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Pricing Equilibria and Graphical Valuations

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We study pricing equilibria for graphical valuations [1, 14], which is a class of valuations that admit a compact representation. These valuations are associated with a value graph, whose nodes correspond to items, and edges encode (pairwise) complementarities/substitutabilities between items. It is known that for graphical valuations a Walrasian equilibrium (a pricing equilibrium that relies on anonymous item prices) does not exist in general. On the other hand, a pricing equilibrium exists when the seller uses an *agent-specific graphical pricing* rule that involves prices for each item and markups/discounts for pairs of items. We study the existence of pricing equilibria with simpler pricing rules which either (i) require anonymity (so that prices are identical for all agents) while allowing for pairwise markups/discounts, or (ii) involve offering prices only for items. We show that a pricing equilibrium with the latter pricing rule exists if and only if a Walrasian equilibrium exists, whereas the former pricing rule may guarantee the existence of a pricing equilibrium even for graphical valuations that do not admit a Walrasian equilibrium. Interestingly, by exploiting a novel connection between the existence of a pricing equilibrium and the partitioning polytope associated with the underlying graph, we also establish that for simple (series-parallel) value graphs a pricing equilibrium with anonymous graphical pricing rule exists if and only if a Walrasian equilibrium exists. These equivalence results imply that simpler pricing rules (i) and (ii) do not guarantee the existence of a pricing equilibrium for all graphical valuations.

CCS Concepts: •**Mathematics of computing** → **Graph theory**; •**Applied computing** → *Operations research*;

Additional Key Words and Phrases: Pricing equilibrium, efficient allocation, graphical valuations

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1 INTRODUCTION

In settings with multiple indivisible items, a central question is how a seller can allocate these items to buyers efficiently, i.e., in a way that maximizes the total welfare generated by the allocation. This question is studied in the context of the well-known package assignment model [8], and it is known that a seller can find an efficient allocation by solving a linear programming formulation of the efficient allocation problem. Moreover, the dual optimal solutions associated with this linear program reveal market prices that support this efficient outcome and clear the market, i.e., at these prices assigning to each buyer a bundle she demands constitutes an efficient allocation, and this assignment also maximizes the revenue of the seller. The tuple of the efficient allocation and the supporting prices is often referred to as a pricing equilibrium.

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A pricing equilibrium is a generalization of the well-known Walrasian equilibrium concept from microeconomic theory. In a Walrasian equilibrium, the price of a bundle of items can be expressed as the sum of prices of the items that are contained in it, i.e., the price offered to an agent m for a bundle S of items is given by $p^m(S) = \sum_{i \in S} p_i$, where $\{p_i\}$ denote *anonymous item prices*. Intuitively, these prices clear the market (by ensuring that each item is demanded by a single agent), and enable the agents to coordinate on an efficient allocation. On the other hand, a Walrasian equilibrium exists mainly under substitutability assumptions on preferences of agents [4, 23, 26].

The pricing equilibrium concept generalizes Walrasian equilibrium by allowing for more general pricing rules, which often (i) are agent-specific, and (ii) involve a distinct price term for each bundle of items (i.e., that specify a price $p^m(S)$ for each agent m and bundle of items S). The existence of pricing equilibria under such pricing rules, referred to as agent-specific bundle pricing, can always be guaranteed, thereby implying that the efficient allocation can always be supported by them. On the other hand, setting such prices requires specifying exponentially many (in the number of items) price terms [8, 20, 29, 33, 34], and hence the complexity of this pricing rule renders it impractical for implementing an efficient outcome and clearing the market.

Intuitively, in settings where the valuations of agents exhibit a special structure, it should be possible to obtain pricing equilibria that rely on simpler pricing rules. As explained above, valuations that satisfy (gross) substitutability constitute such a special class, since for them a Walrasian equilibrium (i.e., a pricing equilibrium that relies on anonymous item prices $\{p_i\}$) exists. On the other hand, in many practical settings (such as spectrum auctions, see, e.g., [17]), items exhibit synergies, or complementarities, hence a Walrasian equilibrium need not exist. Motivated by this observation, in this paper we investigate whether the existence of a pricing equilibrium with a simple structure can be guaranteed in settings where the valuations exhibit complementarities in addition to substitutabilities.

Specifically, we investigate the question of existence of a pricing equilibrium for *graphical valuations*. The value functions that belong to this class are associated with a value graph, nodes of which correspond to the items that are sold by the seller. There are edges between items that can exhibit pairwise value complementarity or substitutability. We associate weights with the nodes and edges of the underlying graph. Positive weights associated with an edge capture value complementarity between the nodes (items) at the end points of this edge, and negative weights capture substitutability. The value a buyer has for a set of items is equal to the sum of the node and edge weights of the induced subgraph obtained by restricting the original value graph to this set of nodes (items). Graphical valuations allow for a compact representation of the value functions of agents, and capture pairwise complementarity/substitutability present in practical settings.

In recent work, [14] established that graphical valuations, where the underlying value graph has a tree structure, and all buyers view a given pair of items (connected by an edge) either as substitutes or as complements (a condition referred to as sign-consistency), always admit a Walrasian equilibrium. Conversely, they established that these assumptions are necessary, and for more general graphical valuations, a Walrasian equilibrium does not exist. It is straightforward to establish that an *agent-specific graphical pricing* pricing rule that has the same structure as valuations (i.e., which involves an agent-specific price for each item, as well as discounts/markups for pairs of items) suffices for existence of a pricing equilibrium (see Section 4). However, having agent-specific prices is undesirable in many practical settings, and it is not known whether simpler pricing rules than agent-specific graphical pricing still lead to existence of a pricing equilibrium for graphical valuations.

Contributions: Our main focus is on understanding whether pricing rules simpler than agent-specific graphical pricing suffice to guarantee the existence of a pricing equilibrium for graphical

valuations (Section 3). We answer this question by investigating two pricing rules: (i) *anonymous graphical pricing* and (ii) *agent-specific item pricing*. The former pricing rule admits a graphical structure that involves pairwise discount/markup terms but restricts attention to prices that are the same for all agents. The latter allows for agent-specific prices for items, but disallows for discounts/markups for pairs of items. We provide a straightforward argument establishing that a pricing equilibrium with agent-specific item pricing exists if and only if a Walrasian equilibrium exists. Moreover, this result holds for all valuations, including those that do not admit a graphical structure. On the other hand, we establish that anonymous graphical pricing guarantees the existence of a pricing equilibrium in a strictly larger set of graphical valuations. However, if the underlying value graphs are “simple” series-parallel graphs, then a Walrasian equilibrium exists if and only if a pricing equilibrium with anonymous graphical pricing exists.¹

By leveraging these results we identify settings where the underlying graphical valuations exhibit *only* pairwise substitutabilities or *only* pairwise complementarities, yet a Walrasian equilibrium, as well as pricing equilibria with agent-specific item pricing rule and anonymous graphical pricing rule do not exist. In particular, we show that this holds for simple graphical valuations (where the underlying graph is series-parallel), and in settings where the value graph of each agent has a (different) tree structure. These results imply that even in special settings with no pairwise complementarities or substitutabilities, agent-specific graphical pricing is needed to guarantee the existence of a pricing equilibrium, and simpler pricing rules (i) and (ii) do not suffice for this purpose.

In our analysis of the anonymous graphical pricing rule, we identify an interesting connection between the efficient allocation problem and the *partitioning polytope* of the underlying value graph, i.e., the convex hull of the incidence vectors of the partitions associated with the graph. We first provide a linear programming formulation of the efficient allocation problem which admits integral optimal solutions if and only if a pricing equilibrium with anonymous graphical pricing exists. Then we observe that feasible allocations of items to agents can be viewed as obtaining partitions of the underlying graph. On the other hand, for series-parallel graphs the associated partitioning polytope can be explicitly characterized (in terms of the so called cycle inequalities - see [15]). We use this explicit characterization to simplify the aforementioned linear programming formulation for series-parallel graphs, and show that whenever it admits an integer optimal solution, the dual of this problem implies the existence of a pricing equilibrium with a pricing rule that does not involve pairwise discounts/markups. Hence, in such settings a Walrasian equilibrium also exists.

Our results can be extended to settings, where the underlying valuations are not necessarily graphical but an anonymous pricing rule that allows for pairwise discounts is employed for pricing the items. In particular, in these settings if the aforementioned pricing rule can be associated with a series-parallel graph (where the edges are present between items/nodes that receive pairwise discount/markups), then it is not possible to obtain a pricing equilibrium with such anonymous graphical pricing, unless a Walrasian equilibrium exists. This result highlights a persistent shortcoming of the (simple) anonymous graphical pricing rule in more general settings.

Related literature: At full generality implementing the efficient outcome in a multi-item setting is a hard problem both from computational complexity and communication complexity points of view [9, 10, 18, 28, 31]. This motivates considering classes of value functions with additional structure [10, 16, 18]. Recently, [16, 41] and [1] studied a graphical valuation model that exhibits pairwise (or k -wise) complementarities/substitutabilities similar to the one we consider in this

¹Series-parallel graphs are equivalent to graphs that do not include a 4-clique as a minor. These graphs also correspond to graphs that have tree-width of at most two, and hence many graph-theoretic decision problems can be efficiently solved for such graphs [35].

paper. They characterized the computational complexity of efficient auction design for (hyper) graphical valuations, and developed approximately efficient sealed-bid auctions for settings where valuations do not exhibit substitutabilities. In this work, we adopt a similar valuation model to the one present in these papers, and investigate whether a pricing equilibrium with a simple structure exists for this class of valuations.

In a recent paper, [36] identify a surprising connection between the existence of a pricing equilibrium and the computational complexity of welfare-maximization, finding demanded bundles, and identifying revenue-maximizing allocations for a seller. Their results suggest that if the welfare-maximization problem is computationally strictly harder than the problem of finding demanded bundles (under anonymous item-pricing) then a Walrasian equilibrium does not exist. Moreover, they establish that similar conditions on the existence of pricing equilibria with more general pricing rules can be obtained. In particular, they show that if a given class of valuations and pricing rule admit a compact structure, the revenue maximization problem (for the aforementioned pricing rule) is tractable, and a pricing equilibrium exists (for this class of valuations and pricing rule), then finding the demanded bundles is computationally at least as hard as finding the efficient allocation. These results provide complexity theoretic certificates that can be used for testing the existence of a pricing equilibrium. Using such certificates [36] identify new subclasses of valuations (e.g., pair-demand valuations, triplet valuations, and positive graphical valuations), where a pricing equilibrium with anonymous graphical pricing does not exist, thereby complementing the results of this paper as well as [13]. We emphasize that in the current paper, we provide an interesting equivalence between Walrasian equilibrium existence and the existence of a pricing equilibrium with anonymous graphical pricing for simple graphical valuations, through a novel connection to the partitioning polytope. This equivalence allows us to establish the nonexistence of a pricing equilibrium with anonymous graphical valuations in new settings, such as graphical valuations that do not involve any complementarities (or positive edge weights).

The existence of a Walrasian equilibrium is important in auction design settings, as it can also be leveraged to obtain iterative auction formats (e.g., generalizations of English/Dutch auctions) that implement the efficient outcome [3, 4]. These auctions rely on setting prices for items, and adjusting them (e.g., by increasing the prices of overdemanded items and decreasing the prices of underdemanded ones) until a Walrasian equilibrium is reached. Thus, iterative auction formats that rely on this termination condition can guarantee efficiency only in settings where a Walrasian equilibrium exists [4, 14, 38]. Walrasian equilibrium, on the other hand, mainly exists in the absence of complementarities [4, 23, 24], or requires restrictive complementarity structures [14, 37]. Extending such auction formats to more general settings where a Walrasian equilibrium does not exist, necessitates focusing on the more general pricing equilibrium concept, and understanding when a pricing equilibrium with a simple pricing rule may exist.

Earlier literature explored generalizations of the Walrasian equilibrium concept to more general pricing rules, and studied the conditions under which pricing equilibria exist. In particular, [8] considered settings with potentially multiple buyers and sellers, and characterized the existence of a pricing equilibrium (using a definition similar to ours), for anonymous/agent-specific pricing rules where the sellers set a price for each item/bundle of items. They established that a pricing equilibrium always exists if (there is a single seller and) the seller offers an agent-specific price for each bundle of items. A similar result is also established by [30], who also studied the relation between the VCG payments and pricing equilibria with agent-specific bundle pricing rule. [27, 39] established that if attention is restricted to superadditive valuations, then anonymous bundle pricing suffices for supporting the efficient allocation. The existence of these more general pricing equilibria is also used for iterative auction design. However, since the focus has been on settings

where valuations do not exhibit any special structure, the pricing equilibria and auction formats proposed by the earlier literature relied on complex bundle pricing schemes [6, 8, 19, 34, 39, 40]. Examples of such auctions include the package bidding auction [5], iBundle auction [33], and auctions that rely on universally competitive equilibria (UCE) [30]. In this work, on the other hand, our objective is to understand whether simpler pricing rules suffice to guarantee the existence of a pricing equilibrium graphical valuations.

2 MODEL AND PRELIMINARIES

In this paper, we focus on settings where a seller sells N (heterogeneous) items to M agents. We denote the set of items by \mathcal{N} and the set of agents by \mathcal{M} . For each agent $m \in \mathcal{M}$, the value function $v^m : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ captures the value $v^m(S)$ this agent has for any set $S \subset \mathcal{N}$ of items. We make two standard assumptions about the value functions:

ASSUMPTION 2.1. *Agents have zero value for not receiving any items, i.e., $v^m(\emptyset) = 0$. Additionally, valuations are monotone, i.e., $v^m(S_1) \leq v^m(S_2)$ if $S_1 \subset S_2$.*

The value an agent has for a set S need not be additive over the items in this set, i.e., $v^m(S) \neq \sum_{i \in S} v^m(\{i\})$. If items i and j are such that $v^m(\{i, j\}) \geq v^m(\{i\}) + v^m(\{j\})$, then we say that these items are *pairwise complementary*. On the other hand, if i and j are such that $v^m(\{i, j\}) \leq v^m(\{i\}) + v^m(\{j\})$, we refer to them as *pairwise substitutes*. In this work we are mainly interested in pairwise complementarity/substitutability, and unless noted otherwise, we refer to pairwise complementarity/substitutability simply as complementarity/substitutability.

Our focus throughout this paper is on value functions that admit a compact graphical representation:²

Definition 2.1 (Graphical Valuations). Let $G = (\mathcal{N}, E)$ be a connected graph such that the set of nodes corresponds to the set of items \mathcal{N} . An edge between two nodes represents value complementarity or substitutability and the set of edges is denoted by E . We refer to G as a value graph for set of items \mathcal{N} . The value function $v : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ is a *graphical valuation* (with respect to G) if:

- there exist nonnegative node weights $w_i \geq 0$ for each $i \in \mathcal{N}$,
- there exist (positive or negative) edge weights w_{ij} for each $(i, j) \in E$,
- v is such that $v(S) = \sum_{i \in S} w_i + \sum_{(i, j) \in E | i, j \in S} w_{ij}$ for all S .

ASSUMPTION 2.2. *There exists a value graph $G = (\mathcal{N}, E)$ such that the value function of each agent is a graphical valuation with respect to G . That is, for each agent $m \in \mathcal{M}$, there exist weights $\{w_i^m\}_{i \in \mathcal{N}}$ and $\{w_{ij}^m\}_{(i, j) \in E}$ such that $v^m(S) = \sum_{i \in S} w_i^m + \sum_{(i, j) \in E | i, j \in S} w_{ij}^m$.*

Graphical valuations can be compactly described by specifying $O(N^2)$ weights for nodes and edges of the underlying graph (unlike general valuations that are described by specifying a value for each bundle, i.e., exponentially many values in N). They naturally capture pairwise complementarity and substitutability in valuations, while providing a first order approximation of more complex complementarities/substitutabilities that bundles of items might exhibit. In particular, a positive (negative) edge weight w_{ij} associated with an edge between two items i and j captures pairwise complementarity (substitutability) between these items.

²In this paper, to simplify exposition, we focus on graphical valuations which have a connected value graph, and analyze their pricing equilibria. Observe that our results on existence of pricing equilibria readily generalize to value graphs that are not connected. This is because in such cases the problem of finding market clearing prices decouples over connected subgraphs. Hence, pricing equilibria for connected subcomponents give pricing equilibria for the entire graph, and vice versa.

The pairwise complementarities/substitutabilities that these valuations capture are prominent in a variety of practical settings. In the context of spectrum auctions [17] argue that local synergies may exist (for bidders in spectrum auctions) to provide seamless roaming to customers, to limit boundary interference, or to exploit advertising spillovers (in neighboring geographical regions). [2] focus on pairwise complementarity (or synergy) in transportation auctions, and study how agents should bid in the presence of such complementarities. Finally, a recent stream of literature uses a pairwise complementarity/substitutability model in the context of course allocation mechanisms, by allowing students to report their valuations for the courses that they want, as well as adjustments for pairs of courses [11, 12, 32]. These pairwise valuations are then used to identify the most preferred bundle for students (at approximately market clearing prices) subject to the constraint that the cost of this bundle does not exceed the available budget for the students. The valuations in these papers can immediately be captured by the graphical valuation model, which also focuses on specifying a value for each item as well as pairwise complementarities/substitutabilities for pairs of items.

2.1 Efficient Allocation and Pricing Equilibria

A natural objective in settings with multiple items is to obtain an efficient allocation of items to agents, i.e., an allocation that maximizes the aggregate value derived by the agents:

Definition 2.2 (Efficient allocation). Given bundles of items $S^m \subset \mathcal{N}$ for all $m \in \mathcal{M}$, we say that $\{S^m\}_{m \in \mathcal{M}}$ is a *feasible allocation* if each item is assigned to at most one agent, i.e., $S^m \cap S^l = \emptyset$ for $m, l \in \mathcal{M}$ with $m \neq l$. An *efficient allocation* is a feasible allocation $\{S^m\}_{m \in \mathcal{M}}$ that maximizes the *welfare* or *total value*, i.e., $\sum_m v^m(S^m) = \max_{\{Z^m\}_m | Z^m \cap Z^l = \emptyset} \sum_m v^m(Z^m)$.

We say that a feasible allocation $\{S^m\}$ is *complete* if for every item i , there exists an agent m such that $i \in S^m$. Observe that Assumption 2.1 guarantees that there exists an efficient allocation that is also complete.

It may be possible to identify market prices for (bundles of) items that support the efficient outcome, i.e., under these prices (i) each agent demands a disjoint bundle of items, and (ii) the resulting allocation is efficient. Existence of such prices is important, as it implies that in competitive markets items can be efficiently allocated to self-interested agents by appropriately pricing them. We refer to such a tuple of prices and the corresponding efficient allocation as a *pricing equilibrium*.

In order to define pricing equilibrium formally, we next introduce some notation. We denote by $p^m(S)$ the price offered to agent m for bundle S . We refer to the quantity $v^m(S) - p^m(S)$ as agent m 's surplus for bundle S . We say that a bundle S^* is *demanded* by agent m if the maximum surplus is achieved for this bundle, i.e., $v^m(S^*) - p^m(S^*) = \max_S (v^m(S) - p^m(S))$. We denote the set of bundles agent m demands by D^m , i.e., $D^m = \arg \max_S (v^m(S) - p^m(S))$. Using this notation, the pricing equilibrium can be defined as follows (see [8] for a discussion of pricing equilibria in more general settings with multiple sellers):

Definition 2.3. The tuple $(\{p^m(S)\}_{m,S}, \{S^m\}_m)$ is a *pricing equilibrium* if

- (i) $\{S^m\}_m$ is a feasible allocation,
- (ii) For each m , S^m is demanded by agent m : $v^m(S^m) - p^m(S^m) \geq v^m(S) - p^m(S)$ for any $S \subset \mathcal{N}$, and $m \in \mathcal{M}$,
- (iii) At the given prices, the seller maximizes revenue using allocation $\{S^m\}$: $\sum_m p^m(S^m) \geq \sum_m p^m(Z^m)$ for any feasible allocation $\{Z^m\}_m$.

It can be seen from Definition 2.3 that at a pricing equilibrium each agent m receives a demanded bundle S^m , and the induced allocation $\{S^m\}_m$ is efficient. To see this note that for any feasible

allocation $\{Z^m\}_m \neq \{S^m\}_m$ we have $\sum_m v^m(S^m) - p^m(S^m) \geq \sum_m v^m(Z^m) - p^m(Z^m)$ (Definition 2.3 (ii)), and $\sum_m p^m(S^m) \geq \sum_m p^m(Z^m)$ (Definition 2.3 (iii)). These two inequalities suggest that $\sum_m v^m(S^m) \geq \sum_m v^m(Z^m)$. Since, this is true for any feasible allocation $\{Z^m\}$, we conclude that $\{S^m\}_m$ is efficient (i.e., the first welfare theorem).

[8] show that in the single seller setting, a pricing equilibrium with a general nonnegative bundle pricing rule $\{p^m(S)\}_{m,S}$ always exists. Moreover, the equilibrium allocation and prices can be obtained as the optimal solutions of the following LP formulations of primal/dual *efficient allocation* problems:³

$$\begin{array}{ll}
 \max & \sum_m \sum_S x^m(S) v^m(S) \\
 \text{s.t.} & \sum_S x^m(S) \leq 1 \quad \forall m \\
 (LP1) & \sum_{\mu} \delta(\mu) \leq 1 \\
 & x^m(S) \leq \sum_{\mu | \mu^m = S} \delta(\mu) \quad \forall m, S \\
 & x^m(S), \delta(\mu) \geq 0 \quad \forall m, S, \mu
 \end{array}
 \quad \left| \quad
 \begin{array}{ll}
 \min & \sum_m \pi^m + \pi^S \\
 \text{s.t.} & \pi^m \geq v^m(S) - p^m(S) \quad \forall m, S \\
 (D1) & \pi^S \geq \sum_m p^m(\mu^m) \quad \forall \mu \\
 & \pi^m, \pi^S, p^m(S) \geq 0 \quad \forall m, S.
 \end{array}$$

In LP1, μ denotes a feasible allocation, and μ^m denotes the bundle this allocation assigns to agent m . In integral feasible solutions of this optimization formulation, having $x^m(S) = 1$ is interpreted as agent m requesting bundle S , and $\delta(\mu) = 1$ is interpreted as the seller choosing allocation μ . The first and second constraints respectively guarantee that each agent requests at most one bundle, and the seller chooses at most one allocation. The third constraint guarantees that the bundles requested by the agents are consistent with the allocations chosen by the seller. At integral solutions where the first constraint holds with equality for all m , the objective function is the total value generated by an assignment of items according to the allocation that corresponds to $\{x^m(S)\}_m$, i.e., the allocation $\{S^m\}$ with S^m such that $x^m(S^m) = 1$.

We denote the multipliers corresponding to the first three constraints of LP1 respectively by $\pi^m, \pi^S, p^m(S)$, and using these variables we provide the corresponding dual problem in D1. In D1, $p^m(S)$ can be interpreted as the price offered to agent m for bundle S . The constraints in the dual problem suggest that for any bundle S , π^m is an upper bound on $v^m(S) - p^m(S)$, i.e., the surplus agent m associates with acquiring bundle S at the given prices. Moreover, at optimality, it can be seen that π^m is equal to the maximum surplus that can be associated with some bundle. This suggests that π^m can be interpreted as the surplus of agent m at the given prices. Similarly, for any allocation μ , $\sum_m p^m(\mu^m)$ captures the associated revenue of the seller, and π^S is an upper bound on this quantity. Thus, at optimality, π^S is equal to the maximum revenue of the seller.

³Observe that Definition 2.3 suggests that pricing equilibria are invariant to constant changes in the prices, i.e., if at a pricing equilibrium all prices of m are increased by a constant, the new prices also support a pricing equilibrium. [8] focus on nonnegative prices $p^m(S) \geq 0$, and show that pricing equilibria with such prices can be obtained through LP1/D1. Note that the nonnegativity assumption has no impact on the optimal objectives of LP1/D1. To see this observe that due to the monotonicity of valuations, the third constraint of LP1 always holds with equality at an optimal solution. Hence, replacing this constraint with equality (or dropping the nonnegativity constraint on prices) does not change the optimal objective value, and pricing equilibria can be obtained through such a formulation. Thus, in this paper, when we provide similar primal/dual LP formulations of the efficient allocation problem for graphical valuations, we do not impose nonnegativity on prices of bundles. Additionally, even in the absence of nonnegativity constraints the prices of bundles allocated by the seller at a pricing equilibrium are nonnegative (provided that $p^m(\emptyset) = 0$) since otherwise Definition 2.3 (iii) suggests that such a bundle would not be assigned to an agent at a pricing equilibrium.

[8] show that LP1 has optimal solutions that are integral and such solutions together with corresponding optimal dual prices $\{p^m(S)\}$ constitute a pricing equilibrium. On the other hand, clearing the market using this pricing rule requires identifying exponentially many prices, which is infeasible in settings with a large number of items. Intuitively, for compact valuation models, such as the graphical valuation model, this full generality should not be needed, and simpler pricing rules should suffice for clearing the market.

In general, when price structure is as expressive as valuation profiles, a pricing equilibrium exists. This insight was implicit in the work of [8], and implies that for graphical valuations, a pricing equilibrium with an *agent-specific graphical pricing rule*, which involves price parameters $\{p_i^m, p_{ij}^m\}_{m \in \mathcal{M}, i \in \mathcal{N}, (i,j) \in E}$ and defines the price of bundle $S \subset \mathcal{N}$ for agent $m \in \mathcal{M}$ by $p^m(S) = \sum_{i \in S} p_i^m + \sum_{i,j \in S | (i,j) \in E} p_{ij}^m$, exists (see Section 4 for a relevant discussion). The agent-specific nature of this pricing rule is undesirable, as it necessitates explicitly differentiating agents. A pricing equilibrium with *anonymous item prices* $\{p_i \geq 0\}$ such that price of bundle $S \subset \mathcal{N}$ for agent $m \in \mathcal{M}$ is given by $p^m(S) = \sum_{i \in S} p_i$, is equivalent to the well-known Walrasian equilibrium concept. As established in [14] a Walrasian equilibrium does not exist for graphical valuations in general. In the next section, we investigate if other anonymous pricing rules, or agent-specific pricing rules with simpler structure than agent-specific graphical pricing rule are sufficient for existence of a pricing equilibrium for graphical valuations.

Remark: To see that a pricing equilibrium exists when “price structure is as expressive as valuation profiles”, note that under prices $p^m(S) = v^m(S)$, each agent is indifferent between offered bundles, and the seller can choose an allocation $\mu = \{\mu^m\}$ that maximizes her revenues $\sum_m p^m(\mu^m)$. It can be readily checked that these prices and the associated allocation constitute a pricing equilibrium.

3 PRICING EQUILIBRIA WITH SIMPLE PRICING RULES

In this section, we investigate the existence of a pricing equilibrium for the following pricing rules:

- Definition 3.1.* (i) *Anonymous graphical pricing* involves price terms $\{p_i, p_{ij}\}_{i \in \mathcal{N}, (i,j) \in E}$ where p_i represents the price the seller offers for item $i \in \mathcal{N}$, and p_{ij} represents the discount/markup the seller offers for a pair of items i, j that are connected by an edge in the underlying value graph, i.e., $p^m(S) = \sum_{i \in S} p_i + \sum_{i,j \in S} p_{ij}$ for all $m \in \mathcal{M}, S \subset \mathcal{N}$.
(ii) *Agent-specific item pricing* involves price terms $\{p_i^m\}_{m \in \mathcal{M}}$ where p_i^m represents the price the seller offers for item $i \in \mathcal{N}$ to agent $m \in \mathcal{M}$, i.e., $p^m(S) = \sum_{i \in S} p_i^m$ for all $m \in \mathcal{M}, S \subset \mathcal{N}$.

In Section 3.1, we show that if the underlying graph has a simple structure (in particular if it is a series-parallel graph), then a pricing equilibrium with the anonymous graphical pricing rule exists if and only if a pricing equilibrium with anonymous item pricing also exists. In Section 3.2, we provide a simple observation that establishes that a pricing equilibrium with agent-specific item pricing rule exists if and only if a pricing equilibrium with anonymous item pricing (hence a Walrasian equilibrium) exists. We provide in Section 3.3 examples of graphical valuations, where the underlying graph has a simple structure, and valuations exhibit *only* complementarities or substitutabilities. Despite this special structure, a Walrasian equilibrium does not exist for these examples. Hence, combined with our results in Sections 3.1 and 3.2, these examples indicate that simple pricing rules (i) and (ii) do not necessarily guarantee the existence of a pricing equilibrium for graphical valuations.

3.1 Anonymous Graphical Pricing

We start by providing a linear programming formulation of the efficient allocation problem (similar to LP1/D1), and establish that this problem has integral optimal solutions if and only if a pricing

equilibrium with anonymous graphical pricing exists. Recall that in D1, we interpret the variables $p^m(S)$ as the price offered to agent m for bundle S of items. To obtain a similar formulation for anonymous graphical prices we require these dual variables to also satisfy: $p^m(S) = \sum_{i \in S} p_i + \sum_{i,j \in S | ij \in E} p_{ij}$. Restating D1 explicitly in term of anonymous graphical prices (while omitting the nonnegativity constraint on prices), we obtain the following formulation:

$$\begin{aligned}
 (D2) \quad & \min \pi^s + \sum_m \pi^m \\
 & s.t. \pi^m \geq v^m(S) - \sum_{i \in S} p_i - \sum_{i,j \in S | ij \in E} p_{ij} \quad \forall S, m \\
 & \pi^s \geq \sum_m \left(\sum_{i \in \mu^m} p_i + \sum_{i,j \in \mu^m | ij \in E} p_{ij} \right) \forall \mu \\
 & \pi^m, \pi^s \geq 0.
 \end{aligned}$$

All other variables have the same interpretation as in D1 (e.g., $\mu = \{\mu^m\}_m$ represents a feasible allocation, and π^m, π^s can respectively be interpreted as the surplus of agent m and the revenue of the seller). The corresponding primal LP can be stated as follows:

$$\begin{aligned}
 (LP2) \quad & \max \sum_m \sum_S x^m(S) v^m(S) \\
 & s.t. \sum_S x^m(S) \leq 1 \quad \forall m \in \mathcal{M} \\
 & \sum_{\mu} \delta(\mu) \leq 1 \\
 & \sum_m \sum_{S | i \in S} x^m(S) = \sum_m \sum_{\mu | i \in \mu^m} \delta(\mu) \quad \forall i \in \mathcal{N} \\
 & \sum_m \sum_{S | ij \in S} x^m(S) = \sum_m \sum_{\mu | ij \in \mu^m} \delta(\mu) \quad \forall ij \in E \\
 & \delta(\mu) \geq 0, x^m(S) \geq 0.
 \end{aligned}$$

As in LP1, the variables $\{x^m(S)\}$ and $\{\delta(\mu)\}$ respectively represent the bundles requested by agent m , and the allocations chosen by the seller. The first two constraints are identical to those in LP1. The third constraint implies that if an item i is requested by an agent, then the allocation chosen by the seller should assign it to one of the agents. Similarly, the fourth constraint guarantees that if two items are jointly requested by an agent, then the seller should choose an allocation that jointly assigns these to the same agent. Note that the objective function of this LP formulation is exactly the same as that of LP1, i.e., maximizing welfare.

Following a similar approach to [7] we next establish that this LP has integral optimal solutions if and only if a pricing equilibrium with anonymous graphical prices exist.

THEOREM 3.2. *LP2 has an optimal solution that is integral if and only if a pricing equilibrium with anonymous graphical pricing exists. Moreover, if a pricing equilibrium exists, then the prices at a dual optimal solution of D2, and the allocation suggested by an integral optimal solution of LP2 (i.e., $\{S^m\}_m$, where S^m is such that $x^m(S^m) = 1$ at an optimal solution of LP2) constitute a pricing equilibrium.*

The proof of this theorem is given in Appendix A. This result follows by establishing that if LP2 has integral optimal solution, then the complementary slackness conditions in LP2/D2 correspond to the conditions of a pricing equilibrium with anonymous graphical pricing. Conversely, if a

pricing equilibrium exists, the equilibrium conditions imply that a feasible integral solution to LP2 that satisfies complementary slackness conditions with a dual solution can be constructed, and its optimality can be verified.

We next strengthen this result by establishing that for characterizing pricing equilibria it suffices to restrict attention to solutions of LP2 whose associated allocations are complete (i.e., allocate all items to agents, see Section 2).

COROLLARY 3.3. *If a pricing equilibrium with anonymous graphical pricing exists, then LP2 has an integral optimal solution such that $\delta(\mu) = x^m(\mu^m) = 1$ for all m , and a complete allocation μ . Moreover, together with prices $\{p_i, p_{ij}\}$ obtained from an optimal solution of D2, this allocation constitutes a pricing equilibrium.*

The proof of the corollary can also be found in Appendix A. This result is a consequence of Assumption 2.1. This assumption implies that agents have larger values for larger bundles, hence there exists an efficient allocation where all items are allocated to agents. Therefore, it can be established that if LP2 has integral optimal solutions (or equivalently if a pricing equilibrium with anonymous graphical valuations exists), then another integral solution (equivalently pricing equilibrium) where all items are assigned also exists.

Note that this corollary allows for restricting attention to solutions of LP2 that have a special structure (i.e., that assign all items to agents). Such solutions constitute a partition of the underlying value graph. Next, we exploit this structure (and the known results about partitions of graphs with simple structure) to further characterize the pricing equilibrium with anonymous graphical pricing.

3.1.1 Pricing Equilibrium with Anonymous Graphical Pricing. Our main result for pricing equilibria with anonymous graphical pricing distinguishes between “simple” graphical valuation structures (where the underlying value graph is series-parallel), and fully general ones. We start by formally introducing some necessary definitions.

Definition 3.4 (Series-Parallel Graph ([15])). A graph is a *series-parallel* graph if it can be obtained from a tree network by repeatedly adding an edge in parallel to an existing one or by replacing an edge by a path.

Observe that Definition 3.4 allows the resulting graph to have multiple edges between two nodes. Such graphs are sometimes referred to as *multigraphs*. In this paper, we restrict attention only to graphs that are not multigraphs (even when we focus on series-parallel graphs).

Definition 3.5 (Minor). An undirected graph G_m is called a *minor* of the graph G if G_m can be obtained from G by deletion of edges, vertices, and contraction of edges.

Definition 3.6 (Complete Graph). A graph is a *complete graph* if there is an edge between any pair of vertices. A complete graph with n nodes is denoted by K_n .

Series-parallel graphs and K_4 are closely related. In particular, series-parallel graphs are equivalent to graphs that do not contain K_4 as a minor [15, 21, 22].

Using these definitions, in Theorem 3.7 we provide a characterization of pricing equilibria with anonymous graphical pricing. Note that this result indicates that the structure of the underlying value graph plays a key role in equilibrium characterization.

THEOREM 3.7. (i) *There exists graphical valuations where the underlying graph is not series-parallel and a Walrasian equilibrium does not exist, yet a pricing equilibrium with anonymous graphical pricing exists.*

- (ii) *Assume that there are at least three agents, an anonymous graphical pricing equilibrium exists, and the underlying pricing graph is series-parallel. Then a pricing equilibrium with anonymous item pricing (Walrasian equilibrium) also exists.*

This result establishes that in general, in terms of existence of a pricing equilibrium, anonymous graphical pricing is strictly more powerful than anonymous item pricing, i.e., a pricing equilibrium with anonymous graphical pricing may exist even in cases where a Walrasian equilibrium does not. However, in such cases, the underlying value graph is not series-parallel (e.g., it includes a 5-clique as in the proof of the theorem). On the other hand, for series-parallel graphs (i.e., graphs which do not include 4-clique as a minor), using the anonymous graphical pricing rule is not valuable, since a pricing equilibrium with this pricing rule exists if and only if a Walrasian equilibrium does. Thus our result characterizes settings where a pricing equilibrium with anonymous graphical pricing can be expected to exist, despite nonexistence of a Walrasian equilibrium. This is not an if and only if characterization of when a pricing equilibrium with anonymous graphical pricing rule exists. Such a characterization is given in Theorem 3.2.

Importantly, Theorem 3.7 implies that a pricing equilibrium with anonymous graphical pricing need not exist even for value graphs that have only three nodes (and hence are necessarily series-parallel/4-clique free), since a Walrasian equilibrium does not exist in general for such value graphs (see Section 3.3).

The proof of the first part of the claim is obtained by explicitly constructing an example with 5 items (and a corresponding value graph that is K_5), and showing that a pricing equilibrium with anonymous graphical pricing rule exists, whereas a Walrasian equilibrium does not. The proof of the second part relies on establishing that for series-parallel graphs if a pricing equilibrium exists, and hence LP2 has integral optimal solution, then some constraints of LP2 can be relaxed without impacting the optimal solution of LP2. However, this implies that for such graphs simpler (dual) prices can be used to support the efficient outcome, which in turn can be used to establish the existence of a Walrasian equilibrium.

We first prove the second part of the theorem in Section 3.1.2, and illustrate how the series-parallel structure can be exploited for obtaining the equivalence of pricing equilibria with anonymous graphical and anonymous item pricing. We then focus on the first part of the theorem in Section 3.1.3, and establish that anonymous graphical pricing leads to equilibrium existence more generally.

Remarks:

- Second part of Theorem 3.7 specifically focuses on valuations whose value graphs are series-parallel. Consider construction of a series-parallel graph (in terms of a sequence of serial and parallel connections), and focus on two edges e, f connected in parallel at some step of this construction. These edges can be later replaced by paths (that possibly involve further parallel connections). The end nodes of e, f represent items, conditional on assignment of which, the values of agents for sets of items on the corresponding two paths are independent from each other. Thus, roughly speaking, series-parallel networks represent settings where conditional on assignment of some items to agents, the valuations additively decompose over smaller bundles.
- It was noted in [36] that it is “rare” to find a class of valuations \mathcal{V} which does not admit a Walrasian equilibrium, yet admits a pricing equilibrium that relies on a pricing rule \mathcal{P} that can be expressed strictly more succinctly than the valuations themselves. Theorem 3.7 offers an explanation as to why this might be the case for graphical valuations (and anonymous graphical pricing). Specifically, the result implies that for series-parallel graphs existence of a Walrasian equilibrium is effectively equivalent to existence of a pricing equilibrium

with anonymous graphical pricing. Thus, in order to find a class of graphical valuations \mathcal{V} that satisfies the aforementioned structure (with the natural anonymous graphical pricing rule), it is necessary to consider nontrivial families of graphical valuations, e.g., valuations that admit K_5 as a minor.

- It can be seen that the analysis provided in the subsequent sections and the conclusion of Theorem 3.7 can be generalized to settings, where the agents do not have graphical valuations, but the seller uses an anonymous graphical pricing rule. In particular, if the pricing rule can be associated with a series-parallel graph G , whose nodes correspond to items, and edges capture the pairs of items which are offered pairwise discounts/markups under the aforementioned pricing rule, then Theorem 3.7 still implies that a pricing equilibrium with this pricing rule exists if and only if a Walrasian equilibrium exists. Thus, Theorem 3.7 can be used as a certificate for establishing the existence/nonexistence of anonymous graphical pricing in more general settings than graphical valuations.

It is worth mentioning that the proof of Theorem 3.7 heavily exploits the fact that the seller uses a graphical pricing rule. In particular, the constraints of LP2 relate to the cuts of the edges of the network G (which encode items that receive pairwise discounts/markups), induced by partitions of items. The proof shows that when G is series-parallel, the cuts admit an explicit characterization, using which LP2 as well as its dual (prices) can be simplified. This approach does not readily extend to more general pricing rules. In order to obtain similar results for more general (additively decomposable) pricing rules (and study whether they lead to existence of a pricing equilibrium in settings that a Walrasian equilibrium does not exist), it may be possible to first associate a hypergraph with the prices (whose hyperedges correspond to markups/discounts associated with larger subsets of items), and then characterize the structure of cuts of the hyperedges induced by different partitions of items.

3.1.2 Proof of Theorem 3.7 (ii). Before we prove Theorem 3.7, we cover some graph-theoretic preliminaries that will be used in the proof, and establish an auxiliary result.

We start by defining a partition of a graph. Formally, a collection $\pi = \{A^m\}_{m \in \mathcal{A}}$ is a partition if (i) $|A^m| \geq 1$, (ii) $A^m \cap A^k = \emptyset$ for all $m, k \in \mathcal{A}$, (iii) $\cup_m A^m$ gives the set of all nodes. We denote the set of all partitions associated with a graph G as Π . Note that when defining Π , we do not restrict the cardinality of \mathcal{A} . A partition where $|\mathcal{A}| = k$, is referred to as a k -partition of G .

Some of the edges of the graph can be cut by a given partition, i.e., the end points of an edge may belong to different subsets of the partition. We denote the set of edges that are cut by a partition π by $E(\pi) \subset E$, i.e., $E(\pi) = \{(i, j) \in E \mid \text{there is no } m \in \mathcal{A} \text{ where } i, j \in A^m\}$. We can associate each partition π with an incidence vector $z(\pi) \in \{0, 1\}^{|E|}$ that captures the edges that are cut by the given partition, i.e.,

$$z_e(\pi) = \begin{cases} 1 & \text{for } e \in E(\pi) \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The graph partitioning polytope $P(G)$ is the polytope associated with these incidence vectors. In particular, for a given graph G , $P(G)$ is the convex hull of the incidence vectors, i.e., $P(G) = \text{conv}\{z(\pi) \mid \pi \in \Pi\}$.

For series-parallel graphs, this polytope admits an alternative characterization. Consider a cycle $C = (V_c, E_c)$ in a given graph. With any edge $e^* \in E_c$, we can associate the following *cycle inequality*,

$$\sum_{e \in E_c - \{e^*\}} z_e - z_{e^*} \geq 0. \quad (2)$$

The set of all such inequalities defines a polytope,

$$LP(G) = \{z \in [0, 1]^{|E|} \mid z \text{ satisfies (2) for all cycles } C = (V_c, E_c), \text{ and edges } e^* \in E_c\}. \quad (3)$$

An interesting result due to [15] shows that for series-parallel graphs

$$P(G) = LP(G), \quad (4)$$

i.e., the cycle inequalities fully characterize the partitioning polytope for this class of graphs.

We next establish another useful property of partitions of series-parallel graphs, and show that for these graphs the incidence vectors of partitions can be characterized by restricting attention to 3-partitions.

LEMMA 3.8. *Let G be a series-parallel graph, and z^* be an incidence vector associated with a k -partition of G , where $k > 3$. Then, there exists a 3-partition of G with the same incidence vector.*

PROOF. Consider a given partition $\{S^m\}$ of G with more than 3 sets, and the associated incidence vector z^* . Obtain another graph \hat{G} by contracting every connected subset of nodes of G that belong to the same S^m (for some m) to a single node. For each node \hat{i} of \hat{G} , denote the set of nodes of G that are contracted to \hat{i} by $\beta(\hat{i})$. Note that \hat{G} is a minor of G . Since G is a series-parallel graph, and series-parallel graphs are equivalent to graphs that do not include 4-clique as a minor, it follows that \hat{G} does not have a 4-clique as a minor, and is a series-parallel graph. Series-parallel graphs have a chromatic number of at most three [25]. Thus it follows that nodes of \hat{G} admit a 3-coloring. Consider a 3-coloring of \hat{G} , and the associated 3-partition $\{\hat{S}^1, \hat{S}^2, \hat{S}^3\}$ of \hat{G} where nodes of same color are assigned to the same set. A corresponding partition $\{\bar{S}^1, \bar{S}^2, \bar{S}^3\}$ of G can be obtained by defining $\bar{S}^k = \cup_{\hat{j} \in \hat{S}^k} \beta(\hat{j})$ for $k \in \{1, 2, 3\}$. We claim that the incidence vector associated with this partition coincides with z^* . To see this consider adjacent nodes (i, j) of G . If $z_{ij}^* = 1$, then it should be the case that i and j belong to different sets in the original partition $\{S^m\}$. Consequently, they correspond to adjacent nodes in \hat{G} as well. Thus in the coloring, they have different colors, and they belong to different sets of partition $\{\hat{S}^m\}$. This implies that in the final partition $\{\bar{S}^k\}$ the nodes i and j belong to different sets, and hence the incidence vector associated with this partition satisfies $z_{ij} = 1$. Conversely, if $z_{ij}^* = 0$, then nodes i and j belong to the same set in the original partition $\{S^m\}$. Consequently, in \hat{G} they are represented by the same node and in the final partition $\{\bar{S}^k\}$ they belong to the same set. Thus, the incidence vector associated with $\{\bar{S}^k\}$ also satisfies $z_{ij} = 0$. Therefore, we conclude that it is possible to construct a 3-partition of G that has the same incidence vector z^* as the original partition, and the claim follows. \square

Using these definitions and Lemma 3.8 we next establish Theorem 3.7 (ii).

PROOF OF THEOREM 3.7 (II). Since a pricing equilibrium with anonymous graphical pricing exists, by Theorem 3.2, it follows that LP2 has optimal solutions that are integral. The claim is proved using this observation, and following three steps:

- Step 1: We reformulate LP2 after restricting attention to complete feasible allocations, and establish that the resulting optimization problem also has integral optimal solutions.
- Step 2: We project the feasible set of this problem onto $\{x^m(S)\}$ variables, and establish that this projection can be expressed in terms of the partitioning polytope and the corresponding problem has integral optimal solutions.
- Step 3: When the underlying graph is series-parallel, we establish that this problem has redundant constraints. Relaxing these constraints we obtain an alternative LP formulation of the efficient allocation problem, which has integral optimal solutions. Existence of

integral optimal solutions to this LP immediately implies that a pricing equilibrium with anonymous item pricing exists, establishing the desired result.

Step 1: Let χ denote the set of all complete feasible allocations, i.e.,

$$\chi = \{\mu \mid \mu \text{ is a feasible allocation and } \cup_{m \in \mathcal{M}} \mu^m = \mathcal{N}\}.$$

We claim that the following optimization problem also has optimal solutions that are integral:

$$\begin{aligned} & \max \sum_m \sum_S x^m(S) v^m(S) \\ & \text{s.t.} \sum_S x^m(S) \leq 1 \quad \forall m \\ & \text{LP2b :} \quad \sum_{\mu \in \chi} \delta(\mu) = 1 \\ & \quad \sum_m \sum_{S \mid i \in S} x^m(S) = 1 \quad \forall i \\ & \quad \sum_m \sum_{S \mid ij \in S} x^m(S) = \sum_m \sum_{\mu \in \chi \mid ij \in \mu^m} \delta(\mu) \quad \forall ij \in E \\ & \quad \delta(\mu) \geq 0, x^m(S) \geq 0 \end{aligned}$$

Here LP2b is obtained after restricting attention in LP2 to $\mu \in \chi$, i.e., imposing $\sum_{\mu \in \chi} \delta(\mu) = 1$ and allowing only for $\delta(\mu)$ variables for $\mu \in \chi$. Since for allocations $\mu \in \chi$, all items are allocated to an agent by definition of complete feasible allocations, it follows from this restriction that $\sum_m \sum_{\mu \in \chi \mid i \in \mu^m} \delta(\mu) = \sum_{\mu \in \chi} \delta(\mu) = 1$. Hence, while formulating LP2b, the third constraint of LP2 is appropriately modified by setting the right hand side equal to 1.

Since LP2 has optimal solutions that are integral, Corollary 3.3 implies that it also has optimal integral solutions associated with complete allocations. Thus, the restriction of LP2 to such allocations, given by LP2b, also has an integral optimal solution.

Step 2: The optimization formulation in LP2b can equivalently be stated in terms of the partitioning polytope as follows:

$$\begin{aligned} & \max \sum_m \sum_S x^m(S) v^m(S) \\ & \text{s.t.} \sum_S x^m(S) \leq 1 \quad \forall m \\ & \text{LP2c :} \quad \sum_m \sum_{S \mid i \in S} x^m(S) = 1 \quad \forall i \\ & \quad \left\{ 1 - \sum_m \sum_{S \mid ij \in S} x^m(S) \right\}_{ij} \in P(G) \\ & \quad x^m(S) \geq 0. \end{aligned}$$

Our next lemma establishes the equivalence of LP2c and LP2b. This result is important as it allows us to characterize the optimal solutions of LP2b and LP2c by exploiting the properties of the partitioning polytope (more precisely the property given in (4)).

LEMMA 3.9. Assume that there are at least three agents and the underlying graph is series-parallel. Then, LP2b and LP2c are equivalent, i.e., if $\{x^m(S)\}$ is feasible in LP2c, then for some $\{\delta(\mu)\}$, $\{x^m(S), \delta(\mu)\}$ is feasible in LP2b, and conversely if $\{x^m(S), \delta(\mu)\}$ is feasible in LP2b, then $\{x^m(S)\}$ is feasible in LP2c.

The proof of this lemma is given after the proof of the theorem, and exploits the fact that any complete feasible allocation μ introduces a partition of the underlying graph. Thus, for integral $\{\delta(\mu)\}$ the vector $\{1 - \sum_m \sum_{S|ij \in S} x^m(S) = 1 - \sum_m \sum_{\mu \in \chi|ij \in \mu^m} \delta(\mu)\}_{ij}$ can be interpreted as an incidence vector of such a partition. This allows for expressing the fourth constraint of LP2b, in terms of the partitioning polytope.

Note that LP2c can be viewed as a projection of the feasible set of LP2b onto $\{x^m(S)\}$. Since the objective is only a function of $\{x^m(S)\}$, it also follows from Lemma 3.9 that LP2b and LP2c have the same optimal objective value. Moreover, since the optimal solution of LP2b is integral, so is the optimal solution of LP2c.

Step 3: It follows from (4) that when G is a series-parallel graph, we can replace the constraint $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} \in P(G)$ in LP2c with $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} \in LP(G)$. The next lemma, proof of which is given after the proof of the theorem, shows that the latter constraint immediately follows from the constraints $\sum_S x^m(S) \leq 1$ and $\sum_m \sum_{S|i \in S} x^m(S) = 1$ in LP2c, and hence can be omitted.

LEMMA 3.10. Assume that $x^m(S) \geq 0$, $\sum_S x^m(S) \leq 1$ for all m, S , and $\sum_m \sum_{S|i \in S} x^m(S) = 1$ for all i . Then, $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} \in LP(G)$.

Thus, for series-parallel graphs, the constraint $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} \in P(G)$ or equivalently $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} \in LP(G)$ in LP2c can be omitted, without any change in the optimal solution. Since LP2c has integral optimal solutions, it follows that the following problem (obtained after omitting the aforementioned constraint) has integral optimal solutions:

$$\begin{aligned}
 & \max \sum_m \sum_S x^m(S) v^m(S) \\
 & \text{s.t.} \quad \sum_S x^m(S) \leq 1 \quad \forall m \\
 & \quad \sum_m \sum_{S|i \in S} x^m(S) = 1 \quad \forall i \\
 & \quad x^m(S) \geq 0.
 \end{aligned} \tag{5}$$

On the other hand, it is known that this optimization problem has integral optimal solutions if and only if a Walrasian equilibrium exists [7].⁴ Thus, we conclude that a Walrasian equilibrium, or a pricing equilibrium with anonymous item pricing, exists for graphical valuations with series-parallel value graphs whenever the valuations admit a pricing equilibrium with anonymous graphical pricing. Hence, the claim follows. \square

PROOF OF LEMMA 3.9. Consider any feasible solution $\{x^m(S), \delta(\mu)\}$ of LP2b. We claim that

$$\{h_{ij}\}_{ij \in E} = \left\{ 1 - \sum_m \sum_{\mu \in \chi|ij \in \mu^m} \delta(\mu) \right\}_{ij \in E},$$

⁴In the formulation of [7], the second constraint is an inequality constraint: $\sum_m \sum_{S|i \in S} x^m(S) \leq 1$. However, it can be readily seen that under Assumption 2.1, there always exists an optimal solution where the inequality is satisfied with equality. Thus, the former formulation has integral optimal solutions, if and only if the latter does, thereby implying that integrality of (5) suffices for establishing the existence of a Walrasian equilibrium.

belongs to $P(G)$. To see this observe that a complete feasible allocation μ induces a partition of items to k different sets, where k is the number of nonempty components of $\{\mu^m\}$. Assume that $\delta(\mu) = 1$ for some $\mu \in \chi$. The definition of h_{ij} suggests that for $(i, j) \in E$, $h_{ij} = 0$ if $i, j \in \mu^m$ for some m , and $h_{ij} = 1$ if $i \in \mu^m, j \in \mu^l$. This implies that $\{h_{ij}\}$ is an incidence vector associated with μ . More generally, for $\{\delta(\mu) \geq 0\}$ such that $\sum_{\mu \in \chi} \delta(\mu) = 1$ (e.g., for feasible solutions of LP2b), it follows that $\{h_{ij}\}$ belongs to the convex hull of these incidence vectors, and is in $P(G)$.

Note that since $\{x^m(S), \delta(\mu)\}$ is feasible in LP2b, we also have

$$1 - \sum_m \sum_{S|ij \in S} x^m(S) = 1 - \sum_m \sum_{\mu \in \chi | ij \in \mu^m} \delta(\mu) = h_{ij}.$$

On the other hand, since $\{h_{ij}\}$ belongs to $P(G)$, it follows that $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} \in P(G)$. Thus, we conclude that $\{x^m(S)\}$ is feasible in LP2c.

Conversely, consider some $z \in P(G)$. Observe that by definition of $P(G)$, we have $z = \sum_k \alpha_k z^k$, where $\alpha_k \in [0, 1]$, $\sum_k \alpha_k = 1$, and each z^k is an incidence vector that corresponds to a partition of the underlying value graph G . Lemma 3.8 implies that each z^k can be associated with an l -partition of G , where $l \leq 3$. Let $\hat{\mu}$ be an allocation of items to $|M|$ agents, which assigns items to first l agents as suggested by the partition z^k , while not assigning any items to the remaining agents (note that since there are at least three agents as per our assumption, and $l \leq 3$ this is always possible). Consider $\{\delta(\mu)\}$ such that $\delta(\hat{\mu}) = 1$, $\sum_{\mu} \delta(\mu) = 1$, and $\delta(\mu) \geq 0$. As before $1 - \sum_m \sum_{\mu \in \chi | ij \in \mu^m} \delta(\mu)$ corresponds to the incidence vector of the partition induced by allocation $\{\hat{\mu}^m\}$, and hence $z_{ij}^k = 1 - \sum_m \sum_{\mu \in \chi | ij \in \mu^m} \delta(\mu)$. Since this is true for any z^k , and z is a convex combination of $\{z^k\}$, it follows that for some choice of $\delta(\mu)$, we have

$$z_{ij} = 1 - \sum_m \sum_{\mu \in \chi | ij \in \mu^m} \delta(\mu), \quad \text{for all } ij. \quad (6)$$

Consider a feasible solution $\{x^m(S)\}$ of LP2c. It follows from (6) that there exists some $\{\delta(\mu)\}$ such that $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} = \{1 - \sum_m \sum_{\mu \in \chi | ij \in \mu^m} \delta(\mu)\}_{ij}$ for all edges (i, j) . This implies that $\{x^m(S), \delta(\mu)\}$ is feasible in LP2b.

These observations show that for any feasible solution $\{x^m(S)\}$ of LP2c we have a feasible solution $\{x^m(S), \delta(\mu)\}$ of LP2b and vice versa. Thus, the claim follows. \square

PROOF OF LEMMA 3.10. To prove that $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij} \in LP(G)$, we need to show (i) $1 - \sum_m \sum_{S|ij \in S} x^m(S) \in [0, 1]$, and (ii) $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij}$ satisfy cycle inequalities in (2). Note that since $\sum_m \sum_{S|i \in S} x^m(S) = 1$, and $x^m(S) \geq 0$ it follows that

$$1 - \sum_m \sum_{S|ij \in S} x^m(S) \geq 1 - \sum_m \sum_{S|i \in S} x^m(S) = 0.$$

This implies that $1 - \sum_m \sum_{S|ij \in S} x^m(S) \in [0, 1]$ for all edges $(i, j) \in E$. Thus, in order to establish the claim it suffices to focus on (ii) and establish that cycle inequalities hold.

Consider any cycle $C = (V_c, E_c)$ and $e^* \in C$. Without loss of generality we assume that E_c consists of ordered tuples $(i_1, j_1), (i_2, j_2), \dots$ so that $j_n = i_{(n+1)}$ for $n = 1, \dots, |V_c| - 1$. The cycle constraint associated with $\{1 - \sum_m \sum_{S|ij \in S} x^m(S)\}_{ij}$ is given below:

$$\sum_{e \in E_c - \{e^*\}} \left(1 - \sum_m \sum_{S|e \in S} x^m(S) \right) \geq 1 - \sum_m \sum_{S|e^* \in S} x^m(S).$$

Rearranging the terms, we need to show

$$|E_c| - 2 \geq \sum_{e \in E_c - \{e^*\}} \sum_m \sum_{S|e \in S} x^m(S) - \sum_m \sum_{S|e^* \in S} x^m(S). \quad (7)$$

Let $\sum_m \sum_{S|e^* \in S} x^m(S) = b$ for some real number b , and edge $e^* = (i, j)$. Note that

$$\sum_m \sum_{S|i \in S} x^m(S) = 1 = \sum_m \sum_{S|e^* \in S} x^m(S) + \sum_m \sum_{S|i \in S, j \notin S} x^m(S). \quad (8)$$

This implies that

$$\sum_m \sum_{S|i \in S, j \notin S} x^m(S) = 1 - b. \quad (9)$$

On the other hand, a similar expression to (8) can be obtained for any other edge $(i', j') \in E_c$, implying $\sum_m \sum_{S|i', j' \in S} x^m(S) = 1 - \sum_m \sum_{S|i' \in S, j' \notin S} x^m(S)$. Summing this expression over $(i', j') \in E_c - \{e^*\}$ we obtain

$$\sum_{(i', j') \in E_c - \{e^*\}} \sum_m \sum_{S|i', j' \in S} x^m(S) = |E_c| - 1 - \sum_{(i', j') \in E_c - \{e^*\}} \sum_m \sum_{S|i' \in S, j' \notin S} x^m(S). \quad (10)$$

Consider again the edge $e^* = (i, j) \in E_c$. Observe that since $\sum_m \sum_{S|\hat{i} \in S} x^m(S) = 1$ for all \hat{i} , it follows that

$$\sum_m \sum_{S|i \in S, j \notin S} x^m(S) = \sum_m \sum_{S|j \in S, i \notin S} x^m(S). \quad (11)$$

On the other hand, any set S that contains j but not i , cuts the cycle C at edge $(i, j) \in E_c$ and at another edge $(i', j') \in E_c$, i.e., if $j \in S, i \notin S$, then there exists an edge $(i', j') \in E_c$ such that $(i, j) \neq (i', j')$, and $i' \in S, j' \notin S$. This observation implies that

$$\sum_m \sum_{S|j \in S, i \notin S} x^m(S) \leq \sum_{(i', j') \in E_c - \{e^*\}} \sum_m \sum_{S|i' \in S, j' \notin S} x^m(S). \quad (12)$$

Combining this with (11), we obtain

$$\sum_{(i', j') \in E_c - \{e^*\}} \sum_m \sum_{S|i' \in S, j' \notin S} x^m(S) \geq \sum_m \sum_{S|i \in S, j \notin S} x^m(S) = 1 - b, \quad (13)$$

where the equality follows from (9). Together with (10) this implies that

$$\sum_{e \in E_c - \{e^*\}} \sum_m \sum_{S|e \in S} x^m(S) \leq |E_c| - 1 - (1 - b).$$

Hence, we conclude

$$\sum_{e \in E_c - \{e^*\}} \sum_m \sum_{S|e \in S} x^m(S) - \sum_m \sum_{S|e^* \in S} x^m(S) \leq |E_c| - 1 - (1 - b) - b = |E_c| - 2. \quad (14)$$

Thus, (7) holds, and the claim follows. \square

3.1.3 Proof of Theorem 3.7 (i). We establish the claim in the first part of Theorem 3.7 by constructing an example (Example 3.11) where a pricing equilibrium with anonymous graphical pricing exists, but a pricing equilibrium with anonymous item pricing (i.e., a Walrasian equilibrium) does not exist.

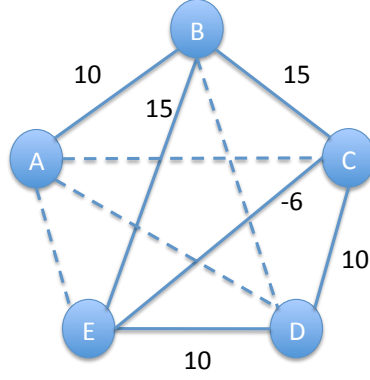


Fig. 1. All agents have the same weights, and the edge weights are as given in the above figure. In order to simplify the figure, some weights are omitted. In particular, the dashed lines have weight -30 , and all nodes have weight 100 .

Example 3.11. Let the underlying value graph be K_5 , and assume that there are 3 agents. We assume that all agents have the same weights for all nodes and edges, which are given in Figure 1.

In this example, the efficient allocation is obtained by assigning item A to the first agent, items B and E to the second one, and items C and D to the third one. The total welfare corresponding to this allocation is 525.⁵ On the other hand, consider the formulation in (5) due to [7]. Observe that setting $x^1(\{A, B\}) = x^1(\{A\}) = 1/2$, $x^2(\{B, E, C\}) = x^2(\emptyset) = 1/2$, $x^3(\{E, D\}) = x^3(\{D, C\}) = 1/2$ and the remaining $x^m(S)$ variables to zero, a feasible solution of this formulation can be obtained. It can be checked that the objective value of (5) associated with this solution is 527. Feasible integral solutions of this problem correspond to feasible allocations, and this problem has integral optimal solutions if and only if a Walrasian equilibrium exists [7].⁶ Since a feasible solution to this problem with larger objective (527) than the optimal welfare (525) exists, it follows that this problem does not have integral optimal solutions, and a Walrasian equilibrium does not exist for the valuations in Figure 1.

On the other hand, a feasible solution to D2 is obtained by setting $p_i = w_i^m$, and $p_{ij} = w_{ij}^m$ for all nodes i and edges (i, j) ; and $\pi^m = 0$ for all m . It can be immediately checked from D2 that by setting π^s equal to the maximum welfare that can be associated with this example (525), the feasibility of the constructed dual solution follows. Hence, we have a dual feasible solution with objective value 525. This implies that the maximum objective value of LP2 is bounded by the value of the dual feasible solution, 525. On the other hand, setting $x^1(\{A\}) = x^2(\{B, E\}) = x^3(\{C, D\}) = \delta(\hat{\mu}) = 1$, for allocation $\hat{\mu}$ that allocates the items according to the efficient allocation, and setting the remaining variables of LP2 to zero, a feasible solution of this problem with objective value 525 can be obtained. Since this solution achieves the dual objective, it follows that it is optimal, and constitutes an integral optimal solution of LP2. Thus, by Theorem 3.2 it follows that for the valuations in Figure 1 a pricing equilibrium with anonymous graphical pricing exists.

⁵The fact that optimal welfare is 525 can be verified by solving an integer program (obtained after restricting the decision variables in LP2 to integers) or via exhaustive search. The result can also be obtained by analyzing different cases associated with the assignments of items (e.g., one agent receives at least 3 items, or two agents receive two items both). The latter argument is detailed, and is available upon request.

⁶It can be checked that by construction the valuations in Figure 1 satisfy Assumption 2.1. Recall that under this assumption the formulation in (5) is equivalent to the formulation of [7], and integral optimal solutions to this problem exist if and only if a Walrasian equilibrium exists.

3.2 Agent-specific Item Pricing

We next focus on the agent-specific item pricing rule, and establish that if $(\{p^m(S)\}_{m,S}, \{S^m\}_m)$ is a pricing equilibrium with the agent-specific item pricing rule $\{p_i^m\}$, then $(\{\hat{p}^m(S)\}_{m,S}, \{S^m\}_m)$ is another pricing equilibrium that relies on anonymous item prices $\hat{p}_i = (\max_{m \in \mathcal{M}} p_i^m)^+$ for all items $i \in \mathcal{N}$. Additionally, if a pricing equilibrium with anonymous item pricing exists, then a pricing equilibrium with agent-specific item pricing trivially exists (since the former pricing rule is a special case of the latter one). These results jointly imply that a pricing equilibrium with one of these pricing rules exists if and only a pricing equilibrium with the other one also exists.

THEOREM 3.12. *A pricing equilibrium with agent-specific item pricing exists if and only if a pricing equilibrium with anonymous item pricing exists.*

The proof of this theorem is given in Appendix A, and relies on verifying that anonymous prices defined by $\hat{p}_i = (\max_{m \in \mathcal{M}} p_i^m)^+$ satisfy the pricing equilibrium conditions (Definition 2.3) and support the efficient outcome, whenever agent-specific prices $\{p_i^m\}$ belong to a pricing equilibrium.

This result does not require a restriction to graphical valuations, and holds for any class of valuations. Intuitively, this result implies that considering agent-specific item prices does not allow for obtaining a pricing equilibrium for settings where a pricing equilibrium with anonymous item pricing (or Walrasian equilibrium) does not exist.

3.3 Nonexistence Examples

In this section we provide examples of graphical valuations, where the underlying graph has a simple structure, and a Walrasian equilibrium does not exist. The results of Sections 3.1 and 3.2 immediately imply that for these examples pricing equilibria with agent-specific item pricing and anonymous graphical pricing also do not exist. In the first example, we focus on a graphical valuation which exhibits only complementarities, whereas in the second one we focus on a graphical valuation which only exhibits substitutabilities.

Example 3.13. Consider a setting with three items A, B, C and three agents. Assume that the underlying valuations are graphical, and can be represented with a value graph that involves edges (A, B) , (B, C) , and (C, A) . Further assume that the first agent has weight one for edge (A, B) , the second one has weight one for edge (B, C) , the third one has weight one for edge (C, A) ; and all remaining weights are zero. It was established in [14] that for this graphical valuation a Walrasian equilibrium does not exist. Note that this graph consists of only three nodes, and hence does not involve a 4-clique as a minor. Thus, it follows from Theorems 3.7 and 3.12 that for this example, pricing equilibria with agent-specific item pricing, and anonymous graphical pricing also do not exist.

Using complexity theoretic certificates, [36] obtained a similar result indicating that for graphical valuations that only exhibit complementarity a pricing equilibrium with anonymous graphical pricing does not exist. Example 3.13 indicates that Theorem 3.7 can also be used to recover this result. Our next example indicates that this nonexistence result is not only restricted to graphical valuations that only exhibit complementarities. In particular, even when all edge weights are negative (and hence valuations only exhibit pairwise substitutabilities), pricing equilibria with agent-specific item pricing and anonymous graphical pricing need not exist.

Example 3.14. Consider a setting with four items A, B, C, D and three agents. The corresponding value graph and weights for the first two agents are as specified in Figure 2. We assume that the third agent has value zero for all bundles (or zero weights for all nodes/edges).

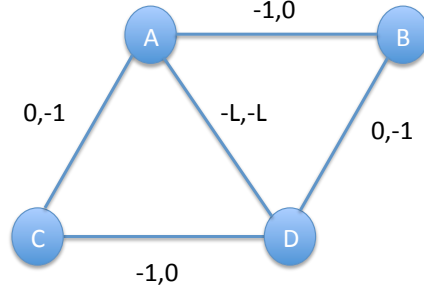


Fig. 2. The first (second) number on each edge corresponds to the weight the first (second) agent associates with it. We assume that the negative weight $-L$ is such that $-L < -10$. Finally, assume that the node weights are equal to $c > 0$, which is a large enough constant that ensures that valuations satisfy Assumption 2.1.

Observe that in this example an efficient allocation is given by assigning items A, C to agent 1, and B, D to agent 2. The corresponding welfare is equal to $4c - 1$.

Consider a fractional assignment of items to agents so that $x^1(\{A, C\}) = x^1(\{B, D\}) = x^2(\{A, B\}) = x^2(\{C, D\}) = 1/2$, $x^3(\emptyset) = 1$, and $x^m(S) = 0$ for the remaining bundles S , and agents $m \in \{1, 2, 3\}$. It can be seen that this constitutes a feasible solution of the formulation of the efficient allocation problem (5) (due to [7]) which has integral solutions if and only if a Walrasian equilibrium exists. Moreover, the corresponding objective value is equal to $4c$, i.e., larger than the maximum welfare of $4c - 1$. Thus, it follows from [7] that the valuations in Figure 2 do not admit a Walrasian equilibrium. Thus, by Theorems 3.7 and 3.12 it also follows that these valuations do not admit pricing equilibria with the other pricing rules considered in this section.

A common feature of Examples 3.13 and 3.14 is that ignoring the edges with zero weight, the underlying value graph for each agent can be viewed as a tree graph. Moreover, the weights agents associate with different edges are also sign-consistent (i.e., it is never the case that an agent has positive weight for an edge, whereas another agent has a negative weight). Thus, these examples indicate that if agents have sign-consistent tree valuations associated with *different* tree graphs, then a Walrasian equilibrium and pricing equilibria with agent-specific item/anonymous graphical pricing do not exist. This is in contrast with [14], which establishes the existence of Walrasian equilibria when sign-consistent valuations can be associated with the same tree network for all agents. This observation indicates that even for simple settings where each agent has a graphical valuation associated with a (distinct) tree graph, and in the presence of only pairwise complementarities or substitutabilities, pricing rules simpler than agent-specific graphical pricing do not necessarily support the efficient outcome.

4 ADDITIVELY DECOMPOSABLE VALUATIONS

In this section, we focus on additively decomposable valuations, which is a class of valuation functions that generalize graphical valuations. The valuations in this class allow for complementarity/substitutability not only for pairs, but also for larger sets of items.

We start by formally defining additively decomposable valuations. Consider a collection of subset of items \mathcal{B} , i.e., $B \in \mathcal{B}$ is such that $B \subset \mathcal{N}$. Assume that the valuations of agents can be additively decomposed over these subsets as follows:

$$v^m(S) = \sum_{B \in \mathcal{B}} w_B^m(S \cap B), \quad (15)$$

where $w_B^m : 2^B \rightarrow \mathbb{R}$, captures the component of the valuation of agent m associated with subset B . We refer to such valuations as *additively decomposable valuations* with collection \mathcal{B} .

We note that any valuation function can be represented using additively decomposable valuations, by considering a collection \mathcal{B} such that $N \in \mathcal{B}$. On the other hand, if \mathcal{B} consists of few sets of small cardinality, then the valuation functions can be compactly represented by specifying their components $\{w_B^m\}$. For instance, graphical valuations are a special class of additively decomposable valuations, where

- \mathcal{B} consists of singletons, and pairs of items that correspond to the edges of the underlying value graph,
- $w_i^m(S) = 0$ if $S \neq \{i\}$, and it equals to the weight associated with node i otherwise,
- $w_{ij}^m(S) = 0$ if $S \neq \{i, j\}$, and it equals to the edge weight for edge (i, j) otherwise.

We next provide an LP formulation of the efficient allocation problem for additively decomposable valuations.

$$\begin{aligned}
 \max \quad & \sum_m \sum_S x^m(S) v^m(S) \\
 \text{s.t.} \quad & \sum_S x^m(S) \leq 1 \quad \forall m \\
 LP - G : \quad & \sum_{\mu} \delta(\mu) \leq 1 \\
 & \sum_{S|S'=S \cap B} x^m(S) = \sum_{\mu|\mu^m \cap B=S'} \delta(\mu) \quad \forall m, B \in \mathcal{B}, S' \subset B \\
 & x^m(S), \delta(\mu) \geq 0 \quad \forall m, S, \mu.
 \end{aligned}$$

The corresponding dual LP (D-G) is given as follows:

$$\begin{aligned}
 \min \quad & \pi^S + \sum_m \pi^m \\
 \text{s.t.} \quad & \pi^m \geq v^m(S) - \sum_B p_B^m(S \cap B) \quad \forall m, S \\
 D - G : \quad & \pi^S \geq \sum_m \sum_B p_B^m(\mu^m \cap B) \quad \forall \mu \\
 & \pi^m, \pi^S \geq 0 \quad \forall m.
 \end{aligned}$$

As discussed in Section 2, it is always possible to obtain a pricing equilibrium which relies on a pricing rule that has the same structure as valuations. Our next theorem establishes that LP-G always has integral optimal solutions, and can be used to identify the efficient outcome and such pricing equilibria for additively decomposable valuations. This result is implicit in [8], which shows that for general valuations (where the valuations are given by specifying a value for each bundle), a pricing rule that is as general suffices for obtaining a pricing equilibrium.

THEOREM 4.1. *Assume that agents have additively decomposable valuations. LP-G has an optimal solution that is integral. Moreover, the prices suggested by the optimal solution of D-G, and the allocation suggested by LP-G constitute a pricing equilibrium.*

Note that when set \mathcal{B} as well as its elements have small cardinality, LP-G has substantially fewer constraints than LP1. Thus, our result suggests that in such cases the efficient allocation (as well as the corresponding prices) can be obtained by solving a more compact linear program.

The proof of this theorem is omitted as it immediately follows by (i) establishing the feasibility of the dual solution $p_B^m = w_B^m$, $\pi^m = 0$, $\pi^s = W^*$ (where W^* denotes the maximum welfare that can be associated with a feasible allocation), and (ii) showing that there exists an integral primal feasible solution (that coincides with the efficient allocation) which also achieves W^* . Using complementary slackness conditions in LP-G and D-G, it can also be established that the primal-dual optimal solutions constitute a pricing equilibrium, where the prices decompose over the collection \mathcal{B} , as suggested by D-G. This result implies that if valuations are additively decomposable over a few sets with small cardinality, a simple pricing rule suffices for obtaining a pricing equilibrium. In particular, for graphical valuations a pricing equilibrium always exists under the agent-specific graphical pricing rule.

Remark: While additively decomposable valuations can be used to represent *any* valuation class, a particularly interesting subset is additively decomposable valuations, where the bundle sizes in collection \mathcal{B} are bounded by some constant $k \geq 2$, independent of the number of items N . Such additively decomposable valuations constitute a class of compact valuations that contain graphical valuations as a subset. [36] provide a generalization of such compact additively decomposable valuations, referred to as *succinct linear valuations*. They offer an extension of the linear programming formulation LP-G/D-G (provided in this section and [13]), which can be used for characterizing pricing equilibrium to succinct linear valuations.

5 CONCLUSIONS

In this paper we study pricing equilibria for graphical valuations. We show that for value graphs that are series-parallel, a pricing equilibrium with anonymous graphical valuations exists if and only if an equilibrium with anonymous item pricing (or equivalently a Walrasian equilibrium) exists. We also establish that a pricing equilibrium with agent-specific item pricing exists if and only if an equilibrium with anonymous item pricing exists. We then establish that even for simple value graphs that only exhibit pairwise complementarities or pairwise substitutabilities, a Walrasian equilibrium need not exist. Thus, our results imply that for such valuations pricing equilibria with pricing rules simpler than agent-specific graphical pricing do not exist.

We also establish that a pricing equilibrium with anonymous graphical pricing may exist even when Walrasian equilibrium does not, in settings where the associated network structure is not simple (e.g., when it admits 5-clique as a subgraph). There are no known sufficient conditions for the existence of a pricing equilibrium with anonymous graphical pricing. Since pairwise discounts are common in practice, obtaining such sufficient conditions is an important direction for future research.

Another future direction is to study when alternative simple pricing rules suffice for existence of pricing equilibria for subsets of graphical valuations. In particular, preserving anonymity, but allowing for discount/markup terms for some bundles of cardinality greater than two, richer pricing rules than anonymous graphical pricing can be obtained. It is of interest to see for what type of value graphs introducing such terms for a small number of bundles suffice for existence of a pricing equilibrium.

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A ADDITIONAL PROOFS

PROOF OF THEOREM 3.2. It can readily be seen from LP2 that given a feasible allocation μ , there is a feasible solution of LP2, such that $x^m(\mu^m) = \delta(\mu) = 1$, and $x^m(S) = \delta(\bar{\mu}) = 0$ for $S \neq \mu^m$, and $\bar{\mu} \neq \mu$. Thus, optimal objective of LP2 is weakly larger than the maximum welfare that can be achieved via a feasible allocation of items to agents.

First, assume that LP2 has an optimal solution that is integral. Define an allocation $\hat{\mu}$ such that for each agent m if in the optimal solution there exist a bundle S^m with $x^m(S^m) = 1$, then $\hat{\mu}^m = S^m$, and otherwise $\hat{\mu}^m = \emptyset$. It can immediately be checked that another feasible solution to LP2 can be constructed by letting $\delta(\hat{\mu}) = x^m(\hat{\mu}^m) = 1$ for all m , and setting the remaining variables to zero. The objective value of LP2 associated with this solution is equal to the welfare of $\hat{\mu}$. Moreover, by construction (i.e., since $\hat{\mu}^m = S^m$ for bundles S^m in the original solution such that $x^m(S^m) = 1$), this solution has the same objective value as the initial one, and is optimal in LP2. Finally, since the optimal objective of LP2 is weakly larger than the maximum welfare (and hence the welfare associated with any feasible allocation), it follows that $\hat{\mu}$ is an efficient allocation.

Observe that the complementary slackness conditions suggest that (i) if $x^m(\hat{\mu}^m) = 1$, then an optimal dual solution of D2 satisfies $\pi^m = v^m(\hat{\mu}^m) - \sum_{i \in \hat{\mu}^m} p_i - \sum_{i, j \in \hat{\mu}^m | ij \in E} p_{ij}$, and (ii) similarly if $\delta(\hat{\mu}) = 1$ then the optimal dual solution satisfies

$$\pi^S = \sum_m \left(\sum_{i \in \hat{\mu}^m} p_i + \sum_{i, j \in \hat{\mu}^m | ij \in E} p_{ij} \right).$$

Thus, together with dual feasibility conditions, these observations imply that allocation $\hat{\mu}$ is revenue maximizing for the seller, and $\hat{\mu}^m$ is surplus maximizing for agent m . Hence, Definition 2.3 implies that the (feasible) allocation $\hat{\mu}$ and prices $\{p^m(S) = \sum_m (\sum_{i \in S} p_i + \sum_{i, j \in S | ij \in E} p_{ij})\}$ constitute a pricing equilibrium with anonymous graphical prices.

Conversely, assume that a pricing equilibrium with anonymous graphical pricing exists. Denote the associated allocation by $\hat{\mu}$, and prices by $\{p_i, p_{ij}\}$. Definition 2.3 immediately implies that choosing $\pi^m = v^m(\hat{\mu}^m) - \sum_{i \in \hat{\mu}^m} p_i - \sum_{i, j \in \hat{\mu}^m | ij \in E} p_{ij}$, and $\pi^S = \sum_m (\sum_{i \in \hat{\mu}^m} p_i + \sum_{i, j \in \hat{\mu}^m | ij \in E} p_{ij})$ with these prices, leads to a feasible solution of D2. Moreover, it can be checked that this solution satisfies complementary slackness conditions with $x^m(\hat{\mu}^m) = \delta(\hat{\mu}) = 1$, and $x^m(S) = \delta(\mu) = 0$ for remaining m, S , and μ . Moreover, the latter solution is feasible in LP2, and hence is optimal, and the claim follows. \square

PROOF OF COROLLARY 3.3. By Theorem 3.2, if a pricing equilibrium with anonymous graphical pricing exists, then LP2 has an optimal solution that is integral. This solution satisfies one of the following cases: (i) $\delta(\mu) = 0$ for all feasible allocations μ , (ii) $\delta(\mu) = 1$ for some feasible allocation μ (that is possibly not complete). We next show that in cases (i) and (ii) another integral optimal solution of LP2 which can be associated with a complete feasible allocation $\hat{\mu}$ can be constructed.

In case (i), we have $x^m(S) = 0$ for all sets $S \neq \emptyset$ (from the feasibility conditions in LP2), and hence the optimal objective value of LP2 is equal to zero. On the other hand, by Assumption 2.1 it follows that any solution of LP2 such that $\delta(\hat{\mu}) = x^m(\hat{\mu}^m) = 1$ for some complete feasible allocation $\hat{\mu}$ has a (weakly) larger objective, and hence is optimal.

In case (ii), consider an optimal solution of LP2 with $\delta(\mu) = 1$. Fix some agent k . Define a complete allocation $\hat{\mu}$ such that for $m \neq k$, $\hat{\mu}^m = S^m$ if there exists a bundle S^m such that $x^m(S^m) = 1$, and $\hat{\mu}^m = \emptyset$ otherwise. On the other hand, for $m = k$, let $\hat{\mu}^m = (\mathcal{N} \setminus \cup_{l \neq k} \mu^l)$. By setting $x^m(\hat{\mu}^m) = 1$ for all m , and $\delta(\hat{\mu}) = 1$ and setting remaining primal variables to zero, another feasible solution of LP2 can be obtained. Note that by construction if $x^m(S^m) = 1$ for some bundle S^m in the initial optimal solution, it can be seen that $S^m \subset \hat{\mu}^m$. Thus, by Assumption 2.1, the latter solution has a (weakly) larger objective, and is optimal in LP2. Finally, since $\hat{\mu}^k = (\mathcal{N} \setminus \cup_{l \neq k} \mu^l)$, it follows that $\hat{\mu}$ is a complete feasible allocation. Hence, the claim follows.

The fact that the dual optimal prices together with the integral optimal solution of LP2 associated with a complete feasible allocation constitute a pricing equilibrium follows from Theorem 3.2. \square

PROOF OF THEOREM 3.12. If $(\{p^m(S)\}_{m,S}, \{S^m\}_m)$ is a pricing equilibrium with anonymous item pricing, it trivially can be generalized to a pricing equilibrium with agent-specific item pricing, since anonymous item pricing is a special case of the latter pricing rule. Thus, to complete the proof, it suffices to establish that when a pricing equilibrium with agent-specific item pricing exists, it is possible to construct a pricing equilibrium with anonymous item pricing.

Assume that $(\{p^m(S)\}_{m,S}, \{S^m\}_m)$ is a pricing equilibrium with agent-specific item pricing rule, i.e., $p^m(S) = \sum_{i \in S} p_i^m$. First we establish that if an item is assigned to an agent at this pricing equilibrium, then the price offered to this agent should be nonnegative, and greater than the prices offered to other agents.

LEMMA A.1. *If $i \in S^m$, then $p_i^m \geq p_i^k$ for all k , and $p_i^m \geq 0$. Moreover, if $i \notin S^m$ for any $m \in \mathcal{M}$, then $p_i^m \leq 0$ for all $m \in \mathcal{M}$.*

The proof of this lemma is given at the end of the proof of the theorem.

Consider a pricing rule $\hat{p}^m(S)$ with anonymous item prices $\hat{p}_i \triangleq \max_{m \in \mathcal{M}} p_i^m$ for all $i \in \cup_m S^m$, and $\hat{p}_i = 0$ otherwise. Observe that $\hat{p}_i \geq 0$, since by Lemma A.1, $p_i^m \geq 0$ for $i \in S^m$, and $\hat{p}_i \geq p_i^m$ for all m and i . We claim that $(\{\hat{p}^m(S)\}_{m,S}, \{S^m\}_m)$ is a pricing equilibrium with anonymous item pricing.

Since $(\{p^m(S)\}_{m,S}, \{S^m\}_m)$ is a pricing equilibrium, Definition 2.3 implies that $\{S^m\}$ is a feasible allocation. Additionally, by Lemma A.1, it follows that

$$\sum_m \hat{p}^m(S^m) \geq \sum_m \hat{p}^m(Z^m) \quad (16)$$

for any allocation $\{Z^m\}$. This can be seen by noting that by construction only items $i \in \cup_m S^m$ have nonnegative prices $\{\hat{p}_i\}$, and the left hand side of (16) contains all such prices.

In addition, for any bundle Z^m , the surplus of agent m under $\{\hat{p}^m(S)\}$ is such that:

$$v^m(S^m) - \hat{p}^m(S^m) = v^m(S^m) - p^m(S^m) \geq v^m(Z^m) - p^m(Z^m) \geq v^m(Z^m) - \hat{p}^m(Z^m). \quad (17)$$

Here the equality follows from Lemma A.1 (since the lemma implies that if $i \in S^m$, then $\hat{p}_i = p_i^m \geq \max_k p_i^k$), the first inequality follows since $(\{p^m(S)\}_{m,S}, \{S^m\}_m)$ is a pricing equilibrium, and the

last one follows from the fact that by construction $\hat{p}^m(Z^m) \geq p^m(Z^m)$ (recall that by definition $\hat{p}_i \geq p_i^m$ for all m and i). Observe that feasibility of $\{S^m\}$, together with (16) and (17), implies that $(\{\hat{p}^m(S)\}_{m,S}, \{S^m\}_m)$ satisfies all conditions of Definition 2.3, and constitutes a pricing equilibrium with anonymous item pricing. \square

PROOF OF LEMMA A.1. We prove the claim by contradiction. First assume that $p_i^k > p_i^m$ for some $i \in S^m$ and agent $k \neq m$. Then, it can be seen that

$$\sum_m p^m(\hat{S}^m) > \sum_m p^m(S^m) \quad (18)$$

for an allocation $\{\hat{S}^m\}$ that assigns item i to agent k while keeping the assignments of all other items intact, i.e., $\{\hat{S}^m\}$ is such that $i \in \hat{S}^k$, and if $j \neq i$ belongs to S^l for some $l \in \mathcal{M}$, then $j \in \hat{S}^l$. Note that (18) violates Condition (iii) in Definition 2.3, thereby leading to a contradiction to our assumption that $(\{p^m(S)\}_{m,S}, \{S^m\}_m)$ is a pricing equilibrium. Analogously, if $0 > p_i^m$ for some $i \in S^m$, the same argument can be repeated by defining $\{\hat{S}^m\}$ after excluding item i from all bundles (while keeping the assignments of other items intact). It can be seen that (18) again holds under this assumption, thereby leading to a contradiction. Thus, we conclude if $i \in S^m$, then $p_i^m \geq p_i^k$ for all k and $p_i^m \geq 0$.

Finally, assume that $i \notin S^m$ for any $m \in \mathcal{M}$. Then Definition 2.3 immediately implies that $p_i^m \leq 0$ for all $m \in \mathcal{M}$, as otherwise defining a new allocation $\{\hat{S}^m\}$ by allocating i to an agent m such that $p_i^m > 0$, increases the seller's revenue, thereby violating the Condition (iii) of the definition. Thus, it follows that if $i \notin S^m$ for any $m \in \mathcal{M}$, then $p_i^m \leq 0$ for all $m \in \mathcal{M}$. \square