

MIT Open Access Articles

A PI degree theorem for quantum deformations

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

Citation: Etingof, Pavel. "A PI Degree Theorem for Quantum Deformations." Journal of Algebra 466 (November 2016): 308–313.

As Published: <http://dx.doi.org/10.1016/J.JALGEBRA.2016.07.026>

Publisher: Elsevier BV

Persistent URL: <http://hdl.handle.net/1721.1/119410>

Version: Original manuscript: author's manuscript prior to formal peer review

Terms of use: Creative Commons Attribution-NonCommercial-NoDerivs License



A PI DEGREE THEOREM FOR QUANTUM DEFORMATIONS

PAVEL ETINGOF

1. INTRODUCTION

Let F be an algebraically closed field. We show that if a quantum formal deformation A of a commutative domain A_0 over F is a PI algebra, then A is commutative if $\text{char}(F) = 0$, and has PI degree a power of p if $\text{char}(F) = p > 0$. This implies the same result for filtered deformations (i.e., filtered algebras A such that $\text{gr}(A) = A_0$).

Note that a quantum formal deformation of a commutative domain A_0 may fail to be PI, even for finitely generated A_0 in characteristic p (Example 3.3(2)). However, we don't know if this is possible for filtered deformations. Thus we propose

Question 1.1. Let $\text{char}(F) = p > 0$, and A be a filtered deformation of a commutative finitely generated domain A_0 over F . Must A be a PI algebra? In other words, must the division ring of quotients of A be a central simple algebra?

This question is closely related to the question asked in the introduction to [CEW], which would have affirmative answer if the answer to Question 1.1 is affirmative. We don't know the answer to either of these questions even when A_0 is a polynomial algebra with generators in positive degrees.

Acknowledgements. The author is grateful to K. Brown, C. Walton and J. Zhang for useful discussions. The work of the author was partially supported by the NSF grant DMS-1502244.

2. DEFORMATIONS OF FIELDS

Let F be an algebraically closed field, and A_0 a field extension of F . Let A be a quantum formal deformation of A_0 over $F[[\hbar]]$, i.e. an $F[[\hbar]]$ -algebra isomorphic to $A_0[[\hbar]]$ as an $F[[\hbar]]$ module and equipped with an isomorphism of algebras $A/(\hbar) \cong A_0$ (for basics and notation on deformations, see [EW], Section 2).

Theorem 2.1. *Suppose that A is a PI algebra of degree d .*

(i) If $\text{char} F = 0$, then $d = 1$ (i.e., A is commutative).

(ii) If $\text{char} F = p > 0$, then d is a power of p .

Proof. Let C be the center of A . It is easy to see that the division algebra of quotients of A is $A[\hbar^{-1}]$ with center $C[\hbar^{-1}]$ (see [EW], Example 2.7). Moreover, by Posner's theorem ([MR], 13.6.5), $A[\hbar^{-1}]$ is a central division algebra over $C[\hbar^{-1}]$ of degree d , so $[A[\hbar^{-1}] : C[\hbar^{-1}]] = d^2$.

Let $C_0 = C/(\hbar)$. It is clear that C_0 is a subfield of A_0 , and C is a (commutative) formal deformation of C_0 .

Lemma 2.2. $[A_0 : C_0] = d^2$.

Proof. Let $a_1^0, \dots, a_m^0 \in A_0$ be linearly independent over C_0 . Let a_1, \dots, a_m be lifts of these elements to A . Then a_1, \dots, a_m are linearly independent over C and hence over $C[\hbar^{-1}]$. Thus $m \leq d^2$. Moreover, if a_1^0, \dots, a_m^0 are a basis of A_0 over C_0 then a_1, \dots, a_m are a free basis of A over C and hence a basis of $A[\hbar^{-1}]$ over $C[\hbar^{-1}]$, so $m = d^2$. \square

Now for every integer $r \geq 0$, let $A_r \subset A_0$ be the field of all $x \in A_0$ which admit a lift to a central element of $A/(\hbar^{r+1})$. Note that $A_r \supset A_{r+1}$, and by Lemma 2.2, this is a finite field extension.

Let us now prove (i). Assume the contrary, i.e. that A is noncommutative. Let r be the largest integer such that $[a, b] \in \hbar^r A$ for all $a, b \in A$. Then we have a nonzero Poisson bracket on A_0 given by $\{a_0, b_0\} = \hbar^{-r}[a, b] \bmod \hbar$, where a, b are any lifts of a_0, b_0 to A . Moreover, by definition $\{, \}$ is bilinear over A_r . Recall that $\{, \}$ is a derivation in each argument, and that any K -linear derivation of a finite extension of a field K of characteristic zero vanishes. Since $[A_0 : A_r] < \infty$, this implies that $\{, \} = 0$, a contradiction. This proves (i).

We now prove (ii).

Lemma 2.3. For large enough r , $A_r = C_0$.

Proof. For nonnegative integers $r \geq s$, let $C_{r,s} \subset A/(\hbar^{s+1})$ be the set of elements liftable to a central element of $A/(\hbar^{r+1})$. It is clear that $C_{s,s}$ is the center of $A/(\hbar^{s+1})$, $C_{r,s} \supset C_{r+1,s}$, and $C_{r,s-1}$ is a quotient of $C_{r,s}$. Also $C_{r,s}$ is a $C_0/(\hbar^{s+1})$ -submodule of $A/(\hbar^{s+1})$. Let $C_{\infty,s}$ be the intersection of $C_{r,s}$ over all r . By Lemma 2.2, $A/(\hbar^{s+1})$ has finite length as a $C_0/(\hbar^{s+1})$ -module, so $C_{\infty,s} = C_{r(s),s}$ for a suitable $r(s)$. This implies that the natural map $C_{\infty,s} \rightarrow C_{\infty,s-1}$ is surjective (as it coincides with the map $C_{r,s} \rightarrow C_{r,s-1}$ for a suitable r). Let $C_{\infty,\infty} = \varprojlim C_{\infty,s} \subset A$.

We claim that any element $a \in C_{\infty,\infty}$ is central in A . Indeed, a projects to $a_s \in C_{\infty,s} \subset C_{s,s}$ which is central in $A/(\hbar^{s+1})$. Hence for any $b \in A$ we have $[a, b] = O(\hbar^{s+1})$. Since this holds for all s , we get that $[a, b] = 0$.

This implies that $C_{\infty, \infty} = C$ (as $C_{\infty, \infty}$ clearly contains C). Hence $C_{\infty, s} = C/(\hbar^{s+1})$ and in particular $C_{\infty, 0} = C_0$. Hence $C_{r, 0} = C_0$ for a large enough r . But by definition $C_{r, 0} = A_r$, which implies the lemma. \square

Lemma 2.4. *For all $r \geq 0$, one has $A_{r+1} \supset A_r^p$.*

Proof. Let $a_0 \in A_r$, and a be its lift to A central modulo \hbar^{r+1} . Let $b \in A$. We have

$$[a^p, b] = \sum_{i=0}^{p-1} a^i [a, b] a^{p-1-i} = p[a, b] a^{p-1} + \sum_{i=0}^{p-1} [a^i, [a, b]] a^{p-1-i} =$$

$$\sum_{i=0}^{p-1} [a^i, [a, b]] a^{p-1-i}$$

(as we are in characteristic p). We have $[a, b] \in \hbar^{r+1}A$, hence $[a^i, [a, b]] \in \hbar^{r+2}A$. Thus $a^p \in A_{r+1}$. \square

Lemma 2.4 implies that A_r is a purely inseparable extension of A_{r+1} . In particular, $[A_r : A_{r+1}]$ is a power of p . Since by Lemma 2.3 $A_r = C_0$ for large r , this implies that $[A_0 : C_0]$, and hence d , is a power of p , as desired. \square

Remark 2.5. Here is another proof of Theorem 2.1 (which deviates from the above proof after Lemma 2.2). By Lemma 2.2, it suffices to show that A_0 is a purely inseparable extension of C_0 (in particular, $A_0 = C_0$ in characteristic zero). To this end, consider the algebra $B := A \otimes_C A^{\text{op}}$. Since $A[\hbar^{-1}]$ is a central division algebra of degree d over $C[\hbar^{-1}]$, we have $B[\hbar^{-1}] \cong \text{Mat}_d(C[\hbar^{-1}])$, hence B does not contain nontrivial central idempotents. Therefore, the same holds for $B_0 := B/(\hbar)$ (otherwise we would have a nontrivial decomposition $B_0 = B'_0 \oplus B''_0$, which would lift to a decomposition $B = B' \oplus B''$, and $1_{B'}$ would be a nontrivial central idempotent in B). But $B_0 = A_0 \otimes_{C_0} A_0$. Hence B_0 has no nontrivial idempotents (i.e., is local). Let $x \in A_0$ be a separable element over C_0 and $K := C_0[x] \subset A_0$. Then $K \otimes_{C_0} K \subset A_0 \otimes_{C_0} A_0$ is reduced and projects onto K , hence contains nontrivial idempotents unless $K = C_0$. Hence $x \in C_0$, and A_0 is purely inseparable over C_0 , as desired.

3. DEFORMATIONS OF DOMAINS

Let us now extend Theorem 2.1 to deformations of domains.

Theorem 3.1. *Theorem 2.1 holds more generally, if A_0 is a domain over F .*

Proof. Following [EW], Subsection 2.2, let $Q(A) = \varprojlim Q(A/(\hbar^{N+1}))$, where $Q(A/(\hbar^{N+1}))$ is the classical quotient ring of $A/(\hbar^{N+1})$.¹ Also, let $Q_*(A) \subset Q(A)[\hbar^{-1}]$ be the quotient division algebra of A (which exists since A is a PI domain). Then $Q_*(A)$ is dense in $Q(A)[\hbar^{-1}]$ in the \hbar -adic topology (although in general $Q_*(A) \neq Q(A)$), and hence satisfies the same polynomial identities as $Q(A)[\hbar^{-1}]$. By Posner's theorem, $Q_*(A)$ is a central division algebra of degree d , hence so is $Q(A)[\hbar^{-1}]$ (as it is a division algebra satisfying the identities of $d \times d$ matrices but not matrices of smaller size). Also, $Q(A)$ is a formal quantum deformation of the quotient field $Q(A_0)$ of A_0 , which is a field extension of F . Thus, Theorem 2.1 applies to $Q(A)$, and the theorem is proved. \square

Corollary 3.2. *Let A be a \mathbb{Z}_+ -filtered deformation of a commutative domain A_0 over F (i.e., $\text{gr}(A) = A_0$). Suppose that A is a PI algebra of degree d .*

- (i) *If $\text{char} F = 0$, then $d = 1$ (i.e., A is commutative).*
- (ii) *If $\text{char} F = p > 0$ then d is a power of p .*

Proof. Let $R(A)$ be the Rees algebra of R and $\widehat{R}(A)$ the completed Rees algebra of A (see e.g. [EW], Subsection 2.1). Then $R(A)$ satisfies the identities of matrices of size $d \times d$ but not smaller (since so does A). Since $R(A)$ is dense in $\widehat{R}(A)$ in the \hbar -adic topology, the same holds for $\widehat{R}(A)$. But $\widehat{R}(A)$ is a formal quantum deformation of A . Thus Theorem 3.1 implies the result. \square

Example 3.3. 1. Suppose $\text{char} F = p > 0$. Let A be the formal n -th Weyl algebra, i.e. the \hbar -adically complete algebra over $F[[\hbar]]$ generated by $x_1, \dots, x_n, y_1, \dots, y_n$ with defining relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad [y_i, x_j] = \hbar \delta_{ij}.$$

Then A is a formal deformation of $A_0 := F[x_1, \dots, x_n, y_1, \dots, y_n]$, which is the completed Rees algebra of its filtered deformation (the usual Weyl algebra $\mathbf{A}_n(F)$). The center A is $C = F[x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p][[\hbar]]$, so A is PI of degree $d = p^n$. Note that if we have infinitely many generators x_i, y_i then A is not PI, so the “finitely generated” assumption in Question 1.1 is needed.

2. A formal quantum deformation of a finitely generated commutative domain does not have to be PI, even in characteristic p . E.g., let A be the formal quantum polynomial algebra, i.e. the \hbar -adically complete algebra generated by x, y with relation $yx = (1 + \hbar)xy$. This algebra is a quantum formal deformation of $A_0 := F[x, y]$. It has trivial center and hence is not PI.

¹The characteristic zero assumption of [EW] is not used in these considerations.

Remark 3.4. Here is a direct proof of Corollary 3.2(i), bypassing localizations and formal deformations.

Let C be the center of A and $C_0 = \text{gr}(C)$. We claim that A_0 is algebraic over C_0 . To show this, let $a_0 \in A_0$ be a homogeneous element, and lift it to an element $a \in A$. Since A is PI, by Posner's theorem it is algebraic over C , so there exists a nonzero $P \in C[t]$ such that $P(a) = 0$. Taking the leading terms of this equation gives a nonzero polynomial $P_0 \in C_0[t]$ such that $P_0(a_0) = 0$, as desired.

Now assume that A is noncommutative. Let $\{, \}$ be the nonzero Poisson bracket on A_0 defined in the proof of Theorem 2.1(i). Given $a_0 \in A_0$, the operator $\{a_0, ?\}$ is a derivation of A_0 which vanishes on C_0 . Since A_0 is algebraic over C_0 and $\text{char} F = 0$, this derivation vanishes, i.e., $\{, \} = 0$, a contradiction.

The same argument works for formal deformations (Theorem 3.1 when $\text{char} F = 0$).

Remark 3.5. Let us say that an algebra A is locally PI if any finitely generated subalgebra of A is PI. An example of such an algebra is the Weyl algebra $\mathbf{A}_{\mathcal{I}}(F)$ generated by x_i, y_i , $i \in \mathcal{I}$ for an infinite set \mathcal{I} and $\text{char} F = p > 0$. Corollary 3.2 immediately implies that if A is a locally PI filtered quantization of a commutative domain A_0 over F then A must be commutative if $\text{char}(F) = 0$, and the PI degree of every finitely generated subalgebra of A is a power of p if $\text{char}(F) = p > 0$. Thus, in the special case when A is a connected Hopf algebra equipped with the coradical filtration and $\text{char} F = 0$, we recover [BGZ], Theorem 4.5.

REFERENCES

- [BGZ] K. Brown, P. Gilmer, J. J. Zhang, Connected (graded) Hopf algebras, arXiv:1601.06687.
- [CEW] J. Cuadra, P. Etingof, and C. Walton, Semisimple Hopf actions on Weyl algebras, *Advances in Math.*, Volume 282, pp. 47–55, arXiv:1409.1644.
- [EW] P. Etingof and C. Walton, Finite dimensional Hopf actions on deformation quantizations, arXiv:1602.00532.
- [MR] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Graduate Studies in Mathematics, 30, American Mathematical Society, Providence, RI, 2001.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, USA

E-mail address: etingof@math.mit.edu