ON THE LOGIC OF FEEDBACK

by

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ABSTRACT

The variables entering an analysis problem are linked together by a set of functional relationships. These relationships may be expressed in the form of a flow graph, a network of directed branches which connect at nodes. Each node is associated with one of the variables of the problem and the branches entering that node indicate the dependence of that node variable upon other variables. Branch jk originates at node j and terminates at node k. The transmission of branch jk is defined as the partial derivative of node-variable k with respect to node-variable j.

A flow graph exhibits certain topological properties which relate to the structure of the functional relationships entering that graph. In particular, a closed path in the graph indicates that an equation must be solved in order to find the values of the node variables whereas the absence of any closed paths means that the values of the node variables are obtainable explicitly by direct substitution in the original functional relationships. The number of nontrivial simultaneous equations which must be handled in the problem solution is evident from the graph on the basis of certain topological principles. Specifically, if at least n nodes must be erased in order to eliminate all closed paths in the graph, then the analysis problem requires the solution of n simultaneous equations.

For linear graphs, which come from linear functional relationships, the analysis problem may be solved by direct manipulation of the graph, according to certain transformations and equivalences. The manipulative algebra of flow graphs permits many relatively complicated linear problems to be solved by inspection. In addition to aiding the solution of practical problems, the flow graph approach provides a convenient language in which the fundamental theory of feedback theorems become exceedingly simple when carried out on the flow graph basis.

The transmission through a flow graph is, in general, affected by changes in the branch transmission of the graph. It is often desirable, in the design of a feedback system, to minimize the sensitivity of the overall transmission T with respect to changes in some particular branch transmission t. The representation of such systems as flow graphs leads to certain structures which are inherently insensitive. The picture of signal flow afforded by a flow graph facilitates the physical interpretation of the basic processes which result in low sensitivity.

When each branch transmission is a function of the complex frequency, the presence of closed paths in the graph may lead to sustained oscillations within the system. The general stability criterion for potentially unstable flow graphs includes criteria previously developed for electronic circuits and automatic control devices. In certain cases the criterion developed for flow graphs results in a simplification of earlier methods. In particular, the number of Nyquist plots required in a stability investigation may be less. As to the determination of stability information from a Nyquist plot, a method depending only upon the concept of conformality, and requiring only simple curve sketching, has been devised.

Thesis Supervisor: Ernst A. Guillemin
Title: Professor of Electrical Engineering

January 6, 1951.

Professor Joseph S. Newell
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge 39, Mass.

Dear Professor Newell:

In accordance with the regulations of the faculty, I hereby submit a thesis entitled, ON THE LOGIC OF FEEDBACK, in partial fulfillment of the requirements for the degree of DOCTOR OF SCIENCE.

Signature of Applicant

ABSTRACT

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For linear graphs, which come from linear functional relationships, the analysis problem may be solved by direct manipulation of the graph, according to certain transformations and equivalences. The manipulative algebra of flow graphs permits many relatively complicated linear problems to be solved by inspection. In addition to aiding the solution of practical problems, the flow graph approach provides a convenient language in which the fundamental theory of feedback in linear systems may be expressed. The proofs of many classical feedback theorems become exceedingly simple when carried out on the flow graph basis.

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ACKNOWLEDGMENTS

The author is indebted to the supervisor of this thesis, Professor E. A. Guillemin, and to the readers, Professor W. K. Linvill and Professor J. B. Wiesner, for their encouragement, constructive suggestions, and patience. Much credit must be given to H. W. Bode, of the Bell Telephone Laboratories, whose influence, through his publications, is clear and present in many portions of this work.

During the initial stages of the investigation, the writer enjoyed many long discussions with Professor G. T. Coate. It was during these discussions that the general direction of the research was established and certain of the topological concepts were crystallized. At that time the writer was teaching a graduate subject dealing with electronic circuit theory which had previously been taught by Professor Coate. It was Professor Coate who first departed somewhat from the formal determinental analysis of general active circuits; suggesting, instead, the application of simpler procedures based upon superposition. The writer then chose flow graphs as a medium and undertook to study their topological properties. Suggestions by Professor Coate provided much stimulation and encouragement during the initiation of that phase of the investigation.

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CHAPTER I

INTRODUCTION

1.1 The Feedback Concept

Feedback is a magical word which means many different things to different people; a certain amplifier circuit to the student, a panacea or a headache to the design engineer, and, with the popularization of the automatic control art and the pronouncement of Cybernetics, an attractive new analytical tool to the psychologist and the economist. As a physical concept, feedback is most meaningful when associated with unilateral devices, such as vacuum-tube amplifiers. If the output of a controlled power source finds a return path to the control point, then feedback is said to exist. Physically, feedback implies a closed loop around which energy flows in a specified direction.

Many definitions of the term "feedback" or "feedback system" have been proposed. Some are restricted by choice to electronic circuits. For example,

> "<u>feedback</u> (fed'bak'),
> <u>adj</u>. 1. Electronics - denoting or pertaining to a system in which some of the energy of the plate circuit of a vacuum tube is returned (fed back) to the grid circuits. When this opposes the input, it is called inverse, when it aids the input it is called regenerative.
> <u>n</u>. 2. A feedback system." [Ref.9]

In contrast, Bode offers a purely mathematical definition of feedback which requires subsequent physical interpretation,

> "Definition: The return difference, or feedback, for any element in a complete circuit is equal to the ratio of the values assumed by the circuit determinant when the specified element has its normal value and when the specified element vanishes." [Ref.4, p.49]

If the specified element mentioned by Bode is the gain μ of a unilateral amplifier, and if a fraction β of the output is fed back in series with the input, then the return difference is equal to $1 - \mu\beta$, where $\mu\beta$ is, of course, the loop gain. The advantage of Bode's generalized definition lies in its applicability to complicated circuits wherein the quantities μ and β are not evident as separate entities. Still another definition is suggested by Wiener in his discussion of control systems involving people,

> "Notice that in this system there is a human link in the chain of the transmission and return of information: in what we shall from now on call the chain of feedback." [Ref.24, p.114]

Worthy of note is the fact that the last two definitions above permit the identification of feedback in passive bilateral systems such as, for example, a transmission line supporting both incident and reflected waves. This generalization, which frees the feedback notion of its traditional bond to unilateral active elements, represents a significant step. As an illustration of a further break with tradition, the writer can not resist the temptation to interject Prof. R. B. Adler's definition of a multiple-loop feedback system as "an octopus biting all of his fingernails."

Although the qualitative concept of feedback had been recognized earlier, it remained for Nyquist and Black, working at the Bell Telephone Laboratories in the early nineteen-thirties, to crystallize an elementary but powerful feedback theory which dealt with the behavior of a unilateral amplifier having a simple external feedback path. That the study of vacuum-tube amplifiers should have cultured the feedback notion is not surprising. Electronic amplifiers were, with certain minor exceptions, the first unilateral signal transmission devices to undergo careful study. The very presence of unilateral elements gave direction and hence physical embodiment to the concept of a feedback loop around which signal energy might flow.

Black analyzed the now familiar mu-beta circuit, for which

$$\mu' = \frac{\mu}{1 - \mu\beta} , \qquad (1.1)$$

where μ' and μ are the amplifications with and without feedback, respectively, and β is the so-called feedback constant. His work brought forth useful design information concerning the quantitative effects of feedback and also served to unify certain disconnected theoretical results previously obtained by others. Nyquist developed a stability criterion, based upon the steady-state performance characteristics of the broken feedback loop, which resolved many an enigma, among them the rule of thumb that a feedback amplifier having more than two stages is likely to support spontaneous oscillations. The value of Nyquist's test lay in the simplicity of its application. Previous connections between transient and steady-state response had been, in the main, sufficiently complicated so that their acceptance by engineers was inhibited.

Under the stimulus of the advances made by Black, Nyquist, Blackman, and others, feedback was employed to advantage in many new circuit designs. The scope of feedback theory, however, remained limited almost entirely to Black's elementary circuit. The nineteen-thirties might well, perhaps, be called the mu-beta era. Nevertheless, much good work was done and the great god Mu-Beta was assumed to be omnipotent until proven otherwise.

With the appearance in the early nineteen-forties of the work of H. W. Bode, the elegant and powerful methods of network theory were brought to play upon the general problem of amplifier analysis and design. Bode treats vacuum-tube amplifiers as passive networks in which are imbedded unilateral active elements representing tube transconductances. Nearly all of his theorems for feedback circuits are stated in terms of the "return difference," which is defined in terms of the mesh or nodal network determinant. Hence calculations and proofs depend upon the manipulation of the determinant and its cofactors. Following the formal analysis of active circuits, Bode considers the relation between the real and imaginary parts of network functions and applies the results to the design of terminal and interstage networks, and finally to the design of single-loop feedback amplifiers.

In the mu-beta era, when telephone repeaters constituted possibly the largest field of application for high quality feedback amplifiers, the Bell Telephone Laboratories fostered much of the research then in progress. During the war, however, radar and servomechanisms applications created an increased demand for feedback techniques. In the servomechanisms field, especially, certain design procedures adapted from feedback amplifier theory were rapidly advanced, refined, and implemented.

At present there exists a mass of literature on the subject of feedback, the large majority of which is devoted to special cases of practical interest. Since the publication of Bode's work on active networks relatively little material of a general nature has appeared. The feedback art has become more detailed and complex without a balance of effort toward the formulation of a simple, inclusive, and unified theory. The feedback notion itself, perhaps one of the most important concepts of our time, has enjoyed a recent growth in stature with the pronouncement of Cybernetics, but this growth has taken place principally at a philosophical level. There remains, in the opinion of the writer, a need for development at the working level so that engineers may find the feedback approach more useful in their dealings with the broad class of technical problems wherein such techniques are applicable.

1.2 Purpose of this Investigation

The underlying goal of this research has been to obtain a better understanding of the properties of feedback systems. In this paper the concept of a signal-flow graph is introduced. The flow graph provides a visual presentation of the relationships entering an analysis problem and facilitates certain manipulations leading to the solution. The exploitation of the signal-flow graph approach, as related to the fundamental theory of feedback and to the practical solution of engineering analysis problems, is the specific purpose of this work.

CHAPTER II

THE SIGNAL-FLOW GRAPH CONCEPT

2.1 Structure of the Flow Graph

A flow graph is a network of directed branches which connect at nodes. Branch j-k originates at node j and terminates upon node k, its direction being indicated by an arrowhead. A simple flow graph is shown in Fig. 2.1. This particular graph contains nodes 1, 2, 3

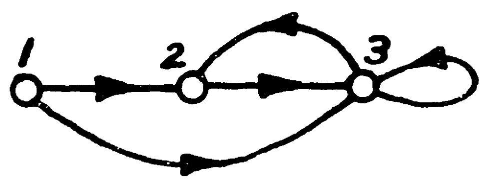


Fig. 2.1

and branches 1-2, 1-3, 2-3, 3-2, and 3-3. The flow graph may be interpreted as a signal transmission system in which each node is a tiny repeater station. Each station receives signals via the incoming branches, combines the information in some manner, and then transmits the result along each outgoing branch.

If the resulting signal at node j is called x_j, then the flow graph of Fig. 2.1 leads to the functional relationships

$$\begin{aligned} x_1 &= \text{a specified quantity or a parameter} \\ x_2 &= f_2(x_1,x_3) \\ x_3 &= f_3(x_1,x_2,x_3). \end{aligned} \tag{2.1}$$

The second equation, for example, simply states that signal x_2 is directly influenced by signals x_1, and x_3, as indicated by the presence of branches 1-2 and 3-2 in the graph. More precisely, the branch transmission of branch j-k may be defined as the partial derivative

$$t_{jk} = \partial f_k / \partial x_j \, . \tag{2.2}$$

If branch j-k does not appear in the graph, then the corresponding branch transmission t_{jk} vanishes identically. Total differentiation of Eqs. 2.1, therefore, yields

$$\begin{aligned} dx_1 &= 0 \\ dx_2 &= t_{12}dx_1 + t_{32}dx_3 \\ dx_3 &= t_{13}dx_1 + t_{23}dx_2 + t_{33}dx_3, \end{aligned} \tag{2.3}$$

where t_{12} and t_{32} are, in general, functions of both x_1 and x_3, and t_{13}, t_{23}, and t_{33} are functions of x_1, x_2, and x_3.

The flow graph shown in Fig. 2.1 may be thought of as a graphical representation of Eqs. 2.3. When the values or actual analytical forms of the branch transmissions are marked upon the graph near the corresponding branches, the graph is said to be explicit. An explicit graph and its associated set of total differential equations are equivalent. Both contain the same information, the difference between them being one of notation alone. As we shall see later, explicit graphs are useful primarily in linear problems since the branch transmissions are then independent of the variables x_j. Under the assumptions of linearity, Eqs. 2.3 integrate directly, so that the original Eqs. 2.1 must be of the form

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 + t_{12}x_1 + t_{32}x_3 \\ x_3 &= y_3 + t_{13}x_1 + t_{23}x_2 + t_{33}x_3, \end{aligned} \tag{2.4}$$

where y_1, y_2, y_3 are constants of integration. If, now, quantities y_1, y_2, y_3 are treated as additional variables, the flow graph takes the explicit linear form shown by Fig. 2.2. Each branch may be likened to a unilateral amplifier having a gain equal to the branch transmission and each node acts as an adder which sums the incoming signals. Since y_j always transmits to x_j through a branch of unity

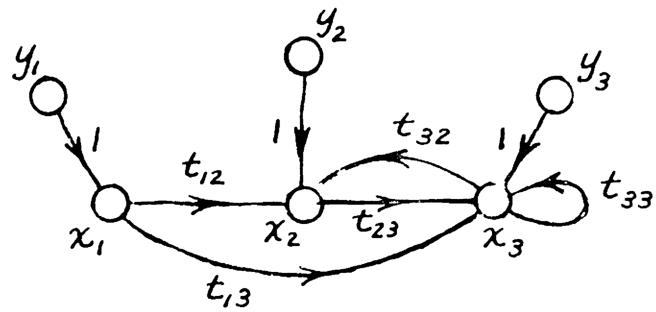


Fig. 2.2

transmission, it is not really necessary to show the additional nodes and branches; they may be omitted provided their effect is automatically implied. The important point is that an explicit set of linear equations may be recast in the form of an entirely equivalent flow graph, whereas a nonlinear set of relationships must be totally differentiated before an explicit flow graph becomes meaningful.

A symbolic flow graph is one on which the branch transmissions are not indicated. Each branch is simply present or absent, as in Fig. 2.1. Such graphs prove useful in the consideration of both linear and nonlinear problems. A symbolic graph shows the structure or "Gestalt" of the associated set of functional relationships but not the precise nature of the functions or operations involved. The associated functions f_k need not be analytic or even single valued in order to be representable as a symbolic flow graph. The presence of branch j-k means only that f_k depends in some fashion upon x_j and does not necessarily imply the mathematical existence of the partial derivative $\partial f_k / \partial x_j$. In short, a branch may be significant even though its transmission, as defined by relation 2.2, is not.

2.2 The Formulation of Analysis Problems in Terms of Flow Graphs

Preparatory to more general considerations, let us glance at Black's linear circuit, shown in Fig. 2.3. Given the input voltage V_o, what is the output voltage V_3? Were we sufficiently familiar

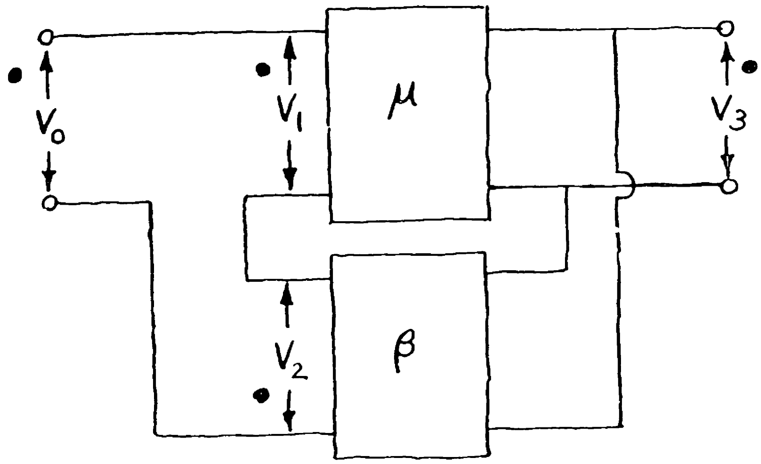


Fig. 2.3

with the circuit it might be possible to write the answer by inspection,

$$V_3 = V_o \frac{\mu}{1 - \mu\beta} . \tag{2.5}$$

Lacking such knowledge, we must introduce additional variables chosen so that the relations among them are evident by inspection. In general, for a given physical problem, these relations become simpler as the number of variables is increased. For the circuit of Fig. 2.3 the introduction of variable V_1 permits us to write $V_3 = \mu V_1$. Since V_1 itself is not known, it must be expressed in terms of other quantities, $V_1 = V_o + V_2$, and so on, $V_2 = \beta V_3$, until the set of relations is complete.

The process is one of tracing a succession of causes and effects through the physical system. One variable is expressed as an explicit effect due to certain causes, which are in turn recognized as effects due to still other causes. Each link in the chain of dependency is limited in extent only by our powers of inspection. The problem may be formulated in a few complicated steps or it may be subdivided into a larger number of simple ones, as determined by our judgment and knowledge of the particular problem under consideration.

The approach outlined above leads to a general formulation which has the form

$$\begin{aligned} x_1 &= f_1(x_1, x_2, \cdots, x_n) \\ x_2 &= f_2(x_1, x_2, \cdots, x_n) \\ &\vdots \\ x_n &= f_n(x_1, x_2, \cdots, x_n), \end{aligned} \tag{2.6}$$

in which one or more of the variables $x_1, x_2, \cdots, x_n$ may be absent from each of the functions f_k. Relations 2.6 are evidently in proper form for association with a flow graph. Moreover, the flow graph could have been constructed directly from the physical problem, without "writing" the equations. The explicit flow graph shown in Fig. 2.4,

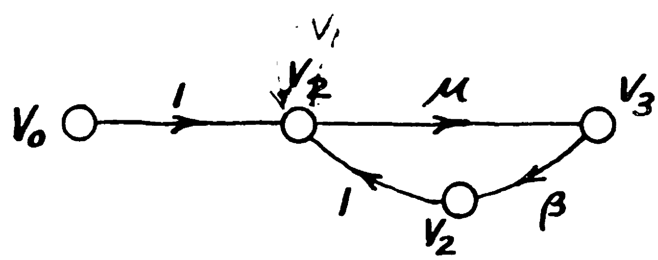


Fig. 2.4

for example, may be sketched by inspection of Black's linear circuit, Fig. 2.3. This graph is quite equivalent to the set of equations

$$\begin{aligned} V_o &= \text{a specified quantity} \\ V_1 &= V_o + V_2 \\ V_2 &= \beta V_3 \\ V_3 &= \mu V_1 \ . \end{aligned} \tag{2.7}$$

As another elementary example, let us attempt to formulate the input impedance of the grounded-grid amplifier shown in Fig. 2.5(a). Under certain simplifying assumptions, the linear incremental equivalent circuit is that of Fig. 2.5(b). When driven by a current source

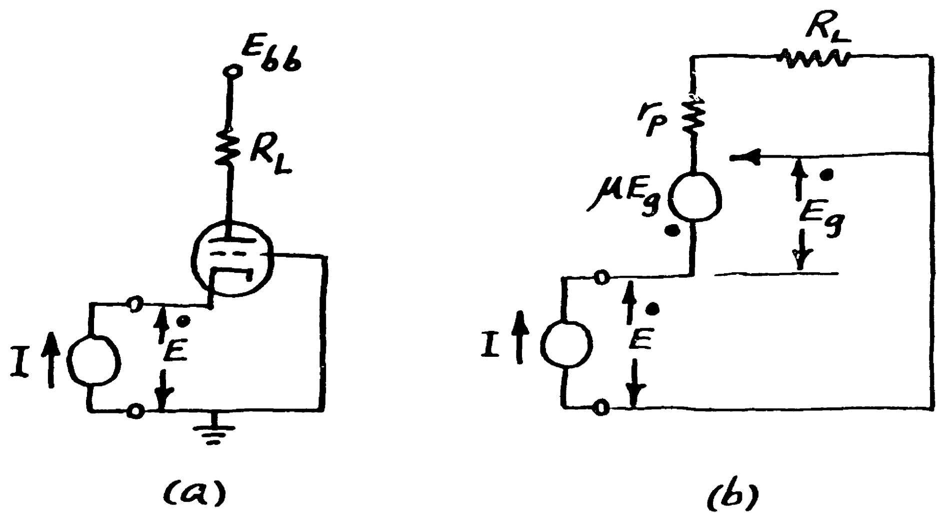


Fig. 2.5

I the circuit responds with a voltage E, and the ratio of E to I is the desired impedance. Voltage E may be expressed as the sum of the generated voltage μE_g and the voltage drop $(r_p + R_L)I$. Hence the flow graph begins as shown in Fig. 2.6(a). The dependency of

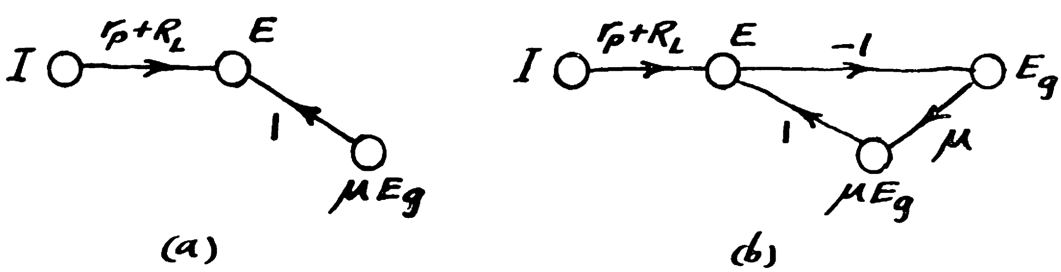


Fig. 2.6

μE_g upon E_g, and E_g in turn upon E, yields two branches which complete the graph as shown in Fig. 2.6(b). The completeness and validity of the graph, of course, must be determined by judgment. It is entirely possible to make errors in formulating a graph just as it is possible to write a set of equations which do not properly describe the physical problem.

Figure 2.7 shows a passive linear circuit which will serve as a third sample problem. A straightforward analysis results from the loop equations

$$E_o - (R_o + R_1)I_1 + R_1 I_2 = 0 \qquad (2.8)$$
$$-R_1 I_1 + (R_1 + R_2)I_2 = 0,$$

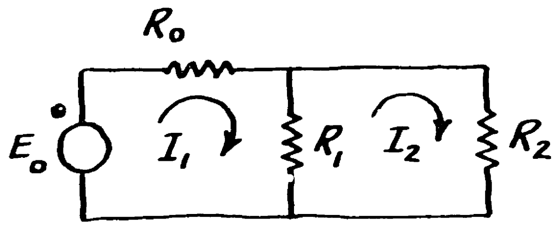


Fig. 2.7

which may be solved simultaneously for I_1 and I_2. Our purpose here, however, is to illustrate the formulation of flow graphs. In order to make way for a graph, the first and second equations may be solved individually for I_1 and I_2, respectively, yielding

$$I_1 = (E_o + R_1 I_2) \frac{1}{R_o + R_1}$$
$$I_2 = (R_1 I_1) \frac{1}{R_1 + R_2} \quad . \tag{2.9}$$

The corresponding flow graph is shown in Fig. 2.8. This graph

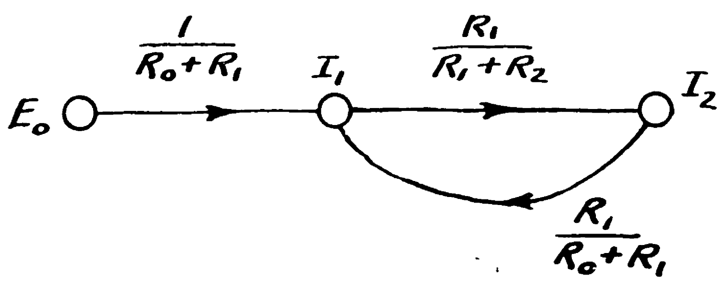


Fig. 2.8

might have been deduced directly from the circuit in the following manner. Effectively, loop current I_1 is caused by the voltages E_o and I_2R_1 which are produced in the first loop by quantities other than I_1. The contribution of each voltage to I_1 is given by the quotient of that voltage and the self-resistance $R_o + R_1$ of the first loop. Hence branches 0-1 and 2-1 in the flow graph. Similarly, current I_1 produces a voltage I_1R_1 in the second loop and division by the second self-resistance $R_1 + R_2$ yields the transmission ratio of branch 1-2, as shown in Fig. 2.8.

The flow graph of Fig. 2.4 bears a striking resemblance to the physical circuit from which it was derived. In Figs. 2.6(b) and 2.8, however, we have the same graphical structure arising from other physical problems wherein the pattern of signal flow is not as clearly related to the physical arrangement of system elements. Later we shall see how the solution of each of these analysis problems may be written by inspection of the graph. It is the pattern of signal flow, rather than the particular circuit configuration, which points the way toward the desired solution. This suggests, perhaps, the freedom and generality to be enjoyed by a theory based upon flow graphs, as contrasted with a theory which is bound to a particular class of physical components or systems.

The foregoing examples indicate the general manner in which a flow graph may be formulated but they do not, perhaps, emphasize the fact that the flow graph of a particular problem is not unique. Any one of a number of alternative graphs may be built by the analyst as he formulates the problem. A knowledge of the current-divider principle, for example, would permit us to represent the circuit of Fig. 2.7 as the graph shown in Fig. 2.9, where the first

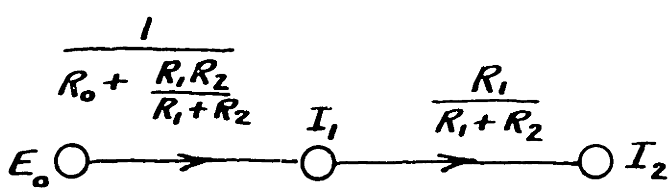


Fig. 2.9

branch transmission is the conductance faced by E_o and the second transmission is the current-divider ratio. A comparison of Figs. 2.8 and 2.9 bears out the rule that more complicated branch transmissions are the price which must be paid for a simpler flow graph structure. In later chapters an algebra is developed by which

Fig. 2.8 is very simply reducible to Fig. 2.9. The process of reduction makes clear the increasing complexity of branch transmissions as the graph is condensed.

2.3 The Flow Matrix

A flow graph implies the set of relations

$$x_k = f_k(x_1, x_2, \cdots, x_n) \qquad k = 1,2, \cdots, n \tag{2.10}$$

where f_k denotes the result of some specified operation upon the set of variables $x_1, x_2, \cdots, x_n$. In this section we shall undertake a straightforward mathematical treatment of the equation-set. This treatment leads to a square matrix whose properties are intimately associated with the structure of the corresponding flow graph. A familiarity with the matrix is not at all necessary for the exploitation of flow graphs. All primary methods and results discussed in this paper can be obtained without ever mentioning the words "matrix" or "determinant". Nevertheless, the formal mathematical treatment provides a background for certain bridge-points and parallelisms which are of sufficient interest to warrant their inclusion.

To proceed, we may define a new set of functions ϕ, where

$$\phi_k = x_k - f_k(x_1, x_2, \cdots, x_n) \qquad k = 1,2, \cdots, n. \tag{2.11}$$

Total differentiation yields the matrix equation

$$d\phi = dx(u - t), \tag{2.12}$$

where

$$dx = [dx_1 \; dx_2 \; \cdots \; dx_n]$$

$$d\phi = [d\phi_1 \; d\phi_2 \; \cdots \; d\phi_n]$$

are the differentials of the row matrices

$$x = [x_1 \; x_2 \; \cdots \; x_n]$$
$$\phi = [\phi_1 \; \phi_2 \; \cdots \; \phi_n],$$

u is the unit matrix, and t is the <u>branch matrix</u>

$$t = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix} \qquad (2.13)$$

having as elements the branch transmissions

$$t_{jk} = \partial f_k / \partial x_j. \qquad (2.2)$$

As mentioned previously, branch j-k appears in the flow graph if and only if t_{jk} does not vanish identically.

It is convenient to define u - t as the <u>flow matrix</u>

$$p = u - t. \qquad (2.14)$$

The elements of p are given by

$$p_{jk} = \frac{\partial \phi_k}{\partial x_j} = \begin{cases} 1 - t_{kk}, & j = k \\ -t_{jk}, & j \neq k, \end{cases} \qquad (2.15)$$

so that the flow matrix has the form

$$p = \begin{bmatrix} 1 - t_{11} & -t_{12} & \cdots & -t_{1n} \\ -t_{21} & 1 - t_{22} & \cdots & -t_{2n} \\ \vdots & \vdots & & \vdots \\ -t_{n1} & -t_{n2} & \cdots & 1 - t_{nn} \end{bmatrix}; \qquad (2.16)$$

The inverse of p is defined as the <u>transmission matrix</u>

$$T = p^{-1} \qquad (2.17)$$

and its elements are called transmissions (not to be confused with branch transmissions). The elements are given by

$$T_{jk} = \frac{P_{kj}}{P}, \tag{2.18}$$

where

P = determinant of the p matrix

P_{kj} = cofactor of p_{kj}.

Quantity P, hereafter called the flow determinant, is recognizable as the Jacobian of Eqs. 2.10. The significance of the Jacobian is brought out by the following geometrical interpretation, in which the quantities ϕ_k are visualized as scalar point-functions in n-dimensional space, the n coordinates being $x_1, x_2, \cdots, x_n$. In such an interpretation, the original Eqs. 2.10 are represented by the surfaces $\phi_k = 0$ $(k = 1, 2, \cdots, n)$ and the intersection of these surfaces is the desired solution. If the intersection is a single point then the solution is unique. Similarly, if the surfaces intersect in a number of separate points, then a number of discrete solutions obtain. When the surfaces intersect in a common curve, however, every point on the curve satisfies the equations and a discrete solution no longer exists. Since the curve may be specified by only n-1 equations, the original equations are not independent.

A curve common to each of the surfaces $\phi_k = 0$ is normal to each of the gradients $\nabla\phi_k$. If s is a vector tangent to the curve, having components $s_1, s_2, \cdots, s_n$, then

$$s \cdot \nabla\phi_k = 0 \qquad k = 1,2, \cdots, n. \tag{2.19}$$

Expansion of the dot product gives

$$\sum_{j=1}^{n} s_j \, \partial\phi_k/\partial x_j = \sum_{j=1}^{n} s_j p_{jk} = 0 \qquad k = 1,2, \cdots, n. \tag{2.20}$$

Now consider the following matrix product

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} . \qquad (2.21)$$

Zeros in the first row of the product matrix are due to relations 2.20. Since the determinant of the product must equal the product of the determinants, we have

$$P\, s_1 = 0. \qquad (2.22)$$

The numbering of coordinates is arbitrary so that we may assume $s_1 = 0$ without loss of generality. Remembering the conditions which led to relation 2.22, we see that the flow determinant vanishes identically when, and only when, a discrete solution of the original equations is impossible. Without specifying the nature of the functional relations, little more can be said about the general nonlinear problem.

When p is independent of x the problem is said to be linear. Integration of Eq. 2.12 yields the linear relation

$$\phi = xp - y, \qquad (2.23)$$

where

$$y = [y_1 y_2 \cdots y_n]$$

contains the constants of integration. In n-dimensional space, the planes $\phi = 0$ represent the original equations. If P is nonvanishing, the intersection of these planes at the single point

$$x = yT \qquad (2.24)$$

gives the desired solution. In the particular case $y = 0$, all planes pass through the origin and the trivial solution $x = 0$ results. If, however, $y = 0$ and $P = 0$, then the planes intersect in a line passing through the origin. Hence the ratio of any two coordinates x_k/x_j is specified but their actual values are indeterminate.

Relation 2.24 states the superposition principle. Since T is independent of x and y in linear problems, the response

$$x_k = \sum_{j=1}^{n} y_j T_{jk} \tag{2.25}$$

due to a set of drives y is the sum of the individual responses due to each drive acting alone. In view of the validity of superposition, no loss of generality results if we consider only a single drive y_j. The corresponding responses x_k are given by

$$\left(\frac{x_k}{y_j}\right)_{y_i=0, i\neq j} = T_{jk} = \frac{P_{kj}}{P}, \tag{2.26}$$

and the ratio of any two responses is

$$\left(\frac{x_k}{x_m}\right)_{y_i=0, i\neq j} = \frac{T_{jk}}{T_{jm}} = \frac{P_{kj}}{P_{mj}} . \tag{2.27}$$

In terms of the linear flow graph, T_{jk} is the signal appearing at node k when a unit signal is <u>injected</u> into node j through an externally driven branch. If, for some reason, P and y_j both become small, then in the limit x_k may still have some nonvanishing finite value,

$$x_k = y_j T_{jk} = (\text{zero} \times \text{infinity}). \tag{2.28}$$

The ratio 2.27 still holds, however, and

$$\left(\frac{x_k}{x_m}\right)_{P=0, y=0} = \frac{P_{kj}}{P_{mj}}, \quad \text{for any j.} \tag{2.29}$$

When P vanishes, therefore, the flow graph supports a self-sustained flow of signals in which only the relative values of the node signal levels are known.

CHAPTER III

THE TOPOLOGY OF FLOW GRAPHS

3.1 Introduction

In the preceding chapter the flow graph concept was introduced and the flow graph representation of physical analysis problems was discussed. The graph was also shown to have an associated flow matrix which enters the formal solution of the problem. In this chapter the topological properties of flow graphs will be considered. Topology has to do with the form and structure of a geometrical entity, but not its precise shape or size. The topology of electrical networks, for example, is concerned with the interconnection pattern of the circuit elements but not with the characteristics of the elements themselves. Similarly, flow graph topology deals with symbolic graphs, for which the branch transmissions are unspecified. Flow graphs differ from electrical network graphs in that the branches are directed. In accounting for branch directions, we shall need to take an entirely different line of approach from that adopted in electrical network theory.

3.2 Classification of Branches and Nodes

As a signal travels through some portion of a flow graph, traversing a number of successive branches in their indicated directions, that signal traces out a <u>flow path</u>. In Fig. 3.1 the succession of

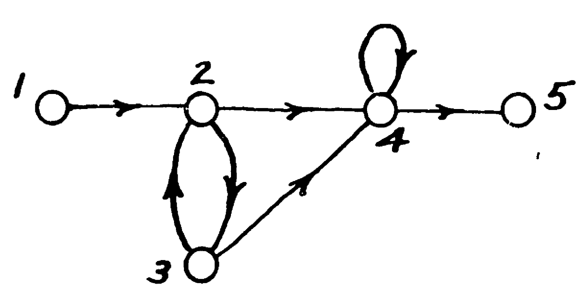


Fig. 3.1

branches (1-2, 2-4, 4-5), or (2-3, 3-2, 2-4), or (2-3, 3-4, 4-4, 4-5), constitutes a flow path, as do many other combinations. In general, there may be many different paths which originate at node j and terminate upon node k, or there may be none. For example, no path from node 4 to node 2 appears in Fig. 3.1.

Feedback now enters directly into our discussion for the first time with the definition of a feedback loop as any flow path which closes upon itself. The flow graph shown in Fig. 3.1 has feedback loops (2-3, 3-2) and (4.4). Additional loops are (2-3, 3-2, 2-3, 3-2), (4-4, 4-4), etc., but these are trivial.

The branches of a flow graph may be classified as either feedback or cascade branches. A feedback branch is one which forms part of a feedback loop. All others are called cascade branches. Returning to Fig. 3.1, we see that 2-3, 3-2, and 4-4 are the only feedback branches present. If each branch in a flow graph is imagined to be a one-way street, then a lost automobilist who obeys the law may drive through Feedback Street any number of times but he can traverse Cascade Boulevard only once as he wanders about in the graph.

The nodes in a flow graph are evidently susceptible to the same classification as branches. Namely, a feedback node is one which forms part of a feedback loop and all others are cascade nodes.

3.3 Feedback Graphs

A feedback graph is a flow graph which contains only feedback branches. Similarly, a cascade graph has only cascade branches. If all cascade branches are removed from a flow graph, the remaining feedback branches form one or more separate feedback graphs, which are said to be imbedded or contained in the original flow graph. The graph shown in Fig. 3.2(a), for example, contains the two feedback graphs indicated in (b) and (c).

Feedback graph (b) has three loops whereas (c) possesses only one. The number of loops, however, is not the most important characteristic

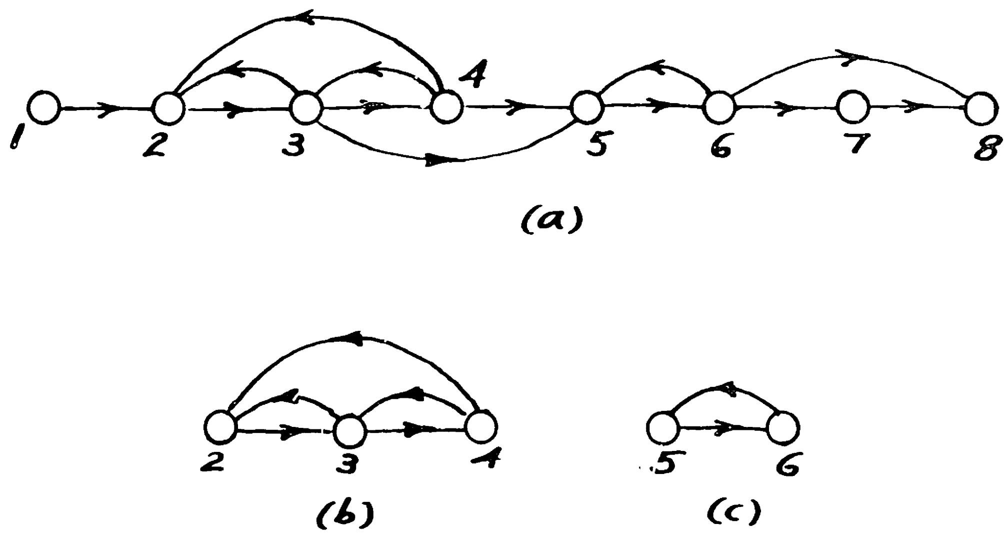


Fig. 3.2

of a graph relative to the difficulty of solution. More significant is a quantity called the index, defined as follows. The <u>index</u> of a feedback graph is the <u>minimum</u> number of nodes which must be blocked in order to interrupt all feedback loops in the graph. The process of node blocking simply interrupts all paths passing through that node. For the determination of index, blocking a particular node is equivalent to removing all branches which connect at that node. Both graph (b) and graph (c) in Fig. 3.2 are of index one, since the blocking of node 3 and either 5 or 6 serves to interrupt all closed paths.

The index notion is sufficiently important to justify its development from another point of view. The alternate approach begins with the expansion of each node, as shown in Fig. 3.3. Node k, likened to a repeater station, may be <u>expanded</u> into a <u>receiver</u> k' and a transmitter k", which are connected by a new branch k'-k" of unity transmission. This process produces a new graph having signal levels x_k' and x_k'' equal to the original signal x_k in the unexpanded graph. In terms of the expanded graph, the index may be defined as the minimum number of branches which must be broken or removed in order to eliminate all feedback loops. The expansion of Fig. 3.2(b) is shown in

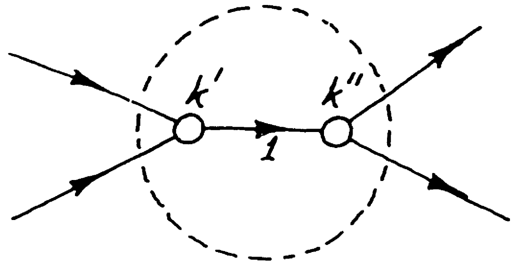


Fig. 3.3

Fig. 3.4 The removal of branch 3'-3" evidently leaves a cascade graph, so that the index is unity, as before.

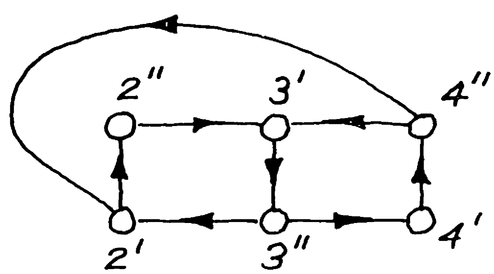


Fig. 3.4

Two nodes are said to be <u>connected</u> if they lie in a common flow path, and <u>coupled</u> if they occur in a common feedback loop. A flow graph is connected (or coupled) if every pair of its nodes is connected (or coupled). A feedback graph may now be defined rigorously as a connected flow graph having only feedback branches. It follows that a feedback graph is also coupled; a flow path exists from j to k and also from k to j for all j and k. For a proof we choose some flow path containing j and k, as shown in Fig. 3.5. Since each branch pq is a feedback branch, a path must exist from q to p and hence, by continuation, from k to j. Nodes j and k, therefore are coupled.

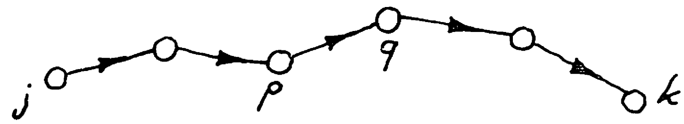


Fig. 3.5

3.4 Cascade Graphs

A cascade graph exhibits a special character which suggests the numbering of its nodes in a certain succession. The nodes will be termed early or late according to their positions in the succession; node j is earlier than node k if k exceeds j. In addition, we shall define a forward path as any flow path from j to k and a backward path as any path from k to j, where k exceeds j. In a cascade graph it is possible to number the nodes in a sequence, called the order of flow, such that no backward paths exist. The proof is as follows. If we trace in the reverse direction (i.e. opposite to the direction indicated by the arrowheads) along any path in a cascade graph, the tracing process must end at a source node, which transmits but does not receive (i.e. a node from which one or more branches radiate but upon which no branches terminate). Otherwise the path could be extended in the reverse direction until it eventually closed upon itself; an impossibility in a cascade structure. Having established the existence of at least one source node, let us choose one of these, designate it as node number one, and then temporarily erase it from the graph, together with its radiating branches. The remaining graph again exhibits a source which may be taken as node number two. By removing node 2 we find one or more sources from which to choose node 3, and so on to completion, at which point all branches, and all nodes from which any branches originally radiated, have been erased. Remaining are one or more sink nodes, which received but did not transmit in the original graph. The sink nodes are numbered last in any desired order. Let us now assume that there is no path from node k to any of the nodes 1, 2, 3, ..., j-1, where $k > j$.

Under this assumption, there is no path from k to j via a node earlier than j. The only possible path from k to j, then, is one which does not contain 1, 2, ..., or j - 1. In the light of the numbering procedure described above, however, no such route is permitted, since j is a source when 1, 2, ..., and j - 1 are removed. Hence there is no path at all from k to j, provided we assume none from k to 1, 2, ..., or j - 1. Since this assumption is valid for j = 2 it follows for all j by induction. Thus, we have proven the existence of an order of flow for cascade graphs, such that there are no backward paths from later to earlier nodes.

Figure 3.6 shows two simple cascade graphs whose nodes have been numbered in the order of flow. The numbering of graph (a) is unique, whereas other possibilities exist for graph (b), the scheme shown in (c) being an example.

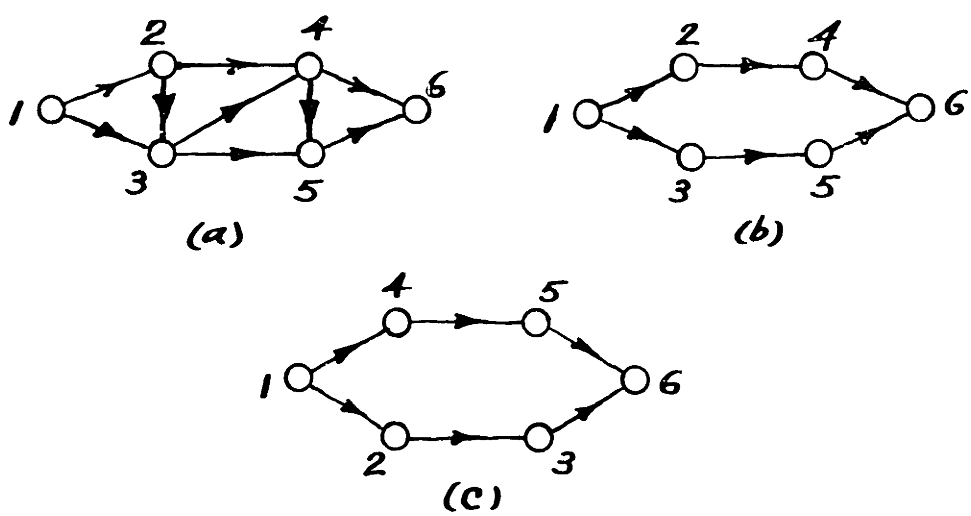


Fig. 3.6

3.5 Partitioning

The notion of an order of flow may be applied, in modified form, to a flow graph having both cascade and feedback branches. As a first step let us consider the graph shown in Fig. 3.7, which would be a cascade graph were it not for the presence of the self-loop t_{22}. An

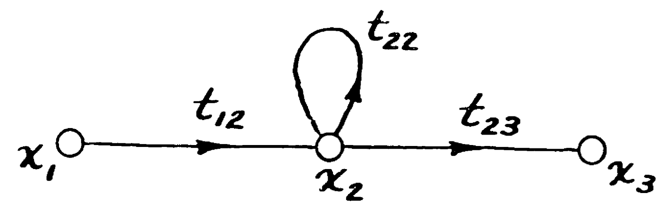


Fig. 3.7

order of flow still prevails, however, since no backward paths exist (i.e. paths from k to j, where $k > j$). Now consider the more complicated structure shown in Fig. 3.8. If the imbedded feedback graph is encircled and treated as a single supernode, then an order of flow

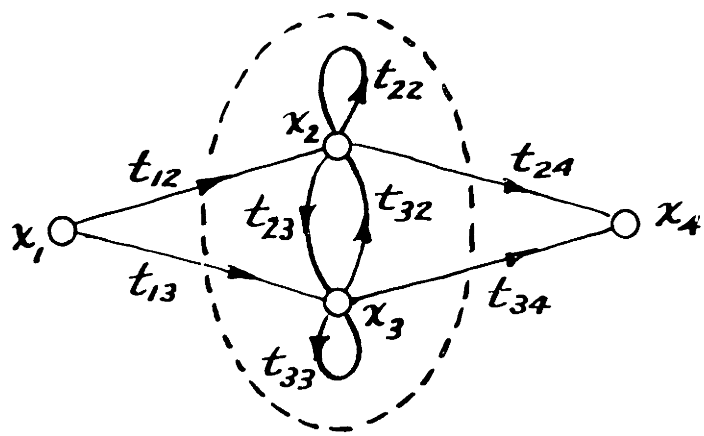


Fig. 3.8

again prevails; from the source x_1 to the supernode x_2x_3 to the sink x_4. With the aid of matrix notation we may condense this graph to the form shown in Fig. 3.9. Coupling within the supernode is

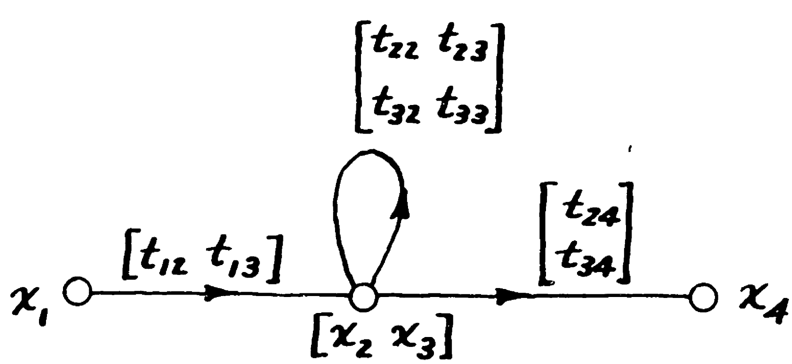


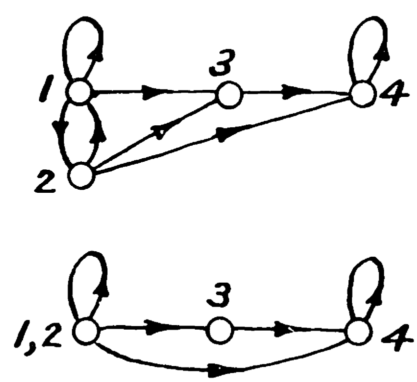
Fig. 3.9

specified by the single self-branch whose branch transmission is a matrix instead of a single term. Similarly, the matrix $t_{12}\ t_{13}$ takes the place of the two branch transmissions t_{12} and t_{13} appearing in Fig. 3.8. In general, a branch originating at a supernode containing b nodes and terminating upon a supernode of c nodes will have a matrix transmission of b rows and c columns. Similarly, the variable associated with any supernode is a row matrix containing the individual node variables. Within each supernode the individual nodes are numbered consecutively in any order.

When a flow graph is condensed and numbered in the order of flow, that graph is said to be <u>partitioned</u>. The process of partitioning places the flow matrix in a particularly significant form. For illustration Fig. 3.10 shows (a) a flow graph, (b) its condensation, and (c) the corresponding flow matrix. Because of the order of flow, the

(a)

(b)



(c)

$1-t_{11}$	$-t_{12}$	$-t_{13}$	0
$-t_{21}$	1	$-t_{23}$	$-t_{24}$
0	0	1	$-t_{34}$
0	0	0	$1-t_{44}$

Fig. 3.10

lower left portion of the flow matrix is empty. No path from a node or supernode to an earlier node or supernode can be present (i.e. $t_{jk} = 0$ for j k, provided j and k are not common to one supernode).

The Laplace development of a determinant shows that

$$P = \left| \begin{array}{c|c} P_{aa} & P_{ab} \\ \hline 0 & P_{bb} \end{array} \right| = \left| P_{aa} \right| \times \left| P_{bb} \right| , \qquad (3.1)$$

where P is partitioned into subdeterminants P_{cd} having c rows and d columns, and where 0 represents an empty subdeterminant. We see immediately from the form of Fig.3.10(c) that the flow determinant of a graph is equal to the product of the flow determinants of its imbedded feedback graphs. The flow determinant of a cascade graph evidently has the value unity.

Having identified feedback graphs with factors of the flow determinant, we may complete the tie by giving a suitable mathematical meaning to the index of a feedback graph. The index is found by removing nodes until only a cascade graph remains, the index being the minimum number of such removals required. When nodes j and k are removed the new flow determinant is the algebraic complement of the diagonal subdeterminant

$$\begin{vmatrix} p_{jj} & p_{jk} \\ p_{kj} & p_{kk} \end{vmatrix} .$$

The complement is obtained by striking out rows and columns j and k in the original flow determinant. For example, the removal of nodes 1 and 3 from a graph originally having nodes 1, 2, 3, 4, changes the flow determinant from

$$\begin{vmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{vmatrix} \quad \text{to} \quad \begin{vmatrix} p_{22} & p_{24} \\ p_{42} & p_{44} \end{vmatrix} ,$$

which is the complement of

$$\begin{vmatrix} p_{11} & p_{13} \\ p_{31} & p_{33} \end{vmatrix} .$$

The index, then, may be identified as the order of the smallest diagonal subdeterminant whose complement is unity, since a unity complement indicates a remaining cascade graph.

3.6 The Residual Flow Graph

A cascade graph represents a set of equations which may be solved by direct substitution. Figure 3.11, for example, has the

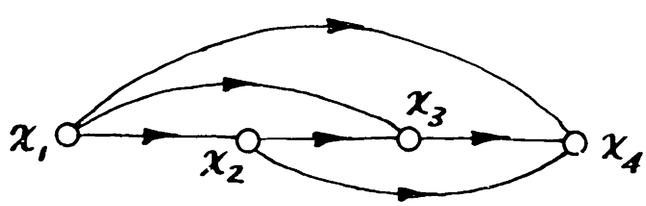


Fig. 3.11

associated equation set

$$\begin{aligned} x_2 &= f_2(x_1) \\ x_3 &= f_3(x_1,x_2) \\ x_4 &= f_4(x_1,x_2,x_3). \end{aligned} \tag{3.2}$$

Given the value of the source x_1, we obtain the value of x_4 by the explicit process

$$x_4 = f_4\left\{ x_1, f_2(x_1),\ f_3[x_1, f_2(x_1)]\right\} = F_4(x_1). \tag{3.3}$$

In general, once the order of flow is established, a knowledge of x_1 fixes x_2 since paths from x_3, $x_4, \cdots$, or x_n to x_2 are nonexistent. Similarly, with x_1 and x_2 known, x_3 is determined explicitly; and so on to x_n. A cascade graph is immediately reducible, therefore, to a <u>residual graph</u> in which only sources and sinks appear.

The residual form of Fig. 3.11 is the single branch shown in Fig. 3.12, which represents Eq. 3.3. The extension to more than

Fig. 3.12

one source or sink is obvious. Had two sources and two sinks appeared in the original cascade graph, the residual graph would have contained a maximum of four branches as indicated by Fig. 3.13.

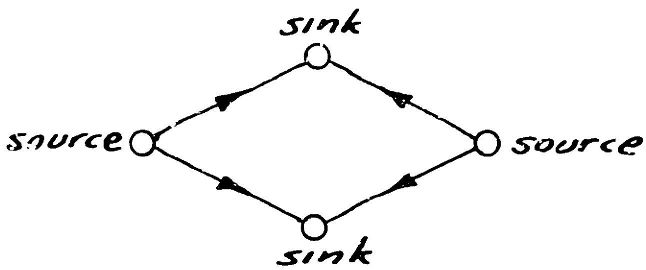


Fig. 3.13

Unlike those associated with a cascade graph, the equations of a feedback graph are not soluble by direct substitution. Consider the simple example shown in Fig. 3.14. An attempt to express x_3 as

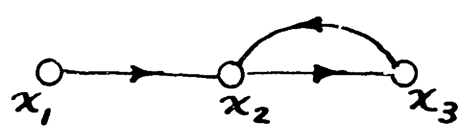


Fig. 3.14

an explicit function of x_1 fails because of the closed chain of dependency between x_2 and x_3. The equations are

$$\begin{aligned} x_2 &= f_2(x_1,x_3) \\ x_3 &= f_3(x_2), \end{aligned} \tag{3.4}$$

so that elimination of x_2 by substitution yields

$$x_3 = f_3\left[f_2(x_1,x_3)\right] = F_3(x_1,x_3). \tag{3.5}$$

With x_1 a specified constant we have x_3 expressed as a function of itself. The implicit relationship may sometimes be solved for x_3 as an explicit function of x_1. In general, however, the solution for x_1 requires a cut and try process of the first order, which is equivalent to plotting $y = F_3(x_1,x_3)$ against x_3 and noting its intersection with the line $y = x_3$.

Nevertheless, certain superflous nodes may be eliminated from a feedback graph by direct substitution, leaving the more bothersome implicit relationships exposed. The definition of index implies the existence of a set of residual nodes, equal in number to the index, whose removal interrupts all feedback loops in the graph. The set is not always unique. In Figs. 3.15(a) and (b) the set consists of a

Fig. 3.15

single node. In graph (a) node 2 is the only possibility, whereas either 1, 2, or 3 may be chosen as the residual node in graph (b). Once a set of residual nodes has been chosen, all other nodes, except sources and sinks, may be eliminated by direct substitution. The presence or absence of particular residual branches in the residual graph may be determined by inspection of the dependencies indicated in the original graph. For the sake of definiteness, we may define a residual path as a path which originates at a source or a residual node and terminates upon a sink or a residual node, but which passes through no residual nodes. Branch j-k will appear in the residual graph if and only if residual path j-k is present in the original flow graph. As an example, consider the reduction shown in Fig. 3.16.

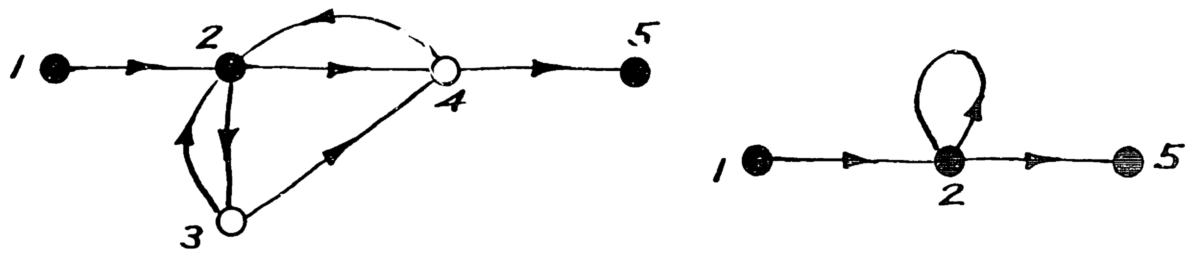


Fig. 3.16

The sources, sinks, and residual nodes, which are retained in the residual graph, are shown darkened. Residual branch 2-5 in (b) accounts for the residual paths 2-4-5 and 2-3-4-5 in graph (a). Similarly, 2-2 in (b) represents the combined effect of 2-3-2, 2-3-4-2, and 2-4-2 in (a). A path from 1 to 5 exists in (a) but only via node 2. Hence there is no residual path 1-5. Accordingly, residual branch 1-5 is absent from (b). For comparison, the equations of graph (a) are

$$\begin{aligned} x_2 &= f_2(x_1,x_3,x_4) \\ x_3 &= f_3(x_2) \\ x_4 &= f_4(x_2,x_3) \\ x_5 &= f_5(x_4) \end{aligned} \tag{3.6}$$

and those associated with (b) are

$$\begin{aligned} x_2 &= f_2\left\{x_1, f_3(x_2),\ f_4[x_2, f_3(x_2)]\right\} = F_2(x_1,x_2) \\ x_5 &= f_5\left\{f_4[x_2, f_3(x_2)]\right\} = F_5(x_2). \end{aligned} \tag{3.7}$$

A minor dilemma arises in the reduction process if we desire, for some reason, to preserve a node which is neither a residual node nor a sink. In Fig. 3.17(a), for example, if we desire an eventual solution for x_3 in terms of x_1, then a node corresponding to x_3 must be retained in the residual graph. Apparently, no further reduction is possible. The simple device shown in (b) may be employed, however,

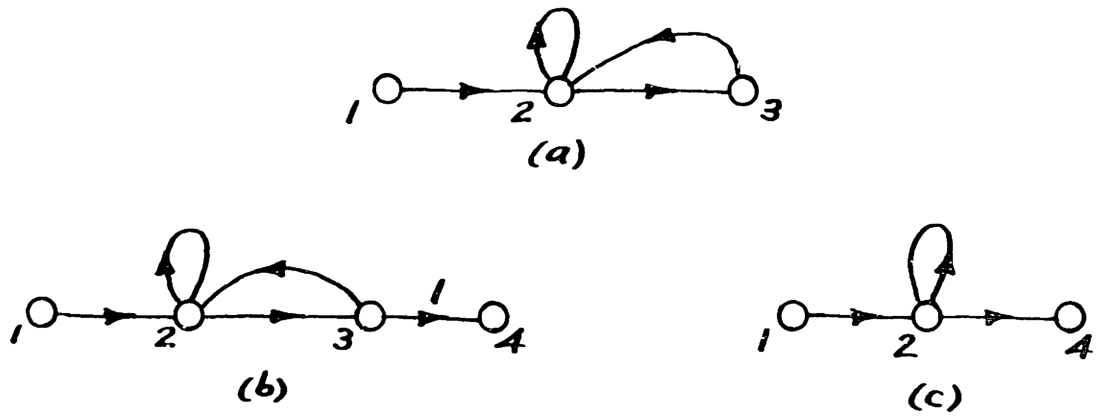


Fig. 3.17

to obtain the reduction (c). The trick is to connect node 3 to an additional node through a branch of unity transmission, so that $x_4 = x_3$. Node 3 then disappears in the reduction, leaving the desired value of x_3 available at the sink.

Since a flow graph and its residual counterpart, though different in form, are both representations of the same analysis problem, it is perhaps not surprising that their flow determinants (i.e. their Jacobians) have the same value. To prove this, we need only show that the elimination of a variable by direct substitution leaves the value of the flow determinant unchanged. The direct elimination of a particular variable, say x_n, is possible only if f_n is independent of x_n,

$$x_n = f_n(x_1, x_2, \cdots, x_{n-1}). \tag{3.8}$$

Substitution into f_k yields

$$x_k = f_k[x_1, x_2, \cdots, x_{n-1}, f_n(x_1, x_2, \cdots, x_{n-1})] \tag{3.9}$$

$$k = 1, 2, \cdots, n-1$$

and total differentiation gives

$$dx_k = \sum_{j=1}^{n-1} dx_j(t_{jk} + t_{jn}t_{nk}), \tag{3.10}$$

$$k = 1, 2, \cdots, n-1.$$

Taking n = 3 for convenience of illustration, we now write the matrix equation

$$\begin{bmatrix} 1-t_{11} & -t_{12} & -t_{13} \\ -t_{21} & 1-t_{22} & -t_{23} \\ -t_{31} & -t_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_{31} & t_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1-t_{11}-t_{13}t_{31} & -t_{12}-t_{13}t_{32} & -t_{13} \\ -t_{21}-t_{23}t_{32} & 1-t_{22}-t_{23}t_{32} & -t_{23} \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.11)$$

(a) (b) (c)

Matrix (a) is the original flow matrix and the upper left portion of (c) is recognizable as the flow matrix of the residual graph. Since the determinant of the product of two matrices equals the product of their determinants, we see that the elimination of a superfluous node leaves the value of the flow determinant unchanged.

3.7 Reciprocation of a Flow Path

A single constraint or relationship among several variables appears topologically as a cascade graph containing one sink and one or more sources. Figure 3.18(a) is an elementary example. In principle,

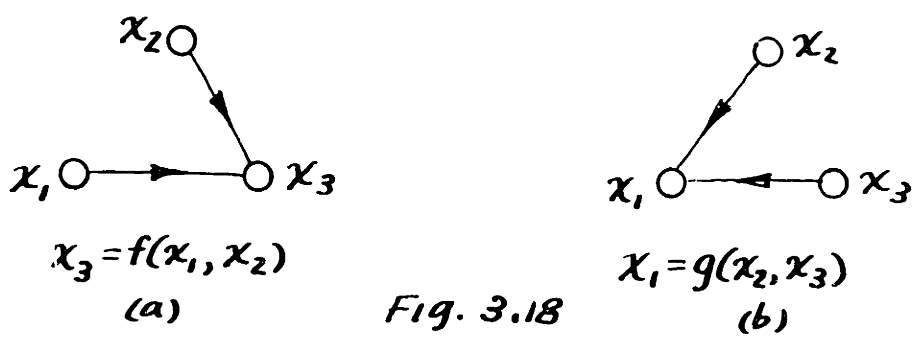


Fig. 3.18

nothing prevents us from solving the equation in (a) for one of the independent variables, say x_1, to obtain the form shown in (b). In terms of the flow graph, we say that branch 1-3 has been reciprocated. The reciprocation of a path is accomplished by reciprocating each of the branches along that path, as shown in Fig. 3.19. Reciprocation of branch 1-3 in (a) yields graph (b). Similarly, reciprocation of branch 3-5 in (b) gives (c). Hence graph (c) is recognizable as the result of reciprocating path 1-3-5 in graph (a).

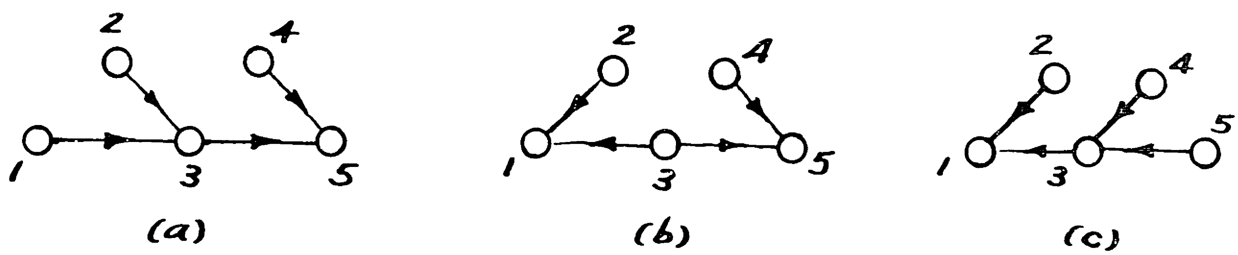


Fig. 3.19

Topologically, the reciprocation of a path has two effects. First, the directions of branches forming that path are reversed, and second, branches entering that path have their entry points shifted to the next downstream node in the new direction of flow. The reciprocation of an open path (i.e. an unclosed path) is significant only if that path starts from a source node. Otherwise, two expressions for the same variable are obtained and two separate graphs are required in the topological representation, contrary to our original hypotheses regarding flow graphs and their associated equations. For illustration, let us reciprocate the single branch 3-5 in Fig. 3.19(a), thereby obtaining an expression for x_3 in terms of x_4 and x_5. Branches 1-3 and 2-3, however, already imply that x_3 is a function of x_1 and x_2. The new graph, then, takes the degenerate form shown in Fig. 3.20, which fits rather poorly into our picture of signal flow.

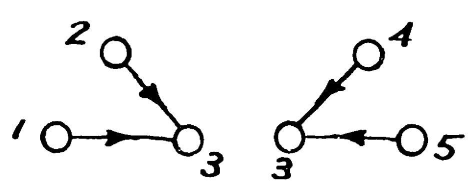


Fig. 3.20

Thus far, the possibility of reciprocating a closed path has not been considered. By a rather simple extension of the foregoing notions we see that feedback loops do lend themselves readily to reciprocation. For example, given the graph of Fig. 3.21(a), we may

$x_2 = f(x_1, x_3)$ $\quad$ $x_3 = F(x_1, x_2)$

$x_3 = g(x_2)$ $\quad$ $x_2 = G(x_3)$

(a) $\quad$ (b)

Fig. 3.21

alternatively solve the first equation for x_3 and the second for x_2, producing a new graph as shown in (b). Branches external to the path which is being reciprocated evidently obey the same shifting rule as that developed for open paths. We may, in fact, define the reciprocation of any path, closed or open, as follows:

> The <u>reciprocation</u> of a flow path containing branch jk but not branch ik is accomplished by transforming each branch jk into branch kj and replacing each branch ik by a new branch ij.

The definition enunciated above evidently excludes open paths which do not originate at sources, since if k is such a point of origination there exists a branch ik but no branch jk and the meaning of branch ij is lost. In addition, reciprocation is not applicable to a closed path which intersects itself (i.e. which touches upon the same node more than once, as does the path 1-2-3-2-1), for if jk and mk are both contained in the chosen closed path and if jk is reciprocated first, then branch mk becomes mj and the original closed path is destroyed.

The process of reciprocation, as might be expected, influences the topological properties of the flow graph. Of greatest interest here is the effect upon index. The reciprocation of a path may yield a new graph whose index is the same as that of the original graph, as in Fig. 3.21. It is also possible that the index may be reduced, as illustrated by the reciprocation of loop 2-3-2 in Fig. 3.22(a),

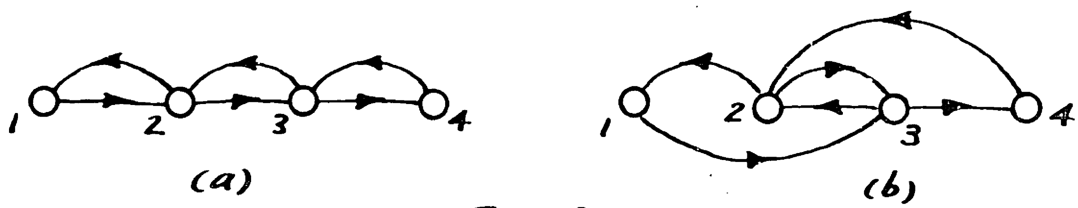


Fig. 3.22

which yields a graph (b) of index one. Similarly, the index may increase, as demonstrated by the reciprocation of path 2-3-2 in (b) which restores graph (a).

In general, paths parallel to a given path contribute to the formation of feedback loops when the given path is reciprocated, and conversely. Hence, if we wish to accomplish a reduction of index we should choose for reciprocation a path having many attached backward paths but few parallel forward paths. No specific rules or tests for the minimum index obtainable by reciprocation will be given here, chiefly because such rules, stemming from a consideration of the flow determinant, are sufficiently unwieldy to be of vanishing practical interest.

The use of reciprocation may be illustrated by the simple analysis problem which follows. Given the cathode-follower circuit shown in Fig. 3.23, it is desired to plot an output-input characteristic;

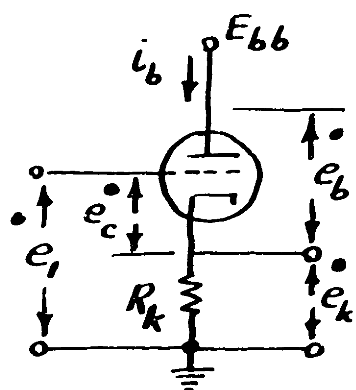


Fig. 3.23

a curve showing e_k as a function of e_1 for a specified tube $i_b = f(e_b, e_c)$ and specified values of E_{bb} and R_k. Since e_1 is the natural driving variable, we might construct the flow graph shown in Fig. 3.24. The uppermost branches, whose transmissions are not indicated, represent the function $i_b = f(e_b, e_c)$; presumably available in the form of graphical plate characteristics.

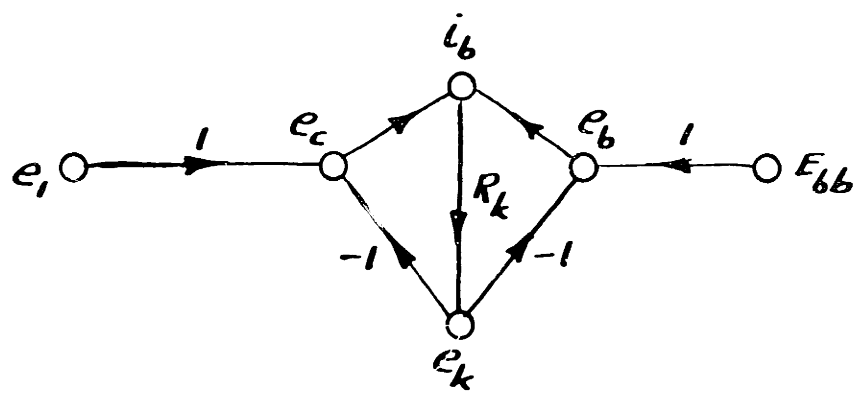


Fig. 3.24

As it stands, the flow graph is of index unity, either i_b or e_k serving as the residual node. For a single specified value of e_1, a first order cut and try process is required for determination of the corresponding value of e_k. Taking i_b as the residual node, we may assume a trial value for i_b and then, knowing e_1 and E_{bb}, compute the values of the remaining variables e_k, e_c, and e_b. As i_b is varied through a range of trial values, a load line $e_b = E_{bb} - i_b R_k$ and a bias line $e_c = e_1 - i_b R_k$ are described upon the plate characteristics. The intersection of the load and bias lines, of course, represents the desired solution.

Since an output-input curve, rather than a single operating point, is desired, we might have simplified the problem somewhat by beginning with an assumed value of e_k (or i_b) and calculating the corresponding value of e_1 required by the equations. A series of such value-pairs specifies the output-input characteristic just as well as a series obtained by choosing each value of e_1 first. If e_k is chosen first, the associated computational process corresponds to a graph in which e_k appears as a source. To obtain such a graph, we may reciprocate path e_1, e_c, i_b, e_k in Fig. 3.24. The resulting form is shown in Fig. 3.25.

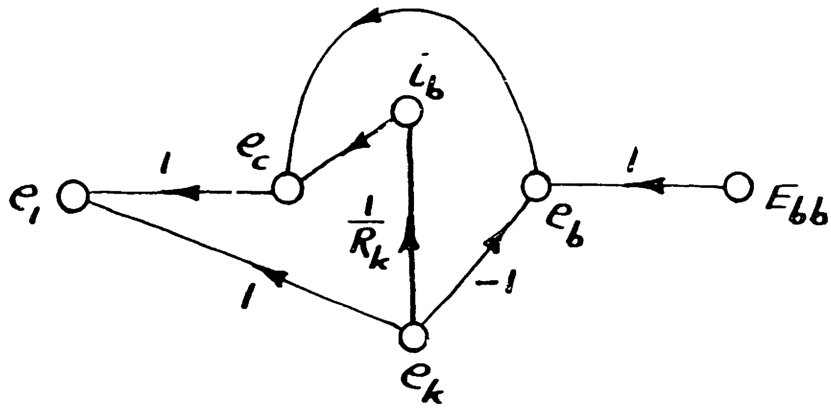


Fig. 3.25

Following the simple rules developed earlier, we change the directions of the branches along the path e_1, e_c, i_b, e_k and shift the tips of branches (e_k, e_c) and (e_b, i_b) along the new direction of flow to obtain the new branches (e_k, e_1) and (e_b, e_c). By inspection of the circuit the new branch transmissions are as shown in the figure. A cut and try process is no longer required since the index has been reduced to zero. Choosing a value of e_k, we compute $e_b = E_{bb} - e_k$, $i_b = e_k/R_k$, $e_c = f(e_b, i_b)$ from the plate characteristics, and $e_1 = e_k + e_c$, all as dictated by the cascade graph of Fig. 3.25.

The foregoing cathode-follower problem was chosen for simplicity of illustration. Having such simplicity and familiarity, it may, of course, be grasped and analyzed without recourse to flow graphs. Nevertheless, the flow graph notion affords an organized representation which the reader may find attractive even in such simple problems. When the problem becomes more complex, the practical advantages of flow graph methods stand out somewhat more strongly. The flow graph tends to extend our powers of perception with regard to the character of a set of functional relationships. If we see a problem clearly, then no graph is needed. As the complexity increases, flow graph notions become a valuable aid to systemization. A still greater increase in complexity, however, may make the graph itself so complicated that perception suffers until the graph has been somewhat condensed by one reduction process or another. In short, the flow graph offers no panacea to the analyst but it does provide a graphical language which facilitates the formulation and logical analysis of a great variety of problems. If a specific problem is totally confounding and mysterious, then a flow graph may or may not help. If, on the other hand, we have at least some familiarity with a device or circuit, if we have some notion as to how to proceed with the problem, then flow graph methods can help to crystallize our thinking. Ultimately, the organization of the problem comes from within our minds, not from the flow diagram, which can not be expected to serve as more than a convenient mode of expression or manipulative tool.

CHAPTER IV

THE ALGEBRA OF LINEAR FLOW GRAPHS

4.1 Introduction

The preceding chapters have dealt largely with the properties of general flow graphs. We shall now specialize our attention to graphs of the linear class. As specified previously, a linear flow graph is one whose branch transmissions are not dependent upon the values of the node variables. Linear circuit problems lead to such graphs, provided the analysis is carried out in the frequency domain where the linear intego-differential equations involving time become linear algebraic equations with frequency as a parameter.

The restriction to linear problems permits us to develop a quantitative algebra for the manipulation of flow graphs. The formal background was set forth in the second chapter, wherein the transmission matrix was identified as the inverse of the flow matrix. In this chapter we shall be concerned with the evaluation of a transmission by direct manipulation of the flow graph. In developing the algebra of linear flow graphs we shall accumulate a body of theorems, relations, and methods which constitute the beginnings of a basic theory of feedback in linear systems. This theory is supplemented in the later chapters dealing with sensitivity and stability.

4.2 Elementary Transformations

Figure 4.1 illustrates several elementary transformations or equivalences. The parallel or multipath transformation (a) reduces the number of branches. The cascade transformation (b) eliminates a node, as

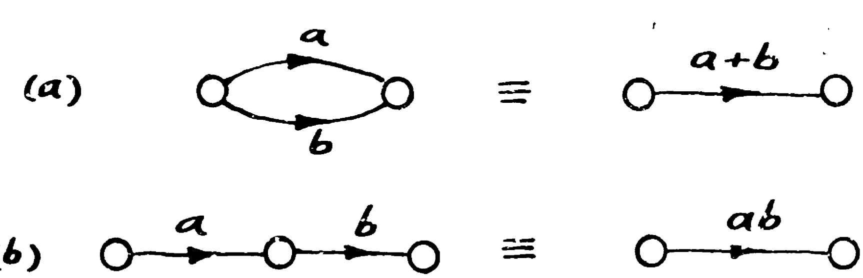


Fig. 4.1

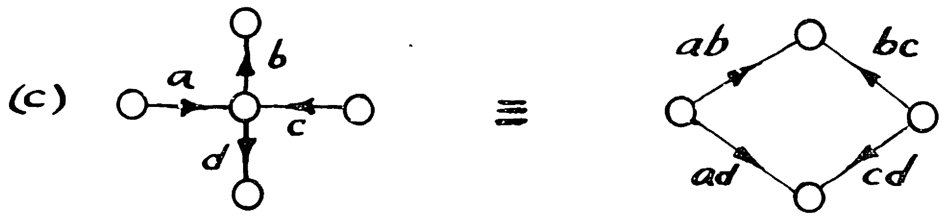


Fig. 4.1

does the star-to-mesh transformation (c), of which (b) is actually a special case. These elementary equivalences permit us to evaluate the branch transmissions appearing in a residual graph by systematic reduction of the original graph.

Consider, for example, the diagram of Fig. 4.2(a) (the explicit counterpart of Fig. 3.16, on which branch transmissions were not indicated.) Elimination of node 3 by means of the star-to-mesh transformation yields graph (b). By the parallel equivalence, we may immediately simplify to the form (c). Subsequent elimination of node 4 and combination of parallel branches leads through (d) to the residual graph (e).

In practice it is possible to recognize the branch transmissions of the residual graph by direct inspection of the original diagram. In order to provide a sound basis for this more direct process, we may define a <u>path transmission</u> as the product of the transmissions of the branches forming that path. In addition, a <u>residual transmission</u> from j to k is defined as the algebraic sum of the transmissions of all different residual paths from j to k. As defined previously, a residual path is one which originates at a source or a residual node and terminates upon a sink or a residual node, but which does not pass through any residual nodes. It follows from the nature of our elementary transformations that the transmission of branch j-k in the residual graph is equal to the residual transmission from j to k in the original graph.

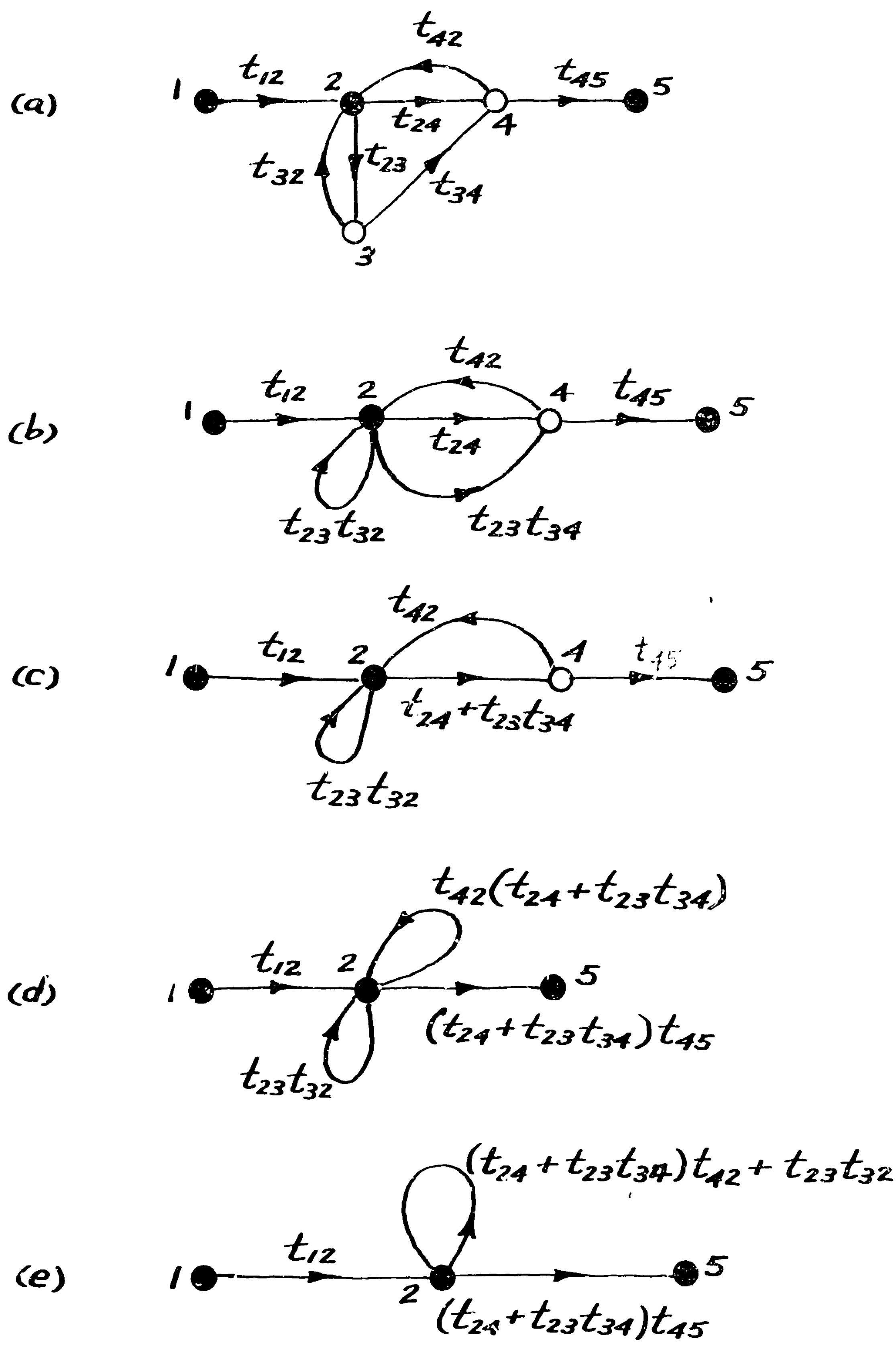

(a)
1
2
3
4
5
t12
t42
t24
t32
t23
t34
t45
(b)
t23t32
t23t34
(c)
t24+t23t34
(d)
t42(t24+t23t34)
(t24+t23t34)t45
(e)
(t24+t23t34)t42+t23t32

Fig. 4.2

Returning to Fig. 4.2(a), we observe that three different residual paths from node 2 to itself are present. The sum of their transmissions is $t_{24}t_{42} + t_{23}t_{32} + t_{23}t_{34}t_{42}$, which checks the value of the self-transmission at node 2 in the residual graph of Fig. 4.2(e). Similarly, two paths from 2 to 5 are present and the residual transmission is evidently given by $t_{24}t_{45} + t_{23}t_{34}t_{45}$. No residual path exists from 1 to 5 in the original graph, since such paths are prohibited from passing through node 2, the residual node. Hence no branch connecting 1 to 5 appears in Fig. 4.2(e).

4.3 <u>Loop Transmission and Loop Difference</u>

When a flow graph is simplified to its residual form, one or more self-loops appear. The effect of a self-loop at node j upon a signal passing through node j may be studied in terms of Fig. 4.3. The signal

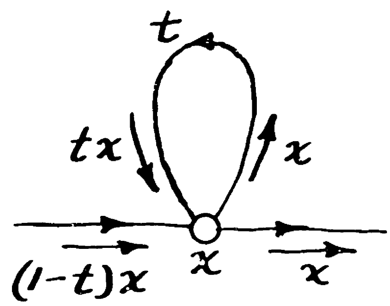


Fig. 4.3

leaving the node along each of the outgoing branches is x, the value of the node variable. The signal returning via the self-loop is tx, where t is the transmission of the self-branch. Since signals entering the node add algebraically to produce x, it follows that the external signal entering from the left must be $(1-t)x$. The node and self-loop, therefore, may be replaced by a single cascade branch whose transmission is the reciprocal of $(1-t)$, as shown in Fig. 4.4.

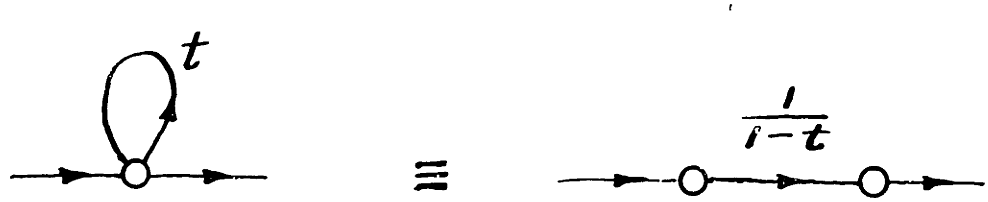


Fig. 4.4

Approaching the self-loop effect from another direction, we may observe that a signal entering the node is confronted with an infinity of paths by which it may eventually exit and pass onward. One path passes directly through the node, the second path traverses the loop once before leaving, the third path circles the loop twice, and so on. Hence, when a unit signal is injected into the node, the total exiting signal is given by the geometrical series

$$1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}, \tag{4.1}$$

which sums to the familiar result. The convergence of this series (for $t < 1$) poses no dilemma in view of the validity of analytic continuation. The result is evidently valid for all values of t except the singular point $t = 1$, near which the transmission $1/(1-t)$ becomes arbitrarily large.

The self-loop to branch transformation evidences the basic effect of feedback as a contribution to the <u>denominator</u> of an expression giving a transmission in terms of branch transmissions. In our algebra, feedback is associated with <u>division</u> or, more generally, with the inversion of a matrix whose determinant is not identically equal to unity.

For the purpose of enlarging upon the notion of circulating signal-flow, we shall now define the <u>loop transmission</u> of a <u>node</u> as the signal received at that node per unit signal transmitted by that node. In Fig. 4.5, for example, a unit signal transmitted from node 1 passes outward along branches a and b. In traversing node 2 the signal is divided by $1 - d$. The loop transmission of node 1, then, is equal to the total

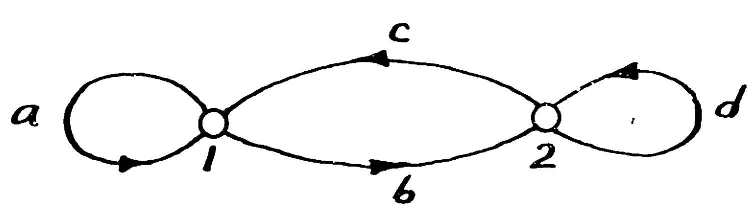


Fig. 4.5

returned signal,

$$\tau_1 = a + \frac{bc}{1-d} . \tag{4.2}$$

An alternative definition of loop transmission is the following, given here for the sake of additional perspective. If node k is expanded into a receiver k' and a transmitter k", and if the internal connecting branch is broken or removed, then the loop transmission of node k may be defined as the transmission from source k" to sink k'. In terms of this definition, Fig. 4.5 may be replaced by Fig. 4.6 for the evaluation of τ_1. The transmission from 1" to 1' is given by expression 4.2, as may be seen by performing a self-loop elimination at node 3 and then applying the cascade and parallel equivalences to obtain a single equivalent branch transmission from 1" to 1'.

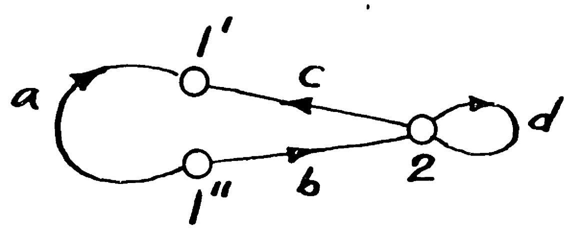


Fig. 4.6

Another quantity, for which we shall find subsequent use, is the loop transmission of a branch. In Fig. 4.5, the loop transmission of branch b would be obtained by breaking or cutting branch b at some point, injecting a unit signal at the forward side of the break, and then measuring the signal returned to the opposite side of the cut, as shown in Fig. 4.7(a). Since the signal traverses the entirety of

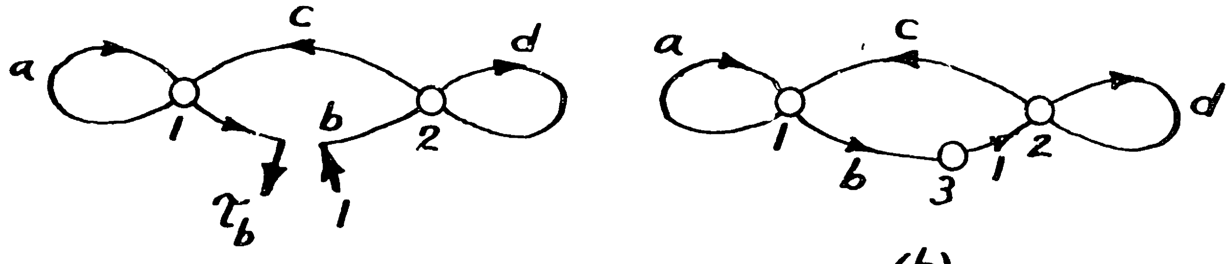


Fig. 4.7

branches b and c, and also passes through two nodes supporting self-loops, the returned signal is given by

$$\tau_b = \frac{bc}{(1-d)(1-a)} . \qquad (4.3)$$

The loop transmission of a branch may also be specified in terms of the existing definition for the loop transmission of a node. We shall, in fact, define the <u>loop transmission</u> of a <u>branch</u> as the loop transmission of a new node artificially introduced at some internal point of that branch. The introduction of the new node i effectively replaces the given branch j-k by two cascaded branches j-i and i-k, such that the product of their transmissions is equal to that of the original branch ($t_{ji}t_{ik} = t_{jk}$). In relation to our running example, the loop transmission of branch b in Fig. 4.5 is identical by definition to the loop transmission of node 3 in Fig. 4.7(b).

With regard to notation, we shall designate all loop transmissions by the symbol τ. Subscripts denote a certain <u>node</u> or <u>branch</u>, the distinction being evident from the manner in which the flow graph is lettered or numbered. In Fig. 4.7(b), to take a specific example, τ_3 is the τ of node 3 and τ_j represents the τ of an arbitrary node. Either τ_{13} or τ_b might be used to designate the τ of the specific branch b, but the τ of an arbitrary branch j-k would probably be given the double subscript τ_{jk}.

To summarize the foregoing definitions and illustrations, we may state that τ_j represents the transmission from node j to itself, with all possible return paths taken into account. It is evident that cascade branches can have no effect upon the value of τ_j. As a consequence, only that imbedded feedback graph which contains node j need be considered in the evaluation of τ_j. In Fig. 4.8, for example, $\tau_2 = \tau_3 = \tau_{23} = t_{23}t_{32}$.

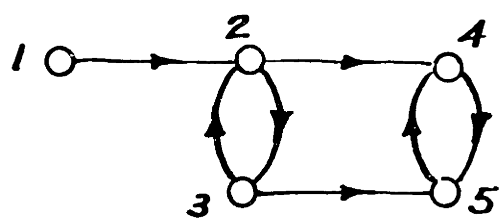


Fig. 4.8

Having defined loop transmission, we may expand the simple self-loop equivalence to a more general form, which may be stated as follows. If a signal y_j is injected into some node j of a general linear flow graph, then the transmission from the external signal source to node j is given by

$$T_{jj} = \frac{x_j}{y_j} = \frac{1}{1-\tau_j} \, . \qquad (4.4)$$

Figure 4.9 shows the associated flow graph. Quantity $1-\tau_j$, which finds its way into all feedback calculations, is defined as the loop

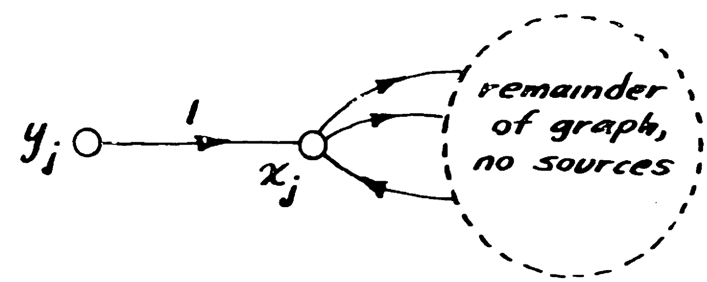


Fig. 4.9

difference of node j,

$$D_j = 1-\tau_j . \qquad (4.5)$$

A corresponding definition applies to the loop difference of a branch D_{jk}.

The loop difference D_j may be interpreted as the fractional part of x_j supplied by y_j, the remaining contribution to x_j coming, of course, from signals which return through the remainder of the graph. A simple mathematical relation exists between the loop difference and the flow determinant. As demonstrated in Chapter II, the transmission from y_j to x_j is given by the quotient of P_{jj} and P, where P_{jj} is the cofactor of p_{jj} in the flow determinant P. Hence, from Eqs. 4.4 and 4.5, we may write

$$D_j = \frac{P}{P_{jj}} \, . \qquad (4.6)$$

Since P_{jj} is just the value which P would assume if node j were absent, we have the relation

$$D_j = \frac{P}{P_o} , \qquad (4.7)$$

where P_o is the flow determinant of the graph remaining when node j (together with all its connecting branches) is erased from the original graph.

The loop difference of a branch D_{jk} bears a similar relation to the flow determinant. As demonstrated in Section 3.6, the value of P is invariant under the elementary transformations. Hence, we may introduce an auxiliary node at some internal point of branch j-k without altering P. In addition, the erasure of this auxiliary node effectively breaks or removes branch j-k. In view of our definition of $_{jk}$ as the of the auxiliary node, it follows directly that

$$D_{jk} = \frac{P}{P_o} , \qquad (4.8)$$

where P_o is the flow determinant of the graph remaining when branch j-k is erased from the original graph.

An illustrative example might prove worthwhile at this point. Consider the simple graph shown in Fig. 4.10(a).

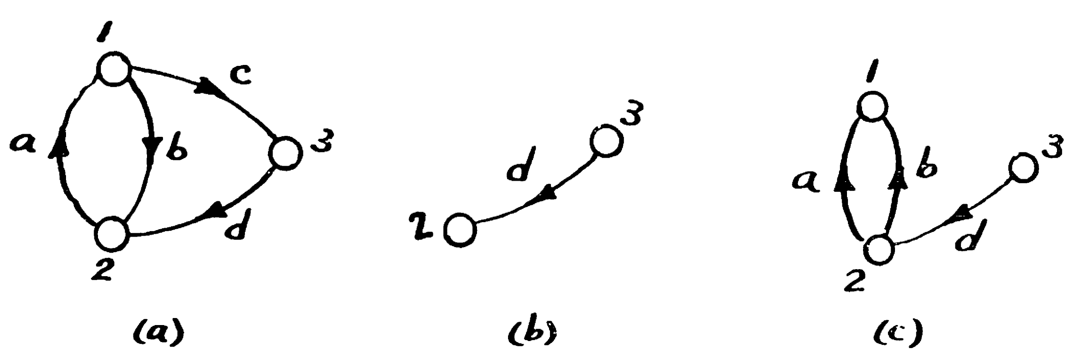


Fig. 4.10

The flow determinant is

$$P = \begin{vmatrix} 1 & -b & -c \\ -a & 1 & 0 \\ 0 & -d & 1 \end{vmatrix} = 1 - a(b + cd). \qquad (4.9)$$

With node 1 erased, the graph takes the form shown in (b), a cascade graph for which the flow determinant is evidently unity. The loop difference of node 1 is therefore

$$D_1 = \frac{P}{P_o} = \frac{1 - a(b + cd)}{1} = 1 - ab - acd. \qquad (4.10)$$

By inspection of graph (a), the loop transmission of node 1 is

$$\mathcal{T}_1 = ba + cda, \qquad (4.11)$$

which checks the relation $D_1 = 1 - \mathcal{T}_1$. With branch c removed, the flow graph remaining is that shown in (c). Its determinant has the value $1 - ab$. Hence the loop difference of branch c is

$$D_{13} = \frac{P}{P_o} = \frac{1 - a(b + cd)}{1 - ab} = 1 - \frac{acd}{1 - ab}. \qquad (4.12)$$

The loop transmission $\mathcal{T}_{13}$ of branch c is the $\mathcal{T}$ of an auxiliary node introduced into that branch. Since branches b and c are in simple cascade, however, node 3 serves as just such an auxiliary node. In other words, $\mathcal{T}_{13} = \mathcal{T}_3$. Loop transmission $\mathcal{T}_3$ is equivalent to the transmission from 3" to 3' in the graph of Fig. 4.11. For a unit signal originating at 3", the signal injected into node 2 has the value d so that

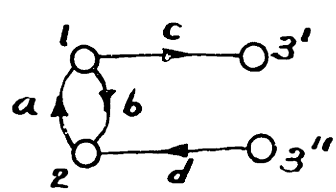


Fig. 4.11

x_2 is given by $\frac{d}{1 - ab}$. The signal appearing at 3' is just acx_2, with the result that

$$\mathcal{T}_{13} = \mathcal{T}_3 = \frac{dac}{1 - ab}. \qquad (4.13)$$

As before, the relation between D and $\mathcal{T}$ is verified, $D_{13} = 1 - \mathcal{T}_{13}$.

In electrical circuit problems the node variables are currents and voltages. Hence the branch transmissions are impedances, admittances, or ratios of impedances or admittances. We may speak, therefore, of the loop transmission around a "tube", meaning the loop transmission of the branch whose transmission is the transconductance or amplification factor of that tube. Figure 4.12(a) shows the current equivalent circuit for a tube, together with a connecting network. The plate resistance and interelectrode capacitances are treated as part of the network. A

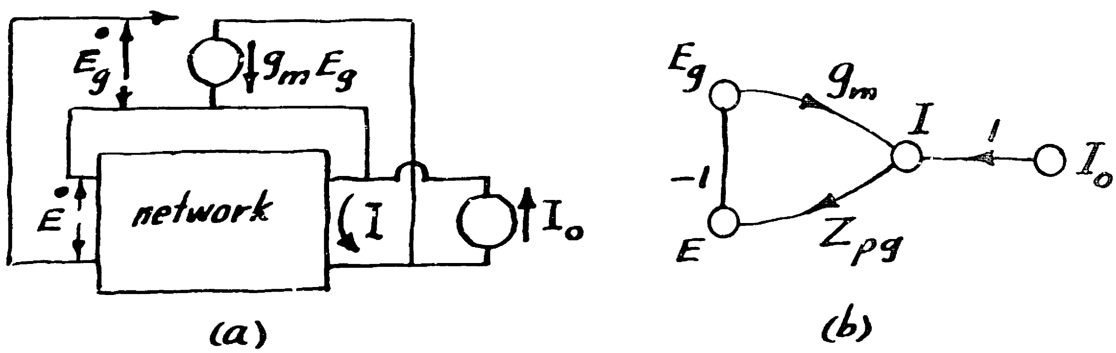


Fig. 4.12

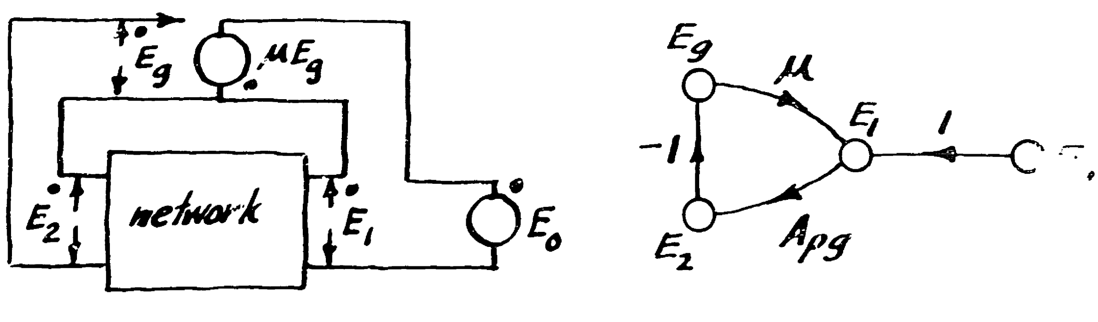
convenient flow graph representation is shown in Fig. 4.12(b). The plate-to-grid transfer impedance is defined by the E to I_o ratio as measured with the tube "dead" (i.e. with $g_m = 0$, but with the plate resistance sustained at its normal value). From the graph, we see that the loop transmission of the tube is

$$\mathcal{T}_{g_m} = -g_m Z_{pg}, \tag{4.14}$$

or

$$\mathcal{T}_{g_m} = \left(\frac{I}{E} \text{ due to tube}\right)\left(\frac{E}{I} \text{ due to network}\right). \tag{4.15}$$

Had the voltage equivalent circuit been employed, the circuit and graph would have appeared as shown in Fig. 4.13. An external voltage drive is provided in this case, rather than a current source. The boxed networks of Figs. 4.12 and 4.13 differ only in the placement of the element representing plate resistance; in parallel with I_o and in series with E_o, respectively. The plate-to-grid transfer function A_{pg}



(a) (b)

Fig. 4.13

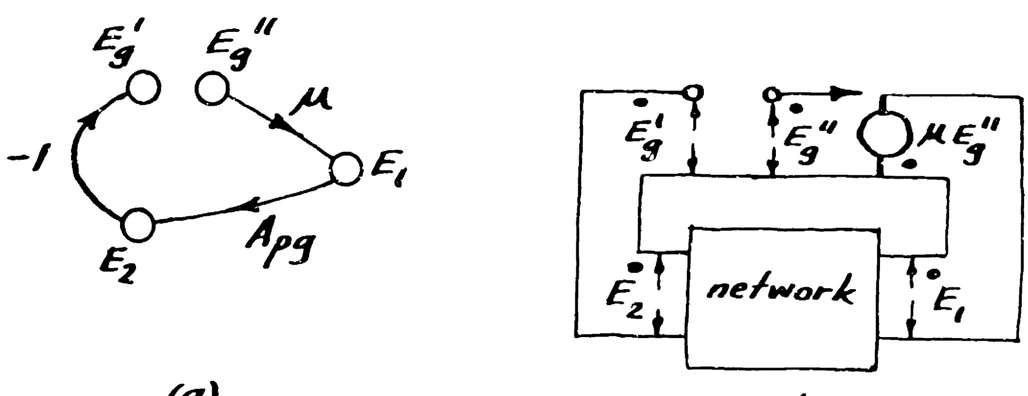
is defined by the ratio of E_2 to E_o as measured with the tube dead (i.e. $\mu = 0$). The loop transmission of the tube is

$$\mathcal{T}_\mu = -\mu A_{pg}, \tag{4.16}$$

or

$$\mathcal{T}_\mu = \left(\frac{E_1}{E_2} \text{ due to tube}\right)\left(\frac{E_2}{E_1} \text{ due to network}\right).$$

Relations 4.14 and 4.16 are actually identical. Their identity becomes apparent when the network element representing plate resistance is taken into account. An alternative demonstration of the identity stems from physical considerations. The $\mathcal{T}$ of the tube is just the $\mathcal{T}$ of node E_g in Fig. 4.13(b). Expanding this node into a transmitter and a receiver, we have the graph shown in Fig. 4.14(a), from which $\mathcal{T}$ may be evaluated as the transmission from E_g'' to E_g'. In the associated physical circuit (b)



(a) (b)

Fig. 4.14

the node expansion appears as a break in the grid lead. If one volt is applied at E''_g then the voltage appearing at E'_g is numerically equal to the $\mathcal{T}$ of the tube. This measurement is obviously independent of the type of equivalent circuit used to represent the actual vacuum-tube circuit; provided, of course, the representation is valid. Incidentally, the loop difference of the tube is just the voltage appearing across the break $(E''_g - E'_g)$ per unit voltage applied at E''_g. Hence D, as well as $\mathcal{T}$, is interpretable in terms of a physical measurement. In an actual circuit this measurement may be difficult to perform, since it requires breaking the grid lead inside the point at which the grid-to-cathode and grid-to-plate capacitances are effectively connected. Nevertheless, such an interpretation is an aid to the evaluation of $\mathcal{T}$ or D by inspection of a circuit.

The circuit shown in Fig. 4.15(a) may be represented by the flow graph (b). Here we have a one-terminal-pair element Z, as contrasted

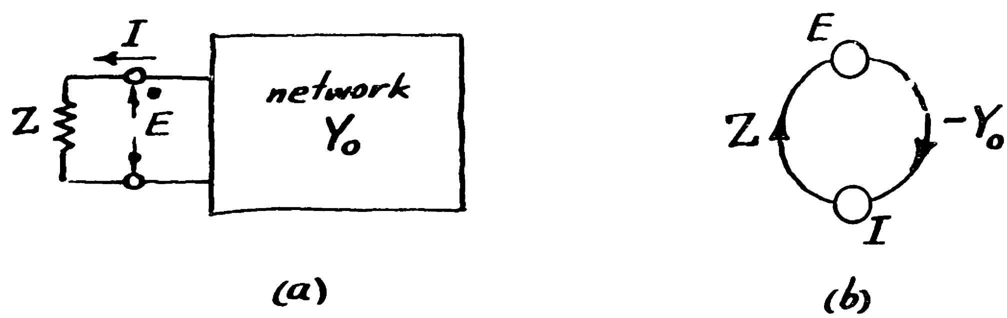


Fig. 4.15

with a transfer element such as μ or g. If the input admittance of the connecting network is Y_o, then the loop transmission of impedance Z is

$$\mathcal{T}_Z = -ZY_o = \left(\frac{E}{I} \text{ due to Z}\right)\left(\frac{I}{E} \text{ due to network}\right). \tag{4.17}$$

The negative sign accounts for the fact that Y_o is defined in terms of a current opposite to that shown in the figure. The expansion of node E or I does not, in this case, lead to the interpretation of $\mathcal{T}$ as the direct result of a single physical measurement. For one-terminal-pair elements, then, we have only the result that the loop transmission of an impedance (or admittance) is the negative of the product of that impedance (or admittance) and the admittance (or impedance) which it faces.

The foregoing discussion is illustrative of the manner in which the concept of loop transmission may be linked to certain physical elements. It must be remembered, however, that the fundamental definition of $\mathcal{T}$ comes from the flow graph. Starting from the graph, we are free to undertake whatever interpretations seem interesting and useful in a given problem.

We shall now prove a theorem which will be of subsequent interest in the chapter on transients. Our chief purpose here is not to obtain a result for immediate use, but rather to illustrate the ease with which flow graph concepts may be applied to the proof or derivation of certain feedback theorems. Such theorems and formulas, usually developed in terms of a circuit or other physical device, and usually proven by manipulation of equations or their determinants, often become aphorisms when stated in the language of flow graphs. For one familiar with flow graph techniques, it is simpler in many cases to derive the result by inspection of an appropriate graph than it is to memorize the theorem or formula and carry it in readiness.

The theorem to be considered is as follows: The ratio of the loop differences of two tubes in a circuit is the same as the ratio obtained by computing each loop difference with the other tube dead. Clearly, the circuit may be represented as a flow graph in which each of the two tubes (i.e. each g or μ) appears as a branch. Introducing an auxiliary internal node in each of these two branches and eliminating all other nodes by means of elementary transformations and self-loop to branch transformations, we are left with the flow graph shown in Fig. 4.16.

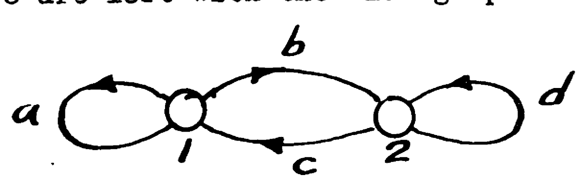


Fig. 4.16

The theorem may now be stated more generally as follows: The ratio of the loop differences of any two nodes in a flow graph is the same as the ratio obtained by computing each loop difference with the other node removed. By inspection of Fig. 4.16,

$$D_1 = 1 - \left[a + \frac{bc}{1-d}\right] \tag{4.18}$$

$$D_2 = 1 - \left[d + \frac{bc}{1-a}\right]. \tag{4.19}$$

Hence,

$$\frac{D_1}{D_2} = \frac{1-a}{1-d}. \tag{4.20}$$

Quantity $(1-a)$ is recognizable as the value of D_1 as computed with node 2 removed from the graph. Similarly, $D_2 = 1-d$ when node 1 is absent. The demonstration is therefore complete. In reducing the original graph to the form of Fig. 4.16 we need only consider the feedback graph containing nodes 1 and 2, since cascade branch transmissions do not enter the calculation of a loop difference. Had nodes 1 and 2 occurred in separate feedback graphs, then no coupling would exist (i.e. branch b or c would be absent from Fig. 4.16) and the theorem would reduce to a trivial special case in which $D_1 = 1-a$ and $D_2 = 1-d$ with or without the opposite node present.

4.4 The Evaluation of a Transmission

We now have four tools which may be applied to the problem of evaluating the transmission from one node to another in a linear flow graph. These tools are (1) the elementary transformations, (2) the concept of residual transmission, (3) the self-loop to branch transformation, and (4) the general concepts of loop transmission and loop difference. Actually, (1) and (2) are closely related, as are (3) and (4). In support of these tools there lies the flow graph topology developed earlier.

To proceed with an illustrative example, we shall find the transmission T_{13} through graph (a) of Fig. 4.17. A convenient means of indicating the desired transmission is shown by graph (b) in which a source

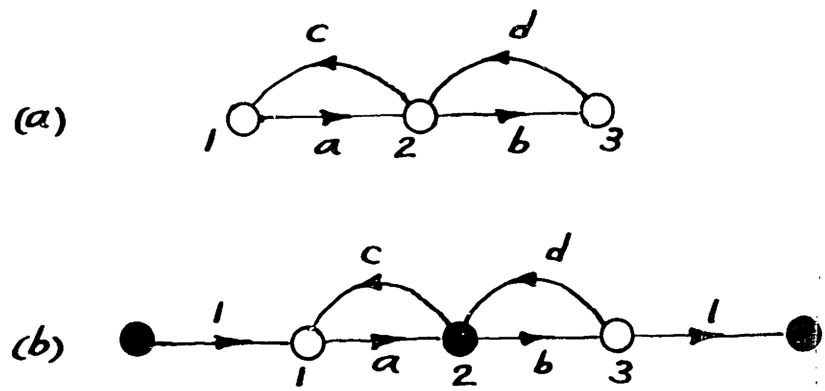


Fig. 4.17

and a sink have been affixed. When only one source and one sink are present we may drop the subscripts and speak, without ambiguity, of the (source-to-sink) transmission T. The T of graph (b) is evidently identical with the T_{13} of graph (a).

With the source providing a signal x_o, the signal at node 1 may be found from the relation

$$\frac{x_1}{x_o} = \frac{1}{D_1} = \frac{1}{1-\tau_1} . \qquad (4.21)$$

By inspection of the graph, τ_1 is the transmission of path ac as modified, of course, by the effective self-loop bd at node 2. Hence

$$\tau_1 = \frac{ac}{1-bd} ,$$

and

$$\frac{x_1}{x_o} = \frac{1}{1-\frac{ac}{1-bd}} . \qquad (4.22)$$

With the value of x_1 known, we may think of node 1 as a source which injects a signal ax_1 into node 2. Branch c evidently can not affect the value of x_2/x_1 and may be removed, if we wish, for this calculation.

Again by inspection,

$$\frac{x_2}{x_1} = \frac{a}{1 - bd} \ . \tag{4.23}$$

Finally,

$$\frac{x_3}{x_2} = b, \tag{4.24}$$

and the over-all transmission is

$$T = \frac{x_1}{x_o} \cdot \frac{x_2}{x_1} \cdot \frac{x_3}{x_2} = \frac{ab}{1 - ac - bd} \ . \tag{4.25}$$

In the foregoing example we have paid in complexity for our failure to recognize the topological characteristics of the graph. Beginning afresh, we identify the graph as one of first index, the only possible residual node being number 2. The residual graph, obtained either by applying the star-to-mesh equivalence at nodes 2 and 3, or by identifying the appropriate residual transmissions, is that shown in Fig. 4.18.

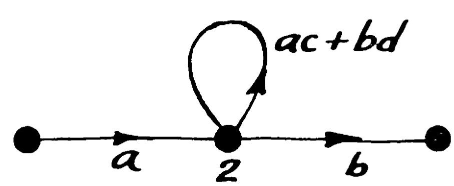


Fig. 4.18

By inspection, the value of T is the same as that given by expression 4.25.

For a second example, we shall take the second-index graph shown in Fig. 4.19. The source-to-sink transmission is just the reciprocal

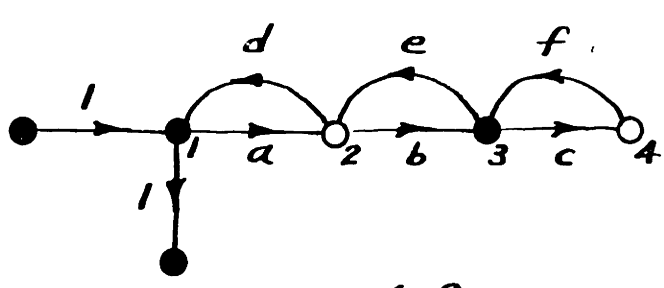


Fig. 4.19

of D_1. Because of the chain structure of the graph, a simple continued fraction results,

$$T = \cfrac{1}{1 - \cfrac{ad}{1-\cfrac{be}{1 - cf}}} \quad . \tag{4.26}$$

As a check, let us also compute T by reduction to a residual graph. We may choose as residual nodes any one of the pairs (1,3), (2,3), (2,4). The choice (1,3) offers the advantage that no direct source-to-sink residual transmission is present. Figure 4.20(a) shows the corresponding residual graph. Eliminating the self-loop at node 3, we have graph (b)

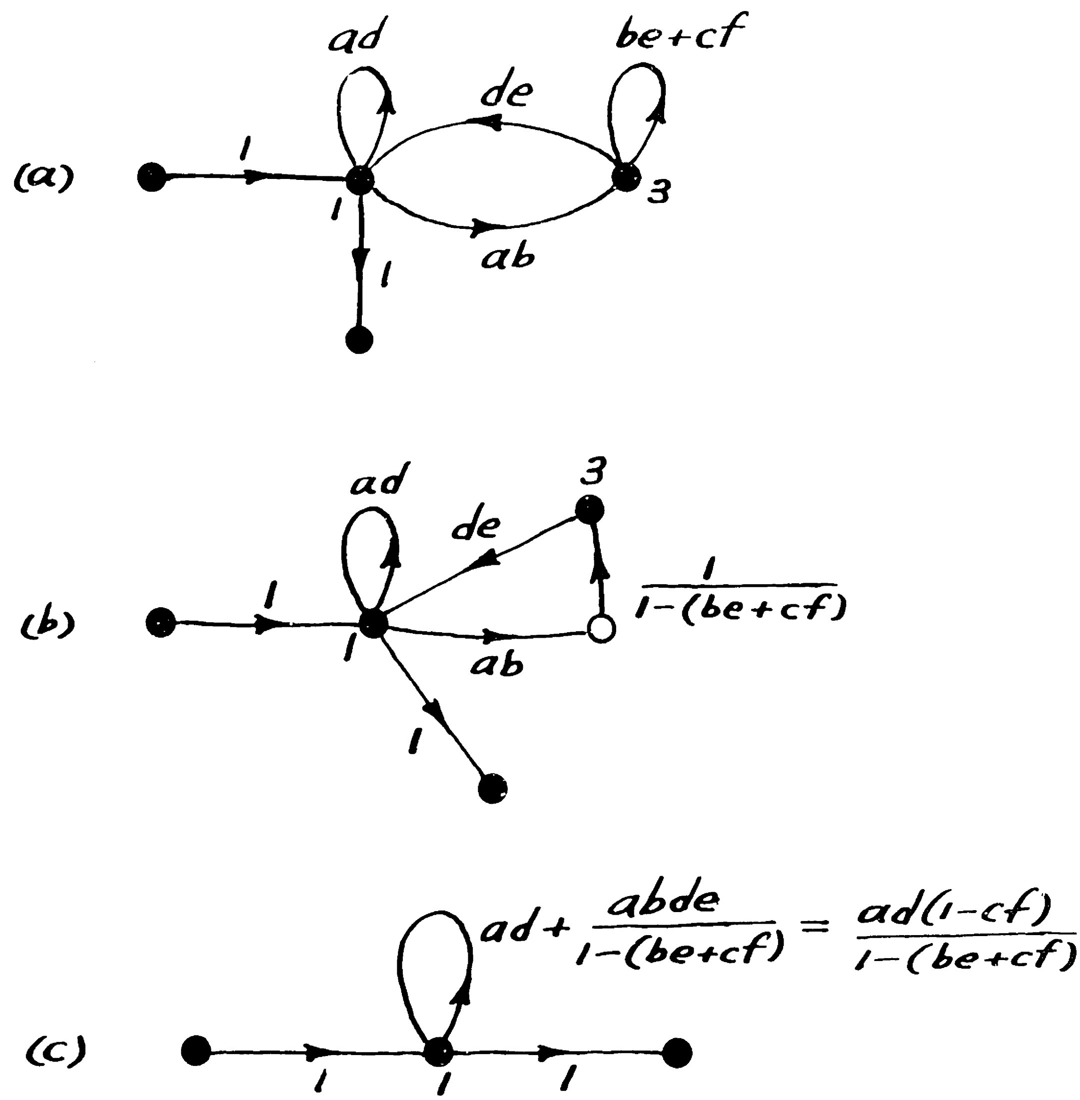


Fig. 4.20

which immediately reduces to the residual form shown in (c). By inspection of graph (c),

$$T = \frac{1}{1 - \frac{ad(1-cf)}{1-(be+cf)}} = \frac{1 - be - cf}{1 - ad - be - cf + acdf}, \tag{4.27}$$

which proves to be identical with expression 4.26.

In quest for a more orderly presentation of the reduction process outlined above, let us examine the most general second-index residual graph which is sketched in Fig. 4.21(a). The substitution of a branch

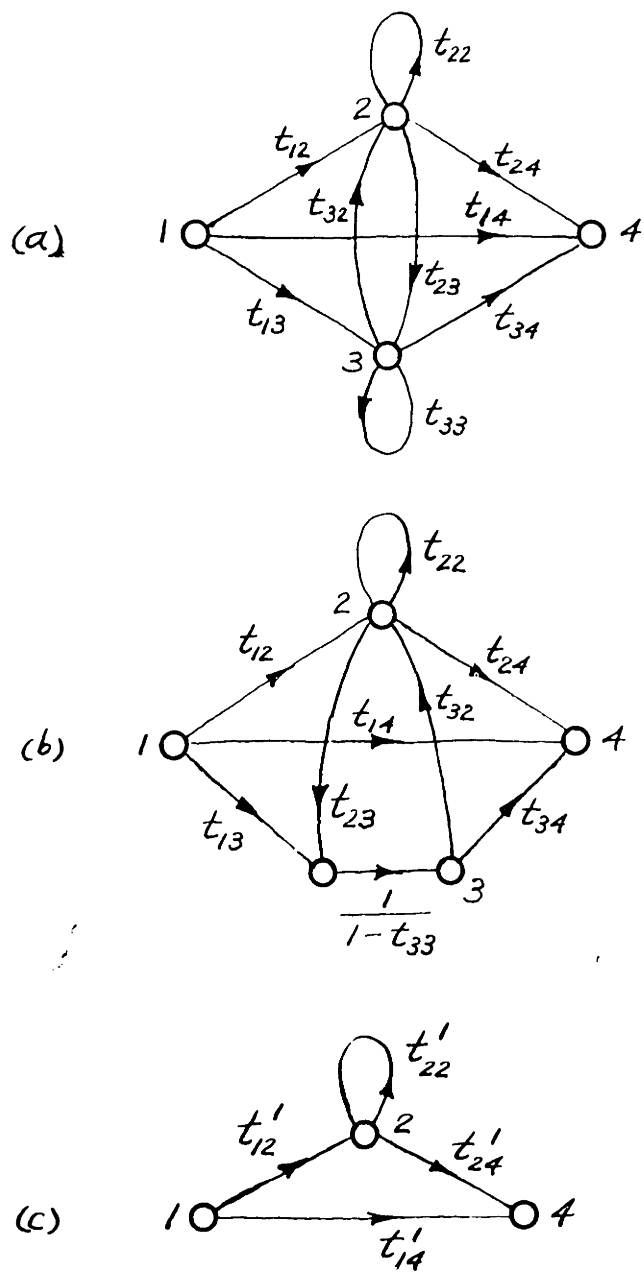


Fig. 4.21

for the self-loop at node 3 leaves a first-index graph (b) which has the residual form shown in (c). Denoting the new residual transmissions by primes, we have

$$
\begin{aligned}
t'_{12} &= t_{12} + \frac{t_{13}t_{32}}{1-t_{33}} \\
t'_{14} &= t_{14} + \frac{t_{13}t_{34}}{1-t_{33}} \\
t'_{22} &= t_{22} + \frac{t_{23}t_{32}}{1-t_{33}} \\
t'_{24} &= t_{24} + \frac{t_{23}t_{34}}{1-t_{33}} ,
\end{aligned}
\qquad (4.28)
$$

and, by inspection of graph (c),

$$T = T_{14} = t'_{14} + \frac{t'_{12}t'_{24}}{1-t'_{22}} . \qquad (4.29)$$

The general scheme, made evident by the orderliness of the subscripts in expressions 4.28 and 4.29, may be stated as the following <u>reduction formula</u>. The elimination of node i from a flow graph leads to a new flow graph whose branch transmissions are given by

$$t'_{jk} = t_{jk} + \frac{t_{ji}t_{ik}}{1-t_{ii}} , \qquad (4.30)$$

where the primed and unprimed transmissions belong to the new and the original graphs, respectively.

The general process for the evaluation of T by graph reduction may now be summarized as follows:

Step (1): A set of residual nodes is chosen and the superfluous nodes are eliminated by either (a) elementary transformations, (b) evaluation of the residual transmissions by inspection, or (c) the repeated application of the reduction formula. For superfluous nodes, of course, t_{ii} vanishes and relation 4.30 assumes a particularly simple form. Alternative (b) is probably the simplest of the three, however, provided we are careful to notice all possible residual paths.

Step (2): One of the residual nodes is eliminated by either (a) the self-loop to branch transformation, or (b) the reduction formula.

Steps (1) and (2) are then repeated in order until the desired transmission is obtainable by inspection. If the formula is employed in step (2), however, the resulting graph is residual and further applications of step (1) are not necessary.

A final example, shown in Fig. 4.22, will illustrate the use of the reduction formula. The residual nodes in graph (a) are 3 and 5.

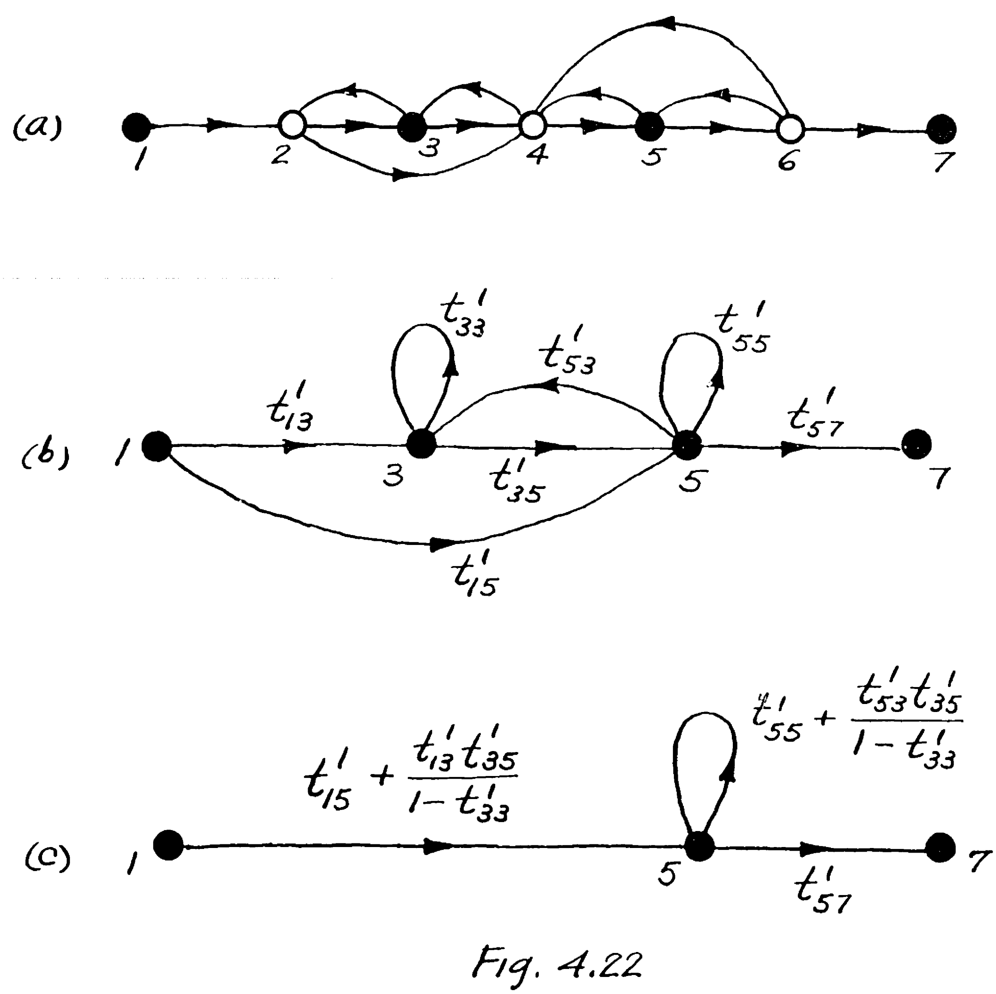


Fig. 4.22

By inspection, the residual transmissions appearing in graph (b) are

$$
\begin{aligned}
t'_{13} &= t_{12}t_{23} + t_{12}t_{24}t_{43} \\
t'_{33} &= t_{32}t_{23} + t_{34}t_{43} + t_{32}t_{24}t_{43} \\
t'_{15} &= t_{12}t_{24}t_{45} \\
t'_{35} &= t_{34}t_{45} + t_{32}t_{24}t_{45} \\
t'_{53} &= t_{54}t_{43} + t_{56}t_{64}t_{43} \\
t'_{55} &= t_{56}t_{65} + t_{54}t_{45} + t_{56}t_{64}t_{45} \\
t_{57} &= t_{56}t_{67}.
\end{aligned}
\tag{4.31}
$$

Now, application of the reduction formula to node 3 yields graph (c). Finally, by inspection of (c), the over-all transmission is

$$
T = \frac{\left(t'_{15} + \dfrac{t'_{13}t'_{35}}{1 - t'_{33}}\right) t'_{57}}{1 - \left(t'_{55} + \dfrac{t'_{53}t'_{35}}{1 - t'_{33}}\right)} . \tag{4.32}
$$

4.5 <u>Reciprocal Transmissions</u>

We have already seen how the form of a flow graph is altered by the reciprocation of a path. For linear graphs it is profitable to continue with an inquiry into the quantitative effects of reciprocation. In particular, we wish to determine the branch transmissions appearing in the reciprocated graph.

Figure 4.23(a) shows two branches which may be imaged in to form part of a larger graph. The signal entering node 2 via branch b is bx_3. The

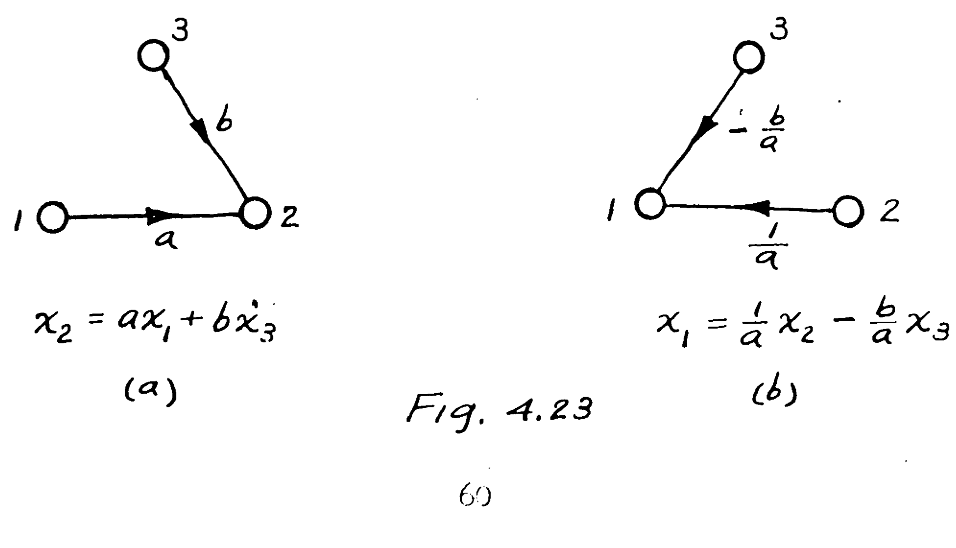


Fig. 4.23

contribution arriving from branch a, then, must be just $x_2 - bx_3$, since the sum of these two contributions is equal to x_2. Hence, given x_2 and x_3, the required value of x, is that indicated by graph (b).

In general, therefore, we may say that the reciprocation of branch j-k is accomplished by (1) reversing that branch and inverting its transmission, (2) shifting any branch i-k not contained in that path to the new position i-j and dividing its transmission by the negative of t_{jk}. The operation is specified by the transformations

$$t'_{kj} = \frac{1}{t_{jk}} \qquad\qquad (4.33)$$

$$t'_{ij} = \frac{-t_{ik}}{t_{jk}} ,$$

where primes denote the new branch transmissions.

Figure 4.24 offers an elementary example of the use of reciprocation in the evaluation of a transmission. Application of the reduction formula to node 3 of the second-index graph shown in (a) leads to the expression

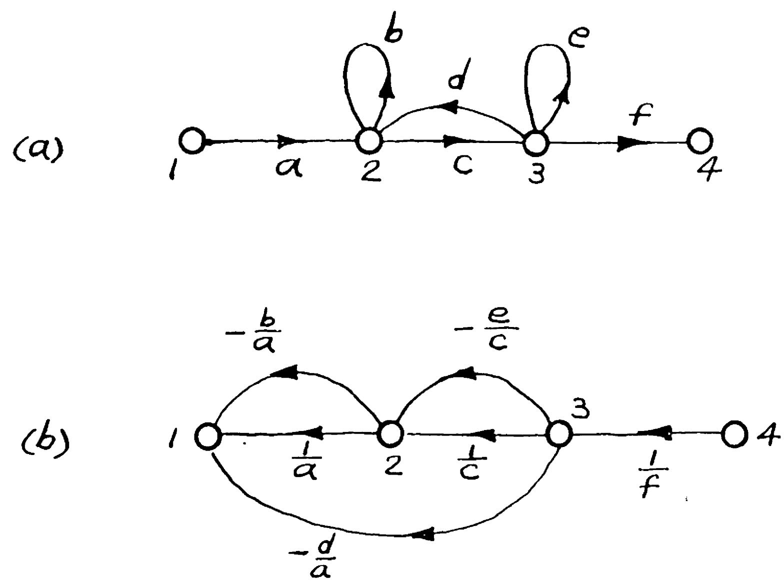


Fig. 4.24

$$T = \frac{x_4}{x_1} = \frac{a\left(\frac{cf}{1-e}\right)}{1-\left(b+\frac{cd}{1-e}\right)} = \frac{acf}{(1-b)(1-e)-cd} \,. \qquad (4.34)$$

Reciprocation of path 1-2-3-4 produces cascade graph (b), for which the transmission is

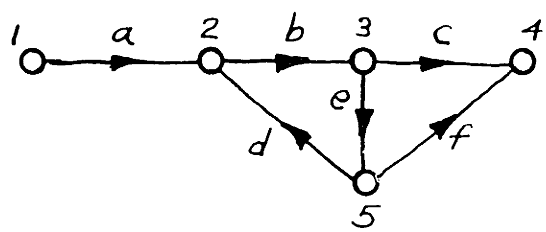
$$T' = \frac{x_1}{x_4} = \frac{1}{T} = \frac{1}{f}\left[\left(\frac{1}{a}-\frac{b}{a}\right)\left(\frac{1}{c}-\frac{e}{c}\right) - \frac{d}{a}\right]. \qquad (4.35)$$

Since the reciprocation is here accompanied by an index reduction (from 2 to 0), T' is more readily obtainable by inspection than is T.

This example is indicative of the general result which may be summarized as follows. If a path joining a source to a sink is reciprocated, the graph obtained has a transmission which is the reciprocal of the original transmission. Moreover, if the reciprocation accomplishes a reduction of index, then the reciprocal transmission may be considerably simpler from the standpoint of evaluation by inspection.

Figure 4.25 illustrates, in principle, another possible use of reciprocation in transmission calculations. The transmission T_{14} of

(a)



(b)

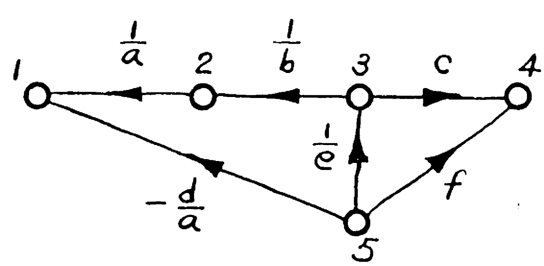


Fig. 4.25

graph (a) is evidently given by

$$T_{14} = \frac{x_4}{x_1} = \frac{ab(c+ef)}{1 - bde} \quad . \tag{4.36}$$

In graph (b), the result of reciprocating path 1-2-3-5, the transmissions of interest are

$$T'_{54} = \frac{x_4}{x_5} = \frac{c}{e} + f \tag{4.37}$$

$$T'_{51} = \frac{x_1}{x_5} = \frac{1}{abe} - \frac{d}{a} \quad . \tag{4.38}$$

It follows that

$$T_{14} = \frac{T'_{54}}{T'_{51}} = \frac{\frac{c}{e} + f}{\frac{1}{abe} - \frac{d}{a}} \quad , \tag{4.39}$$

which checks with the previous result. In general,

$$T_{jk} = \frac{T'_{ik}}{T'_{ij}} \quad , \tag{4.40}$$

where T_{jk} is the transmission from a source node j to any other node k, and where T'_{ik} and T'_{ij} are found from the graph after a path joining j to i has been reciprocated. For i = k, of course, $T'_{ii} = 1$ and we have the simple relation indicated previously.

Had we reciprocated path 1-2-3-4 in Fig. 4.25, we would have obtained a graph with the same index as graph (a), namely unity. Graph (b), however, is a cascade structure having zero index. It is possible, therefore, that a transmission calculation may be simplified somewhat by the reciprocation of a path which does not connect the specified input and output nodes. In the simple example given here, of course, relation 4.36 is very easily obtainable without recourse to special devices such as reciprocation.

As an incidental note we shall mention at this point another flow graph transformation which is, at first glance, apparently related to reciprocation, and yet entirely different in effect. A graph is said to be transposed when the direction of each branch is reversed, leaving the branch transmission unaltered. In short,

$$t'_{jk} = t_{kj}, \tag{4.41}$$

where the prime denotes a branch transmission of the transposed graph. It follows that

$$T'_{jk} = T_{kj}. \tag{4.42}$$

This result, immediately obvious from the matrix relation $T = t^{-1}$, also follows from an elementary consideration of signal flow. Each path from j to k in the original graph (including those paths which traverse a feedback loop once, twice, etc.) is evidently replaced in the transposed graph by a path k-j having the same transmission. Hence T'_{kj}, which is the sum of all such path transmissions, is equal to T_{jk}. The summation must be assumed to converge, of course, but this gives us no trouble. We may simply choose values of t for which the T series is convergent, and then continue the function T(t) throughout the t domain.

Figures 4.26(a) and (b) show a flow graph and its transposition.

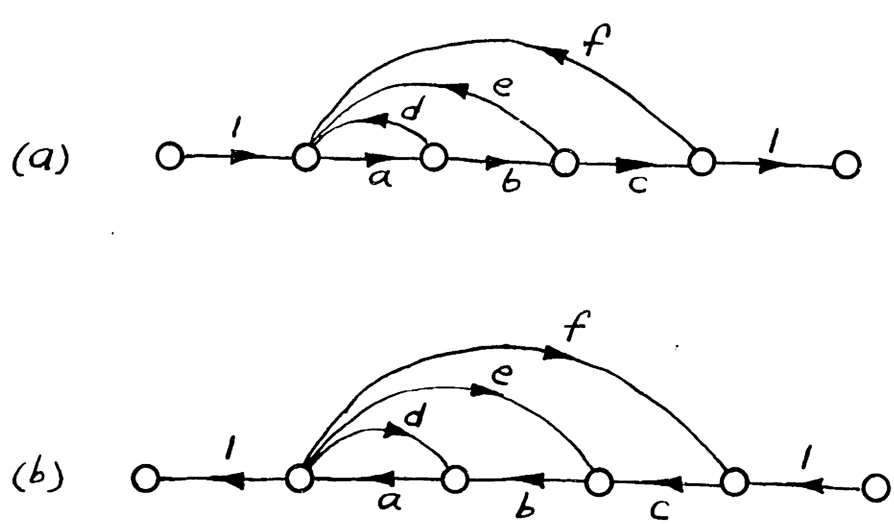


Fig. 4.26

(c)

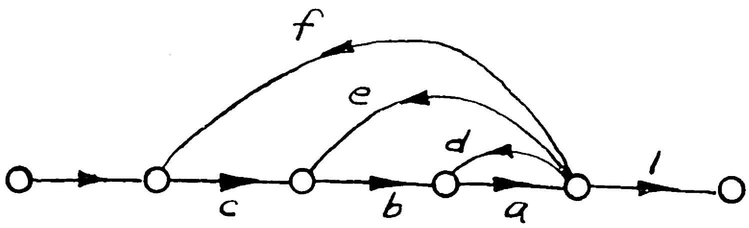


Fig. 4.26

Graph (c) is topologically identical with (b) except for a reversal of the orientation (i.e. (c) is the image of (b) in a vertical line). Given only graphs (a) and (c), a knowledge of transposition helps us to recognize the equality of their transmissions. In general, transposition serves as an equivalence transformation which may be used to obtain alternative flow graph representations of a specified transmission.

4.6 The Transmission Relative to a Particular Branch

It is often desirable to express a transmission T in a form which emphasizes its dependency upon the transmission of a particular branch t, which we shall call the reference branch. In order to accomplish such a representation, we may isolate the specified reference branch by the introduction of two auxiliary nodes and then reduce the graph to a form containing only four nodes; the two auxiliary nodes plus the source and sink which define T. An illustrative example is shown in Fig. 4.27.

(a)

(b)

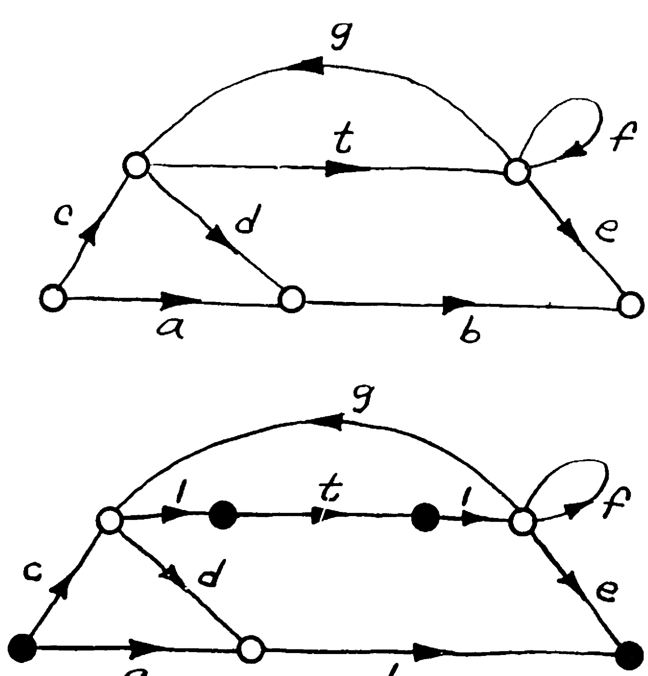


Fig. 4.27

(c)

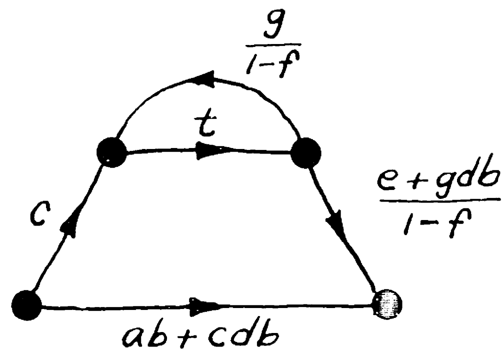


Fig. 4.27

Desirous of focusing our attention upon the chosen reference branch t of graph (a), we place two auxiliary nodes as indicated in graph (b). Elimination of the unwanted nodes leads directly to graph (c). It is clear from the nature of this process that an arbitrary graph may always be reduced to the standard form shown in Fig. 4.28. The relation between the reference transmission $t = t_{jk}$ and the over-all transmission

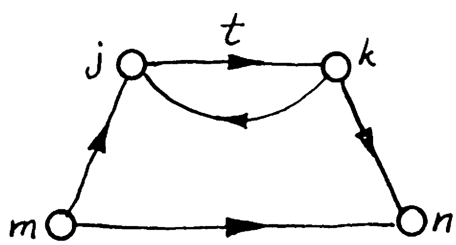


Fig. 4.28

$T = T_{mn}$ is given by

$$T_{mn}(t_{jk}) = t_{mn} + \frac{t_{mj}t_{jk}t_{kn}}{1 - t_{jk}t_{kj}} . \tag{4.43}$$

For convenience of physical interpretation we shall adopt the simpler notation,

$$T(t) = T_o + \frac{T_f}{1-\tau} , \tag{4.44}$$

where by definition,

T_o = the direct transmission (or leakage) past branch t
τ = the loop transmission around branch t (4.45)
T_f = the forward transmission through branch t.

Quantities T_f and τ each contain t as a factor, whereas T_o is independent of t. Incidentally, transmissions T, T_o, and T_f have the same dimensions (i.e. x_n/x_m in Fig. 4.28) while τ (like all loop transmissions) is dimensionless. The feedback formula is similar in nature to the reduction formula 4.30 developed previously for the elimination of a <u>node</u>. Expression 4.43 may be thought of as a reduction process which accounts for the effect of a particular <u>branch</u>.

The use of the feedback formula in circuit analysis is illustrated by the following example. Suppose that we desire to find the voltage amplification E_2/E_1 of the circuit shown in Fig. 4.29. We might, of course,

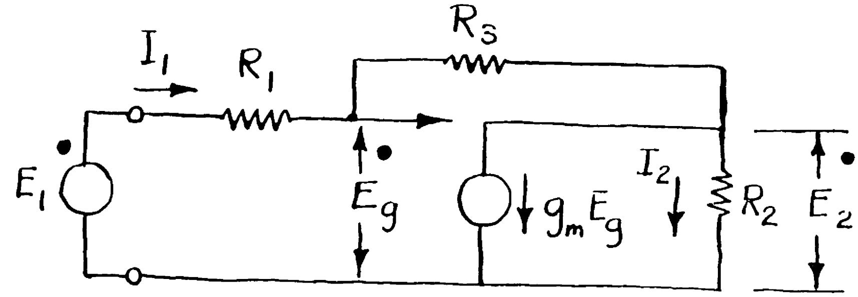


Fig. 4.29

construct a sprawling flow graph such as that shown in Fig. 4.30(a).

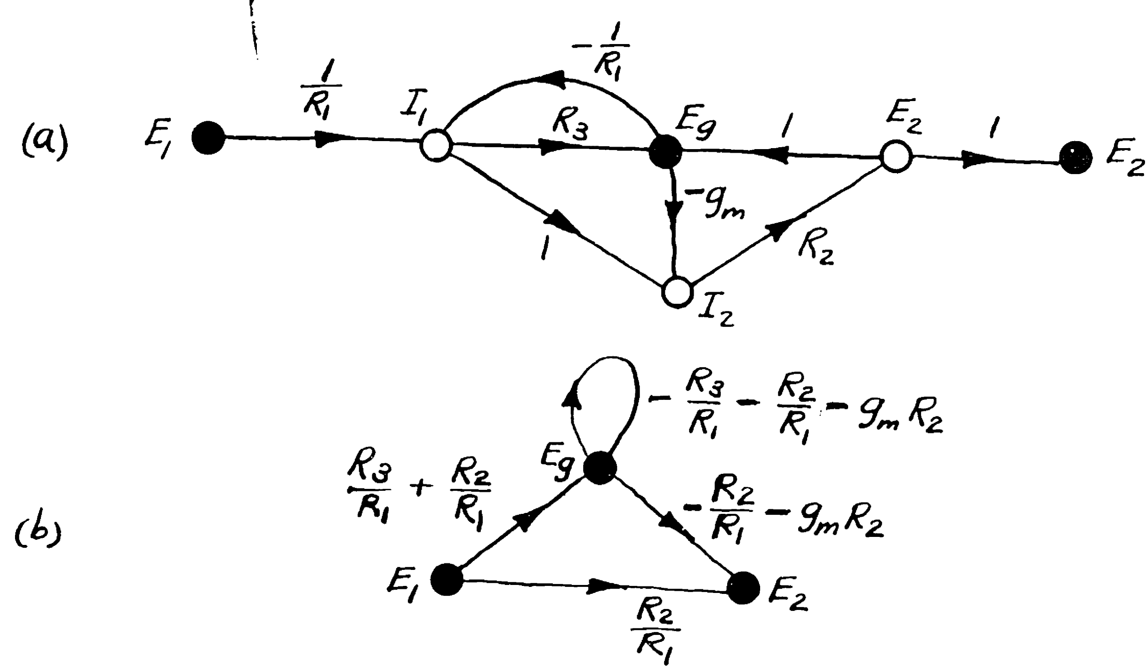


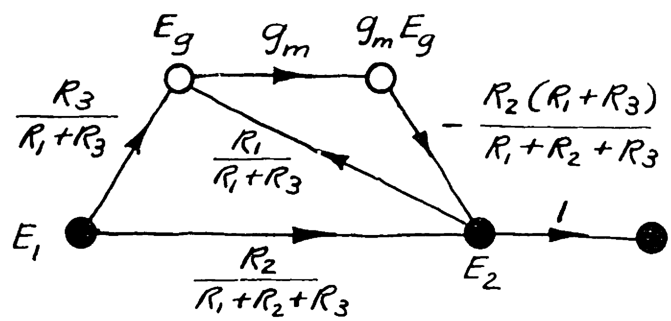
Fig. 4.30

By inspection of the residual graph (b), the transmission is

$$\frac{E_2}{E_1} = \frac{R_2}{R_1} + \frac{\left(\frac{R_3}{R_1} + \frac{R_2}{R_1}\right)\left(\frac{-R_2}{R_1} - g_m R_2\right)}{1 + \frac{R_3}{R_1} + \frac{R_2}{R_1} + g_m R_2} = \frac{R_2(1 - g_m R_3)}{R_1 + R_2 + R_3 + g_m R_1 R_2} . \qquad (4.46)$$

A neater original graph having slightly more complicated branch transmissions is shown in Fig. 4.31(a). From the residual form (b), we see

(a)



(b)

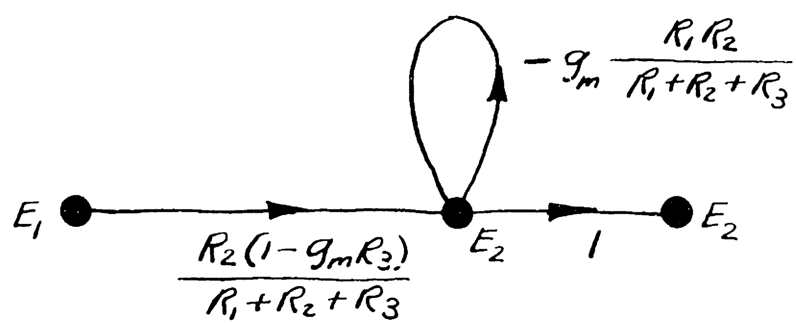


Fig. 4.31

that the transmission is again

$$\frac{E_2}{E_1} = \frac{\frac{R_2(1 - g_m R_3)}{R_1 + R_2 + R_3}}{1 + g_m \frac{R_1 R_2}{R_1 + R_2 + R_3}} = \frac{R_2(1 - g_m R_3)}{R_1 + R_2 + R_3 + g_m R_1 R_2} . \qquad (4.47)$$

If we wish to use the feedback formula, however, we must think in terms of the standard flow graph. For purposes of illustration the formula will be evaluated for two different reference transmissions. Suppose, first, that the transconductance g_m is taken as the reference element. The corresponding flow graph is shown in Fig. 4.32(a). For

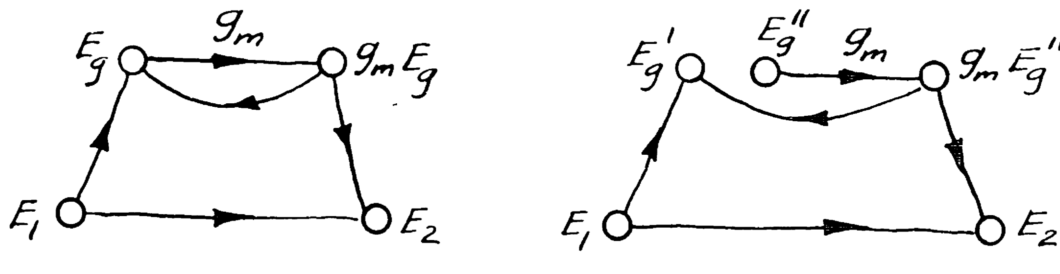


Fig. 4.32

convenience of physical interpretation, we may expand node E_g as shown in (b). By inspection of the graph,

$$T_o = \left(\frac{E_2}{E_1}\right)_{E_g'' = 0} \tag{4.48}$$

$$\tau = \left(\frac{E_g'}{E_g''}\right)_{E_1 = 0} \tag{4.49}$$

$$T_f = \left(\frac{E_g}{E_1}\right)_{E_g = 0} \times \left(\frac{E_2}{E_g}\right)_{E_1 = 0} . \tag{4.50}$$

Now, by inspection of the corresponding circuit, Fig. 4.33, we identify

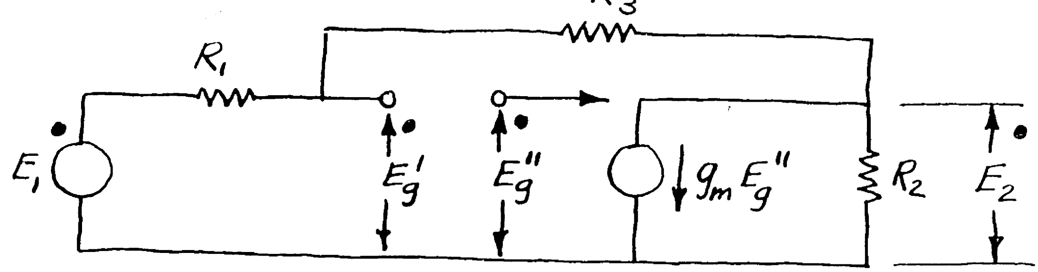


Fig. 4.33

the transmissions,

$$T_o = \frac{R_2}{R_1 + R_2 + R_3} \tag{4.51}$$

$$\tau = -g_m \frac{R_1 R_2}{R_1 + R_2 + R_3} \tag{4.52}$$

$$T_f = \left(\frac{R_2 + R_3}{R_1 + R_2 + R_3}\right)\left(-g_m \frac{R_2(R_1 + R_3)}{R_1 + R_2 + R_3}\right) . \tag{4.53}$$

Upon application of the feedback formula, we find

$$T = \frac{E_2}{E_1} = \frac{R_2}{R_1 + R_2 + R_3} - g_m \frac{\dfrac{R_2(R_1 + R_3)(R_2 + R_3)}{(R_1 + R_2 + R_3)^2}}{1 + g_m \dfrac{R_1 R_2}{R_1 + R_2 + R_3}} , \tag{4.54}$$

which simplifies to the same form as expressions 4.46 and 4.47.

As an alternative, let us now choose resistance R_2 as the reference transmission. As we shall see, T_o vanishes and a particularly simple formulation results. The appropriate flow graph is shown in Fig. 4.34, together with the circuit.

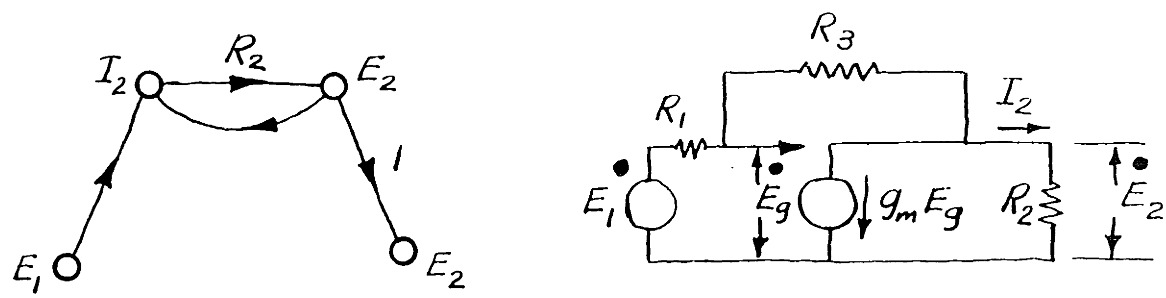


Fig. 4.34

By inspection of Fig. 4.34, we identify

$$T_o = \left(\frac{E_2}{E_1}\right)_{R_2 = 0} = 0 \tag{4.55}$$

$$\mathcal{T} = -R_2(\text{admittance faced by } R_2)_{E_1=0} = -R_2\left(\frac{1+g_m R_1}{R_1+R_3}\right) \qquad (4.56)$$

$$T_f = \left(\frac{I_2}{E_1}\right)_{R_2=0} \times R_2 \times (1) = \left(\frac{1-g_m R_3}{R_1+R_3}\right) R_2. \qquad (4.57)$$

Hence,

$$T = \frac{T_f}{1-\mathcal{T}} = \frac{R_2(1-g_m R_3)}{R_1+R_2+R_3+g_m R_1 R_2}. \qquad (4.58)$$

The feedback formula may be placed in many other forms, each of which finds special use. If, for example, T is known for the particular values of the reference transmission $t = 0$ and $t = \infty$, then the T corresponding to any other value of t is given by

$$T(t) = \frac{T_o - T_\infty \mathcal{T}}{1-\mathcal{T}}, \qquad (4.59)$$

where, of course,

$$T(o) = T_o$$
$$T(\infty) = T_\infty.$$

Since $\mathcal{T}$ contains t as a factor while T_o and T_∞ do not, expression 4.59 points out a well known property of linear systems which may be stated in the language of flow graphs as follows. If all branch transmissions in a flow graph are fixed, except for one which has the arbitrary value t, then the transmission T of the graph is related to t by a linear fractional transformation of the general form

$$T = \frac{a+bt}{1+ct}, \qquad (4.60)$$

where a, b, and c are constants.

Another representation of the feedback formula is especially adaptable to the determination of driving-point impedances or admittances. The derivation of this form follows from Fig. 4.35. Suppose that the input impedance E/I of the system is to be found in terms of a particular reference transmission t. By inspection of the graph

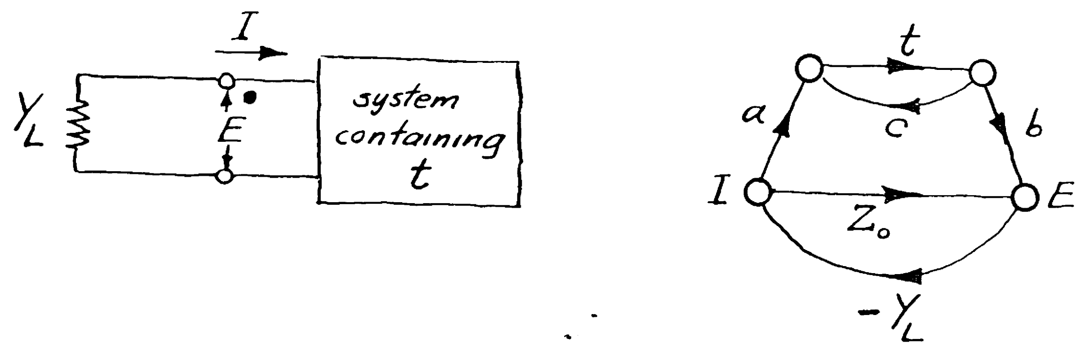


Fig. 4.35

$$Z = \frac{E}{I} = Z_o + \frac{Z_f}{1-\tau}, \tag{4.61}$$

where

$$Z_f = atb \tag{4.62}$$

$$\tau = tc. \tag{4.63}$$

Expression 4.61 may be rewritten as

$$Z = Z_o \frac{1 - (\tau - \frac{Z_f}{Z_o})}{1-\tau}. \tag{4.64}$$

Let us now find the value of the loop transmission of t in the presence of a load Y_L as shown. The flow graph provides two paths around t, one through c and the other through b, $-Y_L$, and a. The second path supports a feedback loop $-Y_L Z_o$. The value of τ, therefore, is

$$\tau = t(c - \frac{bY_L a}{1 + Z_o Y_L}). \tag{4.65}$$

For $Y_L = 0$ and $Y_L = \infty$, the input terminals are open-circuited and short-circuited, respectively. Hence, we may write

$$\tau_{o.c.} = tc, \tag{4.66}$$

$$\tau_{s.c.} = t(c - \frac{ab}{Z_o}), \tag{4.67}$$

and

$$\tau_{s.c.} = \tau_{o.c.} - \frac{Z_f}{Z_o}. \tag{4.68}$$

It follows that

$$Z = Z_o\left(\frac{1-\tau_{s.c.}}{1-\tau_{o.c.}}\right). \tag{4.69}$$

The expression breaks down, of course, when Z_o vanishes or becomes infinite.

An illustrative application of formula 4.69 will be made to the circuit shown in Fig. 4.36. Taking g_m as a reference, we find

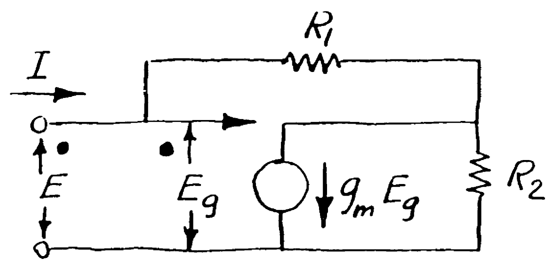


Fig. 4.36

$$\tau_{o.c.}(I = 0) = -\, g_m R_2 \tag{4.70}$$

$$\tau_{s.c.}(E = 0) = 0, \tag{4.71}$$

$$Z_o = \left(\frac{E}{I}\right)_{g_m = 0} = R_1 + R_2, \tag{4.72}$$

so that

$$Z = \frac{E}{I} = \frac{R_1 + R_2}{1 + g_m R_2}\,. \tag{4.73}$$

Our primary purpose in this chapter has been to develop a simple flow graph algebra and to indicate the applicability of this algebra to linear analysis problems. The several formulas and examples are to be taken only as illustrations of this applicability. The main point is that such formulas may be derived very simply on the flow graph basis, often by mere inspection of an appropriate graph.

CHAPTER V

SENSITIVITY CONSIDERATIONS

5.1 Introduction

From the design standpoint, not only the transmission through a system is of interest, but also the manner in which that transmission is affected by changes in the system parameters. In a vacuum-tube amplifier, for example, we may ask for the change in midband gain caused by a variation in the transconductance of one of the tubes, due perhaps to the aging of that tube or to the deviation of that tube from the norm of a manufactured lot. If such variations have little effect, we say that the sensitivity of the gain, relative to the transconductance, is small.

In this chapter the sensitivity concept will be developed for flow graphs. We shall examine certain structures which exhibit low sensitivity, mentioning possible design applications. Our purpose, however, is not to obtain specific design criteria, but rather to obtain a feeling for sensitivity. The flow graph offers a general basis for such considerations.

5.2 The Sensitivity of a Transmission Relative to a Particular Reference Branch

The sensitivity of a transmission T to changes in a particular reference branch t is defined here as

$$S = \frac{\partial \ln T}{\partial \ln t} = \frac{\partial T/T}{\partial t/t}. \qquad (5.1)$$

If we relax expression (5.1) to admit small increments rather than differentials, then S is the percentage change in T produced by a one per cent increase in t.

The sensitivity may be evaluated by direct differentiation of the feedback formula 4.44 discussed previously. However, for the sake of a novel viewpoint which provides additional background, we shall venture a less straightforward approach to the evaluation of S. Consider, first, any two branches A and B of a flow graph. By the introduction of auxiliary nodes (as described previously in connection with the feedback formula) we may reduce the graph to the essential form shown in Fig. 5.1, which places branches A and B in evidence. Cascade branches are dropped,

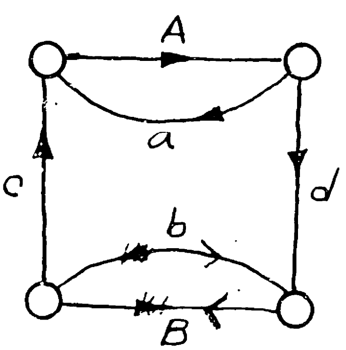


Fig. 5.1

since they do not enter the forthcoming considerations. If continuity of signal flow is postulated, then the loop transmission of every node and branch must equal unity. Branch c, for example, has the loop transmission

$$\mathcal{T}_c = \frac{cAdB}{(1-aA)(1-bB)}, \tag{5.2}$$

as measured by breaking branch c and computing the transmission from one side of the break through the graph to the other side of the breach. Since branch c is actually unbroken in Fig. 5.1, the signal flow is continuous and $\mathcal{T}_c = 1$. Hence the branch transmissions are related by the constraining equation

$$\left(\frac{A}{1-aA}\right)\left(\frac{B}{1-bB}\right) = \frac{1}{cd}. \tag{5.3}$$

Given B,a,b,c,d, we may compute A. Treating a,b,c,d as constants, we have a functional dependence between A and B. Now, since

$$d\,\ln\left(\frac{A}{1-aA}\right) = \frac{d\,\ln A}{1-aA}, \tag{5.4}$$

for a constant, it follows that

$$\frac{\partial \ln A}{\partial \ln B} = -\left(\frac{1-\tau_A}{1-\tau_B}\right), \tag{5.5}$$

where $\tau_A = aA$, $\tau_B = bB$. This result may be stated as a general theorem.

> If A and B are any two branch transmissions in a feedback graph which supports continuous signal flow, then, for small changes, the ratio of the percentage increase in A to the percentage decrease in B is equal to the ratio of their loop differences.

According to a previous theorem, each loop difference may be computed with the other branch removed, without affecting their ratio.

With expression 5.5 at hand, we return to the question of sensitivity. Figure 5.2(a) shows the familiar graph which results from exhaustive reduction about the reference branch t. With a unit signal

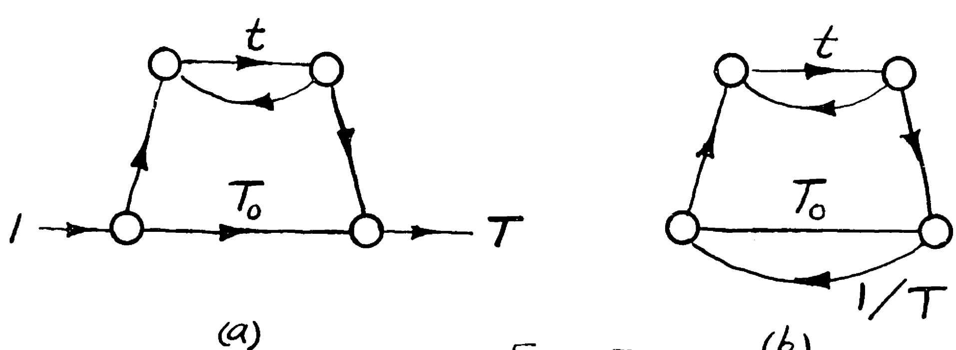


Fig. 5.2

injected at the source, the signal appearing at the sink is just T, the transmission of the graph. If, now, the sink is connected to the source through a branch of transmission 1/T, as shown in (b), the result is a feedback graph in which continuity of signal flow is preserved. The concept here is the same as that usually employed in the description of a feedback oscillator. Taking t and 1/T as the two variable branch transmissions of the feedback graph, we have, from relation 5.5,

$$-\frac{\partial \ln(1/T)}{\partial \ln t} = \frac{1-\frac{T_o}{T}}{1-\tau}$$

or

$$S = \frac{\partial \ln T}{\partial \ln t} = \frac{1-\frac{T_o}{T}}{1-\tau}, \tag{5.6}$$

where τ is the loop transmission of t in Fig. 5.2(a).

Before going on to more general considerations, we might profitably examine a very elementary design problem which illustrates the use of the sensitivity formula. Suppose that the voltage transmission of the circuit shown in Fig. 5.3 has the specified value $T = E_2/E_1 = -1$. What

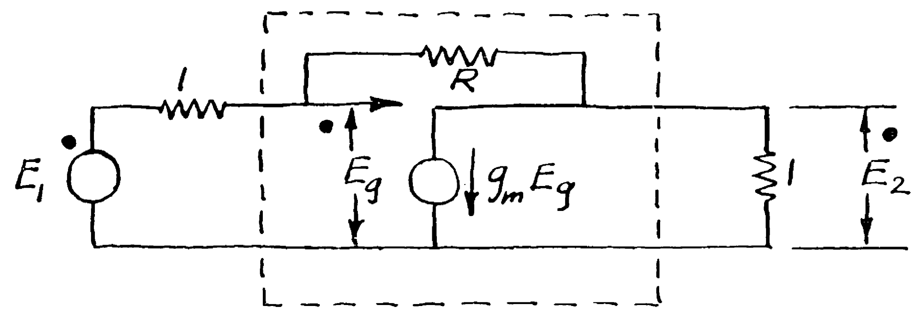


Fig. 5.3

values of g_m and R should be chosen for an "optimum" (minimum) sensitivity of T to changes in g_m? Our first task is the evaluation of T in terms of g_m and R. By inspection of the circuit,

$$T_o = \frac{1}{R+2} \tag{5.7}$$

$$\tau = -g_m\left(\frac{R+1}{R+2}\right)\left(\frac{1}{R+1}\right) = -\frac{g_m}{R+2}\,. \tag{5.8}$$

For variety, let us use the alternative form of the feedback formula 4.59 which requires the evaluation of T for g_m infinite. From Fig. 5.3, we observe that a very large value of g_m will result in a current $g_m E_g$ which tends to oppose any deviation of E_g from the value zero. For $E_g = 0$, the voltage generator must supply a current numerically equal to E_1. Hence $E_2 = -RE_1$ and

$$T_\infty = -R. \tag{5.9}$$

The transmission is given by

$$T = \frac{T_o - T_\infty \tau}{1 - \tau} = \frac{1 - g_m R}{2 + g_m + R}. \tag{5.10}$$

It follows that for T = -1, as postulated, the constraining relation between g_m and R is

$$R = \frac{g_m + 3}{g_m - 1}, \tag{5.11}$$

and that the sensitivity is

$$S = \frac{1 - \frac{T_o}{T}}{1 - \tau} = \frac{4g_m}{(g_m + 1)^2}. \tag{5.12}$$

The "optimum" design, therefore, is obtained by choosing a very large transconductance, whence the sensitivity is very small. The accompanying value of R is very close to unity.

The above example indicates the general rule that the sensitivity is small when the loop transmission of the reference element is large. The sensitivity also vanishes if T is made equal to T_o, a condition which in practice is usually either unobtainable or trivial. In the above example, T_o can be made equal to -1 only by the use of an unallowable negative resistance R = -3. Had the specified transmission been T = 1/3, however, then the available generator power $E_1^2/4$ would have exceeded the desired load power $(E_1/3)^2$ and a passive circuit would have sufficed (i.e. $g_m = 0$, R = 1 is the solution). In short, when $T_o = T$ the reference element does not contribute to the transmission and presumably may be removed from the system.

A large loop transmission results in what is commonly called "negative feedback". This terminology arises from the logarithmic representation of the feedback formula. The logarithm is

$$\ln(T - T_o) = \ln T_f + \ln\left(\frac{1}{1-\tau}\right). \qquad (5.13)$$

The real part of Eq. 5.13 is

$$\ln|T - T_o| = \ln|T_f| + \ln\left|\frac{1}{1-\tau}\right|, \qquad (5.14)$$

where by definition

$$\ln|T - T_o| = \text{the gain due to the reference transmission} \qquad (5.15)$$

$$\ln|T_f| = \text{the forward gain} \qquad (5.16)$$

$$\ln\left|\frac{1}{1-\tau}\right| = \text{the feedback gain.} \qquad (5.17)$$

In words, the gain (nepers or decibels) due to the reference element is the sum of the forward gain through that element and the feedback gain. The feedback is said to be positive or negative according to the algebraic sign of expression 5.17. In general, the term transmission refers to the ratio of a response to a drive, whereas a gain is the logarithm of a transmission. The imaginary part of Eq. 5.13 relates the phases of the factors entering the feedback formula. The transmission $(T - T_o)$ due to the reference element has a phase shift equal to the sum of the phase shifts of the forward transmission (T_f) and the feedback factor $(1/1-\tau)$.

5.3 Insensitive Flow Graphs

The example considered in the preceding section is representative of a particular class of insensitive flow graphs; those in which the reference transmission is made very large in order to achieve low sensitivity. The basic effect is shown in Fig. 5.4. We shall postulate a reference element composed of two cascaded branches, the first being a linear transmission A and the second a nonlinear transmission $x_3 = f(x_2)$ which exhibits a saturation limit for large values of x_2. Figure 5.4(b) illustrates a graphical construction from which the output-input characteristic (x_3 vs x_o) may be determined. For a specified value of x_o, the corresponding values of x_1 and x_2 are fixed by intersection 0 of the straight line $x_1 = x_o + Bx_3$ and the curve $x_3 = f(Ax_1)$.

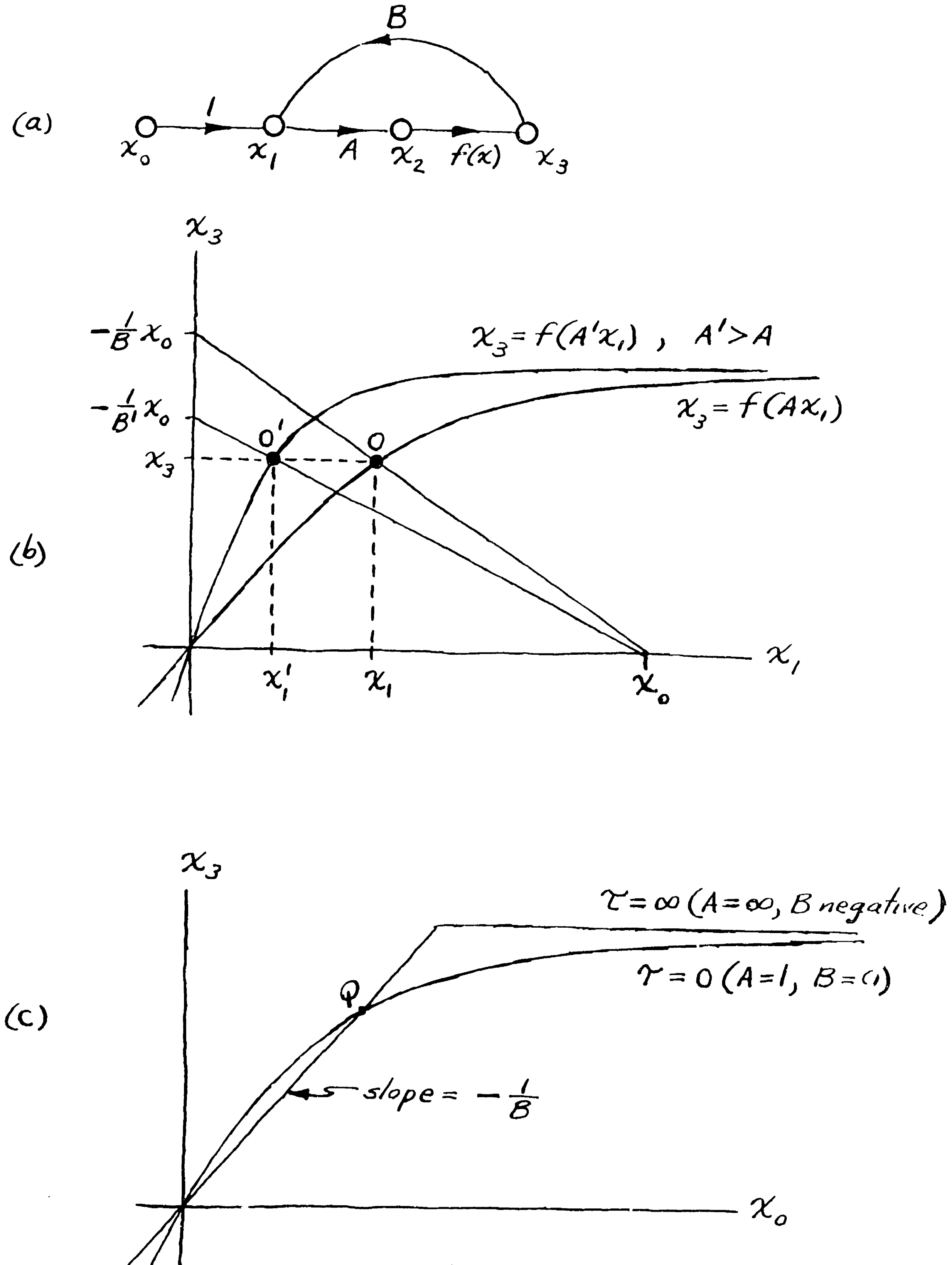

(a)
B
1
x0
x1
A
x2
f(x)
x3
(b)
x3
-1/B x0
-1/B' x0
x3
O'
O
x3 = f(A'x1) , A'>A
x3 = f(Ax1)
x1'
x1
x0
x1
(c)
x3
τ = ∞ (A=∞, B negative)
τ = 0 (A=1, B=0)
Q
slope = -1/B
x0

Fig. 5.4

As transmission A is increased to a new value A', the nonlinear transmission curve shears to the left. If, at the same time, transmission B is changed to the value B', then the output x_3 is unaltered. For all such pairs of values of A and B, we have a family of output-input characteristics, all passing through the origin and the point Q in Fig. 5.4(c) Only two of these are shown. Others, such as those corresponding to points O and O' in (b), lie somewhere between the curves $\tau = 0$ and $\tau = \infty$. Note that the scale of x_0 has been compressed in Fig. 5.4(c), as compared with the scale used in (b). As A becomes very large, the intersection point O moves very close to the vertical axis in (b). Hence, in the region below saturation, $x_3 = -x_0/B$, with the result that the over-all transmission is independent of A and also independent of the form of the nonlinear transmission function f(x). It follows that a time-varying signal injected at x_0 will be reproduced at x_3 without distortion and with an amplification $T = -1/B$, provided the saturation limit of the output element is not exceeded.

Another approach to the question of distortion is afforded by the concept of a "contamination source". Suppose that branch 2-3 in Fig. 5.5 is nonlinear. We may express the output signal from this branch as the

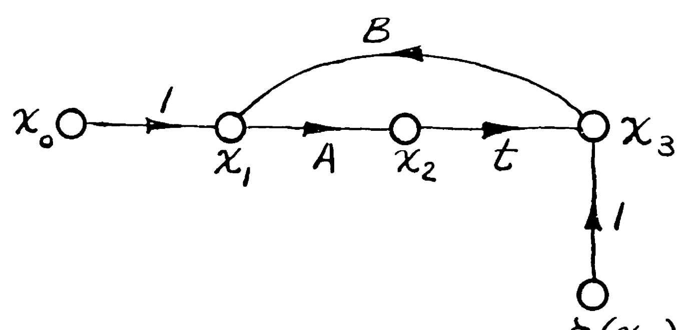


Fig. 5.5

sum of a linear term and a contaminating signal representing the effects of distortion,

$$x_3 = tx_2 + \xi(x_2). \tag{5.18}$$

Equation 5.18 leads to the <u>linear</u> graph shown in Fig. 5.5, containing a contamination source. The notation is actually quite general. Quantities x may stand for the complex frequency spectra (i.e. the Fourier or Laplace transforms) of time-varying signals, in which case t is a complex function of frequency (such as an impedance function) and is an operator which produces a spectrum $\xi(x)$ from a spectrum x according to some definite, though perhaps complicated, rule.

Given a desired output signal x_3, we may always (1) determine the corresponding input x_2 from the known properties of the nonlinear branch, (2) choose an appropriate linear transmission t, which represents the gross effect of the branch, and (3) find the resulting contamination, $\xi(x_2) = x_3 - tx_2$. It matters little whether we write ξ as a function of x_2 or x_3, since x_2 and x_3 are explicitly related by the properties of the branch. If we demand an output signal x_3 whose level exceeds the saturation limit of the nonlinear element, then presumably x_2 and $\xi(x_2)$ must become infinite in order to supply the required output. For the entire class of physically obtainable output signals, however, we may expect the required input x_2 and the contamination $\xi(x_2)$ to remain bounded. Now, by inspection of the graph, the transmission from x_o to x_3 is

$$T_{13} = \frac{At}{1 - ABt}, \tag{5.19}$$

and the transmission from $\xi(x_2)$ to x_3 is given by

$$T_{33} = \frac{1}{1 - ABt}. \tag{5.20}$$

By superposition, the complete output signal is

$$x_3 = \frac{At}{1 - ABt} x_o + \frac{1}{1 - ABt} \xi(x_2). \tag{5.21}$$

For large values of A, expression 5.21 approaches

$$x_3 = -\frac{1}{B} x_o - \frac{1}{ABt} \xi(x_2). \tag{5.22}$$

For unsaturated operation $\xi(x_2)$ is bounded and the last term of relation 5.22 may be made negligible by the choice of a sufficiently large value of A. We have, then, the general result that sensitivity and distortion are both small when the loop transmission is large, provided the system remains unsaturated.

For low distortion systems, wherein $\xi(x_2)$ is a small correction, we may compute the amount of distortion appearing in the output by a rapidly convergent successive approximations method. As mentioned previously, the contamination may be thought of as a function of either the input or output of the nonlinear element. With $\xi(x_3)$ in place of $\xi(x_2)$, Eq. 5.21 suggests the iterated operation,

$$x_3^{(k+1)} = \frac{At}{1-\tau}x_o + \frac{1}{1-\tau}\xi(x_3^{(k)}). \qquad (5.23)$$

Taking

$$\xi(x_3^{(o)}) = 0, \qquad (5.24)$$

we have as a first approximation

$$x_3^{(1)} = \frac{At}{1-\tau}x_o, \qquad (5.25)$$

wherein distortion is ignored. Given $x_3^{(1)}$, the corresponding input $x_2^{(1)}$ may be computed and the "first contamination" is then found from

$$\xi(x_3^{(1)}) = x_3^{(1)} - tx_2^{(1)}. \qquad (5.26)$$

The second approximation, which includes distortion, is

$$x_3^{(2)} = \frac{At}{1-\tau}x_o + \frac{1}{1-\tau}\xi(x_3^{(1)}). \qquad (5.27)$$

The third step will take into account the "first distortion of the distortion", or the "second contamination", and so on to the limit. The practical value of the process stems from the fact that $x_3^{(2)}$ is actually a very good approximation to x_3 in low distortion systems. In addition, the computation of the first contamination 5.26 is not excessively troublesome in practice, whereas the determination of second and higher contaminations is usually much more tedious.

The preceding discussion has been concerned with low-sensitivity systems in which the reference transmission is large. There exists another class of insensitive graphs in which none of the branch transmissions need be large. We shall find, however, that the difference is one of configuration rather than principle. In both classes the sensitivity reduction is accomplished by making the loop transmission of the reference branch very high. The "ideal regulator" shown in Fig. 5.6 provides a transmission which is insensitive to changes in the reference transmission t. The desired transmission from x_o to x_2 is t_o.

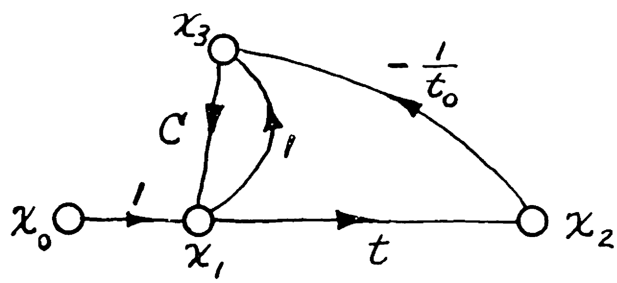


Fig. 5.6

In order to detect any variation of t from the desired value t_o, the output is fed through an attenuator $1/t_o$ and subtracted from the input x_1 to give the error x_3. For $t = t_o$, we have $x_3 = 0$. If t becomes larger than t_o, a negative error signal is produced at x_3. This error is fed back to the input through branch C as a correction signal, thereby reducing the value of x_1 and helping to compensate for the unwanted increase in t. A simple quantitative analysis shows that the choice $C = 1$ yields flat compensation; that is, an increase in t produces a correction which reduces x_1 in the same proportion.

The condition for flat compensation may be deduced by inspection of the flow graph. The loop transmission of node 1 is

$$\tau_1 = C - \frac{tC}{t_o} . \qquad (5.28)$$

Hence, the transmission from x_o to x_2 is given by

$$T_{12} = \frac{t}{1 - \tau_1} = \frac{t}{1 - C + \frac{tC}{t_o}} . \qquad (5.29)$$

For C = 1, this reduces to

$$T_{12} = t_o, \tag{5.30}$$

which is obviously independent of the reference transmission t. The insensitivity to changes in t is also indicated by the loop transmission of t,

$$\mathcal{T}_{12} = -\frac{Ct/t_o}{1 - C}, \tag{5.31}$$

which becomes infinite for C = 1. Contamination, representing distortion or noise, does not contribute to the output x_2 when injected into node 2, since

$$T_{22} = \frac{1}{1-\mathcal{T}_2} = \frac{1}{1-\mathcal{T}_{12}} \tag{5.32}$$

vanishes for C = 1.

The reciprocation of path 0-1-2 in Fig. 5.6 offers still another means of recognizing the properties of the graph. The result of this reciprocation is shown in Fig. 5.7. The reciprocal transmission is

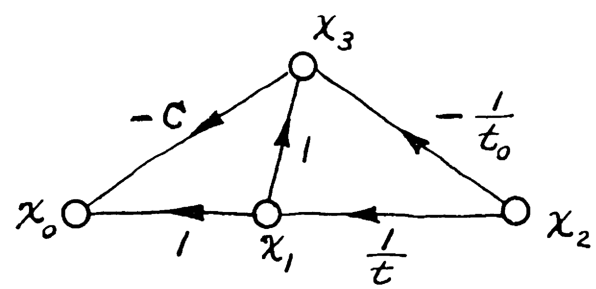


Fig. 5.7

$$\frac{1}{T} = \frac{x_o}{x_2} = \frac{1}{t} - \frac{C}{t} + \frac{C}{t_o}, \tag{5.33}$$

so that T approaches t_o for values of C near unity.

5.4 Insensitive Multistage Graphs

As a somewhat natural extension of the sensitivity considerations undertaken thus far, we shall examine the following problem. Given a flow graph containing a number of reference branches whose transmissions $u, v, w, \cdots, z$ are subject to variation or deviation from the desired operating point $u_o, v_o, w_o, \cdots, z_o$, how should the fixed transmissions of the remaining branches be chosen for (1) a specified value of the over-all transmission $T(u,v,w,\cdots,z)$ at the operating point, and (2) a minimum sensitivity of T to variations in $u,v,w,\cdots,z$? First of all, we shall qualify the problem further by ignoring direct transmission past any of the reference branches. This qualification is equivalent to the condition

$$
\begin{aligned}
T(o,v,w,\cdots,z) &= 0 \\
T(u,o,w,\cdots,z) &= 0 \\
T(u,v,o,\cdots,z) &= 0 \\
\vdots \qquad & \quad \vdots \\
T(u,v,w,\cdots,o) &= 0 .
\end{aligned}
\qquad (5.34)
$$

In other words, every path from the source to the sink passes through all of the reference elements.

If two auxiliary nodes are introduced into each reference branch, after the manner of Fig. 4.27 (page IV-27) and superfluous nodes are eliminated, the resulting simplified graph will have the form (for three reference elements) shown in Fig. 5.8. This is the most general

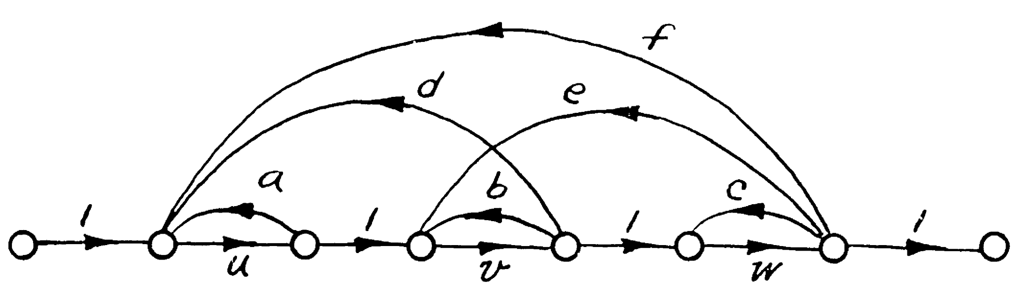


Fig. 5.8

structure which complies with the above simplification procedure and which also satisfies our assumption regarding direct transmission. Unity transmissions may be assigned to the fixed forward branches, as shown, without loss of generality. For justification of such an apparently restrictive assignment, we need only recognize the following general <u>normalization rule</u>.

> If each branch transmission t_{jk} in a flow graph is multiplied by a scale factor α_{jk}, the scale factors being so chosen that the transmissions of all closed paths are unaltered, then the over-all transmission of the graph is multiplied by $\alpha_{12}\alpha_{23}\cdots\alpha_{mn}$, where $1,2,3,\ldots,m,n$ is any path from the source to the sink.

The transmissions of graphs (a) and (b) in Fig. 5.9, for example, are

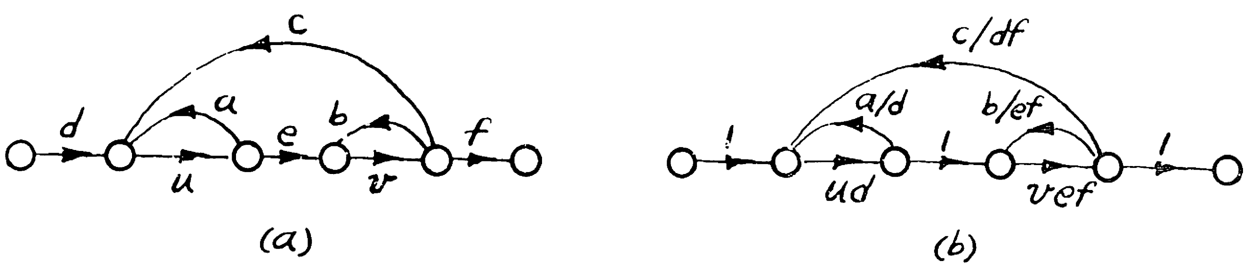


Fig. 5.9

identical. If the branch transmissions are now renamed, $u' = ud$, $a' = ad$, etc., we have a structure similar to that shown in Fig. 5.8. The normalization rule also permit us to choose

$$u_o = v_o = w_o = 1 \tag{5.35}$$

as an operating point for the reference transmissions in Fig. 5.8. Likewise without loss of generality, the desired transmission at the operating point may be taken as unity,

$$T(1,1,1) = 1. \tag{5.36}$$

It remains to determine the values of a,b,c,d,e,f which satisfy requirement 5.36 and which also make T insensitive to changes in u,v,w. Reciprocation of the forward path in Fig. 5.8 yields the cascade structure shown in Fig. 5.10. Without actually evaluating the reciprocal transmission we see that the expression will be of the form

$$\frac{1}{T} = \sum_{i=o}^{1} \sum_{j=o}^{1} \sum_{k=o}^{1} A_{ijk}\left(\frac{1}{u} - a\right)^{i}\left(\frac{1}{v} - b\right)^{j}\left(\frac{1}{w} - c\right)^{k}, \tag{5.37}$$

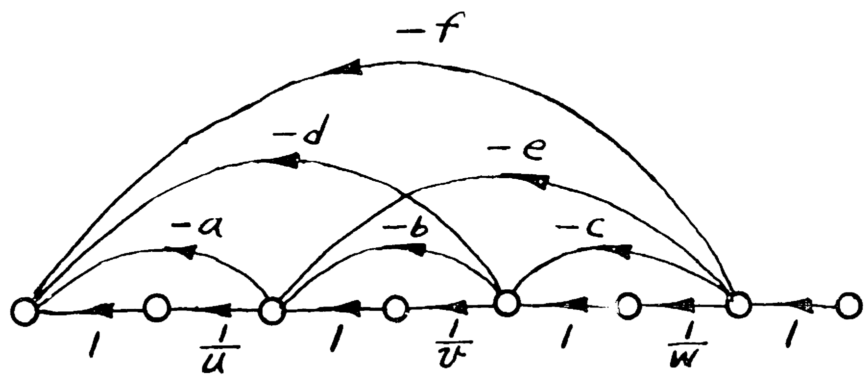


Fig. 5.10

a finite Taylor's series in three independent variables, linear in each. The function

$$T^{-1} = \phi(u^{-1}, v^{-1}, w^{-1}) \tag{5.38}$$

may be interpreted as a surface in four dimensional space, T^{-1} being plotted as the elevation above the u^{-1}, v^{-1}, w^{-1} hyper-plane. If all coefficients in the Taylor's series except A_{ooo} and A_{111} can be made to vanish, then

$$\frac{1}{T} = A_{ooo} + A_{111}\left(\frac{1}{u} - a\right)\left(\frac{1}{v} - b\right)\left(\frac{1}{w} - c\right), \tag{5.39}$$

and the surface will exhibit a saddle point at $u^{-1} = a$, $v^{-1} = b$, $w^{-1} = c$. In the neighborhood of this point the surface is flat and level, so that T^{-1} is insensitive to changes in the reciprocal reference transmissions. As an obvious consequence, T is insensitive to changes in u,v,w in the neighborhood of $u = a^{-1}$, $v = b^{-1}$, $w = c^{-1}$. Since we desire low sensitivity at the operating point, the choice

$$a = b = c = 1 \tag{5.40}$$

is indicated. Now, by inspection of Fig. 5.10, we identify

$$d = e = 0$$
$$f = -A_{000} \qquad (5.41)$$
$$1 = A_{111}.$$

In addition,

$$A_{000} = 1 \qquad (5.42)$$

in order to satisfy condition 5.36. The final result is the flow graph shown in Fig. 5.11, having positive feedback around each stage and over all negative feedback. [Ref:22,p.478] The transmission is given by

$$\left(\frac{1}{T}-1\right) = \left(\frac{1}{u}-1\right)\left(\frac{1}{v}-1\right)\left(\frac{1}{w}-1\right) \qquad (5.43)$$

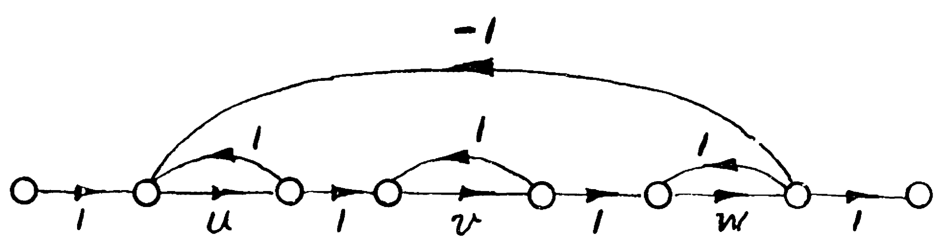


Fig. 5.11

or

$$T(u,v,w) = \frac{uvw}{1-(u+v+w)+(uv+vw+wu)} \,. \qquad (5.44)$$

For small deviations from the operating point, T departs from unity by an approximate amount

$$\Delta T \cong (\Delta u)(\Delta v)(\Delta w). \qquad (5.45)$$

With $u = v = w = 0.9$, for example, relation 5.45 gives $T = 0.999$. The actual value, to the fourth decimal place, is $T = 0.9986$. It is illuminating to compare these results with those obtained from a simple cascaded combination of three reference branches, for which

$$T' = uvw. \qquad (5.46)$$

For small excursions about the operating point, we have, approximately,

$$\Delta T' \cong \Delta u + \Delta v + \Delta w. \qquad (5.47)$$

With u = v = w = 0.9, relation 5.47 gives T' = 0.7. The actual value, of course, is T' = 0.729. Hence the transmission variation is reduced from roughly 30 per cent to 0.1 per cent by the use of feedback.

The extension to more than three reference elements is obvious. In general, for n cascaded reference branches, each having the same maximum variation Δt, the addition of fixed feedback branches as shown in Fig. 5.11 has two effects: (1) the value of T at the operating point remains unchanged, and (2) the maximum per unit variation in T is reduced from $\Delta T'$ to

$$\Delta T \cong \left(\frac{\Delta T'}{n}\right)^n . \qquad (5.48)$$

Approximation 5.48 comes from the exact expression

$$\left(\frac{1}{T} - 1\right) = \prod_{k=1}^{n} \left(\frac{1}{t_k} - 1\right), \qquad (5.49)$$

where $t_1, t_2, \cdots, t_n$ are the reference transmissions.

It may be noted from the form of relation 5.49 that if any one of the reference transmissions is maintained precisely at the desired operating value (unity), then the transmission T is completely independent of the values taken by the other reference transmissions. Suppose, now, that Fig. 5.12 represents a two-stage feedback amplifier. We are presented with the apparent enigma that if the gain of one stage is held exactly at the desired value (unity), then the other stage is unimportant and may, presumably, be removed from the system without affecting

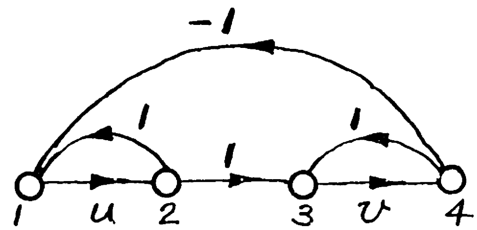


Fig. 5.12

the over-all transmission. This enigma may be resolved by consideration of the following results. Let

$$T_{jk}(u,v) = x_k \text{ per unit signal injected at node } j. \qquad (5.50)$$

Then

$$T_{11}(u,1) = T_{12}(u,1) = 0 \qquad (5.51)$$

$$T_{13}(u,1) = T_{14}(u,1) = 1 \qquad (5.52)$$

$$T_{34}(u,1) = T_{44}(u,1) = \frac{1}{u} - 1. \qquad (5.53)$$

On the basis of relations 5.51 and 5.52, the signal levels throughout the graph do not change as u approaches zero. Relation 5.53, however, shows that the slightest noise or distortion injected at the second stage will appear greatly amplified in the output for small values of u. Hence the amplifier may be expected to perform poorly for small values of u, even when very close tolerances on v are maintained. Now suppose that u is held at unity while v varies. It follows that

$$T_{11}(1,v) = T_{12}(1,v) = \frac{1}{v} - 1 \qquad (5.54)$$

$$T_{13}(1,v) = \frac{1}{v} \qquad (5.55)$$

$$T_{14}(1,v) = 1 \qquad (5.56)$$

$$T_{34}(1,v) = T_{44}(1,v) = 0. \qquad (5.57)$$

Relations 5.56 and 5.57 show that the gain remains unchanged and the output remains free of contamination as v is made smaller. Equations 5.54 and 5.55, however, point out the fact that the signal levels within the amplifier grow very large as v decreases, so that saturation may be expected to occur.

The insensitive structure shown in Fig. 5.11 has, of course, many alternate forms which may arise in the analysis of particular physical problems. Figure 5.13(a) indicates one of these possibilities. The introduction of auxiliary nodes (b) leads to a residual graph (c) which is identical with the residual form of Fig. 5.11. Hence the transmissions of the graphs shown in Figs. 5.11 and 5.13(a) are the same.

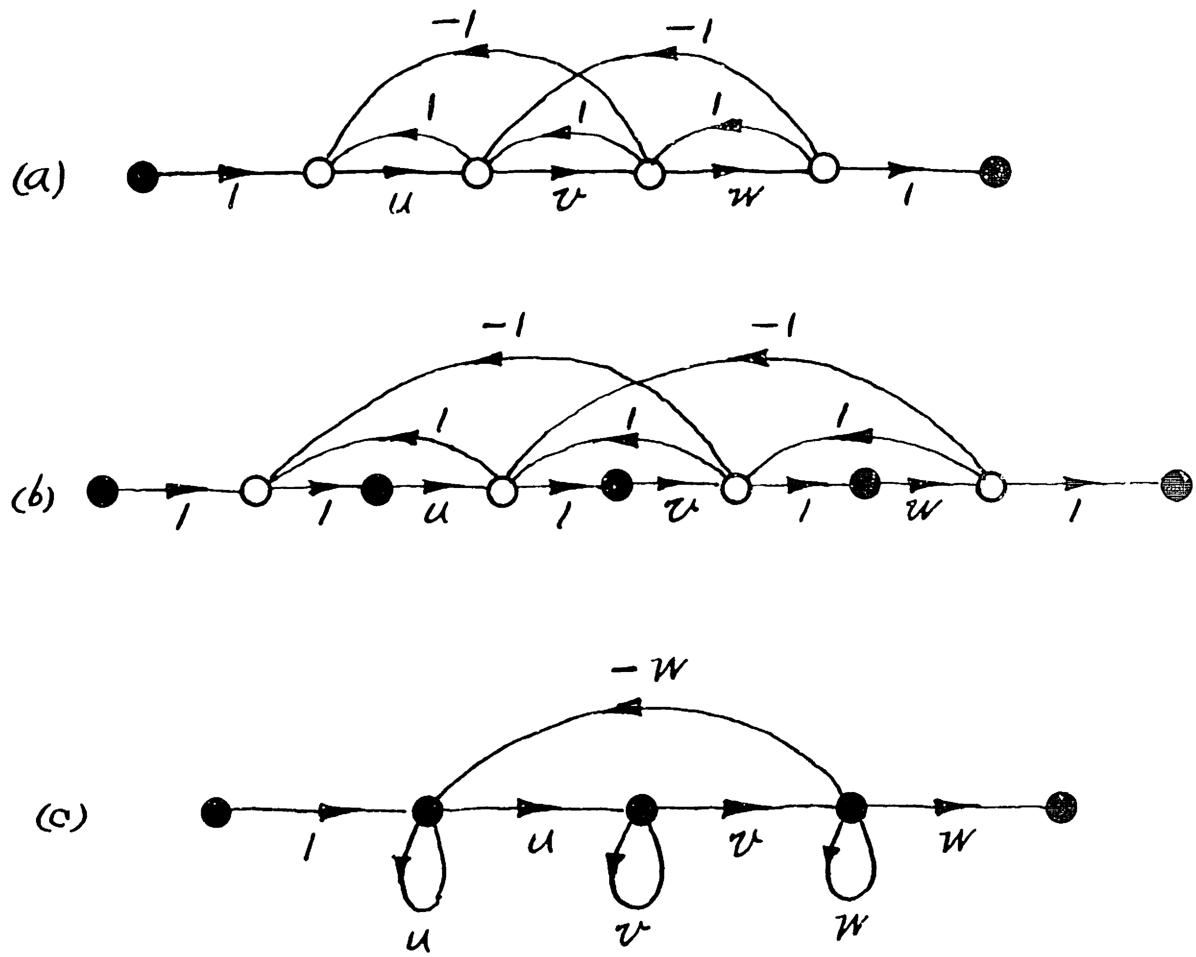


Fig. 5.13

The general residual structure of an insensitive multistage graph is indicated by Fig. 5.14. At the operating point all branch transmissions are unity, with the exception of t_{n1} which is negative, as

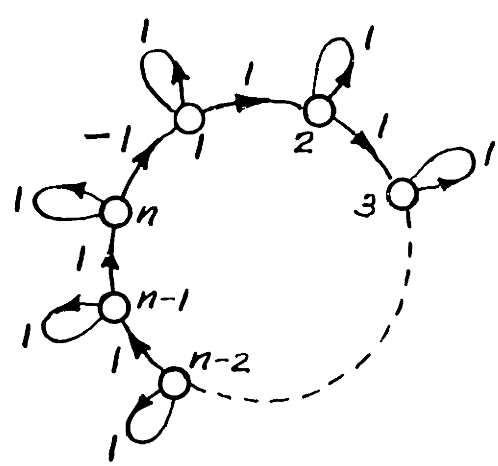


Fig. 5.14

shown. This feedback ring possesses the special property that a signal injected at any given node will appear only at the next counterclockwise node. The ring has, in fact, an injection transmission matrix T which is just the transposition of the flow matrix p. For n = 4,

$$p = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{5.58}$$

Incidentally, each element of p or T is equal to its cofactor and both determinants have unity value. The insensitivity of T_{14} is indicated by the presence of zeros in the last column of the T matrix. These zeros represent discrimination against contaminating signals which enter the system at points other than the input.

The sensitivity considerations undertaken in this chapter show that an arbitrarily small sensitivity to incremental changes in stage transmission may be had by employing a sufficiently large number of stages. In the next chapter, which deals with the general topic of transients in linear flow graphs, we shall find that the number of stages is limited, in practice, by stability and bandwidth requirements.

CHAPTER VI

TRANSIENTS IN LINEAR FLOW GRAPHS

6.1 Transmission Functions

When excited by a time-dependent drive $y(t)$, a dynamical system reacts with a response

$$x(t) = \phi[y(t)], \tag{6.1}$$

where ϕ is an operator characterizing the system. We shall postulate that a linear system is one which obeys the additive law

$$\phi[ay_1(t) + by_2(t)] = a\phi[y_1(t)] + b\phi[y_2(t)], \tag{6.2}$$

where $y_1(t)$ and $y_2(t)$ are any two driving functions weighted by constants a and b. Expression 6.2 is simply a statement of the superposition principle. As a natural extension we may write

$$y(t) = \sum_{k=-\infty}^{\infty} a_k f_k(t) \tag{6.3}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi[f_k(t)], \tag{6.4}$$

provided, of course, the series converge properly. Now consider the particular expressions

$$y(t) = \sum_{k=-\infty}^{\infty} y(kt_o)\ f(t - kt_o)t_o \tag{6.5}$$

$$x(t) = \sum_{k=-\infty}^{\infty} y(kt_o)\ \phi[f(t - kt_o)]t_o, \tag{6.6}$$

wherein

$$f(t) = \frac{\sin \pi(t/t_o)}{\pi t} \tag{6.7}$$

Given $y(kt_o)$ for all k, series 6.5 interpolates an analytic function $y(t)$. Since $f(t)$ is relatively small except in the vicinity of $t = 0$, and since

$$\int_{-\infty}^{\infty} f(t)dt = 1, \tag{6.8}$$

we see that y(t) is, in effect, represented as the sum of a succession of slightly overlapping pulses. If limits are approached by series 6.5 and 6.6 as t_o becomes arbitrarily small, then these same limits are approached by the integrals

$$y(t) = \lim_{t_o \to 0} \int_{-\infty}^{\infty} y(\tau)\, f(t - \tau)\, d\tau \tag{6.9}$$

$$x(t) = \lim_{t_o \to 0} \int_{-\infty}^{\infty} y(\tau)\, \phi[f(t - \tau)]\, d\tau . \tag{6.10}$$

It is convenient to denote 6.9 and 6.10 by the more compact expressions

$$y(t) = \int_{-\infty}^{\infty} y(\tau)\, \delta(t - \tau)\, d\tau \tag{6.11}$$

$$x(t) = \int_{-\infty}^{\infty} y(\tau)\, \phi(t, \tau)\, d\tau , \tag{6.12}$$

wherein function $\delta(t)$ is the unit impulse

$$\delta(t) = \lim_{t_o \to 0} \frac{\sin \pi(t/t_o)}{\pi t} , \tag{6.13}$$

a pulse of unit area whose effective duration is so short that the actual shape and duration of the pulse do not enter into consideration. Expression 6.9 is valid everywhere except at points of discontinuity of y(t), where a spike or overshoot of the Gibbs variety appears. For piecewise continuous functions we may say that 6.9 holds "almost everywhere." Function $\phi(t,\tau)$ is just the response at time t due to a unit impulse drive applied at time τ. The integral form of the superposition principle, therefore, is

$$\phi\left[\int_{-\infty}^{\infty} y(\tau)\delta(t-\tau)d\tau \right] = \int_{-\infty}^{\infty} y(\tau)\phi[\delta(t-\tau)]d\tau . \tag{6.14}$$

Thus far only linearity has been assumed. Let us now postulate that the system is <u>time-invariant</u>,

$$x(t - t_1) = \phi[y(t - t_1)] \text{ for any } t_1 \tag{6.15}$$

and <u>realizable</u>,

$$x(t) = 0 \text{ for } t < t_1$$
$$\text{if } y(t) = 0 \text{ for } t < t_1. \tag{6.16}$$

As a direct consequence of 6.15,

$$\phi(t,\tau) = \phi[\delta(t-\tau)] = h(t-\tau) \tag{6.17}$$

and we are permitted to write

$$x(t) = \int_{-\infty}^{\infty} y(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)y(t-\tau)d\tau. \tag{6.18}$$

Now, in view of the realizability condition 6.16, function $h(\tau)$ vanishes for negative τ so that the lower limit of the right-hand integral may be changed to zero. The final result is the familiar <u>convolution integral</u>

$$x(t) = \int_{0}^{\infty} h(\tau)\, y(t-\tau)d\tau, \tag{6.19}$$

wherein $h(\tau)$ is the <u>weighting function</u> or <u>memory</u> of the system. Although a Riemann integral is defined only for piecewise continuous integrands, the presence of impulses in $h(t)$ or $y(t)$ offers no real practical difficulty. If, for example, $y(t) = \delta(t)$, then it is only necessary to visualize $y(t)$ as a very short pulse of unit area, whereupon we see that $y(t-\tau)$ is large in the neighborhood of $\tau = t$ and ignorable elsewhere. Quantity $h(\tau)$ is essentially constant in this neighborhood and may be replaced by $h(t)$, leaving

$$x(t) = h(t)\int_{0}^{\infty} y(t-\tau)d\tau = h(t). \tag{6.20}$$

Relation 6.20 exhibits the obvious result that the unit impulse response of the system is identical with the memory function. Similarly, for the singularity functions

$$\delta_k(t) = \lim_{f(t)\to\delta(t)}\left[\frac{d^k}{dt^k}\, f(t)\right], \tag{6.21}$$

it follows directly that (almost everywhere) for any (piecewise continuous) function $f(t)$,

$$\frac{d^k f(t)}{dt^k} = \int_{-\infty}^{\infty} f(\tau)\delta_k(t-\tau)d\tau = \int_{-\infty}^{\infty}\delta_k(\tau)f(t-\tau)d\tau. \tag{6.22}$$

Also, by allowing f(t) to approach $\delta_j(t)$, we obtain

$$\delta_{j+k}(t) = \int_{-\infty}^{\infty} \delta_j(\tau)\, \delta_k(t-\tau)\,d\tau. \tag{6.23}$$

With the aid of these elementary relations 6.22 and 6.23, we may speak of singularity functions in the language of ordinary integrals.

Henceforth we shall write the convolution integral in the generalized form

$$x(t) = \int_{0}^{\infty} h(\tau)y(t-\tau)d\tau + \sum_{k=0}^{n} \sum_{j=0}^{\infty} a_{jk} y^{(k)}(t-b_{jk}), \tag{6.24}$$

$$y^{(k)}(t) = \frac{d^k y(t)}{dt^k}\,.$$

The effect of singularity functions in the system memory is accounted for by the series, thereby leaving a piecewise continuous function $h(\tau)$ within the Riemann integral. This form will be useful in the forthcoming discussion of stability. For the present we shall simply assert that the time-domain behavior of a very large class of linear physical systems is describable in terms of a piecewise continuous weighting function $h(\tau)$, together with a series of derivative coefficients a_{jk} and time lags b_{jk}, as shown in Eq. 6.24.

Suppose, now, that attention is restricted to drives of the form

$$y(t) = Y \exp(st), \tag{6.25}$$

where s is the <u>complex</u> <u>frequency</u>

$$s = \sigma + j\omega; \tag{6.26}$$

σ and ω being real parameters. No essential loss of generality results from this restriction since any time-function may be analyzed into exponential components by Fourier methods. It is to be understood, of course, that the actual physical variable is the real part of expression 6.25,

$$Re[y(t)] = |Y| \exp(\sigma t) \cos(\omega t + \phi), \tag{6.27}$$

where

$$Y = |Y| \exp(j\phi) \tag{6.28}$$

is the so called <u>complex amplitude</u> of the drive. Operation upon the exponential function 6.25 by the generalized operator 6.24 yields a response

$$x(t) = \left[\int_0^\infty h(\tau) \exp(-s\tau) d\tau + \sum_{k=0}^{n} \sum_{j=0}^{\infty} a_{jk} s^k \exp(-b_{jk} s) \right] Y \exp(st). \tag{6.29}$$

Hence the response is of the form

$$x(t) = X \exp(st), \tag{6.30}$$

where the ratio of the complex amplitudes X and Y is given by the <u>transmission function</u>

$$T(s) = \frac{X}{Y} = \int_0^\infty h(\tau) \exp(-s\tau) d\tau + \sum_{k=0}^{n} s^k \sum_{j=0}^{\infty} a_{jk} \exp(-b_{jk} s). \tag{6.31}$$

The right hand side of Eq. 6.31 is recognizable as the sum of a Laplace integral and n different Dirichlet series, each series multiplied by a different power of s. Postulating convergence of the integral for a sufficiently large finite value of σ, assuming convergence of the series, and remembering that $h(\tau)$ is piecewise continuous, we observe that Eq. 6.31 defines an analytic function T(s). Since $h(\tau)$, a_{jk}, and b_{jk} are real, the transmission function exhibits <u>conjugate symmetry</u>,

$$T(\bar{s}) = \bar{T}(s), \tag{6.32}$$

where the bar directs us to take the complex conjugate of the quantity beneath it.

Physical systems having a finite number of lumped parameters, (resistances, self and mutual inductances, capacitances, rigid masses, weightless springs, and so forth) always yield transmission functions which are <u>rational</u>. A rational transmission is a ratio of polynomials in s and therefore may be placed in the factored form

$$T(s) = \frac{\prod_{j=1}^{m} (s - s_j')}{\prod_{j=1}^{n} (s - s_j)}, \qquad (6.33)$$

where s_j and s_j' are the poles and zero frequencies of $T(s)$, respectively. A factor may appear more than once in the numerator or denominator of $T(s)$, producing a pole or zero of order or multiplicity other than unity. In general, if

$$\lim_{s \to s_o} (s - s_o)^n \, T(s) = K, \qquad (6.34)$$

where K is finite and nonzero, then $T(s)$ is said to exhibit a pole of order n (for n positive) or a zero of order $-n$ (for n negative) at the point $s = s_o$. The existence of a pole or zero at $s = \infty$ may be determined by examining $T(1/s)$ in the neighborhood of the origin. Now let N_p and N_z be the numbers of poles and zeros, respectively, counting multiple poles or zeros according to their multiplicity, and let a possible pole or zero at infinity be included in the count. It follows from the form of expression 6.33 that

$$N_z = N_p. \qquad (6.35)$$

As an illustration, the transmission

$$T = \frac{1}{s(s + 1)}, \qquad \text{(see 6.56)}$$

exhibits a double zero at $s = \infty$ and poles at $s = 0, -1$. Hence $N_z = N_p = 2$. Another result of interest here concerns the placement of poles and zeros. From relation 6.32 we have

$$|T(s)| = |T(\bar{s})|. \qquad (6.36)$$

It follows that if a pole (or zero) of multiplicity p occurs at s_o, then one of the same order must be located at $\bar{s}_o$. Hence poles and zeros which are not real must occur in complex conjugate pairs.

Physical systems having distributed parameters or hypothetical systems having an infinte number of finite lumped parameters lead to transmissions which are analytic but irrational. The distributed parameter systems are of greater interest here since they represent devices within the realm of experience.

Consider, for example, a unit length of transmission line having distributed parameters R = L = G = C = 1. If this line is terminated in its characteristic impedance

$$Z_o = \sqrt{\frac{R + Ls}{G + Cs}} = 1, \tag{6.37}$$

then the voltage transmission ratio is the irrational function

$$T(s) = \exp(-s - 1). \tag{6.38}$$

In the absence of shunt conductance (G = 0, R = L = C = 1), we may again postulate that the line be terminated in its characteristic impedance, whence the transmission is

$$T(s) = \exp[\,-\sqrt{s(s + 1)}\,]. \tag{6.39}$$

The multivalued character of expression 6.39 arises from the fact that the terminal impedance is unrealizable as a finite system. In order to provide the postulated termination we must attach an infinite length of the same transmission line. Conditions at the far end of an infinite line are, of course, beyond our control. Identification of the two branches of function 6.39 with inward and outward traveling waves, together with the intuitive assumption that only the outward wave will exist, leads us to the choice of that branch for which $|T(s)| < 1$. If the line termination is realizable as a finite system, however, then the transmission function is again single valued. In particular, let us take R = L = C = 1, G = 0, and a unit resistive load. The associated voltage transmission ratio is

$$T(s) = \frac{1}{\cosh\sqrt{s(s+1)} + \frac{1}{s}\sqrt{s(s+1)}\,\sinh\sqrt{s(s+1)}}. \tag{6.40}$$

Since sinh (-x) = - sinh x and cosh (-x) = cosh x, the choice of sign of the radical has no effect upon the value of T(s), provided the same sign is used consistently in each of the three places where the radical appears.

On the basis of experience, it appears that linear physical systems constructible in a finite space lead to (single-valued) meromorphic transmission functions. A <u>meromorphic</u> function has no singularities in the finite s-plane other than poles and hence only a finite number of poles (or zeros) in any finite region of the s-plane. Rational functions are, of course, meromorphic.

Certain restrictions are implied by the <u>Paley-Wiener theorem</u>, which will be rephrased here as follows:

> Given the gain function $G(j\omega)$, where
>
> $$G(s) = \ln \left| T(s) \right|, \tag{6.41}$$
>
> a necessary and sufficient condition for the existence of an associated phase function such that $T(s)$ is <u>realizable</u>, i.e. such that the system represented by $T(s)$ does not respond before the initial application of the drive, is that
>
> $$\int_{-\infty}^{\infty} \frac{|G(j\omega)|}{1 + \omega^2}\, d\omega < \infty \tag{6.42}$$

As a result of this elegant theorem we may state the following corollary.

> A necessary (but not sufficient) condition for the physical realizability of a given transmission $T(s)$ is that its gain function must satisfy the Paley-Wiener criterion.

Transmissions $\exp(-s)$, $\exp(s)$, $\tanh s$, and s^n, for example, satisfy criterion 6.42, whereas $\exp(s^2)$ does not. As an illustration of the insufficiency of the criterion, we need only notice that $\exp(-s)$ is realizable as the transmission of a lossless matched transmission line, while $\exp(s)$, representing a perfect negative time delay, is not physically obtainable.

6.9 <u>The Stability of a Transmission</u>

Consider a drive containing two discrete complex frequencies s_1 and s_2,

$$y(t) = Y_1 \exp(s_1 t) + Y_2 \exp(s_2 t). \quad (6.43)$$

By superposition, the associated response is

$$x(t) = T(s_1)\, Y_1 \exp(s_1 t) + T(s_2)\, Y_2 \exp(s_2 t). \quad (6.44)$$

Now suppose that s_2 is a pole of $T(s)$. If Y_2 vanishes, we have a drive containing only one complex frequency,

$$y(t) = Y_1 \exp(s_1 t), \quad (6.45)$$

but the response may still contain two complex frequencies,

$$x(t) = X_1 \exp(s_1 t) + X_2 \exp(s_2 t), \quad (6.46)$$

where

$$X_1 = T(s_1)\, Y_1 \quad (6.47)$$

$$X_2 = T(s_2) \cdot Y_2 = \infty \cdot 0 = \text{indeterminate}. \quad (6.48)$$

In general, the response will exhibit components at the driving frequencies and also at the pole frequencies of $T(s)$. The terms involving pole frequencies are usually referred to as <u>transients</u>, since in dissipative passive systems these components eventually die out, leaving the so called <u>steady state</u> response due to the drive.

The complex amplitudes of transient terms are indeterminate on the basis of the analysis given here but they may be evaluated from the initial conditions in any particular problem. For our purposes, the actual values of these amplitudes are unimportant. We may assume, however, that all possible transients of a physical system will have been excited at some time in the finite history of the device, so that the complex amplitudes of transients, though possibly very small, are nonvanishing. This excitation may be attributed to initial switching, thermal noise, or other incidental disturbances.

A <u>stable transmission</u> will now be defined as follows:

> If, by placing a bound upon the drive $y(t)$ and its first n derivatives, a bounded response $x(t)$ is assured, then the transmission from $y(t)$ to $x(t)$ is stable. If such assurance is impossible for

any finite n, then the transmission is unstable.

Under this definition, the poles of a transmission are necessarily excluded from the right half s-plane and the finite $j\omega$ axis. In other words, poles of T(s) may lie only in the interior of the left half s-plane or possibly at infinity. The proof follows from the fact that the response contains transients of the form X $\exp(s_o t)$, where $s_o = \sigma_o + j\omega_o$ is a pole of T(s) and where coefficient X is nonvanishing. The transmission is evidently unstable if σ_o is positive since the transient then grows exponentially. Moreover, if a pole occurs on the finite $j\omega$ axis, then a bounded steady-state drive of the same frequency will produce an infinite (steady-state) response. As to multiple poles in the finite s plane, we find by allowing n simple poles to coalesce that the transient response due to an nth order pole at $s = s_o$ contains time-dependent terms of the form

$$t^k \exp(s_o t); \quad k = 0, 1, 2, \ldots, n-1. \qquad (6.23)$$

Hence the same restrictions apply to poles of multiplicity greater than one.

Nothing yet has been said about the permissibility of singularities at infinity in a stable transmission function. The effect of a pole at infinity upon the time-response is particularly simple. The voltage-to-current transmission of an ideal capacitance C, for example, is just $T(s) = Cs$. Any applied voltage wave $y(t)$ yields a current response $x(t) = C\,dy(t)/dt$. Hence a bounded square wave voltage produces a current composed of a sequence of impulses. Under the definition of stability adopted here, however, we are permitted to place a bound on $dy(t)/dt$ so that the transmission $T(s) = s$ is stable. With an nth order pole at infinity, the response contains, in general, the first n derivatives of the drive. It follows that a pole at infinity does not produce instability, although it may result in unbounded response for certain bounded drives.

Returning to the generalized convolution integral,

$$x(t) = \int_0^\infty h(\tau)y(t-\tau)d\tau + \sum_{k=0}^{n}\sum_{j=0}^{\infty} a_{jk}y^{(k)}(t-b_{jk}), \qquad (6.24)$$

we infer that the necessary and sufficient condition for stability is that

$$\int_0^\infty |h(\tau)|\, d\tau + \sum_{k=0}^{n} \sum_{j=0}^{\infty} |a_{jk}| < \infty , \qquad (6.50)$$

For if we choose

$$|y^{(k)}(t)| = M, \quad k = 0, 1, 2, \ldots, n, \qquad (6.51)$$

give $y(-\tau)$ the same sign as $h(\tau)$ for (almost) all τ, and let $y^{(k)}(-b_{jk})$ take the sign of a_{jk}, then

$$x(0) = M \left[\int_0^\infty |h(\tau)|\, d\tau + \sum_{k=0}^{n} \sum_{j=0}^{\infty} |a_{jk}| \right]. \qquad (6.52)$$

It follows directly from 6.50 and

$$T(s) = \int_0^\infty h(\tau) \exp(-s\tau) d\tau + \sum_{k=0}^{n} s^k \sum_{j=0}^{\infty} a_{jk} \exp(-b_{jk} s) \qquad (6.21)$$

that a necessary stability condition upon T(s) is that

$$|T(s)| < \infty, \text{ for } \sigma \geq 0, \ s \neq \infty. \qquad (6.53)$$

This means that a stable transmission may have no singularities in the interior of the right half s-plane or upon the finite $j\omega$ axis.

For rational transmission condition 6.53 is also sufficient. For irrational meromorphic transmissions, however, which exhibit essential singularities at infinity, the sufficiency is questionable unless certain additional restrictions are imposed upon h(t) or T(s). The memory function $h(t) = \cos(t^2)$, for example, fails to satisfy criterion 6.50, although its Laplace transform is free of singularities in the right half s-plane and upon the $j\omega$ axis. Instability is demonstrable by the choice of the particular drive $y(t) = \cos(t^2)$, which results in an unbounded output at $t = 0$,

$$x(0) = \int_0^\infty \cos^2(\tau^2) d\tau = \infty . \qquad (6.54)$$

Under our definition of stability, however, a bound may be placed upon the derivative of y(t), so that the density of zero-crossings of y(t) is limited and drives such as $y(t) = \cos(t^2)$ are excluded. Hence the time-domain and frequency domain stability criteria agree in this example. We shall now assert without proof that if the density of zero-crossings of h(t) and the density of points $t = b_{jk}$ are bounded as t becomes large, then conditions 6.50 and 6.53 are each necessary and sufficient for stability. This assertion is supported here only by experience. It is known to be true for rational transmissions and for each of many meromorphic transmission functions which have come under the consideration of the writer. The proof (or disproof) is left as a suggestion for further investigation.

6.3 Nyquist Mapping

Under certain conditions, the presence or absence of right half plane poles or zeros in a transmission T(s) may be ascertained from the behavior of T(s) on the jω axis. Given the analytic form of T(s) it is often much less tedious to compute and plot the complex locus of T(jω) than it is to solve directly for the pole frequencies. Moreover, the function T(jω) is sometimes available only in graphical form as the direct or derived result of experimental steady-state tests upon a physical device.

A complex plot of T(jω) is called a Nyquist diagram. The information may also be presented as a complex locus of the logarithm of T(jω) or as separate curves of the gain and phase of T(jω), each plotted against the logarithm of ω. For our purposes the simple Nyquist plot will suffice. Let us assume, for the present, that T(s) is free of singularities on the jω axis, including the point at infinity. The Nyquist plot is then a simple closed curve which may intersect itself as shown in Fig. 6.1. The direction of increasing ω is indicated by the arrow-heads. In

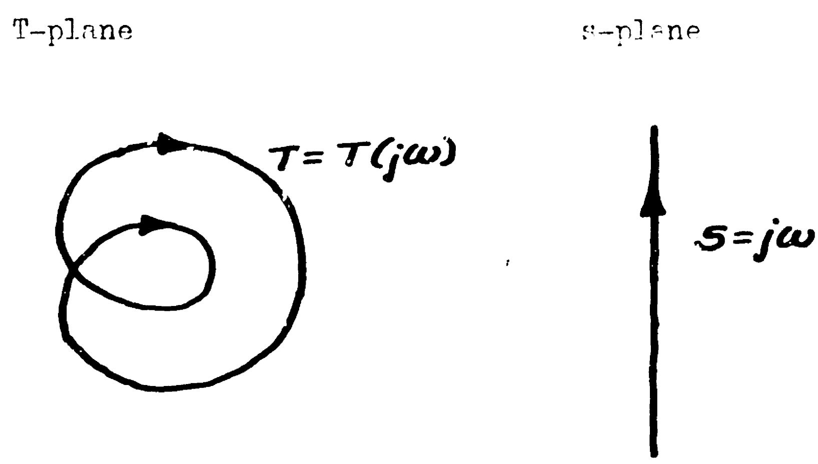


Fig. 6.1

mathematical language, the $j\omega$ axis in the s-plane <u>maps</u> into the Nyquist plot in the T-plane. Any other curve in the s-plane would map into the T-plane in a similar fashion. Since T(s) is single-valued, the mapping is unique; any s-curve maps into a single T-curve (perhaps self intersecting). If, however, we attempt to map a T-curve into an s-curve, the mapping will in general be multiple-valued. Consider, for example, that segment of the real T-axis lying between +1 and +4. If $T(s) = s^2$, then the s-plane map of this segment is a pair of segments, one between -2 and -1 and the other running from +1 to +2. Each separate branch of such an s-map, of course, is itself a simple curve and these curves do not cross each other.

Before deriving a result which relates to pole and zero locations, we shall need to state one additional property of the analytic mapping transformation T(s). This property is <u>conformality</u> and it may be described as follows:

> If a point b, moving about in the s-plane, suddenly changes its course to the right or to the left, then the corresponding T-plane point, B = T(b) will at the same time alter its course by the same angle and in the same direction relative to its original motion.

Conformality follows directly from the fact that an analytic function, by definition, possesses a unique derivative.

Now consider the curve AB joining points A and B in the T-plane, as shown in Fig. 6.2. The corresponding s-plane map of AB is shown by the

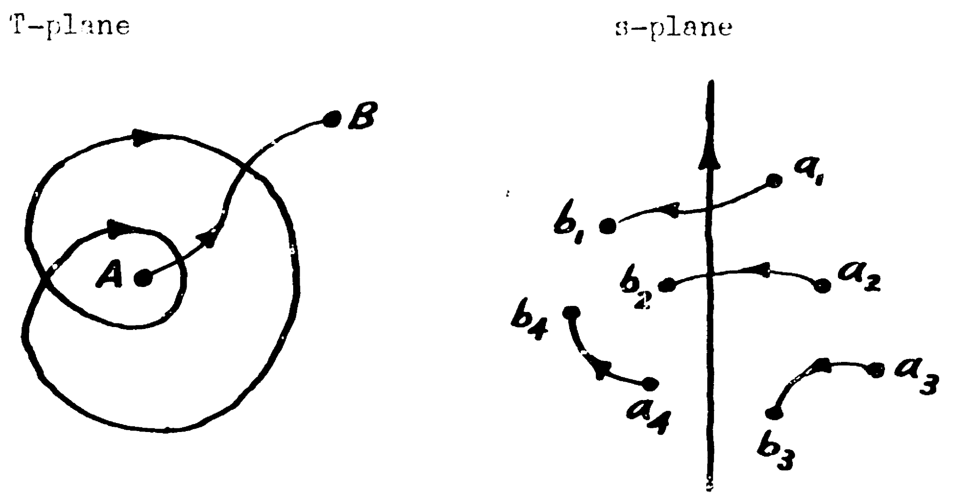


Fig. 6.2

separate curves a_1b_1, a_2b_2, etc. By conformality, if the $j\omega$ axis crosses two of the ab curves from left-to-right, as seen by an observer facing along ab in the indicated direction, then the Nyquist plot must cut across the AB curve twice from left to right, as shown. Other possible branches of ab, such as a_3b_3 and a_4b_4, are not intersected by the $j\omega$ axis since such intersections would necessarily show in the T-plane. If we let

n_a = total multiplicity of a-points in the right half s-plane
n_b = total multiplicity of b-points in the right half s-plane
n = net number of left-to-right crossings of AB by the Nyquist plot (right-to-left crossings counted as negative)

then evidently

$$n = n_a - n_b. \qquad (6.55)$$

A multiple a-point (a multiple root of the equation T(s) = A) must, of course, be counted according to its multiplicity, since p different ab curves would radiate from such a point of multiplicity p. For Fig. 6.2, $n = 2$ and $n_a \geq 2$. Figure 6.3 provides another illustration. Given points A and B, we sketch any curve AB joining them and then indicate corresponding s-plane intersections as shown in (b). Two possibilities consistent with (b) are (c) and (d). For both (c) and (d), of course, $n = n_a - n_b = 0$.

Suppose, now, that the origin in the T-plane coincides with point A in Fig. 6.3(a), so that the a-points are zeros of T(s). Suppose, also, that point B is allowed to move far out in the T-plane, causing the b-points to approach the poles of T(s). It follows that the net number of clockwise encirclements of the origin by the Nyquist plot is equal to the excess of zeros over poles in the right half s-plane. Figure 6.3(a) indicates equal numbers of poles and zeros in the right half plane. By joining point C in Fig. 6.3(a) to a distant B-point, we should find that T(s) has a total right half plane pole multiplicity of at least two. In general, if any point in the finite T-plane is encircled counterclockwise by the Nyquist plot, then at least one right half plane pole is assured.

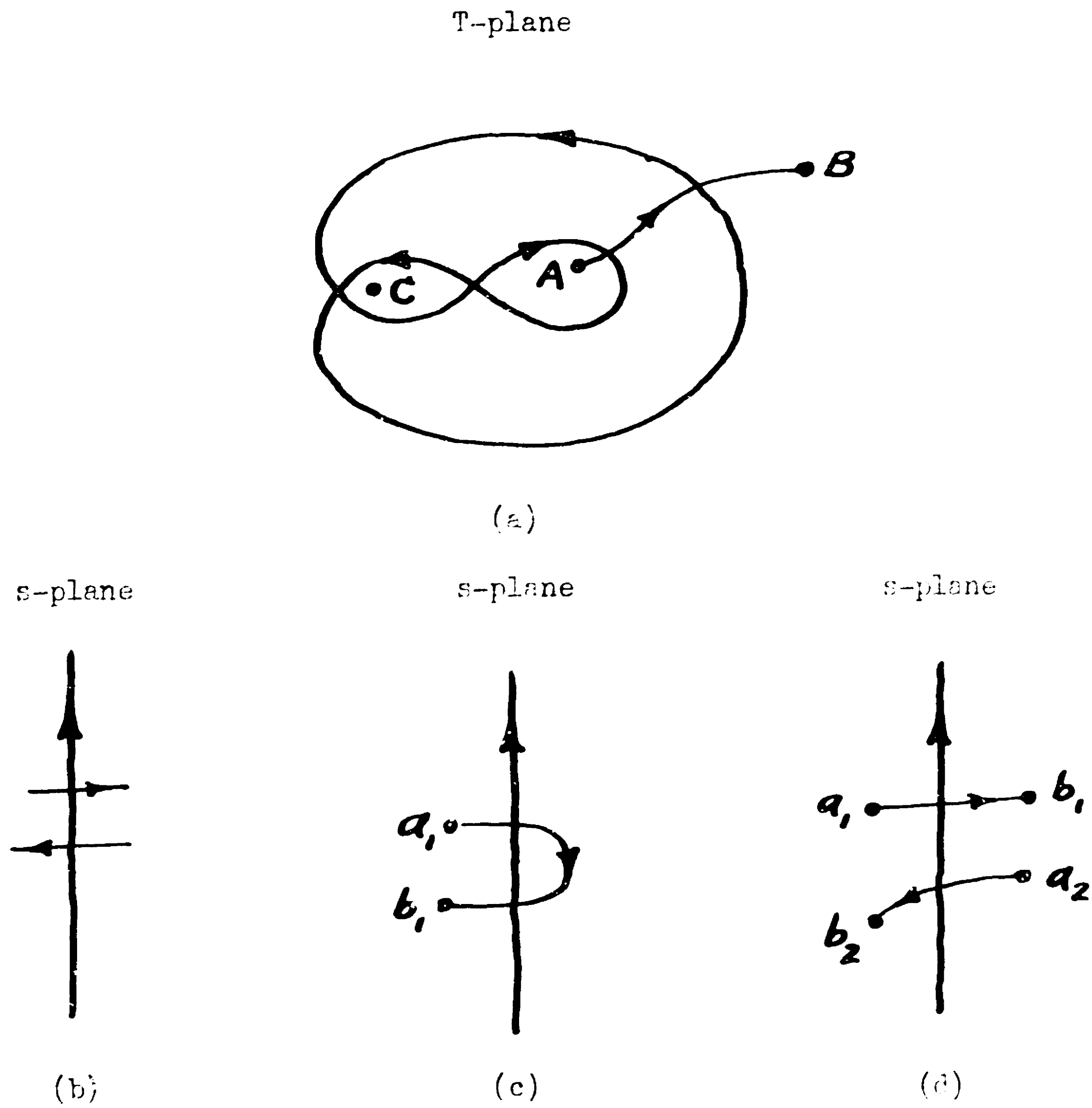


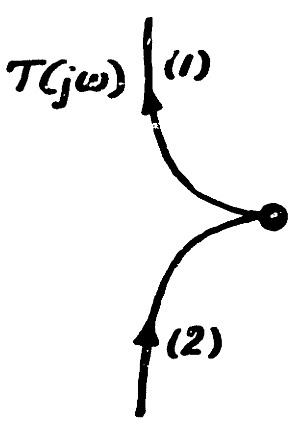
Fig. 6.3

Theorems such as 6.55 and the corollary results dealing with poles and zeros are the basis for all stability criteria which depend upon Nyquist plots. Our purpose in reiterating these well-known theorems has been to expose a proof based upon simple curve-sketching and depending only upon elementary concepts of continuity and conformality. The graphical simplicity of this process makes it unnecessary to remember the details of a particular theorem. Instead, a moment of pencil work gives the desired result directly. For emphasis, we shall set down the process in detail. Given the Nyquist plot of a transmission T(s) for which n_a is known, we find n_b by (1) sketching a directed curve AB, (2) noting the intersections of this curve with the Nyquist plot, (3) sketching conformal intersections on the $j\omega$ axis in the s-plane, (4) assigning a possible set of a and b points which properly terminate the s-plane curves in a manner consistent with the specified value of n_a, and (5) noting n_b from the sketch. When n_a is unknown, of course, the Nyquist plot can tell us only the value of the difference $n_a - n_b$.

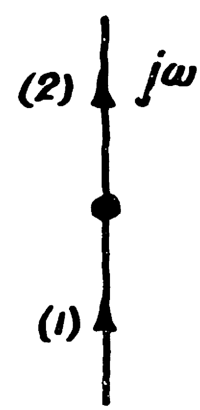
At the outset of our discussion of Nyquist diagrams T(s) was assumed regular on the entire $j\omega$ axis, including the point at infinity. Now let us take up a simple example which illustrates the modifications necessary when the $j\omega$ axis passes through a pole or zero of T(s). Consider the particular transmission

$$T(s) = \frac{1}{s(s+1)}, \tag{6.56}$$

whose Nyquist plot is the T-plane contour shown in Fig. 6.4(a), the map of the s-contour indicated in (b). Corresponding portions of the two contours are denoted by numbers in parentheses and the origin in each plane is marked by a heavy dot. To obtain a closed T-contour which does not



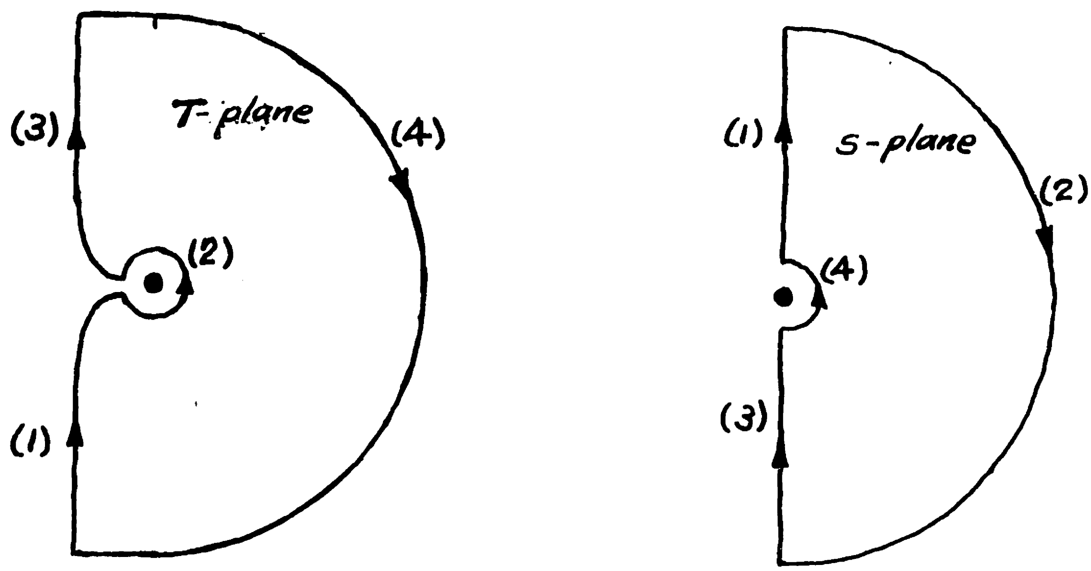
(a)



(b)

Fig. 6.4

pass through the origin, we need only alter the s-contour as shown in Fig. 6.5(b). If the s-plane semicircles (2) and (4) are made very large and very small, respectively, all right half plane zeros and poles of T(s) fall within the s-contour. By joining the origin to the point at infinity in the T-plane, we have an AB curve which cuts the T-contour the same number of times in each direction. Hence T(s) has the same number of zeros and poles in the <u>interior</u> of the right half s-plane. The



(a) (b)

Fig. 6.5

orientation of circular arcs (2) and (4) in the T-plane is determined by conformality. In passing from (1) to (2), (2) to (3), (3) to (4), and (4) to (1), the s-contour turns right through 90°. T-contour must do the same, as shown. Since T(s) exhibits a double zero at $s = \infty$, the asymptotic behavior is

$$T(s) \rightarrow \frac{1}{s^2}, \text{ as } s \rightarrow \infty. \qquad (6.57)$$

By writing s and T in polar form, we observe that T must traverse a (nearly) complete small circle as s progresses along the large semicircle (2). Were the analytic form of T(s) unknown, then a log-log plot of the experimental data, $|T(j\omega)|$ versus ω, would exhibit a slope of -2 for large ω, indicating a double zero. In any case the multiplicity of a pole or zero must be determined before the T-plane arcs are shown. A multiplicity p always leads to a T-plane arc which encompasses, in the limit $p\pi$ radians.

An essential singularity at infinity, such as that appearing in T(s) = exp(-s), requires special attention if the Nyquist plot is to be closed properly. The same approach applies, however. Figure 6.6(a) shows the Nyquist plot of T(s) = exp(-s). Here it is convenient to choose a rectangular rather than a semicircular detour about the point

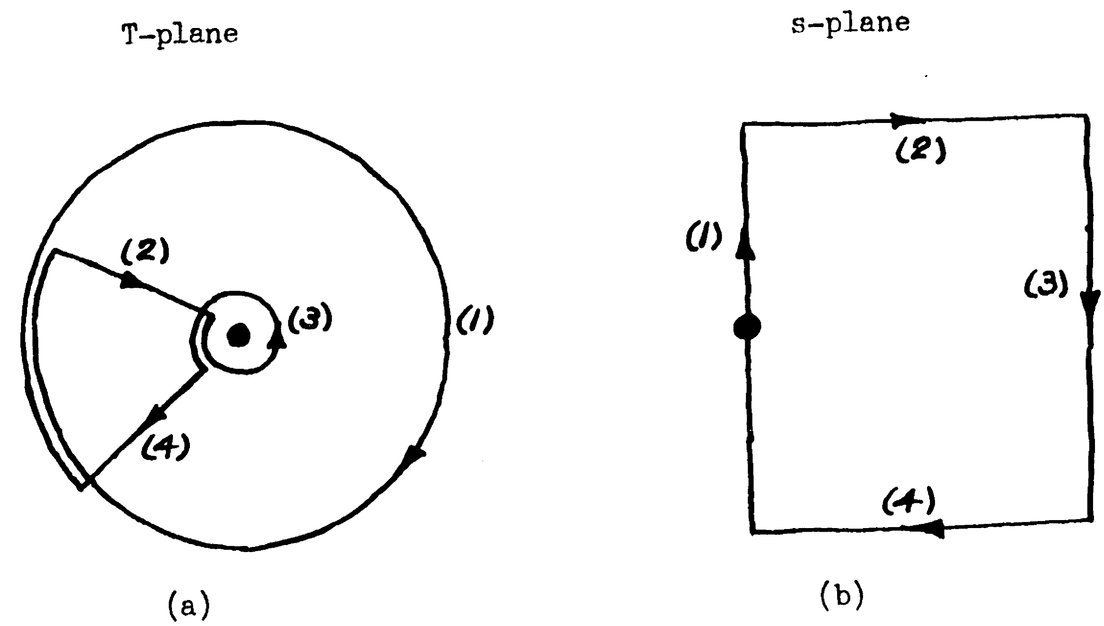


Fig. 6.6

at infinity in the s-plane. Evidently T(s) has the same number of zeros and poles in the right half plane. The simple function exp(-s), of course, possesses neither poles nor zeros anywhere.

Nyquist techniques do not enter certain active system synthesis problems where stability is assured by the original choice of pole positions. Nevertheless, the Nyquist plot of a transmission function offers a means of presentation which is often attractive to the designer, even when his system is known to be stable. In particular, the Q associated with a pole close to the $j\omega$ axis may be estimated [Ref:25] from a plot of 1/T(s). The "M-circle criterion," which has found use in servomechanisms design, is an adaptation of this principle. Such techniques serve as a semi-quantitative tie between the steady-state and transient behaviors of the transmission system.

6.4 The Stability of a Transmission Relative to a Particular Branch

The flow graph shown in Fig. 6.7 represents a generalization of the elementary stability problem originally posed by Nyquist. It is a generalization in the same sense that the feedback formula 4.44 is a generalization of Black's formula 2.5.

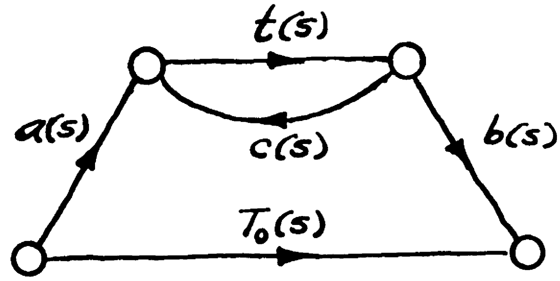


Fig. 6.7

The transmission is given by

$$T(s) = T_o(s) + \frac{T_f(s)}{1 - \mathcal{T}(s)} \tag{6.58}$$

where

$$T_f(s) = a(s)t(s)b(s)$$

$$\mathcal{T}(s) = t(s)c(s).$$

Suppose that all branch transmissions are known to be stable. Since the product of two transmissions can have only the poles of each factor, it follows that $T_f(s)$ and $\mathcal{T}(s)$ are also stable. From the form of 6.58 we see that the only possible right half s-plane poles of T(s) are the 1-points of $\mathcal{T}(s)$. It remains, therefore, to count the number of times which $\mathcal{T}(s)$ assumes the value unity in the right half s-plane. Figure 6.8(a) shows a possible Nyquist plot of $\mathcal{T}(s)$. Allowing point B to recede toward infinity in the $\mathcal{T}$-plane causes points b to approach the poles of $\mathcal{T}(s)$. For the plot shown, T(s) is evidently unstable since $\mathcal{T}(s)$ assumes the

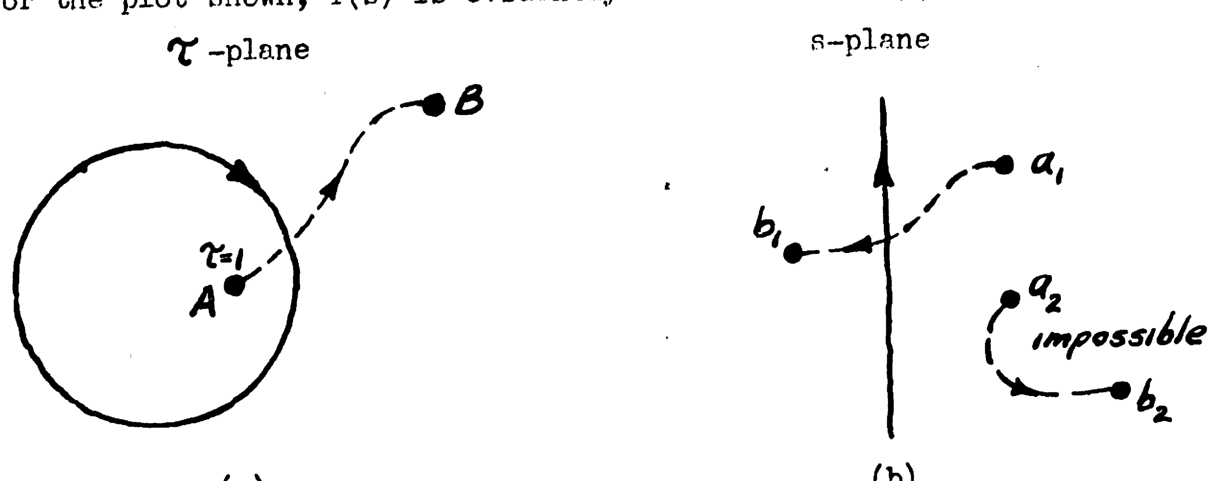


Fig. 6.8

value unity exactly once in the right half s-plane. A branch such as a_2b_2 is impossible since $\tau(s)$ has been assumed stable (i.e. no right half plane poles). Had the Nyquist plot not encircled point A, then there would have been no a-points at all in the right half plane, and T(s) would have been definitely stable.

In some feedback systems $\tau(s)$ is purposely made unstable in order to achieve certain performance characteristics. This does not mean that T(s) is also unstable, as the following example will show. Let us postulate that $\tau(s)$ has no right half plane zeros but may, perhaps, be unstable. Suppose that the Nyquist plot of $\tau(s)$ is shown by Fig. 6.9(a). For this plot no right half plane b-points are possible so that T(s) is stable.

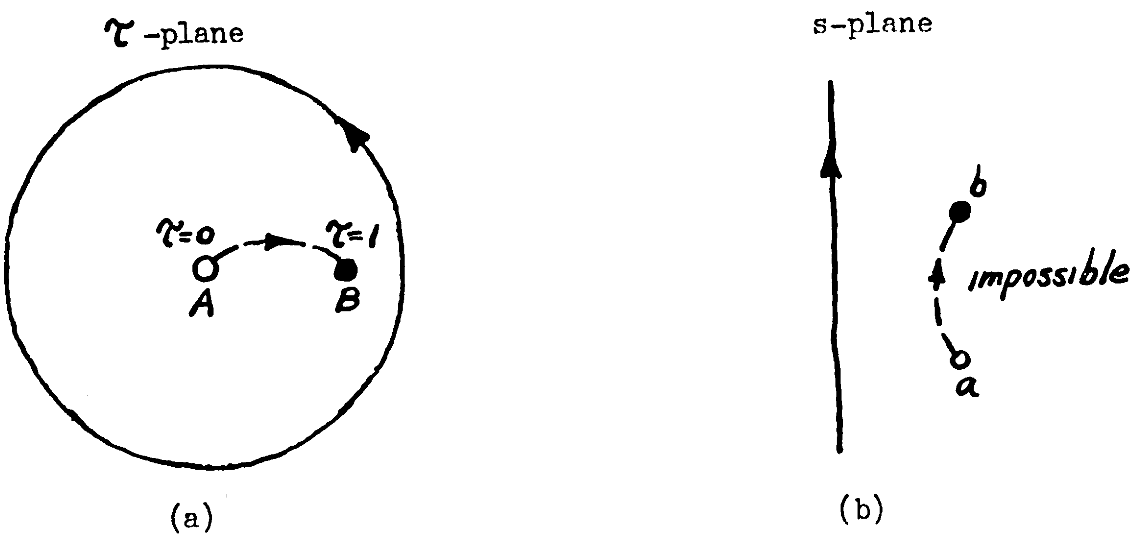


Fig. 6.9

Figure 6.10, however, indicates an unstable transmission T(s), since exactly one b-point occurs in the right half plane.

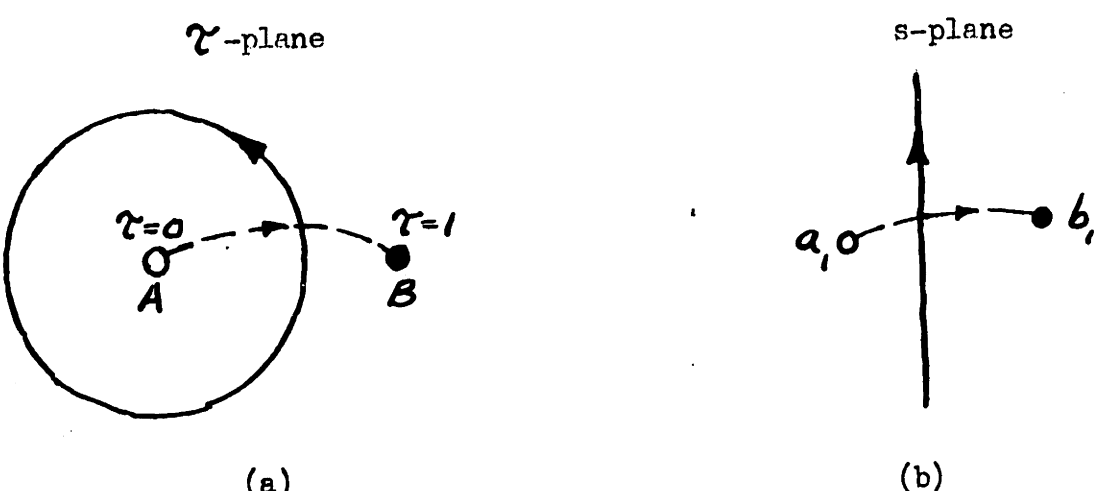


Fig. 6.10

In either of the above examples we might equally well have plotted the Nyquist diagram of the loop difference $D(s) = 1 - \mathcal{T}(s)$ and taken $D = 0$ as the critical point B. In servomechanisms analysis the negative of $\mathcal{T}$ is usually plotted, making -1 the critical point. The choice is trivial. Whatever the critical point, we join it to another point whose right half plane multiplicity is known, and then deduce the stability of the system from a simple conformal sketch.

6.5 The Stability of a General Flow Graph

In this section we shall consider the stability of the transmission through a general flow graph, postulating at the outset that the branch transmissions are all stable. We shall find that each imbedded feedback graph is a potential oscillator, which may produce an unstable pole in the transmission. The simple example shown by Fig. 6.11 is instructive

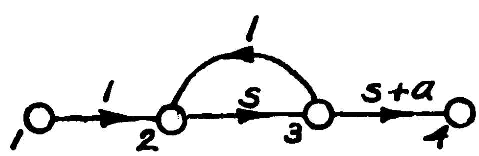


Fig. 6.11

at this point. By inspection of the graph,

$$T_{14} = \frac{s(s+a)}{1-s} . \tag{6.59}$$

This transmission exhibits a pole at $s = 1$ where the loop differences D_2 and D_3 have zeros. If, perchance, $a = -1$, then the pole disappears and the transmission is apparently equal to s. Nevertheless, a growing transient still appears at nodes 2 and 3, so that the system eventually saturates and fails to operate as a linear transmission device. Mathematically, the stability of a particular transmission depends only upon the locations of its poles. In practice, however, that transmission is stable and linear only if every node in the graph is free of undamped transients. If no undamped transients appear at any node in the graph, we shall say that the graph is stable.

The various signals entering a particular node may be separated into feedback signals and cascade signals, according to the type of branch through which they arrive. The complex frequencies present in the feedback signal at node j are just the zeros of the loop difference D_j, for then $\tau_j = 1$ and the condition for continuity of signal flow at node j is satisfied. A cascade signal at node j may arise either from a pole of the transmission of a cascade branch entering node j from some other node i or from a signal of the same frequency at node i. An undamped cascade signal, however, can come only from another node, for the simple reason that all branch transmissions are stable. It follows that an undamped signal at any node arises either from a zero of the loop difference of that node or from a zero of the loop difference of some other node. Hence, a flow graph is stable if and only if the loop differences of its nodes exhibit no zeros in the right half s-plane or upon the finite $j\omega$ axis. Since the loop transmission of a node depends only upon the transmissions of branches in the feedback graph containing that node, it is sufficient to consider separately the stability of each feedback graph. The flow graph is stable if and only if each of its imbedded feedback graphs is stable. Moreover, we may begin with the residual graph is we so desire, for its branch transmissions, being sums of products of branch transmissions appearing in the original graph, are likewise stable.

To proceed with the development of a stability criterion, let the nodes of a feedback graph be numbered 1, 2, ... , n in any desired order and let

> D'_j = loop difference of node j as computed with nodes 1, 2, ... , j present but nodes j+1, j+2, ..., n temporarily erased from the graph.

Evidently $D'_n = D_n$. We shall first show that the product

$$P = D'_1 D'_2 \dots D'_n \tag{6.60}$$

is independent of the order in which the nodes are numbered. The proof follows from a theorem mentioned in Chapter 4. We may restate Eq. 4.20 in the form

$$D'_j D'_{j-1} = D''_j D'''_{j-1} \tag{6.61}$$

Where D''_j is computed with only nodes 1, 2, ... , j-2 and j present, and D'''_{j-1} is found from the graph containing only nodes 1, 2, ... , j. The right hand side of Eq. 6.61 is recognizable as just the $D'_jD'_{j-1}$ product which results when the numbers of nodes j and j-1 are interchanged. Since any permutation is obtainable by such interchanges, the demonstration is complete.

Loop transmission τ'_j is equal to the returned signal at node j when a unit signal is transmitted from node j, as computed with nodes j+1, j+2, ... , n temporarily erased from the graph. A right half plane pole of τ'_j at some frequency s_o means that an undamped signal returns to node j even when no signal is transmitted from j. It follows that at least one of the nodes 1, 2, ... , j-1 will oscillate at frequency s_o if node j is also erased. Since the numbering of nodes is arbitrary, we may assume that the oscillation appears at node j-1 when j is obliterated. Hence a pole of $D'_j = 1 - \tau'_j$ at s_o is a zero of D'_{j-1} at s_o, provided we number the nodes properly. A zero of D'_{j-1}, however, is not necessarily a pole of D'_j, for the branches connecting node j to lower numbered nodes may result in zero coupling between nodes j-1 and j at frequency s_o. Figure 6.12 shows the result of erasing nodes j+1,j+2, ... , n and then eliminating nodes 1, 2, ... , j-2 by the reduction processes described in Chapter 4.

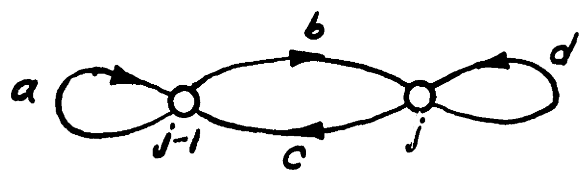


Fig. 6.12

By inspection of the figure

$$D'_{j-1} = 1 - a \tag{6.62}$$

$$D'_j = 1 - d - \frac{bc}{1-a} . \tag{6.63}$$

If the coupling transmission bc vanishes at s_o, then a zero of D'_{j-1} at s_o does not appear as a pole of D'_j. In this case, however, an oscillation of frequency s_o persists at node j-1 even with node j present, for then

$$D_{j-1}(j \text{ present}) = 1-a-\frac{bc}{1-d} = 1-a. \qquad (6.64)$$

As a result of the foregoing discussion we may state that

> If a new node j is added to a graph containing nodes 1, 2, ... , j-1, and if the original graph oscillated at a frequency s_o, then either (1) the zero of D'_{j-1} at s_o appears as a pole of D'_j and the new graph oscillates only at the zero frequencies of D'_j, or (2) the zero of D'_{j-1} at s_o does not appear as a pole of D'_j and the new graph oscillates at both s_o and the zeros of D'_j.

The product $P = D'_1D'_2 \dots D'_n$, therefore, contains only the poles of D'_1 and has zeros at each natural frequency of the complete graph. Since $\mathcal{T}'_1$, being a residual transmission of the original graph, is stable, it follows that

> The product $P = D'_1D'_2 \dots D'_n$ has no poles in the right half s-plane or upon the finite $j\omega$ axis and contains all the zeros of the loop differences $D_1, D_2, \dots, D_n$.

Hence a Nyquist plot of P will detect the presence of right half plane zeros in any loop difference of the complete feedback graph. Alternatively, we may make a Nyquist plot of each of the loop differences D'_j. If

z_j = number of right half plane zeros of D'_j
p_j = number of right half plane poles of D'_j,
$n_j = z_j - p_j$

then the Nyquist plot of D'_j yields the difference n_j. Now, since $p_{j+1} = z_j$ for j = 1, 2, ... , n-1, and since $p_1 = 0$, we have

$$\sum_{j=1}^{n} n_j = p_n. \qquad (6.65)$$

A positive p_n means instability, a p_n equal to zero indicates stability, and a negative p_n is impossible.

The feedback graph shown in Fig. 6.13 will serve as an illustrative

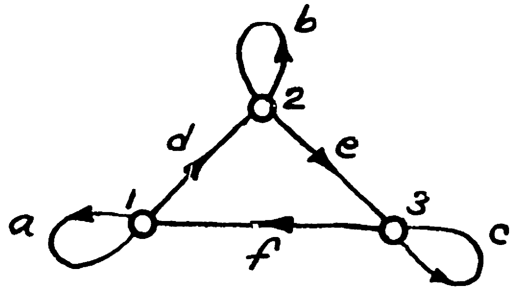


Fig. 6.13

example, By inspection of this graph,

$$D_1' = 1 - a \tag{6.66}$$

$$D_2' = 1 - b \tag{6.67}$$

$$D_3' = 1 - c - \frac{def}{(1-a)(1-b)} \tag{6.68}$$

$$P = (1-a)(1-b)(1-c) - def \tag{6.69}$$

$$D_1 = 1 - a - \frac{def}{(1-b)(1-c)} = \frac{P}{(1-b)(1-c)} \tag{6.70}$$

$$D_2 = \frac{P}{(1-a)(1-c)} \tag{6.71}$$

$$D_3 = \frac{P}{(1-a)(1-b)}. \tag{6.72}$$

Loop differences D_1, D_2, and D_3, all have the same zeros and these are just the zeros of P. The zeros of D_1' and D_2' appear later as poles of D_3'. This is also true if the numbering of nodes 1 and 2 is interchanged. If perchance,

$$e = 1 - a, \tag{6.73}$$

then the coupling among the three nodes vanishes at the zeros of D_1'. Under condition 6.73, D_1' and D_2' are unchanged but

$$D_3' = 1 - c \frac{df}{1-b}, \tag{6.74}$$

$$P = (1-a)[(1-b)(1-c) - df] \tag{6.75}$$

$$D_1 = \frac{[(1-b)(1-c) - df](1-s)}{(1-b)(1-c)} \tag{6.76}$$

$$D_2 = \frac{(1-b)(1-c) - df}{1-c} \tag{6.77}$$

$$D_3 = \frac{(1-b)(1-c) - df}{1-b} \tag{6.78}$$

Here D_1 has zeros not contained in D_2 or D_3 but the product P still exhibits every zero of D_1, D_2, or D_3. The zeros of D_1' do not appear later as poles of D_2' or D_3', since the system oscillates at the zero frequencies of D_1' even when nodes 2 and 3 are present.

Relation 4.7 of Chapter 4 leads to a very simple relation between product P and the flow determinant. They are, in fact, identical. If we let

P_j = value of the flow determinant as computed with only nodes 1,2, ... , j present,

then evidently

$$D_j' = \frac{P_j}{P_{j-1}} , \tag{6.79}$$

where $P_o = 1$. Hence

$$P = P_n = D_1'D_2' \ldots D_n' . \tag{6.80}$$

The flow determinant P, whose value is the sum of products of branch transmissions, has no right half plane or finite $j\omega$ axis poles and its zeros are the natural frequencies of the graph. Hence a Nyquist plot of P (or the succession of Nyquist plots of its factors $D_1', D_2', \ldots, D_n'$) determines the stability of the graph. The practice of treating each imbedded feedback graph separately in a stability analysis is justified by the fact that the flow determinant of a graph is the product of the flow determinants of the imbedded feedback graphs.

As a final example we shall undertake to determine the stability of the vacuum-tube amplifier shown in Fig. 6.14.

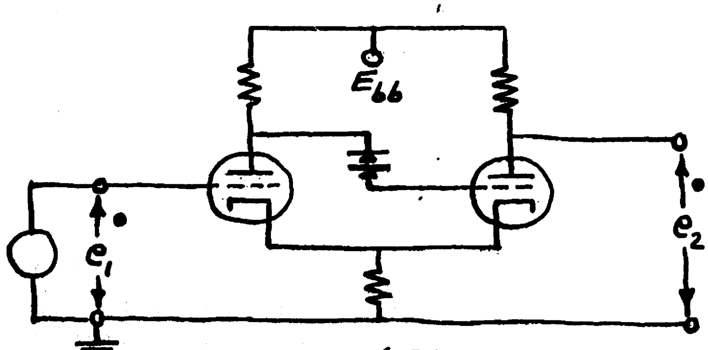


Fig. 6.14

Under the usual assumptions the linear incremental equivalent circuit is that shown by Fig. 6.15.

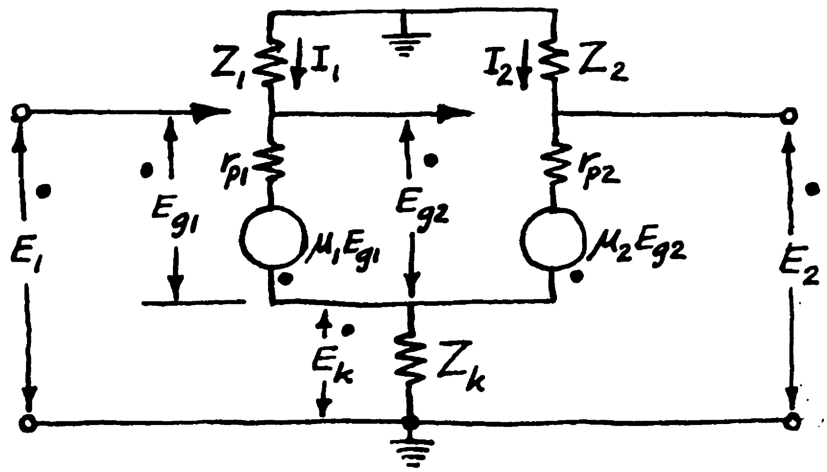


Fig. 6.15

An appropriate flow graph appears in Fig. 6.16. Voltage V_1 is just the

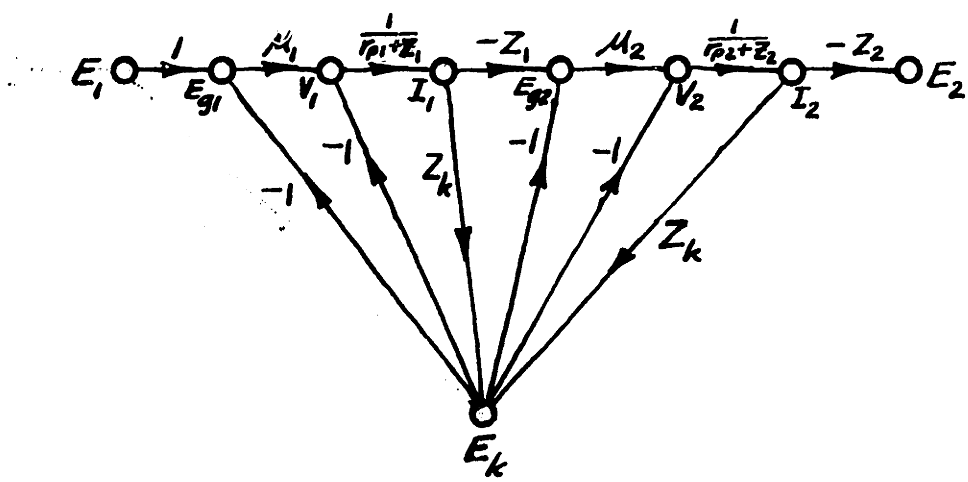


Fig. 6.16

net force $\mu_1 E_{g1} - E_k$ tending to draw current downward through r_{p1} and R_1. The interpretation of this graph is simplified if we eliminate nodes E_{g1} and E_{g2} by means of the star to mesh transformation to obtain the slightly more condensed form shown in Fig. 6.17.

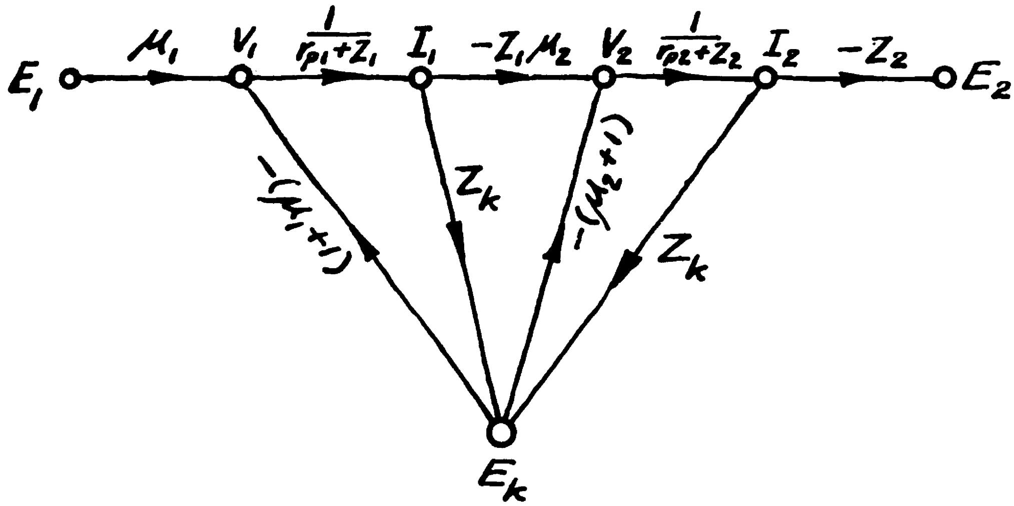


Fig. 6.17

Loops $E_kV_1I_1E_k$ and $E_kV_2I_2E_k$ represent the degenerative feedback in each stage due to cathode loading, whereas loop $E_kV_1I_1V_2I_2E_k$ accounts for positive feedback around both stages due to cathode coupling. The graph is of index unity and E_k is the residual node. By inspection of the graph, the residual transmission from node k to itself is

$$\tau_k = \frac{(\mu_1+1)\mu_2 Z_1 Z_k}{(r_{p1}+Z_1)(r_{p2}+Z_2)} - \frac{(\mu_1+1)Z_k}{r_{p1}+Z_1} - \frac{(\mu_2+1)Z_k}{r_{p2}+Z_2}\,. \tag{6.81}$$

If Z_1, Z_2, and Z_k are passive, then τ_k is stable and a Nyquist plot of $D_k = 1 - \tau_k$ determines the stability of the system. Suppose that Z_1 and Z_2 are purely resistive but Z_k is a parallel RC combination. Factoring Z_k in expression 6.81 and setting $D_k = 0$ ($\tau_k = 1$), we have

$$1 = \frac{M}{G + Cs} \tag{6.82}$$

where

$$M = \frac{(\mu_1+1)\mu_2 Z_1}{(r_{p1}+Z_1)(r_{p2}+Z_2)} - \frac{\mu_1+1}{r_{p1}+Z_1} - \frac{\mu_2+1}{r_{p2}+Z_2} \tag{6.83}$$

The transient complex frequency of the system is, from 6.82,

$$s = \frac{1}{C}\,(M - G). \tag{6.84}$$

Instability results when

$$M \geqq G. \qquad (6.85)$$

Incidentally, when M is approximately equal to G this circuit is useful as a level selector, the static output-input characteristic being that shown in Fig. 6.18.

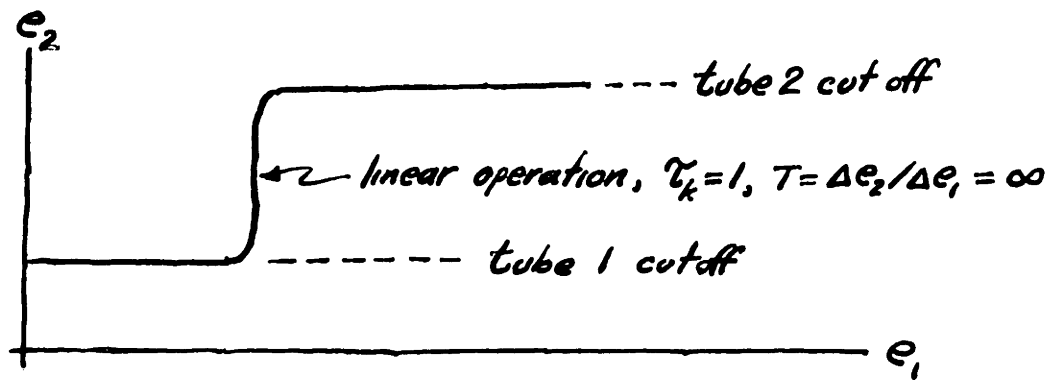


Fig. 6.18

6.6 The Stability of a Multistage Insensitive Graph

In Chapter 5 it was shown that the sensitivity of a cascaded structure could be markedly reduced by the use of feedback. We may now investigate the stability of such a structure, making certain assumptions as to the frequency dependence of its branches. The example to be considered here although not sufficiently general to encompass all low-sensitivity design problems, is nevertheless illustrative of the limitations upon sensitivity which may be imposed by stability and bandwidth requirements. Figure 6.19 shows a three stage structure having positive feedback around each stage and negative feedback around the complete system. We shall assume that only the forward elements are frequency dependent. In particular,

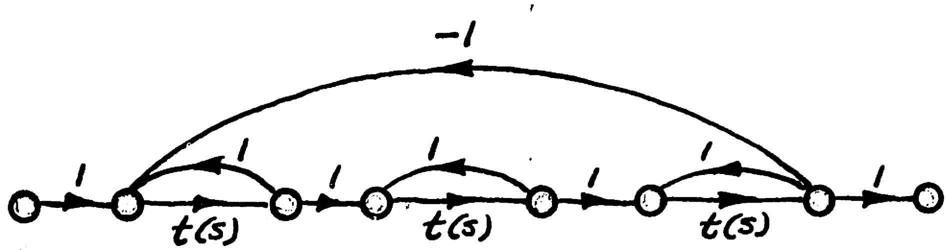


Fig. 6.19

let

$$t(s) = \frac{1}{s+1} , \qquad (6.86)$$

yielding an asymptotic frequency characteristic of the type associated with parasitic shunt capacitance in a voltage amplifier. The transmission of this graph is insensitive to changes in the stage gain t(s) at the operating point t = 1. Hence s = 0 is the operating frequency. It follows from Eq. 5.43 that the transmission T of an n-stage graph of this type may be found from

$$(\frac{1}{T} - 1) = (\frac{1}{t} - 1)^n. \qquad (6.87)$$

Hence T(s) has poles where

$$(\frac{1}{t} - 1)^n = -1. \qquad (6.88)$$

Taking the nth root of Eq. 6.88, we find the critical values of t to be

$$t_m = \frac{1}{1 + e^{j(2m+1)\pi/n}} \; ; \; m = 0, \pm 1, \pm 2, \ldots \, . \qquad (6.89)$$

The location of points t_m in the t-plane is shown in Fig. 6.20. Nyquist

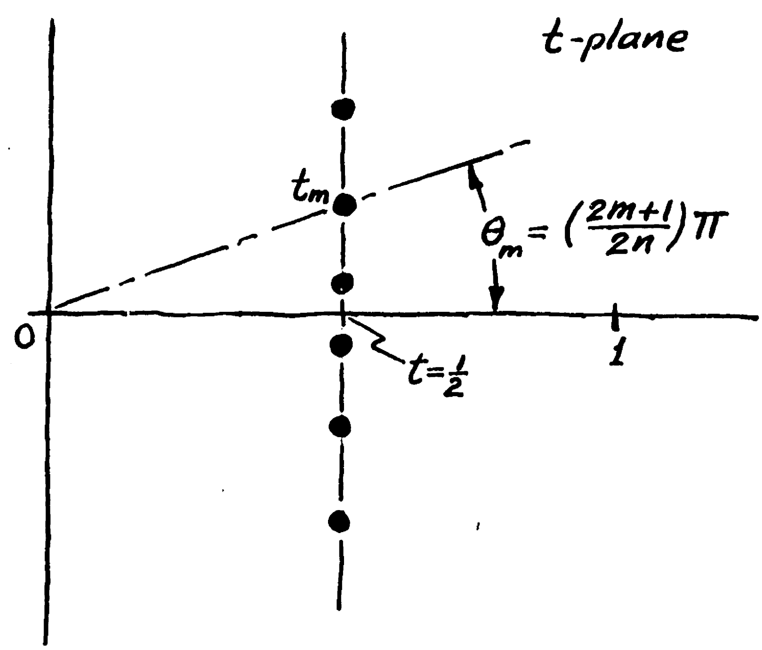


Fig. 6.20

plots of t(s) are made in Fig. 6.21. The arrows indicate the direction

t-plane

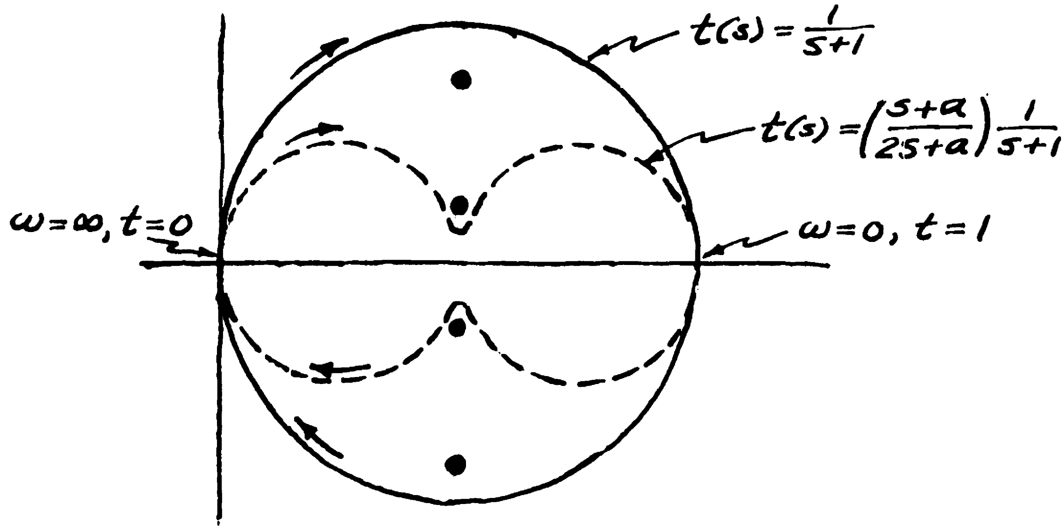


Fig. 6.21

of increasing ω. With two or more stages in the graph, critical points fall within the Nyquist plot of t(s) = 1/s+1. Hence t(s) takes on a critical value at some right half plane frequency and T(s) is unstable. By modifying the low frequency behavior of t(s) it is possible to alter the shape of the Nyquist plot as shown by the dashed locus in Fig. 6.21. One possible scheme of low frequency compensation yields

$$t(s) = \left(\frac{s+a}{2s+a}\right) \frac{1}{s+1} \quad . \tag{6.90}$$

This compensating factor reduces the phase angle of t(jω) in a certain region but leaves the high frequency asymptotic character of t(s) unchanged. The critical point closest to the real t axis falls at an angle

$$\theta_o = \pi/2n. \tag{6.91}$$

For a ≪ 1, the Nyquist plot of function 6.90 dips to a minimum phase angle of

$$\theta_{min} \cong \sqrt{2a} \quad \text{at } \omega_o \cong \sqrt{\frac{a}{2}} \tag{6.92}$$

at a point where the real part of $t(j\omega)$ is very nearly equal to one half. By the choice of a sufficiently small value of a, the plot may be made to exclude all critical points. For a large number of stages the useful bandwidth of $T(j\omega)$ may be taken as the phase-dip frequency ω_o. It follows from 6.91 and 6.92 that the maximum stable bandwidth obtainable with this particular type of compensation is given by the approximate expression

$$\omega_o \cong \frac{\pi}{4n} \, . \tag{6.93}$$

For comparison, a cascaded n-stage structure having no feedback and no compensation has an over-all bandwidth equal to

$$\omega_o' = \frac{1}{\sqrt{n}} \, . \tag{6.94}$$

Relation 6.93 points out a practical limitation upon the number of stages. By increasing the number of stages we may reduce the sensitivity, provided we are at the same time willing to sacrifice bandwidth.

With the aid of a more sophisticated compensation function it is perhaps possible to increase the gain-bandwidth product somewhat. Nevertheless, the example given here suggests some general relationship among gain, bandwidth, and sensitivity which requires that one be sacrificed if the others are to be enhanced. The establishment of such a relationship appears to be an end worthy of future investigation.

CHAPTER VII

CONCLUDING REMARKS

The flow graph approach offers a visual structure, a universal graphical language, a common ground, on the basis of which all analysis problems involving relationships among a number of variables may be laid out and compared. The similarity between two physical problems arises not from the arrangement of physical elements or the dimensions of the variables but rather from the structure of the set of relationships which we care to write. The challenge facing us at the start of an analysis problem is to express the pertinent relationships as a flow graph having simplicity and beauty. The degree to which this challenge is met is limited by our experience, judgement, and fundamental knowledge; in short, by our ability to perceive the problem. That poor perception of a problem may lead to an ugly flow graph is to be expected. Such ugliness usually fades with a few revisions of the graph as familiarity and confidence increase. It is hoped, however, and this hope is the very motivation for the work presented here, that a knowledge of flow graph techniques will enhance and extend our powers of perception, no matter how weak these powers may be at the beginning of a problem. An ugly flow graph is, perhaps, better than none. At the other extreme, a carefully composed flow graph, exhibiting to a trained glance the very essence of a problem, is, in the opinion of the writer, worth a thousand words or a hundred equations.

The underlying purpose of this paper is the exposition and illustration of a tool, rather than the statement of factual results. Many of the feedback theorems proven here are also proven in Bode's treatise. The use of positive stage feedback and over-all negative feedback for low distortion amplification is also mentioned in the literature [Ref:22,p.478]. The proofs given here, however, are novel, as are certain results. In particular, the general stability criterion advanced by Bode directs us to plot the Nyquist diagram of the loop difference of the jth tube (i.e. the jth g_m or μ) in a circuit, as computed with only tubes 1,2, ... , j active. The succession of such diagrams determines the stability of the circuit. For the circuit of Fig. 6.14, two plots would be required. The analysis presented here, however, needs but one plot, that of the loop difference of the cathode impedance Z_k.

Another new result is the use of conformality alone (curve sketching) to obtain stability information from a Nyquist plot. Previous methods depend upon the "principle of the argument," a theorem from the theory of functions of a complex variable, which states that the net number of clockwise encirclements of the origin by the Nyquist plot of T(s) measures the excess of zeros of T(s) over poles of T(s) in the right half plane. In recalling such a theorem from memory, it is all too easy to replace clockwise by counterclockwise or to interchange zeros and poles.

The flow graph concepts outlined in this paper have been presented in part, to graduate students in subject 6.633, Electronic Circuit Theory, at the Massachusetts Institute of Technology. The response has been, in general, favorable. The students appear to absorb the material of Chapters III and IV with some relish. Whether this is mass evidence of the usefulness of flow graphs or merely a symptom of the "new toy" complex, only time will reveal.

BIOGRAPHICAL SKETCH

Samuel Jefferson Mason was born in New York City on June 16, 1921. Two weeks later he moved to Runyon, N.J., a town of twenty-six inhabitants, three miles from the nearest store, policeman, or post office. He attended Willis Grammar School in Old Bridge, Perth Amboy High School, and Rutgers University, receiving the degree of Bachelor of Science in Electrical Engineering on May 12, 1942. On May 16th, 1942, he joined the staff of the Radiation Laboratory at M.I.T., becoming a member of the antenna group under Dr. L. C. Van Atta. In June and July of that year he attended the M.I.T. Radar School. Mr. Mason was engaged in microwave research and the development of microwave radar antennas until the Fall of 1945, when he entered graduate school in the Department of Electrical Engineering at M.I.T., working as a Research Assistant in the Research Laboratory of Electronics. His marriage, in September, 1944, to Miss Jean P. Stinson, of West Newton, was undoubtedly responsible for the inspiration which sustained him during the long hours of study toward a distant goal. The Master of Science degree was received in June, 1947, and a doctorate program was begun. On July 1, 1949, Mr. Mason was appointed an Assistant Professor of Electrical Engineering. Since 1945, he has been a staff member of the Research Laboratory of Electronics. He is a member of Tau Beta Pi, Sigma Xi, and Lambda Chi Alpha, the last being a general fraternity.

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