$

□

Recall that $\mathcal{A} := \mathbf{k}[E^\# - \pi^{-1}(0)]$ is given by (36).

Lemma 5.10. *The addition law on $E^\#$ gives rise to the coproduct morphism*

$$\Delta_{\mathcal{A}} : \mathcal{A} \rightarrow (\mathcal{A} \otimes \mathcal{A}) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right],$$

extending Δ_A by

$$\tilde{c} \mapsto \tilde{c}^{(1)} + \tilde{c}^{(2)} + \frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}}.$$

Proof. This follows from the description of this addition law in §5.2.2. □

We now describe the compatibility of these morphisms with $A \rightarrow \mathbf{k}((p))$, $\mathcal{A} \rightarrow \mathbf{k}[t]((p))$.

Lemma 5.11. *There are well-defined morphisms*

$$\begin{aligned}(A \otimes A) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right] &\rightarrow \mathbf{k}[[p, p']][1/p, 1/p', 1/(p + p')], \\ x^{(1)} &\mapsto \wp_{E, \omega}(p), \quad x^{(2)} \mapsto \wp_{E, \omega}(p'), \quad y^{(1)} \mapsto \wp'_{E, \omega}(p), \quad y^{(2)} \mapsto \wp'_{E, \omega}(p'), \\ \frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} &\mapsto \frac{1}{pp'(p + p')} \cdot (\text{element of } \mathbf{k}[[p, p']])\end{aligned}$$

and

$$(\mathcal{A} \otimes \mathcal{A}) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right] \rightarrow \mathbf{k}[t, t'][[p, p']][1/p, 1/p', 1/(p + p')],$$

extending the previous morphism by

$$\tilde{c}^{(1)} \mapsto t - \frac{d \log \tilde{\theta}_{E, \omega}(p)}{dp}, \quad 1 \otimes \tilde{c} \mapsto t' - \frac{d \log \tilde{\theta}_{E, \omega}(p')}{dp'}.$$

Proof. The expansion $A \rightarrow \mathbf{k}((p))$ takes x, y to series $1/p^2 + \sum_{n \geq 0, \text{even}} a_n p^n$ and $-2/p^3 + \sum_{n \geq 0, \text{odd}} b_n p^n$. Then the ratio

$$\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}}$$

expands as

$$\frac{(-2/p^3 + 2/p'^3) + \sum_{n \geq 0, \text{odd}} b_n(p^n - p'^n)}{(1/p^2 - 1/p'^2) + \sum_{n \geq 0, \text{even}} a_n(p^n - p'^n)}$$

which is

$$\frac{1}{pp'} \frac{-2(p'^3 - p^3) + \sum_{n \geq 0, \text{odd}} b_n(p^n - p'^n)(pp')^3}{(p'^2 - p^2) + \sum_{n \geq 0, \text{even}} a_n(p^n - p'^n)(pp')^2}.$$

In the second fraction, the numerator factorizes as $(p - p') \cdot (\text{element of } \mathbf{k}[[p, p']])$, while the denominator factorizes as $(p^2 - p'^2) \cdot (\text{element of } \mathbf{k}[[p, p']] \text{ with constant term } -1)$. It follows that the overall fraction expands as $\frac{1}{pp'(p+p')} \cdot (\text{element of } \mathbf{k}[[p, p']])$. All this implies the result. \square

One checks:

Lemma 5.12. 1) *The following diagrams are commutative*
(45)

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & (A \otimes A) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right] \\ \downarrow & & \downarrow \\ \mathbf{k}((p)) & \longrightarrow & \mathbf{k}[[p, p']][1/p, 1/p', 1/(p+p')] \end{array} \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\Delta_A} & (\mathcal{A} \otimes \mathcal{A}) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right] \\ \downarrow & & \downarrow \\ \mathbf{k}[t]((p)) & \longrightarrow & \mathbf{k}[t, t'][[p, p']][1/p, 1/p', 1/(p+p')] \end{array}$$

where the bottom maps are given by $p \mapsto p + p'$, $t \mapsto t + t'$.

2) *The maps $a \mapsto a \otimes 1$, $a \mapsto 1 \otimes a$ define morphisms from A (resp., \mathcal{A}) to $A \otimes A$ (resp., $\mathcal{A} \otimes \mathcal{A}$), and therefore also to the localizations $(A \otimes A) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right]$ (resp., $(\mathcal{A} \otimes \mathcal{A}) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right]$), such that the following diagrams commute*

$$(46) \quad \begin{array}{ccc} A & \xrightarrow{-\otimes 1 \text{ (resp., } 1 \otimes -)} & (A \otimes A) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right] \\ \downarrow & & \downarrow \\ \mathbf{k}((p)) & \longrightarrow & \mathbf{k}[[p, p']][1/p, 1/p', 1/(p+p')] \end{array} \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{-\otimes 1 \text{ (resp., } 1 \otimes -)} & (\mathcal{A} \otimes \mathcal{A}) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right] \\ \downarrow & & \downarrow \\ \mathbf{k}[t]((p)) & \longrightarrow & \mathbf{k}[t, t'][[p, p']][1/p, 1/p', 1/(p+p')] \end{array}$$

where the bottom maps are given by substitution to p, t of p, t (resp., p', t').

5.3.3. *Algebraic Fay identity on $(E^\#)^2$.* Recall $\tilde{F}_{E, \omega}(p, z) \in \frac{1}{pz} \mathbb{C}[[p, z]]$ from (32). Let also t, t' be two additional formal variables.

Lemma 5.13. *One has*

$$(47) \quad \begin{aligned} & \tilde{F}_{E, \omega}(p, z) e^{-tz} \tilde{F}_{E, \omega}(p + p', z') e^{-(t+t')z'} - \tilde{F}_{E, \omega}(p', z') e^{-t'z'} \tilde{F}_{E, \omega}(p, z + z') e^{-t(z+z')} \\ & + \tilde{F}_{E, \omega}(p + p', z + z') e^{-(t+t')(z+z')} \tilde{F}_{E, \omega}(p', -z) e^{t'z} = 0 \end{aligned}$$

(identity in $\frac{1}{pp'zz'(p+p')(z+z')} \mathbf{k}[t, t'][[p, p', z, z']]$).

Proof. Follows from identity (33) in $\frac{1}{pp'zz'(p+p')(z+z')} \mathbf{k}[[p, p', z, z']]$. \square

Let

$$\mathbf{f}(z) := \frac{1}{z} + \sum_{\alpha \geq 0} f_\alpha z^\alpha \in \frac{1}{z} \mathcal{A}[[z]]$$

be the series defined in §5.3.1.

Lemma 5.14. *(Algebraic Fay identity.) We have*

$$(48) \quad (\mathbf{f}(z) \otimes 1) \Delta_{\mathcal{A}}(\mathbf{f}(z')) - \mathbf{f}(z + z') \otimes \mathbf{f}(z') + (1 \otimes \mathbf{f}(-z)) \Delta_{\mathcal{A}}(\mathbf{f}(z + z')) = 0$$

(identity in $\frac{1}{zz'(z+z')} (\mathcal{A} \otimes \mathcal{A}) \left[\frac{y^{(1)} - y^{(2)}}{x^{(1)} - x^{(2)}} \right] [[z, z']]$).

This identity gives a family of relations between the $(f_\alpha)_{\alpha \geq 0}$, which are rational functions on $E^\#$, regular except at the preimage of $0 \in E$; each of these relations is an identity of functions on $(E^\#)^2$, regular except at the preimages of $E \times \{0\}$, $\{0\} \times E$ and $E_{\text{antidiag}} := \{(x, y) \in E^2 \mid x + y = 0\}$.

Proof. According to (43), the image of $\mathbf{f}(z) \in \frac{1}{z}\mathcal{A}[[z]]$ under $\text{can} : \frac{1}{z}\mathcal{A}[[z]] \rightarrow \frac{1}{z}\mathbf{k}[t((p))][[z]]$ is $\tilde{F}_{E,\omega}(p, z)e^{-tz}$. This and the right sides of (45) and (46) imply that the elements

$$\tilde{F}_{E,\omega}(p, z)e^{-tz}, \quad \tilde{F}_{E,\omega}(p + p', z')e^{-(t+t')z'}, \quad \tilde{F}_{E,\omega}(p', z')e^{-t'z'},$$

$$\tilde{F}_{E,\omega}(p, z + z')e^{-t(z+z')}, \quad \tilde{F}_{E,\omega}(p + p', z + z')e^{-(t+t')(z+z')}, \quad \tilde{F}_{E,\omega}(p', -z)e^{t'z}$$

of $\frac{1}{pp'zz'(p+p')(z+z')}\mathbf{k}[t, t'][[p, p', z, z']]$ are the images under the canonical map

$$\frac{1}{zz'(z+z')}\mathcal{A} \otimes \mathcal{A} \Big|_{\frac{y^{(1)}-y^{(2)}}{x^{(1)}-x^{(2)}}} [[z, z']] \rightarrow \frac{1}{zz'(z+z')}\mathbf{k}[t, t'][[p, p']][1/p, 1/p', 1/(p+p')][[z, z']]$$

of

$$\mathbf{f}(z) \otimes 1, \quad \Delta_{\mathcal{A}}(\mathbf{f}(z')), \quad 1 \otimes \mathbf{f}(z'), \quad \mathbf{f}(z + z') \otimes 1, \quad \Delta_{\mathcal{A}}(\mathbf{f}(z + z')), \quad 1 \otimes \mathbf{f}(-z).$$

(48) now follows from the fact that this map is injective, together with identity (47). \square

5.3.4. *An identity on $(E^\#)^n$.* For $i < j \in [n]$, we denote by $m_{ij} : (E^\#)^n \rightarrow E^\#$ the morphism $(e_1, \dots, e_n) \mapsto e_i - e_j$. For $i < j < k \in [n]$, we denote by $m_{ijk} : (E^\#)^n \rightarrow (E^\#)^2$ the morphism $(e_1, \dots, e_n) \mapsto (e_i - e_j, e_j - e_k)$. We denote by $s : (E^\#)^2 \rightarrow E^\#$ the sum morphism. Then $s \circ m_{ijk} = m_{ik}$.

The morphism m_{ij} restricts to a morphism of affine varieties

$$\{(e_1, \dots, e_n) \in (E^\#)^n \mid p_1 \neq 0, \dots, p_n \neq 0, p_i \neq p_j\} \xrightarrow{m_{ij}} E^\# - \pi^{-1}(0)$$

where $p_1 := \pi(e_1), \dots, p_n := \pi(e_n)$. Similarly, m_{ijk} restricts to a morphism

$$\{(e_1, \dots, e_n) \in (E^\#)^n \mid p_1 \neq 0, \dots, p_n \neq 0, p_i \neq p_j, p_i \neq p_k, p_j \neq p_k\}$$

$$\xrightarrow{m_{ijk}} \{(e, e') \in (E^\#)^2 \mid p \neq 0, p' \neq 0, p + p' \neq 0\}$$

where $p := \pi(e)$, $p' := \pi(e')$ and s restricts to a morphism

$$\{(e, e') \in (E^\#)^2 \mid p \neq 0, p' \neq 0, p + p' \neq 0\} \xrightarrow{s} E^\# - \pi^{-1}(0).$$

For $a \in \mathcal{A}$ and $i \in [n]$, set $a^{(i)} := 1^{i-1} \otimes a \otimes 1^{\otimes n-i}$. The regular function rings of the varieties involved in the above morphisms are the following

$E^\# - \pi^{-1}(0)$	\mathcal{A}
$\{(e, e') \in (E^\#)^2 \mid p \neq 0, p' \neq 0, p + p' \neq 0\}$	$(\mathcal{A} \otimes \mathcal{A}) \Big _{\frac{y^{(1)}-y^{(2)}}{x^{(1)}-x^{(2)}}}$
$\{(e_1, \dots, e_n) \in (E^\#)^n \mid p_1 \neq 0, \dots, p_n \neq 0, p_i \neq p_j\}$	$\mathcal{A}^{\otimes n} \Big _{\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}}}$
$\{(e_1, \dots, e_n) \in (E^\#)^n \mid p_1 \neq 0, \dots, p_n \neq 0, p_i \neq p_j, p_i \neq p_k, p_j \neq p_k\}$	$\mathcal{A}^{\otimes n} \Big _{\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}}, \frac{y^{(i)}+y^{(k)}}{x^{(i)}-x^{(k)}}, \frac{y^{(j)}+y^{(k)}}{x^{(j)}-x^{(k)}}}$

The dual to the morphism s is the morphism

$$\Delta_{\mathcal{A}} : \mathcal{A} \rightarrow (\mathcal{A} \otimes \mathcal{A}) \Big|_{\frac{y^{(1)}-y^{(2)}}{x^{(1)}-x^{(2)}}}$$

defined in §5.3.2.

The dual to the morphism m_{ij} is the morphism

$$S_{\mathcal{A}}^{(j)} \Delta_{\mathcal{A}}^{(ij)} : \mathcal{A} \rightarrow \mathcal{A}^{\otimes n} \Big|_{\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}}},$$

where $\Delta_{\mathcal{A}}^{(ij)}$ is the composition of $\Delta_{\mathcal{A}}$ with the morphism induced by $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}^{\otimes n}$, $a \otimes b \mapsto 1^{\otimes i-1} \otimes a \otimes 1^{j-i-1} \otimes b \otimes 1^{\otimes n-j}$ and $\frac{y^{(1)}-y^{(2)}}{x^{(1)}-x^{(2)}} \mapsto \frac{y^{(i)}-y^{(j)}}{x^{(i)}-x^{(j)}}$, $S_{\mathcal{A}}$ is the automorphism of \mathcal{A} induced

by $x \mapsto x$, $y \mapsto -y$, $\tilde{c} \mapsto -\tilde{c}$, and $S_{\mathcal{A}}^{(i)}$ is the isomorphism $\mathcal{A}^{\otimes n}[\frac{y^{(i)}-y^{(j)}}{x^{(i)}-x^{(j)}}] \rightarrow \mathcal{A}^{\otimes n}[\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}}]$ extending the automorphism $\text{id}^{\otimes i-1} \otimes S_{\mathcal{A}} \otimes \text{id}^{\otimes n-i}$ of $\mathcal{A}^{\otimes n}$ by $\frac{y^{(i)}-y^{(j)}}{x^{(i)}-x^{(j)}} \rightarrow -\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}}$.

The dual of the morphism m_{ijk} is the morphism

$$m_{ijk}^* : (\mathcal{A} \otimes \mathcal{A})[\frac{y^{(1)}-y^{(2)}}{x^{(1)}-x^{(2)}}] \rightarrow \mathcal{A}^{\otimes n}[\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}}, \frac{y^{(i)}+y^{(k)}}{x^{(i)}-x^{(k)}}, \frac{y^{(j)}+y^{(k)}}{x^{(j)}-x^{(k)}}]$$

induced by $\mathcal{A} \otimes \mathcal{A} \ni a \otimes b \mapsto S_{\mathcal{A}}^{(j)} \Delta_{\mathcal{A}}^{(ij)}(a) \cdot S_{\mathcal{A}}^{(k)} \Delta_{\mathcal{A}}^{(jk)}(b)$, and

$$\frac{y^{(1)}-y^{(2)}}{x^{(1)}-x^{(2)}} \mapsto -(\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}} + \frac{y^{(j)}+y^{(k)}}{x^{(j)}-x^{(k)}} + \frac{y^{(k)}+y^{(i)}}{x^{(k)}-x^{(i)}}).$$

We then have $m_{ijk}^* \circ \Delta_{\mathcal{A}} = S_{\mathcal{A}}^{(k)} \Delta_{\mathcal{A}}^{(ik)}$.

Applying m_{ijk}^* to identity (48), we obtain (see [Fay, Mum, Po]):

Lemma 5.15. *(Algebraic Fay identities on $(E^{\#})^n$) One has*

$$(49) \quad (S_{\mathcal{A}}^{(j)} \Delta_{\mathcal{A}}^{(ij)})(\mathbf{f}(z))(S_{\mathcal{A}}^{(k)} \Delta_{\mathcal{A}}^{(ik)})(\mathbf{f}(z')) - (S_{\mathcal{A}}^{(j)} \Delta_{\mathcal{A}}^{(ij)})(\mathbf{f}(z+z'))(S_{\mathcal{A}}^{(k)} \Delta_{\mathcal{A}}^{(jk)})(\mathbf{f}(z')) \\ + (S_{\mathcal{A}}^{(k)} \Delta_{\mathcal{A}}^{(jk)})(\mathbf{f}(-z))(S_{\mathcal{A}}^{(k)} \Delta_{\mathcal{A}}^{(ik)})(\mathbf{f}(z+z')) = 0$$

(identity in $\frac{1}{zz'(z+z')}\mathcal{A}^{\otimes n}[\frac{y^{(i)}+y^{(j)}}{x^{(i)}-x^{(j)}}, \frac{y^{(i)}+y^{(k)}}{x^{(i)}-x^{(k)}}, \frac{y^{(j)}+y^{(k)}}{x^{(j)}-x^{(k)}}][[z, z']]$).

Remark 5.16. Assume that $\mathbf{k} = \mathbb{C}$ and that $(E, 0, \omega) = (\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), 0, dp)$. The image of $\mathbf{f}(z)$ in $\frac{1}{z}\mathbb{C}[t]((p))[[z]]$ is then

$$\frac{\theta(p+z|\tau)}{\theta(p|\tau)\theta(z|\tau)} e^{-(t-\frac{1}{2}G_2(\tau)p)z},$$

so the image of $(S_{\mathcal{A}}^{(j)} \Delta_{\mathcal{A}}^{(ij)})(\mathbf{f}(z))$ in $\frac{1}{z(p_i-p_j)}\mathbb{C}[t_1, \dots, t_n][[p_1, \dots, p_n, z]]$ is

$$\frac{\theta(p_{ij}+z|\tau)}{\theta(p_{ij}|\tau)\theta(z|\tau)} e^{-c_{ij}z},$$

where $c_i := t_i - \frac{1}{2}G_2(\tau)p_i$ and $c_{ij} := c_i - c_j$. □

Identity (49) then translates as

$$\frac{\theta(p_{ij}+z|\tau)}{\theta(p_{ij}|\tau)\theta(z|\tau)} e^{-c_{ij}z} \frac{\theta(p_{ik}+z'|\tau)}{\theta(p_{ik}|\tau)\theta(z'|\tau)} e^{-c_{ik}z'} - \frac{\theta(p_{jk}+z'|\tau)}{\theta(p_{jk}|\tau)\theta(z'|\tau)} e^{-c_{jk}z'} \frac{\theta(p_{ij}+(z+z')|\tau)}{\theta(p_{ij}|\tau)\theta(z+z'|\tau)} e^{-c_{ij}(z+z')} \\ + \frac{\theta(p_{ik}+(z+z')|\tau)}{\theta(p_{ik}|\tau)\theta(z+z'|\tau)} e^{-c_{ik}(z+z')} \frac{\theta(p_{jk}-z|\tau)}{\theta(p_{jk}|\tau)\theta(z|\tau)} e^{c_{jk}z} = 0.$$

(identity in $\frac{1}{zz'(z+z')(p_i-p_j)(p_i-p_k)(p_j-p_k)}\mathbb{C}[t_1, \dots, t_n][[p_1, \dots, p_n, z, z']]$).

6. COHOMOLOGICAL COMPUTATIONS RELATED TO $E^{\#}$

In this §, we choose an elliptic curve E with fixed origin 0, defined over a field \mathbf{k} of char. 0.

6.1. Spaces of differentials on $E^{\#}$.

6.1.1. *Computation of $\Gamma(E^\#, \mathcal{O}_{E^\#}(\pi^{-1}(0)))$.* In §5.3.1, we defined elements $f_\alpha \in \mathcal{A}$ and a morphism $\text{can} : \mathcal{A} \rightarrow \mathbf{k}[t]((p))$. As the left-hand side of (43) belongs to $\frac{1}{p}\mathbf{k}[t][[p, z]]$, one has $\text{can}(f_\alpha) \in \frac{1}{p}\mathbf{k}[t][[p]]$ for any $\alpha \geq 0$. Since the divisor $\pi^{-1}(0)$ is locally defined by the equation $p = 0$ in the plane (p, t) , this implies that the pole of f_α is simple, therefore

$$\forall \alpha \geq 0, \quad f_\alpha \in \Gamma(E^\#, \mathcal{O}_{E^\#}(\pi^{-1}(0))).$$

There is an exact sequence

$$0 \rightarrow \Gamma(E^\#, \mathcal{O}_{E^\#}) \rightarrow \Gamma(E^\#, \mathcal{O}_{E^\#}(\pi^{-1}(0))) \xrightarrow{\mu} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}),$$

where the last map takes a rational function f on $E^\#$ with a simple pole at $\pi^{-1}(0)$ to the restriction at $\pi^{-1}(0) \simeq \mathbb{A}^1$ of the product $f \cdot p$ ($p = 0$ being a local equation of $\pi^{-1}(0)$).

Lemma 6.1. *Set $f_{-1} := 1$. The family $(f_\alpha)_{\alpha \geq -1}$ is a basis of $\Gamma(E^\#, \mathcal{O}_{E^\#}(\pi^{-1}(0)))$. The map μ is such that*

$$(50) \quad \forall \alpha \geq 0, \quad \mu(f_\alpha) = (-t)^\alpha / \alpha!,$$

therefore μ is surjective.

Proof. The map μ fits in the diagram

$$\begin{array}{ccccc} & & \mu & & \\ & \searrow & & \searrow & \\ \Gamma(E^\#, \mathcal{O}_{E^\#}(\pi^{-1}(0))) & \longrightarrow & \frac{1}{p}\mathbf{k}[t][[p]] & \longrightarrow & \mathbf{k}[t] \xrightarrow{\sim} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \\ \downarrow & & \downarrow & & \\ \Gamma(E^\#, \mathcal{O}_{E^\#}(*\pi^{-1}(0))) & \xrightarrow{\text{can}} & \mathbf{k}[t]((p)) & & \end{array}$$

the map $\frac{1}{p}\mathbf{k}[t][[p]] \rightarrow \mathbf{k}[t]$ being $f \mapsto (f \cdot p)|_{p=0}$. (43) implies

$$e^{-tz} \frac{\tilde{\theta}_{E,\omega}(p+z)}{\tilde{\theta}_{E,\omega}(p)} = \tilde{\theta}_{E,\omega}(z) \left(\frac{1}{z} + \sum_{\alpha \geq 0} \text{can}(f_\alpha) z^\alpha \right)$$

(identity in $\frac{1}{p}\mathbf{k}[t][[p, z]]$), therefore

$$e^{-tz} \cdot \frac{p}{\tilde{\theta}_{E,\omega}(p)} \cdot \tilde{\theta}_{E,\omega}(p+z) = p \cdot \frac{\tilde{\theta}_{E,\omega}(z)}{z} + \sum_{\alpha \geq 0} p \text{can}(f_\alpha) \tilde{\theta}_{E,\omega}(z) z^\alpha$$

(identity in $\mathbf{k}[t][[p, z]]$). Evaluating at $p = 0$, we get

$$e^{-tz} \cdot \tilde{\theta}_{E,\omega}(z) = \sum_{\alpha \geq 0} (p \text{can}(f_\alpha))|_{p=0} \tilde{\theta}_{E,\omega}(z) z^\alpha$$

(identity in $\mathbf{k}[t][[z]]$), therefore $\sum_{\alpha \geq 0} (p \text{can}(f_\alpha))|_{p=0} z^\alpha = e^{-tz}$, therefore for any $\alpha \geq 0$, $\mu(f_\alpha) = (p \text{can}(f_\alpha))|_{p=0} = (-t)^\alpha / \alpha!$, proving (50).

So the image of $(f_\alpha)_{\alpha \geq 0}$ by μ is a basis of $\Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$. This implies both that μ is surjective, and that a basis of $\Gamma(E^\#, \mathcal{O}_{E^\#}(\pi^{-1}(0)))$ is the union of $(f_\alpha)_{\alpha \geq 0}$ and a basis of $\Gamma(E^\#, \mathcal{O}_{E^\#})$. The result follows from the fact that this latter space is $\simeq \mathbf{k} \cdot 1$. \square

Remark 6.2. One computes $f_{-1} = 1$, $f_0 = -\tilde{c}$, $f_1 = \frac{1}{2}(\tilde{c}^2 - x)$.

6.1.2. *Computation of $\Gamma(E^\#, \Omega_{E^\#}^1(\log \pi^{-1}(0)))$.* As $E^\#$ is an algebraic group, its sheaf $\Omega_{E^\#}^1$ of differentials is isomorphic to a direct sum of $\dim E^\# = 2$ copies of the trivial bundle, generated by a basis of invariant differentials.

The differentials

$$\underline{dc} := d\tilde{c} - xdx/y \quad \text{and} \quad \underline{dp} := dx/y$$

form such a basis. Note that for $\mathbf{k} = \mathbb{C}$ and when a uniformization of E_{an} is fixed, they correspond to the differentials dc, dp on $E_{an}^\#$. We have therefore

$$\Omega_{E^\#}^1 \simeq \mathcal{O}_{E^\#} \cdot \underline{dc} \oplus \mathcal{O}_{E^\#} \cdot \underline{dp}.$$

One computes

$$\Omega_{E^\#}^1(\log \pi^{-1}(0)) \simeq \mathcal{O}_{E^\#} \cdot \underline{dc} \oplus \mathcal{O}_{E^\#}(\pi^{-1}(0)) \cdot \underline{dp}.$$

Lemma 6.3. 1) *The family*

$$(51) \quad d\tilde{c} - xdx/y, \quad \omega_\alpha := f_\alpha \cdot \frac{dx}{y}, \quad \alpha \geq -1,$$

where f_α is as in (42), is a basis of $\Gamma(E^\#, \Omega_{E^\#}^1(\log \pi^{-1}(0)))$.

2) *The residue map*

$$\Gamma(E^\#, \Omega_{E^\#}^1(\log \pi^{-1}(0))) \rightarrow \Gamma(\pi^{-1}(0), \mathcal{O}_{\pi^{-1}(0)}) = \mathbf{k}[t]$$

along $\pi^{-1}(0)$ is given by

$$\text{res}(\underline{dc}) = \text{res}(\omega_{-1}) = 0, \quad \text{res}(\omega_\alpha) = (-t)^\alpha / \alpha! \quad \text{for } \alpha \geq 0.$$

Proof. 1) follows from Lemma 6.1. 2) follows from (50). \square

Remark 6.4. If $\mathbf{k} = \mathbb{C}$ and a uniformization of $E_{an} \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is fixed as in §5.2.4 (i.e., such that $dx/y \leftrightarrow dp$), the elements (51) correspond to the following differentials on $E_{an}^\#$

$$(52) \quad dc, \quad \omega_{-1} := dp, \quad \omega_\alpha := \left[\left(\frac{\theta(p+z|\tau)}{\theta(z|\tau)\theta(p|\tau)} e^{-cz} - \frac{1}{z} \right) dp | z^\alpha \right], \quad \alpha \geq 0,$$

where z is a formal variable and the notation $[-|z^\alpha]$ denotes the coefficient of z^α in the expansion in power series at $z = 0$ (see §6.1.1).

6.2. Spaces of differentials on $(E^\#)^n$.

6.2.1. *The setup.* Let n be an integer ≥ 1 ; we set $[n] := \{1, \dots, n\}$ and $I := \{(i, j) | i, j \in [n] \text{ and } i < j\}$.

We assume (X, D) to be as in §2.2. To emphasize the dependence of D in n , we sometimes denote it $D^{(n)}$.

6.2.2. *Computation of $\Gamma(X, \Omega_X^1(\log D_{ij}))$.* Let $(i, j) \in I$. According to §1.2.3, the space $\Gamma(X, \Omega_X^1(\log D_{ij}))$ consists of all forms α in $\Gamma_{rat}(X, \Omega_X^1)$ such that both α and $d\alpha$ are regular except for simple poles along D_{ij} . Moreover, the residue along D_{ij} induces an exact sequence

$$(53) \quad 0 \rightarrow \Gamma(X, \Omega_X^1) \rightarrow \Gamma(X, \Omega_X^1(\log D_{ij})) \xrightarrow{\mu_{ij}} \Gamma(D_{ij}, \mathcal{O}_{D_{ij}}).$$

The automorphism map_{ij} of $(E^\#)^n$ given by

$$\text{map}_{ij} : (e_1, \dots, e_n) \mapsto (e_i - e_j, e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_n)$$

induces an isomorphism $D_{ij} \simeq \pi^{-1}(0) \times (E^\#)^{n-1}$. The isomorphism $\pi^{-1}(0) \simeq \mathbb{A}^1$ then induces an isomorphism

$$(54) \quad D_{ij} \simeq \mathbb{A}^1 \times (E^\#)^{n-1}.$$

The isomorphisms $\Gamma(E^\#, \mathcal{O}_{E^\#}) \simeq \mathbf{k}$, $\Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \simeq \mathbf{k}[t]$ then induce an isomorphism

$$(55) \quad \Gamma(D_{ij}, \mathcal{O}_{D_{ij}}) \simeq \mathbf{k}[t]$$

(in the analytic context, $t = c_{ij}$, see Remark 5.16).

Definition 6.5. For $\alpha \geq 0$ and $(i, j) \in I$, define ω_{ij}^α as the image of ω_α under the composed map

$$\begin{aligned} \Gamma(E^\#, \Omega_{E^\#}^1(\log \pi^{-1}(0))) &\xrightarrow{-\otimes 1^{\otimes n-1}} \Gamma(E^\# \times (E^\#)^{n-1}, \Omega_{E^\#}^1(\log \pi^{-1}(0)) \boxtimes \mathcal{O}_{E^\#}^{\boxtimes n-1}) \\ &\xrightarrow{\text{map}_{ij}^*} \Gamma((E^\#)^n, \Omega_{(E^\#)^n}^1(\log D_{ij})). \end{aligned}$$

Lemma 6.6. *The map μ_{ij} in (53) is surjective.*

Proof. One checks that the image of ω_{ij}^α by the residue map $\Gamma(X, \Omega_X^1(\log D_{ij})) \rightarrow \Gamma(D_{ij}, \mathcal{O}_{D_{ij}}) \simeq \mathbf{k}[t]$ equals $(-t)^\alpha / \alpha!$. So the composed map $\text{Span}((\omega_{ij}^\alpha)_{\alpha \geq 0}) \rightarrow \Gamma(X, \Omega_X^1(\log D_{ij})) \rightarrow \Gamma(D_{ij}, \mathcal{O}_{D_{ij}})$ is surjective. The result follows. \square

For $i \in [n]$, set

$$\underline{dc}_i := d\tilde{c}_{(i)} - x_{(i)} dx_{(i)} / y_{(i)}, \quad \underline{dp}_i := dx_{(i)} / y_{(i)}.$$

(see §6.2.1).

Lemma 6.7. *The family $(\underline{dc}_i, \underline{dp}_i)_{i \in [n]}$ is a basis of $\Gamma(X, \Omega_X^1)$.*

Proof. As $\Gamma(E^\#, \mathcal{O}_{E^\#}) \simeq \mathbf{k}$, we have $\Gamma(X, \Omega_X^1) \simeq \Gamma(E^\#, \Omega_{E^\#}^1)^{\oplus n}$, which implies the result. \square

It follows from the definition of ω_{ij}^α and from Lemma 6.7 that

$$(56) \quad (\omega_{ij}^\alpha)_{\alpha \geq 0}, \quad (\underline{dc}_i)_{i \in [n]}, \quad (\underline{dp}_i)_{i \in [n]}$$

constitutes a family in $\Gamma(X, \Omega_X^1(\log D_{ij}))$.

Lemma 6.8. *Family (56) is a basis of $\Gamma(X, \Omega_X^1(\log D_{ij}))$.*

Proof. Since the composed map

$$\Gamma(X, \Omega_X^1) \simeq \text{Span}((\underline{dc}_i)_{i \in [n]}, (\underline{dp}_i)_{i \in [n]}) \rightarrow \text{Span}((\omega_{ij}^\alpha)_{\alpha \geq 0}, (\underline{dc}_i)_{i \in [n]}, (\underline{dp}_i)_{i \in [n]}) \rightarrow \Gamma(X, \Omega_X^1(\log D_{ij}))$$

is the canonical map $\Gamma(X, \Omega_X^1) \rightarrow \Gamma(X, \Omega_X^1(\log D_{ij}))$, the map

$$\text{Span}((\omega_{ij}^\alpha)_{\alpha \geq 0}, (\underline{dc}_i)_{i \in [n]}, (\underline{dp}_i)_{i \in [n]}) \rightarrow \Gamma(X, \Omega_X^1(\log D_{ij}))$$

is surjective. Moreover, the image of $(\omega_{ij}^\alpha)_{\alpha \geq 0}$ under the residue map is a linearly independent family in $\Gamma(D_{ij}, \mathcal{O}_{D_{ij}})$, and the family $((\underline{dc}_i)_{i \in [n]}, (\underline{dp}_i)_{i \in [n]})$ is also linearly independent, which implies that family (56) is linearly independent. All this proves the result. \square

It follows from the definition of ω_{ij}^α that

$$\frac{dp_{ij}}{z} + \sum_{\alpha \geq 0} \omega_{ij}^\alpha z^\alpha = (S_{\mathcal{A}}^{(j)} \Delta_{\mathcal{A}}^{(ij)})(\mathbf{f}(z)) \cdot \underline{dp}_{ij}.$$

where the notation is as in §§5.3.3, 5.3.4 (identity in $\Gamma_{rat}((E^\#)^n, \Omega_{(E^\#)^n}^1)$).

Remark 6.9. Assume that $\mathbf{k} = \mathbb{C}$ and a uniformization $E_{an} \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ as in §5.2.4 (i.e., such that $dx/y \leftrightarrow dp$) is fixed. Then a uniformization of $(E^\#)^n$ is $(E^\#)_{an}^n = \text{Coker}(\mathbb{Z}^{2n} \rightarrow \mathbb{C}^n)$, where the morphism $\mathbb{Z}^{2n} \rightarrow \mathbb{C}^n$ is $\delta_{2i-1} \mapsto e_i$, $\delta_{2i} \mapsto \tau e_i$, where $i \in [n]$ (the canonical bases of \mathbb{Z}^{2n} and \mathbb{C}^n being $(\delta_i)_{i \in [2n]}$ and $(e_i)_{i \in [n]}$). Let $(p_1, c_1, \dots, p_n, c_n)$ be the standard coordinates on \mathbb{C}^n . Then

$$(S_{\mathcal{A}}^{(j)} \Delta_{\mathcal{A}}^{(ij)})(\mathbf{f}(z)) = \frac{\theta(p_{ij} + z|\tau)}{\theta(z|\tau)\theta(p_{ij}|\tau)} e^{-c_{ij}z},$$

so

$$(57) \quad \omega_{ij}^\alpha = \left[\left(\frac{\theta(p_{ij} + z|\tau)}{\theta(z|\tau)\theta(p_{ij}|\tau)} e^{-c_{ij}z} - \frac{1}{z} \right) dp_{ij} | z^\alpha \right]$$

and $p_{ij} := p_i - p_j$. The elements dc_i, dp_i ($i \in [n]$) form a basis of $\Gamma((E^\#)^n, \Omega_{(E^\#)^n}^1)$ and are the images of the elements $\underline{dc}_i, \underline{dp}_i$.

6.2.3. *Computation of $\sum_{(i,j) \in I} \Gamma(X, \Omega_X^1(\log D_{ij}))$.* Recall that for each $(i, j) \in I$, $\Gamma(X, \Omega_X^1(\log D_{ij}))$ is a vector subspace of $\Gamma_{\text{rat}}(X, \Omega_X^1)$. Then according to §3.5.5,

$$\Omega^1 = \sum_{(i,j) \in I} \Gamma(X, \Omega_X^1(\log D_{ij}))$$

is the sum of these subspaces. It follows from Lemma 6.8 that the following family

$$(58) \quad \underline{dc}_i \quad (i \in [n]), \quad \underline{dp}_i \quad (i \in [n]), \quad \omega_{ij}^\alpha \quad (i < j \in [n], \quad \alpha \geq 0),$$

lies in, and spans, this vector space.

According to §1.3.2, the residue along D_{ij} induces a linear map $\text{Res}_{D_{ij}}^{(1)} : \Omega^1 \rightarrow \Gamma(D_{ij}, \mathcal{O}_{D_{ij}})$. Composing this map with the isomorphism $\Gamma(D_{ij}, \mathcal{O}_{D_{ij}}) \simeq \mathbf{k}[t]$ (see (55)), we obtain a linear map

$$(59) \quad \varrho_{ij} : \Omega^1 \rightarrow \mathbf{k}[t].$$

Lemma 6.10. *The map ϱ_{ij} is given by*

$$\underline{dc}_{i'} \mapsto 0 \quad (i' \in [n]), \quad \underline{dp}_{i'} \mapsto 0 \quad (i' \in [n]), \quad \omega_{i'j'}^\alpha \mapsto \delta_{(i',j'),(i,j)} \cdot (-t)^\alpha / \alpha! \quad ((i,j) \in I).$$

Proof. This follows from the fact that ϱ_{ij} is the sum of the residue maps computed in §6.2.2, from (50), and from the fact that if $(i,j) \neq (i',j') \in I$, then the residue along D_{ij} of $\omega_{i'j'}^\alpha$ is zero. \square

Lemma 6.11. *The family (58) is a basis of $\Omega^1 = \sum_{(i,j) \in I} \Gamma((E^\#)^n, \Omega_{(E^\#)^n}^1(\log D_{ij}))$.*

Proof. Lemma 6.10 implies that the family $(\omega_{ij}^\alpha)_{(i,j) \in I, \alpha \geq 0}$ maps to a basis under the map $\Omega^1 \rightarrow \oplus_{(i,j) \in I} \Gamma(D_{ij}, \mathcal{O}_{D_{ij}})$. The remaining elements in family (58) form a basis in the kernel $\Gamma(X, \Omega_X^1)$ of this map. It follows that the family (58) is also linearly independent. \square

Recall that $\Omega^1 \hookrightarrow \Gamma_{\text{rat}}((E^\#)^n, \Omega_{(E^\#)^n}^1)$ and that the bundle $\Omega_{(E^\#)^n}^1$ is trivial, namely

$$(60) \quad \Omega_{(E^\#)^n}^1 \simeq \mathcal{O}_{(E^\#)^n} \otimes \left(\oplus_{i=1}^n (\mathbf{k} \underline{dp}_i \oplus \mathbf{k} \underline{dc}_i) \right).$$

This induces an isomorphism

$$\Gamma_{\text{rat}}((E^\#)^n, \Omega_{(E^\#)^n}^1) \simeq \mathbf{k}((E^\#)^n) \otimes \left(\oplus_{i=1}^n (\mathbf{k} \underline{dp}_i \oplus \mathbf{k} \underline{dc}_i) \right) = \Gamma_{\text{rat}}^p \oplus \Gamma_{\text{rat}}^c,$$

where

$$\Gamma_{\text{rat}}^p := \mathbf{k}((E^\#)^n) \otimes \left(\oplus_{i=1}^n \mathbf{k} \underline{dp}_i \right), \quad \Gamma_{\text{rat}}^c := \mathbf{k}((E^\#)^n) \otimes \left(\oplus_{i=1}^n \mathbf{k} \underline{dc}_i \right).$$

Lemma 6.11 then implies:

Lemma 6.12. • Ω^1 is graded for this decomposition, that is

$$\Omega^1 = \Omega_p^1 \oplus \Omega_c^1, \quad \text{where} \quad \Omega_p^1 := \Omega^1 \cap \Gamma_{rat}^p, \quad \Omega_c^1 := \Omega^1 \cap \Gamma_{rat}^c,$$

- the elements \underline{dp}_i ($i \in [n]$), ω_{ij}^α ($i < j \in [n]$, $\alpha \geq 0$) form a \mathbf{k} -basis of Ω_p^1 and therefore a linearly independent family of $\mathbf{k}((E^\#)^n) \otimes (\oplus_{i \in [n]} \mathbf{k} \cdot \underline{dp}_i)$,
- the elements \underline{dc}_i ($i \in [n]$) form a \mathbf{k} -basis of Ω_c^1 .

6.2.4. *Rational 2-forms on $(E^\#)^n$.* It follows from (60) that

$$\Omega_{(E^\#)^n}^2 \simeq \mathcal{O}_{(E^\#)^n} \otimes \Lambda^2 \left(\bigoplus_{i \in [n]} \mathbf{k} \cdot \underline{dc}_i \oplus \bigoplus_{i \in [n]} \mathbf{k} \cdot \underline{dp}_i \right),$$

therefore

$$\Gamma_{rat}((E^\#)^n, \Omega_{(E^\#)^n}^2) \simeq \mathbf{k}((E^\#)^n) \otimes \Lambda^2 \left(\bigoplus_{i \in [n]} \mathbf{k} \cdot \underline{dc}_i \oplus \bigoplus_{i \in [n]} \mathbf{k} \cdot \underline{dp}_i \right).$$

We set

$$(61) \quad \begin{aligned} \Gamma_{rat}^{cc} &:= \mathbf{k}((E^\#)^n) \otimes \Lambda^2(\oplus_{i \in [n]} \mathbf{k} \cdot \underline{dc}_i), & \Gamma_{rat}^{cp} &:= \mathbf{k}((E^\#)^n) \otimes (\oplus_{i,j \in [n]} \mathbf{k} \cdot (\underline{dc}_i \wedge \underline{dp}_j)), \\ \Gamma_{rat}^{pp} &:= \mathbf{k}((E^\#)^n) \otimes \Lambda^2(\oplus_{i \in [n]} \mathbf{k} \cdot \underline{dp}_i). \end{aligned}$$

Then

$$\Gamma_{rat}((E^\#)^n, \Omega_{(E^\#)^n}^2) = \Gamma_{rat}^{cc} \oplus \Gamma_{rat}^{cp} \oplus \Gamma_{rat}^{pp}.$$

It follows from (61) that

$$(62) \quad \text{the family } (\underline{dc}_i \wedge \underline{dc}_j)_{i < j \in [n]} \text{ is linearly independent over } \mathbf{k} \text{ in } \Gamma_{rat}^{cc},$$

and from (61) and Lemma 6.12 that

$$(63) \quad \begin{aligned} &\text{the union of two families } (\underline{dc}_i \wedge \underline{dp}_j)_{i,j \in [n]}, (\underline{dc}_i \wedge \omega_{jk}^\alpha)_{i \in [n], j < k \in [n], \alpha \geq 0} \\ &\text{is linearly independent over } \mathbf{k} \text{ in } \Gamma_{rat}^{cp}, \end{aligned}$$

where \wedge is the product in the algebra $\mathbf{k}((E^\#)^n) \otimes \Lambda^\bullet(\oplus_{i \in [n]} \mathbf{k} \underline{dp}_i \oplus \mathbf{k} \underline{dc}_i)$.

6.3. Computation of the kernel of $\odot : \Lambda^2(\Omega^1) \rightarrow \Omega^2$. In this section, we drop $(E^\#)^n$ from the global section notation, so $\Gamma(\mathcal{L})$ will mean $\Gamma((E^\#)^n, \mathcal{L})$. Likewise, $\Gamma_{rat}(\mathcal{L})$ means $\Gamma_{rat}((E^\#)^n, \mathcal{L})$.

6.3.1. *Construction of linear maps.* According to §1.3.2, the wedge product of differential forms induces a linear map

$$(64) \quad \Lambda^2(\Omega^1) \rightarrow \Omega^2 \hookrightarrow \Gamma_{rat}(\Omega_{(E^\#)^n}^2)$$

According to Lemma 6.12, (1), the source of this morphism decomposes as follows

$$\Lambda^2(\Omega^1) \simeq \Lambda^2(\Omega_c^1) \oplus \Omega_c^1 \otimes \Omega_p^1 \oplus \Lambda^2(\Omega_p^1),$$

while the target decomposes as $\Gamma_{rat}(\Omega_{(E^\#)^n}^2) \simeq \Gamma_{rat}^{cc} \oplus \Gamma_{rat}^{cp} \oplus \Gamma_{rat}^{pp}$. The linear map (64) then decomposes as the direct sum of linear maps

$$(65) \quad \Lambda^2(\Omega_c^1) \rightarrow \Gamma_{rat}^{cc},$$

$$(66) \quad \Omega_c^1 \otimes \Omega_p^1 \rightarrow \Gamma_{rat}^{cp},$$

$$(67) \quad \Lambda^2(\Omega_p^1) \rightarrow \Gamma_{rat}^{pp}.$$

and (62) and (63) imply:

Lemma 6.13. *The linear maps (65) and (66) are injective.*

We will now compute the kernel of the map (67).

6.3.2. *Construction of a subspace \mathbf{K} of $\text{Ker}(\Lambda^2(\Omega_p^1) \rightarrow \Gamma_{rat}^{pp})$.* By Lemma 6.12, a basis of Ω_p^1 is given by the family

$$\underline{dp}_i \quad (i \in [n]), \quad \omega_{ij}^\alpha \quad (i < j \in [n], \quad \alpha \geq 0).$$

One derives from there the following basis of $\Lambda^2(\Omega_p^1)$:

$$(68) \quad P(i, j) := \underline{dp}_i \underline{\Delta} \underline{dp}_j \quad (i < j \in [n]),$$

$$(69) \quad Q(i, j, k, \alpha) := \underline{dp}_i \underline{\Delta} \omega_{jk}^\alpha \quad (i, j, k \in [n] \text{ all different}, \quad j < k, \quad \alpha \geq 0),$$

$$(70) \quad Q'(i, j, \alpha) := \underline{dp}_i \underline{\Delta} \omega_{ij}^\alpha \quad (i < j \in [n], \quad \alpha \geq 0),$$

$$(71) \quad Q''(i, j, \alpha) := \underline{dp}_j \underline{\Delta} \omega_{ij}^\alpha \quad (i < j \in [n], \quad \alpha \geq 0),$$

$$(72) \quad S(i, j, k, l, \alpha, \beta) := \omega_{ij}^\alpha \underline{\Delta} \omega_{kl}^\beta \quad (i, j, k, l \in [n] \text{ all different}, \quad i < j, \quad k < l, \quad i < k, \quad \alpha, \beta \geq 0),$$

$$(73) \quad S'(i, j, k, \alpha, \beta) := \omega_{ij}^\alpha \underline{\Delta} \omega_{ik}^\beta \quad (i < j < k \in [n], \quad \alpha, \beta \geq 0),$$

$$(74) \quad S''(i, j, k, \alpha, \beta) := \omega_{ij}^\alpha \underline{\Delta} \omega_{jk}^\beta \quad (i < j < k \in [n], \quad \alpha, \beta \geq 0),$$

$$(75) \quad S'''(i, j, k, \alpha, \beta) := \omega_{ik}^\alpha \underline{\Delta} \omega_{jk}^\beta \quad (i < j < k \in [n], \quad \alpha, \beta \geq 0),$$

$$(76) \quad S(i, j, \alpha, \beta) := \omega_{ij}^\alpha \underline{\Delta} \omega_{ij}^\beta \quad (i < j \in [n], \quad 0 \leq \alpha < \beta),$$

where $\underline{\Delta}$ is the product in the algebra $\Lambda^\bullet(\Omega_p^1)$.

Definition 6.14. $\mathbf{K} \subset \Lambda^2(\Omega_p^1)$ is the subspace spanned by

$$(77) \quad R(i, j, \alpha) := (Q' - Q'')(i, j, \alpha), \quad i < j \in [n], \quad \alpha \geq 0,$$

$$S(i, j, \alpha, \beta), \quad i < j, \quad \alpha > \beta \geq 0,$$

$$(78) \quad \begin{aligned} T(i, j, k, \alpha, \beta) := & S'(i, j, k, \alpha, \beta) - \sum_{\substack{\gamma, \delta \geq 0, \\ \gamma + \delta = \alpha + \beta}} \binom{\delta}{\alpha} S''(i, j, k, \delta, \gamma) + \sum_{\substack{\gamma, \delta \geq 0, \\ \gamma + \delta = \alpha + \beta}} (-1)^\delta \binom{\gamma}{\beta} S'''(i, j, k, \gamma, \delta) \\ & - \binom{\alpha + \beta + 1}{\beta} (Q'(i, k, \alpha + \beta + 1) - Q(j, i, k, \alpha + \beta + 1)) \\ & + \binom{\alpha + \beta + 1}{\alpha} (Q''(i, j, \alpha + \beta + 1) - Q(k, i, j, \alpha + \beta + 1)) \\ & + (-1)^{\alpha+1} (Q(i, j, k, \alpha + \beta + 1) - Q''(j, k, \alpha + \beta + 1)), \quad i < j < k \in [n], \quad \alpha, \beta \geq 0. \end{aligned}$$

The wedge product of forms defines a linear map $\Lambda^2(\Omega_p^1) \rightarrow \Gamma_{rat}^{pp}$. If $i < j \in [n]$ and $\alpha \geq 0$, the image of $R(i, j, \alpha)$ is then $\underline{dp}_{ij} \oslash \omega_{ij}^\alpha$, where \oslash is the operation introduced in §1.3.2, and $\underline{dp}_{ij} := \underline{dp}_i - \underline{dp}_j$. According to (51), $\omega_\alpha = f_\alpha \cdot \omega_{-1}$ for $\alpha \geq -1$, so that $\omega_{ij}^\alpha = \text{map}_{ij}^*(f_\alpha) \cdot \omega_{ij}^{-1}$ for $\alpha \geq -1$. As $\omega_{ij}^{-1} = \underline{dp}_{ij}$, this implies $\underline{dp}_{ij} \oslash \omega_{ij}^\alpha = 0$. The image of $S(i, j, \alpha, \beta)$ is $\omega_{ij}^\alpha \oslash \omega_{ij}^\beta$, which is 0 for the same reason.

The product of identity (48) by $\underline{dp}_{ij} \odot \underline{dp}_{ik}$ expresses as the statement that the element

$$\begin{aligned} & \left(\frac{dp_{ij}}{z} + \sum_{\alpha \geq 0} \omega_{ij}^\alpha z^\alpha \right) \triangle \left(\frac{dp_{ik}}{u} + \sum_{\beta \geq 0} \omega_{ik}^\beta u^\beta \right) \\ & + \left(\frac{dp_{jk}}{u} + \sum_{\alpha \geq 0} \omega_{jk}^\alpha u^\alpha \right) \triangle \left(\frac{dp_{ij}}{z+u} + \sum_{\beta \geq 0} \omega_{ij}^\beta (z+u)^\beta \right) \\ & + \left(\frac{dp_{ik}}{z+u} + \sum_{\alpha \geq 0} \omega_{ik}^\alpha (z+u)^\alpha \right) \triangle \left(\frac{dp_{jk}}{-z} + \sum_{\beta \geq 0} \omega_{jk}^\beta (-z)^\beta \right) \end{aligned}$$

of the space $\frac{1}{zu(z+u)} \Lambda^2(\Omega_p^1)[[z, u]]$ belongs to the kernel of the map

$$\frac{1}{zu(z+u)} \Lambda^2(\Omega_p^1)[[z, u]] \rightarrow \frac{1}{zu(z+u)} \Gamma_{rat}^{pp}[[z, u]]$$

induced by (67). As this element is equal to $\sum_{\alpha, \beta \geq 0} T(i, j, k, \alpha, \beta) z^\alpha u^\beta$, this implies that each $T(i, j, k, \alpha, \beta)$ belongs to the kernel of the map $\Lambda^2(\Omega_p^1) \rightarrow \Gamma_{rat}^{pp}$ for $i < j < k \in [n]$, $\alpha, \beta \geq 0$. All this implies:

Lemma 6.15. *\mathbf{K} is contained in the kernel $\text{Ker}(\Lambda^2(\Omega_p^1) \rightarrow \Gamma_{rat}^{pp})$.*

6.3.3. *A complementary subspace of \mathbf{K} in $\Lambda^2(\Omega_p^1)$.* Let \mathbf{P}_2 (resp., $\mathbf{Q}_3, \mathbf{Q}'_2, \mathbf{Q}''_2, \mathbf{S}_4, \mathbf{S}'_3, \mathbf{S}''_3, \mathbf{S}'''_3, \mathbf{S}_2$) be the linear span of elements (68) (resp., (69), (70), (71), (72), (73), (74), (75), (76)); the notation of these vector spaces has been chosen in agreement with the notation for their generating sets, the index indicating the number of free Latin indices.

According to the beginning of §6.3.2, the space $\Lambda^2(\Omega_p^1)$ decomposes as a direct sum

$$\Lambda^2(\Omega_p^1) = \mathbf{P}_2 \oplus \mathbf{Q}_3 \oplus \mathbf{Q}'_2 \oplus \mathbf{Q}''_2 \oplus \mathbf{S}_4 \oplus \mathbf{S}'_3 \oplus \mathbf{S}''_3 \oplus \mathbf{S}'''_3 \oplus \mathbf{S}_2.$$

The subspace $\mathbf{K} \subset \Lambda^2(\Gamma(\mathcal{E}))$ decomposes as

$$\mathbf{K} = \mathbf{S}_2 + \mathbf{Y} + \mathbf{Z},$$

where \mathbf{Y} is the image of the linear map

$$y : \mathbf{Q}'_2 \rightarrow \mathbf{Q}'_2 \oplus \mathbf{Q}''_2, \quad Q'(i, j, \alpha) \mapsto Q'(i, j, \alpha) \oplus (-Q''(i, j, \alpha)),$$

and \mathbf{Z} is the image of the linear map

$$z : \mathbf{S}'_3 \rightarrow \mathbf{S}'_3 \oplus \mathbf{S}''_3 \oplus \mathbf{S}'''_3 \oplus \mathbf{Q}_3 \oplus \mathbf{Q}'_2 \oplus \mathbf{Q}''_2,$$

$$\begin{aligned} S'(i, j, k, \alpha, \beta) & \mapsto S'(i, j, k, \alpha, \beta) \oplus \left(- \sum_{\gamma, \delta \geq 0, \gamma + \delta = \alpha + \beta} \binom{\delta}{\alpha} S''(i, j, k, \delta, \gamma) \right) \\ & \oplus \left(\sum_{\gamma, \delta \geq 0, \gamma + \delta = \alpha + \beta} (-1)^\delta \binom{\gamma}{\beta} S'''(i, j, k, \gamma, \delta) \right) \oplus \left(\binom{\alpha + \beta + 1}{\beta} Q(j, i, k, \alpha + \beta + 1) \right. \\ & \left. - \binom{\alpha + \beta + 1}{\alpha} Q(k, i, j, \alpha + \beta + 1) + (-1)^{\alpha+1} Q(i, j, k, \alpha + \beta + 1) \right) \\ & \oplus \left(- \binom{\alpha + \beta + 1}{\beta} Q'(i, k, \alpha + \beta + 1) \right) \\ & \oplus \left(\binom{\alpha + \beta + 1}{\alpha} Q''(i, j, \alpha + \beta + 1) + (-1)^\alpha Q''(j, k, \alpha + \beta + 1) \right). \end{aligned}$$

We recall that if $\phi : X \rightarrow Y$ is a linear map between the vector spaces X and Y , then the graph $\text{Graph}(\phi)$ of ϕ is the subspace of $X \oplus Y$, image of the linear map $X \rightarrow X \oplus Y$, $x \mapsto x \oplus \phi(x)$.

The spaces \mathbf{Y} and \mathbf{Z} are therefore the graphs of the linear maps $\tilde{y} : \mathbf{Q}'_2 \rightarrow \mathbf{Q}''_2$ and $\tilde{z} : \mathbf{S}'_3 \rightarrow \mathbf{S}''_3 \oplus \mathbf{S}'''_3 \oplus \mathbf{Q}_3 \oplus \mathbf{Q}'_2 \oplus \mathbf{Q}''_2$ obtained from y and z by composition with the direct sum of all the summands but the first of their target spaces.

Lemma 6.16. *Let A, B, C, D be vector spaces and $f : A \rightarrow B \oplus C \oplus D$, $g : C \rightarrow D$ be linear maps. Then the direct sum $A \oplus B \oplus C \oplus D$ admits a direct sum decomposition*

$$A \oplus B \oplus C \oplus D \simeq (B \oplus D) \oplus \text{Graph}(f) \oplus \text{Graph}(g).$$

Proof. Recall that if $\phi : X \rightarrow Y$ is a linear map, then $\text{Graph}(\phi)$ is a complement of X in $X \oplus Y$. It follows that there is a direct sum decomposition

$$(79) \quad A \oplus B \oplus C \oplus D \simeq (B \oplus C \oplus D) \oplus \text{Graph}(f).$$

One similarly has a direct sum decomposition

$$C \oplus D \simeq D \oplus \text{Graph}(g),$$

therefore

$$B \oplus C \oplus D \simeq (B \oplus D) \oplus \text{Graph}(g).$$

The result follows from the combination of this decomposition with (79). \square

Applying this lemma with $A := \mathbf{S}'_3$, $B := \mathbf{S}''_3 \oplus \mathbf{S}'''_3 \oplus \mathbf{Q}_3$, $C := \mathbf{Q}'_2$, $D := \mathbf{Q}''_2$, $f := \tilde{z}$, $g := \tilde{y}$, we obtain a direct sum decomposition

$$\mathbf{Q}_3 \oplus \mathbf{Q}'_2 \oplus \mathbf{Q}''_2 \oplus \mathbf{S}'_3 \oplus \mathbf{S}''_2 \oplus \mathbf{S}'''_3 \simeq \mathbf{K} \oplus \mathbf{T} \oplus (\mathbf{S}''_3 \oplus \mathbf{S}'''_3 \oplus \mathbf{Q}_3 \oplus \mathbf{Q}''_2).$$

Taking the direct sum with $\mathbf{P}_2 \oplus \mathbf{S}_4 \oplus \mathbf{S}_2$ and distributing the summands, we obtain

$$\Lambda^2(\Gamma(\mathcal{E})) \simeq (\mathbf{K} \oplus \mathbf{S}_2 \oplus \mathbf{T}) \oplus (\mathbf{S}''_3 \oplus \mathbf{S}'''_3 \oplus \mathbf{Q}_3 \oplus \mathbf{Q}''_2 \oplus \mathbf{P}_2 \oplus \mathbf{S}_4).$$

This implies:

Lemma 6.17. *A complement of \mathbf{K} in $\Lambda^2(\Omega_p^1)$ is*

$$\Sigma := \mathbf{S}''_3 \oplus \mathbf{S}'''_3 \oplus \mathbf{Q}_3 \oplus \mathbf{Q}''_2 \oplus \mathbf{P}_2 \oplus \mathbf{S}_4.$$

A basis of Σ is

$$(80) \quad P(i, j) = \underline{dp}_i \wedge \underline{dp}_j, \quad i < j \in [n],$$

$$(81) \quad Q(i, j, k, \alpha) = \underline{dp}_i \wedge \omega_{jk}^\alpha, \quad j < k, \quad i \neq j, k, \quad i, j, k \in [n], \quad \alpha \geq 0,$$

$$(82) \quad Q'(i, j, \alpha) = \underline{dp}_i \wedge \omega_{ij}^\alpha, \quad i < j \in [n], \quad \alpha \geq 0,$$

$$(83) \quad S''(i, j, k, \alpha, \beta) = \omega_{ij}^\alpha \wedge \omega_{jk}^\beta, \quad i < j < k \in [n], \quad \alpha, \beta \geq 0,$$

$$(84) \quad S'''(i, j, k, \alpha, \beta) = \omega_{ik}^\alpha \wedge \omega_{jk}^\beta, \quad i < j < k \in [n], \quad \alpha, \beta \geq 0,$$

$$(85) \quad S(i, j, k, l, \alpha, \beta) = \omega_{ij}^\alpha \wedge \omega_{kl}^\beta, \quad i < j, \quad k < l, \quad i < k, \quad i, j, k, l \text{ all different in } [n], \\ \alpha, \beta \geq 0.$$

6.3.4. *Residue maps.* According to §1.3.2, there exists for any $(i, j) \in I$ a linear map

$$(86) \quad \text{Res}_{D_{ij}}^{(2)} : \Omega^2 \rightarrow \sum_{(k, l) \in I \mid (k, l) \neq (i, j)} \Gamma(D_{ij}, \Omega_{D_{ij}}^1(\log D_{ij} \cap D_{kl})) =: \mathbf{D}_{ij}^1$$

and for any $(k, l) \neq (i, j) \in I$, a linear map

$$\text{Res}_{D_{ij} \cap D_{kl}}^{(1)} : \mathbf{D}_{ij}^1 \rightarrow \Gamma(D_{ij} \cap D_{kl}, \mathcal{O}_{D_{ij} \cap D_{kl}}) =: \mathbf{D}_{ij, kl}^0.$$

The restriction of this map to the summand corresponding to (k', l') is 0 unless $D_{ij} \cap D_{kl} = D_{ij} \cap D_{k'l'}$, which happens only if $(k, l) = (k', l')$ or $\{k, l\} \cup \{i, j\} = \{k', l'\} \cup \{i, j\}$.

6.3.5. *Computation of residue maps.* Let $(i, j) \in I$. For any $(k, l) \neq (i, j) \in I$, the isomorphism $D_{ij} \simeq \mathbb{A}^1 \times (E^\#)^{n-1}$ (see (54)) takes the divisor $D_{ij} \cap D_{kl}$ to $\mathbb{A}^1 \times D_{f_{ij}(k)f_{ij}(l)}^{(n-1)}$, where $f_{ij} : [n] \rightarrow [n-1]$ is the map given by $i, j \mapsto j$ and to induce an increasing bijection $[n] - \{i, j\} \rightarrow [n-1] - \{j\}$ (the exponent $(n-1)$ means that the divisor is in $(E^\#)^{n-1}$). This induces an isomorphism of the summand (k, l) of \mathbf{D}_{ij}^1 with

$$\Gamma(\mathbb{A}^1 \times (E^\#)^{n-1}, \Omega_{\mathbb{A}^1 \times (E^\#)^{n-1}}^1(\log(\mathbb{A}^1 \times D_{f_{ij}(k)f_{ij}(l)}^{(n-1)}))).$$

For X, Y nonsingular varieties and $D \subset Y$ a nonsingular divisor, one has $\Omega_{X \times Y}^1(\log(X \times D)) \simeq \Omega_X^1 \boxtimes \mathcal{O}_Y \oplus \mathcal{O}_X \boxtimes \Omega_Y^1(\log D)$. Using the identifications $\Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \simeq \mathbf{k}[t]$, $\Gamma(\mathbb{A}^1, \Omega_{\mathbb{A}^1}^1) \simeq \mathbf{k}[t]dt$, the latter space then identifies with

$$\mathbf{k}[t]dt \oplus \Gamma((E^\#)^{n-1}, \Omega_{(E^\#)^{n-1}}^1(\log D_{f_{ij}(k)f_{ij}(l)}^{(n-1)}))[t].$$

Let $I_{(n-1)}$, $\Omega_{(n-1)}^1$ be analogues of I, Ω^1 with n replaced by $n-1$. The map $(k, l) \mapsto (f_{ij}(k), f_{ij}(l))$ induces a surjective map $I - \{(i, j)\} \rightarrow I_{(n-1)}$. It follows that

$$(87) \quad \mathbf{D}_{ij}^1 \simeq \mathbf{k}[t]dt \oplus \Omega_{(n-1)}^1[t].$$

If $(i, j) \neq (k, l) \in I$, then the composition of the isomorphism $D_{ij} \cap D_{kl} \simeq \mathbb{A}^1 \times D_{f_{ij}(k)f_{ij}(l)}^{(n-1)}$ induced by (54) and of the isomorphism $D_{f_{ij}(k)f_{ij}(l)}^{(n-1)} \simeq \mathbb{A}^1 \times (E^\#)^{n-2}$ induces an isomorphism $D_{ij} \cap D_{kl} \simeq (\mathbb{A}^1)^2 \times (E^\#)^{n-2}$ and therefore an isomorphism

$$\mathbf{D}_{ij,kl}^0 \simeq \mathbf{k}[t, t'].$$

Let $\odot : \Lambda^2(\Omega_p^1) \rightarrow \Omega^2$ be the linear map induced by the wedge product. One checks that the composed map

$$\Lambda^2(\Omega_p^1) \xrightarrow{\odot} \Omega^2 \xrightarrow{\text{Res}_{D_{ij}}^{(2)}} \mathbf{D}_{ij}^1 \simeq \Omega_{(n-1)}^1[t] \oplus \mathbf{k}[t]dt$$

has its image contained in $\Omega_{(n-1)}^1[t]$. We denote by $\varrho_{ij}^{(2)} : \Lambda^2(\Omega_p^1) \rightarrow \Omega_{(n-1)}^1[t]$ the resulting corestricted map. Then the diagram

$$(88) \quad \begin{array}{ccccc} \Lambda^2(\Omega_p^1) & \xrightarrow{\odot} & \Omega^2 & \xrightarrow{\text{Res}_{D_{ij}}^{(2)}} & \mathbf{D}_{ij}^1 \\ & \searrow \varrho_{ij}^{(2)} & & & \uparrow \sim \\ & & & & \Omega_{(n-1)}^1[t] \oplus \mathbf{k}[t]dt \\ & & & & \uparrow \cup \\ & & & & \Omega_{(n-1)}^1[t] \end{array}$$

commutes.

This diagram implies:

Lemma 6.18. *The kernel $\text{Ker}(\odot : \Lambda^2(\Omega_p^1) \rightarrow \Omega^2)$ is contained in the intersection over all $(i, j) \in I$ of the kernels of the maps $\varrho_{ij}^{(2)} : \Lambda^2(\Omega_p^1) \rightarrow \Omega_{(n-1)}^1[t]$.*

This implies:

Corollary 6.1. *The kernel $\text{Ker}(\odot : \Lambda^2(\Omega_p^1) \rightarrow \Omega^2)$ is contained in the intersection over all pairs $(i, j) \neq (k, l) \in I$ of the kernels of the maps*

$$\varrho_{f_{ij}(k)f_{ij}(l)} \circ \varrho_{ij}^{(2)} : \Lambda^2(\Omega_p^1) \rightarrow \mathbf{k}[t, t'].$$

We now turn to the computation of $\varrho_{ij}^{(2)}$. As \mathbb{O} vanishes on $\mathbf{K} \subset \Lambda^2(\Omega_p^1)$, so does $\varrho_{ij}^{(2)}$. The restriction of $\varrho_{ij}^{(2)}$ to Σ can be computed using Lemma 6.10 and §1.2.4. One gets:

Lemma 6.19. *Let $(i_0, j_0) \in I$. The map*

$$\varrho_{i_0 j_0}^{(2)} : \Lambda^2(\Omega_p^1) \rightarrow \Omega_{(n-1)}^1[t]$$

is given by $R \mapsto 0$, and the following formulas:

- for $i < j \in [n]$, $P(i, j) \mapsto 0$,
- for $i, j, k \in [n]$, $j < k$, $i \neq j, k$, and $\alpha \geq 0$,

$$Q(i, j, k, \alpha) \mapsto \begin{cases} -\frac{(-t)^\alpha}{\alpha!} \cdot \underline{dp}_{f_{i_0 j_0}(i)} & \text{if } (j, k) = (i_0, j_0), \\ 0 & \text{else,} \end{cases}$$

- for $i < j \in [n]$, $\alpha \geq 0$, $Q'(i, j, \alpha) \mapsto \begin{cases} -(-t)^\alpha / \alpha! \cdot \underline{dp}_{i_0} & \text{if } (i, j) = (i_0, j_0), \\ 0 & \text{else,} \end{cases}$
- for $i < j < k \in [n]$, $\alpha, \beta \geq 0$,

$$S''(i, j, k, \alpha, \beta) \mapsto \begin{cases} \frac{(-t)^\alpha}{\alpha!} \cdot \omega_{i_0 f_{i_0 j_0}(k)}^\beta & \text{if } (i, j) = (i_0, j_0), \\ -\frac{(-t)^\beta}{\beta!} \cdot \omega_{f_{i_0 j_0}(i) i_0}^\alpha & \text{if } (j, k) = (i_0, j_0), \\ 0 & \text{else,} \end{cases}$$

- for $i < j < k \in [n]$, $\alpha, \beta \geq 0$,

$$S'''(i, j, k, \alpha, \beta) \mapsto \begin{cases} \frac{(-t)^\alpha}{\alpha!} \cdot (-1)^\beta \omega_{i_0 f_{i_0 j_0}(j)}^\beta & \text{if } (i, k) = (i_0, j_0), \\ -\frac{(-t)^\beta}{\beta!} \cdot \omega_{f_{i_0 j_0}(i) i_0}^\alpha & \text{if } (j, k) = (i_0, j_0), \\ 0 & \text{else,} \end{cases}$$

- for i, j, k, l all different in $[n]$, $i < j$, $k < l$, $i < k$, $\alpha, \beta \geq 0$,

$$S(i, j, k, l, \alpha, \beta) \mapsto \begin{cases} \frac{(-t)^\alpha}{\alpha!} \cdot \omega_{f_{i_0 j_0}(k) f_{i_0 j_0}(l)}^\beta & \text{if } (i, j) = (i_0, j_0), \\ -\frac{(-t)^\beta}{\beta!} \cdot \omega_{f_{i_0 j_0}(i) f_{i_0 j_0}(j)}^\alpha & \text{if } (k, l) = (i_0, j_0), \\ 0 & \text{else,} \end{cases}$$

where $f_{i_0 j_0} : [n] \rightarrow [n-1]$ is as in the beginning of §6.3.5.

Remark 6.20. Let $(k, l) \neq (i, j) \in I$. The commutative diagram

$$\begin{array}{ccc} D_{ij} \cap D_{kl} & \xrightarrow{\sim} & \mathbb{A}^1 \times D_{f_{ij}(k) f_{ij}(l)}^{(n-1)} \\ \downarrow & & \downarrow \\ D_{ij} & \xrightarrow{\sim} & \mathbb{A}^1 \times (E^\#)^{n-1} \end{array}$$

implies that the following diagram commutes

$$\begin{array}{ccc} \Omega_{(n-1)}^1[t] & \xrightarrow{\varrho_{f_{ij}(k) f_{ij}(l)}} & \mathbf{k}[t', t] \\ \downarrow & & \uparrow \\ \mathbf{D}_{ij}^1 & \xrightarrow{\text{Res}_{D_{ij} \cap D_{kl}}^{(1)}} & \mathbf{D}_{ij, kl}^0 \end{array}$$

where the upper map the tensor product with $\mathbf{k}[t]$ of the analogue of the map from (59), with (n, i, j) replaced by $(n-1, f_{ij}(k), f_{ij}(l))$ and t replaced by t' .

Combining diagram (88) with the above diagram, one gets, for any $(i, j) \neq (k, l) \in I$, a commutative diagram

$$\begin{array}{ccccccc}
\Lambda^2(\Omega_p^1) & \xrightarrow{\oplus} & \Omega^2 & \xrightarrow{\text{Res}_{D_{ij}}^{(2)}} & \mathbf{D}_{ij}^1 & \xrightarrow{\text{Res}_{D_{ij} \cap D_{kl}}^{(1)}} & \mathbf{D}_{ij,kl}^0 \\
& & & & \uparrow \sim & & \uparrow \sim \\
& & & & \Omega_{(n-1)}^1[t] \oplus \mathbf{k}[t]dt & & \\
& \searrow \varrho_{ij}^{(2)} & & & \uparrow & & \\
& & & & \Omega_{(n-1)}^1[t] & \xrightarrow{\varrho_{f_{ij}(k)f_{ij}(l)}} & \mathbb{C}[t, t']
\end{array}$$

which gives an interpretation of the map in Corollary 6.1. \square

6.3.6. *Compositions of residue maps.* Computation yields:

Lemma 6.21. *Assume that $i_0, j_0, k_0, l_0 \in [n]$ are all distinct, such that $i_0 < j_0$, $k_0 < l_0$, $i_0 < k_0$. Then*

$$\varrho_{f_{i_0 j_0}(k_0)f_{i_0 j_0}(l_0)} \circ \varrho_{i_0 j_0|\Sigma}^{(2)} : \Sigma \rightarrow \mathbf{k}[t, t']$$

takes $P_2 \oplus Q_3 \oplus Q_2 \oplus S_3'' \oplus S_3'''$ to 0, and its restriction to S_4 is given by

$$S(i, j, k, l, \alpha, \beta) \mapsto \begin{cases} \frac{(-t)^\alpha (-t')^\beta}{\alpha! \beta!} & \text{if } (i, j, k, l) = (i_0, j_0, k_0, l_0), \\ 0 & \text{else,} \end{cases}$$

for distinct $i, j, k, l \in [n]$, such that $i < j$, $k < l$, $i < k$, and $\alpha, \beta \geq 0$.

Lemma 6.22. *Assume that $i_0 < j_0 < k_0 \in [n]$. Then*

$$\varrho_{i_0 f_{i_0 k_0}(j_0)} \circ \varrho_{i_0 k_0|\Sigma}^{(2)} : \Sigma \rightarrow \mathbf{k}[t, t']$$

takes $P_2 \oplus Q_3 \oplus Q_2 \oplus S_3'' \oplus S_4$ to 0, and its restriction to S_3''' is given by

$$S'''(i, j, k, \alpha, \beta) \mapsto \begin{cases} \frac{(-t)^\alpha (t')^\beta}{\alpha! \beta!} & \text{if } (i, j, k) = (i_0, j_0, k_0), \\ 0 & \text{else,} \end{cases}$$

for $i < j < k \in [n]$, and $\alpha, \beta \geq 0$.

Lemma 6.23. *Assume that $i_0 < j_0 < k_0 \in [n]$. Then*

$$\varrho_{i_0 f_{i_0 j_0}(k_0)} \circ \varrho_{i_0 j_0|\Sigma}^{(2)} : \Sigma \rightarrow \mathbf{k}[t, t']$$

takes $P_2 \oplus Q_3 \oplus Q_2 \oplus S_3''' \oplus S_4$ to 0, and its restriction to S_3'' is given by

$$S''(i, j, k, \alpha, \beta) \mapsto \begin{cases} \frac{(-t)^\alpha (-t')^\beta}{\alpha! \beta!} & \text{if } (i, j, k) = (i_0, j_0, k_0), \\ 0 & \text{else,} \end{cases}$$

for $i < j < k \in [n]$, and $\alpha, \beta \geq 0$.

6.3.7. *Computation of $\text{Ker}(\Lambda^2(\Omega_p^1) \rightarrow \Gamma_{rat}^{pp})$.*

Lemma 6.24. *The map $\odot|_{\Sigma} : \Sigma \subset \Lambda^2(\Omega_p^1) \rightarrow \Omega^2$ is injective.*

Proof. Let $\sigma \in \Sigma$ be an element of $\text{Ker}(\odot : \Lambda^2(\Omega_p^1) \rightarrow \Omega^2)$. Then σ can be decomposed as

$$\begin{aligned} \sigma = & \sum_{i < j \in [n]} p(i, j)P(i, j) + \sum_{\substack{i, j, k \in [n], \\ \#\{i, j, k\}=3, j < k \\ \alpha \geq 0}} q(i, j, k, \alpha)Q(i, j, k, \alpha) + \sum_{\substack{i < j \in [n] \\ \alpha \geq 0}} q'(i, j, \alpha)Q'(i, j, \alpha) \\ & + \sum_{\substack{i < j < k \in [n] \\ \alpha, \beta \geq 0}} s''(i, j, k, \alpha, \beta)S''(i, j, k, \alpha, \beta) + \sum_{\substack{i < j < k \in [n] \\ \alpha, \beta \geq 0}} s'''(i, j, k, \alpha, \beta)S'''(i, j, k, \alpha, \beta) \\ & + \sum_{\substack{i, j, k, l \in [n], \#\{i, j, k, l\}=3, \\ i < j, k < l, i < k \\ \alpha, \beta \geq 0}} s(i, j, k, l, \alpha, \beta)S(i, j, k, l, \alpha, \beta), \end{aligned}$$

where $p(i, j)$, etc., are suitable scalars.

Assume that $i_0, j_0, k_0, l_0 \in [n]$ are all distinct, and that $i_0 < j_0, k_0 < l_0, i_0 < k_0$. By Corollary 6.1, $\varrho_{f_{i_0 j_0}(k_0) f_{i_0 j_0}(l_0)} \circ \varrho_{i_0 j_0}^{(2)}(\sigma) = 0$, so $\sum_{\alpha, \beta} s(i_0, j_0, k_0, l_0, \alpha, \beta) \frac{(-t)^\alpha (-t')^\beta}{\alpha! \beta!} = 0$, therefore $s(i_0, j_0, k_0, l_0, \alpha, \beta) = 0$.

Assume that $i_0 < j_0 < k_0 \in [n]$.

By Corollary 6.1, $\varrho_{i_0 f_{i_0 j_0}(k_0)} \circ \varrho_{i_0 j_0}^{(2)}(\sigma) = 0$, so

$$\sum_{\alpha, \beta} s'''(i_0, j_0, k_0, \alpha, \beta) \frac{(-t)^\alpha (t')^\beta}{\alpha! \beta!} = 0,$$

therefore $s'''(i_0, j_0, k_0, \alpha, \beta) = 0$.

By Corollary 6.1, $\varrho_{i_0 f_{i_0 j_0}(k_0)} \circ \varrho_{i_0 j_0}^{(2)}(\sigma) = 0$, so

$$\sum_{\alpha, \beta} s''(i_0, j_0, k_0, \alpha, \beta) \frac{(-t)^\alpha (-t')^\beta}{\alpha! \beta!} = 0,$$

therefore $s''(i_0, j_0, k_0, \alpha, \beta) = 0$.

It follows that

$$\sigma = \sum_{i < j \in [n]} p(i, j)P(i, j) + \sum_{\substack{i, j, k \in [n], \\ \#\{i, j, k\}=3, j < k \\ \alpha \geq 0}} q(i, j, k, \alpha)Q(i, j, k, \alpha) + \sum_{\substack{i < j \in [n] \\ \alpha \geq 0}} q'(i, j, \alpha)Q'(i, j, \alpha).$$

Assume that $i_0 < j_0 < k_0 \in [n]$.

By Lemma 6.18, $\varrho_{i_0 j_0}^{(2)}(\sigma) = 0$, so

$$\sum_{\alpha \geq 0} \left(\sum_{i \in [n] - \{i_0, j_0\}} q(i, i_0, k_0, \alpha) \underline{dp}_{f_{i_0 j_0}(i)} + q'(i_0, j_0, \alpha) \underline{dp}_{i_0} \right) \left(-\frac{(-t)^\alpha}{\alpha!} \right) = 0.$$

As the restriction of the map $f_{i_0 j_0}$ to $[n] - \{i_0, j_0\}$ is injective and as its image does not contain i_0 , we have $q(i, i_0, k_0, \alpha) = 0$ for any $i \in [n] - \{i_0, j_0\}, \alpha \geq 0$ and $q'(i_0, j_0, \alpha) = 0$ for any $\alpha \geq 0$.

This implies that $\sigma = \sum_{i < j \in [n]} p(i, j)P(i, j)$. The relation $\odot(\sigma) = 0$ yields

$$\sum_{i < j \in [n]} p(i, j) \underline{dp}_i \odot \underline{dp}_j = 0.$$

As the family $(\underline{dp}_i \otimes \underline{dp}_j)$ is linearly independent over \mathbf{k} in Ω^2 , we obtain $p(i, j) = 0$ for any i, j , therefore $\sigma = 0$. \square

Lemma 6.25. *The kernel of $\otimes : \Lambda^2(\Omega_p^1) \rightarrow \Omega^2$, and therefore also of the map $\Lambda^2(\Omega_p^1) \xrightarrow{\otimes} \Omega^2 \hookrightarrow \Gamma_{rat}^{pp}$, is equal to \mathbf{K} .*

Proof. Recall that $\Lambda^2(\Omega_p^1) = \mathbf{K} \oplus \Sigma$, that \otimes is a map $\Lambda^2(\Omega_p^1) \rightarrow \Omega^2$ and that $\mathbf{K} \subset \text{Ker}(\otimes)$. The result now follows from Lemma 6.24. \square

Combining this result with those from §6.3.1, we obtain:

Proposition 6.1. *The kernel of the wedge product map $\Lambda^2(\Omega^1) \rightarrow \Omega^2 \hookrightarrow \Gamma_{rat}(\Omega_{(E^\#)^n}^2)$ is equal to the subspace \mathbf{K} of $\Lambda^2(\Omega_p^1) \subset \Lambda^2(\Omega^1)$.*

7. PRESENTATION AND COMPUTATION OF THE LIE ALGEBRA \mathfrak{G} (EQUIVS. (f) AND (g))

7.1. Grading on Ω^1 . Taking into account Lemma 6.11 which makes explicit a basis of Ω^1 , we may define a grading on Ω^1 by

$$\deg(\omega_{ij}^\alpha) = \alpha + 1 \quad \text{for } \alpha \geq 0, \quad i < j \in [n], \quad \deg(\underline{dc}_i) = 1, \quad \deg(\underline{dp}_i) = 0 \quad \text{for } i \in [n].$$

Then Ω^1 is graded in degrees ≥ 0 . For $d \geq 0$, we denote by $\Omega^1[d]$ the degree d part of Ω^1 . So

$$\begin{aligned} \Omega^1 &= \oplus_{d \geq 0} \Omega^1[d], \quad \Omega^1[0] = \oplus_{i \in [n]} \mathbf{k} \underline{dp}_i, \quad \Omega^1[1] = (\oplus_{i \in [n]} \mathbf{k} \underline{dc}_i) \oplus (\oplus_{i < j \in [n]} \mathbf{k} \omega_{ij}^0), \\ \Omega^1[d] &= \oplus_{i < j \in [n]} \mathbf{k} \omega_{ij}^{d-1} \quad \text{for } d \geq 2. \end{aligned}$$

7.2. A graded space \mathbf{I} . The linear map $\otimes : \Lambda^2(\Omega^1) \rightarrow \Omega^2$ has been defined in §1.3.2 and its kernel has been identified with \mathbf{K} (see Def. 6.14) in Lemma 6.25. The grading of Ω^1 from §7.1 induces a grading on $\Lambda^2(\Omega^1)$. The generators of $\mathbf{K} \subset \Lambda^2(\Omega^1)$ are homogeneous for this grading: $R(i, j, \alpha)$ is pure of degree $\alpha + 1$, and $S(i, j, \alpha, \beta)$ and $T(i, j, k, \alpha, \beta)$ are pure of degree $\alpha + \beta + 2$. This implies that \mathbf{K} is a graded subspace of $\Lambda^2(\Omega^1)$. It follows that the quotient space $\Lambda^2(\Omega^1)/\mathbf{K}$ inherits a grading from $\Lambda^2(\Omega^1)$.

We make the following definition:

Definition 7.1. $\mathbf{I} := \text{im}(\otimes : \Lambda^2(\Omega^1) \rightarrow \Omega^2)$

As \mathbf{I} is isomorphic to $\Lambda^2(\Omega^1)/\mathbf{K}$, one defines a grading on \mathbf{I} by transport of structure. Then the composed map $\Lambda^2(\Omega^1) \rightarrow \Lambda^2(\Omega^1)/\mathbf{K} \simeq \mathbf{I}$ is compatible with the gradings. Therefore:

Lemma 7.2. *The space \mathbf{I} is equipped with a grading which is compatible with the map $\otimes : \Lambda^2(\Omega^1) \rightarrow \mathbf{I}$.*

Recall that

$$(89) \quad \Lambda^2(\Omega^1) \simeq \Lambda^2(\Omega_c^1) \oplus (\Omega_c^1 \otimes \Omega_p^1) \oplus \Lambda^2(\Omega_p^1),$$

that $\mathbf{K} \subset \Lambda^2(\Omega_p^1)$ and that a complement Σ of \mathbf{K} in $\Lambda^2(\Omega_p^1)$ has been constructed in Lemma 6.17. It follows that *there is an isomorphism*

$$(90) \quad \mathbf{I} \simeq \Lambda^2(\Omega_c^1) \oplus (\Omega_c^1 \otimes \Omega_p^1) \oplus \Sigma.$$

One derives from there:

the images under $\Lambda^2(\Omega^1)/\mathbf{K} \simeq \mathbf{I}$ of the classes in $\Lambda^2(\Omega^1)/\mathbf{K}$ of the elements of the family
 $(\underline{dc}_i \wedge \underline{dc}_j)_{i < j \in [n]}, (\underline{dc}_i \wedge \underline{dp}_j)_{i, j \in [n]}, (\underline{dc}_i \wedge \omega_{jk}^\alpha)_{i \in [n], j < k \in [n], \alpha \geq 0}, (P(i, j))_{i < j \in [n]}, (Q(i, j, k, \alpha))_{i \neq j, k \in [n], \alpha \geq 0},$
 $(Q'(i, j, \alpha))_{i < j \in [n], \alpha \geq 0}, (S''(i, j, k, \alpha, \beta))_{i < j < k \in [n], \alpha, \beta \geq 0}, (S'''(i, j, k, \alpha, \beta))_{i < j < k \in [n], \alpha, \beta \geq 0},$
 $(S(i, j, k, l, \alpha, \beta))_{i < j, k < l, i < k \in [n], \#\{i, j, k, l\} = 4, \alpha, \beta \geq 0},$
(see (80)-(85)) of elements in $\Lambda^2(\Omega^1)$ form a graded basis of \mathbf{I} .

7.3. Computation of the map $d : \Omega^1 \rightarrow \Omega^2$. The map $d : \Omega^1 \rightarrow \Omega^2$ may be computed as follows:

$$\underline{dc}_i \mapsto 0, \quad i \in [n], \quad \underline{dp}_i \mapsto 0, \quad i \in [n], \quad \omega_{ij}^\alpha \mapsto \begin{cases} -\underline{dc}_{ij} \odot \omega_{ij}^{\alpha-1} & \text{if } \alpha > 0, \\ -\underline{dc}_{ij} \odot \underline{dp}_{ij} & \text{if } \alpha = 0, \end{cases} \quad i < j \in [n].$$

where $\underline{dc}_{ij} := \underline{dc}_i - \underline{dc}_j$, $\underline{dp}_{ij} := \underline{dp}_i - \underline{dp}_j$.

7.4. Grading on the coalgebra \mathbf{C} and the Lie coalgebra \mathfrak{C} . It follows from §7.3 that the image of d is contained in \mathbf{I} , and that the resulting corestricted map $d : \Omega^1 \rightarrow \mathbf{I}$ is graded. Recall also that $\odot : \Lambda^2(\Omega^1) \rightarrow \Omega^2$ corestricts to a graded map $\odot : \Lambda^2(\Omega^1) \rightarrow \mathbf{I}$. All this implies that the map $\mu : T(\Omega^1) \rightarrow T(\Omega^1) \otimes \Omega^2 \otimes T(\Omega^1)$ from §3.6 corestricts to a map $T(\Omega^1) \rightarrow T(\Omega^1) \otimes \mathbf{I} \otimes T(\Omega^1)$. This implies that the bialgebra $\mathbf{C} := \text{Ker}(T(\Omega^1) \rightarrow T(\Omega^1) \otimes \mathbf{I} \otimes T(\Omega^1))$ is graded. Since Ω^1 has finite dimensional graded parts, so does \mathbf{C} .

It follows that the Lie coalgebra $\mathfrak{C} := \text{Coprime}(\mathbf{C})$ is also graded with finite dimensional graded parts. We denote by \mathfrak{G} the graded dual Lie algebra.

7.5. Computation of the Lie algebra \mathfrak{G} . Denote by V^* the graded dual of a graded vector space V ; so for $V = \oplus_{d \in \mathbb{Z}} V[d]$, $V^* = \oplus_{d \in \mathbb{Z}} V^*[d]$, where $V^*[d] := V[-d]^*$. For $V = \oplus_{d \in \mathbb{Z}} V[d]$, $W = \oplus_{d \in \mathbb{Z}} W[d]$ two such spaces, we denote by $V \hat{\otimes} W$ their completed tensor product, equal to $\prod_{d, d'} V[d] \otimes W[d']$.

Denote also by \mathfrak{L} the free Lie algebra functor and by \mathfrak{L}_k is degree k component.

Then \mathfrak{G} may be presented as the quotient

$$\mathfrak{G} = \mathfrak{L}((\Omega^1)^*) / (\mathbf{R}),$$

where

$$\mathbf{R} := \text{im}(\mathbf{I}^* \xrightarrow{d^* \oplus \frac{1}{2} \odot^*} (\Omega^1)^* \oplus (\Lambda^2(\Omega^1))^* \simeq \mathfrak{L}_1((\Omega^1)^*) \oplus \mathfrak{L}_2((\Omega^1)^*))$$

and (\mathbf{R}) is the ideal generated by \mathbf{R} .

We now compute the space \mathbf{R} . Let

$$(X_i)_{i \in [n]}, \quad (Y_i)_{i \in [n]}, \quad (T_{ij}^\alpha)_{i < j \in [n], \alpha \geq 0}$$

be the basis of $(\Omega^1)^*$ dual to the basis (58) of Ω^1 .

Lemma 7.3. *The space \mathbf{R} decomposes as*

$$(91) \quad \mathbf{R} = \mathbf{R}_{cc} + \mathbf{R}_{cp} + \mathbf{R}_{pp},$$

where

$$(92) \quad \mathbf{R}_{cc} = \text{Span}\{[X_i, X_j] \mid i < j \in [n]\},$$

$$(93) \quad \begin{aligned} \mathbf{R}_{cp} = \text{Span}\{ & [X_i, Y_i] - \sum_{j \in [n] \mid j > i} T_{ij}^0 - \sum_{j \in [n] \mid j < i} T_{ji}^0, \quad i \in [n]; \\ & [X_i, Y_j] + T_{ij}^0, \quad i < j \in [n]; \quad [X_i, Y_j] + T_{ji}^0, \quad j < i \in [n]; \\ & [X_k, T_{ij}^\alpha], \quad i < j \in [n], \quad k \in [n] - \{i, j\}, \quad \alpha \geq 0; \\ & [X_i, T_{ij}^\alpha] - T_{ij}^{\alpha+1}, \quad i < j \in [n], \quad \alpha \geq 0; \\ & [X_j, T_{ij}^\alpha] + T_{ij}^{\alpha+1}, \quad i < j \in [n], \quad \alpha \geq 0\}, \end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{pp} = \text{Span}\{ & \pi(i, j), \quad i < j \in [n], \\
& \sigma(i, j, k, l, \alpha, \beta), \quad i < j \in [n], \quad k < l \in [n], \quad i < k, \quad \#\{i, j, k, l\} = 4, \quad \alpha, \beta \geq 0, \\
& \sigma''(i, j, k, \alpha, \beta), \quad i < j < k \in [n], \quad \alpha, \beta \geq 0, \\
& \sigma'''(i, j, k, \alpha, \beta), \quad i < j < k \in [n], \quad \alpha, \beta \geq 0, \\
& \kappa(k, i, j, \alpha), \quad i < j \in [n], \quad k \in [n] - \{i, j\}, \quad \alpha \geq 0, \\
(94) \quad & \kappa'(i, j, \alpha), \quad i < j \in [n], \quad \alpha \geq 0\},
\end{aligned}$$

where

$$(95) \quad \pi(i, j) := [Y_i, Y_j] \quad \text{for } i < j \in [n],$$

$$(96) \quad \sigma(i, j, k, l, \alpha, \beta) := [T_{ij}^\alpha, T_{kl}^\beta] \quad \text{for } i < j \in [n], \quad k < l \in [n], \quad i < k, \quad \#\{i, j, k, l\} = 4, \quad \alpha, \beta \geq 0,$$

$$(97) \quad \sigma''(i, j, k, \alpha, \beta) := [T_{ij}^\alpha, T_{jk}^\beta] + \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha + \beta}} \binom{\alpha}{\gamma} [T_{ij}^\gamma, T_{ik}^\delta] \quad \text{for } i < j < k \in [n], \quad \alpha, \beta \geq 0,$$

$$(98) \quad \sigma'''(i, j, k, \alpha, \beta) := [T_{ik}^\alpha, T_{jk}^\beta] + \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha + \beta}} (-1)^{\beta+1} \binom{\alpha}{\delta} [T_{ij}^\gamma, T_{ik}^\delta] \quad \text{for } i < j < k \in [n], \quad \alpha, \beta \geq 0,$$

$$(99) \quad \kappa(k, i, j, \alpha) = \begin{cases} [Y_k, T_{ij}^\alpha] + \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha + \beta}} (-1)^\gamma [T_{ki}^\gamma, T_{kj}^\delta] & \text{if } k < i, \\ [Y_k, T_{ij}^\alpha] + \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha + \beta}} (-1) \cdot \binom{\alpha}{\delta} [T_{ik}^\gamma, T_{ij}^\delta] & \text{if } i < k < j, \\ [Y_k, T_{ij}^\alpha] + \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha + \beta}} \binom{\alpha}{\gamma} [T_{ij}^\gamma, T_{ik}^\delta] & \text{if } k > j, \end{cases} \quad \text{for } i < j \in [n], \quad \alpha \geq 0,$$

$$\begin{aligned}
\kappa'(i, j, \alpha) = & [Y_i + Y_j, T_{ij}^\alpha] + \sum_{\substack{i < k < j \\ \gamma + \delta = \alpha - 1}} \binom{\alpha}{\delta} [T_{ik}^\gamma, T_{ij}^\delta] + \sum_{\substack{k > j \\ \gamma + \delta = \alpha - 1}} (-1) \cdot \binom{\alpha}{\gamma} [T_{ij}^\gamma, T_{ik}^\delta] + \sum_{\substack{k < i \\ \gamma + \delta = \alpha - 1}} (-1)^{\gamma+1} [T_{ki}^\gamma, T_{kj}^\delta] \\
(100) \quad & \text{for } i < j \in [n], \quad \alpha \geq 0.
\end{aligned}$$

Proof. Set

$$(101) \quad \omega := \sum_{i \in [n]} X_i \otimes \underline{dc}_i + \sum_{i \in [n]} Y_i \otimes \underline{dp}_i + \sum_{i < j \in [n], \alpha \geq 0} T_{ij}^\alpha \otimes \omega_{ij}^\alpha \in (\Omega^1)^* \hat{\otimes} \Omega^1;$$

this is the canonical element of $(\Omega^1)^* \hat{\otimes} \Omega^1$. For any Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, define the product $(\mathfrak{g} \otimes \Omega^1)^2 \rightarrow \mathfrak{g} \otimes \Omega^2$ by

$$(102) \quad (x \otimes h) \cdot (x' \otimes h') := [x, x']_{\mathfrak{g}} \otimes (h \otimes h').$$

Then $d\omega + \frac{1}{2}\omega^2$ is an element of $\mathfrak{L}((\Omega^1)^*) \hat{\otimes} \mathbf{I}$, and \mathbf{R} is the image of the map $\mathbf{I}^* \rightarrow \mathfrak{L}((\Omega^1)^*)$ induced by this element. So modding out by (\mathbf{R}) corresponds to formally imposing the relation $d\omega + \frac{1}{2}\omega^2 = 0$.

Recall that Ω^1 decomposes as $\Omega^1 = \Omega_c^1 \oplus \Omega_p^1$. This induces a decomposition of $(\Omega^1)^* \hat{\otimes} \Omega^1$. The corresponding decomposition of ω is

$$\omega = \omega_c + \omega_p,$$

where

$$\omega_c = \sum_{i \in [n]} X_i \otimes \underline{dc}_i, \quad \omega_p = \sum_{i \in [n]} Y_i \otimes \underline{dp}_i + \sum_{i < j \in [n], \alpha \geq 0} T_{ij}^\alpha \otimes \omega_{ij}^\alpha.$$

It follows from the construction of \mathbf{I} that the map $\otimes : \Lambda^2(\Omega^1) \rightarrow \mathbf{I}$ is compatible with the decompositions (89) and (90) of both sides. Namely,

$$\otimes(\Lambda^2(\Omega_c^1)) = \Lambda^2(\Omega_c^1), \quad \otimes(\Omega_c^1 \otimes \Omega_p^1) = \Omega_c^1 \otimes \Omega_p^1, \quad \otimes(\Lambda^2(\Omega_p^1)) = \Sigma.$$

On the other hand, $d : \Omega^1 \rightarrow \mathbf{I}$ is such that

$$d(\Omega_c^1) = 0, \quad d(\Omega_p^1) \subset \Omega_c^1 \otimes \Omega_p^1.$$

Then $d\omega + \frac{1}{2}\omega^2$ decomposes as

$$d\omega + \frac{1}{2}\omega^2 = \frac{1}{2}\omega_c^2 + (d\omega_p + \omega_p \cdot \omega_c) + \frac{1}{2}\omega_p^2$$

according to the decomposition of $\mathfrak{L}((\Omega^1)^*) \hat{\otimes} \mathbf{I}$ induced by (90). The decomposition (90) induces a decomposition of \mathbf{I}^* , and therefore of the map $\mathbf{I}^* \rightarrow \mathfrak{L}((\Omega^1)^*)$ as the sum of three maps

$$(103) \quad \Lambda^2(\Omega_c^1)^* \rightarrow \mathfrak{L}((\Omega^1)^*), \quad (\Omega_c^1 \otimes \Omega_p^1)^* \rightarrow \mathfrak{L}((\Omega^1)^*), \quad \Sigma^* \rightarrow \mathfrak{L}((\Omega^1)^*).$$

These maps correspond respectively to $\frac{1}{2}\omega_c^2$, $d\omega_p + \omega_c \cdot \omega_p$, and $\frac{1}{2}\omega_p^2$. One then has the decomposition (91) where \mathbf{R}_{cc} , \mathbf{R}_{cp} and \mathbf{R}_{pp} , are the images of the maps induced by (103).

One computes

$$\omega_c^2 = \frac{1}{2} \sum_{i, j \in [n]} [X_i, X_j] \otimes (\underline{dc}_i \otimes \underline{dc}_j) = \sum_{i < j \in [n]} [X_i, X_j] \otimes (\underline{dc}_i \otimes \underline{dc}_j);$$

since $(\underline{dc}_i \otimes \underline{dc}_j)_{i < j \in [n]}$ is a linearly independent family of $\Lambda^2(\Omega_c^1)$, one obtains (92).

One computes

$$\begin{aligned} d\omega_p + \frac{1}{2}\omega_p \cdot \omega_c &= \sum_{i < j \in [n], \alpha \geq 0} T_{ij}^\alpha \otimes d(\omega_{ij}^\alpha) + \sum_{i, j \in [n]} [X_i, Y_j] \otimes (\underline{dc}_i \otimes \underline{dp}_j) + \frac{1}{2} \sum_{k \in [n], i < j \in [n], \alpha \geq 0} [X_k, T_{ij}^\alpha] \otimes (\underline{dc}_k \otimes \omega_{ij}^\alpha) \\ &= \sum_{i \in [n]} ([X_i, Y_i] - \sum_{j|j>i} T_{ij}^0 - \sum_{j|j<i} T_{ji}^0) \otimes (\underline{dc}_i \otimes \underline{dp}_i) \\ &\quad + \sum_{i < j \in [n]} ([X_i, Y_j] + T_{ij}^0) \otimes (\underline{dc}_i \otimes \underline{dp}_j) + \sum_{i > j \in [n]} ([X_i, Y_j] + T_{ji}^0) \otimes (\underline{dc}_i \otimes \underline{dp}_j) \\ &\quad + \sum_{i < j \in [n], k \neq i, j, \alpha \geq 0} [X_k, T_{ij}^\alpha] \otimes (\underline{dc}_k \otimes \omega_{ij}^\alpha) \\ &\quad + \sum_{i < j \in [n]} ([X_i, T_{ij}^\alpha] - T_{ij}^{\alpha+1}) \otimes (\underline{dc}_i \otimes \omega_{ij}^\alpha) + \sum_{i < j \in [n]} ([X_j, T_{ij}^\alpha] + T_{ij}^{\alpha+1}) \otimes (\underline{dc}_j \otimes \omega_{ij}^\alpha). \end{aligned}$$

As the second factors in the last expressions form a basis of $\Omega_c^1 \otimes \Omega_p^1$, one gets (93).

One computes

$$\begin{aligned} \frac{1}{2}\omega_p^2 &= \frac{1}{2} \sum_{i, j \in [n]} [Y_i, Y_j] \otimes (\underline{dp}_i \otimes \underline{dp}_j) + \sum_{i < j \in [n], k \in [n], \alpha \geq 0} [Y_k, T_{ij}^\alpha] \otimes (\underline{dp}_k \otimes \omega_{ij}^\alpha) \\ &\quad + \frac{1}{2} \sum_{i < j \in [n], k < l \in [n], \alpha, \beta \geq 0} [T_{ij}^\alpha, T_{kl}^\beta] \otimes (\omega_{ij}^\alpha \otimes \omega_{kl}^\beta) \end{aligned}$$

which expresses as follows

$$\begin{aligned}
\frac{1}{2}\omega_p^2 &= \sum_{i < j \in [n]} \pi(i, j) \otimes P(i, j) + \sum_{\substack{i < j \in [n], k < l \in [n], i < k, \\ \#\{i, j, k, l\} = 4, \\ \alpha, \beta \geq 0}} \sigma(i, j, k, l, \alpha, \beta) \otimes S(i, j, k, l, \alpha, \beta) \\
&+ \sum_{\substack{i < j < k \in [n], \\ \alpha, \beta \geq 0}} \sigma''(i, j, k, \alpha, \beta) \otimes S''(i, j, k, \alpha, \beta) + \sum_{\substack{i < j < k \in [n], \\ \alpha, \beta \geq 0}} \sigma'''(i, j, k, \alpha, \beta) \otimes S'''(i, j, k, \alpha, \beta) \\
(104) \quad &+ \sum_{\substack{i < j \in [n], \\ k \neq i, j, \\ \alpha \geq 0}} \kappa(k, i, j, \alpha) \otimes Q(k, i, j, \alpha) + \sum_{\substack{i < j \in [n], \\ \alpha \geq 0}} \kappa'(i, j, \alpha) \otimes Q'(i, j, \alpha),
\end{aligned}$$

where $\pi(i, j)$, $\sigma(i, j, k, l, \alpha, \beta)$, $\sigma''(i, j, k, \alpha, \beta)$, $\sigma'''(i, j, k, \alpha, \beta)$, $\kappa(k, i, j, \alpha)$, and $\kappa'(i, j, \alpha)$ are given by (95), (96), (97), (98), (99), (100).

As the second factors in (104) form a basis of Σ , one gets (94). \square

7.6. An isomorphism $\mathfrak{t}_{1,n}^{\mathbb{C}} \simeq \mathfrak{G}$. Recall that $\mathfrak{G} = \mathcal{L}((\Omega^1)^*)/(\mathbf{R})$, where $\mathbf{R} = \mathbf{R}_{pp} + \mathbf{R}_{cp} + \mathbf{R}_{pp}$, and \mathbf{R}_{pp} , \mathbf{R}_{cp} , \mathbf{R}_{pp} are given by (92), (93), (94).

Recall that $\mathfrak{t}_{1,n}$ is the Lie algebra with generators x_i, y_i ($i \in [n]$), t_{ij} ($i \neq j \in [n]$), and relations (1), (2), (3), (4), (5).

Lemma 7.4. *There is a unique morphism of Lie algebras $\mathfrak{t}_{1,n} \rightarrow \mathfrak{G}$, given by*

$$\begin{aligned}
(105) \quad &x_i \mapsto X_i \quad (i \in [n]), \quad y_i \mapsto Y_i \quad (i \in [n]), \quad t_{ij} \mapsto -T_{ij}^0 \quad (i < j \in [n]), \quad t_{ij} \mapsto -T_{ji}^0 \quad (i > j \in [n]).
\end{aligned}$$

Proof. There is a unique morphism from the free Lie algebra with generators x_i, y_i ($i \in [n]$), t_{ij} ($i \neq j \in [n]$) to \mathfrak{G} , defined by (105). By relation (92) of \mathfrak{G} , this morphism takes the first part of relation (1) of $\mathfrak{t}_{1,n}$ to 0. By the first line of relation (94) of \mathfrak{G} , it takes the second part of relation (1) of $\mathfrak{t}_{1,n}$ to 0. By the second line of relation (93) of \mathfrak{G} , it satisfies the first part of relation (2) of $\mathfrak{t}_{1,n}$. The second part of relation (2) of $\mathfrak{t}_{1,n}$ is satisfied by construction. By the first line of relation (93) of \mathfrak{G} , it satisfies relation (3) of $\mathfrak{t}_{1,n}$. By the third line of relation (93) of \mathfrak{G} for $\alpha = 0$, it satisfies the first part of relation (4) of \mathfrak{G} . By the fifth line of relation (94) of \mathfrak{G} for $\alpha = 0$, it satisfies the second part of relation (4) of \mathfrak{G} . By the sum of the two last lines of relation (93) of \mathfrak{G} for $\alpha = 0$, it satisfies the first part of relation (5) of \mathfrak{G} . By the last line of relation (94) of \mathfrak{G} for $\alpha = 0$, it satisfies the second part of relation (5) of \mathfrak{G} .

All this implies that the above morphism factors through a morphism $\mathfrak{t}_{1,n} \rightarrow \mathfrak{G}$. \square

Lemma 7.5. *There is a unique Lie algebra morphism $\mathfrak{G} \rightarrow \mathfrak{t}_{1,n}$, such that*

$$(106) \quad X_i \mapsto x_i \quad (i \in [n]), \quad Y_i \mapsto y_i \quad (i \in [n]), \quad T_{ij}^\alpha \mapsto -(\text{ad} x_i)^\alpha(t_{ij}) \quad (i < j \in [n], \alpha \geq 0).$$

Proof. There is a unique morphism from the free Lie algebra with generators X_i, Y_i ($i \in [n]$), T_{ij}^α ($i < j \in [n], \alpha \geq 0$) to $\mathfrak{t}_{1,n}$, defined by (106). By the first part of relation (1) of $\mathfrak{t}_{1,n}$, this morphism takes relation (92) of \mathfrak{G} to 0. By relation (3) of $\mathfrak{t}_{1,n}$, this morphism takes the first line of relation (93) of \mathfrak{G} to 0. By relation (2) of $\mathfrak{t}_{1,n}$, this morphism takes the second line of relation (93) of \mathfrak{G} to 0. By combining relations $[x_k, x_i] = 0$ and $[x_k, t_{ij}] = 0$ ($i < j \in [n], k \in [n] - \{i, j\}$) of $\mathfrak{t}_{1,n}$, we see that this morphism takes the third line of relation (93) of \mathfrak{G} to 0. By the definitions of the images of X_i and T_{ij}^α , we see that this morphism takes the fourth line of relation (93) of \mathfrak{G} to 0. Combining the definitions of the images of X_j and T_{ij}^α with the relations $[x_i, x_j] = 0$ and $[x_i + x_j, t_{ij}] = 0$ of $\mathfrak{t}_{1,n}$, we see that this morphism takes the last line

of relation (93) of \mathfrak{G} to 0. By the second relations of (1) of $\mathfrak{t}_{1,n}$, this morphism takes the first line of relation (94) of \mathfrak{G} to 0.

Let $i, j, k, l, \alpha, \beta$ be as in the second line of relation (94) of \mathfrak{G} . It follows from relations $[x_i, t_{kl}] = [y_j, t_{kl}] = 0$ and $[x_i, y_j] = t_{ij}$ in $\mathfrak{t}_{1,n}$ that $[t_{ij}, t_{kl}] = 0$ holds in $\mathfrak{t}_{1,n}$. Moreover, relations $[x_i, t_{kl}] = 0$, $[x_k, t_{ij}] = 0$ and $[x_i, x_k] = 0$ imply that relation $[(\text{ad} x_i)^\alpha(t_{ij}), (\text{ad} x_k)^\beta(t_{kl})] = 0$ holds in $\mathfrak{t}_{1,n}$. This implies that the morphism takes the second line of relation (94) of \mathfrak{G} to 0.

Let i, j, k, α, β be as in the third line of relation (94) of \mathfrak{G} . The following equalities hold in $\mathfrak{t}_{1,n}$:

$$\begin{aligned}
& [(\text{ad} x_i)^\alpha(t_{ij}), (\text{ad} x_j)^\beta(t_{jk})] \\
&= [(\text{ad} x_i)^\alpha(t_{ij}), (-\text{ad} x_k)^\beta(t_{jk})] \text{ (by } [x_j + x_k, t_{jk}] = 0 \text{ and } [x_j, x_k] = 0) \\
&= (\text{ad} x_i)^\alpha(-\text{ad} x_k)^\beta([t_{ij}, t_{jk}]) \text{ (by } [x_i, t_{jk}] = [x_k, t_{ij}] = 0) \\
&= -(\text{ad} x_i)^\alpha(-\text{ad} x_k)^\beta([t_{ij}, t_{ik}]) \text{ (by } [t_{ij}, x_i + x_j] = [t_{ij}, y_k] = 0 \text{ and } [x_i, y_k] = t_{ik}, [x_j, y_k] = t_{jk}, \\
&\text{which imply } [t_{ij} + t_{ik}, t_{jk}] = 0) \\
&= -\sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} [(\text{ad} x_i)^\gamma(t_{ij}), (\text{ad} x_i)^{\alpha-\gamma}(-\text{ad} x_k)^\beta(t_{ik})] \text{ (by } [x_k, t_{ij}] = 0) \\
(107) \quad &= -\sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} [(\text{ad} x_i)^\gamma(t_{ij}), (\text{ad} x_i)^{\alpha+\beta-\gamma}(t_{ik})] \text{ (by } [x_i + x_k, t_{ik}] = 0 \text{ and } [x_i, x_k] = 0).
\end{aligned}$$

It follows that the morphism takes the third line of relation (94) of \mathfrak{G} to 0.

Let i, j, k, α, β be as in the fourth line of relation (94) of \mathfrak{G} . Taking into account that identity (107) holds more generally under the assumption $\#\{i, j, k\} = 3$, exchanging j and k in this identity, and replacing the mute index γ by δ , one gets

$$[(\text{ad} x_i)^\alpha(t_{ik}), (\text{ad} x_k)^\beta(t_{jk})] = -\sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} [(\text{ad} x_i)^\delta(t_{ik}), (\text{ad} x_i)^{\alpha+\beta-\delta}(t_{ij})],$$

which using $[x_j, x_k] = 0$ and $[x_j + x_k, t_{jk}] = 0$ for rewriting the second factor of the first bracket, gives

$$[(\text{ad} x_i)^\alpha(t_{ik}), (\text{ad} x_j)^\beta(t_{jk})] = (-1)^\beta \sum_{\delta=0}^{\alpha} \binom{\alpha}{\delta} [(\text{ad} x_i)^{\alpha+\beta-\delta}(t_{ij}), (\text{ad} x_i)^\delta(t_{ik})].$$

It follows that the morphism takes the fourth line of relation (94) of \mathfrak{G} to 0.

Let i, j, k, α be as in the fifth line of relation (94) of \mathfrak{G} . Then

$$\begin{aligned}
[y_k, (\text{ad} x_i)^\alpha(t_{ij})] &= - \sum_{\substack{\alpha', \alpha'' \geq 0 \\ \alpha' + \alpha'' = \alpha - 1}} (\text{ad} x_i)^{\alpha'}([t_{ik}, (\text{ad} x_i)^{\alpha''}(t_{ij})]) \text{ (by } [y_k, t_{ij}] = 0 \text{ and } [y_k, x_i] = -t_{ik}) \\
&= - \sum_{\substack{\alpha', \alpha'' \geq 0 \\ \alpha' + \alpha'' = \alpha - 1}} (\text{ad} x_i)^{\alpha'}([t_{ik}, (-\text{ad} x_j)^{\alpha''}(t_{ij})]) \text{ (by } [x_i, x_j] = [x_i + x_j, t_{ij}] = 0) \\
&= - \sum_{\substack{\alpha', \alpha'', \alpha''' \geq 0 \\ \alpha' + \alpha'' + \alpha''' = \alpha - 1}} \binom{\alpha' + \alpha''}{\alpha'} [(\text{ad} x_i)^{\alpha'}(t_{ik}), (\text{ad} x_i)^{\alpha''}(-\text{ad} x_j)^{\alpha'''}(t_{ij})] \\
&= - \sum_{\substack{\alpha', \alpha'', \alpha''' \geq 0 \\ \alpha' + \alpha'' + \alpha''' = \alpha - 1}} \binom{\alpha' + \alpha''}{\alpha'} [(\text{ad} x_i)^{\alpha'}(t_{ik}), (\text{ad} x_i)^{\alpha'' + \alpha'''}(t_{ij})] \text{ (by } [x_i, x_j] = [x_i + x_j, t_{ij}] = 0) \\
&= - \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha - 1}} \left(\sum_{\alpha''=0}^{\delta} \binom{\gamma + \alpha''}{\gamma} \right) [(\text{ad} x_i)^\gamma(t_{ik}), (\text{ad} x_i)^\delta(t_{ij})] \text{ (replacing } \alpha', \alpha' + \alpha'' \text{ by } \gamma, \delta) \\
(108) \quad &= - \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha - 1}} \binom{\alpha}{\delta} [(\text{ad} x_i)^\gamma(t_{ik}), (\text{ad} x_i)^\delta(t_{ij})] \text{ (using } \sum_{\alpha''=0}^{\delta} \binom{\gamma + \alpha''}{\gamma} = \binom{\gamma + \delta + 1}{\gamma + 1} = \binom{\alpha}{\delta})
\end{aligned}$$

It follows that the morphism takes the fifth line of relation (94) of \mathfrak{G} to 0 when $i < k < j$. Exchanging the roles of γ and δ , equality (108) can be rewritten as follows

$$[y_k, (\text{ad} x_i)^\alpha(t_{ij})] = \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha - 1}} \binom{\alpha}{\gamma} [(\text{ad} x_i)^\gamma(t_{ij}), (\text{ad} x_i)^\delta(t_{ik})],$$

which implies that the morphism takes the fifth line of relation (94) of \mathfrak{G} to 0 when $k > j$.

One also has

$$\begin{aligned}
[y_k, (\text{ad} x_i)^\alpha(t_{ij})] &= - \sum_{\substack{\alpha', \alpha'' \geq 0 \\ \alpha' + \alpha'' = \alpha - 1}} (\text{ad} x_i)^{\alpha'}([t_{ik}, (-\text{ad} x_j)^{\alpha''}(t_{ij})]) \text{ (by the beginning of (108))} \\
&= - \sum_{\substack{\alpha', \alpha'' \geq 0 \\ \alpha' + \alpha'' = \alpha - 1}} (\text{ad} x_i)^{\alpha'}(-\text{ad} x_j)^{\alpha''}([t_{ik}, t_{ij}]) \text{ (using } [x_j, t_{ik}] = 0) \\
&= \sum_{\substack{\alpha', \alpha'' \geq 0 \\ \alpha' + \alpha'' = \alpha - 1}} (\text{ad} x_i)^{\alpha'}(-\text{ad} x_j)^{\alpha''}([t_{ki}, t_{kj}]) \text{ (using } t_{ik} = t_{ki} \text{ and } [t_{ik}, t_{ij} + t_{jk}] = 0) \\
&= \sum_{\substack{\alpha', \alpha'' \geq 0 \\ \alpha' + \alpha'' = \alpha - 1}} [(\text{ad} x_i)^{\alpha'}(t_{ki}), (-\text{ad} x_j)^{\alpha''}(t_{kj})] \text{ (using } [x_i, t_{kj}] = [x_j, t_{ki}] = 0) \\
&= \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha - 1}} [(-\text{ad} x_k)^\gamma(t_{ki}), (\text{ad} x_k)^\delta(t_{kj})] \text{ (replacing } \alpha', \alpha'' \text{ by } \gamma, \delta \text{ and using } [x_i, x_k] = [x_i + x_k, t_{ik}] = 0),
\end{aligned}$$

which implies that the morphism takes the fifth line of relation (94) of \mathfrak{G} to 0 when $k < j$. All this implies that the morphism takes the fifth line of relation (94) of \mathfrak{G} to 0 in all cases.

Let i, j, α be as in the last line of relation (94) of \mathfrak{G} . One has

$$\begin{aligned}
[y_i + y_j, (\text{ad} x_i)^\alpha(t_{ij})] &= \sum_{\alpha' + \alpha'' = \alpha - 1} (\text{ad} x_i)^{\alpha'} \left(\left[- \sum_{k \neq i, j} t_{ik}, (\text{ad} x_i)^{\alpha'' - 1}(t_{ij}) \right] \right) \\
(109) \quad &= \sum_{k \neq i, j} \sum_{\alpha' + \alpha'' = \alpha - 1} -(\text{ad} x_i)^{\alpha'} (-\text{ad} x_j)^{\alpha''}([t_{ik}, t_{ij}]) = \sum_{k \neq i, j} \text{Term}_k,
\end{aligned}$$

where the second equality relies on $[y_i + y_j, x_i] = -\sum_{k \neq i, j} t_{ik}$ and where Term_k is the summand corresponding to k in the last expression. One has

$$\begin{aligned}
\text{Term}_k &= \sum_{\alpha' + \alpha'' = \alpha - 1} (\text{ad} x_i)^{\alpha'} (-\text{ad} x_j)^{\alpha''}([t_{ki}, t_{kj}]) = \sum_{\gamma + \delta = \alpha - 1} [(\text{ad} x_i)^\gamma(t_{ki}), (-\text{ad} x_j)^\delta(t_{kj})] \\
(110) \quad &= \sum_{\gamma + \delta = \alpha - 1} (-1)^\gamma [(\text{ad} x_k)^\gamma(t_{ki}), (\text{ad} x_k)^\delta(t_{kj})],
\end{aligned}$$

where the first equality relies on $[t_{ki}, t_{ij} + t_{kj}] = 0$ and the second equality is obtained by replacing α', α'' by γ, δ and by using $[x_j, t_{ki}] = [x_i, t_{kj}] = 0$.

One also has

$$\begin{aligned}
\text{Term}_k &= \sum_{\substack{\alpha', \alpha'', \alpha''' \geq 0 \\ \alpha' + \alpha'' + \alpha''' = \alpha - 1}} - \binom{\alpha' + \alpha''}{\alpha'} [(\text{ad} x_i)^{\alpha'}(t_{ik}), (\text{ad} x_i)^{\alpha''}(-\text{ad} x_j)^{\alpha'''}(t_{ij})] \text{ (using } [x_j, t_{ik}] = 0) \\
&= \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha - 1}} \sum_{\alpha'' = 0}^{\delta} - \binom{\gamma + \alpha''}{\gamma} [(\text{ad} x_i)^\gamma(t_{ik}), (\text{ad} x_i)^\delta(t_{ij})] \text{ (renaming } \alpha', \alpha'' + \alpha''' \text{ as } \gamma, \delta) \\
(111) \quad &= \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha - 1}} - \binom{\alpha}{\delta} [(\text{ad} x_i)^\gamma(t_{ik}), (\text{ad} x_i)^\delta(t_{ij})] \text{ (using } \sum_{\alpha'' = 0}^{\delta} \binom{\gamma + \alpha''}{\gamma} = \binom{\gamma + \delta + 1}{\gamma + 1} = \binom{\alpha}{\delta}),
\end{aligned}$$

which implies

$$(112) \quad \text{Term}_k = \sum_{\substack{\gamma, \delta \geq 0 \\ \gamma + \delta = \alpha - 1}} \binom{\alpha}{\gamma} [(\text{ad} x_i)^\gamma(t_{ij}), (\text{ad} x_i)^\delta(t_{ik})].$$

Substituting in (109) the identities (110) when $k < i$, (111) when $i < k < j$ and (112) when $k > j$, one sees that the morphism takes the last line of line of relation (94) of \mathfrak{G} to 0. All this proves Lemma 7.5. \square

Combining Lemmas 7.4 and 7.5, one obtains:

Proposition 7.1. *Formula (106) gives rise to an isomorphism of Lie algebras $\mathfrak{G} \rightarrow \mathfrak{t}_{1,n}^{\mathbb{C}}$.*

8. ELEMENTS OF A DESCRIPTION OF $\text{VBFC}(X, D)_{\text{unip}}$ (EQUIV. (e))

8.1. Reduction of a space of forms to Σ_{\log} . Let \mathfrak{g} be a finite dimensional nilpotent Lie algebra over \mathbb{C} , G the corresponding group. Let X be a smooth complex algebraic variety with a divisor D .

Recall also that a 1-form α on $X - D$ is called logarithmic at D if both α and $d\alpha$ have simple poles at D (see §1.2.3). This is equivalent to saying that if D is locally defined by the equation $z = 0$ near its generic point then $\alpha = f \frac{dz}{z} + \beta$, where f is a regular function and β a regular 1-form.

Let $\omega \in \Omega^1(X - D, \mathfrak{g})$ be a 1-form satisfying the Maurer-Cartan equation

$$d\omega + \frac{1}{2}\omega^2 = 0$$

where ω^2 is defined using (102). In other words, $d + \omega$ is a flat connection on the trivial G -bundle on $X - D$.

Lemma 8.1. *If ω has a first order pole at D then it is logarithmic.*

Proof. Let z, x_1, \dots, x_n be local coordinates such that D is locally defined by $z = 0$ near a smooth point. Let $\omega = f \frac{dz}{z} + \sum_i \frac{g_i}{z} dx_i$, where f, g_i are regular functions. Our job is to show that g_i vanish at $z = 0$.

Let $\psi = z\omega$. Then $\psi = f dz + \sum_i g_i dx_i$, a regular 1-form. We have $d\psi = dz \wedge \omega + z d\omega$. Thus from the Maurer-Cartan equation we have that

$$z d\psi = dz \wedge \psi - \frac{1}{2}\psi^2.$$

Thus, $dz \wedge \psi - \frac{1}{2}\psi^2$ vanishes coefficientwise at $z = 0$. In particular, for the coefficient of $dz \wedge dx_i$ we get that $g_i - [f, g_i]$ vanishes at $z = 0$. In other words, if g_{i0}, f_0 are restrictions of g_i, f to $z = 0$, then

$$g_{i0} = [f_0, g_{i0}].$$

But $f_0 \in \mathfrak{g}$, which is a nilpotent Lie algebra. Thus, $g_{i0} = 0$, as desired. \square

Now assume that $D = \cup_{j=1}^N D_j$ is a union of smooth irreducible divisors, intersecting pairwise transversally.

Lemma 8.2. *Let $\omega \in \Omega_{\log}^1(X - D)$. Then for any $1 \leq m \leq N$, the residue of ω at D_m (originally defined generically on D_m) extends to a regular function on the whole D_m .*

Proof. For each m , let f_{m0} be the residue of ω at D_m . This is a regular function on $D_m - \cup_{j:j \neq m} D_j$.

Let $p \in D_m$ be a generic point and z, x_1, \dots, x_n be local coordinates near p such that D_m is locally defined by $z = 0$. Using that $\omega \in \Omega_{\log}^1(X - D)$, on a small ball B_p around p one may write ω as $f \frac{dz}{z} + \beta$, where β is regular. Let β_0 be the restriction of β to $B_p \cap D_m$. Then β_0 is a regular 1-form. Moreover, it is easy to check that β_0 is canonically attached to ω , i.e., it does not depend on the choice of coordinates and the representation $\omega = f \frac{dz}{z} + \beta$. Therefore, β_0 is defined globally on $D_m - \cup_{i \neq m} D_i$.

Moreover, the Maurer-Cartan equation for ω implies the Maurer-Cartan equation for β_0 , i.e., $d + \beta_0$ is a flat connection on the trivial bundle over $D_m - \cup_{i \neq m} D_i$. Namely, $d + \beta_0$ is the "restriction" of the flat connection $d + \omega$ to the pole divisor D_m . Concretely, if D_m^ε is a perturbation of D_m (defined locally near some point) then $d + \beta_0$ is the limit of the restriction of $d + \omega$ to D_m^ε as $\varepsilon \rightarrow 0$.

Since the connection $d + \omega$ has first order poles, it follows from the work of Deligne ([Del]) that so does the connection $d + \beta_0$. Indeed, pick $q \in D_m \cap D_i$ for some i , and let us work on a small ball B_q around q . Let C be a generic smooth curve in $D_m \cap B_q$ passing through q . We need to show that the restriction of $d + \beta_0$ to C has a first order pole at q . To this end, consider a generic perturbation C_ε of C in X , still passing through q (i.e., C_ε is not contained in D_m). Then the restriction of $d + \omega$ to C_ε has a pole only at q inside B_q (because of pairwise transversality of D_j). Moreover, this pole is simple, since by [Del], flat sections of $d + \omega|_{C_\varepsilon}$ have logarithmic growth near q (as ω has first order poles). Hence, taking the limit $\varepsilon \rightarrow 0$, we find that $d + \beta_0|_C$ has a simple pole at q , as claimed.

Also, f_{m0} is a flat section of the adjoint bundle for the trivial G -bundle with the connection $d + \beta_0$, i.e.,

$$df_{m0} + [\beta_0, f_{m0}] = 0.$$

Since G is unipotent and β_0 has simple poles, this means that f_{m0} has logarithmic growth when approaching $D_m \cap (\cup_{i|i \neq n} D_i)$. Hence f_{m0} cannot have a pole at $D_m \cap (\cup_{i|i \neq n} D_i)$, i.e., it extends to a regular function on D_m , as desired. \square

Remark 8.3. One can also prove that ω can locally be written as $\sum_j \omega_j$, where ω_j is a logarithmic form with pole only on D_j . Indeed, according to Lemma 8.2, locally near each point $p \in D_j$ we may extend f_{j0} to a regular function f_j on a neighborhood of p . Also near p the divisor D_j may be defined by the equation $z_j = 0$. Then since $\omega \in \Omega_{log}^1(X - D)$, $\omega - \sum_{j:p \in D_j} f_j \frac{dz_j}{z_j}$ is regular near p , which implies the required statement. \square

8.2. Equality $\Omega^1 = \Sigma_{log}$. Recall that Ω^1 is the subspace

$$\sum_{i \in I} \Gamma(X, \Omega_X^1(\log D_i))$$

of $\Gamma_{rat}(X, \Omega_X^1(D))$.

On the other hand, Σ_{log} is the subspace

$$\{\alpha \in \Gamma_{rat}(X, \Omega_X^1(D)) | \alpha \in \Gamma(X, \Omega_X^1(\log D)) \text{ and } \forall i \in I, \text{ res}_{D_i}(\alpha) \in \Gamma(D_i, \mathcal{O}_{D_i})\}$$

of $\Gamma_{rat}(X, \Omega_X^1(D))$.

Lemma 8.4. *One has $\Omega^1 = \Sigma_{log}$.*

Proof. For each $i \in I$, one has the inclusion $\Gamma(X, \Omega_X^1(\log D_i)) \subset \Gamma(X, \Omega_X^1(\log D))$ and the map $\text{res}_{D_i} : \Gamma(X, \Omega_X^1(\log D)) \rightarrow \Gamma(D_i - (\cup_{j|j \neq i} D_j), \mathcal{O}_{D_i})$ maps $\Gamma(X, \Omega_X^1(\log D_j))$ to 0 if $j \neq i$, and to $\Gamma(D_i, \mathcal{O}_{D_i})$ if $j = i$. All this implies that

$$\Omega^1 \subset \Sigma_{log}.$$

It follows from the definition of Σ_{log} that this space fits in an exact sequence

$$0 \rightarrow \Gamma(X, \Omega_X^1) \rightarrow \Sigma_{log} \xrightarrow{\oplus_{i \in I} \text{res}_{D_i}} \oplus_{i \in I} \Gamma(D_i, \mathcal{O}_{D_i}).$$

In §6.2.2, it is proved that for each $i \in I$, the map $\text{res}_{D_i} : \Gamma(X, \Omega_X^1(\log D_i)) \rightarrow \Gamma(D_i, \mathcal{O}_{D_i})$ is surjective (Lemma 6.6). Together with the fact that the restriction of res_{D_i} to $\Gamma(D_j, \mathcal{O}_{D_j})$ is 0 if $i \neq j$, this implies that the composed map

$$\Omega^1 \hookrightarrow \Sigma_{log} \xrightarrow{\oplus_{i \in I} \text{res}_{D_i}} \oplus_{i \in I} \Gamma(D_i, \mathcal{O}_{D_i})$$

is surjective.

On the other hand, we also have an inclusion $\Gamma(X, \Omega_X^1) \subset \Omega^1$. We have therefore a commutative diagram

$$\begin{array}{ccc} & \Sigma_{log} & \\ \nearrow & & \searrow^{\oplus_{i \in I} \text{res}_{D_i}} \\ \Gamma(X, \Omega_X^1) & & \oplus_{i \in I} \Gamma(D_i, \mathcal{O}_{D_i}) \\ \searrow & \downarrow & \nearrow_a \\ & \Omega^1 & \end{array}$$

where the map a is surjective. It follows that the map $\oplus_{i \in I} \text{res}_{D_i}$ is surjective as well and that the map $\Omega^1 \hookrightarrow \Sigma_{log}$ is an isomorphism. This proves the result. \square

9. RELATION WITH THE UNIVERSAL KZB CONNECTION

Let us fix $\tau \in \mathfrak{H}$. In §3.8, we attach to τ a principal bundle with flat connection over \mathbb{C}^n

$$(\text{trivial } \exp(\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}})\text{-bundle}, d + A_{\text{KZB}}),$$

a principal bundle with flat connection over $(E_{\tau}^{\#})^n$,

$$(\text{trivial } \exp(\mathfrak{G})\text{-bundle}, d + \omega),$$

and maps $\mathbb{C}^{2n} \rightarrow \mathbb{C}^n$, $\mathbb{C}^{2n} \rightarrow (E_{\tau}^{\#})^n$.

We will construct an isomorphism between the lifts to \mathbb{C}^{2n} of these two pairs of bundles with flat connection, thereby proving Theorem 3.1.

9.1. A flat connection on $(E_{\tau}^{\#})^n$. Recall from (101) the element $\omega \in \Gamma_{\text{rat}}((E_{\tau}^{\#})^n, \Omega_{(E_{\tau}^{\#})^n}^1) \hat{\otimes} \mathfrak{G}$. It follows from Lemma 3.19 that $d + \omega$ is a flat connection on the trivial bundle over $(E_{\tau}^{\#})^n$ with group $\exp(\mathfrak{G})$. It follows from (101) and (57) that the lift to \mathbb{C}^{2n} of ω is

$$\tilde{\omega} = \sum_{i \in [n]} X_i \cdot dc_i + \sum_{i \in [n]} Y_i \cdot dp_i + \sum_{i < j \in [n], \alpha \geq 0} T_{ij}^{\alpha} \left[\left(\frac{\theta(p_{ij} + z|\tau)}{\theta(z|\tau)\theta(p_{ij}|\tau)} - \frac{1}{z} \right) e^{-c_{ij}z} dp_{ij} | z^{\alpha} \right]$$

where $(p_1, c_1, \dots, p_n, c_n)$ are the standard coordinates on \mathbb{C}^n .

9.2. The universal KZB system. The corresponding configuration space for E_{τ} is

$$C(E_{\tau}, n) := \{(p_1, \dots, p_n) \in E_{\tau} | p_i \neq p_j \text{ for } i < j \in [n]\}.$$

In [CEE], we defined a pair $(\mathcal{P}_{\text{KZB}}, \nabla_{\text{KZB}})$ of a principal $\exp(\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}})$ -bundle \mathcal{P}_{KZB} over $C(E_{\tau}, n)$ and of a flat connection ∇_{KZB} over it. The projection $\mathbb{C} \rightarrow E_{\tau}$ gives rise to a fibered product

$$\tilde{C}(E_{\tau}, n) := C(E_{\tau}, n) \times_{(E_{\tau})^n} \mathbb{C}^n.$$

Then

$$\tilde{C}(E_{\tau}, n) = \{(p_1, \dots, p_n) \in \mathbb{C}^n | p_i - p_j \notin \mathbb{Z} + \tau\mathbb{Z} \text{ for } i < j \in [n]\}.$$

There is a natural projection $\tilde{C}(E_{\tau}, n) \xrightarrow{p} C(E_{\tau}, n)$ with covering group \mathbb{Z}^{2n} . There is a natural isomorphism of $p^*\mathcal{P}_{\text{KZB}}$ with the trivial principal bundle over $C(E_{\tau}, n)$ with group $\exp(\hat{\mathfrak{t}}_{1,n}^{\mathbb{C}})$. The pull-back $p^*\nabla_{\text{KZB}}$ of the KZB connection over $\tilde{C}(E_{\tau}, n)$ is then the operator

$$d + A_{\text{KZB}},$$

where $A_{\text{KZB}} \in \Gamma(\tilde{C}(E_{\tau}, n), \Omega_{\tilde{C}(E_{\tau}, n)}^1 \otimes \hat{\mathfrak{t}}_{1,n}^{\mathbb{C}})$ is given by

$$A_{\text{KZB}} := - \sum_{i \in [n]} \left(-y_i + \sum_{j | j \in [n], j \neq i} \left(\frac{\theta(p_{ij} + \text{adx}_i|\tau)\text{adx}_i}{\theta(p_{ij}|\tau)\theta(\text{adx}_i|\tau)} - 1 \right) (y_j) \right) dp_i,$$

where $p_{ij} := p_i - p_j$; the expression under the second sum sign should be computed as $\sum_{\alpha \geq 0} F_{\alpha}(p_{ij})(\text{adx}_i)^{\alpha}(y_j)dp_i$, where the function $(p, x) \mapsto \frac{\theta(p+x|\tau)x}{\theta(p|\tau)\theta(x|\tau)} - 1$ is viewed as formal in x and meromorphic in p , expanding as $\sum_{\alpha \geq 0} F_{\alpha}(p)x^{\alpha}$. Note also that the function $F_0(p)$ in this expansion is 0. Therefore

$$A_{\text{KZB}} = - \sum_{i \in [n]} \left(-y_i + \sum_{j | j \in [n], j \neq i} \left(\frac{\theta(p_{ij} + \text{adx}_i|\tau)}{\theta(p_{ij}|\tau)\theta(\text{adx}_i|\tau)} - \frac{1}{\text{adx}_i} \right) (t_{ij}) \right) dp_i,$$

with the same conventions as above, based on the fact that $(p, x) \mapsto \frac{\theta(p+x|\tau)}{\theta(p|\tau)\theta(x|\tau)} - \frac{1}{x} =: g(p, x)$ may be viewed as formal in x and meromorphic in p . As $g(p, x) = g(-p, -x)$, one has

$$\text{if } i \neq j \in [n], \text{ then } \left(\frac{\theta(p_{ij} + \text{adx}_i|\tau)}{\theta(p_{ij}|\tau)\theta(\text{adx}_i|\tau)} - \frac{1}{\text{adx}_i} \right) (t_{ij}) + \left(\frac{\theta(p_{ji} + \text{adx}_j|\tau)}{\theta(p_{ji}|\tau)\theta(\text{adx}_j|\tau)} - \frac{1}{\text{adx}_j} \right) (t_{ij}) = 0,$$

so

$$A_{\text{KZB}} := \sum_{i \in [n]} y_i dp_i - \frac{1}{2} \sum_{i \neq j \in [n]} \left(\frac{\theta(p_{ij} + \text{ad} x_i | \tau)}{\theta(p_{ij} | \tau) \theta(\text{ad} x_i | \tau)} - \frac{1}{\text{ad} x_i} \right) (t_{ij}) \cdot dp_{ij}.$$

9.3. Relation between the two systems. The image of $\tilde{\omega}$ under the isomorphism $\mathfrak{t}_{1,n} \simeq \mathfrak{G}$ is

$$\text{im}(\tilde{\omega}) = \sum_{i \in [n]} x_i \cdot dc_i + \sum_{i \in [n]} y_i \cdot dp_i - \sum_{i < j \in [n], \alpha \geq 0} e^{-c_{ij} \text{ad} x_i} \left(\frac{\theta(p_{ij} + \text{ad} x_i | \tau)}{\theta(\text{ad} x_i | \tau) \theta(p_{ij} | \tau)} - \frac{1}{\text{ad} x_i} \right) (t_{ij}) dp_{ij}$$

The expression $(p_1, \dots, c_n) \mapsto e^{\sum_{i \in [n]} c_i x_i}$ defines a holomorphic map $\mathbb{C}^{2n} \rightarrow \exp(\mathfrak{G})$. One may therefore conjugate $d + \text{im}(\tilde{\omega})$ by this map. One has

$$e^{\sum_{i \in [n]} c_i x_i} d e^{-\sum_{i \in [n]} c_i x_i} = d - \sum_{i \in [n]} x_i \cdot dc_i,$$

and

$$e^{\sum_{i \in [n]} c_i x_i} t_{ij} e^{-\sum_{i \in [n]} c_i x_i} = e^{c_{ij} \text{ad} x_i} (t_{ij}),$$

moreover $e^{\sum_{i \in [n]} c_i x_i}$ commutes with all the x_k , $k \in [n]$, so

$$\begin{aligned} e^{\sum_{i \in [n]} c_i x_i} (d + \text{im}(\tilde{\omega})) e^{-\sum_{i \in [n]} c_i x_i} &= d + \sum_{i \in [n]} x_i \cdot dc_i - \sum_{i < j \in [n], \alpha \geq 0} \left(\frac{\theta(p_{ij} + \text{ad} x_i | \tau)}{\theta(\text{ad} x_i | \tau) \theta(p_{ij} | \tau)} - \frac{1}{\text{ad} x_i} \right) (t_{ij}) \\ &= d + A_{\text{KZB}}. \end{aligned}$$

We have proved:

Theorem 9.1. *The map $\mathbb{C}^{2n} \rightarrow \exp(\mathfrak{G})$, $(p_1, \dots, c_n) \mapsto e^{\sum_{i \in [n]} c_i x_i}$ sets up an isomorphism between the following principal bundles with flat connections over \mathbb{C}^{2n} :*

- the pull-back under $\mathbb{C}^{2n} \rightarrow \mathbb{C}^n \rightarrow (E_\tau)^n$ of $(\mathcal{P}_{\text{KZB}}, \nabla_{\text{KZB}})$;
- the pull-back under $\mathbb{C}^{2n} \rightarrow (E_\tau^\#)^n$ of $((E_\tau^\#)^n \times \exp(\mathfrak{G}), d + \omega)$ given by Lemma 3.19.

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