

MIT Open Access Articles

*Representations of affine superalgebras
and mock theta functions II*

The MIT Faculty has made this article openly available. **Please share**
how this access benefits you. Your story matters.

Citation: Kac, Victor G. and Minoru Wakimoto. "Representations of Affine Superalgebras and Mock Theta Functions II." *Advances in Mathematics* 300 (September 2016): 17–70 © 2016 Elsevier Inc

As Published: <http://dx.doi.org/10.1016/J.AIM.2016.03.015>

Publisher: Elsevier BV

Persistent URL: <http://hdl.handle.net/1721.1/118408>

Version: Original manuscript: author's manuscript prior to formal peer review

Terms of use: Creative Commons Attribution-NonCommercial-NoDerivs License



Representations of affine superalgebras and mock theta functions II

Victor G. Kac* and Minoru Wakimoto†

To the memory of Andrei Zelevinsky.

Abstract

We show that the normalized supercharacters of principal admissible modules, associated to each integrable atypical module over the affine Lie superalgebra $\widehat{sl}_{2|1}$ can be modified, using Zwegers' real analytic corrections, to form an $SL_2(\mathbb{Z})$ -invariant family of functions. Using a variation of Zwegers' correction, we obtain a similar result for $\widehat{osp}_{3|2}$. Applying the quantum Hamiltonian reduction, this leads to new families of positive energy modules over the $N = 2$ (resp. $N = 3$) superconformal algebras with central charge $c = 3(1 - \frac{2m+2}{M})$, where $m \in \mathbb{Z}_{\geq 0}, M \in \mathbb{Z}_{\geq 2}, \gcd(2m+2, M) = 1$ if $m > 0$ (resp. $c = -3\frac{2m+1}{M}$, where $m \in \mathbb{Z}_{\geq 0}, M \in \mathbb{Z}_{\geq 2}, \gcd(4m+2, M) = 1$), whose modified supercharacters form an $SL_2(\mathbb{Z})$ -invariant family of functions.

0 Introduction

This paper is the second in the series of our papers on modular invariance of modified normalized characters of irreducible highest weight representations $L(\Lambda)$ over an affine Lie superalgebra $\widehat{\mathfrak{g}}$, associated to a simple finite-dimensional Lie superalgebra \mathfrak{g} . We shall keep the notation and conventions of the first paper [KW].

We assume that \mathfrak{g} is endowed with a non-degenerate invariant supersymmetric bilinear form $(\cdot|\cdot)$ and that its even part \mathfrak{g}_0 is a reductive subalgebra. (These properties hold if the Killing form κ on \mathfrak{g} is non-degenerate.) The associated affine Lie superalgebra is $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$, where K is a central element, $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ is a central extension of the loop algebra $\mathfrak{g}[t, t^{-1}]$:

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m, -n}(a|b)K, \quad a, b \in \mathfrak{g}, \quad m, n \in \mathbb{Z},$$

and $d = t\frac{d}{dt}$.

Choosing a Cartan subalgebra \mathfrak{h} of \mathfrak{g}_0 , one defines the Cartan subalgebra of $\widehat{\mathfrak{g}}$:

*Department of Mathematics, M.I.T, Cambridge, MA 02139, USA. kac@math.mit.edu

† wakimoto@r6.dion.ne.jp Supported in part by Department of Mathematics, M.I.T.

$$\widehat{\mathfrak{h}} = \mathbb{C}d + \mathfrak{h} + \mathbb{C}K.$$

The restriction of the bilinear form $(\cdot|\cdot)$ to \mathfrak{h} is symmetric non-degenerate, and one extends it from \mathfrak{h} to $\widehat{\mathfrak{h}}$, letting

$$(\mathfrak{h}|\mathbb{C}K + \mathbb{C}d) = 0, \quad (K|K) = (d|d) = 0, \quad (K|d) = 1.$$

One identifies $\widehat{\mathfrak{h}}^*$ with $\widehat{\mathfrak{h}}$ via this bilinear form. Traditionally, the elements of $\widehat{\mathfrak{h}}^*$ corresponding to K and d are denoted by δ and Λ_0 , respectively. One uses the following coordinates on $\widehat{\mathfrak{h}}$:

$$(0.1) \quad h = 2\pi i(-\tau d + z + tK) =: (\tau, z, t), \quad \text{where } \tau, t \in \mathbb{C}, z \in \mathfrak{h}.$$

Given a set of simple roots $\widehat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$ of $\widehat{\mathfrak{g}}$ and $\Lambda \in \widehat{\mathfrak{h}}^*$, one defines the *highest weight module* $L(\Lambda)$ over $\widehat{\mathfrak{g}}$ as the irreducible module, which admits a non-zero vector v_Λ , such that

$$hv_\Lambda = \Lambda(h)v_\Lambda \text{ for } h \in \widehat{\mathfrak{h}} \text{ and } \widehat{\mathfrak{g}}_{\alpha_i}v_\Lambda = 0 \text{ for } i = 0, \dots, \ell,$$

where $\widehat{\mathfrak{g}}_{\alpha_i}$ denotes the root subspace of $\widehat{\mathfrak{g}}$, attached to the simple root α_i . Since K is a central element of $\widehat{\mathfrak{g}}$, it is represented by a scalar $\Lambda(K)$, called the *level* of $L(\Lambda)$ (or Λ).

The *character* ch^+ and the *supercharacter* ch^- of $L(\Lambda)$ are defined as the following series, corresponding to the weight space decomposition of $L(\Lambda)$ with respect to $\widehat{\mathfrak{h}}$, cf. (0.1):

$$\text{ch}_{L(\Lambda)}^\pm(\tau, z, t) = \text{tr}_{L(\Lambda)}^\pm e^{2\pi i(-\tau d + z + tK)},$$

where tr^+ (resp. tr^-) denotes the trace (resp. supertrace). It is easy to see (as in [K], Chapter 10) that these series converge absolutely in the domain $\{h \in \widehat{\mathfrak{h}} \mid \text{Re } \alpha_i(h) > 0, i = 0, 1, \dots, \ell\}$ to holomorphic functions. In all examples these functions extend to meromorphic functions in the domain

$$(0.2) \quad X = \left\{ h \in \widehat{\mathfrak{h}} \mid \text{Re}(K|h) > 0 \right\} = \{(\tau, z, t) \mid \text{Im } \tau > 0\}.$$

Note that, as a $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ -module, $L(\Lambda)$ remains irreducible and it is unchanged if we replace Λ by $\Lambda + a\delta$, $a \in \mathbb{C}$, and the character of the $\widehat{\mathfrak{g}}$ -module gets multiplied by q^a . Here and further $q = e^{2\pi i\tau} = e^{-\delta}$.

In the case when \mathfrak{g} is a simple Lie algebra, there exists an important collection of integrable and, more generally, admissible $\widehat{\mathfrak{g}}$ -modules $L(\Lambda)$, whose normalized characters have modular invariance property [K], [KW1]. Recall that the *normalized (super)character* ch_Λ^\pm is defined as

$$\text{ch}_\Lambda^\pm(\tau, z, t) = q^{m_\Lambda} \text{ch}_{L(\Lambda)}^\pm(\tau, z, t),$$

where $m_\Lambda \in \mathbb{Q}$ is the "modular anomaly" (cf. formula (2.3)). Recall that $\text{ch}_{\Lambda+aK}^\pm = \text{ch}_\Lambda^\pm$, $a \in \mathbb{C}$.

Recall the action of $SL_2(\mathbb{R})$ in the domain X in coordinates (0.1):

$$(0.3) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z, t) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t - \frac{c(z|z)}{2(c\tau + d)} \right).$$

By definition, modular invariance of the normalized (super)character of the $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ means that $L(\Lambda)$ is a member of a finite collection of irreducible highest weight $\widehat{\mathfrak{g}}$ -modules, such that the \mathbb{C} -span of their (super)characters is $SL_2(\mathbb{Z})$ -invariant.

If the Killing form κ on \mathfrak{g} is non-degenerate (only such \mathfrak{g} are considered in the present paper), a $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ (and the highest weight Λ) is called (partially) *integrable* if for any root α of $\widehat{\mathfrak{g}}$, such that $\kappa(\alpha, \alpha) > 0$, the elements from the root space $\widehat{\mathfrak{g}}_\alpha$ act locally nilpotently on $L(\Lambda)$. (If \mathfrak{g} is a Lie algebra, this property is equivalent to integrability, as defined in [K].)

The conjectural Kac–Wakimoto (super)character formula for a tame $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ (see [KW], Definition 3.5(c)) reads:

$$(0.4) \quad \widehat{R}^\pm \text{ch}_{L(\Lambda)}^\pm = \sum_{w \in \widehat{W}^\#} \varepsilon_\pm(w) w \frac{e^{\Lambda + \widehat{\rho}}}{\prod_{\beta \in T_{\bar{\Lambda}}} (1 \pm e^{-\beta})}.$$

Here and further on $\bar{\Lambda}$ denotes the restriction of Λ to \mathfrak{h} ; \widehat{R}^\pm is the affine (super)denominator:

$$\widehat{R}^\pm = e^{\widehat{\rho}} \frac{\prod_{\alpha \in \widehat{\Delta}_{\bar{0},+}} (1 - e^{-\alpha})}{\prod_{\alpha \in \widehat{\Delta}_{\bar{1},+}} (1 \pm e^{-\alpha})},$$

$\widehat{\rho}$ is the affine Weyl vector defined by $2(\widehat{\rho}|\alpha_i) = (\alpha_i|\alpha_i)$, $i = 0, 1, \dots, \ell$; $\widehat{\Delta}_{\bar{0},+}$ and $\widehat{\Delta}_{\bar{1},+}$ are the sets of positive even and odd roots of $\widehat{\mathfrak{g}}$ (counting multiplicities); $T_{\bar{\Lambda}} \subset \widehat{\Pi}$ is a subset of the set of positive roots of \mathfrak{g} , consisting of pairwise orthogonal isotropic roots, orthogonal to $\bar{\Lambda}$, and maximal with this property in this set; $\widehat{W} = W \ltimes t_L$ is the affine Weyl group, where W is the Weyl group of $\mathfrak{g}_{\bar{0}}$, and the subgroup t_L consists of *translations* t_γ , $\gamma \in L$, where L is the coroot lattice of $\mathfrak{g}_{\bar{0}}$, which are defined by

$$(0.5) \quad t_\gamma(\lambda) = \lambda + \lambda(K)\gamma - ((\lambda|\gamma) + \frac{1}{2}\lambda(K)(\gamma|\gamma))\delta, \quad \lambda \in \widehat{\mathfrak{h}}^*;$$

$\widehat{W}^\# = W^\# \ltimes t_{L^\#}$ (resp. $W^\#$) is the subgroup of the affine (resp. finite) Weyl group, generated by reflections in $\alpha \in \widehat{\Delta}_+$ (resp. $\alpha \in \Delta_+$) with $\kappa(\alpha, \alpha) > 0$, where $L^\#$ is the sublattice of L , spanned by the coroots α with $\kappa(\alpha, \alpha) > 0$ of $\mathfrak{g}_{\bar{0}}$; finally, $\varepsilon_\pm(w) = (-1)^{s_\pm(w)}$, for a decomposition of w in a product of s_+ reflections, with respect to non-isotropic even roots, and s_- is the number of those of them, for which the half is not a root (note that $\varepsilon_+(t_\alpha) = 1$). Formula (0.4) is a slightly more precise version of formula (3.14) from [KW].

In all cases studied in the present paper, formula (0.4) is proven in [GK]. Note that it can be rewritten, after multiplying both sides by a suitable power of q , as

$$(0.6) \quad q^{\frac{\text{sdim} \mathfrak{g}}{24}} \widehat{R}^\pm \text{ch}_\Lambda^\pm = \sum_{w \in W^\#} \varepsilon^\pm(w) w(\Theta_{\Lambda + \widehat{\rho}, T}^\pm),$$

where $T = T_{\bar{\Lambda}}$ and $\Theta_{\Lambda + \widehat{\rho}, T}^\pm$ is a mock theta function of degree $(\Lambda + \widehat{\rho})(K)$.

Recall [KW] that, for $\lambda \in \widehat{\mathfrak{h}}^*$, such that $\lambda(K) > 0$, a *mock theta function* $\Theta_{\lambda, T}^\pm$ of degree $n = \lambda(K)$ is defined by the following series:

$$(0.7) \quad \Theta_{\lambda, T}^\pm = q^{\frac{(\lambda|\lambda)}{2n}} \sum_{\gamma \in M} \varepsilon_\pm(t_\gamma) t_\gamma \frac{e^\lambda}{\prod_{\beta \in T} (1 \pm e^{-\beta})},$$

where $M \subset \mathfrak{h}$ is a positive definite integral lattice ($= L^\#$ in (0.6)), t_γ are the translations, defined by (0.5), and $T \subset \mathfrak{h}$ is a finite subset, consisting of pairwise orthogonal isotropic

vectors, orthogonal to λ . This series converges to a meromorphic function in the domain X , which in coordinates (0.1) takes the form

$$(0.8) \quad \Theta_{\lambda,T}(\tau, z, t) = e^{2\pi i n t} \sum_{\gamma \in \frac{\tilde{\lambda}}{n} + M} \varepsilon^\pm(t_\gamma) \frac{q^{n \frac{(\gamma|\gamma)}{2}} e^{2\pi i n \gamma(z)}}{\prod_{\beta \in T} (1 \pm q^{-(\gamma|\beta)} e^{-2\pi i \beta(z)})}.$$

Of course, if $T = \emptyset$, we get the usual Jacobi form, which is modular invariant, up to a weight factor. The normalized (super)denominator $q^{\frac{\text{sdim} \mathfrak{g}}{24}} \widehat{R}^\pm$ is modular invariant, up to the same weight factor, since it can be expressed as a ratio of products of four standard Jacobi forms ϑ_{ab} ($a, b = 0$ or 1) of degree 2 (see [KW], Section 4).

In particular, if \mathfrak{g} is a simple Lie algebra and $L(\Lambda)$ is an integrable $\widehat{\mathfrak{g}}$ -module, then formula (0.4) turns into the usual Weyl-Kac character formula, where $\widehat{W}^\# = \widehat{W}$, and $T_{\bar{\Lambda}} = \emptyset$ (in this case, of course, $\text{ch}^+ = \text{ch}^-$ and $\varepsilon_+(w) = \varepsilon_-(w) = \det(w)$). Therefore (0.6) holds with the usual Jacobi forms, hence ch_Λ is modular invariant.

However, modular invariance fails for mock theta functions, but sometimes it can be achieved by adding non-holomorphic real analytic corrections, discovered by Zwegers [Z].

In our previous paper [KW] we studied the first non-trivial case of $\mathfrak{g} = \mathfrak{sl}_{2|1}$ and the non-typical (i.e. $T \neq \emptyset$) integrable $\widehat{\mathfrak{g}}$ -module $L(m\Lambda_0)$, where m is a positive integer. Namely, we found a modification of the numerator of $\text{ch}_{m\Lambda_0}^\pm$ (i.e. the RHS of (0.6)) in the spirit of Zwegers, so that the corresponding modified normalized supercharacter $\widetilde{\text{ch}}_{m\Lambda_0}^-$ is modular invariant (see [KW], Theorem 7.3 for $M = 1$, $\varepsilon = \varepsilon' = 0$).

Moreover, since the numerators of the normalized supercharacters of admissible modules and their modifications are expressed by a simple substitution via that for the integrable modules, and their denominators remain the same, we obtain modular invariance for the modified normalized supercharacters of admissible modules, associated to $L(m\Lambda_0)$ ([KW], Theorem 7.3 for $\varepsilon = \varepsilon' = 0$). Finally, it turns out that the modified normalized supercharacters and characters, along with their Ramond twisted analogues, again form a modular invariant family ([KW], Theorem 7.3).

In the present paper we show that similar results hold for arbitrary atypical integrable highest weight $\widehat{\mathfrak{g}}$ -modules and the associated principal admissible modules in the case when \mathfrak{g} is a simple Lie superalgebra of rank $\ell = 2$, i.e. \mathfrak{g} is either $\mathfrak{sl}_{2|1}$ or $\mathfrak{osp}_{3|2}$.

For $\mathfrak{g} = \mathfrak{sl}_{2|1}$ an arbitrary atypical integrable weight is of the form $\Lambda_{m;s} = (m-s)\Lambda_0 + s\alpha$, where $m, s \in \mathbb{Z}_{\geq 0}$, $s \leq m$, and α is an isotropic root of \mathfrak{g} . We show that for each $\Lambda_{m;s}$ the corresponding modified normalized supercharacter $\widetilde{\text{ch}}_{\Lambda_{m;s}}^-$ is modular invariant, and consequently, for each $M \in \mathbb{Z}_{\geq 1}$, such that $\gcd(M, 2m+2) = 1$ if $m > 0$, the associated family of principal admissible modified normalized supercharacters is modular invariant (Theorem 2.9). It turns out more convenient to use slightly changed Zwegers' real analytic functions $R_{j,m+1}(\tau, v)$; they are given by (1.6). The method of the present paper is simpler than that of [KW], but still the key result is that $\Phi^{[m;s]} - \Phi^{[m;s]}|_S$ is a holomorphic function, where $\Phi^{[m;s]}$ is the numerator of $\text{ch}_{\Lambda_{m;s}}^-$. It is in the proof of the latter fact that the restriction $\ell = 2$ is essential.

For $\mathfrak{g} = \mathfrak{osp}_{3|2}(= B(1,1))$ the only atypical integrable weight is $m\Lambda_0$, where $m \in \mathbb{Z}_{\geq 0}$. The method is similar to that for $\mathfrak{g} = \mathfrak{sl}_{2|1}$. The corresponding Zwegers' type real analytic functions $R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v)$ are introduced in Section 4. As a result, we construct a modular invariant

modified normalized supercharacter for the $\widehat{\mathfrak{osp}}_{3|2}$ -module $L(m\Lambda_0)$, and, for each $M \in \mathbb{Z}_{\geq 1}$, such that $\gcd(M, 4m+2) = 1$, the associated modular invariant family of principal admissible modified normalized supercharacters (Theorem 5.12).

As in [KW], in all cases considered, modified normalized supercharacters and characters, along with their twisted analogues, again form a modular invariant family (Theorems 2.9 and 5.12).

Next, as in [KW], we apply the quantum Hamiltonian reduction to the principal admissible $\widehat{\mathfrak{g}}$ -modules. As usual, the integrable and a few admissible $\widehat{\mathfrak{g}}$ -modules get erased (i.e. give zero), but what remains of the modular invariant families of modified normalized characters of $\widehat{\mathfrak{g}}$ -modules, produce modular invariant families of modified characters and supercharacters of Neveu-Schwarz and Ramond $N = 2$ (resp. $N = 3$) superconformal algebra positive energy modules. Namely, in the case $\mathfrak{g} = \mathfrak{sl}_{2|1}$ we obtain new families of $N = 2$ superconformal algebra modules with central charge $c = 3(1 - \frac{2m+2}{M})$, where $m \in \mathbb{Z}_{\geq 1}$, $M \in \mathbb{Z}_{\geq 2}$ and $\gcd(M, 2m+2) = 1$ (Theorem 3.1); and in the case of $\mathfrak{g} = \mathfrak{osp}_{3|2}$ we obtain new families of $N = 3$ superconformal algebra modules with central charge $c = -3\frac{2m+1}{M}$, where $m \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 2}$ and $\gcd(M, 4m+2) = 1$ (Theorem 6.3).

In our subsequent paper, we will consider the case $\mathfrak{g} = D(2, 1; a)$ and the corresponding big $N = 4$ superconformal algebras, obtained from $\widehat{\mathfrak{g}}$ by the quantum Hamiltonian reduction [KW2], [KW3].

1 Transformation properties of the mock theta functions $\Phi^{[m;s]}$ and their modifications $\widetilde{\Phi}^{[m;s]}$.

In this section we study modular and elliptic transformation properties of supercharacters $ch_{L(\Lambda)}^-$ of integrable highest weight modules $L(\Lambda)$ over the affine Lie superalgebra $\widehat{\mathfrak{sl}}_{2|1}$. We choose the set of simple roots $\{\alpha_0, \alpha_1, \alpha_2\}$, where α_0 is even and α_1, α_2 are odd, and the scalar products are :

$$(\alpha_0|\alpha_0) = 2, (\alpha_1|\alpha_1) = (\alpha_2|\alpha_2) = 0, (\alpha_0|\alpha_1) = (\alpha_0|\alpha_2) = -1, (\alpha_1|\alpha_2) = 1.$$

Then the highest weight Λ of an integrable module $L(\Lambda)$, such that $(\Lambda|\alpha_1) = 0$, is of the form (up to adding a multiple of the imaginary root δ):

$$\Lambda_{m;s} = (m-s)\Lambda_0 + s\Lambda_2 = m\Lambda_0 + s\alpha_1, \text{ where } m, s \in \mathbb{Z}_{\geq 0}, 0 \leq s \leq m.$$

It is easy to see that the highest weight Λ of any atypical integrable level m $\widehat{\mathfrak{sl}}_{2|1}$ -module can be brought to this form by odd reflections (up to adding a multiple of an imaginary root).

In our paper [KW] we studied transformation properties of the supercharacter of only the vacuum $\widehat{\mathfrak{sl}}_{2|1}$ -module $L(m\Lambda_0)$, by a more complicated method, close to the original approach of [Z].

Since the transformation properties of the superdenominator \widehat{R}^- are well understood [KW], it suffices to study those of the numerator $\widehat{R}^- ch_{L(\Lambda)}^-$.

The numerator of the supercharacter of $L(\Lambda_{m;s})$ is given by the formula (see Conjecture 3.8 in [KW] and its proof in [S] or in [GK]):

$$\widehat{R}^- ch_{L(\Lambda_{m;s})}^- = e^{(m+1)\Lambda_0} \sum_{j \in \mathbb{Z}} \left(\frac{e^{-j(m+1)(\alpha_1+\alpha_2)} e^{s\alpha_1} q^{j^2(m+1)-js}}{1 - e^{-\alpha_1} q^j} - \frac{e^{j(m+1)(\alpha_1+\alpha_2)} e^{-s\alpha_2} q^{j^2(m+1)-js}}{1 - e^{\alpha_2} q^j} \right),$$

where, as usual, $q = e^{-\delta}$. Hence in coordinates

$$h = 2\pi i(-\tau\Lambda_0 - z_1\alpha_2 - z_2\alpha_1 + t\delta) \quad (\text{i.e. } e^{-\alpha_1(h)} = e^{2\pi iz_1}, \quad e^{-\alpha_2(h)} = e^{2\pi iz_2}, \quad e^{-\delta(h)} = e^{2\pi i\tau} = q)$$

it is given by the function

$$\left(\widehat{R}^- \text{ch}_{L(\Lambda_{m;s})}^-\right)(h) = \Phi^{[m;s]}(\tau, z_1, z_2, t) = e^{2\pi i(m+1)t} \left(\Phi_1^{[m;s]}(\tau, z_1, z_2) - \Phi_1^{[m;s]}(\tau, -z_2, -z_1) \right),$$

where

$$(1.1) \quad \Phi_1^{[m;s]}(\tau, z_1, z_2) = \sum_{j \in \mathbb{Z}} \frac{e^{2\pi i(m+1)j(z_1+z_2)} e^{-2\pi isz_1} q^{j^2(m+1)-js}}{1 - e^{2\pi iz_1} q^j}.$$

It is a holomorphic function in the domain $X = \{(\tau, z_1, z_2, t) \in \mathbb{C}^4 \mid \text{Im } \tau > 0\}$.

Recall the formula for the classical theta functions $\Theta_{j,m}(\tau, z, t)$ of degree m , where m is a positive integer and $j \in \mathbb{Z} \bmod 2m\mathbb{Z}$:

$$(1.2) \quad \Theta_{j,m}(\tau, z, t) = e^{2\pi imt} \sum_{n \in \mathbb{Z}} e^{2\pi imz(n + \frac{j}{2m})} q^{m(n + \frac{j}{2m})^2}.$$

These are holomorphic functions in the domain $X_0 = \{(\tau, z, t) \in \mathbb{C}^3 \mid \text{Im } \tau > 0\}$. Let $\Theta_{j,m}(\tau, z) = \Theta_{j,m}(\tau, z, 0)$. In the study of the functions $\Phi^{[m;s]}$ we shall use, in particular, the following obvious properties

$$(1.3) \quad \Theta_{j,m}(\tau, -z) = \Theta_{-j,m}(\tau, z),$$

$$(1.4) \quad \Theta_{j,m}(\tau, z + b) = (-1)^{bj} \Theta_{j,m}(\tau, z) \text{ if } b \in \mathbb{Z}.$$

Lemma 1.1. *The functions $\Phi^{[m;s]}$ satisfy the following properties :*

(0) $\Phi^{[m;s]} - \Phi^{[m;s]}|_S$ is a holomorphic function in the domain X .

$$(1) \quad \Phi^{[m;s]}(\tau + 1, z_1, z_2, t) = \Phi^{[m;s]}(\tau, z_1, z_2, t).$$

$$(2) \quad \Phi^{[m;s]}(\tau, z_1 + a, z_2 + b, t) = \Phi^{[m;s]}(\tau, z_1, z_2, t) \text{ if } a, b \in \mathbb{Z}.$$

$$(3) \quad \Phi^{[m;s]}(\tau, z_1, z_2, t) - e^{2\pi i(m+1)z_1} \Phi^{[m;s]}(\tau, z_1, z_2 + \tau, t) \\ = e^{2\pi i(m+1)t} \sum_{k=0}^m e^{\pi i(k-s)(z_1-z_2)} q^{-\frac{(k-s)^2}{4(m+1)}} \left(\Theta_{k-s, m+1} - \Theta_{-(k-s), m+1} \right) (\tau, z_1 + z_2).$$

$$(4) \quad \Phi^{[m;s]}(\tau, z_1, z_2, t) - e^{-2\pi i(m+1)z_2} \Phi^{[m;s]}(\tau, z_1 - \tau, z_2, t) \\ = e^{2\pi i(m+1)t} \sum_{k=0}^m e^{\pi i(k-s)(z_1-z_2)} q^{-\frac{(k-s)^2}{4(m+1)}} \left(\Theta_{k-s, m+1} - \Theta_{-(k-s), m+1} \right) (\tau, z_1 + z_2).$$

$$(5) \quad \Phi^{[m;s]}(\tau, z_1 + j\tau, z_2 + j\tau, t) = q^{-j^2(m+1)} e^{-2\pi ij(m+1)(z_1+z_2)} \Phi^{[m;s]}(\tau, z_1, z_2, t) \text{ if } j \in \mathbb{Z}.$$

$$(6) \quad \Phi^{[m;s]}(\tau, -z_2, -z_1, t) = -\Phi^{[m;s]}(\tau, z_1, z_2, t).$$

Proof. Recall that, by definition of the action of $SL(2, \mathbb{Z})$ (see [KW], Section 4):

$$(1.5) \quad \Phi_1^{[m;s]}|_S(\tau, z_1, z_2) = \frac{1}{\tau} e^{-\frac{2\pi i(m+1)z_1 z_2}{\tau}} \Phi_1^{[m;s]} \left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau} \right).$$

In order to prove (0), it suffices to show that

$$\Phi_1^{[m;s]}(\tau, z_1, z_2) - \Phi_1^{[m;s]}|_S(\tau, z_1, z_2),$$

viewed as a function in z_1 , has zero residues at all poles. The poles of both $\Phi_1^{[m;s]}$ and $\Phi_1^{[m;s]}|_S$ are the points $z_1 \in \mathbb{Z} + \tau\mathbb{Z}$, and it is easy to see that :

$$\text{Res}_{z_1 = n+j\tau} \Phi_1^{[m;s]} = \text{Res}_{z_1 = n+j\tau} \Phi_1^{[m;s]}|_S = -\frac{1}{2\pi i} e^{-2\pi i j(m+1)z_2},$$

as required.

Claims (1), (2) and (6) are obvious. In order to prove (3), we rewrite :

$$\begin{aligned} & \Phi_1^{[m;s]}(\tau, z_1, z_2) - e^{2\pi i(m+1)z_1} \Phi_1^{[m;s]}(\tau, z_1, z_2 + \tau) \\ &= \sum_{j \in \mathbb{Z}} e^{2\pi i j(m+1)(z_1+z_2)} e^{-2\pi i s z_1} q^{j^2(m+1)-js} \frac{1 - (e^{2\pi i z_1} q^j)^{m+1}}{1 - e^{2\pi i z_1} q^j} \\ &= e^{-2\pi i s z_1} \sum_{k=0}^m e^{\pi i k(z_1-z_2)} \sum_{j \in \mathbb{Z}} (e^{2\pi i j(m+1)(z_1+z_2)} e^{\pi i k(z_1+z_2)}) q^{j^2(m+1)+j(k-s)} \\ &= \sum_{k=0}^m e^{\pi i(k-s)(z_1-z_2)} q^{-\frac{(k-s)^2}{4(m+1)}} \Theta_{k-s, m+1}(\tau, z_1 + z_2). \end{aligned}$$

Using (1.3), we obtain the claim.

Claim (4) follows easily from (6) by sending (z_1, z_2) to $(-z_2, -z_1)$ in (3) and using (1.3).

In order to prove claim (5), note that

$$\Phi_1^{[m;s]}(\tau, z_1 + \tau, z_2 + \tau) = q^{-(m+1)} e^{-2\pi i(m+1)(z_1+z_2)} \Phi_1^{[m;s]}(\tau, z_1, z_2),$$

via replacing j by $j - 1$ in the sum, defining the LHS (see (1.1)). The same formula holds for $\Phi_1^{[m;s]}(\tau, -z_2 - \tau, -z_1 - \tau)$ via replacing j by $j + 1$ in the sum. Hence (5) holds for $j = 1$, and by induction on j it holds for arbitrary $j \geq 1$. \square

We change the coordinates, letting

$$z_1 = v - u, \quad z_2 = -v - u, \quad \text{i.e.} \quad u = -\frac{z_1 + z_2}{2}, \quad v = \frac{z_1 - z_2}{2},$$

and denote

$$\varphi^{[m;s]}(\tau, u, v, t) = \Phi^{[m;s]}(\tau, z_1, z_2, t).$$

Formula (1.5) becomes :

$$\varphi^{[m;s]}|_S(\tau, u, v, t) = \frac{1}{\tau} \varphi^{[m;s]} \left(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t - \frac{u^2 - v^2}{\tau} \right).$$

The following lemma is immediate by Lemma 1.1.

Lemma 1.2. *The function $\varphi^{[m;s]}$ has the following properties:*

- (0) $\varphi^{[m;s]} - \varphi^{[m;s]}|_S$ is a holomorphic function in the domain X .
- (1) $\varphi^{[m;s]}(\tau + 1, u, v, t) = \varphi^{[m;s]}(\tau, u, v, t)$.
- (2) $\varphi^{[m;s]}(\tau, u + a, v + b, t) = \varphi^{[m;s]}(\tau, u, v, t)$ if $a, b \in \frac{1}{2}\mathbb{Z}$ are such that $a + b \in \mathbb{Z}$.
- (3) $\varphi^{[m;s]}(\tau, u, v, t) - e^{2\pi i(m+1)(v-u)} \varphi^{[m;s]}(\tau, u - \frac{\tau}{2}, v - \frac{\tau}{2}, t)$
 $= -e^{2\pi i(m+1)t} \sum_{j=0}^m e^{2\pi i(j-s)v} q^{-\frac{(j-s)^2}{4(m+1)}} (\Theta_{j-s, m+1} - \Theta_{-(j-s), m+1}) (\tau, 2u).$
- (4) $\varphi^{[m;s]}(\tau, u, v, t) - e^{2\pi i(m+1)(u+v)} \varphi^{[m;s]}(\tau, u + \frac{\tau}{2}, v - \frac{\tau}{2}, t)$
 $= -e^{2\pi i(m+1)t} \sum_{j=0}^m e^{2\pi i(j-s)v} q^{-\frac{(j-s)^2}{4(m+1)}} (\Theta_{j-s, m+1} - \Theta_{-(j-s), m+1}) (\tau, 2u).$
- (5) $\varphi^{[m;s]}(\tau, u - \tau, v, t) = e^{4\pi i(m+1)u} q^{-(m+1)} \varphi^{[m;s]}(\tau, u, v, t)$
- (6) $\varphi^{[m;s]}(\tau, -u, v, t) = -\varphi^{[m;s]}(\tau, u, v, t).$

From claims (3) and (4) of Lemma 1.2, we obtain the following:

Lemma 1.3.

$$\begin{aligned} & \varphi^{[m;s]}(\tau, u, v, t) - e^{2\pi i(m+1)(2v-\tau)} \varphi^{[m;s]}(\tau, u, v - \tau, t) \\ &= -e^{2\pi i(m+1)t} \sum_{j=0}^{2m+1} e^{2\pi i(j-s)v} q^{-\frac{(j-s)^2}{4(m+1)}} (\Theta_{j-s, m+1} - \Theta_{-(j-s), m+1}) (\tau, 2u). \end{aligned}$$

□

We put

$$G^{[m;s]}(\tau, u, v, t) = \varphi^{[m;s]}(\tau, u, v, t) - \varphi^{[m;s]}|_S(\tau, u, v, t).$$

Lemma 1.4. *The function $G^{[m;s]}$ satisfies the following properties:*

- (1) $G^{[m;s]}(\tau, u, v, t)$ is holomorphic with respect to v .
- (2) $G^{[m;s]}(\tau, u, v, t) - e^{2\pi i(m+1)(2v-\tau)} G^{[m;s]}(\tau, u, v - \tau, t)$
 $= -e^{2\pi i(m+1)t} \sum_{j=-s}^{-s+2m+1} e^{2\pi i j v} q^{-\frac{j^2}{4(m+1)}} (\Theta_{j, m+1} - \Theta_{-j, m+1}) (\tau, 2u).$
- (3) $G^{[m;s]}(\tau, u, v + 1, t) - G^{[m;s]}(\tau, u, v, t) = i e^{2\pi i(m+1)t} \frac{1}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}}$
 $\times \sum_{j,k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau} \left(v + \frac{k}{2(m+1)}\right)^2} (\Theta_{j, m+1} - \Theta_{-j, m+1}) (\tau, 2u).$
- (4) $G^{[m;s]}$ is determined uniquely by the above three properties.

Proof. Without loss of generality we can let $t = 0$. We have, using Lemma 1.2 (2) for $a = 0$, $b = 1$:

$$\begin{aligned}
& G^{[m;s]}(\tau, u, v, 0) - e^{2\pi i(m+1)(2v-\tau)} G^{[m;s]}(\tau, u, v - \tau, 0) \\
&= \varphi^{[m;s]}(\tau, u, v, 0) - \tau^{-1} e^{-\frac{2\pi i(m+1)}{\tau}(u^2-v^2)} \varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, 0) \\
&\quad - e^{2\pi i(m+1)(2v-\tau)} \left(\varphi^{[m;s]}(\tau, u, v - \tau, 0) - \tau^{-1} e^{-\frac{2\pi i(m+1)}{\tau}[u^2-(v-\tau)^2]} \varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v-\tau}{\tau}, 0) \right) \\
&= \varphi^{[m;s]}(\tau, u, v, 0) - e^{2\pi i(m+1)(2v-\tau)} \varphi^{[m;s]}(\tau, u, v - \tau, 0) \\
&= - \sum_{j=0}^{2m+1} e^{2\pi i(j-s)v} q^{-\frac{(j-s)^2}{4(m+1)}} (\Theta_{j-s, m+1} - \Theta_{-(j-s), m+1}) (\tau, 2u).
\end{aligned}$$

The last equality holds by Lemma 1.3. This proves (2).

In order to prove (3), note that after the substitution $(\tau, u, v) \rightarrow (-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau})$ in Lemma 1.3 and using the modular transformation formula of $\Theta_{\pm(j-s), m+1}$ (see e.g. the Appendix in [KW]), we obtain :

$$\begin{aligned}
& \varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, 0) - e^{\frac{2\pi i(m+1)}{\tau}(2v+1)} \varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v+1}{\tau}, 0) \\
&= - \sum_{j=0}^{2m+1} e^{\frac{2\pi i(m+1)}{\tau}(j-s)v} e^{\frac{2\pi i(j-s)^2}{4(m+1)\tau}} (\Theta_{j-s, m+1} - \Theta_{-(j-s), m+1}) (-\frac{1}{\tau}, \frac{2u}{\tau}) \\
&= \frac{-(-i\tau)^{\frac{1}{2}}}{\sqrt{2(m+1)}} e^{\frac{2\pi i(m+1)}{\tau}(u^2-v^2)} \sum_{\substack{0 \leq j \leq 2m+1 \\ k \in \mathbb{Z}/2(m+1)\mathbb{Z}}} e^{-\frac{\pi i(j-s)k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau}(v + \frac{j-s}{2(m+1)})^2} (\Theta_{k, m+1} - \Theta_{-k, m+1}) (\tau, 2u).
\end{aligned}$$

Using this and Lemma 1.2 (2), we obtain :

$$\begin{aligned}
& G^{[m;s]}(\tau, u, v+1, 0) - G^{[m;s]}(\tau, u, v, 0) \\
&= \varphi^{[m;s]}(\tau, u, v+1, 0) - \tau^{-1} e^{-\frac{2\pi i(m+1)}{\tau}(u^2-(v+1)^2)} \varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v+1}{\tau}, 0) \\
&\quad - \left(\varphi^{[m;s]}(\tau, u, v, 0) - \tau^{-1} e^{-\frac{2\pi i(m+1)}{\tau}(u^2-v^2)} \varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, 0) \right) \\
&= \tau^{-1} e^{-\frac{2\pi i(m+1)}{\tau}(u^2-v^2)} \left(\varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, 0) - e^{\frac{2\pi i(m+1)}{\tau}(2v+1)} \varphi^{[m;s]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v+1}{\tau}, 0) \right) \\
&= \frac{-i}{-i\tau} \cdot \frac{-(-i\tau)^{\frac{1}{2}}}{\sqrt{2(m+1)}} \sum_{\substack{0 \leq j \leq 2m+1 \\ k \in \mathbb{Z}/2(m+1)\mathbb{Z}}} e^{-\frac{\pi i(j-s)k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau}(v + \frac{j-s}{2(m+1)})^2} (\Theta_{k, m+1} - \Theta_{-k, m+1}) (\tau, 2u),
\end{aligned}$$

which proves (3).

The proof of claim (4) is the same as that Proposition 5.4(c) in [KW]. Namely, if $F(\tau, u, v, t)$ is the difference of two functions, satisfying (1)-(3), then the function

$$P(\tau, u, v, t) = F(\tau, u, v, t) \vartheta_{11}((m+1)\tau, (m+1)v)^2$$

is holomorphic and doubly periodic in v , and vanishes at $v = 0$, hence is zero. \square

Lemma 1.5. *Let $a_j(\tau, v)$ ($j \in \mathbb{Z}$, $-s \leq j \leq -s+2m+1$) be functions satisfying the following conditions (i), (ii), (iii):*

(i) $a_j(\tau, v)$ is holomorphic with respect to v ,

$$(ii) \ a_j(\tau, v) - e^{2\pi i(m+1)(2v-\tau)} a_j(\tau, v - \tau) = -e^{2\pi i j v} e^{-\frac{\pi i \tau}{2(m+1)} j^2},$$

$$(iii) \ a_j(\tau, v + 1) - a_j(\tau, v) \\ = \frac{i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau} \left(v + \frac{k}{2(m+1)}\right)^2}.$$

(By the argument in the proof of Lemma 1.4(4), $a_j(\tau, v)$ is uniquely determined by the properties (i), (ii), (iii).) Then the function

$$G^{[m; s]}(\tau, u, v, t) := e^{2\pi i(m+1)t} \sum_{j=-s}^{-s+2m+1} a_j(\tau, v) (\Theta_{j, m+1} - \Theta_{-j, m+1})(\tau, 2u)$$

satisfies the properties (1), (2), (3) of Lemma 1.4.

Proof. It is straightforward. \square

In order to construct functions $a_j(\tau, v)$ satisfying the conditions (i), (ii), (iii) of Lemma 1.5, we define the (modified) Zwegers functions $R_{j; m+1}(\tau, v)$ ($j \in \mathbb{Z}$) as follows:

$$(1.6) \quad R_{j; m+1}(\tau, v) := \sum_{\substack{n \in \mathbb{Z} \\ n \equiv -j \pmod{2(m+1)}}} \left(\operatorname{sgn}\left(n + \frac{1}{2} + j - 2(m+1)\right) - E\left(\left(n + 2(m+1) \frac{\operatorname{Im} v}{\operatorname{Im} \tau}\right) \sqrt{\frac{\operatorname{Im} \tau}{m+1}}\right) \right) e^{-\frac{\pi i n^2}{2(m+1)} \tau - 2\pi i n v},$$

where $E(x) = 2 \int_0^x e^{-\pi u^2} du$. In the same way as in [Z] one shows that this series converges to a real analytic function for all $v, \tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$.

Lemma 1.6. *The functions $R_{j; m+1}$ have the following properties:*

$$(1) \ R_{j; m+1}(\tau, v + \frac{1}{2}) = (-1)^j R_{j; m+1}(\tau, v), \text{ hence } R_{j; m+1}(\tau, v + 1) = R_{j; m+1}(\tau, v).$$

$$(2) \ R_{j; m+1}(\tau, v) = -R_{2m+2-j; m+1}(\tau, -v).$$

$$(3) \ R_{j; m+1}(\tau, v) - e^{2\pi i(m+1)(2v-\tau)} R_{j; m+1}(\tau, v - \tau) = -2 e^{\frac{\pi i \tau}{2(m+1)} j^2} e^{2\pi i j v}.$$

Proof. Claims (1) and (2) are immediate from the definition of the function $R_{j; m+1}$. In order to prove (3), we first prove

$$(1.7) \quad R_{j; m+1}(\tau, v - \tau) - e^{2\pi i(m+1)(\tau-2v)} R_{j; m+1}(\tau, v) = 2e^{-\frac{\pi i \tau}{2(m+1)}(2m+2-j)^2 + 2\pi i(2m+2-j)\tau} e^{-2\pi i(2m+2-j)v}.$$

We have :

$$R_{j; m+1}(\tau, v - \tau) = \sum_{n \equiv -j \pmod{2(m+1)}} \left(\operatorname{sign}\left(n + \frac{1}{2} + j - 2(m+1)\right) - E\left(\left(n - 2(m+1) + 2(m+1) \frac{\operatorname{Im} v}{\operatorname{Im} \tau}\right) \sqrt{\frac{\operatorname{Im} \tau}{m+1}}\right) \right) e^{-\frac{\pi i n^2 \tau}{2(m+1)} - 2\pi i n(v - \tau)}.$$

Letting $n = n' + 2(m+1)$, the RHS can be rewritten as follows :

$$\sum_{n' \equiv -j \pmod{2(m+1)}} \left(\text{sign}\left(n' + \frac{1}{2} + j\right) - E \left(\left(n' + 2(m+1) \frac{\text{Im } v}{\text{Im } \tau} \right) \sqrt{\frac{\text{Im } \tau}{m+1}} \right) \right) \\ \times e^{-\frac{\pi i \tau}{2(m+1)}(n'+2(m+1))^2 - 2\pi i(n'+2(m+1))(v-\tau)}.$$

This can be rewritten, using that

$$\text{sign} \left(n + \frac{1}{2} + j \right) = \text{sign} \left(n + \frac{1}{2} + j - 2(m+1) \right) + 2\delta_{n, -j},$$

as follows :

$$2e^{-\frac{\pi i \tau}{2(m+1)}(2m+2-j)^2 + 2\pi i(2m+2-j)\tau} e^{-2\pi i(2m+2-j)v} + (I),$$

where

$$(I) = \sum_{n \equiv -j \pmod{2(m+1)}} \left(\text{sign} \left(n + \frac{1}{2} + j - 2(m+1) \right) - E \left(\left(n + 2(m+1) \frac{\text{Im } v}{\text{Im } \tau} \right) \sqrt{\frac{\text{Im } \tau}{m+1}} \right) \right) e^{-\frac{\pi i \tau n^2}{2(m+1)} - 2\pi i n v} e^{2\pi i(m+1)(\tau-2v)} \\ = e^{2\pi i(m+1)(\tau-2v)} R_{j; m+1}(\tau, v).$$

This completes the proof of formula (1.7). Multiplying both sides of (1.7) by $-e^{2\pi i(m+1)(2v-\tau)}$, we obtain claim (3). \square

We put

$$(1.8) \quad y = \text{Im } \tau (> 0), \quad v = a\tau - b \text{ (so that } \bar{v} = a\bar{\tau} - b), \text{ where } a, b \in \mathbb{R}.$$

Then in the real coordinates $(a, b, y, \text{Re } \tau)$ we have :

$$(1.9) \quad \frac{\partial}{\partial a} + \bar{\tau} \frac{\partial}{\partial b} = 2iy \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} = -2iy \frac{\partial}{\partial \bar{v}}$$

and

$$(1.10) \quad 2iy = \tau - \bar{\tau}, \quad 2iya = v - \bar{v}, \quad 2iyb = \bar{\tau}v - \tau\bar{v}.$$

Lemma 1.7. *We have:*

$$(1) \quad \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) R_{j; m+1}(\tau, v) = -4\sqrt{(m+1)y} e^{-4\pi(m+1)a^2y} \Theta_{j, m+1}(-\bar{\tau}, 2\bar{v}).$$

$$(2) \quad \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) R_{j; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) = -4 \frac{\tau}{|\tau|} \sqrt{(m+1)y} e^{-4\pi(m+1) \frac{b^2}{|\tau|^2} y} \Theta_{j, m+1} \left(\frac{1}{\bar{\tau}}, \frac{2\bar{v}}{\bar{\tau}} \right).$$

Proof. Due to the second formula in (1.9), we have:

$$\left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) e^{\frac{-\pi i n^2}{2(m+1)} \tau - 2\pi i n v} = 0,$$

and since

$$\left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b}\right) E\left((n+2(m+1)a) \sqrt{\frac{y}{m+1}}\right) = 4e^{-\pi(n+2(m+1)a)^2 \frac{y}{m+1}} (m+1) \sqrt{\frac{y}{m+1}},$$

we obtain, using formula (1.10) :

$$\begin{aligned} \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b}\right) R_{j;m+1}(\tau, v) &= -4\sqrt{(m+1)y} \sum_{n \equiv -j \pmod{2(m+1)}} e^{-\frac{\pi y}{m+1}(n+2(m+1)a)^2 - \frac{\pi i n^2}{2(m+1)}\tau - 2\pi i n(a\tau - b)} \\ &= -4\sqrt{(m+1)y} \sum_{n \equiv -j \pmod{2(m+1)}} e^{-\pi i n^2 \frac{\bar{\tau}}{4(m+1)} - 2\pi i n \bar{v}}. \end{aligned}$$

Replacing n by $-n$ in the last sum, we obtain claim (1) (cf (1.2)).

In order to prove claim (2), let

$$\tau' = -\frac{1}{\tau}, \quad v' = \frac{v}{\tau}, \quad y' = \text{Im } \tau' = \frac{y}{|\tau|^2},$$

and introduce the coordinates $a', b' \in \mathbb{R}$ by

$$v' = a'\tau' - b', \quad \bar{v}' = a'\bar{\tau}' - b'.$$

Then we have :

$$a = -b', \quad b = a'; \quad \frac{\partial}{\partial a'} + \tau' \frac{\partial}{\partial b'} = \frac{1}{\tau} \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right).$$

Using this and claim (1), we obtain :

$$\begin{aligned} \frac{1}{\tau} \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) R_{j;m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) &= \left(\frac{\partial}{\partial a'} + \tau' \frac{\partial}{\partial b'} \right) R_{j;m+1}(\tau', v') \\ &= -4\sqrt{(m+1)y'} e^{-4\pi(m+1)a'^2 y'} \Theta_{j,m+1}(-\bar{\tau}', 2\bar{v}') \\ &= -4\sqrt{\frac{(m+1)y}{|\tau|^2}} e^{-4\pi(m+1)b^2 \frac{y}{|\tau|^2}} \Theta_{j,m+1} \left(\frac{1}{\bar{\tau}}, \frac{2\bar{v}}{\bar{\tau}} \right), \end{aligned}$$

proving (2) □

The formula in Lemma 1.7 (2) together with the modular transformation formula for $\Theta_{j,m+1}$ gives the following :

Lemma 1.8.

$$\begin{aligned} &\left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b}\right) R_{j;m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \\ &= -4i(-i\tau)^{\frac{1}{2}} \sqrt{\frac{y}{2}} e^{\frac{\pi(m+1)}{y\tau}(\bar{\tau}v^2 - 2\tau v\bar{v} + \tau\bar{v}^2)} \sum_{k \in \mathbb{Z}/2(m+1)\mathbb{Z}} e^{-\frac{\pi i j k}{m+1}} \Theta_{k,m+1}(-\bar{\tau}, -2\bar{v}). \end{aligned}$$

□

For $j \in \mathbb{Z}$ such that $-s \leq j \leq -s + 2m + 1$, we define the following functions :

$$\tilde{a}_j(\tau, v) := R_{j;m+1}(\tau, v) + \frac{i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+1)v^2}{\tau}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} R_{k;m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right).$$

Lemma 1.9. *The functions $\tilde{a}_j(\tau, v)$, $-s \leq j \leq -s + 2m + 1$, satisfy the following properties :*

- (1) $\tilde{a}_j(\tau, v)$ are holomorphic in v ,
- (2) $\tilde{a}_j(\tau, v) - e^{2\pi i(m+1)(2v-\tau)} \tilde{a}_j(\tau, v - \tau) = -2e^{-\frac{\pi i \tau}{2(m+1)} j^2} e^{2\pi i j v}$,
- (3) $\tilde{a}_j(\tau, v + 1) - \tilde{a}_j(\tau, v) = \frac{2i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau} \left(v + \frac{k}{2(m+1)}\right)^2}$.

Proof. In view of the second formula in (1.9), we can rewrite Lemma 1.8 as follows :

$$(1.11) \quad \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) \left((-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+1)}{\tau} v^2} R_{j; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \right) \\ = -4i \sqrt{\frac{y}{2}} e^{-4\pi(m+1)a^2 y} \sum_{k \in \mathbb{Z}/2(m+1)\mathbb{Z}} e^{-\frac{\pi i j k}{m+1}} \Theta_{k, m+1}(-\bar{\tau}, -2\bar{v}).$$

In order to prove (1), it suffices to show the following :

$$(1.12) \quad \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) \left(\frac{-i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+1)}{\tau} v^2} \sum_{k=-s}^{-s+2m-1} e^{-\frac{\pi i j k}{m+1}} R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \right) \\ = \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) R_{j; m+1}(\tau, v).$$

The LHS of (1.12) is equal to

$$\frac{-i}{\sqrt{2(m+1)}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) \left((-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+1)}{\tau} v^2} R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \right),$$

which by (1.11) is equal to

$$\frac{-i}{\sqrt{2(m+1)}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} \left(-\sqrt{\frac{y}{2}} e^{-4\pi(m+1)ya^2} \sum_{n \in \mathbb{Z}/2(m+1)\mathbb{Z}} e^{-\frac{\pi i k n}{m+1}} \Theta_{n, m+1}(-\bar{\tau}, -2\bar{v}) \right) \\ = -2\sqrt{\frac{y}{m+1}} e^{-4\pi(m+1)ya^2} \sum_{n \in \mathbb{Z}/2(m+1)\mathbb{Z}} \left(\sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{-\frac{\pi i k n}{m+1}} \right) \Theta_{n, m+1}(-\bar{\tau}, -2\bar{v}).$$

Since $\sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{-\frac{\pi i k n}{m+1}} = 2(m+1)$ if $j + n \equiv 0 \pmod{2(m+1)}$, and $= 0$ otherwise, we obtain

that the LHS of (1.12) is equal to $-4\sqrt{(m+1)y} e^{-4\pi(m+1)ya^2} \Theta_{-j, m+1}(-\bar{\tau}, -2\bar{v})$. Hence, using (1.3), we deduce from Lemma 1.7 (1) that (1.12) holds.

Next, we prove (2). We have :

$$\tilde{a}_j(\tau, v - \tau) = R_{j; m+1}(\tau, v - \tau) \\ + \frac{i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+1)}{\tau} (v-\tau)^2} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v - \tau}{\tau} \right) = R_{j; m+1}(\tau, v - \tau) \\ + e^{2\pi i(m+1)(\tau-2v)} \frac{i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+1)}{\tau} v^2} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \\ = R_{j; m+1}(\tau, v - \tau) + e^{2\pi i(m+1)(\tau-2v)} \left(\tilde{a}_j(\tau, v) - R_{j; m+1}(\tau, v) \right).$$

Multiplying both sides by $e^{2\pi i(m+1)(2v-\tau)}$, we obtain

$$\tilde{a}_j(\tau, v) - e^{2\pi i(m+1)(2v-\tau)} \tilde{a}_j(\tau, v - \tau) = R_{j; m+1}(\tau, v) - e^{2\pi i(m+1)(2v-\tau)} R_{j; m+1}(\tau, v - \tau).$$

Applying Lemma 1.6 (3) to the RHS, we obtain claim (2).

In order to prove (3), note that, replacing (τ, v) by $(-\frac{1}{\tau}, \frac{v}{\tau})$, Lemma 1.6(3) gives

$$(1.13) \quad R_{j; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) - e^{\frac{2\pi i(m+1)}{\tau}(2v+1)} R_{j; m+1} \left(-\frac{1}{\tau}, \frac{v+1}{\tau} \right) = -2e^{\frac{\pi i j^2}{2(m+1)\tau}} e^{\frac{2\pi i j v}{\tau}}.$$

By definition of $R_{j; m+1}$ and using Lemma 1.6 (1), we have

$$\begin{aligned} \tilde{a}_j(\tau, v+1) - \tilde{a}_j(\tau, v) &= \frac{i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} \left(e^{\frac{2\pi i(m+1)}{\tau}(v+1)^2} R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v+1}{\tau} \right) \right. \\ &\quad \left. - e^{\frac{2\pi i(m+1)}{\tau}v^2} R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \right) = \frac{i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau}v^2} \\ &\quad \times \left(e^{\frac{2\pi i(m+1)}{\tau}(2v+1)} R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v+1}{\tau} \right) - R_{k; m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \right). \end{aligned}$$

Using (1.13), we deduce:

$$\begin{aligned} \tilde{a}_j(\tau, v+1) - \tilde{a}_j(\tau, v) &= \frac{2i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau}v^2 + \frac{\pi i k^2}{2(m+1)\tau} + \frac{2\pi i k v}{\tau}} \\ &= \frac{2i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} e^{\frac{2\pi i(m+1)}{\tau} \left(v + \frac{k}{2(m+1)} \right)^2}, \end{aligned}$$

proving claim (3). □

Lemma 1.9 shows that the functions $a_j(\tau, v) := \frac{1}{2}\tilde{a}_j(\tau, v)$ satisfy the conditions (i), (ii), (iii) of Lemma 1.5, hence, by Lemma 1.5 and Lemma 1.4(4), we have

$$\textbf{Proposition 1.10.} \quad G^{[m; s]}(\tau, u, v, t) = \frac{1}{2}e^{2\pi i(m+1)t} \sum_{j=-s}^{-s+2m+1} \tilde{a}_j(\tau, v) (\Theta_{j, m+1} - \Theta_{-j, m+1})(\tau, 2u).$$

We put

$$(1.14) \quad \varphi_{\text{add}}^{[m; s]}(\tau, u, v, t) := -\frac{1}{2}e^{2\pi i(m+1)t} \sum_{j=-s}^{-s+2m+1} R_{j; m+1}(\tau, v) (\Theta_{j, m+1} - \Theta_{-j, m+1})(\tau, 2u)$$

and

$$(1.15) \quad \tilde{\varphi}^{[m; s]} := \varphi^{[m; s]} + \varphi_{\text{add}}^{[m; s]}.$$

Then we obtain the following theorem.

Theorem 1.11.

- (1) $\tilde{\varphi}^{[m;s]}|_S = \tilde{\varphi}^{[m;s]}$; $\tilde{\varphi}^{[m;s]}(\tau+1, u, v, t) = \tilde{\varphi}^{[m;s]}(\tau, u, v, t)$.
- (2) $\tilde{\varphi}^{[m;s]}(\tau, u+a, v+b, t) = \tilde{\varphi}^{[m;s]}(\tau, u, v, t)$ for all $a, b \in \frac{1}{2}\mathbb{Z}$ such that $a+b \in \mathbb{Z}$.
- (3) $\tilde{\varphi}^{[m;s]}(\tau, u+a\tau, v+b\tau, t) = q^{(m+1)(b^2-a^2)} e^{4\pi i(m+1)(bv-au)} \tilde{\varphi}^{[m;s]}(\tau, u, v, t)$ for all $a, b \in \frac{1}{2}\mathbb{Z}$ such that $a+b \in \mathbb{Z}$.

Proof. We have by definition of $\varphi_{\text{add}}^{[m;s]}$:

$$\begin{aligned} \varphi_{\text{add}}^{[m;s]}|_S(\tau, u, v, t) &:= \frac{1}{\tau} \varphi_{\text{add}}^{[m;s]} \left(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t - \frac{u^2 - v^2}{\tau} \right) \\ &= -\frac{1}{2\tau} e^{2\pi i(m+1)\left(t - \frac{u^2 - v^2}{\tau}\right)} \sum_{j=-s}^{-s+2m+1} R_{j,m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) (\Theta_{j,m+1} - \Theta_{-j,m+1}) \left(-\frac{1}{\tau}, \frac{2u}{\tau} \right). \end{aligned}$$

Using the modular transformation formula of $\Theta_{j,m+1}$, the RHS is rewritten as follows :

$$\begin{aligned} &\frac{1}{2} e^{2\pi i(m+1)t} \sum_{k=-s}^{-s+2m+1} \frac{i}{\sqrt{2(m+1)}} (-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+1)}{\tau} v^2} \\ &\times \sum_{k=-s}^{-s+2m+1} e^{-\frac{\pi i j k}{m+1}} R_{j,m+1} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) (\Theta_{k,m+1} - \Theta_{-k,m+1})(\tau, 2u). \end{aligned}$$

By the definition of $\tilde{a}_k(\tau, v)$, this is equal to

$$\frac{1}{2} e^{2\pi i(m+1)t} \sum_{k=-s}^{-s+2m+1} (\tilde{a}_k(\tau, v) - R_{k,m+1}(\tau, v)) (\Theta_{k,m+1} - \Theta_{-k,m+1})(\tau, 2u).$$

Using Proposition 1.10 and the definition of $\varphi_{\text{add}}^{[m;s]}$, this can be rewritten as $G^{[m;s]}(\tau, u, v, t) + \varphi_{\text{add}}^{[m;s]}(\tau, u, v, t)$. Thus, by definition of $G^{[m;s]}$ we have

$$\left(\varphi^{[m;s]} + \varphi_{\text{add}}^{[m;s]} \right) |_S = \varphi^{[m;s]} + \varphi_{\text{add}}^{[m;s]},$$

proving the first formula in (1). The second formula in (1) is straightforward.

Next, let $a, b \in \frac{1}{2}\mathbb{Z}$ be such that $a+b \in \mathbb{Z}$. Using Lemma 1.6 (1) and equation (1.4), we obtain that $\varphi_{\text{add}}^{[m;s]}(\tau, u+a, v+b, t) = \varphi_{\text{add}}^{[m;s]}(\tau, u, v, t)$. This together with Lemma 1.2 (2) proves claim (2).

Claim(3) is easily deduced from claims (1) and (2) as follows. By (1) we have:

$$(1.16) \quad \tau \tilde{\varphi}^{[m;s]}(\tau, u, v, t) = e^{-\frac{2\pi i(m+1)}{\tau}(u^2-v^2)} \tilde{\varphi}^{[m;s]} \left(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t \right).$$

Replacing (u, v) by $(u+a\tau, v+b\tau)$ in this formula, and using claim (2), we obtain :

$$\tau \tilde{\varphi}^{[m;s]}(\tau, u+a\tau, v+b\tau) = e^{-\frac{2\pi i(m+1)}{\tau}((u+a\tau)^2 - (v+b\tau)^2)} \tilde{\varphi}^{[m;s]} \left(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t \right).$$

By (1.16), the RHS of this formula is equal to $\tau q^{(m+1)(b^2-a^2)} e^{4\pi i(m+1)(bv-au)} \tilde{\varphi}^{[m;s]}(\tau, u, v, t)$, proving (3). \square

Translating the formulas for $\varphi^{[m;s]}$ and its modification to $\Phi^{[m;s]}$ and its modification, and using (1.3), we obtain

(1.17)

$$\Phi_{\text{add}}^{[m;s]}(\tau, z_1, z_2, t) = \frac{1}{2} e^{2\pi i(m+1)t} \sum_{j=-s}^{-s+2m+1} R_{j;m+1} \left(\tau, \frac{z_1 - z_2}{2} \right) (\Theta_{j,m+1} - \Theta_{-j,m+1})(\tau, z_1 + z_2)$$

and

$$\tilde{\Phi}^{[m;s]} = \Phi^{[m;s]} + \Phi_{\text{add}}^{[m;s]},$$

where $\Phi_{\text{add}}^{[m;s]}$ is obtained from $\varphi_{\text{add}}^{[m;s]}$ by the change of variables $u = -\frac{z_1+z_2}{2}$, $v = \frac{z_1-z_2}{2}$. Then Theorem 1.11 implies

Theorem 1.12. *The function $\tilde{\Phi}^{[m;s]}$ has the following modular and elliptic transformation properties:*

- (1) $\tilde{\Phi}^{[m;s]}|_S = \tilde{\Phi}^{[m;s]}$, i.e. $\tilde{\Phi}^{[m;s]} \left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t \right) = \tau e^{\frac{2\pi i(m+1)z_1 z_2}{\tau}} \tilde{\Phi}^{[m;s]}(\tau, z_1, z_2, t)$.
- (2) $\tilde{\Phi}^{[m;s]}(\tau + 1, z_1, z_2, t) = \tilde{\Phi}^{[m;s]}(\tau, z_1, z_2, t)$.
- (3) $\tilde{\Phi}^{[m;s]}(\tau, z_1 + a, z_2 + b, t) = \tilde{\Phi}^{[m;s]}(\tau, z_1, z_2, t)$ if $a, b \in \mathbb{Z}$.
- (4) $\tilde{\Phi}^{[m;s]}(\tau, z_1 + a\tau, z_2 + b\tau, t) = q^{-(m+1)ab} e^{-2\pi i(m+1)(bz_1 + az_2)} \tilde{\Phi}^{[m;s]}(\tau, z_1, z_2, t)$ if $a, b \in \mathbb{Z}$.

□

Remark 1.13. After replacing (τ, z_1, z_2, t) by $(\frac{\tau}{M}, \frac{z_1}{M}, \frac{z_2}{M}, t)$, where $M \neq 0$, Theorem 1.12 (1) gives:

$$\tilde{\Phi}^{[m;s]} \left(\frac{\tau}{M}, \frac{z_1}{M}, \frac{z_2}{M}, t \right) = \frac{M}{\tau} \tilde{\Phi}^{[m;s]} \left(-\frac{M}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau M} \right).$$

2 Modular transformation formulae for modified normalized characters of admissible $\widehat{sl}_{2|1}$ -modules.

Recall an explicit description of principal admissible weights Λ of an affine Lie superalgebra $\widehat{\mathfrak{g}}$ (see [KW], Section 3).

Let M be a positive integer and let $S_M = \{(M-1)\delta + \alpha_0, \alpha_1, \dots, \alpha_\ell\}$, where $\{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$ is the set of simple roots of $\widehat{\mathfrak{g}}$ and δ is the primitive imaginary root. Let Λ^0 be a partially integrable weight of level m . Then all principal admissible weights, associated to the pair (M, Λ^0) are obtained as follows. Let \mathfrak{h} be the Cartan subalgebra of $\widehat{\mathfrak{g}}$ and W the (finite) Weyl group. Let $\beta \in \mathfrak{h}^*$ and $y \in W$ be such that the set $S := t_\beta y(S_M)$ lies in the subset of positive roots of $\widehat{\mathfrak{g}}$. Such subsets are called simple. All principal admissible weights with respect to a simple subset S , associated to the pair (M, Λ^0) are of the form (up to adding a multiple of δ) :

$$(2.1) \quad \Lambda = (t_\beta y)(\Lambda^0 - (M-1)(K + h^\vee)\Lambda_0 + \widehat{\rho}) - \widehat{\rho},$$

and all of them have level

$$(2.2) \quad k = \frac{m + h^\vee}{M} - h^\vee,$$

where h^\vee is the dual Coxeter number.

For the Lie superalgebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_{2|1}$ the dual Coxeter number $h^\vee = 1$ and $\widehat{\rho} = \Lambda_0$. There are two kinds of simple subsets:

$$S_{k_1, k_2}^+ = \{k_0\delta + \alpha_0, k_1\delta + \alpha_1, k_2\delta + \alpha_2\}, \quad M = k_0 + k_1 + k_2 + 1, \quad k_i \in \mathbb{Z}_{\geq 0}, \quad y = 1, \quad \beta = -k_1\alpha_2 - k_2\alpha_1;$$

$$S_{k_1, k_2}^- = \{k_0\delta - \alpha_0, k_1\delta - \alpha_1, k_2\delta - \alpha_2\}, \quad M = k_0 + k_1 + k_2 - 1, \quad k_i \in \mathbb{Z}_{> 0}, \quad y = r_{\alpha_1 + \alpha_2}, \quad \beta = k_1\alpha_2 + k_2\alpha_1.$$

By (2.1) all principal admissible weights with respect to S_{k_1, k_2}^+ and S_{k_1, k_2}^- , associated to the pair $(M, \Lambda^0 = \Lambda_{m; s})$ are respectively:

$$\Lambda_{k_1, k_2}^{[s]+} = k\Lambda_0 - (k+1)(k_2 - \frac{s}{k+1})\alpha_1 - (k+1)k_1\alpha_2 - (k+1)k_1(k_2 - \frac{s}{k+1})\delta;$$

$$\Lambda_{k_1, k_2}^{[s]-} = k\Lambda_0 + (k+1)k_2\alpha_1 + (k+1)(k_1 - \frac{s}{k+1})\alpha_2 - (k+1)k_2(k_1 - \frac{s}{k+1})\delta,$$

and, by (2.2), the level of all of them is $k = \frac{m+1}{M} - 1$.

Recall that the normalized character and supercharacter ch_Λ^\pm of a $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ of level k is given by

$$(2.3) \quad ch_\Lambda^\pm = e^{-\left(\frac{|\Lambda + \widehat{\rho}|^2}{2(k+h^\vee)} - \frac{\text{sdim } \mathfrak{g}}{24}\right)\delta} ch_{L(\Lambda)}^\pm.$$

Formula (3.28) from [KW] (see [GK] for its proof) gives in the case of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_{2|1}$ the following expressions for the normalized characters and supercharacters of principal admissible modules, in terms of the functions $\Phi^{[m; s]}$:

Lemma 2.1. *Let Λ be a principal admissible weight of level $k = \frac{m+1}{M} - 1$ for $\widehat{\mathfrak{sl}}_{2|1}$, associated to the pair $(M, \Lambda_{m; s})$. Then the normalized supercharacters ch_Λ^- are given by the following formulae (where \widehat{R}^- is the affine $\widehat{\mathfrak{sl}}_{2|1}$ superdenominator):*

$$\Lambda = \Lambda_{k_1, k_2}^{[s]+} : (\widehat{R}^- ch_\Lambda^-)(\tau, z_1, z_2, t) = q^{\frac{m+1}{M}k_1k_2} e^{\frac{2\pi i(m+1)}{M}(k_2z_1 + k_1z_2)} \Phi^{[m; s]}(M\tau, z_1 + k_1\tau, z_2 + k_2\tau, \frac{t}{M});$$

$$\begin{aligned} \Lambda = \Lambda_{k_1, k_2}^{[s]-} : (\widehat{R}^- ch_\Lambda^-)(\tau, z_1, z_2, t) = & -q^{\frac{m+1}{M}(M-k_1)(M-k_2)} e^{\frac{2\pi i(m+1)}{M}((M-k_2)z_1 + (M-k_1)z_2)} \\ & \times \Phi^{[m; s]}(M\tau, z_1 + (M-k_1)\tau, z_2 + (M-k_2)\tau, \frac{t}{M}). \end{aligned}$$

□

In order to derive the modular transformation formulae for the modified admissible $\widehat{\mathfrak{sl}}_{2|1}$ -characters we need the following theorem.

Theorem 2.2. *Let M be a positive integer and let m be a non-negative integer, such that $\gcd(M, 2m+2) = 1$ if $m > 0$. Then*

$$\begin{aligned} & \tilde{\Phi}^{[m; s]} \left(-\frac{M}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau M} \right) \\ &= \frac{\tau}{M} \sum_{j, k \in \mathbb{Z}/M\mathbb{Z}} q^{\frac{m+1}{M}jk} e^{\frac{2\pi i(m+1)}{M}(kz_1 + jz_2)} \tilde{\Phi}^{[m; s]}(M\tau, z_1 + j\tau, z_2 + k\tau, t). \end{aligned}$$

(By Theorem 1.12(4), each term in the RHS depends only on $j, k \pmod{M}$.) \square

Given coprime positive integers p and q , for each integer $n \in [s+1, s+q]$ there exist unique integers $n' \in [s+1, s+p]$ and b_n , such that

$$(2.4) \quad n = n'q + b_n p.$$

Furthermore, the set

$$(2.5) \quad I_{q,p}^{[s]} := \{b_n \mid n \in [s+1, s+q]\}$$

consists of q distinct integers. Any $n \in \mathbb{Z}$ can be uniquely represented in the form (2.4), where $n' \in \mathbb{Z}$ and $b_n \in I_{q,p}^{[s]}$, and this decomposition has the following properties :

- (i) $n \geq s+1$ iff $n' \geq s+1$;
- (ii) if $j, j_0 \in \mathbb{Z}$ are such that $j \equiv qj_0 \pmod{p}$, then $n \equiv \pm j \pmod{p}$ iff $n' \equiv \pm j_0 \pmod{p}$.

We shall apply this setup to $p = 2m+2$, $q = M$, and let $I^{[s]} = I_{M, 2m+2}^{[s]}$. The same proof as that of Lemma 6.2 (a) in [KW] (except that we expand $\Phi_1^{[m; s]}(\tau, -z_2, -z_1)$ in the domain $\text{Im } z_2 > 0$) gives the following result.

Lemma 2.3. *Let M be a positive integer and m be a non-negative integer, such that $\gcd(M, 2m+2) = 1$. Then*

$$\begin{aligned} & \Phi^{[m; s]} \left(\frac{\tau}{M}, \frac{z_1}{M}, \frac{z_2}{M}, t \right) \\ &= \sum_{\substack{0 \leq a < M \\ b \in I^{[s]}}} e^{\frac{2\pi i(m+1)}{M}(az_1 + (a+2b)z_2)} q^{\frac{m+1}{M}a(a+2b)} \Phi^{[m; s]}(M\tau, z_1 + (a+2b)\tau, z_2 + a\tau, t). \end{aligned}$$

\square

Remark 2.4. Since Remark 6.6 from [KW] holds for arbitrary s , we see that Lemma 2.3 holds if $\gcd(M, m+1) = 1$, the set $I^{[s]}$ is replaced by the set $I_{M, m+1}^{[s]}$, and $2b$ is replaced in each summand by b .

The proof of the following lemma is the same as that of Lemma 6.3 from [KW].

Lemma 2.5. *Let M and m be as in Theorem 2.2. For each integer $j \in [-s, -s+2m+1]$ take the unique integer j_0 in the same interval, such that $j \equiv Mj_0 \pmod{2m+2}$. Then*

$$(1) \quad R_{j; m+1} \left(\frac{\tau}{M}, \frac{v}{M} \right) = \sum_{b \in I^{[s]}} q^{-\frac{m+1}{M}b^2} e^{-\frac{4\pi i(m+1)}{M}bv} R_{j_0; m+1}(M\tau, v + b\tau).$$

$$(2) \Theta_{\pm j, m+1} \left(\frac{\tau}{M}, \frac{2u}{M} \right) = \sum_{a \in I^{[s]}} q^{\frac{m+1}{M} a^2} e^{\frac{4\pi i(m+1)}{M} au} \Theta_{\pm j_0, m+1}(M\tau, 2(u + a\tau)).$$

□

From the definition (1.13) of $\varphi_{\text{add}}^{[m; s]}$ and Lemmas 1.6 (2) and 2.5 we obtain

Lemma 2.6. *Let M and m be as in Lemma 2.3. Then*

$$\varphi_{\text{add}}^{[m; s]} \left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, t \right) = \sum_{a, b \in I^{[s]}} q^{\frac{m+1}{M}(a^2 - b^2)} e^{\frac{4\pi i(m+1)}{M}(au - bv)} \varphi_{\text{add}}^{[m; s]}(M\tau, u + a\tau, v + b\tau, t).$$

□

Proof. of Theorem 2.2. It follows from Lemma 1.1 (5) and Theorem 1.11 (3) that each term in the RHS of the equation in Lemma 2.3 remains unchanged if we add to a or to b an integer multiple of M . Similarly, it follows from Theorem 1.12 (4) that each term in the RHS of the equation in Lemma 2.6 remains unchanged if we add to a or to b an integer multiple of M . Hence, by Remark 1.13, Theorem 2.2 follows from Lemmas 2.3 and 2.6 and, for $m = 0$, from Remark 2.4 and the observation that $\varphi_{\text{add}}^{[0; s]} = 0$. □

As in [KW], let $\xi = -\frac{1}{2}(\alpha_1 + \alpha_2)$ and consider the twisted normalized admissible characters and supercharacters $t_\xi ch_\Lambda^\pm$ and their denominators and superdenominators $t_\xi \widehat{R}^\pm$. As in [KW], we shall use the following notation :

$$(2.6) \quad ch_{\Lambda; 0}^{(0)} = ch_\Lambda^-, \quad ch_{\Lambda; 0}^{(\frac{1}{2})} = ch_\Lambda^+, \quad ch_{\Lambda; \frac{1}{2}}^{(0)} = t_\xi ch_\Lambda^-, \quad ch_{\Lambda; \frac{1}{2}}^{(\frac{1}{2})} = t_\xi ch_\Lambda^+,$$

and similarly for their (super)denominators :

$$(2.7) \quad \widehat{R}_0^{(0)} = \widehat{R}^-, \quad \widehat{R}_0^{(\frac{1}{2})} = \widehat{R}^+, \quad \widehat{R}_{\frac{1}{2}}^{(0)} = t_\xi \widehat{R}^-, \quad \widehat{R}_{\frac{1}{2}}^{(\frac{1}{2})} = t_\xi \widehat{R}^+.$$

Recall that the untwisted and twisted (super)denominators for $\widehat{s\ell}_{2|1}$ can be conveniently written in terms of Jacobi's four theta functions of degree 2 (see (7.4) in [KW]) :

$$(2.8) \quad \widehat{R}_{\varepsilon'}^{(\varepsilon)}(\tau, z_1, z_2, t) = (-1)^{2\varepsilon(1-2\varepsilon')} i e^{2\pi i t} \frac{\eta(\tau)^3 \vartheta_{11}(\tau, z_1 + z_2)}{\vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, z_1) \vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, z_2)},$$

where $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$.

As in the case $s = 0$ in [KW], Section 7, the non-twisted and twisted normalized admissible (super)characters can be written in terms of the following functions :

$$(2.9) \quad \Psi_{a, b; \varepsilon'}^{[M, m, s; \varepsilon]}(\tau, z_1, z_2, t) = q^{\frac{(m+1)ab}{M}} e^{\frac{2\pi i(m+1)}{M}(bz_1 + az_2)} \Phi^{[m; s]}(M\tau, z_1 + a\tau + \varepsilon, z_2 + b\tau + \varepsilon, \frac{t}{M}),$$

where $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$, and $a, b \in \varepsilon' + \mathbb{Z}$. Namely, Lemma 2.1, along with Lemma 1.1 (5) and (6), implies the following character formulae.

Proposition 2.7. (1) If $\Lambda = \Lambda_{j,k}^{[s]+}$, where $j, k \in \mathbb{Z}$, $0 \leq j, k, j+k \leq M-1$, then

$$\left(\widehat{R}_{\varepsilon'}^{(\varepsilon)} ch_{\Lambda; \varepsilon'}^{(\varepsilon)} \right) (\tau, z_1, z_2, t) = \Psi_{j+\varepsilon', k+\varepsilon', \varepsilon'}^{[M, m, s; \varepsilon]} (\tau, z_1, z_2, t).$$

(2) If $\Lambda = \Lambda_{j,k}^{[s]-}$, where $j, k \in \mathbb{Z}$, $1 \leq j, k, j+k \leq M$, then

$$\left(\widehat{R}_{\varepsilon'}^{(\varepsilon)} ch_{\Lambda; \varepsilon'}^{(\varepsilon)} \right) (\tau, z_1, z_2, t) = -\Psi_{M+\varepsilon'-j, M+\varepsilon'-k; \varepsilon'}^{[M, m, s; \varepsilon]} (\tau, z_1, z_2, t).$$

□

As in [KW], introduce the modification $\widetilde{\Psi}_{a,b; \varepsilon'}^{[M, m, s; \varepsilon]}$ of the function $\Psi_{a,b; \varepsilon'}^{[M, m, s; \varepsilon]}$ by replacing $\Phi^{[m; s]}$ in the RHS of (2.9) by its modification $\widetilde{\Phi}^{[m; s]}$. The following theorem is immediate by Theorem 2.2 and Theorem 1.12 (2).

Theorem 2.8. Let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$, and let $j, k \in \varepsilon' + \mathbb{Z}/M\mathbb{Z}$. Then

$$\widetilde{\Psi}_{j, k; \varepsilon'}^{[M, m, s; \varepsilon]} \left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau} \right) = \frac{\tau}{M} \sum_{a, b \in \varepsilon + \mathbb{Z}/M\mathbb{Z}} e^{-\frac{2\pi i(m+1)}{M}(ak+bj)} \widetilde{\Psi}_{a, b; \varepsilon}^{[M, m, s; \varepsilon']} (\tau, z_1, z_2, t);$$

$$\widetilde{\Psi}_{j, k; \varepsilon'}^{[M, m, s; \varepsilon]} (\tau + 1, z_1, z_2, t) = e^{\frac{2\pi i(m+1)}{M}jk} \widetilde{\Psi}^{[M, m, s; |\varepsilon - \varepsilon'|]} (\tau, z_1, z_2, t).$$

□

As in [KW], in order to state a unified modular transformation formula for modified normalized admissible characters, it is convenient, for each $s \in \mathbb{Z}$, $0 \leq s \leq m$, to introduce the following notations :

$$ch_{j+\varepsilon', k+\varepsilon'; \varepsilon'}^{[M, m, s; \varepsilon]} := ch_{\Lambda_{j,k}^{[s]+}; \varepsilon'}^{(\varepsilon)}, \quad ch_{M+\varepsilon'-j, M+\varepsilon'-k; \varepsilon'}^{[M, m, s; \varepsilon]} = -ch_{\Lambda_{j,k}^{[s]-}; \varepsilon'}^{(\varepsilon)}.$$

Then

$$\{ch_{j+\varepsilon', k+\varepsilon'; \varepsilon'}^{[M, m, s; \varepsilon]} | j, k \in \mathbb{Z}, 0 \leq j, k \leq M-1\}$$

is (up to a sign) precisely the set of all admissible characters (resp. supercharacters), associated to the pair $(M, \Lambda_m; s)$, if $\varepsilon' = 0$ and $\varepsilon = \frac{1}{2}$ (resp. $\varepsilon = 0$), and it is the set of all twisted admissible characters (resp. supercharacters), associated to the pair $(M, \Lambda_m; s)$, if $\varepsilon' = \frac{1}{2}$ and $\varepsilon = \frac{1}{2}$ (resp. $\varepsilon = 0$).

In view of these observations, introduce the *modified* normalized characters ($\varepsilon = \frac{1}{2}, \varepsilon' = 0$), supercharacters ($\varepsilon = 0, \varepsilon' = 0$), twisted characters ($\varepsilon = \frac{1}{2}, \varepsilon' = \frac{1}{2}$), and twisted supercharacters ($\varepsilon = 0, \varepsilon' = \frac{1}{2}$), letting

$$(2.10) \quad ch_{j, k; \varepsilon'}^{[M, m, s; \varepsilon]} (\tau, z_1, z_2, t) = \frac{\widetilde{\Psi}_{j, k; \varepsilon'}^{[M, m, s; \varepsilon]} (\tau, z_1, z_2, t)}{\widehat{R}_{\varepsilon'}^{(\varepsilon)} (\tau, z_1, z_2, t)}, \quad j, k \in \varepsilon' + \mathbb{Z}, 0 \leq j, k < M.$$

Then from formulae (7.5) and (7.6) in [KW] and Theorem 2.8 we obtain the following theorem.

Theorem 2.9. *Let M be a positive integer and let m be a non-negative integer, such that $\gcd(M, 2m+2) = 1$ if $m > 0$. One has the following modular transformation formulae for each $s \in \mathbb{Z}$, $0 \leq s \leq m$ ($\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$; $j, k \in \varepsilon' + \mathbb{Z}$, $0 \leq j, k < M$) :*

$$\begin{aligned} & \sim_{[M, m, s; \varepsilon]} ch_{j, k; \varepsilon'} \left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau} \right) \\ &= (-1)^{4\varepsilon\varepsilon'} \frac{1}{M} \sum_{\substack{a, b \in \varepsilon + \mathbb{Z} \\ 0 \leq a, b < M}} e^{-\frac{2\pi i(m+1)}{M}(ak+bj)} \sim_{[M, m, s; \varepsilon']} ch_{a, b; \varepsilon}(\tau, z_1, z_2, t); \\ & \sim_{[M, m, s; \varepsilon]} ch_{j, k; \varepsilon'}(\tau + 1, z_1, z_2, t) = e^{2\pi i \frac{(m+1)jk}{M} - \pi i \varepsilon'} \sim_{[M, m, s; |\varepsilon - \varepsilon'|]} ch_{j, k; \varepsilon'}(\tau, z_1, z_2, t). \end{aligned}$$

As in [KW], Section 9, in order to perform the quantum Hamiltonian reduction for $\widehat{sl}_{2|1}$, we need a different choice of the twisting vector : $\xi' = \frac{1}{2}(\alpha_1 - \alpha_2)$. Then the obtained twisted denominators differ only by a sign, and only when $\varepsilon = \varepsilon' = \frac{1}{2}$, so we keep for them the same notation. Their modular transformation formulae also differ by a sign (see equations (9.16) and (9.17) in [KW]). The character formulae differ (in the twisted case) from those, given by Proposition 2.7, and are as follows (cf Proposition 9.2 from [KW]).

Proposition 2.10. (a) *If $\Lambda = \Lambda_{j, k}^{[s]^+}$, where $j, k \in \mathbb{Z}$, $0 \leq j, k, j+k \leq M-1$, then*

$$\left(\widehat{R}_{\varepsilon'}^{(\varepsilon)} ch_{\Lambda; \varepsilon'}^{(\varepsilon)} \right) (\tau, z_1, z_2, t) = \Psi_{j+\varepsilon', k-\varepsilon'; \varepsilon'}^{[M, m, s; \varepsilon]}(\tau, z_1, z_2, t).$$

(b) *If $\Lambda = \Lambda_{j, k}^{[s]^-}$, where $j, k \in \mathbb{Z}$, $1 \leq j, k, j+k \leq M$, then*

$$\left(\widehat{R}_{\varepsilon'}^{(\varepsilon)} ch_{\Lambda; \varepsilon'}^{(\varepsilon)} \right) (\tau, z_1, z_2, t) = -\Psi_{M+\varepsilon'-j, M-\varepsilon'-k; \varepsilon'}^{[M, m, s; \varepsilon]}(\tau, z_1, z_2, t).$$

□

3 Modular transformation formulae for modified characters of admissible $N = 2$ modules

Recall (see [KW], Section 9) that the quantum Hamiltonian reduction associates to a principal admissible $\widehat{sl}_{2|1}$ -module $L(\Lambda)$ of level $\frac{m+1}{M} - 1$, where $m, M \in \mathbb{Z}$, $m \geq 0$, $M \geq 1$, and $\gcd(M, 2m+2) = 1$ if $m > 0$, a module $H(\Lambda)$ over the $N = 2$ superconformal algebra with central charge

$$(3.1) \quad c = 3\left(1 - \frac{2m+2}{M}\right).$$

In [KW] we considered the quantum Hamiltonian reduction for the principal admissible $\widehat{sl}_{2|1}$ -modules $L(\Lambda)$, such that $\Lambda^0 = \Lambda_{m; s}$ with $s = 0$. Here we consider the case of arbitrary $s \in \mathbb{Z}$, $0 \leq s \leq m$, so we may assume that $m \geq 1$.

Recall that $H(\Lambda) = 0$ iff $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$ and that $H(\Lambda)$ is irreducible otherwise. If $M = 1$, then a principal admissible $\widehat{sl}_{2|1}$ -module $L(\Lambda)$ is partially integrable, hence $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$ and therefore $H(\Lambda) = 0$. If $M > 1$, it follows from the formulas for the principal admissible weights $\Lambda = \Lambda_{k_1, k_2}^{[s]\pm}$ in Section 2, that $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$ iff $k_0 = 0$ (recall that $k_0 = M - k_1 - k_2 \mp 1$).

Thus, in what follows we may assume that $M \geq 2$ and $k_0 > 0$. Then $H(\Lambda)$ is an irreducible module over the NS type $N = 2$ superconformal algebra. Using the formulas for principal admissible weights in Section 2 and formulas (9.12) and (9.13) from [KW] for the lowest energy h_Λ and spin s_Λ of $H(\Lambda)$, we obtain for them the following explicit formulas :

$$(3.2) \quad h_{\Lambda_{k_1, k_2}^{[s]\pm}} = \frac{m+1}{M} \left((k_1 \pm \frac{1}{2})(k_2 \pm \frac{1}{2}) - \frac{1}{4} \right) - s \left(k_{(3\mp 1)/2} \pm \frac{1}{2} \right),$$

$$(3.3) \quad s_{\Lambda_{k_1, k_2}^{[s]\pm}} = \pm \frac{m+1}{M} (k_2 - k_1) - s.$$

Recall that the twisted quantum Hamiltonian reduction associates to a principal admissible twisted $\widehat{sl}_{2|1}$ -module $L^{\text{tw}}(\Lambda)$, where $\Lambda = \Lambda_{k_1, k_2}^{[s]\pm}$ has level $\frac{m+1}{M} - 1$, a module $H^{\text{tw}}(\Lambda)$ over the Ramond type $N = 2$ superconformal algebra with central charge (3.1). As before, $\gcd(M, 2m+2) = 1$ and we may assume that $m \geq 1$, $M \geq 2$; also the module $H^{\text{tw}}(\Lambda) = 0$ iff $k_0 = 0$, and it is irreducible otherwise. Again, it is easy to compute the corresponding characteristic numbers :

$$(3.4) \quad h_{\Lambda_{k_1, k_2}^{[s] +}}^{\text{tw}} = \frac{m+1}{M} k_2 (k_1 + 1) - \frac{m+1}{4M} + \frac{1}{8} - s(k_1 + 1),$$

$$(3.5) \quad h_{\Lambda_{k_1, k_2}^{[s] -}}^{\text{tw}} = \frac{m+1}{M} k_2 (k_1 - 1) - \frac{m+1}{4M} + \frac{1}{8} - s k_2,$$

$$(3.6) \quad s_{\Lambda_{k_1, k_2}^{[s]\pm}}^{\text{tw}} = \pm \frac{m+1}{M} (k_2 - k_1) - \frac{m+1}{M} + \frac{1}{2} - s.$$

Note that for each s the set of all principal admissible weights (with $k_0 > 0$) of level $\frac{m+1}{M} - 1$ is a union of two sets :

$$\begin{aligned} \mathcal{A}^+ &= \left\{ \Lambda_{k_1, k_2}^{[s] +} \mid k_1, k_2 \in \mathbb{Z}_{\geq 0}, k_1 + k_2 \leq M - 2 \right\}, \\ \mathcal{A}^- &= \left\{ \Lambda_{k_1, k_2}^{[s] -} \mid k_1, k_2 \in \mathbb{Z}_{\geq 1}, k_1 + k_2 \leq M \right\}. \end{aligned}$$

Note also that we have the following bijective map :

$$\nu : \mathcal{A}^+ \rightarrow \mathcal{A}^-, \quad \nu \left(\Lambda_{k_1, k_2}^{[s] +} \right) = \Lambda_{k_2+1, k_1+1}^{[s] -}.$$

It is immediate to see from (3.2), (3.3) (resp. (3.4)-(3.6)) that

$$h_{\Lambda_{k_1, k_2}^{[s] +}} = h_{\nu(\Lambda_{k_1, k_2}^{[s] +})}, \quad s_{\Lambda_{k_1, k_2}^{[s] +}} = s_{\nu(\Lambda_{k_1, k_2}^{[s] +})} \text{ for } \Lambda_{k_1, k_2}^{[s] +} \in \mathcal{A}^+,$$

and the same holds for h^{tw} and s^{tw} . Since the irreducible $N = 2$ modules are uniquely determined by their central charges and the characteristic numbers, the sets \mathcal{A}^+ and \mathcal{A}^- correspond to the same set of irreducible $N = 2$ modules. Hence, it suffices to consider only the highest weights $\Lambda_{k_1, k_2}^{[s] +} \in \mathcal{A}^+$.

In order to compute the characters and supercharacters of the corresponding $N = 2$ modules $H(\Lambda)$ and $H^{\text{tw}}(\Lambda)$, we use formulae (9.4) and (9.8) from [KW].

First, we have formula (9.21) from [KW] for the $N = 2$ normalized denominators :

$$R_{\varepsilon'}^{(2)}(\tau, z) = \frac{(-1)^{(1-2\varepsilon)(1-2\varepsilon')} \eta(\tau)^3}{\vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, z)},$$

where $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$. Here the superscript ε refers to the denominator if $\varepsilon = \frac{1}{2}$ and to the superdenominator if $\varepsilon = 0$, and the subscript ε' refers to the Neveu-Schwarz sector if $\varepsilon' = \frac{1}{2}$, and to the Ramond sector if $\varepsilon' = 0$.

It is convenient to introduce the following two reindexings of the set $\mathcal{A}^{(+)}$:

$$\begin{aligned} \mathcal{A}_{NS} &= \left\{ \Lambda_{j,k} = \Lambda_{j-\frac{1}{2}, k-\frac{1}{2}}^{[s]+} \mid j, k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}, j+k \leq M-1 \right\}, \\ \mathcal{A}_R &= \left\{ \Lambda_{j,k} = \Lambda_{j-1, k}^{[s]+} \mid j \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 0}, j+k \leq M-1 \right\}. \end{aligned}$$

We let

$$H_{NS}(\Lambda_{j,k}) = H\left(\Lambda_{j-\frac{1}{2}, k-\frac{1}{2}}^{[s]+}\right), \Lambda_{j,k} \in \mathcal{A}_{NS}; \quad H_R(\Lambda_{j,k}) = H\left(\Lambda_{j-1, k}^{[s]+}\right), \Lambda_{j,k} \in \mathcal{A}_R.$$

It follows from (3.2)-(3.6) that the lowest energy and the spin of these $N = 2$ modules (with central charge (3.1)) are as follows :

$$(3.7) \quad h_{jk}^{NS} = \frac{m+1}{M}jk - \frac{m+1}{4M} - sj, \quad s_{jk}^{NS} = \frac{m+1}{M}(k-j) - s;$$

$$(3.8) \quad h_{jk}^R = \frac{m+1}{M}jk - \frac{m+1}{4M} + \frac{1}{8} - sj, \quad s_{jk}^R = \frac{m+1}{M}(k-j) + \frac{1}{2} - s.$$

Introduce the following notation for the characters and supercharacters of these $N = 2$ modules :

$$\begin{aligned} ch_{j,k;\frac{1}{2}}^{N=2[M,m,s;\varepsilon]}(\tau, z) &= ch_{H_{NS}(\Lambda_{j,k})}^{\pm}(\tau, z), \Lambda_{j,k} \in \mathcal{A}_{NS}; \\ ch_{j,k;0}^{N=2[M,m,s;\varepsilon]}(\tau, z) &= ch_{H_R(\Lambda_{j,k})}^{\pm}(\tau, z), \Lambda_{j,k} \in \mathcal{A}_R. \end{aligned}$$

Formulae (9.4) and (9.8) from [KW] imply the following expressions for these characters :

$$(3.9) \quad \left(R_{\varepsilon'}^{(2)} ch_{j,k;\varepsilon'}^{N=2[M,m,s;\varepsilon]} \right) (\tau, z) = \Psi_{j,k;\varepsilon'}^{[M,m,s;\varepsilon]}(\tau, -z, z, 0),$$

where the functions $\Psi_{j,k;\varepsilon}^{[M,m,s;\varepsilon]}(\tau, z_1, z_2, t)$ are defined by (2.9) and $j, k \in \varepsilon' + \mathbb{Z}_{\geq 0}$, subject to restrictions $j+k \leq M-1$, $j > 0$.

Introduce the *modified* $N = 2$ characters and supercharacters, letting

$$\left(R_{\varepsilon'}^{(2)} \sim ch_{j,k;\varepsilon'}^{N=2[M,m,s;\varepsilon]} \right) (\tau, z) = \tilde{\Psi}_{j,k;\varepsilon}^{[M,m,s;\varepsilon]}(\tau, -z, z, 0),$$

where the modification $\tilde{\Psi}$ of Ψ was introduced in Section 2 (before Theorem 2.8). Theorem 2.8 along with Lemma 9.3 from [KW] give the following modular transformation properties of the modified $N = 2$ characters and supercharacters, in the Neveu-Schwarz and Ramond sectors.

Theorem 3.1. Let $M \in \mathbb{Z}_{\geq 2}$ and let $m \in \mathbb{Z}_{\geq 0}$ be such that $\gcd(M, 2m+2) = 1$ if $m > 0$. Let $c = 3(1 - \frac{2m+2}{M})$. Let $s \in \mathbb{Z}$ be such that $0 \leq s \leq m$. Let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$, and let

$$\Omega_{\varepsilon}^{(M)} = \{(j, k) \in (\varepsilon + \mathbb{Z}_{\geq 0})^2 \mid j + k \leq M - 1, j > 0\}.$$

Then we have the following modular transformation formulae for $ch_{j,k;\varepsilon'}^{\sim N=2[M,m,s;\varepsilon]}$, $(j, k) \in \Omega_{\varepsilon'}^{(M)}$:

$$ch_{j,k;\varepsilon'}^{\sim N=2[M,m,s;\varepsilon]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{\frac{\pi ic}{6\tau} z^2} \sum_{(a,b) \in \Omega_{\varepsilon}^{(M)}} S_{(j,k),(a,b)}^{[M,m,\varepsilon,\varepsilon']} ch_{a,b;\varepsilon}^{\sim N=2[M,m,s;\varepsilon']}(\tau, z),$$

where

$$S_{(j,k),(a,b)}^{[M,m,\varepsilon,\varepsilon']} = (-i)^{(1-2\varepsilon)(1-2\varepsilon')} \frac{2}{M} e^{\frac{\pi i(m+1)}{M}(j-k)(a-b)} \sin \frac{m+1}{M}(j+k)(a+b)\pi;$$

$$ch_{j,k;\varepsilon'}^{\sim N=2[M,m,s;\varepsilon]}(\tau+1, z) = e^{\frac{2\pi i(m+1)}{M}jk - \frac{\pi i\varepsilon'}{2}} ch_{j,k;\varepsilon'}^{\sim N=2[M,m,s;|\varepsilon-\varepsilon'|]}(\tau, z).$$

This theorem coincides with Theorem 9.4 in [KW] in the case $s = 0$. For an arbitrary $s \in [0, m]$ the proof is the same, and the modular transformation formulae are the same. Note that, unlike in the Lie algebra case, modular transformations do not mix characters with different s .

4 Transformation properties of the mock theta functions $\Phi^{[B;m]}$.

In this section we study modular and elliptic transformation properties of supercharacters $ch_{L(\Lambda)}^-$ of partially integrable highest weight modules $L(\Lambda)$ over the affine Lie superalgebra $\widehat{\mathfrak{g}}$, where $\mathfrak{g} = osp_{3|2}$. We choose the set of simple roots of $\widehat{\mathfrak{g}}$ to be $\widehat{\Pi} = \{\alpha_0, \alpha_1, \alpha_2\}$, where α_0 and α_2 are even and α_1 is odd, and the scalar products are :

$$(4.1) \quad (\alpha_0|\alpha_0) = 2, \quad (\alpha_1|\alpha_1) = 0, \quad (\alpha_2|\alpha_2) = -\frac{1}{2}, \quad (\alpha_0|\alpha_1) = -1, \quad (\alpha_1|\alpha_2) = \frac{1}{2}, \quad (\alpha_0|\alpha_2) = 0.$$

For the underlying finite-dimensional Lie superalgebra $osp_{3|2}$ we have: its set of positive even roots is $\Delta_{\bar{0},+} = \{\alpha_2, \theta\}$, where $\theta = 2(\alpha_1 + \alpha_2)$, its set of positive odd roots is $\Delta_{\bar{1},+} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$, hence the highest root is θ , the Weyl vector $\rho = -\frac{1}{2}\alpha_1$, and therefore, by (3.1) from [KW], the dual Coxeter number $h^\vee = \frac{1}{2}$. Furthermore, the Weyl group of \mathfrak{g} is $W = \{1, r_{\alpha_2}, r_\theta, r_{\alpha_2}r_\theta\}$, and also $\Delta_0^\# = \{\pm\theta\}$ (see (3.3) from [KW]) and $L^\# = \mathbb{Z}\theta$, since $(\theta|\theta) = 2$.

By the definition of a partially integrable module (see [KW], Definition 3.2), an $\widehat{osp}_{3|2}$ -module $L(\Lambda)$ is partially integrable iff $\Lambda = m\Lambda_0$, where m is a non-negative integer. The numerator $\widehat{R}^- ch_{L(m\Lambda_0)}^-$ of the supercharacter of $L(m\Lambda_0)$ is given by Conjecture 3.8 in [KW], proved in [GK].

In order to write down an explicit formula, note that

$$\widehat{\rho} = h^\vee \Lambda_0 + \rho = \frac{1}{2}\Lambda_0 - \frac{1}{2}\alpha_1, \quad \varepsilon^-(t_{n\theta}) = (-1)^n, \quad \varepsilon^-(r_{\alpha_2}) = -1, \quad \varepsilon^-(r_\theta) = 1.$$

We choose $T_0 = \{\alpha_1\}$. Then we get the following formula :

$$\widehat{R}^- ch_{L(m\Lambda_0)}^- = \sum_{w \in \widehat{W}^\#} \varepsilon^-(w) w \frac{e^{(m+\frac{1}{2})\Lambda_0 - \frac{1}{2}\alpha_1}}{1 - e^{-\alpha_1}},$$

where $\widehat{W}^\# = \langle 1, r_\theta \rangle \ltimes t_{L^\#} \subset \widehat{W}$. Explicitly:

$$(4.2) \quad \widehat{R}^- ch_{L(m\Lambda_0)}^- = e^{(m+\frac{1}{2})\Lambda_0} \times \sum_{j \in \mathbb{Z}} (-1)^j \left(\frac{e^{-(2m+1)j(\alpha_1+\alpha_2) - \frac{1}{2}\alpha_1} q^{(m+\frac{1}{2})j^2 + \frac{j}{2}}}{1 - e^{-\alpha_1} q^j} - \frac{e^{-(2m+1)j(\alpha_1+\alpha_2) - \frac{1}{2}(\alpha_1+2\alpha_2)} q^{(m+\frac{1}{2})j^2 + \frac{j}{2}}}{1 - e^{-(\alpha_1+2\alpha_2)} q^j} \right).$$

Introduce the following coordinates in the Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{osp}_{3|2}$:

$$(4.3) \quad h = 2\pi i(-\tau\Lambda_0 - z_1(\alpha_1 + 2\alpha_2) - z_2\alpha_1 + t\delta) := (\tau, z_1, z_2, t),$$

so that for $z = -z_1(\alpha_1 + 2\alpha_2) - z_2\alpha_1$ we have :

$$(4.4) \quad (z|z) = 2z_1z_2.$$

Let

$$\Phi^{[B;m]}(\tau, z_1, z_2, t) = \left(\widehat{R}^- ch_{L(m\Lambda_0)}^- \right) (h)$$

be the numerator of $ch_{L(m\Lambda_0)}^- = ch_{m\Lambda_0}^-$ in coordinates (4.3). By (4.2) we have:

$$(4.5) \quad \Phi^{[B;m]}(\tau, z_1, z_2, t) = \Phi_1^{[B;m]}(\tau, z_1, z_2, t) - \Phi_1^{[B;m]}(\tau, z_2, z_1, t),$$

where

$$(4.6) \quad \Phi_1^{[B;m]}(\tau, z_1, z_2, t) = e^{2\pi i(m+\frac{1}{2})t} \sum_{j \in \mathbb{Z}} (-1)^j \frac{e^{\pi i j(2m+1)(z_1+z_2) + \pi i z_1} q^{(m+\frac{1}{2})j^2 + \frac{j}{2}}}{1 - e^{2\pi i z_1} q^j}.$$

Let $\Phi_1^{[B;m]}(\tau, z_1, z_2) = \Phi_1^{[B;m]}(\tau, z_1, z_2, 0)$.

Recall the right action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ on the space of meromorphic functions in the domain $X = \{h \in \widehat{\mathfrak{h}}^* \mid \mathrm{Re} \delta(h) > 0 \text{ (or } \mathrm{Im} \tau > 0) \}$ for $\ell = 2 (= \mathrm{rank} osp_{3|2})$:

$$(4.7) \quad F|_A(\tau, z, t) = (c\tau + d)^{-1} F\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t - \frac{c(z|z)}{2(c\tau + d)}\right).$$

For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have, by (4.4):

$$(4.8) \quad F|_S(\tau, z_1, z_2, t) = \tau^{-1} F\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau}\right).$$

Note that the meromorphic function $\Phi_1^{[B;m]}(\tau, z_1, z_2)$, viewed as a function of z_1 , has poles at $z_1 \in \mathbb{Z} + \tau\mathbb{Z}$, all of them are simple and have residues :

$$\text{Res}_{z_1=n+j\tau} \Phi_1^{[B;m]}(\tau, z_1, z_2) = \frac{(-1)^{(j+1)(n+1)}}{2\pi i} e^{-\pi i j(2m+1)z_2}.$$

The function $(\Phi_1^{[B;m]}|_S)(\tau, z_1, z_2, t=0)$ has the same poles, all of them are simple, and have the same residues at these poles. Hence for the functions $\Phi^{[B;m]}$ and $\Phi_1^{[B;m]}$ we have an analogue of Lemma 1.1(0):

Lemma 4.1. *The functions $\Phi_1^{[B;m]} - \Phi_1^{[B;m]}|_S$ and $\Phi^{[B;m]} - \Phi^{[B;m]}|_S$ are holomorphic in the domain X . \square*

Lemma 4.2. *The functions $\Phi^{[B;m]}$ (and $\Phi_1^{[B;m]}$) satisfy the following properties :*

- (1) $\Phi^{[B;m]}(\tau, -z_1, -z_2, t) = -\Phi^{[B;m]}(\tau, z_1, z_2, t).$
- (2) $\Phi^{[B;m]}(\tau, z_1 + a, z_2 + b, t) = (-1)^a \Phi^{[B;m]}(\tau, z_1, z_2, t)$ if $a, b \in \mathbb{Z}$ have the same parity.
- (3) $\Phi^{[B;m]}(\tau, z_1 + j\tau, z_2 + j\tau, t) = (-1)^j e^{-\pi i j(2m+1)(z_1+z_2)} q^{-j^2(m+\frac{1}{2})} \Phi^{[B;m]}(\tau, z_1, z_2, t).$

Proof. The proof of (1) is straightforward by changing j to $-j$ in the RHS of (4.6). The proof of (2) and (3) is the same as that of Lemma 1.1 (2) and (5). \square

As in Section 1, we change the coordinates, letting

$$z_1 = u + v, \quad z_2 = u - v, \quad \text{i.e.} \quad u = \frac{z_1 + z_2}{2}, \quad v = \frac{z_1 - z_2}{2},$$

and denote $\varphi^{[B;m]}(\tau, u, v, t) = \Phi^{[B;m]}(\tau, z_1, z_2, t)$, and similarly for $\varphi_1^{[B;m]}$ and $\Phi_1^{[B;m]}$.

Then we get :

$$\varphi^{[B;m]}(\tau, u, v, t) = \varphi_1^{[B;m]}(\tau, u, v, t) - \varphi_1^{[B;m]}(\tau, u, -v, t),$$

where

$$(4.9) \quad \varphi_1^{[B;m]}(\tau, u, v, t) = e^{2\pi i(m+\frac{1}{2})t} \sum_{j \in \mathbb{Z}} (-1)^j \frac{e^{2\pi i j(2m+1)u + \pi i(u+v)} q^{(m+\frac{1}{2})j^2 + \frac{j}{2}}}{1 - e^{2\pi i(u+v)} q^j}.$$

In order to state an analogue of Lemma 1.2 (3), (4), we shall need the following ‘‘alternate’’ analogues of the theta functions $\Theta_{j,m}$, defined by (1.2) :

$$(4.10) \quad \Theta_{j,m}^-(\tau, z, t) = e^{2\pi i m t} \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi i m z(n + \frac{j}{2m})} q^{m(n + \frac{j}{2m})^2}.$$

Here $m \in \frac{1}{4}\mathbb{Z}_{\geq 1}$ is the degree and $j \in \frac{1}{2}\mathbb{Z}$. These are holomorphic functions in the domain X_0 . As before, let $\Theta_{j,m}^-(\tau, z) = \Theta_{j,m}^-(\tau, z, 0)$. We obviously have :

$$(4.11) \quad \Theta_{j+2ma,m}^-(\tau, z) = (-1)^a \Theta_{j,m}^-(\tau, z) \text{ if } a \in \mathbb{Z}.$$

In particular, $\Theta_{j,m}^-$ depends only on $j \bmod 4m$.

The functions $\Theta_{j,m}^-(\tau, z)$ satisfy the following modular transformation properties.

Proposition 4.3. *Let $m \in \frac{1}{4}\mathbb{Z}_{\geq 1}$. Then we have*

- (1) $\Theta_{j,m}^{-}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \left(\frac{-i\tau}{2m}\right)^{\frac{1}{2}} e^{\frac{\pi i m z^2}{2\tau}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{4m-1} e^{\frac{-\pi i j k}{2m}} \Theta_{\frac{k}{2},m}^{-}(\tau, z)$, provided that $j \in \frac{1}{2} + \mathbb{Z}$.
- (2) If $j \in \mathbb{Z}$, then the formula in (1) holds if we replace $\Theta_{\frac{k}{2},m}^{-}$ in the RHS by $\Theta_{\frac{k}{2},m}$.
- (3) If $j \in \frac{1}{2} + \mathbb{Z}$, then the formula in (1) holds if we replace $\Theta_{j,m}^{-}$ (resp $\Theta_{\frac{k}{2},m}^{-}$) in the LHS (resp. RHS) by $\Theta_{j,m}^{-} + \Theta_{-j,m}^{-}$ (resp. by $\Theta_{\frac{k}{2},m}^{-} + \Theta_{-\frac{k}{2},m}^{-}$).
- (4) If $m \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}$, then

$$\Theta_{\pm(j+\frac{1}{2}),m+\frac{1}{2}}^{-}(\tau+1, z) = e^{\frac{\pi i}{2m+1}(j+\frac{1}{2})^2} \Theta_{\pm(j+\frac{1}{2}),m+\frac{1}{2}}^{-}(\tau, z).$$

Proof. Formulae (1)-(3) follow from (A.5) in the Appendix of [KW], using the obvious identity

$$\Theta_{j,m}^{\pm}(\tau, z) = \Theta_{2j,4m}(\tau, \frac{z}{2}) \pm \Theta_{2j-4m,4m}(\tau, \frac{z}{2}).$$

Formula (4) is a special case of Proposition A.3 from the Appendix to the present paper. \square

Lemma 4.4. *The functions $\varphi_1^{[B;m]}$ and $\varphi^{[B;m]}$ satisfy the following properties :*

- (1) $\varphi_1^{[B;m]}(\tau, u+a, v+b, t) = (-1)^{a+b} \varphi_1^{[B;m]}(\tau, u, v, t)$ if $a, b \in \mathbb{Z}$, and the same holds for the functions $\varphi^{[B;m]}$.
- (2) $\varphi_1^{[B;m]}(\tau, u, v, t) + e^{2\pi i(m+\frac{1}{2})(2v-\tau)} \varphi_1^{[B;m]}(\tau, u, v-\tau, t)$
 $= e^{2\pi i(m+\frac{1}{2})t} \sum_{k=0}^{2m} e^{2\pi i(k+\frac{1}{2})v} q^{-\frac{(k+\frac{1}{2})^2}{2(2m+1)}} \Theta_{k+\frac{1}{2},m+\frac{1}{2}}^{-}(\tau, 2u).$
- (3) $\varphi^{[B;m]}(\tau, u, v, t) + e^{2\pi i(m+\frac{1}{2})(2v-\tau)} \varphi^{[B;m]}(\tau, u, v-\tau, t)$
 $= e^{2\pi i(m+\frac{1}{2})t} \sum_{k=0}^{2m} e^{2\pi i(k+\frac{1}{2})v} q^{-\frac{(k+\frac{1}{2})^2}{2(2m+1)}} (\Theta_{k+\frac{1}{2},m+\frac{1}{2}}^{-} + \Theta_{-(k+\frac{1}{2}),m+\frac{1}{2}}^{-})(\tau, 2u).$

Proof. Claim (1) is obvious, and claim (3) follows from claim (2), using Lemma 4.2 (1) and (4.9). The proof of (2) is the same as that of Lemma 1.1 (3). Namely, taking the shift $j \rightarrow j+1$ in the RHS of (4.9), and assuming without loss of generality, that $t=0$, we have :

$$\begin{aligned} \varphi_1^{[B;m]}(\tau, u, v-\tau, 0) &= - \sum_{j \in \mathbb{Z}} (-1)^j \frac{e^{2\pi i(j+1)(2m+1)u + \pi i(u+v)}}{1 - e^{2\pi i(u+v)} q^j} q^{(m+\frac{1}{2})(j+1)^2 + \frac{j}{2}} \\ &= -e^{-2\pi i(2m+1)v} q^{m+\frac{1}{2}} \sum_{j \in \mathbb{Z}} (-1)^j \frac{e^{2\pi i j(2m+1)u + \pi i(u+v)} q^{(m+\frac{1}{2})j^2 + \frac{j}{2}}}{1 - e^{2\pi i(u+v)} q^j} (e^{2\pi i(u+v)} q^j)^{2m+1}. \end{aligned}$$

Multiplying both sides by $e^{2\pi i(2m+1)v} q^{-(m+\frac{1}{2})}$, we obtain :

$$\begin{aligned} (4.12) \quad & e^{2\pi i(2m+1)v} q^{-(m+\frac{1}{2})} \varphi_1^{[B;m]}(\tau, u, v-\tau, 0) \\ &= - \sum_{j \in \mathbb{Z}} (-1)^j \frac{e^{2\pi i j(2m+1)u + \pi i(u+v)} q^{(m+\frac{1}{2})j^2 + \frac{j}{2}}}{1 - e^{2\pi i(u+v)} q^j} \left(e^{2\pi i(u+v)} q^j \right)^{2m+1}. \end{aligned}$$

Adding (4.10) and (4.12), we obtain :

$$\begin{aligned}
& \varphi_1^{[B;m]}(\tau, u, v, 0) + e^{2\pi i(2m+1)v} q^{-(m+\frac{1}{2})} \varphi_1^{[B;m]}(\tau, u, v - \tau, 0) \\
&= \sum_{j \in \mathbb{Z}} (-1)^j e^{2\pi i j(2m+1)u + \pi i(u+v)} q^{(m+\frac{1}{2})j^2 + \frac{j}{2}} \frac{1 - (e^{2\pi i(u+v)} q^j)^{2m+1}}{1 - e^{2\pi i(u+v)} q^j} \\
&= \sum_{k=0}^{2m} e^{2\pi i(k+\frac{1}{2})v} q^{-\frac{(k+\frac{1}{2})^2}{2(2m+1)}} \sum_{j \in \mathbb{Z}} (-1)^j e^{2\pi i(2m+1)\left(j+\frac{k+\frac{1}{2}}{2m+1}\right)u} q^{(m+\frac{1}{2})\left(j+\frac{k+\frac{1}{2}}{2m+1}\right)^2} \\
&= \sum_{k=0}^{2m} e^{2\pi i(k+\frac{1}{2})v} q^{-\frac{(k+\frac{1}{2})^2}{2(2m+1)}} \Theta_{k+\frac{1}{2}, m+\frac{1}{2}}^-(\tau, 2u),
\end{aligned}$$

proving claim (2). \square

Next, in analogy with Section 1, introduce the following function :

$$G^{[B;m]}(\tau, u, v, t) := \varphi^{[B;m]}(\tau, u, v, t) - \varphi^{[B;m]}|_S(\tau, u, v, t),$$

where (cf. (4.8)) :

$$\varphi^{[B;m]}|_S(\tau, u, v, t) = \frac{1}{\tau} \varphi^{[B;m]} \left(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t - \frac{u^2 - v^2}{\tau} \right).$$

Lemma 4.5. *The function $G^{[B;m]}$ satisfies the following three properties (1), (2), (3) :*

- (1) $G^{[B;m]}$ is a holomorphic function in the domain X .
- (2) $G^{[B;m]}(\tau, u, v, t) + e^{\pi i(2m+1)(2v-\tau)} G(\tau, u, v - \tau, t)$
 $= e^{2\pi i(m+\frac{1}{2})t} \sum_{k=0}^{2m} e^{2\pi i(k+\frac{1}{2})v} q^{-\frac{(k+\frac{1}{2})^2}{2(2m+1)}} \left(\Theta_{k+\frac{1}{2}, m+\frac{1}{2}}^- + \Theta_{-(k+\frac{1}{2}), m+\frac{1}{2}}^- \right) (\tau, 2u).$
- (3) $G^{[B;m]}(\tau, u, v + 1, t) + G^{[B;m]}(\tau, u, v, t) = \frac{i}{\sqrt{2m+1}} (-i\tau)^{-\frac{1}{2}} e^{2\pi i(m+\frac{1}{2})t}$
 $\times \sum_{k=0}^{2m} e^{\frac{\pi i(2m+1)}{\tau} \left(v + \frac{k+\frac{1}{2}}{2m+1} \right)^2} \sum_{\substack{1 \leq j \leq 4m+1 \\ j \text{ odd}}} e^{-\frac{\pi i j}{2m+1} (k+\frac{1}{2})} \left(\Theta_{\frac{j}{2}, m+\frac{1}{2}}^- + \Theta_{-\frac{j}{2}, m+\frac{1}{2}}^- \right) (\tau, 2u).$
- (4) *The function $G^{[B;m]}(\tau, u, v, t)$ is uniquely determined by the above three properties (1), (2), (3).*

Proof. Property (1) of $G^{[B;m]}$ is immediate by Lemma 4.1. The proof of property (2) is straightforward (as that of the analogous Lemma 1.4 (2)).

The proof of property (3) follows the same lines as that of Lemma 1.4 (3), using Proposition 4.3 (3) and Lemma 4.4 (1).

The proof of the uniqueness of the function, satisfying properties (1), (2), (3) is the same as that of Lemma 1.4 (4). \square

We keep proceeding in the same way as in Section 1. The omitted proofs are the same.

Lemma 4.6. Let $a_j(\tau, v)$ ($j \in \mathbb{Z}$, $0 \leq j \leq 2m$) be functions satisfying the following conditions (i), (ii), (iii) :

(i) $a_j(\tau, v)$ is holomorphic with respect to v ,

$$(ii) \quad a_j(\tau, v) + e^{\pi i(2m+1)(2v-\tau)} a_j(\tau, v-\tau) = e^{2\pi i(j+\frac{1}{2})v} q^{-\frac{1}{2(2m+1)}(j+\frac{1}{2})^2}$$

$$(iii) \quad a_j(\tau, v) + a_j(\tau, v+1) = \frac{i}{\sqrt{2m+1}}(-i\tau)^{-\frac{1}{2}} \sum_{k=0}^{2m} e^{-\frac{2\pi i}{2m+1}(j+\frac{1}{2})(k+\frac{1}{2})} e^{\frac{2\pi i(m+\frac{1}{2})}{\tau} \left(v + \frac{k+\frac{1}{2}}{2m+1}\right)^2}$$

($a_j(\tau, v)$ is uniquely determined by these properties).

Then the function

$$(4.13) \quad G^{[B;m]}(\tau, u, v, t) := e^{2\pi i(m+\frac{1}{2})t} \sum_{j=0}^{2m} a_j(\tau, v) \left(\Theta_{j+\frac{1}{2}, m+\frac{1}{2}}^- + \Theta_{-(j+\frac{1}{2}), m+\frac{1}{2}}^- \right) (\tau, 2u)$$

satisfies the properties (1), (2), (3) of Lemma 4.5.

Now we introduce the following functions ($j \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$):

$$R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v) := \sum_{\substack{n \in \mathbb{Z} + \frac{1}{2} \\ n \equiv j+\frac{1}{2} \pmod{2m+1}}} (-1)^{n-(j+\frac{1}{2})} \left\{ \text{sgn}(n) - E \left(\left(n + (2m+1) \frac{\text{Im}(v)}{\text{Im}(\tau)} \right) \sqrt{\frac{\text{Im}(\tau)}{m+\frac{1}{2}}} \right) \right\} e^{-\frac{\pi i n^2 \tau}{2m+1} - 2\pi i n v}.$$

Lemma 4.7. The function $R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}$ has the following properties :

$$(1) \quad R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v+1) = -R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v).$$

(2) For $0 \leq j \leq 2m$ we have :

$$R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v-\tau) = -e^{\pi i(2m+1)(\tau-2v)} R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v) + 2e^{-\frac{\pi i \tau}{2m+1}(j+\frac{1}{2})^2 + 2\pi i \tau(j+\frac{1}{2})} e^{-2\pi i(j+\frac{1}{2})v}.$$

$$(3) \quad R_{2m-j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v) = R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, -v).$$

$$(4) \quad R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau+1, v) = e^{\frac{-\pi i}{2m+1}(j+\frac{1}{2})^2} R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v).$$

□

Letting $(\tau, v) \rightarrow (-\frac{1}{\tau}, \frac{v}{\tau})$ in Lemma 4.7 (1), (2), we obtain the following :

Lemma 4.8.

$$(1) \quad R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(-\frac{1}{\tau}, \frac{v-\tau}{\tau}) = -R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(-\frac{1}{\tau}, \frac{v}{\tau}).$$

(2) For $0 \leq j \leq 2m$, we have :

$$R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]} \left(-\frac{1}{\tau}, \frac{v+1}{\tau} \right) = -e^{-\frac{\pi i(2m+1)}{\tau}(2v+1)} R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) \\ + 2e^{\frac{\pi i}{(2m+1)\tau}(j+\frac{1}{2})^2 - \frac{2\pi i}{\tau}(j+\frac{1}{2})} e^{-\frac{2\pi i}{\tau}(j+\frac{1}{2})v}.$$

□

Let (a, b) and y be the real coordinates defined by (1.8). Then we have the following analogue of Lemma 1.7 and Lemma 1.8.

Lemma 4.9.

$$(1) \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v) = -\sqrt{2(2m+1)y} e^{-2\pi(2m+1)a^2 y} \Theta_{j+\frac{1}{2}, m+\frac{1}{2}}^{-}(-\bar{\tau}, -2\bar{v}).$$

$$(2) \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) = -\frac{\tau}{|\tau|} \sqrt{2(2m+1)y} e^{-2\pi(2m+1)\frac{b^2 y}{|\tau|^2}} \Theta_{j+\frac{1}{2}, m+\frac{1}{2}}^{-} \left(\frac{1}{\tau}, -\frac{2\bar{v}}{\tau} \right).$$

$$(3) \left(\frac{\partial}{\partial a} + \tau \frac{\partial}{\partial b} \right) R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) = -i(-i\tau)^{\frac{1}{2}} \sqrt{2y} e^{\frac{\pi(2m+1)}{2y\tau}(\bar{\tau}v^2 - 2\tau v\bar{v} + \tau\bar{v}^2)} \\ \times \sum_{k=0}^{2m} e^{-\frac{2\pi i}{2m+1}(j+\frac{1}{2})(k+\frac{1}{2})} \Theta_{k+\frac{1}{2}, m+\frac{1}{2}}^{-}(-\bar{\tau}, 2\bar{v}).$$

□

Let

$$\tilde{a}_j(\tau, v) =$$

$$R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v) + \frac{i}{\sqrt{2m+1}} (-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i(m+\frac{1}{2})}{\tau}v^2} \sum_{k=0}^{2m} e^{-\frac{2\pi i}{2m+1}(j+\frac{1}{2})(k+\frac{1}{2})} R_{k+\frac{1}{2}, m+\frac{1}{2}}^{[B]} \left(-\frac{1}{\tau}, \frac{v}{\tau} \right).$$

Then we have the following analogue of Lemma 1.9.

Lemma 4.10. *The functions $\tilde{a}_j(\tau, v)$, $0 \leq j \leq 2m$, satisfy the following properties :*

$$(1) \tilde{a}_j(\tau, v) \text{ is holomorphic with respect to } v,$$

$$(2) \tilde{a}_j(\tau, v) + e^{\pi i(2v-\tau)} \tilde{a}_j(\tau, v-\tau) = 2e^{-\frac{\pi i\tau}{2m+1}(2m-j+\frac{1}{2})^2} e^{2\pi i(2m-j+\frac{1}{2})v},$$

$$(3) \tilde{a}_j(\tau, v) + \tilde{a}_j(\tau, v+1) = \frac{2i}{\sqrt{2m+1}} (-i\tau)^{-\frac{1}{2}} \sum_{k=0}^{2m} e^{-\frac{2\pi i}{2m+1}(2m-j+\frac{1}{2})(k+\frac{1}{2})} e^{\frac{\pi i(2m+1)}{\tau}(v+\frac{k+\frac{1}{2}}{2m+1})^2}.$$

□

Lemma 4.10 shows that the functions $a_j(\tau, v) := \frac{1}{2}\tilde{a}_{2m-j}(\tau, v)$ ($0 \leq j \leq 2m$) satisfy the conditions (i), (ii), (iii) of Lemma 4.6. Hence, by Lemma 4.6 and Lemma 4.5(4), we have the following analogue of Proposition 1.10

Proposition 4.11.

$$G^{[B;m]}(\tau, u, v, t) = \frac{1}{2} e^{2\pi i(m+\frac{1}{2})t} \sum_{j=0}^{2m} \tilde{a}_{2m-j}(\tau, v) (\Theta_{j+\frac{1}{2}, m+\frac{1}{2}}^- + \Theta_{-(j+\frac{1}{2}), m+\frac{1}{2}}^-)(\tau, 2u).$$

□

Finally, let (cf. (1.13) and (1.14)) :

$$(4.14) \quad \varphi_{add}^{[B;m]}(\tau, u, v, t) := \frac{1}{2} e^{2\pi i(m+\frac{1}{2})t} \sum_{j=0}^{2m} R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, v) (\Theta_{j+\frac{1}{2}, m+\frac{1}{2}}^- + \Theta_{-(j+\frac{1}{2}), m+\frac{1}{2}}^-)(\tau, 2u)$$

and

$$(4.15) \quad \tilde{\varphi}^{[B;m]} := \varphi^{[B;m]} + \varphi_{add}^{[B;m]}.$$

Then we have, as in the proof of Theorem 1.11:

Lemma 4.12.

- (1) $\varphi_{add}^{[B;m]} - \varphi_{add}^{[B;m]}|_S = -G^{[B;m]}$; $\varphi_{add}^{[B;m]}(\tau + 1, u, v, t) = \varphi_{add}^{[B;m]}(\tau, u, v, t)$.
- (2) $\varphi_{add}^{[B;m]}(\tau, u + a, v + b, t) = (-1)^{a+b} \varphi_{add}^{[B;m]}(\tau, u, v, t)$ if $a, b \in \mathbb{Z}$.
- (3) $\varphi_{add}^{[B;m]}(\tau, u, -v, t) = -\varphi_{add}^{[B;m]}(\tau, u, v, t)$; $\varphi_{add}^{[B;m]}(\tau, -u, v, t) = \varphi_{add}^{[B;m]}(\tau, u, v, t)$.

□

Thus we obtain the following analogue of Theorem 1.11.

Theorem 4.13.

- (1) $\tilde{\varphi}^{[B;m]}|_S = \tilde{\varphi}^{[B;m]}$, i.e. $\tilde{\varphi}^{[B;m]}(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t + \frac{v^2 - u^2}{\tau}) = \tau \tilde{\varphi}^{[B;m]}(\tau, u, v, t)$.
- (2) $\tilde{\varphi}^{[B;m]}(\tau + 1, u, v, t) = \tilde{\varphi}^{[B;m]}(\tau, u, v, t)$.
- (3) For $a, b \in \mathbb{Z}$, we have :
 $\tilde{\varphi}^{[B;m]}(\tau, u + a, v + b, t) = (-1)^{a+b} \tilde{\varphi}^{[B;m]}(\tau, u, v, t)$;
 $\tilde{\varphi}^{[B;m]}(\tau, u + a\tau, v + b\tau, t) = (-1)^{a+b} e^{2\pi i(2m+1)(bv-au)} q^{(m+\frac{1}{2})(b^2-a^2)} \tilde{\varphi}^{[B;m]}(\tau, u, v, t)$.
- (4) $\tilde{\varphi}^{[B;m]}(\tau, u, -v, t) = -\tilde{\varphi}^{[B;m]}(\tau, u, v, t)$, $\tilde{\varphi}^{[B;m]}(\tau, -u, v, t) = \tilde{\varphi}^{[B;m]}(\tau, u, v, t)$.

□

Translating the formulas for $\varphi^{[B;m]}$ and its modification to $\Phi^{[B;m]}$ and its modification, we obtain :

$$\Phi_{add}^{[B;m]}(\tau, z_1, z_2, t) = \frac{1}{2} e^{2\pi i(m+\frac{1}{2})t} \sum_{j=0}^{2m} R_{j+\frac{1}{2}, m+\frac{1}{2}}^{[B]}(\tau, \frac{z_1 - z_2}{2}) (\Theta_{j+\frac{1}{2}, m+\frac{1}{2}}^- + \Theta_{-(j+\frac{1}{2}), m+\frac{1}{2}}^-)(\tau, z_1 + z_2)$$

and

$$\tilde{\Phi}^{[B;m]} := \Phi^{[B;m]} + \Phi_{add}^{[B;m]},$$

where $\Phi_{add}^{[B;m]}$ is obtained from $\varphi_{add}^{[B;m]}$ by the change of variables $u = \frac{z_1+z_2}{2}$, $v = \frac{z_1-z_2}{2}$. Then Theorem 4.13 implies

Theorem 4.14. *The function $\tilde{\Phi}^{[B;m]}$ ($m \in \mathbb{Z}_{\geq 0}$) has the following modular and elliptic transformation properties :*

- (1) $\tilde{\Phi}^{[B;m]}|_S = \tilde{\Phi}^{[B;m]}$, i.e. $\tilde{\Phi}^{[B;m]}(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau}) = \tau \tilde{\Phi}^{[B;m]}(\tau, z_1, z_2, t)$.
- (2) $\tilde{\Phi}^{[B;m]}(\tau + 1, z_1, z_2, t) = \tilde{\Phi}^{[B;m]}(\tau, z_1, z_2, t)$.
- (3) For $a, b \in \mathbb{Z}$ such that $a + b \in 2\mathbb{Z}$, we have :
 $\tilde{\Phi}^{[B;m]}(\tau, z_1 + a, z_2 + b, t) = (-1)^a \tilde{\Phi}^{[B;m]}(\tau, z_1, z_2, t);$
 $\tilde{\Phi}^{[B;m]}(\tau, z_1 + a\tau, z_2 + b\tau, t) = (-1)^a e^{-2\pi i(m+\frac{1}{2})(az_2+bz_1)} q^{-(m+\frac{1}{2})ab} \tilde{\Phi}^{[B;m]}(\tau, z_1, z_2, t).$
- (4) $\tilde{\Phi}^{[B;m]}(\tau, z_2, z_1, t) = -\tilde{\Phi}^{[B;m]}(\tau, z_1, z_2, t);$ $\tilde{\Phi}^{[B;m]}(\tau, -z_1, -z_2, t) = -\tilde{\Phi}^{[B;m]}(\tau, z_1, z_2, t)$

□

Remark 4.15. It follows from (4.11) that $\Phi_{add}^{[B;0]} = 0$, hence $\tilde{\Phi}^{[B;0]} = \Phi^{[B;0]}$.

5 Modular transformation formulae for modified normalized characters of admissible $\widehat{osp}_{3|2}$ -modules.

Throughout this section $\mathfrak{g} = osp_{3|2}$. Let $m \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 1}$ be such that $\gcd(M, 4m+2) = 1$. Recall that $h^\vee = \frac{1}{2}$ and $\hat{\rho} = \frac{1}{2}\Lambda_0 - \frac{1}{2}\alpha_1$. Hence, by (2.1), all principal admissible weights with respect to a simple subset $S = t_\beta y(S_M)$, associated to the pair $(M, m\Lambda_0)$, are as follows :

$$(5.1) \quad \Lambda = k\Lambda_0 + (k + \frac{1}{2})\beta + \frac{1}{2}(\alpha_1 - y\alpha_1) + \frac{1}{2}((\beta|y\alpha_1) - (k + \frac{1}{2})(\beta|\beta))\delta,$$

and all of them have level

$$(5.2) \quad k = \frac{2m+1}{2M} - \frac{1}{2}.$$

There are four kinds of simple subsets :

$$S_{k_1, k_2}^{(1)} = \{k_0\delta + \alpha_0, k_1\delta + \alpha_1, k_2\delta + \alpha_2\}, M = k_0 + 2(k_1 + k_2) + 1, k_i \in \mathbb{Z}_{\geq 0}, \\ y = 1, \beta = -k_1(\alpha_1 + 2\alpha_2) - (k_1 + 2k_2)\alpha_1;$$

$$S_{k_1, k_2}^{(2)} = \{k_0\delta - \alpha_0, k_1\delta - \alpha_1, k_2\delta - \alpha_2\}, M = k_0 + 2(k_1 + k_2) - 1, k_i \in \mathbb{Z}_{\geq 1}, \\ y = r_{\alpha_2}r_{\theta}, \beta = k_1(\alpha_1 + 2\alpha_2) + (k_1 + 2k_2)\alpha_1;$$

$$S_{k_1, k_2}^{(3)} = \{k_0\delta + \alpha_0, k_1\delta + \alpha_1 + 2\alpha_2, k_2\delta - \alpha_2\}, M = k_0 + 2(k_1 + k_2) + 1, k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_2 \in \mathbb{Z}_{\geq 1}, \\ y = r_{\alpha_2}, \beta = -(k_1 + 2k_2)(\alpha_1 + 2\alpha_2) - k_1\alpha_1;$$

$$S_{k_1, k_2}^{(4)} = \{k_0\delta - \alpha_0, k_1\delta - \alpha_1 - 2\alpha_2, k_2\delta + \alpha_2\}, M = k_0 + 2(k_1 + k_2) - 1, k_0, k_1 \in \mathbb{Z}_{\geq 1}, k_2 \in \mathbb{Z}_{\geq 0}, \\ y = r_{\theta}, \beta = (k_1 + 2k_2)(\alpha_1 + 2\alpha_2) + k_1\alpha_1.$$

By (5.1), all principal admissible weights with respect to the simple subsets $S_{k_1, k_2}^{(s)}$, associated to the pair $(M, \Lambda^0 = m\Lambda_0)$, are as follows, where k is given by (5.2) :

$$\Lambda_{k_1, k_2}^{(1)} = k\Lambda_0 - (k + \frac{1}{2})k_1(\alpha_1 + 2\alpha_2) - (k + \frac{1}{2})(k_1 + 2k_2)\alpha_1 - \varphi(k_1, k_2)\delta; \\ \Lambda_{k_1, k_2}^{(2)} = k\Lambda_0 + (k + \frac{1}{2})k_1(\alpha_1 + 2\alpha_2) + ((k + \frac{1}{2})(k_1 + 2k_2) + 1)\alpha_1 - \varphi(k_1, k_2)\delta; \\ \Lambda_{k_1, k_2}^{(3)} = k\Lambda_0 - ((k + \frac{1}{2})(k_1 + 2k_2) + \frac{1}{2})(\alpha_1 + 2\alpha_2) + (\frac{1}{2} - (k + \frac{1}{2})k_1)\alpha_1 - \varphi(k_1, k_2)\delta; \\ \Lambda_{k_1, k_2}^{(4)} = k\Lambda_0 + ((k + \frac{1}{2})(k_1 + 2k_2) + \frac{1}{2})(\alpha_1 + 2\alpha_2) + ((k + \frac{1}{2})k_1 + \frac{1}{2})\alpha_1 - \varphi(k_1, k_2)\delta,$$

where

$$\varphi(k_1, k_2) = \frac{1}{2}(\frac{2m+1}{M}k_1(k_1 + 2k_2) + k_1).$$

As in the $\widehat{sl}_{2|1}$ case in Section 2, the non-twisted and twisted normalized admissible (super)characters can be written in terms of the following functions :

$$(5.3) \quad \Psi_{a, b; \varepsilon'}^{[B; M, m, \varepsilon]}(\tau, z_1, z_2, t) = q^{\frac{2m+1}{2M}ab} e^{\frac{2\pi i(2m+1)}{2M}(bz_1 + az_2)} \Phi^{[B; m]}(M\tau, z_1 + a\tau + \varepsilon, z_2 + b\tau + \varepsilon, \frac{t}{M}),$$

where $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$, $a, b \in \varepsilon' + \mathbb{Z}$, $a - b \in 2\mathbb{Z}$.

Formula (3.28) from [KW] (see [GK] for its proof) gives, using (4.2), the following expressions for the normalized supercharacters of principal admissible modules over $\widehat{osp}_{3|2}$ in terms of the functions $\Psi_{a, b; \varepsilon'}^{[B; M, m, \varepsilon]}$.

Lemma 5.1. *Let $\Lambda = \Lambda_{k_1, k_2}^{(s)}$ be a principal admissible weight (of level (5.2)) with respect to the simple subset $S_{k_1, k_2}^{(s)}$, associated to the pair $(M, m\Lambda_0)$. Then the normalized supercharacters ch_{Λ}^{-} are given by the following formulae (where \widehat{R}^{-} is the affine $osp_{3|2}$ superdenominator):*

$$(1) \quad \Lambda = \Lambda_{k_1, k_2}^{(1)} : \widehat{R}^{-} ch_{\Lambda}^{-} = \Psi_{k_1, k_1 + 2k_2; 0}^{[B; M, m; 0]} ;$$

$$\begin{aligned}
(2) \quad \Lambda &= \Lambda_{k_1, k_2}^{(2)} : \widehat{R}^- ch_{\Lambda}^- = -\Psi_{-k_1, -(k_1+2k_2); 0}^{[B; M, m; 0]} ; \\
(3) \quad \Lambda &= \Lambda_{k_1, k_2}^{(3)} : \widehat{R}^- ch_{\Lambda}^- = -\Psi_{k_1+2k_2, k_1; 0}^{[B; M, m; 0]} ; \\
(4) \quad \Lambda &= \Lambda_{k_1, k_2}^{(4)} : \widehat{R}^- ch_{\Lambda}^- = \Psi_{-(k_1+2k_2), -k_1; 0}^{[B; M, m; 0]} .
\end{aligned}$$

□

Corollary 5.2. Let $\Lambda_{k_1, k_2}^{(s)}$ be a principal admissible weight with respect to the simple subset $S_{k_1, k_2}^{(s)}$, $s = 1, 2, 3, 4$. Let $\varepsilon_s = (-1)^{\frac{(s-1)(s-2)}{2}}$. Introduce the following reparametrization of indices (k_1, k_2) :

$$\begin{aligned}
s = 1 : \quad & j = k_1, k = k_1 + 2k_2, \text{ so that } 0 \leq j \leq k, j + k \leq M - 1; \\
s = 2 : \quad & j = M - k_1, k = M - (k_1 + 2k_2), \text{ so that } 1 \leq k < j \leq M - 1, j + k \geq M; \\
s = 3 : \quad & j = k_1 + 2k_2, k = k_1, \text{ so that } 0 \leq k < j, j + k \leq M - 1; \\
s = 4 : \quad & j = M - (k_1 + 2k_2), k = M - k_1, \text{ so that } 1 \leq j < k \leq M - 1, j + k \geq M.
\end{aligned}$$

Then, as s runs over $1, 2, 3, 4$, the set of pairs (j, k) fills exactly the set of points with integer coordinates, such that $j - k$ is even, in the square $0 \leq j, k \leq M - 1$, and the supercharacter formula becomes :

$$\widehat{R}^- ch_{\Lambda_{k_1, k_2}^{(s)}}^- = \varepsilon_s \Psi_{j, k; 0}^{[B; M, m; 0]}.$$

Proof. We use the following simple observation, which follows from Lemma 4.2(3):

$$(5.4) \quad \Psi_{j+aM, k+aM; \varepsilon'}^{[B; M, m; \varepsilon]} = (-1)^{a(1+2\varepsilon)} \Psi_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]} \text{ if } a \in \mathbb{Z}.$$

□

Formula (5.3) and Lemma 5.1 show that, in order to construct a modular invariant family of characters, we need to study modular transformation properties of the functions $\Phi^{[B; M]}(M\tau, z_1, z_2, t)$, or rather their modifications, $\Phi_{\sim[B; M]}(M\tau, z_1, z_2, t)$.

Let $m \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 1}$ be such that $\gcd(M, 4m + 2) = 1$, and let (see (2.5)) :

$$I = I_{M, 4m+2}^{[-1]}.$$

$$\begin{aligned}
\textbf{Lemma 5.3.} \quad \Phi^{[B; m]}(\frac{\tau}{M}, \frac{z_1}{M}, \frac{z_2}{M}, 0) &= \sum_{\substack{0 \leq a < M \\ b \in I}} (-1)^a e^{\frac{\pi i(2m+1)}{M}((a+2b)z_1 + az_2)} \\
&\times q^{\frac{2m+1}{2M}a(a+2b)} \Phi^{[B; m]}(M\tau, z_1 + a\tau, z_2 + (a+2b)\tau, 0).
\end{aligned}$$

Proof. Letting $t = 0$ and expanding the RHS of (4.6) in the geometric progression in the domain $\text{Im } z_1 > 0$ and replacing (τ, z_1, z_2) by $\frac{1}{M}(\tau, z_1, z_2)$, we obtain :

$$(5.5) \quad \Phi_1^{[B; m]}(\frac{\tau}{M}, \frac{z_1}{M}, \frac{z_2}{M}) = \left(\sum_{\substack{j, k \geq 0 \\ k \text{ odd}}} - \sum_{\substack{j, k < 0 \\ k \text{ odd}}} \right) (-1)^j e^{\frac{\pi i(2m+1)}{M}j(z_1 + z_2) + \frac{\pi i}{M}kz_1} q^{\frac{1}{M}((m + \frac{1}{2})j^2 + \frac{ik}{2})}.$$

Decomposing j and k as

$$j = j'M + a, k = k'M + (4m + 2)b, \text{ where } 0 \leq a < M, b \in I,$$

we have (cf. (i) and (ii) after (2.5)):

$$(-1)^j = (-1)^{j'+a}, k \text{ odd iff } k' \text{ odd}, j \geq 0 \text{ iff } j' \geq 0, k \geq 0 \text{ iff } k' \geq 0.$$

Hence the RHS of (5.5) becomes:

$$\begin{aligned} & \sum_{\substack{0 \leq a < M \\ b \in I}} (-1)^a e^{\frac{\pi i(2m+1)}{M}((a+2b)z_1 + az_2)} q^{\frac{2m+1}{2M}a(a+2b)} \\ & \times \left(\sum_{\substack{j', k' \geq 0 \\ k' \text{ odd}}} - \sum_{\substack{j', k' < 0 \\ k' \text{ odd}}} \right) (-1)^{j'} e^{\pi i(2m+1)j'((z_1 + a\tau) + (z_2 + (a+2b)\tau))} e^{\pi i k' (z_1 + a\tau)} q^{M((m+\frac{1}{2})j'^2 + \frac{j'k'}{2})}. \end{aligned}$$

Therefore

$$\begin{aligned} (5.6) \quad & \Phi_1^{[B;m]} \left(\frac{\tau}{M}, \frac{z_1}{M}, \frac{z_2}{M} \right) \\ & = \sum_{\substack{0 \leq a < M \\ b \in I}} (-1)^a e^{\frac{\pi i(2m+1)}{M}((a+2b)z_1 + az_2)} q^{\frac{2m+1}{2M}a(a+2b)} \Phi_1^{[B;m]}(M\tau, z_1 + a\tau, z_2 + (a+2b)\tau). \end{aligned}$$

Next, using Lemma 4.2(1), we obtain from (5.6):

$$\begin{aligned} (5.7) \quad & \Phi_1^{[B;m]} \left(\frac{\tau}{M}, \frac{z_2}{M}, \frac{z_1}{M} \right) \\ & = \sum_{\substack{0 \leq a < M \\ b \in I}} (-1)^a e^{\frac{-\pi i(2m+1)}{M}((a+2b)z_2 + az_1)} q^{\frac{2m+1}{2M}a(a+2b)} \Phi_1^{[B;m]}(M\tau, z_2 - a\tau, z_1 - (a+2b)\tau). \end{aligned}$$

Decomposing $-(a+2b) = nM + a'$, where $0 \leq a' < M$, we obtain, using Lemma 4.2(3) :

$$\begin{aligned} & \Phi_1^{[B;m]}(M\tau, z_2 - a\tau, z_1 - (a+2b)\tau) \\ & = (-1)^n e^{-\pi i(2m+1)n(z_1 + z_2)} q^{-(2m+1)n(a'+b) - M(n+\frac{1}{2})n^2} \Phi_1^{[B;m]}(M\tau, z_2 + (a'+2b)\tau, z_1 + a'\tau). \end{aligned}$$

Plugging this in (5.7), we obtain :

$$\begin{aligned} & \Phi_1^{[B;m]} \left(\frac{\tau}{M}, \frac{z_2}{M}, \frac{z_1}{M} \right) \\ & = \sum_{\substack{0 \leq a' < M \\ b \in I}} (-1)^{a'} e^{\frac{\pi i(2m+1)}{M}((a'+2b)z_1 + a'z_2)} q^{\frac{2m+1}{2M}a'(a'+2b)} \Phi_1^{[B;m]}(M\tau, z_2 + (a'+2b)\tau, z_1 + a'\tau). \end{aligned}$$

This formula together with (5.6) concludes the proof. \square

Translating Lemma 5.3 into the language of $\varphi^{[B;m]}$, we obtain :

$$(5.8) \quad \varphi^{[B;m]} \left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, 0 \right) = \sum_{\substack{0 \leq a < M \\ b \in I}} (-1)^{a+b} e^{\frac{2\pi i(2m+1)}{M}(au + bv)} q^{\frac{2m+1}{2M}(a^2 - b^2)} \varphi^{[B;m]}(M\tau, u + a\tau, v - b\tau, 0).$$

Next, we derive a similar formula for the additional term, hence for the modified function $\tilde{\varphi}^{[B;m]}$:

Lemma 5.4. Formula (5.8) holds if we replace $\varphi^{[B;m]}$ by $\varphi_{add}^{[B;m]}$, hence, by $\tilde{\varphi}^{[B;m]}$. □

Translating Lemma 5.4 back, into the language of $\tilde{\Phi}^{[B;m]}$, we obtain

Proposition 5.5. Let $m \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 1}$ be such that $\gcd(M, 4m+2) = 1$. Then

$$\begin{aligned} \tilde{\Phi}^{[B;m]} \left(\frac{\tau}{M}, \frac{z_1}{M}, \frac{z_2}{M}, 0 \right) &= \sum_{\substack{0 \leq a < M \\ b \in I}} (-1)^a e^{\frac{\pi i(2m+1)}{M}((a+2b)z_1 + az_2)} \\ &\times q^{\frac{2m+1}{2M}a(a+2b)} \tilde{\Phi}^{[B;m]} (M\tau, z_1 + a\tau, z_2 + (a+2b)\tau, 0). \end{aligned}$$
□

Let

$$\Omega^{[B;M;\varepsilon]} = \{(j, k) \in (\varepsilon + \mathbb{Z})^2 \mid -M \leq j < M, 0 \leq k < M, j \equiv k \pmod{2}\}; \quad \Omega^{[B;M]} = \Omega^{[B;M;0]}.$$

Then, after applying Theorem 4.14, Proposition 5.5 leads to the following result.

Proposition 5.6. Let $m \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 1}$ be such that $\gcd(M, 4m+2) = 1$. Then

$$\begin{aligned} \tilde{\Phi}^{[B;m]} \left(-\frac{M}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau M} \right) \\ = \frac{\tau}{M} \sum_{(a,b) \in \Omega^{[B;M]}} (-1)^a e^{\frac{2\pi i(2m+1)}{2M}(bz_1 + az_2)} q^{\frac{2m+1}{2M}ab} \tilde{\Phi}^{[B;m]} (M\tau, z_1 + a\tau, z_2 + b\tau, t). \end{aligned}$$
□

Using Proposition 5.6, we obtain the modular transformation formulae for the functions $\tilde{\Psi}_{j,k;\varepsilon'}^{[B;M,m;\varepsilon]}$, obtained from $\Psi_{j,k;\varepsilon'}^{[B;M,m;\varepsilon]}$ in (5.3) by replacing $\Phi^{[B;m]}$ by $\tilde{\Phi}^{[B;m]}$.

Theorem 5.7. Let $m \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 1}$ be such that $\gcd(M, 4m+2) = 1$, and let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$. Then for $j, k \in \Omega^{[B;M;\varepsilon']}$ we have :

$$\begin{aligned} (1) \quad & \tilde{\Psi}_{j,k;\varepsilon'}^{[B;M,m;\varepsilon]} \left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau M} \right) \\ &= \frac{\tau}{M} (-1)^{\varepsilon' + j} \sum_{(a,b) \in \Omega^{[B;M;\varepsilon]}} (-1)^{a-\varepsilon} e^{-\frac{2\pi i(2m+1)}{2M}(ak+bj)} \tilde{\Psi}_{a,b;\varepsilon}^{[B;M,m;\varepsilon']} (\tau, z_1, z_2, t). \\ (2) \quad & \tilde{\Psi}_{j,k;\varepsilon'}^{[B;M,m;\varepsilon]} (\tau + 1, z_1, z_2, t) = (-1)^{j-\varepsilon'+4\varepsilon\varepsilon'} e^{\frac{2\pi i(2m+1)}{2M}jk} \tilde{\Psi}_{j,k;\varepsilon'}^{[B;M,m;[\varepsilon-\varepsilon']]} (\tau, z_1, z_2, t). \end{aligned}$$
□

Comparing the set $\Omega^{[B;m]}$ with Corollary 5.2, we see that points $(j, k) \in \Omega^{[B;m]}$ with j negative are missing. The reason is that the set $\{\Lambda_{k_1, k_2}^{(s)}\}$, that occurs in Lemma 5.1, does not exhaust all principal admissible weights. This happens because the set of simple roots $\widehat{\Pi}$ of $\widehat{osp}_{3|2}$ with scalar products (4.1) is not the only set of simple roots, up to equivalence. The other one is $r_{\alpha_1} \widehat{\Pi}$, where r_{α_1} is the odd reflection with respect to the isotropic simple root α_1 .

The corresponding simple subsets are $S'_{k_1, k_2}{}^{(s)} := r_{\gamma_s} S_{k_1, k_2}^{(s)}$ ($s = 1, 2, 3, 4$), where γ_s is the isotropic root in $S_{k_1, k_2}^{(s)}$. Taking into account the condition that a simple subset should consist of positive roots (with respect to $\widehat{\Pi}$), we obtain the following new four kinds of simple subsets :

$$\begin{aligned}
S'_{k_1, k_2}{}^{(1)} &= \{(k_0 + k_1 + 1)\delta - \alpha_1 - 2\alpha_2, -k_1\delta - \alpha_1, (k_1 + k_2)\delta + \alpha_1 + \alpha_2\}, \\
M &= k_0 + 2(k_1 + k_2) + 1, k_1 + 2k_2 \leq M - 1, k_1 \leq -1, k_1 + k_2 \geq 0; \\
S'_{k_1, k_2}{}^{(2)} &= \{(k_0 + k_1 - 1)\delta + \alpha_1 + 2\alpha_2, -k_1\delta + \alpha_1, (k_1 + k_2)\delta - \alpha_1 - \alpha_2\}, \\
M &= k_0 + 2(k_1 + k_2) - 1, k_1 + 2k_2 \leq M, k_1 \leq 0, k_1 + k_2 \geq 1; \\
S'_{k_1, k_2}{}^{(3)} &= \{(k_0 + k_1 + 1)\delta - \alpha_1, -k_1\delta - \alpha_1 - 2\alpha_2, (k_1 + k_2)\delta + \alpha_1 + \alpha_2\}, \\
M &= k_0 + 2(k_1 + k_2) + 1, k_1 + 2k_2 \leq M - 1, k_1 \leq -1, k_1 + k_2 \geq 0; \\
S'_{k_1, k_2}{}^{(4)} &= \{(k_0 + k_1 - 1)\delta + \alpha_1, -k_1\delta + \alpha_1 + 2\alpha_2, (k_1 + k_2)\delta - \alpha_1 - \alpha_2\}, \\
M &= k_0 + 2(k_1 + k_2) - 1, k_1 + 2k_2 \leq M, k_1 \leq 0, k_1 + k_2 \geq 1.
\end{aligned}$$

The corresponding principal admissible weights are again $\Lambda_{k_1, k_2}^{(s)}$, but the range of the pairs (k_1, k_2) is different. We shall denote them by $\Lambda'_{k_1, k_2}{}^{(s)}$, in order to distinguish them from those, corresponding to the simple subsets $S_{k_1, k_2}^{(s)}$. For these new principal admissible weights we have the following analogue of Lemma 5.1.

Lemma 5.8. *Let $\Lambda = \Lambda'_{k_1, k_2}{}^{(s)}$ be a principal admissible weight (of level (5.2)) with respect to the simple subset $S'_{k_1, k_2}{}^{(s)}$, associated to the pair $(M, m\Lambda_0)$. Then the normalized supercharacters are given by the following formulae :*

$$\begin{aligned}
(1) \quad \Lambda = \Lambda'_{k_1, k_2}{}^{(1)} &: \widehat{R}^- ch_{\Lambda}^- = \Psi_{k_1, k_1 + 2k_2; 0}^{[B; M, m; 0]}; \\
(2) \quad \Lambda = \Lambda'_{k_1, k_2}{}^{(2)} &: \widehat{R}^- ch_{\Lambda}^- = -\Psi_{-k_1, -(k_1 + 2k_2); 0}^{[B; M, m; 0]}; \\
(3) \quad \Lambda = \Lambda'_{k_1, k_2}{}^{(3)} &: \widehat{R} ch_{\Lambda}^- = -\Psi_{k_1 + 2k_2, k_1; 0}^{[B; M, m; 0]}; \\
(4) \quad \Lambda = \Lambda'_{k_1, k_2}{}^{(4)} &: \widehat{R} ch_{\Lambda}^- = \Psi_{-(k_1 + 2k_2), -k_1; 0}^{[B; M, m; 0]}.
\end{aligned}$$

□

Corollary 5.9. *Let $\Lambda = \Lambda'_{k_1, k_2}{}^{(s)}$ be a principal admissible weight with respect to the simple subset $S'_{k_1, k_2}{}^{(s)}$, $s = 1, 2, 3, 4$. Introduce the following reparametrization of indices (k_1, k_2) :*

$$\begin{aligned}
s = 1 &: j = k_1, k = k_1 + 2k_2, \text{ so that } j \leq -1, k \leq M - 1, j + k \geq 0; \\
s = 2 &: j = -k_1, k = -k_1 - 2k_2, \text{ so that } j \geq 0, k \geq -M, j + k \leq -2; \\
s = 3 &: j = k_1 + 2k_2, k = k_1, \text{ so that } j \leq M - 1, k \leq -1, j + k \geq 0; \\
s = 4 &: j = -k_1 - 2k_2, k = -k_1, \text{ so that } j \geq -M, k \geq 0, j + k \leq -2.
\end{aligned}$$

Then, as s runs over 1 and 4 (resp. 3 and 2) the set of pairs (j, k) fills exactly the points with integer coordinates, such that $j - k$ is even, in the square $-M \leq j < 0, 0 \leq k \leq M - 1$ (resp. $0 \leq j \leq M - 1, -M \leq k < 0$), and we obtain a unified supercharacter formula for the principal admissible highest weights $\Lambda = \Lambda_{k_1, k_2}^{(s)}$ and $\Lambda'_{k_1, k_2}^{(s)}$:

$$\widehat{R}^- ch_{\Lambda}^- = \pm \Psi_{j, k; 0}^{[B; M, m; 0]}.$$

□

As in [KW] and Section 2 of the present paper, we introduce the twisted normalized admissible characters and supercharacters $t_{\xi} ch_{\Lambda}^{\pm}$ of $\widehat{osp}_{3|2}$ and their denominators and superdenominators $t_{\xi} \widehat{R}^{\pm}$.

Choose $\xi = -(\alpha_1 + \alpha_2)$. Then we have in the coordinates (4.3) :

$$t_{-\xi}(h) = (\tau, z_1 + \frac{\tau}{2}, z_2 + \frac{\tau}{2}, t + \frac{z_1 + z_2}{2} + \frac{\tau}{4}).$$

The proof of the following lemma is straightforward.

Lemma 5.10.

$$\Psi_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]}(\tau, z_1 + \frac{\tau}{2}, z_2 + \frac{\tau}{2}, t + \frac{z_1 + z_2}{2} + \frac{\tau}{4}) = \Psi_{j + \frac{1}{2}, k + \frac{1}{2}, \frac{1}{2} - \varepsilon'}^{[B; M, m; \varepsilon]}(\tau, z_1, z_2, t),$$

and the same formula holds if we replace Ψ by $\widetilde{\Psi}$.

□

We shall use the same notation (2.6) and (2.7) as before, and let

$$ch_{j + \varepsilon', k + \varepsilon'; \varepsilon'}^{[B; M, m; \varepsilon]} = ch_{\Lambda_{j, k; \varepsilon'}^{(s)}}^{(\varepsilon)}.$$

Then

$$(5.9) \quad \{ch_{j + \varepsilon', k + \varepsilon'; \varepsilon'}^{[B; M, m; \varepsilon]} \mid j, k \in \Omega^{[B; M]}\} \cup \{ch_{j + \varepsilon', k + \varepsilon'; \varepsilon'}^{[B; M, m; \varepsilon]} \mid j, k \in \mathbb{Z}, 0 \leq j \leq M - 1, -M \leq k < 0\}$$

is (up to a sign) precisely the set of all normalized principal admissible characters (resp. supercharacters), associated to the pair $(M, m\Lambda_0)$, if $\varepsilon' = 0$ and $\varepsilon = \frac{1}{2}$ (resp. $\varepsilon = 0$), and it is the set of all normalized twisted principal admissible characters (resp. supercharacters), associated to the pair $(M, m\Lambda_0)$, if $\varepsilon' = \frac{1}{2}$ and $\varepsilon = \frac{1}{2}$ (resp. $\varepsilon = 0$).

By Corollaries 5.2 and 5.9, and Lemma 5.10, we obtain the following unified formula :

$$(5.10) \quad \widehat{R}_{\varepsilon'}^{(\varepsilon)} ch_{j + \varepsilon', k + \varepsilon'; \varepsilon'}^{[B; M, m; \varepsilon]} = \widetilde{\varepsilon}_s \Psi_{j + \varepsilon', k + \varepsilon'; \varepsilon'}^{[B; M, m; \varepsilon]}, \quad j, k \in \Omega^{[B; M]},$$

where $\widetilde{\varepsilon}_s = \varepsilon_s$ if $j \geq 0$ and $\widetilde{\varepsilon}_s = 1$ if $j < 0$.

In view of these observations, introduce the modified normalized untwisted and twisted (super)characters, letting

$$(5.11) \quad ch_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]}(\tau, z_1, z_2, t) = (-1)^{j - \varepsilon'} \frac{\widetilde{\Psi}_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]}(\tau, z_1, z_2, t)}{\widehat{R}_{\varepsilon'}^{(\varepsilon)}(\tau, z_1, z_2, t)}, \quad j, k \in \Omega^{[B; M; \varepsilon]}.$$

Since, by Theorem 4.14(3), we have: $\overset{\sim}{\Psi}_{j+aM, k+bM; \varepsilon'}^{[B; M, m; \varepsilon]} = (-1)^{a+\varepsilon(a+b)} \overset{\sim}{\Psi}_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]}$ for $a, b \in \mathbb{Z}$, $a+b$ even, in view of Corollary 5.9, after the modification the second subset in (5.9) coincides with a half of the first one, hence the first subset produces, after the modification, all untwisted and twisted modified normalized principal admissible (super)characters, associated to the pair $(M, m\Lambda_0)$.

Using (4.1), (4.3), (4.5), (4.6) from [KW], we can write the following unified formula for the untwisted and twisted (normalized) (super)denominators for $\widehat{osp}_{3|2}$:

$$(5.12) \quad \widehat{R}_{\varepsilon'}^{(\varepsilon)}(\tau, z_1, z_2, t) = c_{\varepsilon, \varepsilon'} e^{\pi i t} \frac{\vartheta(\tau)^3 \vartheta_{11}(\tau, z_1 + z_2) \vartheta_{11}(\tau, \frac{z_1 - z_2}{2})}{\vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, z_1) \vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, z_2) \vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, \frac{z_1 + z_2}{2})},$$

where $c_{\varepsilon, \varepsilon'} = e^{\frac{\pi i}{2}(2(\varepsilon - \varepsilon') - 4\varepsilon\varepsilon' - 1)}$.

The modular transformation formulae for the functions $\widehat{R}_{\varepsilon'}^{(\varepsilon)}$ follow from Theorem 4.1 in [KW] :

Lemma 5.11.

$$(1) \quad \widehat{R}_{\varepsilon'}^{(\varepsilon)}(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau}) = i^{-(2\varepsilon + 2\varepsilon' + 4\varepsilon\varepsilon')} \tau \widehat{R}_{\varepsilon}^{(\varepsilon')}(\tau, z_1, z_2, t).$$

$$(2) \quad \widehat{R}_{\varepsilon'}^{(\varepsilon)}(\tau + 1, z_1, z_2, t) = e^{\frac{3\pi i}{2}\varepsilon'} \widehat{R}_{\varepsilon'}^{(|\varepsilon - \varepsilon'|)}(\tau, z_1, z_2, t).$$

□

Modular transformation properties of the modified normalized (super)characters (5.11) follow from Theorem 5.7 and Lemma 5.11 :

Theorem 5.12. *Let $m \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 1}$ be such that $\gcd(M, 4m+2) = 1$, and let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$. Then for $j, k \in \Omega^{[B; M; \varepsilon']}$ we have :*

$$(1) \quad \overset{\sim}{ch}_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]}(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau})$$

$$= \frac{i^{4\varepsilon\varepsilon' + 2\varepsilon - 2\varepsilon'}}{M} \sum_{(a, b) \in \Omega^{[B; M; \varepsilon]}} e^{-\frac{\pi i(2m+1)}{M}(bj+ak)} \overset{\sim}{ch}_{a, b; \varepsilon}^{[B; M, m; \varepsilon']}(\tau, z_1, z_2, t).$$

$$(2) \quad \overset{\sim}{ch}_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]}(\tau + 1, z_1, z_2, t) = (-1)^{4\varepsilon\varepsilon'} e^{\frac{\pi i(2m+1)}{M}jk + \pi i(j - \frac{3}{2}\varepsilon')} \overset{\sim}{ch}_{j, k; \varepsilon'}^{[B; M, m; |\varepsilon - \varepsilon'|]}(\tau, z_1, z_2, t).$$

□

Remark 5.13. Due to Remark 4.15, Theorem 5.12 holds if we replace the modified normalized characters $\overset{\sim}{ch}$ by the non-modified ones ch for any odd $M \in \mathbb{Z}_{\geq 1}$. In particular the normalized characters

$$\{ch_{j, k; \varepsilon'}^{[B; M, 0; \varepsilon]} \mid \varepsilon, \varepsilon' = 0 \text{ or } \frac{1}{2}, (j, k) \in \Omega^{[B; M; \varepsilon']}\}$$

span an $SL_2(\mathbb{Z})$ -invariant space for any odd $M \in \mathbb{Z}_{\geq 1}$.

6 Modular transformation formulae for modified characters of admissible $N = 3$ modules.

Let $\mathfrak{g} = osp_{3|2}$ and let \mathfrak{h} be its Cartan subalgebra, so that $l = \dim \mathfrak{h} = 2$.

Choose the set of positive roots Δ_+ as in Section 4 :

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \theta = 2\alpha_1 + 2\alpha_2\}.$$

Let $x = \alpha_1 + \alpha_2 \in \mathfrak{h}^*$, which is identified with \mathfrak{h} via the bilinear form $(\cdot|\cdot)$. The eigenspace decomposition of \mathfrak{g} with respect to $ad x$ is

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_{-\frac{1}{2}} + \mathfrak{g}_0 + \mathfrak{g}_{\frac{1}{2}} + \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$, $\mathfrak{g}_{\pm\frac{1}{2}}$ are purely odd of dimension 3, and $\mathfrak{g}_0 = \mathbb{C}x + \mathfrak{g}^\#$, where $\mathfrak{g}^\#$ is the orthogonal complement to $\mathbb{C}x$ in \mathfrak{g}_0 with respect to $(\cdot|\cdot)$, isomorphic to sl_2 . The subspace $\mathfrak{g}^\# \cap \mathfrak{h}$ is spanned by the element $J_0 = -2\alpha_2$.

Recall that the quantum Hamiltonian reduction associates to a $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ of level $k \neq -\frac{1}{2}$ ($= -h^\vee$), a module $H(\Lambda)$ over the $N = 3$ superconformal algebra of Neveu-Schwarz type, such that the following properties hold [KRW], [KW2], [A] :

- (i) the module $H(\Lambda)$ is either 0 or an irreducible positive energy module ;
- (ii) $H(\Lambda) = 0$ iff $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$;
- (iii) the irreducible module $H(\Lambda)$ is characterized by three numbers:
 - (α) the central charge

$$(6.1) \quad c_k = -3(2k + 1),$$

(β) the lowest energy

$$(6.2) \quad h_\Lambda = \frac{(\Lambda + 2\widehat{\rho}|\Lambda)}{2k + 1} - (x + d|\Lambda),$$

(γ) the spin

$$(6.3) \quad s_\Lambda = \Lambda(J_0).$$

- (iv) the character ch^+ and the supercharacter ch^- of the module $H(\Lambda)$ are given by the following formula :

$$(6.4) \quad ch_{H(\Lambda)}^\pm(\tau, z) := tr_{H(\Lambda)}^\pm q^{L_0 - \frac{c_k}{24}} e^{2\pi i z J_0} = (\widehat{R}^\pm ch_\Lambda^\pm)(\tau, -\tau x + J_0 z, \frac{\tau}{4}) R^{\pm 3}(\tau, z)^{-1},$$

where

$$(6.5) \quad R^+(\tau, z) = \frac{\eta(\frac{\tau}{2})\eta(2\tau)\vartheta_{11}(\tau, z)}{\vartheta_{00}(\tau, z)} \quad \text{and} \quad R^-(\tau, z) = \frac{\eta(\tau)^3\vartheta_{11}(\tau, z)}{\eta(\frac{\tau}{2})\vartheta_{01}(\tau, z)}$$

are the $N = 3$ superconformal algebra normalized denominator and superdenominator.

Now we turn to the Ramond twisted sector. For each $\alpha \in \Delta_+$ choose $s_\alpha \in \mathbb{Z}$ (resp. $\in \frac{1}{2} + \mathbb{Z}$) if the root α is even (resp. odd), such that $s_\theta = 0$ and $s_\alpha + s_{\theta-\alpha} = \delta_{\alpha, \theta/2}$ if both α and $\theta - \alpha$ are odd roots. Recall [KW3], [A] that, given such a suitable choice of s_α 's, the twisted quantum Hamiltonian reduction associates to a $\widehat{\mathfrak{g}}^{\text{tw}}$ -module $L^{\text{tw}}(\Lambda)$ of level $k \neq -\frac{1}{2}$, a positive energy module $H^{\text{tw}}(\Lambda)$ over the corresponding Ramond $N = 3$ superconformal algebra, for which the properties (i) and (ii) hold with H replaced by H^{tw} . A suitable choice of the s_α is (see [KW3]) :

$$(6.6) \quad s_{\alpha_1} = s_{\alpha_1 + \alpha_2} = -s_{\alpha_1 + 2\alpha_2} = \frac{1}{2}, \quad s_{\alpha_2} = s_\theta = 0.$$

It is not difficult to see that, choosing the element $w = t_{\frac{1}{2}\theta} r_\theta \in \widehat{W}$, and a lifting \tilde{r}_θ of r_θ in the corresponding $SL_2(C)$, we can lift w to an isomorphism

$$\tilde{w} = t_{\frac{1}{2}\theta} \tilde{r}_\theta : \widehat{\mathfrak{g}}^{\text{tw}} \xrightarrow{\sim} \widehat{\mathfrak{g}}, \text{ where } \widehat{\mathfrak{g}}^{\text{tw}} = \mathfrak{g}_0[t, t^{-1}] \oplus \mathfrak{g}_1[t, t^{-1}]t^{\frac{1}{2}} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

such that the set of positive roots $\widehat{\Delta}_+$ of $\widehat{\mathfrak{g}}$ corresponds to the set of positive roots $\widehat{\Delta}_+^{\text{tw}}$ of $\widehat{\mathfrak{g}}^{\text{tw}}$, associated to the choice (6.6) of the s_α 's. The set of simple roots of $\widehat{\Delta}_+^{\text{tw}}$ is

$$(6.7) \quad -\frac{1}{2}\delta + \alpha_1 + 2\alpha_2, \quad \frac{1}{2}\delta + \alpha_1, \quad \frac{1}{2}(\delta - \theta).$$

It is easy to see that in the coordinates (4.3) we have:

$$(6.8) \quad w(h) = w(\tau, z_1, z_2, t) = (\tau, -z_2 + \frac{\tau}{2}, -z_1 + \frac{\tau}{2}, t - \frac{z_1 + z_2}{2} + \frac{\tau}{4}).$$

Note also that $w^2 = 1$.

Via the isomorphism \tilde{w} , the $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ becomes a $\widehat{\mathfrak{g}}^{\text{tw}}$ -module, denoted by $L^{\text{tw}}(\Lambda)$; its highest weight is

$$(6.9) \quad \Lambda^{\text{tw}} = w(\Lambda),$$

and its normalized character and supercharacter are :

$$(6.10) \quad ch_\Lambda^{\text{tw}, \pm} = w(ch_\Lambda^{\text{tw}, \pm}),$$

their denominator and superdenominator being $\widehat{R}^{\text{tw}, \pm} = w(\widehat{R}^\pm)$.

The irreducible module $H^{\text{tw}}(\Lambda)$ is again characterized by three numbers : the central charge c_k , given by (6.1), the lowest energy

$$(6.11) \quad h_\Lambda^{\text{tw}} = \frac{(\Lambda^{\text{tw}} + 2\widehat{\rho}^{\text{tw}} | \Lambda^{\text{tw}})}{2k + 1} - (x + d | \Lambda^{\text{tw}}) - \frac{3}{16},$$

and the spin

$$(6.12) \quad s_\Lambda^{\text{tw}} = \Lambda^{\text{tw}}(J_0) - \frac{1}{2}.$$

Furthermore, the (super)character of the module $H^{\text{tw}}(\Lambda)$ is given by the following formula :
(6.13)

$$ch_{H^{\text{tw}}(\Lambda)}^{\pm}(\tau, z) := tr_{H^{\text{tw}}(\Lambda)}^{\pm} q^{L_0^{\text{tw}} - \frac{c_k}{24}} e^{2\pi i z J_0^{\text{tw}}} = (\hat{R}^{\text{tw}, \pm} ch_{\Lambda}^{\text{tw}, \pm})(\tau, -\tau x + J_0 z, \frac{\tau}{4}) \hat{R}^{3^{\text{tw}, \pm}}(\tau, z)^{-1},$$

where

$$(6.14) \quad \hat{R}^{3^{\text{tw}, +}}(\tau, z) = \frac{\eta(\tau)^3 \vartheta_{11}(\tau, z)}{\eta(2\tau) \vartheta_{10}(\tau, z)}$$

is the Ramond $N = 3$ superconformal algebra normalized denominator, and $\hat{R}^{3^{\text{tw}, -}}(\tau, z)^{-1} = 0$, hence $ch_{H^{\text{tw}}(\Lambda)}^{-} = 0$.

As in the $N = 2$ case, we introduce a more convenient notation: $\hat{R}_{\frac{1}{2}}^{3^{\frac{1}{2}}} = \hat{R}^{3^+}$, $\hat{R}_{\frac{1}{2}}^{3^{(0)}} = \hat{R}^{3^-}$, and $\hat{R}_0^{3^{\frac{1}{2}}} = 2^{-\frac{1}{2}} \hat{R}^{3^{\text{tw}, +}}$ (cf. (6.5) and (6.14)). Using the modular transformation formulae for the four Jacobi theta functions (see e.g. [KW], Proposition A.7) we obtain the modular transformation formulae for these functions.

Proposition 6.1. *Let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$ be such that $\varepsilon + \varepsilon' \neq 0$. Then*

$$\begin{aligned} (1) \quad \hat{R}_{\varepsilon'}^{3^{(\varepsilon)}}(-\frac{1}{\tau}, \frac{z}{\tau}) &= -\tau \hat{R}_{\varepsilon}^{3^{(\varepsilon')}}(\tau, z). \\ (2) \quad \hat{R}_{\varepsilon'}^{3^{(\varepsilon)}}(\tau + 1, z) &= e^{\frac{\pi i}{12}(1+9\varepsilon')} \hat{R}_{\varepsilon'}^{3^{(|\varepsilon-\varepsilon'|)}}(\tau, z). \end{aligned}$$

□

Let $\Lambda = \Lambda_{k_1, k_2}^{(s)}$ and $\Lambda = \Lambda_{k_1, k_2}'^{(s)}$ be one of the principal admissible weights (for the simple subsets $S_{k_1, k_2}^{(s)}$ and $S_{k_1, k_2}'^{(s)}$), described by Corollaries 5.2 and 5.9, along with the reparametrization (j, k) of the pairs (k_1, k_2) . Then we have :

$$(\hat{R}^{\text{tw}, +} ch_{\Lambda}^{\text{tw}, +})(h) = (\hat{R}^{+} ch_{\Lambda}^{+})(w^{-1}h) = (\hat{R}^{+} ch_{\Lambda}^{+})(wh).$$

Using (6.8) and Lemma 5.10, we deduce :

$$(6.15) \quad (\hat{R}^{\text{tw}, +} ch_{\Lambda}^{\text{tw}, +})(h) = \tilde{\varepsilon}_s \Psi_{j+\frac{1}{2}, k+\frac{1}{2}; \frac{1}{2}}^{[B; M, m; \frac{1}{2}]}(\tau, -z_2, -z_1, t).$$

Furthermore, we have in coordinates (4.1) :

$$(6.16) \quad 2\pi i(-\tau \Lambda_0 - \tau x + z J_0) = (\tau, z + \frac{\tau}{2}, -z + \frac{\tau}{2}, 0).$$

Now we can apply formulae (6.4) and (6.15). Using (6.16) and Lemma 5.10, we obtain the following.

Proposition 6.2. *Let $m \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 1}$ be such that $\gcd(M, 4m+2) = 1$. Let $\Lambda = \Lambda_{k_1, k_2}^{(s)}$ or $\Lambda = \Lambda_{k_1, k_2}'^{(s)}$ be one of the principal admissible weights, described in Corollaries 5.2 and 5.9, along with the reparametrization (j, k) of the pairs (k_1, k_2) . Then*

$$(1) (R^{3\pm} ch_{H(\Lambda)}^\pm)(\tau, z) = \Psi_{j+\frac{1}{2}, k+\frac{1}{2}, \frac{1}{2}}^{[B; M, m; \frac{1}{4}(1\pm 1)]}(\tau, z, -z, 0).$$

$$(2) (R^{3\text{tw}, +} ch_{H^{\text{tw}}(\Lambda)}^+)(\tau, z) = \Psi_{j, k; 0}^{[B; M, m; \frac{1}{2}]}(\tau, z, -z, 0).$$

By (6.1), the central charge of the $N = 3$ modules $H(\Lambda)$ and $H^{\text{tw}}(\Lambda)$, where $\Lambda = \Lambda_{k_1, k_2}^{(s)}$ or $\Lambda_{k_1, k_2}'^{(s)}$ have level k , given by formula (5.2), is equal to

$$(6.17) \quad c_k = -3 \frac{2m+1}{M}.$$

Recall that $H(\Lambda)$ and $H^{\text{tw}}(\Lambda)$ are zero iff $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$, and that $H(\Lambda)$ is irreducible otherwise. If $M = 1$, then a principal admissible $\widehat{osp}_{3|2}$ -module is partially integrable, hence $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$ and therefore $H(\Lambda) = 0 = H^{\text{tw}}(\Lambda)$. If $M > 1$, it follows from the formulas for the principal admissible weights, listed in Section 5, that $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$ iff $k_0 = 0$ and $s = 1$ or 3 .

Thus, in what follows we may assume that $M \geq 2$ and $k_0 > 0$ when $s = 1$ or 3 . Using the formulas for the principal admissible weights in Section 5 and formulas (6.2), (6.3), (6.11) and (6.12), we obtain the following explicit formulae for the lowest energy h_Λ (resp. h_Λ^{tw}) and spin (resp. s_Λ^{tw}) of $H(\Lambda)$ (resp. $H^{\text{tw}}(\Lambda)$), and the same formulae for $\Lambda_{k_1, k_2}'^{(s)}$:

$$(6.18) \quad h_{\Lambda_{k_1, k_2}^{(1 \text{ or } 3)}} = (k + \frac{1}{2})((k_1 + \frac{1}{2})(k_1 + 2k_2 + \frac{1}{2}) - \frac{1}{4}) + \frac{k_1}{2} = h_{\Lambda_{k_1+1, k_2}^{(4 \text{ or } 2)}},$$

$$(6.19) \quad s_{\Lambda_{k_1, k_2}^{(1)}} = 2(k + \frac{1}{2})k_2 = s_{\Lambda_{k_1+1, k_2}^{(4)}},$$

$$(6.20) \quad s_{\Lambda_{k_1, k_2}^{(3)}} = -2(k + \frac{1}{2})k_2 - 1 = s_{\Lambda_{k_1+1, k_2}^{(2)}},$$

$$(6.21) \quad h_{\Lambda_{k_1, k_2}^{(s)}}^{\text{tw}} = (k + \frac{1}{2})(k_1(k_1 + 2k_2) - \frac{1}{4}) + \frac{k_1}{2} - \frac{1}{16}, \quad s = 1, 2, 3, 4,$$

$$(6.22) \quad s_{\Lambda_{k_1, k_2}^{(1)}}^{\text{tw}} = 2(k + \frac{1}{2})k_2 - \frac{1}{2} = s_{\Lambda_{k_1, k_2}^{(4)}}^{\text{tw}},$$

$$(6.23) \quad s_{\Lambda_{k_1, k_2}^{(3)}}^{\text{tw}} = -2(k + \frac{1}{2})k_2 - \frac{3}{2} = s_{\Lambda_{k_1, k_2}^{(2)}}^{\text{tw}}.$$

Since the irreducible $N = 3$ positive energy modules are uniquely determined by their characteristic numbers, we obtain the following isomorphisms of the $N = 3$ Neveu-Schwarz type superconformal algebra modules:

$$(6.24) \quad H(\Lambda_{k_1, k_2}^{(4)}) \simeq H(\Lambda_{k_1+1, k_2}^{(4)}), \quad H(\Lambda_{k_1, k_2}^{(3)}) \simeq H(\Lambda_{k_1+1, k_2}^{(2)}), \quad H(\Lambda_{k_1, k_2}'^{(1)}) \simeq H(\Lambda_{k_1+1, k_2}'^{(4)}),$$

and of the $N = 3$ Ramond type modules :

$$(6.25) \quad H^{\text{tw}}(\Lambda_{k_1, k_2}^{(1)}) \simeq H^{\text{tw}}(\Lambda_{k_1, k_2}^{(4)}), \quad H^{\text{tw}}(\Lambda_{k_1, k_2}^{(3)}) \simeq H^{\text{tw}}(\Lambda_{k_1, k_2}^{(2)}), \quad H^{\text{tw}}(\Lambda_{k_1, k_2}'^{(1)}) \simeq H^{\text{tw}}(\Lambda_{k_1, k_2}'^{(4)}).$$

Thus, we may consider only the $N = 3$ modules $H(\Lambda_{k_1, k_2}^{(s)})$ and $H^{\text{tw}}(\Lambda_{k_1, k_2}^{(s)})$ for $s = 1$ and 3 , and $H(\Lambda_{k_1, k_2}'^{(s)})$ and $H^{\text{tw}}(\Lambda_{k_1, k_2}'^{(s)})$ for $s = 1$.

In the reindexing of Corollaries 5.2 and 5.9 we let :

$$H_{NS}(\Lambda_{j, k}) = H\left(\Lambda_{j-\frac{1}{2}, k-\frac{1}{2}}^{(1 \text{ or } 3)}\right), \quad H_{NS}(\Lambda_{j, k}) = H\left(\Lambda_{j-\frac{1}{2}, k-\frac{1}{2}}'^{(1)}\right),$$

$$H_R(\Lambda_{j, k}) = H^{\text{tw}}\left(\Lambda_{j, k}^{(1 \text{ or } 3)}\right), \quad H_R(\Lambda_{j, k}) = H^{\text{tw}}\left(\Lambda_{j, k}'^{(1)}\right).$$

Using (6.24), (6.25) and Lemma 5.10, it is easy to see that the set of all pairs (j, k) that occur in $H_{NS}(\Lambda_{j, k})$ (resp. $H_R(\Lambda_{j, k})$) is the following, where, as before $\varepsilon = \frac{1}{2}$ (resp. $= 0$) in the Neveu-Schwarz case (resp. Ramond case) :

$$\Omega_\varepsilon^{[3; M]} = \{(j, k) \in (\varepsilon + \mathbb{Z})^2 \mid 0 \leq k < M, 0 < j + k < M, j - k \in 2\mathbb{Z}\}.$$

It is easy to see from (6.18)-(6.23) that the characteristic numbers of these $N = 3$ modules are given by the following unified formulae for $\Lambda = \Lambda_{k_1+1, k_2}^{(s)}$ or $\Lambda = \Lambda_{k_1+1, k_2}'^{(s)}$:

$$(6.26) \quad h_\Lambda = (k + \frac{1}{2})(jk - \frac{1}{4}) + \frac{1}{2}(j - \frac{1}{2}), \quad s_\Lambda = (k + \frac{1}{2})(k - j) \text{ if } j \leq k,$$

$$(6.27) \quad h_\Lambda = (k + \frac{1}{2})(jk - \frac{1}{4}) + \frac{1}{2}(k - \frac{1}{2}), \quad s_\Lambda = (k + \frac{1}{2})(j - k) - 1 \text{ if } j > k,$$

$$(6.28) \quad h_\Lambda^{\text{tw}} = (k + \frac{1}{2})(jk - \frac{1}{4}) + \frac{j}{2} - \frac{1}{16}, \quad s_\Lambda^{\text{tw}} = (k + \frac{1}{2})(k - j) - \frac{1}{2}, \text{ if } j \leq k,$$

$$(6.29) \quad h_\Lambda^{\text{tw}} = (k + \frac{1}{2})(jk - \frac{1}{4}) + \frac{k}{2} - \frac{1}{16}, \quad s_\Lambda^{\text{tw}} = (k + \frac{1}{2})(j - k) - \frac{3}{2}, \text{ if } j > k.$$

As in Section 3, introduce the following notation for the characters and supercharacters of these modules :

$$ch_{j, k; \frac{1}{2}}^{N=3[M, m; \varepsilon]}(\tau, z) = ch_{H_{NS}(\Lambda_{j, k})}^\pm(\tau, z),$$

$$ch_{j, k; 0}^{N=3[M, m; \frac{1}{2}]}(\tau, z) = ch_{H_R(\Lambda_{j, k})}^+(\tau, z).$$

Now the character formulae, given by Proposition 6.2, can be rewritten in a unified way as follows ($\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$, $\varepsilon + \varepsilon' \neq 0$) :

$$(6.30) \quad \left(R_{\varepsilon'}^{(3)} ch_{j, k; \varepsilon'}^{N=3[M, m; \varepsilon]} \right) (\tau, z) = \Psi_{j, k; \varepsilon'}^{[B; M, m; \varepsilon]}(\tau, z, -z, 0), \quad j, k \in \Omega_{\varepsilon'}^{[3; M]}.$$

Introduce the following modified $N = 3$ characters \tilde{ch} by the formula :

$$\left(\begin{matrix} 3 \\ R_{\varepsilon'} \end{matrix} \right)^{\sim N=3[M,m;\varepsilon]}_{ch_{j,k;\varepsilon'}}(\tau, z) = (-1)^{j-\varepsilon'} \tilde{\Psi}_{j,k;\varepsilon'}^{[B;M,m;\varepsilon]}(\tau, z, -z, 0), \quad j, k \in \Omega_{\varepsilon'}^{[3;M]}.$$

Theorem 5.7 and Proposition 6.1 imply the following modular transformation properties of these modified characters.

Theorem 6.3. *Let $m \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 2}$ be such that $\gcd(M, 4m+2) = 1$, and let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$ be such that $\varepsilon + \varepsilon' \neq 0$. Then for $j, k \in \Omega_{\varepsilon'}^{[3;M]}$ we have the following transformation formula of the modified characters of the $N = 3$ modules with central charge c given by (6.1) :*

$$(1) \quad \tilde{ch}_{j,k;\varepsilon'}^{\sim N=3[M,m;\varepsilon]} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{\frac{\pi i c z^2}{3\tau}} \sum_{(a,b) \in \Omega_{\varepsilon}^{[3;M]}} S_{(j,k),(a,b)}^{[\varepsilon',\varepsilon]} \tilde{ch}_{a,b;\varepsilon}^{\sim N=3[M,m;\varepsilon']}(\tau, z),$$

where

$$S_{(j,k),(a,b)}^{[\frac{1}{2},\frac{1}{2}]} = \frac{2}{M} e^{\frac{\pi i(2m+1)}{2M}(a-b)(j-k)} \cos \frac{2m+1}{2M}(a+b)(j+k)\pi,$$

$$S_{(j,k),(a,b)}^{[\varepsilon',\varepsilon]} = -\frac{2}{M} e^{\frac{\pi i(2m+1)}{2M}(a-b)(j-k)} \sin \frac{2m+1}{2M}(a+b)(j+k)\pi \quad \text{if } \varepsilon' \neq \varepsilon.$$

$$(2) \quad \tilde{ch}_{j,k;\varepsilon'}^{\sim N=3[M,m;\varepsilon]}(\tau + 1, z) = e^{-\pi i(j + \frac{1}{12} - \frac{\varepsilon'}{4} + 4\varepsilon\varepsilon')} e^{\frac{2\pi i(2m+1)}{2M}jk} \tilde{ch}_{j,k;\varepsilon'}^{\sim N=3[M,m;|\varepsilon-\varepsilon'|]}(\tau, z).$$

□

A Appendix. A brief review of theta functions

In this appendix we review some basic facts about theta functions in a slightly more general setup than in [K], Chapter 13, or [KW].

Let \mathfrak{h} be an ℓ -dimensional vector space over \mathbb{C} , endowed with a non-degenerate symmetric bilinear form $(\cdot|\cdot)$; we shall identify \mathfrak{h} with \mathfrak{h}^* via this form. Let k be a positive real number and let L be a lattice of rank ℓ (i.e. a free rank ℓ abelian subgroup) in \mathfrak{h} , such that

$$k(\alpha|\beta) \in \mathbb{Z} \quad \text{for all } \alpha, \beta \in L,$$

i.e. $kL \subset L^*$, where $L^* = \{\lambda \in \mathfrak{h} \mid (\lambda|L) \subset \mathbb{Z}\}$ is the dual lattice, and the restriction of the bilinear form $(\cdot|\cdot)$ to L is positive-definite.

Let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the $\ell+2$ -dimensional vector space over \mathbb{C} with the (non-degenerate) symmetric bilinear form $(\cdot|\cdot)$, extended from \mathfrak{h} by letting $\mathfrak{h} \perp (\mathbb{C}K + \mathbb{C}d)$, $(K|K) = 0 = (d|d)$, $(K|d) = 1$. We shall identify $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$ using this form. Given $\lambda \in \hat{\mathfrak{h}}$, we denote by $\bar{\lambda}$ its projection on \mathfrak{h} . Let $X = \{h \in \hat{\mathfrak{h}} \mid \operatorname{Re}(K|h) > 0\}$.

Define the following representation of the additive group of the vector space \mathfrak{h} on the vector space $\hat{\mathfrak{h}}$ (cf. (0.5)):

$$t_{\alpha}(h) = h + (K|h)\alpha - ((\alpha|h) + \frac{(\alpha|\alpha)}{2}(K|h))K, \quad \alpha \in \mathfrak{h}.$$

This action leaves the bilinear form $(\cdot|\cdot)$ on $\widehat{\mathfrak{h}}$ invariant and fixes K , hence leaves the domain X invariant.

The additive group of \mathfrak{h} also acts on $\widehat{\mathfrak{h}}$ by the affine transformations

$$p_\alpha(h) = h + 2\pi i\alpha, \quad \alpha \in \mathfrak{h},$$

which leave X invariant.

Denote by $N'_\mathbb{Z}$ the subgroup of affine transformations of the domain X , generated by the transformations t_α and p_β for all $\alpha, \beta \in L$. (This is a subgroup of the group $N_\mathbb{Z}$, considered in [K], Chapter 13.)

A theta function of degree k is a holomorphic function F in the domain X , satisfying the following two properties ($h \in X$):

- (i) $F(n(h)) = F(h)$ for all $n \in N'_\mathbb{Z}$,
- (ii) $F(h + aK) = e^{ka}F(h)$ for all $a \in \mathbb{C}$.

Denote by $\widetilde{\text{Th}}_k$ the space of all theta functions of degree k . Then $\widetilde{\text{Th}}_0$ is the algebra of holomorphic functions in τ ($\text{Im } \tau > 0$), [K], Lemma 3.2, and $\widetilde{\text{Th}} := \bigoplus_{\substack{k \geq 0 \\ kL \subset L^*}} \widetilde{\text{Th}}_k$ is a $\mathbb{R}_{\geq 0}$ -graded algebra over the subalgebra $\widetilde{\text{Th}}_0$.

Let D be the Laplace operator on $\widehat{\mathfrak{h}}$, associated to the bilinear form $(\cdot|\cdot)$, i.e. $De^h = (h|h)e^h$, $h \in \widehat{\mathfrak{h}}$, and let $\text{Th} = \bigoplus_{\substack{k \geq 0 \\ kL \subset L^*}} \text{Th}_k$ denote the kernel of D in $\widetilde{\text{Th}}$. This is an $\mathbb{R}_{\geq 0}$ -graded algebra over \mathbb{C} . Elements of Th_k are called classical theta functions (or Jacobi forms) of degree k .

For $k > 0$, such that $kL \subset L^*$, let

$$P_k = \left\{ \lambda \in \widehat{\mathfrak{h}} \mid (\lambda|K) = k \text{ and } \bar{\lambda} \in L^* \right\}.$$

Given $\lambda \in P_k$, let

$$\Theta_\lambda = e^{-\frac{(\lambda|\lambda)}{2k}K} \sum_{\alpha \in L} e^{t_\alpha(\lambda)}.$$

This series converges to a holomorphic function in the domain X , which is an example of a theta function (=Jacobi form). Note that

$$\Theta_{\lambda+k\alpha+aK} = \Theta_\lambda \quad \text{for } \alpha \in L, a \in \mathbb{C}.$$

Proposition A.1. *The set $\{\Theta_\lambda \mid \lambda \in P_k \bmod(kL + \mathbb{C}K)\}$ is a \mathbb{C} -basis of Th_k (resp. $\widetilde{\text{Th}}_0$ -basis of $\widetilde{\text{Th}}_k$) if $k > 0$, and $\text{Th}_0 = \mathbb{C}$.*

Proof. It is the same as that of Proposition 13.3 and Lemma 13.2 in [K]. □

Introduce coordinates (τ, z, t) on $\widehat{\mathfrak{h}}$ by (0.1), so that $X = \{(\tau, z, t) \mid \text{Im } \tau > 0\}$ and $q := e^{2\pi i\tau} = e^{-K}$. In these coordinates we have the usual formula for the Jacobi form Θ_λ , $\lambda \in P_k$, of degree $k > 0$:

$$(A.1) \quad \Theta_\lambda(\tau, z, t) = e^{2\pi ikt} \sum_{\gamma \in L + \frac{\bar{\lambda}}{k}} q^{\frac{k(\gamma|\gamma)}{2}} e^{2\pi ik(\gamma|z)}.$$

Proposition A.2. *One has the following transformation formulae of a Jacobi form Θ_λ of degree $k \in \mathbb{R}_{>0}$, such that $kL \subset L^*$, where $\lambda \in P_k$:*

$$(a) \quad \Theta_\lambda(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{(z|z)}{2\tau}) = (-i\tau)^{\frac{\ell}{2}} |L^*/kL|^{-\frac{1}{2}} \sum_{\mu \in P_k \bmod (kL + \mathbb{C}K)} e^{-\frac{2\pi i}{k}(\bar{\lambda}|\bar{\mu})} \Theta_\mu(\tau, z, t).$$

$$(b) \quad \Theta_\lambda(\tau + 1, z, t) = e^{2\pi i k t} e^{\frac{\pi i(\bar{\lambda}|\bar{\lambda})}{k}} \sum_{\alpha \in L} (-1)^{k(\alpha|\alpha)} e^{2\pi i(\bar{\lambda} + k\alpha|z)} q^{\frac{|\bar{\lambda} + k\alpha|^2}{2k}}.$$

In particular,

$$\Theta_\lambda(\tau + 1, z, t) = e^{\frac{\pi i(\bar{\lambda}|\bar{\lambda})}{k}} \Theta_\lambda(\tau, z, t),$$

hence the \mathbb{C} -span of $\{\Theta_\lambda\}_{\lambda \in P_k \bmod (kL + \mathbb{C}K)}$ is $\text{SL}_2(\mathbb{Z})$ -invariant (up to the weight factor), provided that $k(\alpha|\alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$.

Proof. It is the same as in [KW], using the same argument as in [K], Theorem 13.5, where in place of Lemma 13.4 we use that (S, j) normalizes the group $N'_\mathbb{Z}$. \square

Let $L_{\bar{0}}$ (resp. $L_{\bar{1}}$) = $\{\alpha \in L \mid k(\alpha|\alpha) \in 2\mathbb{Z}$ (resp. $\in 1 + 2\mathbb{Z}\})$. Proposition A.2 says that the \mathbb{C} -span of $\{\Theta_\lambda\}_{\lambda \in P_k \bmod (kL + \mathbb{C}K)}$ is $\text{SL}_2(\mathbb{Z})$ -invariant, provided that $L = L_{\bar{0}}$.

Now we construct an $\text{SL}_2(\mathbb{Z})$ -invariant family of classical theta functions in the case when $L \neq L_{\bar{0}}$. In this case $L_{\bar{1}} = \beta_0 + L_{\bar{0}}$ for any $\beta_0 \in L$, such that $k(\beta_0|\beta_0) \in 1 + 2\mathbb{Z}$. In what follows we shall often write $\Theta_\lambda = \Theta_{\lambda, L}$ in order to emphasize the dependence on L . We let:

$$\Theta_\lambda^\pm = \Theta_{\lambda, L_{\bar{0}}} \pm \Theta_{\lambda + k\beta_0, L_{\bar{0}}}.$$

Note that $\Theta_\lambda^+ = \Theta_{\lambda, L}$ and that Θ_λ^- is an alternate analogue of a Jacobi form:

$$\Theta_\lambda^- = e^{-\frac{(\lambda|\lambda)}{2k}K} \sum_{\alpha \in L} (-1)^{k(\alpha|\alpha)} e^{t_\alpha(\lambda)}.$$

Fix $\gamma_0 \in L_{\bar{0}}^* \setminus L^*$, and let

$$\Theta_\lambda^{\pm, \gamma_0} = \Theta_{\lambda + \gamma_0, L_{\bar{0}}} \pm \Theta_{\lambda + \gamma_0 + k\beta_0, L_{\bar{0}}}.$$

Note that the \mathbb{C} -span of the classical theta functions Θ_λ^\pm and $\Theta_\lambda^{\pm, \gamma_0}$ of degree k is the span of all classical theta functions of degree k for the lattice $L_{\bar{0}}$.

It is easy to deduce the next proposition from Proposition A.2.

Proposition A.3. *Let $k \in \mathbb{R}_{>0}$ be such that $kL \subset L^*$ and assume that $L_{\bar{0}} \neq L$, and fix $\gamma_0 \in L_{\bar{0}}^* \setminus L^*$. Let $\lambda \in P_k$. Then:*

$$(a) \quad \Theta_\lambda^-(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{(z|z)}{2\tau}) =$$

$$(-i\tau)^{\frac{\ell}{2}} |L^*/kL|^{-\frac{1}{2}} \sum_{\mu \in P_k \bmod kL + \mathbb{C}K} e^{-\frac{2\pi i}{k}(\bar{\lambda}|\bar{\mu} + \gamma_0)} \Theta_\mu^{-, \gamma_0}(\tau, z, t);$$

$$\Theta_\lambda^{+, \gamma_0}(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{(z|z)}{2\tau}) =$$

$$(-i\tau)^{\frac{\ell}{2}} |L^*/kL|^{-\frac{1}{2}} \sum_{\mu \in P_k \bmod kL + \mathbb{C}K} e^{-\frac{2\pi i}{k}(\bar{\lambda} + \gamma_0|\bar{\mu})} \Theta_\mu^-(\tau, z, t);$$

$$\Theta_{\lambda}^{-,\gamma_0}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{(z|z)}{2\tau}\right) =$$

$$(-i\tau)^{\frac{\ell}{2}} |L^*/kL|^{-\frac{1}{2}} \sum_{\mu \in P_k \bmod kL + \mathbb{C}K} e^{-\frac{2\pi i}{k}(\bar{\lambda} + \gamma_0|\bar{\mu} + \gamma_0)} \Theta^{-,\gamma_0}(\tau, z, t).$$

$$(b) \quad \Theta_{\lambda}^{\pm}(\tau + 1, z, t) = e^{\frac{\pi i}{k}(\bar{\lambda}|\bar{\lambda})} \Theta_{\lambda}^{\mp}(\tau, z, t); \quad \Theta_{\lambda}^{\pm, \gamma_0}(\tau + 1, z, t) = e^{\frac{\pi i}{k}(\bar{\lambda} + \gamma_0|\bar{\lambda} + \gamma_0)} \Theta_{\lambda}^{\pm, \gamma_0}(\tau, z, t).$$

Consequently, the \mathbb{C} -span of the classical theta functions $\left\{ \Theta_{\lambda}^{+}, \Theta_{\lambda}^{-}, \Theta_{\lambda}^{+, \gamma_0}, \Theta_{\lambda}^{-, \gamma_0} \right\}_{\lambda \in P_k \bmod kL + \mathbb{C}K}$ is $\text{SL}_2(\mathbb{Z})$ -invariant. \square

Proposition A.4. *One has the following elliptic transformation properties:*

$$\Theta_{\lambda}^{\pm}(\tau, z + \beta, t) = \Theta_{\lambda}^{\pm}(\tau, z, t); \quad \Theta_{\lambda}^{\pm, \gamma_0}(\tau, z + \beta, t) = (-1)^{k(\beta|\beta)} \Theta_{\lambda}^{\pm, \gamma_0}(\tau, z, t) \quad \text{if } \beta \in L.$$

$$\Theta_{\lambda}^{\pm}(\tau, z + \tau\beta, t) = e^{-2\pi i k(\beta|z)} q^{-\frac{k}{2}(\beta|\beta)} \Theta_{\lambda+k\beta}^{\mp}(\tau, z, t);$$

$$\Theta_{\lambda}^{\pm, \gamma_0}(\tau, z + \tau\beta, t) = e^{-2\pi i k(\beta|z)} q^{-\frac{k}{2}(\beta|\beta)} \Theta_{\lambda+k\beta}^{\mp, \gamma_0}(\tau, z, t) \quad \text{if } \beta \in \frac{1}{k}L^*.$$

\square

References

- [A] T. Arakawa, Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture, *Duke Math. J.* 130 (2005), 435-478.
- [GK] M. Gorelik, V. G. Kac, On admissible modules over affine Lie superalgebras, in preparation.
- [K] V. G. Kac, *Infinite-dimensional Lie algebras*, Third edition, Cambridge University press, 1990.
- [KRW] V. G. Kac, S.-S. Roan, M. Wakimoto, Quantum reduction of affine superalgebras, *Comm. Math. Phys.* 241 (2003), 307-342.
- [KW] V. G. Kac, M. Wakimoto, Representations of affine superalgebras and mock theta functions, *arXiv:1308.1261*
- [KW1] V. G. Kac, M. Wakimoto, Classification of modular invariant representations of affine algebras, *Adv. Ser. Math. Phys.* 7, World Sci. 1989, pp 138-177.
- [KW2] V. G. Kac, M. Wakimoto, Quantum reduction and representation theory of superconformal algebras, *Adv. Math.* 185 (2004), 400-458. Corrigendum *Adv. Math.* 193 (2005), 453-455.
- [KW3] V. G. Kac, M. Wakimoto, Quantum reduction in the twisted case, *Progress in Math.* 237, Birkhauser, 2005, 85-126.

- [S] V. Serganova, Kac-Moody superalgebras and integrability, in Developments and trends in infinite-dimensional Lie theory, Progress in Math. 288, Birkhauser, Boston, 2011, pp 4281-4299.
- [Z] S. P. Zwegers, Mock theta functions, arXiv:0807.4834