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Pursuit-Evasion Games in Dynamic Flow Fields via Reachability Set Analysis

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Abstract— **In this paper, we adopt a reachability-based approach to deal with the pursuit-evasion differential game between two players in the presence of dynamic environmental disturbances (e.g., winds, sea currents). We give conditions for the game to be terminated in terms of reachable set inclusions. Level set equations are defined and solved to generate the reachable sets of the pursuer and the evader. The corresponding time-optimal trajectories and optimal strategies can be readily retrieved afterwards. We validate our method by applying it to a pursuit-evasion game in a simple flow field, for which an analytical solution is available. We then implement the proposed scheme to a problem with a more realistic flow field.**

I. Introduction

Pursuit-evasion games is a subclass of differential games that has received a great deal of attention since the early 1960's mainly owing to its application for air combat scenarios. Starting from the seminal work by Isaacs in his book *Differential Games* [1], a large literature exists on the subject. From a theoretical point of view, the optimal strategies of both players (the pursuer and the evader) are given from the solution a nonlinear, partial differential equation (the Hamilton-Jacobi-Issacs equation). From a practical point of view, the problem is far from being solved, solutions of HJI equations are not readily available. This is especially the case for problems with multiple players having non-trivial dynamics. Much of the effort has been therefore devoted to establishing numerical techniques for the solution of pursuit-evasion problems under a minimal set of assumptions.

Despite the formidable character of the underlying HJI, several pursuit-evasion problems admit closed-form solutions. The solutions are often geometric in nature (thus are limited to problems of just two players on the plane) and involve the back-propagation of certain singular fronts from the terminal surface. Such methods are outlined in great detail in Isaacs's book. The Homicidal Chauffeur game [1], for instance, deals with a pursuit-evasion game between an evader having a finite maximum turning radius and an agile pursuer. A converse version of the Homicidal Chauffeur game, also known as the Suicidal Pedestrian game, was studied in [2], [3]. The game between two players with curvature constraints is studied in the Game of Two Cars [4]. A general result for this problem was presented in [5]. Other pursuit-evasion games under some specific conditions include the isotropic rocket problem [1] and the Lion and Man problem [6]. An extension of the game of pursuit with curvature constraints to the three-dimensional space was addressed in [7]. Stochastic differential games of two players have also been explored, including a stochastic version of the Homicidal Chauffeur game, addressed in [8].

Another framework several researchers have used when dealing with pursuit-evasion problems is based on reachable set analysis [9]–[11]. According to this approach, the reachable state space of both players is utilized to find the optimal controls of the pursuer and/or the evader. Reachable set analysis has been applied for performing missile/sensor trade-offs in homing guidance [12], for obtaining escape strategy under pursuit [13], and for finding pursuer control under control constraints [14].

Despite the previous work in this area, few approaches have taken into consideration how dynamic environmental conditions may affect the outcome of the game. For instance, when either the pursuer or the evader (or both) is a small autonomous underwater vehicle (AUV) or small unmanned aerial vehicle (UAV), the presence of sea currents or winds, respectively, may significantly affect the vehicle motion. As a result, during pursuit-evasion, the optimal behavior of these vehicles, as the solution of a differential game, may be greatly affected by the existence of the external dynamic flow field.

Some optimal control problems have taken into account the effect of an external flow field. For example, in [15] the authors address the problem of optimal guidance of a Dubins vehicle [16] in a flow field to a specified position. The minimum-time guidance problem for the isotropic rocket in the presence of wind has been studied in [17]. The problem of minimizing the expected time to steer a Dubins vehicle to a target set in a stochastic wind field has also been discussed in [18]. However, the same level of attention has not been shared in the literature for pursuit-evasion games under the influence of external disturbances.

In this paper, we consider a two-player pursuit-evasion game in an external dynamic flow field. Due to the generality of the external flow, Issacs' approach cannot be readily used. Instead, we find the optimal trajectories of the players

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through the evolution of their reachable sets. We utilize the level set method [19], [20] to generate the reachable sets and retrieve the corresponding optimal control actions at the current location of the players by backward propagation of their respective reachable sets. Repeated application of the procedure thus results in the calculation of the optimal feedback strategies of both players. Since the computation of the reachable sets can be performed independently for each player, the proposed procedure leads to a decentralized computation of the feedback strategies of all the players.

Level set methods have been previously applied by Tomlin et al. to solve pursuit-evasion games [21], [22]. The authors of [21] first reduce the degrees of freedom of the problem by reformulating it in terms of the relative distance between the pursuer and the evader. Then the level set method is applied to the corresponding Hamilton-Jacobi-Isaacs (HJI) equation to back-propagate the *backward reachable set* to solve the differential game directly. Our approach differs from those in [21], [22] since we do not attempt to solve the pursuit-evasion game directly by solving the HJI equation. Instead, we generate the *forward* reachable sets of the players separately and find the optimal time-to-capture as the first time when the reachable set of the evader is fully covered by the reachable set of the pursuer [10]. We then identify the first rendezvous point of the players and retrieve the optimal trajectories and controls of both players through backtracking of their respective trajectories [23]–[25]. The reason we choose this approach instead of the more direct approach in [21], [22] is due to the dimensionality of our problem. When we introduce dynamic environmental effects into the system, the pursuit-evasion problem cannot be reduced to a problem described solely in terms of their relative distance, unless some very restrictive assumptions are imposed on the structure of the external flow field [26]. We also note that an advantage of the forward reachable set approach is that it is efficient, even in realistic simulations with dynamic ocean currents that can be much larger than vehicle speeds [27]. On the other hand, working directly with the HJI equation is not easily generalizable to multiple players. The approach can also be combined with distance-based coordination of multiple vehicles and with dynamic obstacles [28].

II. Problem Formulation

Consider a pursuit-evasion game in an external dynamic flow field with a single pursuer P and a single evader E. The dynamics of the pursuer P is given by

$$\dot{X}_P(t) = u_P(t) + w(X_P(t), t), \qquad X_P(0) = X_{P_0}, \tag{1}$$

where $X_P(t) = [x_P(t), y_P(t)]^\intercal \in \mathbb{R}^2$ denotes the position of the pursuer, $u_P(t) \in \mathbb{R}^2$ is the control input of the pursuer that satisfies the piecewise constraint $u_P(t) \in U_P$, where $U_P = \{u \in \mathbb{R}^2, |u| \leqslant \bar{u}\}$, and $|\cdot|$ represents the 2-norm. In (1), $w(X,t) \in \mathbb{R}^2$ represents the external dynamic flow. It is reasonable to assume that the magnitude of this flow (e.g. winds or currents) is bounded from above by some constant, hence there exists a constant $\bar{w}$ such that $|w(X,t)| \leq \bar{w}$, for all (X,t). Here we assume that the effects of the external dynamic flow field on the pursuer and evader are identical.

The objective of the pursuer is to intercept an evader, whose kinematics is given by

$$\dot{X}_E(t) = u_E(t) + w(X_E(t), t), \qquad X_E(0) = X_{E_0}, \tag{2}$$

where $X_E(t) = [x_E(t), y_E(t)]^\intercal \in \mathbb{R}^2$ is the position of the evader, and $u_E(t)$ is its control input such that $u_E(t) \in U_E$, where $U_E = \{v \in \mathbb{R}^2, |v| \leqslant \bar{v}\}$.

Let $\bar{X} = [X_E^\intercal, X_P^\intercal]^\intercal \in \mathbb{R}^4$ denote the state of the game. Then the game begins at initial time $t_0 = 0$ with initial positions $\bar{X}_0 = [X_{E_0}^\intercal, X_{P_0}^\intercal]^\intercal$, and terminates when the pursuer reaches the location of the evader. The terminal time T of the game is defined by

$$T = \inf\{t \in \mathbb{R}^+ : X_P(t) = X_E(t)\}. \tag{3}$$

Let $J(\gamma_P, \gamma_E) = T$ be the cost function of the game, where $\gamma_P, \gamma_E : \mathbb{R}^+ \times \mathbb{R}^4 \mapsto \mathbb{R}^2$ denote the feedback strategies of the pursuer and the evader, respectively, such that $\gamma_P(t, \bar{X}) = u_P(t)$ and $\gamma_E(t, \bar{X}) = u_E(t)$. We assume that each player has perfect knowledge of the dynamics of the system represented by (1) and (2), the constraint sets U_P and U_E, the cost function J, as well as the initial state $\bar{X}_0$. It is also assumed that the value V of the game [1] exists, that is,

$$V = \min_{\gamma_P} \max_{\gamma_E} J = \max_{\gamma_E} \min_{\gamma_P} J. \tag{4}$$

The objective of this paper is to find the open-loop representation of the optimal strategies of the pursuer and the evader. In particular, we utilize a reachability-based method to obtain optimal controls $u_P^\star(t) = \gamma_P^\star(t, \bar{X}^\star(t))$ and $u_E^\star(t) = \gamma_E^\star(t, \bar{X}^\star(t))$, with $\bar{X}^\star(t)$ denoting the corresponding optimal state trajectory. Henceforth, we consider the control of the pursuer $u_P \in \mathcal{U}_P$, where $\mathcal{U}_P$ consists of all piecewise continuous functions, whose range is included in U_P, and call u_P an *admissible* control of the pursuer. Similarly, the control u_E is an *admissible* control of the evader if $u_E \in \mathcal{U}_E$, which consists of all piecewise continuous functions whose range is included in U_E.

III. Problem Analysis

A. Reachable Sets

A reachable (or attainable) set at a given time is defined as the set of points that can be visited by the agent at a particular time [29]. The boundary of the reachable set is the *reachability front*. In particular, the reachable set of the pursuer at time $t \geq 0$, denoted by $\mathcal{R}_P(X_{P_0}, t)$, is the set of all points $X \in \mathbb{R}^2$ such that there exists a trajectory satisfying (1), with initial position X_{P_0} and terminal position X at time t. Similarly, the reachable set $\mathcal{R}_E(X_{E_0}, t)$ of the evader at time $t \geq 0$ is the set of all points $X \in \mathbb{R}^2$ such that there exists a trajectory satisfying (2), with initial position X_{E_0} and terminal position X at time t. The reachability fronts of the pursuer and the evader at time $t \geq 0$ are denoted by $\partial\mathcal{R}_P(X_{P_0}, t)$ and $\partial\mathcal{R}_E(X_{E_0}, t)$, respectively. We also denote by $\mathcal{R}_E^\star(X_{E_0}, t)$ the *usable reachable set* of the evader, which is the set of all terminal points (at time t) of admissible trajectories of the evader that do not pass through the reachable set of the pursuer at *any* time in the interval $[0, t]$. In other words, $\mathcal{R}_E^\star(X_{E_0}, t)$ is the set of terminal points of all the 'safe' evader trajectories.

These definitions with respect to the reachable sets lead to the following proposition, which is an extension of Theorem I in [10], where the authors derive the condition for capture under the assumption of linear dynamics for both players and a finite energy constraint for the controls.

Proposition 3.1: Let $T = \inf\{t \in \mathbb{R} : \mathcal{R}_E^\star(X_{E_0}, t) = \emptyset\}$. If $T < \infty$, then capture is guaranteed for any time greater than T, while the evader can always escape within a time smaller than T. That is, T is the optimal time-to-capture. Moreover, let X_f denote the location where the evader is captured by the pursuer. Then X_f lies on the intersection of the reachability front of the pursuer $\partial\mathcal{R}_P(X_{P_0}, T)$ and the reachability front of the evader $\partial\mathcal{R}_E(X_{E_0}, T)$.

Proof: Since $\mathcal{U}_P$ is compact and convex, it follows that, for each (t, x), the set $\{u_P + w(X_P, t) : u_P \in \mathcal{U}_P\}$ is compact and convex. Also, since u_P and $w(X_P, t)$ are bounded by assumption, the solution of (1) exists on $[0, t_f]$, for all finite t_f. Therefore, by Filippov's Theorem [30], the reachable set $\mathcal{R}_P(X_{P_0}, t)$ is compact, for all $t \in [0, t_f]$. Similarly, $\mathcal{R}_E(X_{E_0}, t)$ is compact, for all $t \in [0, t_f]$. Since $\mathcal{R}_E^\star(X_{P_0}, t) \subseteq \mathcal{R}_E(X_{E_0}, t)$, $\mathcal{R}_E^\star(X_{P_0}, t)$ is bounded for all $t \in [0, t_f]$.

Since $\mathcal{R}_E^\star(X_{E_0}, T) = \emptyset$, it follows that for any point $X \in \mathcal{R}_E(X_{E_0}, T)$ that can be visited by the evader at time T through an admissible evading control $u_E \in \mathcal{U}_E$, it is also true that $X \in \mathcal{R}_P(X_{P_0}, T)$. In other words, there exists an admissible control of the pursuer $u_P \in \mathcal{U}_P$ such that $X_P(T) = X$. Therefore, regardless of the strategy it picks, the evader can be captured by the pursuer at time T. This implies that capture is also guaranteed for any time greater than T.

On the other hand, since $t = T$ is the first time such that $\mathcal{R}_E^\star(X_{E_0}, t) = \emptyset$ is satisfied, it follows that $\mathcal{R}_E^\star(X_{E_0}, t) \neq \emptyset$ for all $0 \le t < T$. Hence, for all $t \in [0, T)$, there exists $X_t \in \mathcal{R}_E(X_{E_0}, t)$ such that $X_t \notin \mathcal{R}_P(X_{P_0}, t)$. That is, for any time $t \in [0, T)$, there exist an admissible control for the evader to reach X_t such that X_t cannot be visited by the pursuer at time t through any admissible control. It follows that the evader can always avoid capture before time T.

From the two previous statements, we can conclude that T is the optimal time-to-capture.

Let X be the point that is the intersection of the reachability front of the pursuer $\partial\mathcal{R}_P(X_{P_0}, T)$ and the reachability front of the evader $\partial\mathcal{R}_E(X_{E_0}, T)$. Then X is a point in $\mathcal{R}_E(X_{E_0}, T)$ that the pursuer cannot reach before time T (by definition of $\mathcal{R}_E^\star$). This implies that X should be the destination of the evader if the latter aims to maximize the time-to-capture. On the other hand, the pursuer also needs to reach X if (s)he would like to capture the evader. Therefore, the location X_f where the evader is captured by the pursuer must coincide with X, which completes the proof. ■

Remark 1: In cases when $\bar{u} \ge \bar{v}$, the safe reachable set of the evader $\mathcal{R}_E^\star(X_{E_0}, t)$ satisfies

$$\mathcal{R}_E^\star(X_{E_0}, t) = \mathcal{R}_E(X_{E_0}, t) \backslash \mathcal{R}_P(X_{P_0}, t), \tag{5}$$

for all $t \ge 0$. In such cases, the condition $\mathcal{R}_E^\star(X_{E_0}, t) = \emptyset$ is equivalent to the condition $\mathcal{R}_E(X_{E_0}, t) \subseteq \mathcal{R}_P(X_{P_0}, t)$. Then, the optimal time-to-capture is the first time when the reachable set of the pursuer $\mathcal{R}_P(X_{P_0}, t)$ completely covers the reachable set of the evader $\mathcal{R}_E(X_{E_0}, t)$.

Proposition 3.1 is valid under the assumption that capture is guaranteed at some finite time. We provide a sufficient condition for this to occur in the next theorem.

Theorem 3.2: Assume $w(X, t)$ satisfies the triangle inequality in X and its norm is bounded from above by a constant $\lambda > 0$, where $\lambda < \bar{u} - \bar{v}$. Then the game terminates in finite time regardless of the initial positions between the pursuer and the evader. Furthermore, the time-to-capture satisfies the upper bound

$$T \le \frac{|X_{E_0} - X_{P_0}|}{\bar{u} - \bar{v} - \lambda}. \tag{6}$$

Proof: Let $\Delta X = X_E - X_P$. We have that

$$\begin{aligned}\frac{\mathrm{d}|\Delta X|}{\mathrm{d}t} &= \frac{\mathrm{d}\Delta X}{\mathrm{d}t}^{\intercal} \frac{\Delta X}{|\Delta X|} \\ &= \big(u_E - u_P + w(X_E, t) - w(X_P, t)\big)^{\intercal} \frac{\Delta X}{|\Delta X|}.\end{aligned} \tag{7}$$

Since $w(X, t)$ satisfies the triangle inequality and is bounded, it follows that

$$|w(X_E, t) - w(X_P, t)| \le |w(X_E - X_P, t)| \le \lambda.$$

Therefore,

$$\begin{aligned}&\frac{\mathrm{d}|\Delta X|}{\mathrm{d}t} = \big(u_E - u_P + w(X_E, t) - w(X_P, t)\big)^{\intercal} \frac{\Delta X}{|\Delta X|} \\ &\le \big(u_E - u_P\big)^{\intercal} \frac{\Delta X}{|\Delta X|} + |w(X_E, t) - w(X_P, t)| \left|\frac{\Delta X}{|\Delta X|}\right| \\ &\le \min_{u_P} \max_{u_E} \left\{ \big(u_E - u_P\big)^{\intercal} \frac{\Delta X}{|\Delta X|} \right\} + \lambda \\ &\le \bar{v} - \bar{u} + \lambda.\end{aligned} \tag{8}$$

Note that (8) implies that the right-hand side of (7) is strictly negative, since it is assumed that $\lambda < \bar{u} - \bar{v}$. Thus, $|\Delta X|$ can be driven to 0 in finite time, for all initial conditions of the pursuer and the evader. Finally, (6) follows after integrating both sides of (8). ■

IV. Numerical Solution

A. Level Set Method

In order to construct the forward reachable sets of the pursuers and the evader, we utilize the level set method. The level set method is a convenient tool to track the evolution of the reachability front. It evolves an interface (front) by embedding it as a hyper-surface in a higher dimension, where time is the augmented dimension. Automatic handling of merging and pinching of fronts and other topological changes are made possible by such higher dimensional embedding. Level sets provide an implicit representation of the front, which offers several advantages over an explicit representation.

For any $c \in \mathbb{R}$, the c-level set of a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is the set $\{X \in \mathbb{R}^2 | \phi(X) = c\}$. We consider the *signed distance function*

$$\phi(X) = \begin{cases} \min_{Y \in \partial\mathcal{R}} |X - Y|, & \text{if } X \text{ is outside the front,} \\ -\min_{Y \in \partial\mathcal{R}} |X - Y|, & \text{if } X \text{ is inside the front.} \end{cases} \tag{9}$$

The signed distance function is one of the most commonly used implicit functions in level sets. It is smooth and monotonic across the interface. It also keeps fixed amplitude gradients in the field. For all $X \in \partial\mathcal{R}$, we have $\phi(X) = 0$. That is, the zero level set implicitly represents the reachability front. Moreover, the reachable set can be represented by $\{X \in \mathbb{R}^2 | \phi(X) \le 0\}$.

The reachability front $\partial\mathcal{R}_P(X_{P_0}, t)$ of the pursuer is governed by the viscosity solution of the Hamilton-Jacobi (HJ) equation [24], [25], [31]

$$\frac{\partial \phi_P(X,t)}{\partial t} + \bar{u}\, |\nabla \phi_P| + w(X,t) \nabla \phi_P = 0, \tag{10}$$

with initial condition $\phi_P(X, 0) = |X - X_{P_0}|$. Moreover, the reachable set of the pursuer coincides with the region(s) where ϕ_P is non-positive. Similarly, the reachability front $\partial\mathcal{R}_E(X_{E_0}, t)$ of the evader is given by the HJ equation

$$\frac{\partial \phi_E(X,t)}{\partial t} + \bar{v}\, |\nabla \phi_E| + w(X,t) \nabla \phi_E = 0, \tag{11}$$

with initial conditions $\phi_E(X, 0) = |X - X_{E_0}|$.

In the case when $\bar{v} > \bar{u}$, we need to track the propagation of $\partial\mathcal{R}^\star_E(X_{E_0}, t)$. Its reachability front can be computed by solving the following modified version of the Hamilton-Jacobi equation (11):

$$\frac{\partial \phi^\star_E(X,t)}{\partial t} + \tilde{v}(t)\, |\nabla \phi^\star_E| + w(X,t) \nabla \phi^\star_E = 0, \tag{12}$$

where

$$\tilde{v}(t) = \begin{cases} \bar{v}, & \text{if } \phi_P(X,t) \ge 0, \\ \bar{u}, & \text{otherwise.} \end{cases} \tag{13}$$

The main idea is to propagate $\mathcal{R}^\star_E(X_{E_0}, t)$ with the maximum speed of the evader $\bar{v}$ when it is outside the reachable set of the pursuer, and to keep pace with the propagation of $\partial\mathcal{R}_P(X_{P_0}, t)$ when the front of the evader enters the reachable set of the pursuer to make sure that it never grows out of the reachable set of the pursuer again. Note that $\mathcal{R}^\star_E(X_{E_0}, t)$ is represented by $\{X \in \mathbb{R}^2 | \phi^\star_E(X,t) \le 0 \text{ and } \phi_P(X,t) \ge 0\}$.

B. Time-Optimal Paths

As was shown in Proposition 3.1, the location X_f where the evader is captured by the pursuer is the first intersection of $\partial\mathcal{R}_P(X_{P_0}, T)$ and $\partial\mathcal{R}_E(X_{E_0}, T)$ when $\mathcal{R}_E(X_{E_0}, T) \subset \mathcal{R}_P(X_{P_0}, T)$. An example is presented in Figure 1. After we have identified the (common) terminal position of the pursuer and the evader, we can retrieve the optimal trajectories and optimal controls of both players by backward propagation along the reachable sets.

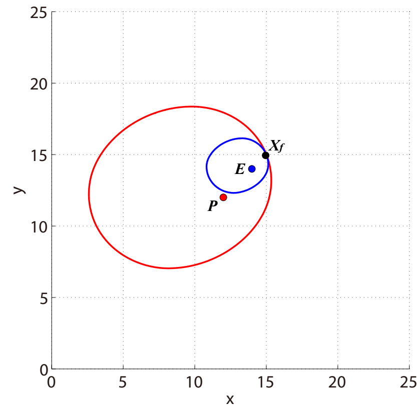


Fig. 1: Level sets of the pursuer in red and the evader in blue at time T, which is the first time such that $\mathcal{R}_E(X_{E_0}, T) \subseteq \mathcal{R}_P(X_{P_0}, T)$. X_f is the point common to both fronts, $\partial\mathcal{R}_P(X_{P_0}, T)$ and $\partial\mathcal{R}_E(X_{E_0}, T)$. The initial positions of the pursuer and the evader are depicted by red and blue dots, respectively.

In particular, the time-optimal trajectories $X^\star_P$ and $X^\star_E$ satisfy the following differential equations [24], when ϕ_P and ϕ_E are differentiable:

$$\frac{\mathrm{d}X^\star_P}{\mathrm{d}t} = \bar{u} \frac{\nabla \phi_P}{|\nabla \phi_P|} + w(X^\star_P, t), \tag{14}$$

$$\frac{\mathrm{d}X^\star_E}{\mathrm{d}t} = \bar{v} \frac{\nabla \phi_E}{|\nabla \phi_E|} + w(X^\star_E, t). \tag{15}$$

Hence, the corresponding time-optimal controls of the pursuer and the evader are

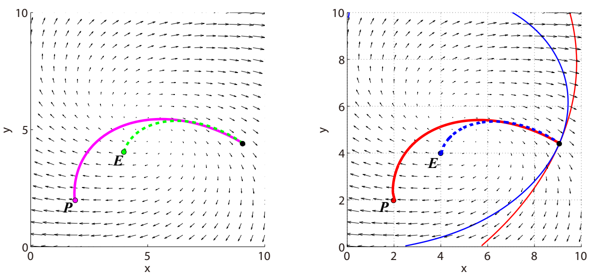
$$u^\star_P = \bar{u} \frac{\nabla \phi_P}{|\nabla \phi_P|}, \qquad u^\star_E = \bar{v} \frac{\nabla \phi_E}{|\nabla \phi_E|}. \tag{16}$$

C. Numerical Implementation

In this section, we present an algorithm to solve the pursuit-evasion game in an external flow field.

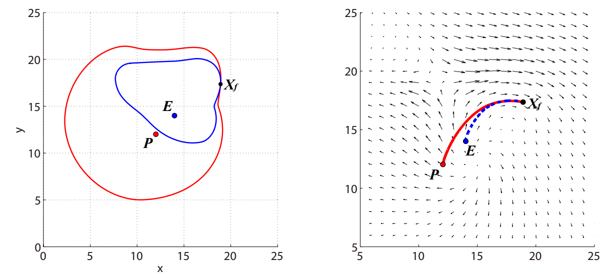
The algorithm contains the following three steps:

1. *Evolution of Forward Reachable Sets*: In cases when $\bar{u} \ge \bar{v}$, the forward reachable sets of the pursuer and the evader are evolved by computing the viscosity solutions to the unsteady HJ equations (10) and (11) respectively. These evolutions are carried out until the reachable set of the evader is fully covered by that of the pursuer. Otherwise ($\bar{v} > \bar{u}$), , we propagate $\partial\mathcal{R}_P(X_{P_0}, t)$ and $\partial\mathcal{R}^\star_E(X_{P_0}, t)$ with (10) and (12) until $\mathcal{R}^\star_E(X_{P_0}, t) = \emptyset$.
2. *End Point Identification*: When $\bar{u} \ge \bar{v}$, find the location of X_f where the pursuer captures the evader by identifying the intersection of the reachable fronts between the pursuer and the evader. Another numerical way to find X_f is by identifying the point on the reachability front of the evader at the terminal time that has the highest value of the pursuer's signed distance function. Otherwise ($\bar{v} > \bar{u}$), , X_f can be approximated by the point in $\mathcal{R}^\star_E(X_{P_0}, t)$ one time step before becomes the empty set.



(a) (b)

Fig. 2: (a) Optimal trajectories of the pursuer and the evader in magenta and dotted green respectively, generated from the analytical solution. (b) Optimal trajectories of the pursuer and the evader in red and dotted blue lines, respectively, generated from the reachable set approach. The red and blue curves on the right of the figure are part of the reachable fronts of the pursuer and the evader at the terminal time, respectively.



(a) (b)

Fig. 3: (a) Red and blue curves represent the reachable fronts of the pursuer and the evader at the terminal time, respectively. They intersect at X_f, where the pursuer captures the evader eventually. (b) Optimal trajectories of the pursuer and the evader in red and dotted blue lines, respectively, generated from the reachable set approach. The wind field is depicted in the background.

3. *Backward Trajectory Tracking*: When $\bar{u} \geq \bar{v}$, and after the reachability fronts of the pursuer and the evader meet at X_f, the optimal controls of the pursuer and the evader can be achieved through (16). Also, we can compute the optimal trajectories $X_P^\star$ and $X_E^\star$ of the pursuer and the evader, respectively, by solving (14) and (15) backwards starting from X_f at time $t = T$. Otherwise ($\bar{v} > \bar{u}$), we simply replace $\nabla\phi_E$ with $\nabla\phi_E^\star$ and follow the same procedure to find the optimal trajectories.

For more details about the numerical schemes for the propagation of level sets and for backtracking of the optimal trajectories, please refer to [23], [24], [31]–[33].

V. Simulation Results

In this section, we present simulation results of the pursuit-evasion problem under an external flow field. We first verify our numerical solution with an analytical solution under an affine flow field. We then apply our method to a problem with a more realistic representation of the flow field. Note that in both cases, we assume $\bar{u} > \bar{v}$.

When the external wind field is approximated by an affine function $w(X) = A(X - S_0) + b$, where $A \in \mathbb{R}^{2\times 2}$ and $S_0, b \in \mathbb{R}^2$ are constant matrix and constant vectors, respectively, then the problem can be solved through the standard Isaacs' differential game approach [26]. This wind field can be seen as a flow generated from a single singularity point located at S_0, with its characteristics captured by A and b. We set

$$A = \begin{bmatrix} 0.2 & 0.3 \\ -0.15 & 0.1 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The initial conditions of the pursuer and the evader are given by $X_{P_0} = [2, 2]^\intercal$ and $X_{E_0} = [4, 4]^\intercal$. The maximum speeds of the pursuer and the evader are set to $\bar{u} = 2$ and $\bar{v} = 1$, respectively. The optimal trajectories of the pursuer and the evader calculated from [26] are presented in Figure 2a, and the result generated by the method in this paper is shown in Figure 2b. They are identical to each other, as expected.

Next, we consider a wind field approximation generalized from the Rankine model of vortex [34]:

$$w(X) = w_0 + \sum_{i=1}^{n_s} \omega_i A_i (X - x_{s_i}), \tag{17}$$

where $\omega_i = 1/\max\{r_{s_i}^2, \|X - x_{s_i}\|^2\}$. In (17) n_s is the number of flow singularities, x_{s_i} is the location of the i-th flow singularity and r_{s_i} denotes the singularity radius. A_i is a 2×2 matrix, whose structure captures the local characteristics of the i-th flow singularity. The model approximates the velocity field of a vortex with a linear vector field inside a disk and the velocity outside of the disk decreases as the inverse squared distance to the center of the disk.

For our numerical simulation, we set the number of flow singularities to $n_s = 3$. The locations of the flow singularities are $x_{s_1} = [18, 18]^\intercal$, $x_{s_2} = [12, 19]^\intercal$, $x_{s_3} = [14, 12]^\intercal$, and the corresponding radii are $r_{s_1} = 3$, $r_{s_2} = 2$, $r_{s_3} = 3$, respectively. The local wind field matrices are given by

$$A_1 = \begin{bmatrix} 0 & 0.3 \\ -0.15 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.4 & 0.2 \\ 0 & -0.2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & 0.2 \end{bmatrix}.$$

We also choose $w_0 = [0.2, -0.3]$.

The reachable fronts of the pursuer and the evader at the terminal time are shown in Figure 3a. The corresponding optimal trajectories of the pursuer and the evader are shown in Figure 3b.

In order to demonstrate that the proposed numerical procedure results in feedback strategies that take advantage of a suboptimal play by either one of the players, in Fig. 4 we show the resulting trajectories of a game in which the evader follows a constant bearing strategy.

On the other hand, the pursuer determines its control action at each instant of time using the reachability set analysis outlined in Section III. Capture occurs at $T = 0.93$, whereas if the evader had acted optimally, capture would have occurred at $T = 1.08$, which is the value of this game. For this example, the maximum speeds are $\bar{u} = 4$ and $\bar{v} = 1$ and the initial conditions are $X_{P_0} = [2, 2]^\intercal$ and $X_{E_0} = [4, 4]^\intercal$ for the pursuer and the evader, respectively.

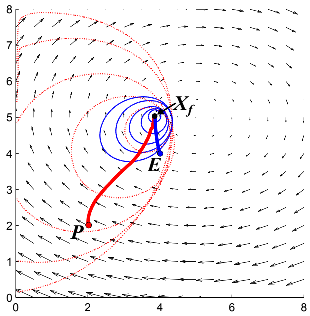


Fig. 4: Evolution of the reachability fronts and optimal trajectories at the optimal time-to-capture. Evader plays suboptimally. The red and blue closed curves represent the reachable fronts of the pursuer and the evader in distinct time steps.

VI. Conclusion

In this paper, we consider a differential game between a pursuer and an evader in an external dynamic flow field. It is shown that the game terminates when the usable reachable set of the evader is the empty set for the first time. A sufficient condition for the existence of finite termination time is presented. The level set method is adopted to generate the reachable sets of both players, and the optimal trajectories and controls of both agents are retrieved by backward propagation of the corresponding reachable sets. We have tested our method on a pursuit-evasion game whose optimal controls and trajectories can be computed analytically. We then applied our scheme to a more realistic flow field.

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