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(Received 18 October 2019; accepted 21 January 2020; published 26 February 2020)

In this paper, we prove a family of Lieb-Robinson bounds for discrete spin systems with long-range interactions. Our results apply for arbitrary k-body interactions, so long as they decay with a power law greater than kd, where d is the dimension of the system. More precisely, we require that the sum of the norm of terms with diameter greater than or equal to R, acting on any one site, decays as a power law $1/R^\alpha$, with $\alpha > d$. These bounds allow us to prove that, at any fixed time, the spatial decay of a time evolved operator follows arbitrarily closely to $1/r^\alpha$. Moreover, we introduce an alternative light cone definition for power-law interacting quantum systems which captures the region of the system where changing the Hamiltonian can affect the evolution of a local operator. In short-range interacting systems, this light cone agrees with the conventional definition. However, in long-range interacting systems, our definition yields a stricter light cone, which is more relevant in most physical contexts.

DOI: 10.1103/PhysRevA.101.022333

In a relativistic quantum field theory, information can never travel faster than the speed of light. A *Lieb-Robinson bound* [1–7] establishes a similar "light cone" for the spread of quantum information in a nonrelativistic discrete system. However, the information spread outside the light cone is not strictly vanishing but, instead, has nonzero tails. Such constraints on the spread of information, in addition to being physically important in their own right, have also been used as ingredients in the rigorous mathematical proof of key results about nonrelativistic discrete quantum systems [2–4,8–17], including the exponential decay of correlations in the ground states of gapped Hamiltonians [2,3] and the stability of topological order [4,13,14,16].

More recently, numerical and analytical works have investigated the existence of analogous Lieb-Robinson bounds in discrete spin systems where interactions *do not* have a finite range, but rather fall off as a power of the spin separation [3,18–28]. Such long-range interactions arise in a wide variety of experimental platforms, ranging from solid-state spin defects [29–31] to quantum optical systems of trapped ions [32], polar molecules [33], and Rydberg atoms [34]. While the majority of previous studies have focused on few-body physics, recent advances have enabled a number of these platforms to begin probing the many-body dynamics and information propagation of strongly interacting, long-range systems [20,35,36].

Motivated by the development of these physical platforms, in this paper, we improve Lieb-Robinson bounds for generic power-law interactions. Specifically, let us consider a system of spins on a set of sites Λ governed by a Hamiltonian H, which can be written as a sum $H = \sum_Z H_Z$ of terms acting on subsets of sites $Z \subseteq \Lambda$ in d-dimensional space. Moreover, we assume (among other conditions described in Sec. I A) that there exists a constant J such that

$$\sup_{z\in\Lambda} \sum_{Z\ni z:\mathrm{diam}(Z)\geqslant R} ||H_Z|| \leqslant \frac{J}{R^\alpha}, \tag{1}$$

where $\mathrm{diam}(Z)$ is the greatest distance between any two points in Z. A familiar example [37,38] is the long-range Ising interaction,

$$H = H_{\text{short-range}} + \tilde{J}\sum_{i\neq j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|^{d+\alpha}} \sigma_i^z \sigma_j^z. \tag{2}$$

An early result on Lieb-Robinson bounds in power-law interacting systems was proved in Ref. [3], which demonstrated the existence of a light cone whose size grows exponentially in time for any $\alpha > 0$. More recently, this result was improved in Refs. [21,24], where it was shown that a *power-law* light cone emerges for $\alpha > d$, where d is the spatial dimension.

However, each of these results has certain limitations (Table I). On the one hand, Ref. [21] assumes a two-body Hamiltonian, where each term acts on at most two spins [39]. This assumption limits the usage of this result in analyzing multibody effective Hamiltonians of broad interest

TABLE I. Summary of power-law Lieb-Robinson bounds for $\alpha > d$. Note that the LC1 and LC2 columns describe the power-law regime where these light cones exist and are power-law.

Reference	Multibody Hamiltonians	Asymptotic spatial decay	LC1	LC2
Ref. [21]	✗	$r^{-(\alpha+d)}$	$\alpha > d$	$\alpha > d$
Ref. [24]	✓	$r^{-(\alpha-d)/(\eta+1)}$	$\alpha > d$	$\alpha > 2d$
Our paper	✓	$r^{-\alpha}$	$\alpha > d$	$\alpha > d$

in condensed-matter physics. Such Hamiltonians can arise in a number of different contexts: for example, ring-exchange interactions may be important in solid ^{3}He [40] and are known to stabilize certain topological phases [41,42]; multi-body Hamiltonians arise in explicit constructions of various results in mathematical physics [9,13,14,17]; and higher-body interactions naturally emerge in the effective description of periodically driven two-body Hamiltonians [17,43].

On the other hand, while Ref. [24] overcomes this two-body assumption, it proves a significantly weaker result regarding the power-law decay of information outside the light cone (Table I) [44]. In particular, for $\alpha \gtrsim d$, the bounds of Ref. [24] ensure only a relatively slow decay outside the light cone, which can limit its applicability to some important results, e.g., bounding the difference in operators time evolved under slightly different Hamiltonians.

In this paper, we prove a Lieb-Robinson bound that addresses both of the above concerns. We demonstrate that, for multibody interactions with $\alpha > d$, the spatial decay of a time evolved operator, outside the light cone, scales arbitrarily closely to $\sim 1/r^\alpha$ (Table I). While this bound is not as strong as the $\sim 1/r^{\alpha+d}$ decay obtained in Ref. [21], our combination of an improved scaling (over Ref. [24]) and applicability to arbitrary multibody Hamiltonians enables the usage of this Lieb-Robinson bound to prove new results in mathematical physics [45].

An important comment is in order. Unlike either short-range or exponentially decaying interactions, power-law interactions are characterized by Lieb-Robinson bounds with power-law tails which lack a natural notion of a length scale. This implies that one must be particularly careful when defining an associated light cone for such long-range interacting systems. One possible definition of a light cone (used in Refs. [21,24]) is the following: at late times, the propagation of a local operator to any one point outside the light cone is small. From here on, we will refer to this as light cone 1 (LC1). For short-range interacting systems, LC1 is the only length scale associated with time evolution. For power-law interacting systems, one can already get a sense of the insufficiency of LC1 by noting the following: despite the differences between the asymptotic spatial decays obtained in Refs. [21,24] and this paper (Table I), they all yield the same LC1 (Table II).

To this end, we introduce a second light cone, LC2, which properly captures these differences. In particular, LC2 ensures that, at late times, the evolution of a local operator is not affected by changes to the Hamiltonian outside of LC2. For short-range interacting systems, LC1 and LC2 coincide, but for long-range interacting systems they can be quite different. More specifically, Ref. [21] exhibits a finite, power-law LC2 for $\alpha > d$, while Ref. [24] only has a finite LC2 for $\alpha > 2d$, despite both having the same power-law LC1. Intuitively, the lack of an LC2 for $2d > \alpha > d$ in Ref. [24] stems from the aforementioned slow asymptotic spatial decay of quantum information. This highlights the importance of our improved decay; it enables us to prove our second main result, which is the existence of a power-law LC2 for $\alpha > d$ for arbitrary multibody Hamiltonians (Table I) [46].

This paper is divided into two main sections. In Sec. I, we present an improved Lieb-Robinson bound for multibody long-range interacting systems. In Sec. I A, we introduce the necessary notation and assumptions used in its derivation. We state the final bound in Sec. I B, and present its detailed proof in Sec. I C. In Sec. II, we introduce the definition of an alternative light cone (LC2), discussing its differences from the light cone usually considered in the literature (LC1), as well as its physical motivation and how it relates to previous work. We conclude with a brief summary and discussion in Sec. III.

I. IMPROVED LIEB-ROBINSON BOUND

A. Assumptions and notation

Our notation will be similar to that of Ref. [24]. We consider a set of sites Λ with a metric $d(x, y)$ for $x, y \in \Lambda$, and a Hamiltonian H written as a sum of terms $H = \sum_Z H_Z$, where H_Z is supported on the set $Z \subseteq \Lambda$. We extend the notation of the metric to sets, denoting $d(X, Y)$ as the minimum distance between any two elements of the sets $X, Y \subseteq \Lambda$, as well as between sets and sites, denoting $d(X, y) = d(X, \{y\})$. We define a function $f(R)$ that captures the power-law decay of interactions:

$$f(R) := \sup_{z\in\Lambda} \sum_{Z\ni z:\mathrm{diam}(Z)\geqslant R} \|H_Z\|, \tag{3}$$

where

$$\mathrm{diam}(Z) = \sup_{x,y\in Z} d(x, y), \tag{4}$$

and we assume there are constants J and $\alpha > d$ (the dimensionality of the system) such that $f(R) \leqslant JR^{-\alpha}$. We also require that the sum of the operator norms of all of the terms involving any site be finite:

$$\mathcal{C}_0 := \sup_{x\in\Lambda} \sum_{y\in\Lambda} \sum_{Z\ni x,y} \|H_Z\| < \infty. \tag{5}$$

Finally, we assume certain conditions on the set of sites Λ and its metric. Specifically, we assume that Λ can be embedded in Euclidean space $\mathbb{R}^d$, so that for each $z \in \Lambda$ there is a corresponding $\mathbf{r}_z \in \mathbb{R}^d$, such that $d(x, y) = |\mathbf{r}_x - \mathbf{r}_y|$. Moreover, we assume there is a smallest separation a such that $d(x, y) \geqslant a$ for any $x, y \in \Lambda$ unless $x = y$. We choose to work in units such that $a = 1$. Despite an emphasis on this class of physically motivated sets of sites and metrics, the strategy and arguments developed in this paper should extend to more general Λ and $d(x, y)$, as in Ref. [24].

Let us also define $\tau_t^H(O)$ as the operator O time evolved according to the Heisenberg representation

$$\tau_t^H(O) = e^{itH}\, O\, e^{-itH}. \tag{6}$$

Throughout the paper we will use "C" to refer to any constants that depends only on σ (the parameter introduced in the statement of the theorem in the next section) and Λ. It will not necessarily be the same constant each time it appears.

B. Statement of the main result

Theorem 1. Given the assumptions stated in Sec. I A, let observables A and B be supported on sets X and Y, respectively. Then, for any $(d+1)/(\alpha+1) < \sigma < 1$,

$$\|[\tau_t^H(A), B]\| \leqslant \|A\|\|B\|\left\{2|X|e^{vt-r^{1-\sigma}} + C_1\frac{\mathfrak{G}(vt)}{r^{\sigma\alpha}}\right\}, \quad (7)$$

where $r = d(X, Y)$ and $v = C_2 \max(J, \mathcal{C}_0)$. Moreover, there exists a constant C_3 such that

$$\mathfrak{G}(\tau) \leqslant C_3(\tau + \tau^{1+d/(1-\sigma)})|X|^{n^*+2}, \quad (8)$$

where

$$n^* = \left\lceil \frac{\sigma d}{\sigma\alpha - d}\right\rceil. \quad (9)$$

Here, all C_i are constants only dependent on σ and Λ.

By choosing σ arbitrarily close to 1 we obtain a decay of the Lieb-Robinson bound that approaches $\sim r^{-\alpha}$ for large r.

C. Proof

1. Iteration procedure

The main challenge in understanding the spread of a local operator in long-range interacting systems is being able to differentiate the contribution from strong "short" range terms and the weak "long" range terms in a problem with no natural length scale. As a result, there is no single separation between short- and long-range terms of the Hamiltonian that yields a strict bound. To this end, we develop a construction that iteratively introduces a short scale [21,24], enabling us to better account for the spatial decay of interactions in the Hamiltonian and obtain an improved Lieb-Robinson bound.

As a starting point, we consider a truncated version of our long-range Hamiltonian with a cutoff R, $H^{\leqslant R}$:

$$H^{\leqslant R} = \sum_{Z:\mathrm{diam}(Z)\leqslant R} H_Z. \quad (10)$$

At the end of our construction we can make $R \to \infty$, recovering the full Hamiltonian. Because $H^{\leqslant R}$ has finite range R, a Lieb-Robinson bound for short-ranged Hamiltonians can be applied. However, this is clearly not the optimal bound, as it assumes all interactions of range up to R are equally strong, ignoring their decay with range. Nevertheless, this provides the starting point for our iterative process.

An outline of this procedure is as follows. At each iteration step, the Hamiltonian $H^{\leqslant R}$ is split into a short- and a long-range piece using a new cutoff R':

$$H^{\leqslant R} = H^{\leqslant R'} + H^{R';R} \quad (11)$$

$$\text{where} \quad H^{R';R} = \sum_{Z:R'<\mathrm{diam}(Z)\leqslant R} H_Z. \quad (12)$$

Then, following the strategy of Refs. [21,24], the time evolution of an operator A is separated into a contribution from the short-range part $H^{\leqslant R'}$ and the long-range part $H^{R';R}$. The role of these two terms can be intuited by considering the long-range part as a weak perturbation on top of the short-range part: under evolution via $H^{\leqslant R'}$ alone, the operator spreads with a linear light cone as per short-range Lieb-Robinson bounds, Fig. 1(a); the weak $H^{R';R}$ part then leads to a faster spreading by directly connecting this growing operator with the outside of its light cone, Fig. 1(b).

This picture is made precise in Lemma 3.1 in Ref. [24], where the total spread of the operator is bounded as a contribution from the short-range part $H^{\leqslant R'}$, as well as an *additional* contribution due to the long-range part $H^{R';R}$:

$$\|[\tau_t^{H^{\leqslant R}}(A), B]\| \leqslant \|[\tau_t^{H^{\leqslant R'}}(A), B]\| + 2\|B\|\int_0^t \|[\tau_{t-s}^{H^{\leqslant R'}}(A), H^{R';R}]\|ds. \quad (13)$$

This procedure enables us to better distinguish the contribution of the strong short-range terms and the weak long-range terms of the evolution, improving upon the initial naive bound. Once this iteration step is concluded and an improvement is obtained, one can perform the procedure again, further reducing the contribution from the long-range piece of Eq. (13) and improving the spatial decay of the Lieb-Robinson bound. We note this iterative process recovers the argument of Ref. [24] after one iteration; by iterating multiple times we can improve on those results. We make this iterative construction more precise with the following lemma.

Lemma 1. Fix a set $X \subseteq \Lambda$ and a time t. Suppose that we have a function $\lambda^{(R)}(r)$ such that for all $0 \leqslant s \leqslant t$, $Y \subseteq \Lambda$, and observables A and B supported on sets X and Y, respectively, the bound

$$\|[\tau_s^{H^{\leqslant R}}(A), B]\| \leqslant \lambda^{(R)}(d(X,Y))\|A\|\|B\| \quad (14)$$

is satisfied. We assume that $\lambda^{(R)}(r)$ is monotonically increasing in R and decreasing in r. Then, for any $R' > 0$, Eq. (14) is also satisfied with λ replaced by $\widetilde{\lambda}$, defined according to

$$\widetilde{\lambda}^{(R)}(r) = \lambda^{(R')}(r) + C\Theta(R - R')\,|X|\,t\,f(R')\mathcal{I}[\lambda^{(R')}], \quad (15)$$

where $f(R)$ is given in Eq. (3); C is a constant independent of R, R', $|X|$, and t; $\Theta(x)$ is the Heaviside theta function, and

$$\mathcal{I}[\lambda] = \lambda(0) + \int_{1/2}^{\infty} \rho^{d-1}\lambda(\rho)\,d\rho. \quad (16)$$

Proof. For $R' \geqslant R$, the result follows directly from the monotonicity with respect to R. On the other hand, for $R' < R$ we have, from Eq. (13),

$$\begin{aligned}
\|[\tau_t^{H^{\leqslant R}}(A), B]\| &\leqslant \|[\tau_t^{H^{\leqslant R'}}(A), B]\| + 2\|B\|\int_0^t \|[\tau_{t-s}^{H^{\leqslant R'}}(A), H^{R';R}]\|\,ds \\
&\leqslant \|[\tau_t^{H^{\leqslant R'}}(A), B]\| + 2\|B\|\int_0^t \sum_{Z:R'<\mathrm{diam}(Z)\leqslant R} \|[\tau_s^{H^{\leqslant R'}}(A), H_Z]\|\,ds \\
&\leqslant \lambda^{(R')}(d(X,Y))\|A\|\|B\| + 2t\|B\| \sum_{Z:R'<\mathrm{diam}(Z)\leqslant R} \lambda^{(R')}(d(X,Z))\|H_Z\|\|A\|
\end{aligned}$$

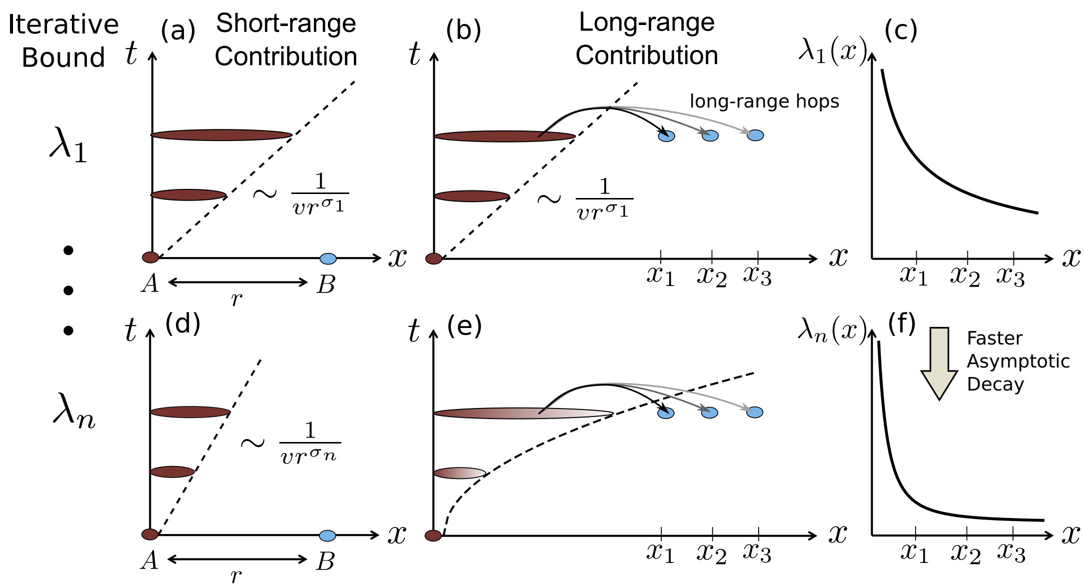

FIG. 1. The Lieb-Robinson bound captures the spread of quantum information during evolution by bounding the commutator of a time evolved local operator A, with another operator B a distance r away. The spread of the operator A can be apportioned into the spread due to interactions of range shorter than R' (left column) and long-range hops due to interactions of range larger than R' (center column). The long-range hops connect the short-range time evolved operator A with strength at most $f(R')$ but they can originate from any location that A has spread to, so the total contribution of these long-range hops is weighted by the integral $\mathcal{I}[\lambda]$ (see Lemma 1). At the first iterative step, which yields λ_1, the short-range interactions can always be characterized by an exponentially decaying bound with a sharp light cone with slope $(vR')^{-1}$, Eq. (21). This corresponds exactly to the short-range contribution to λ_1 (a). The long-range contribution arises from the long-range hops that connect the inside of the light cone to the support of B (b). By choosing the cutoff R' as a function of the operator distance, $R' = r^{\sigma_1}$, the resulting bound becomes the sum of exponential and power-law decaying terms, Eq. (28), the latter of which dominate the long-distance decay of the bound (c). This choice of R' leads to the light-cone slope of $(vr^{\sigma_1})^{-1}$ of panel (a). At the nth iteration step, which yields λ_n, we choose a new cutoff $\tilde{R}'$. As before, we obtain a short-range contribution that yields a linear light cone with slope $(v\tilde{R}')^{-1}$ (d). More importantly, the long-range hops will now be weighted by the power-law decay of the previous bound λ_{n-1}, illustrated by the dark shading (e). It is the combination of these two power-law decays that enables our iterative procedure to improve the asymptotic decay of the bound λ_n after specifying the cutoff as $\tilde{R}' = r^{\sigma_n}$ (f) (see Sec. III B). This choice of $\tilde{R}'$ leads to the light-cone slope of $(vr^{\sigma_n})^{-1}$ of panel (d).

$$
\begin{aligned}
&\leqslant \lambda^{(R')}(d(X,Y))\|A\|\|B\| + 2t\|A\|\|B\| \sum_{z\in\Lambda} \sum_{Z\ni z: R'<\mathrm{diam}(Z)\leqslant R} \lambda^{(R')}(d(X,z))\|H_Z\| \\
&\leqslant \lambda^{(R')}(d(X,Y))\|A\|\|B\| + 2t\|A\|\|B\| f(R') \sum_{z\in\Lambda} \lambda^{(R')}(d(X,z)) \\
&\leqslant \|A\|\|B\|\lambda^{(R')}(d(X,Y)) + 2\|A\|\|B\| t f(R')|X| \sup_{x\in X} \sum_{z\in\Lambda} \lambda^{(R')}(d(x,z)) \\
&\leqslant \|A\|\|B\|\lambda^{(R')}(d(X,Y)) + 2\|A\|\|B\| t f(R')|X|\mathcal{I}[\lambda^{(R')}].
\end{aligned} \tag{17}
$$

In going from the second to the third inequality, it is helpful to recall that $\lambda^{(R')}(d(X,Y))$ is independent of s (but dependent on t). In going from the fourth to the fifth inequality, we used

$$\sum_{z\in\Lambda} \sum_{Z\ni z, R'<\mathrm{diam}(Z)\leqslant R} \lambda^{(R')}(d(X,z))\|H_Z\| \tag{18}$$

$$= \sum_{z\in\Lambda} \lambda^{(R')}(d(X,z)) \sum_{Z\ni z, R'<\mathrm{diam}(Z)\leqslant R} \|H_Z\| \tag{19}$$

$$\leqslant f(R') \sum_{z\in\Lambda} \lambda^{(R')}(d(X,z)). \tag{20}$$

To obtain the final result, we have used Lemma 2 in Appendix A to replace the sum by an integral in the last inequality of Eq. (17).

Finally, let us emphasize that the simplest bound for $\lambda^{(R')}(d(X,Y))$ corresponds to the short-range Lieb-Robinson bound where the interactions have at most range R' and, thus, can always be used as the first term of Eq. (13). ■

We now iteratively apply Lemma 1. Equation (15) says that a Lieb-Robinson bound $\lambda^{(R)}$ for an interaction with maximum range R can be rewritten as the sum of two contributions: a Lieb-Robinson bound $\lambda^{(R')}$ for an interaction of maximum range R', which can be interpreted as the short-range part of the evolution; and an additional contribution due to long-range hops, which have range between R' and R and maximum strength $f(R')$. However, these "long-range hops" need not originate in the support of the original A itself but, rather, in the support of the time-evolved A under the short-range part of the interaction. This additional effect, depicted in Fig. 1, is captured by the $\mathcal{I}[\lambda^{(R')}]$ term.

At each iteration we replace the short-range contribution by the short-range Lieb-Robinson bound. We make use of the bound proven in Theorem A.1 of Ref. [24] which states that, for observables A and B supported on sets X and Y, respectively,

$$\left\|\left[\tau_t^{H^{\leqslant R'}}(A), B\right]\right\| \leqslant 2|X| \exp[vt - d(X,Y)/R']\|A\|\|B\|. \tag{21}$$

Finally, we are free to choose R' in Eq. (15). In particular, we choose it to be a function of r; specifically, at the nth iteration we take $R' = r^{\sigma_n}$, with $d/\alpha < \sigma_n < 1$. The resulting bound no longer depends on any cutoff R' and, when used again in Eq. (15), leads to a faster decaying $\mathcal{I}[\lambda]$ and an improved bound.

Therefore, at the nth iteration we obtain the bound

$$\left\|\left[\tau_s^{H^{\leqslant R}}(A), B\right]\right\| \leqslant \lambda_n^{(R)}(d(X,Y))\|A\|\|B\|, \tag{22}$$

where the iteration equation is

$$\lambda_n^{(R)}(r) = \Delta_r\left\{2|X|\exp[vt - r^{1-\sigma_n}] + C\Theta(R - r^{\sigma_n})|X|t\, f(r^{\sigma_n})\mathcal{I}\left[\lambda_{n-1}^{(r^{\sigma_n})}\right]\right\}, \tag{23}$$

where

$$\Delta_r(u) = \begin{cases} 2 & r < 1 \text{ or } u > 2 \\ u & \text{otherwise} \end{cases}. \tag{24}$$

This choice of Δ_r ensures that we always use the trivial bound on the commutator when $r = 0$ or when it is the most stringent bound. Now, it only remains to carry out the iteration.

2. Analyzing the iteration

To begin the iterative process we can invoke the generic Lieb-Robinson bound for finite-range Hamiltonians, as described in Eq. (21). Taking into account the trivial case,

$$\left\|\left[\tau_t^{H^{\leqslant R}}(A), B\right]\right\| \leqslant 2\|A\|\|B\|, \tag{25}$$

we begin the iteration with the initial bound:

$$\lambda_0^{(R)}(r) = \Delta_r(2|X|e^{vt - r/R}). \tag{26}$$

We then find (calculation in Appendix B1)

$$\mathcal{I}\left[\lambda_0^{(R')}\right] \leqslant C|X|[1 + (vtR')^d]. \tag{27}$$

Taking Eq. (23) and setting $R' = r^{\sigma_1}$, we have

$$\lambda_1^{(R)}(r) \leqslant \Delta_r\{2|X|e^{vt - r^{1-\sigma_1}} + C\Theta(R - r^{\sigma_1}) \times |X|^2 Jtr^{-\sigma_1\alpha}[1 + r^{\sigma_1 d}(vt)^d]\}, \tag{28}$$

which recovers the results in Ref. [24] with an appropriate choice of σ_1. From this point, we proceed by induction. Indeed, suppose at the nth iteration we have

$$\lambda_n^{(R)}(r) \leqslant \Delta_r\left[2|X|e^{vt - r^{1-\sigma_n}} + C\Theta(R - r^{\sigma_n})\sum_{i=1}^{2}\mathfrak{F}_i^{(n)}(vt)\, r^{\mu_i^{(n)}}\right]. \tag{29}$$

Note that, according to Eq. (28), this is satisfied for $n = 1$ if we take

$$\mu_1^{(1)} = \sigma_1(-\alpha + d), \tag{30}$$

$$\mu_2^{(1)} = -\sigma_1\alpha, \tag{31}$$

$$\mathfrak{F}_1^{(1)}(\tau) = C\tau^{d+1}|X|^2, \tag{32}$$

$$\mathfrak{F}_2^{(1)}(\tau) = C\tau|X|^2. \tag{33}$$

(Here we used the fact that $J/v \leqslant C$ given the definitions of these quantities.) Then, so long as $\mu_1^{(n)} + d > 0$ and $\mu_2^{(n)} + d < 0$ we have (computed in Appendix B 2)

$$\mathcal{I}\left[\lambda_n^{(R')}\right] \leqslant C\left\{|X|[1 + (vt)^{d/(1-\sigma_n)}] + \mathfrak{F}_2^{(n)}(vt)\,(vt)^{(d+\mu_2^{(n)})/(1-\sigma_n)} + \mathfrak{F}_1^{(n)}(vt)\,(R')^{(\mu_1^{(n)}+d)/\sigma_n}\right\}, \tag{34}$$

and therefore, using Eq. (23) and setting $R' = r^{\sigma_{n+1}}$,

$$\begin{aligned}\lambda_{n+1}^{(R)}(r) \leqslant \Delta_r\Big(&2|X|e^{vt - r^{1-\sigma_{n+1}}} \\ &+ C\Theta(R - r^{\sigma_{n+1}})|X|Jtr^{-\sigma_{n+1}\left(\alpha - \frac{\mu_1^{(n)}+d}{\sigma_n}\right)}\mathfrak{F}_1^{(n)}(vt) \\ &+ C\Theta(R - r^{\sigma_{n+1}})|X|Jtr^{-\sigma_{n+1}\alpha} \\ &\times \left\{|X|[1 + (vt)^{d/(1-\sigma_n)}] \right. \\ &\left. + \mathfrak{F}_2^{(n)}(vt)\,(vt)^{(d+\mu_2^{(n)})/(1-\sigma_n)}\right\}\Big).\end{aligned} \tag{35}$$

By choosing $\sigma_{n+1} \leqslant \sigma_n$ we ensure that the spatial decay of the exponential term does not increase in performing the iterative procedure.

So at the next iteration we have

$$\mu_1^{(n+1)} = \sigma_{n+1}\left[-\alpha + \left(\mu_1^{(n)} + d\right)/\sigma_n\right], \tag{36}$$

$$\mu_2^{(n+1)} = -\sigma_{n+1}\alpha, \tag{37}$$

$$\mathfrak{F}_1^{(n+1)}(\tau) = C\tau|X|\,\mathfrak{F}_1^{(n)}(\tau), \tag{38}$$

$$\mathfrak{F}_2^{(n+1)}(\tau) = C\tau|X|\left\{|X|[1 + \tau^{d/(1-\sigma_n)}] + \mathfrak{F}_2^{(n)}(\tau)\,\tau^{(d+\mu_2^{(n)})/(1-\sigma_n)}\right\}. \tag{39}$$

Iteratively applying Eq. (36) to the initial condition of Eq. (30) yields

$$\mu_1^{(n)} = \left(1 + \frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \dots \frac{1}{\sigma_{n-1}}\right)\sigma_n d - n\sigma_n\alpha. \tag{40}$$

At each iteration, $\mu_1^{(n)}$ is made smaller (i.e., more negative) at the cost of increasing the leading power of τ in $\mathfrak{F}_1^{(n)}(\tau)$, so long as $\mu_1^{(n)} > -d$. By choosing appropriate σ_j, we eventually reach an iteration step $n = n^*$ such that $\mu_1^{(n^*)} + d < 0$ and Eq. (34) no longer holds [and neither will the iteration equations Eqs. (36)–(39)]. For $n > n^*$, $\mathcal{I}[\lambda_n^{R'}]$ becomes independent of R':

$$\mathcal{I}\left[\lambda_{n\geqslant n^*}^{R'}\right] \leqslant C\left\{|X|[1 + (vt)^{d/(1-\sigma_n)}] + \mathfrak{F}_1^{(n)}(vt)\,(vt)^{(d+\mu_1^{(n)})/(1-\sigma_n)} + \mathfrak{F}_2^{(n)}(vt)\,(vt)^{(d+\mu_2^{(n)})/(1-\sigma_n)}\right\}, \tag{41}$$

which leads to new iterative steps where the spatial decays of both polynomial terms are the same:

$$\mu_1^{(n+1)} = \mu_2^{(n+1)} = -\sigma_{n+1}\alpha, \tag{42}$$

$$\mathfrak{F}_1^{(n+1)}(\tau) = \tau^{1+(d+\mu_1^{(n)})/(1-\sigma_n)}|X|\,\mathfrak{F}_1^{(n)}(\tau), \tag{43}$$

$$\mathfrak{F}_2^{(n+1)}(\tau) = \tau|X|\{|X|[1+\tau^{d/(1-\sigma_n)}] + \mathfrak{F}_2^{(n)}(\tau)\tau^{(d+\mu_2^{(n)})/(1-\sigma_n)}\}. \tag{44}$$

At this point in the iterative procedure, further iterations do not improve on the power-law decay of the Lieb-Robinson bound since they are set by $-\sigma_n\alpha$.

With regards to the time dependence of the bound, at each iteration step n, one can choose $\sigma_n > (1+d)/(1+\alpha)$, reducing the time dependence of $\mathfrak{F}_i^{(n)}(vt)$ in Eqs. (39), (43), and (44). For such choices of σ_n and enough iteration steps, the leading temporal dependence arises from the $\tau^{1+d/(1-\sigma_n)}$ term introduced in each iteration step in Eq. (44). As a result, there is some iteration number $m > n^*$ above which the most meaningful terms of the bound do not change. At this point, the bound $\lambda_m^{(R)}(r)$ is given by

$$\lambda_m^{(R)} \leqslant \Delta_r\big[2|X|e^{vt-r^{1-\sigma_m}} + C\Theta(R-r^{\sigma_m}) \times r^{-\sigma_m\alpha}\{|X|^2(vt)^{1+d/(1-\sigma_m)} + \dots\}\big], \tag{45}$$

where the ellipses denote terms with lower power in vt, but higher power in $|X|$.

We can make the previous considerations more concrete by analyzing the case where σ_j are all made equal, $\sigma_j = \sigma > (d+1)/(\alpha+1)$. This inequality ensures the reduction of the time dependence of $\mathfrak{F}_i^{(n)}(vt)$.

For this choice of σ_j, Eq. (40) simplifies to

$$\mu_1^{(n)} = (n-1+\sigma)d - n\sigma\alpha, \tag{46}$$

further leading to $n^* = \lceil \sigma d/(\sigma\alpha - d)\rceil$.

For $n > n^*$, the time dependence is encoded in

$$\mathfrak{F}_1^{(n)}(\tau) \sim \tau^{1+d/(1-\sigma)}[\tau^{[1+d-\sigma(1+\alpha)]/[1-\sigma]}]^{n-1} + \dots, \tag{47}$$

$$\mathfrak{F}_2^{(n)}(\tau) \sim \tau^{1+d/(1-\sigma)} + \dots \tag{48}$$

where ellipses correspond to lower power of τ. Then, $\mathfrak{F}_2(\tau)$ becomes the dominant term immediately for iteration step n^*+1 as the term $[\,\cdot\,]^{n-1}$ reduces the leading term of $\mathfrak{F}_1^{(n)}(\tau)$ to be smaller than $\mathfrak{F}_2^{(n)}(\tau)$. Because different terms have different dependences on $|X|$, to ensure all constants are independent of $|X|$, we include the largest power of $|X|$ emerging from our construction in front of the time dependence. Finally, taking $R \to \infty$ yields the final result as expressed in Theorem 1.

II. POWER-LAW LIGHT CONES

In short-range interacting systems, the length scale associated with the exponential decay of the Lieb-Robinson bound, Eq. (21), provides a natural definition for a light cone. In contrast, Lieb-Robinson bounds in long-range interacting systems are characterized by power-law spatial decays that lack a natural length scale [47]. As a result, the precise notion of a light cone will depend on which properties we wish to capture.

One way to define a light cone is in terms of the "spread of information": that is, suppose we consider the time evolution of two states $|\psi\rangle$ and $O|\psi\rangle$, where O is a local operator. The light cone is the region of radius $R_{\mathrm{LC1}}(t)$ around the support of O, outside which both time-evolved states yield nearly identical local observables. It is a direct measure of the spread of the influence of the perturbation O across the system as a function of time t. We refer to this light cone as LC1.

A different way to define a light cone is in terms of the region of the system that can affect the evolution of local observables appreciably. More specifically, consider the time evolution of an operator O under two different Hamiltonians, H and $H+\Delta H$. Intuitively, if ΔH only acts very far away from O, it will not have a significant impact on the evolution of O at short times. One can make this intuition precise and guarantee that the evolution of O does not change appreciably, until time t, if ΔH only acts a distance $R_{\mathrm{LC2}}(t)$ away from O. $R_{\mathrm{LC2}}(t)$ then characterizes the "zone of influence" of the evolution of operator O. We refer to this light cone as LC2. Strictly speaking, LC2 is not a light cone. However, this zone of influence is intimately connected with a *modified* notion of the past light cone. Our usual understanding of such a past light cone consists of all events (points in space-time) where acting with a *local operator* can influence the current event. The modified past light cone that is naturally associated with LC2 corresponds to all events where a change in the *Hamiltonian* can influence the current event. In long-range systems, these two light cones need not be equal, as even a local change to the Hamiltonian can affect the system nonlocally.

In general, in power-law interacting systems, LC2 will be greater than LC1. Intuitively, as the operator O expands outwards, the number of terms of ΔH it can interact with increases dramatically. As a result, it is not only necessary that the operator is mostly localized to a particular region, but also that the spatial profile of the operator spread decays fast enough to counteract the increasing number of terms that can modify its dynamics.

We now make these definitions more precise. In order to simplify the notation in this section, we write the Lieb-Robinson bound between two operators A and B, such that $d(A,B) = r$, and with at least one of $|A|$ or $|B|$ bounded by a constant C, as

$$\frac{\|[\tau_t^H(A), B]\|}{\|A\|\|B\|} \leqslant \mathcal{C}(r,t). \tag{49}$$

This allows us to formally define LC1 as the light cone used in previous literature.

Definition 1. Let LC1 be defined as a relation $r = f(t)$ such that

$$\lim_{t\to\infty} \mathcal{C}(f(t),t) = 0. \tag{50}$$

The meaning of LC1 is that the propagation of an operator outside the light cone is small and gets smaller as $t \to \infty$ [48]. Because we are interested in the asymptotic behavior, we focus on power-law light cones, $f(t) = t^\gamma$, which characterize the Lieb-Robinson bounds considered here. The smallest light cone is characterized by the exponent β^{LC1}, the infimum of the γ which satisfy Eq. (50).

In contrast we wish to define LC2 as the region outside which changing the Hamiltonian of the system has no significant impact on the evolution of the operator. To obtain a precise condition for LC2, we consider how changing the Hamiltonian H to $H + \Delta H$ impacts the evolution of an operator. More specifically, we consider modifying the Hamiltonian only a distance $r_{\min}$ away from the operator of interest O. In Appendix C, we show that the difference in the time evolved operators is bounded by

$$\|e^{iHt}Oe^{-iHt} - e^{i(H+\Delta H)t}Oe^{-i(H+\Delta H)t}\| \leqslant C\Delta J\|O\|t\int_{r_{\min}}^{\infty} dr\, r^{d-1}\mathcal{C}(r,t) \quad (51)$$

where ΔJ quantifies the local norm of ΔH.

LC2 is then given by the relationship between $r_{\min}$ and t that ensures that operator difference, bound in Eq. (51), remains small and goes to zero in the long-time limit. This immediately motivates the definition of LC2 as follows.

Definition 2. Let LC2 be defined as a relation $r = f(t)$ such that

$$\lim_{t\to\infty} t\int_{f(t)}^{\infty} dr\, r^{d-1}\mathcal{C}(r,t) = 0, \quad (52)$$

where d is the dimensionality of the system.

Again, we will focus on polynomial light cones, $f(t) = t^{\gamma}$, and define $\beta^{\rm LC2}$ as the infimum of the γ which satisfy Eq. (52).

In short-ranged interacting systems, where $\mathcal{C}(r,t) \propto e^{vt-r/R}$, the exponential suppression of $\mathcal{C}(r,t)$ at large r is insensitive to the extra volume term in the definition of LC2, Eq. (52), leading to the same linear light cone for both LC1 and LC2. This result is an immediate consequence of the natural length scale in $\mathcal{C}(r,t)$.

However, in long-range interacting systems, $\mathcal{C}(r,t)$ has a power-law decay in space which *is* sensitive to the extra volume term in LC2. For example, for Eq. (52) to converge and ensure a power-law LC2, the Lieb-Robinson bound must decay *faster* than r^{-d}; for LC1 there is no such requirement. As a result, for slowly decaying Lieb-Robinson bounds one may have a power-law LC1 but no LC2, i.e., there is no power-law $f(t)$ that satisfies Eq. (52). This is the case for the bound in Matsuta *et al.* [24], where LC2 does not exist for $d < \alpha < 2d$, yet LC1 matches that of Foss-Feig *et al.* [21]. LC2 is able to capture the difference between these two results.

By comparison, our result supports both an LC1 and LC2 for $\alpha > d$, extending the existence of an LC2 in long-range multibody Hamiltonians to $d < \alpha < 2d$. In this regime both our Lieb-Robinson bound and that of Ref. [21] lead to a finite

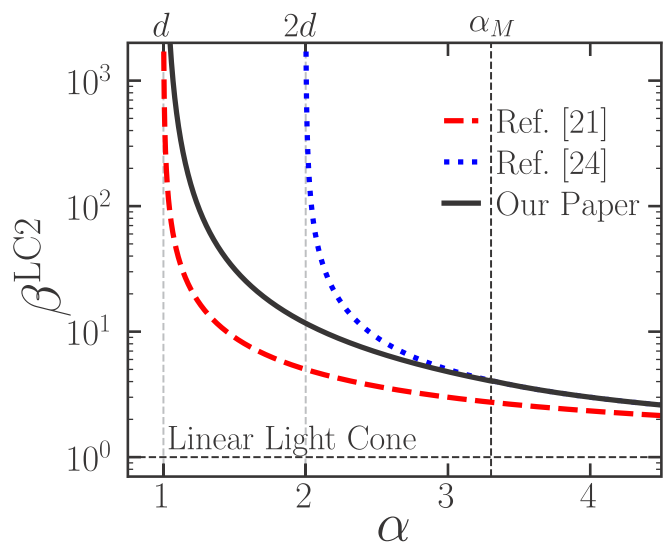

FIG. 2. Power-law LC2 exponent for the present paper and Refs. [21,24] for $d = 1$ as a function of α. While Ref. [24] has a finite power-law LC2 for $\alpha > 2d$, Ref. [21] and our paper have a power-law LC2 for all $\alpha > d$. For $\alpha < \alpha_M$ our paper leads to a better LC2 than Ref. [24], while matching it for $\alpha \geqslant \alpha_M$. The horizontal dashed line corresponds to a linear light cone. More details about the calculation can be found in Appendix D.

LC2, albeit our bound exhibits a larger light-cone exponent. Much like the difference in decay profile, this might be inherent to our treatment of the more general case of arbitrary multibody interactions.

In Table II and Fig. 2, we compare the different light-cone exponents obtained from both our paper and previous literature for different values of α. In Fig. 2, we plot the exponent of LC2 of the different works as a function of α for dimension $d = 1$. The general formulas for all space dimensions d are summarized in Table II and the details of the calculation can be found in Appendix D.

III. DISCUSSION

In this paper, we have proven an improved Lieb-Robinson bound for generic multibody long-range interactions, characterized by a faster asymptotic spatial decay. The importance of this improvement is captured by the notion of LC2, a definition of light cone that provides a stricter definition of locality for the growth of operators, in particular, that their

TABLE II. Summary of the power-law light-cone exponents of LC1 and LC2 for both previous literature and our paper. We use the subscripts FF and M to refer to the light-cone exponents from the bounds of Refs. [21] and [24], respectively. Here $\tilde{\beta} = \frac{2}{(\alpha-d)^2} \times [\alpha - d + \alpha d(1 + \sqrt{1 + 2/d - 2/\alpha})]$ and $\alpha_M = \frac{3d}{2}[1 + \sqrt{1 + \frac{8}{9d}}]$. For a detailed calculation see Appendix D.

Reference	LC1 ($d < \alpha$)	LC2 ($d < \alpha \leqslant 2d$)	LC2 ($2d < \alpha < \alpha_M$)	LC2 ($\alpha_M \leqslant \alpha$)
Ref. [21]	$\beta_{\rm FF}^{\rm LC1} = \frac{\alpha+1}{\alpha-d}$	$\beta_{\rm FF}^{\rm LC2} = \frac{\alpha+d}{\alpha}\frac{\alpha+1}{\alpha-d} + \frac{1}{\alpha}$	$\beta_{\rm FF}^{\rm LC2} = \frac{\alpha+d}{\alpha}\frac{\alpha+1}{\alpha-d} + \frac{1}{\alpha}$	$\beta_{\rm FF}^{\rm LC2} = \frac{\alpha+d}{\alpha}\frac{\alpha+1}{\alpha-d} + \frac{1}{\alpha}$
Ref. [24]	$\beta_{\rm M}^{\rm LC1} = \frac{\alpha+1}{\alpha-d}$	✗	$\beta_{\rm M}^{\rm LC2} = \frac{\alpha+2}{\alpha-2d}$	$\beta_{\rm M}^{\rm LC2} = \frac{\alpha+2}{\alpha-2d}$
Our paper	$\beta^{\rm LC1} = \frac{\alpha+1}{\alpha-d}$	$\beta_{\rm FF}^{\rm LC2} < \beta^{\rm LC2} = \tilde{\beta}$	$\beta_{\rm FF}^{\rm LC2} < \beta^{\rm LC2} = \tilde{\beta} < \beta_{\rm M}^{\rm LC2}$	$\beta^{\rm LC2} = \frac{\alpha+2}{\alpha-2d}$

evolution is not affected by the outside region for large t. Our paper extends the existence of an LC2 light cone for generic multibody interacting systems for $d < \alpha < 2d$.

This improvement has important implications for understanding prethermalization and Floquet phases of matter in periodically driven systems. In such systems (especially in the high-frequency regime), one can capture the evolution under a time-dependent Hamiltonian $H(t)$ using a time-independent approximation. Even when the original $H(t)$ has strictly two-body terms, the time-independent approximation will naturally exhibit multibody terms. The results which establish the accuracy and limitations of such approximations require Lieb-Robinson bounds for multibody power-law interactions with a rapid decay outside the light cone [45].

Note added. Recently, the authors became aware of a new Lieb-Robinson bound [49] that improves upon Ref. [21]. The bound in Ref. [49] has an LC1 exponent of $\alpha/(\alpha - d)$ under similar assumptions as Ref. [21], namely, two-body interactions. However, the authors' result (phrased in terms of commutators) does not yield a finite LC2 for $d < \alpha < 2d$. Nevertheless, the structure of their arguments is intriguing, and understanding how to generalize their results to multibody interactions is a promising direction for future study.

ACKNOWLEDGMENTS

We thank A. Gorshkov, G. Meyer, S. Michalakis, and C. Olund for helpful discussions. This paper was supported by the NSF (Grant No. PHY-1654740), the DARPA DRINQS Program (D18AC00033), and the A.P. Sloan Foundation. D.V.E. was supported by the Microsoft Corporation and by the EPiQS Initiative of the Gordon and Betty Moore Foundation, Grant No. GBMF4303.

APPENDIX A: TECHNICAL RESULTS

Lemma 2. Let $f(r)$ be a monotonically decreasing function of r, and fix an $x \in \Lambda$. Then

$$\sum_{\substack{z \in \Lambda : \\ a \leqslant d(z,x) \leqslant R}} f(d(z,x)) \leqslant \frac{C}{a^d} \int_{a/2}^{R} f(r) r^{d-1}\, dr, \tag{A1}$$

where a is the minimum separation between sites.

Proof. Around each site z, consider a ball B_z of radius $a/2$. Given our assumption that a is the smallest separation between sites, these balls are pairwise disjoint (up to sets of measure zero). Now, for any $\mathbf{r}$ in B_z, we have that $f(|\mathbf{r} - \mathbf{r}_x| - a/2) \geqslant f(d(z,x))$. Therefore,

$$V f(d(z,x)) \leqslant \int_{B_z} f(|\mathbf{r} - \mathbf{r}_x| - a/2)\, d^d\mathbf{r}, \tag{A2}$$

where V is the volume of the ball B_z. In the case that $d(x,z) < 3a/2$, we will use the tighter bound:

$$V f(d(z,x)) \leqslant \int_{B_z : |\mathbf{r}-\mathbf{r}_x| < a} f(|\mathbf{r} - \mathbf{r}_x|)\, d^d\mathbf{r} + \int_{B_z : |\mathbf{r}-\mathbf{r}_x| > a} f(|\mathbf{r} - \mathbf{r}_x| - a/2)\, d^d\mathbf{r}. \tag{A3}$$

Now using the fact that $\cup B_z \subseteq \mathbb{R}^d$, we find that

$$\sum_{\substack{z \in \Lambda : \\ a \leqslant d(z,x) \leqslant R}} f(d(z,x)) \leqslant \frac{C}{a^d} \left[\int_{a/2}^{a} r^{d-1} f(r)\, dr + \int_{a}^{R+a/2} r^{d-1} f(r - a/2)\, dr \right]. \tag{A4}$$

We can bound the second integral by

$$\int_{a}^{R+a/2} r^{d-1} f(r - a/2)\, dr \tag{A5}$$

$$= \int_{a/2}^{R} (u + a/2)^{d-1} f(u)\, du \tag{A6}$$

$$\leqslant C' \int_{a/2}^{R} u^{d-1} f(u)\, du. \tag{A7}$$

This immediately proves the lemma. ■

Lemma 3. For any μ and ν and positive ρ then the following inequality holds for a constant C independent of ρ:

$$\int_{\rho}^{\infty} e^{-x^{\nu}} x^{\mu}\, dx \leqslant C e^{-\rho^{\nu}} (1 + \rho^{\mu - \nu + 1}). \tag{A8}$$

Proof. It is sufficient to consider the case of $\nu = 1$, since we can reduce to this case by a change of variables. Let us first consider $\mu < 0$; then

$$\int_{\rho}^{\infty} e^{-x} x^{\mu}\, dx \leqslant \rho^{\mu} \int_{\rho}^{\infty} e^{-x}\, dx = \rho^{\mu} e^{-\rho}. \tag{A9}$$

We are now left with the case $\mu > 0$. In that case, if $\rho \leqslant 1$ then we can certainly bound the left-hand side of Eq. (A8) by

$$\int_{\rho}^{\infty} e^{-x} x^{\mu}\, dx \leqslant e^{1-\rho} \left[\int_{0}^{\infty} e^{-x} x^{\mu} \right] = C_1 e^{-\rho}. \tag{A10}$$

On the other hand, for $\rho \geqslant 1$, we have

$$\int_{\rho}^{\infty} e^{-x} x^{\mu}\, dx \tag{A11}$$

$$= e^{-\rho} \rho^{\mu} \int_{\rho}^{\infty} e^{-(x-\rho)} (x/\rho)^{\mu}\, dx \tag{A12}$$

$$= e^{-\rho} \rho^{\mu} \int_{0}^{\infty} e^{-u} (u/\rho + 1)^{\mu}\, du \tag{A13}$$

$$\leqslant e^{-\rho} \rho^{\mu} \int_{0}^{\infty} (u + 1)^{\mu} e^{-u}\, du \tag{A14}$$

$$= C_2 e^{-\rho} \rho^{\mu} \leqslant C_2 e^{-\rho} \rho^{\mu + \epsilon} \tag{A15}$$

for any $\epsilon > 0$. Adding both bounds with $C = \max(C_1, C_2)$ ensures it holds for all values of ρ. ■

APPENDIX B: CALCULATION OF $\mathcal{I}[\lambda]$

In this section we perform the calculation of $\mathcal{I}[\lambda_n^{(R')}]$, defined in Eq. (16) of the main text. We divide this calculation into two cases, $n = 0$ and $n > 0$, where $\lambda_n^{(R')}$ takes different functional forms.

1. $n = 0$ case

In the initial bound, given in Eq. (26) of the main text, one can define a light cone inside which the trivial bound is best, described by

$$|X|e^{vt-r/R'} = 1 \quad \Rightarrow \quad r = R'(\ln|X| + vt). \tag{B1}$$

As a result, one can bound $\mathcal{I}[\lambda_0^{(R')}]$ by the less stringent light cone $r = R'vt$ as follows:

$$\mathcal{I}\big[\lambda_0^{(R')}\big] = \lambda_0^{(R')}(0) + \int_{1/2}^{\infty} \rho^{d-1}\lambda_0^{(R')}(\rho)\,d\rho \tag{B2}$$

$$\leqslant 2 + 2\int_{1/2}^{R'vt} \rho^{d-1}\,d\rho + 2|X|\int_{R'vt}^{\infty} \rho^{d-1}e^{vt-\rho/R'}\,d\rho \tag{B3}$$

$$\leqslant 2\left\{1 + \frac{1}{d}(R'vt)^d + |X|e^{vt}e^{-vt}C[1+(R'vt)^{d-1}]\right\} \tag{B4}$$

$$\leqslant C[|X| + |X|(R'vt)^{d-1} + (R'vt)^d] \tag{B5}$$

where we made use of Lemma 3 to bound the second integral.

This bound can be made less stringent as follows:

$$\mathcal{I}\big[\lambda_0^{(R')}\big] \leqslant C|X|[1+(R'vt)^d]. \tag{B6}$$

This simplification leads to one less polynomial term in our iterative analysis but does not affect the spatial or temporal asymptotic behavior of the bound. In contrast, it increases $\mathfrak{F}_1$ by a factor of $|X|$ in our construction.

2. $n \geqslant 1$ case

For $n \geqslant 1$, the bound $\lambda_n^{(R')}(r)$ is composed of an exponential term and two polynomial terms, as described in Eq. (29) of the main text.

Similar to the calculation for $n = 0$, there exists a light cone inside of which the trivial bound is best. Such a light cone, in principle, will depend on the polynomial terms of the bound; however, it must be at least as big as the length scale of the exponential term of the bound given by

$$vt - r^{1-\sigma_n} = 0 \Rightarrow r = (vt)^{1/(1-\sigma_n)}. \tag{B7}$$

One can then bound $\mathcal{I}[\lambda_n^{(R')}]$ as

$$\mathcal{I}\big[\lambda_n^{(R')}\big] = \lambda_n^{(R')}(0) + \int_{1/2}^{\infty} \rho^{d-1}\lambda_n^{(R')}(\rho)\,d\rho \tag{B8}$$

$$\leqslant 2 + 2\int_{1/2}^{(vt)^{1/(1-\sigma_n)}} \rho^{d-1}\,d\rho + \int_{(vt)^{1/(1-\sigma_n)}}^{\infty} \rho^{d-1}\left[2|X|e^{vt-\rho^{1-\sigma_n}} + C\Theta(R'-\rho^{\sigma_n})\sum_{i=1}^{2}\mathfrak{F}_i^{(n)}(vt)\rho^{\mu_i^{(n)}}\right]d\rho \tag{B9}$$

$$\leqslant 2 + 2\int_{1/2}^{(vt)^{1/(1-\sigma_n)}} \rho^{d-1}\,d\rho + 2|X|\int_{(vt)^{1/(1-\sigma_n)}}^{\infty} \rho^{d-1}e^{vt-\rho^{1-\sigma_n}}\,d\rho + \sum_{i=1}^{2} C\int_{(vt)^{1/(1-\sigma_n)}}^{R'^{1/\sigma_n}} \rho^{d-1}\mathfrak{F}_i^{(n)}(vt)\rho^{\mu_i^{(n)}}\,d\rho \tag{B10}$$

$$\leqslant 2 + \frac{2}{d}(vt)^{d/(1-\sigma_n)} + 2|X|C[1+(vt)^{(d)/(1-\sigma_n)-1}] + C\sum_{i=1}^{2}\mathfrak{F}_i^{(n)}(vt)\left.\frac{\rho^{d+\mu_i^{(n)}}}{d+\mu_i^{(n)}}\right|_{(vt)^{1/(1-\sigma_n)}}^{(R')^{1/\sigma_n}}. \tag{B11}$$

The sign of $d + \mu_i^{(n)}$ becomes important in bounding the polynomial terms [50]: if $d + \mu_i^{(n)} > 0$, we can bound the term solely by the upper limit of integration; if $d + \mu_i^{(n)} < 0$, then we can bound using the lower limit. The final bound on $\mathcal{I}[\lambda_n^{(R')}]$ then becomes

$$\mathcal{I}\big[\lambda_n^{(R')}\big] \leqslant C[|X| + |X|(vt)^{d/(1-\sigma_n)-1} + (vt)^{d/(1-\sigma_n)}] + C\sum_{i=1}^{2}\mathfrak{F}_i^{(n)}(vt)\times\begin{cases}(R')^{(d+\mu_i^{(n)})/\sigma_n} & d+\mu_i^{(n)} > 0\\ (vt)^{(d+\mu_i^{(n)})/(1-\sigma_n)} & d+\mu_i^{(n)} < 0\end{cases}. \tag{B12}$$

This bound can be slightly simplified at the expense of a higher dependence of the $(vt)^{d/(1-\sigma_n)}$ term on $|X|$. Nevertheless, this simplification does not change the asymptotic spatial or temporal decay of our results:

$$\mathcal{I}\big[\lambda_n^{(R')}\big] \leqslant C|X|[1+(vt)^{d/(1-\sigma_n)}] + C\sum_{i=1}^{2}\mathfrak{F}_i^{(n)}(vt)\times\begin{cases}(R')^{(d+\mu_i^{(n)})/\sigma_n} & d+\mu_i^{(n)} > 0\\ (vt)^{(d+\mu_i^{(n)})/(1-\sigma_n)} & d+\mu_i^{(n)} < 0\end{cases}. \tag{B13}$$

APPENDIX C: BOUNDING THE OPERATOR DIFFERENCE UNDER TWO DIFFERENT HAMILTONIANS

Consider a local operator O, which is time evolved under two different Hamiltonians H_1 and H_2. Let us consider $\Delta H = H_1 - H_2$ such that it is only nonzero at sites outside some radius $r_{\min}$ around O and quantify its difference in terms of the largest local difference:

$$\sup_{x\in\Lambda}\sum_{Z:x\in Z}\|(H_1)_Z - (H_2)_Z\| \tag{C1}$$

$$= \sup_{x\in\Lambda}\sum_{Z:x\in Z}\|\Delta H_Z\| < \Delta J, \tag{C2}$$

where Λ is the set of sites of the system and $\|\cdot\|$ corresponds to the norm of the H_Z term.

The goal of this section is to bound how much O will differ when evolved under the two different Hamiltonians. In particular, we consider the following norm:

$$\big\|U_1^\dagger OU_1 - U_2^\dagger OU_2\big\| = \big\|O - U_1U_2^\dagger OU_2U_1^\dagger\big\| \tag{C3}$$

$$\text{where}\quad U_n = e^{-iH_nt}$$

where the time dependence of U_n is implicit to simplify the notation. Let us also note the similarities to results in Loschmidt echoes, where one evolves the system forwards with one Hamiltonian H_1 and then backwards with a slightly different Hamiltonian H_2 [51,52].

We begin by noting the following property:

$$f(t) = O - U_1(t)U_2^\dagger(t)OU_2(t)U_1^\dagger(t), \tag{C4}$$

$$\frac{d}{dt}f(t) = -iU_1(t)[U_2(t)OU_2^\dagger(t), \Delta H]U_1^\dagger(t) \tag{C5}$$

where we used the fact that $[U_n, H_n] = 0$. One can now bound the difference as

$$\begin{aligned}\|f(t)\| &= \left\| \int_0^t ds\, iU_1(s)[U_2(s)OU_2^\dagger(s), \Delta H]U_1^\dagger(s)\right\| \\ &\leqslant \int_0^t ds\, \|[U_2(s)OU_2^\dagger(s), \Delta H]\| \\ &\leqslant \int_0^t ds \sum_Z \|[U_2(s)OU_2^\dagger(s), \Delta H_Z]\|. \end{aligned}\tag{C6}$$

We now focus our attention to the inner sum. Because ΔH is only nonzero on sites at $r_{\min}$ away from the operator O we can bound

$$\sum_Z \|[U_2(s)OU_2^\dagger(s), \Delta H_Z]\| \tag{C7}$$

$$\leqslant \sum_{z:d(z,O)\geqslant r_{\min}} \sum_{\substack{Z:z\in Z,\\ d(Z,O)=d(z,O)}} \|[U_2(s)OU_2^\dagger(s), \Delta H_Z]\| \tag{C8}$$

$$\leqslant \sum_{z:d(z,O)\geqslant r_{\min}} \|O\| \sum_{\substack{Z:z\in Z\\ d(Z,O)=d(z,O)}} \|\Delta H_Z\|\, \mathcal{C}(d(O,z), s) \tag{C9}$$

$$= \|O\| \sum_{z:d(z,O)\geqslant r_{\min}} \mathcal{C}(d(O,z), s) \sum_{\substack{Z:z\in Z\\ d(Z,O)=d(z,O)}} \|\Delta H_Z\| \tag{C10}$$

$$\leqslant \|O\| \sum_{z:d(z,O)\geqslant r_{\min}} \mathcal{C}(d(O,z), s)\Delta J \tag{C11}$$

$$\leqslant C\|O\|\Delta J \int_{r_{\min}}^\infty dr\, r^{d-1}\mathcal{C}(r, s), \tag{C12}$$

where we used Lemma 3 to turn the sum into an integral.

Using the fact that $\mathcal{C}(r, s)$ is an increasing function in s we can obtain the final bound:

$$\|U_1^\dagger OU_1 - U_2^\dagger OU_2\| \leqslant C\|O\|\Delta Jt \int_{r_{\min}}^\infty dr\, r^{d-1}\mathcal{C}(r, t). \tag{C13}$$

APPENDIX D: CALCULATION OF LIGHT CONES

Our task in this section is to determine the LC1 and LC2 light cones for Refs. [21,24] and our paper. In order to simplify the notation, let us write the Lieb-Robinson bounds in terms of $\mathcal{C}(r, t)$ as defined in Eq. (49) of the main text. For the different results, $\mathcal{C}(r, t)$ contains a combination of exponential and power-law terms which need to be considered in determining LC1 and LC2.

Let us note that the iterative construction that leads to the bound in Theorem 1 of the main text depends on two parameters: σ, the scaling of the inner cutoff in the iterative procedure, and n, the number of iterations performed. While the fastest spatial decay occurs for $\sigma \to 1$, this does not necessarily lead to the smallest light cone, as the spatial decrease occurs at the expense of an increased growth in the temporal dependency. The same is true for the number of iterations n. As a result, one has to optimize both σ and n to find the smallest light cone.

1. LC1 for power-law interactions

a. Foss-Feig et al. [21] and Matsuta et al. [24]

The computation of LC1 for Refs. [21,24] is performed in those works, leading to a matching light-cone power law:

$$\beta_{\rm FF}^{\rm LC1} = \beta_{\rm M}^{\rm LC1} = \frac{\alpha+1}{\alpha-d}, \tag{D1}$$

where the subscripts FF and M refer to Refs. [21] and [24], respectively.

b. Our paper

As described in the main text, our proposed iterative construction matches the result of Ref. [24] for $n = 1$. As a result, for $n = 1$ we have $\beta^{\rm LC1} = \beta_{\rm M}^{\rm LC1}$.

We now can show that performing further iterative steps does not change the value of the LC1 exponent. In the iterative construction of our Lieb-Robinson bound, for $n > 1$ and $\sigma_j = \sigma$ for all j, we have

$$\mathcal{C}(r, t) \leqslant C\{e^{vt-r^{1-\sigma}} + \mathfrak{F}_1^{(n)}(vt)r^{\mu_1^{(n)}} + \mathfrak{F}_2^{(n)}(vt)r^{\mu_2^{(n)}}\}, \tag{D2}$$

where we have absorbed any $|X|$ dependence into the constant C as it does not affect the light cone calculation, and

$$\mathfrak{F}_i^{(n)}(\tau) = \tau^{\gamma_i^{(n)}} + \dots, \tag{D3}$$

where ellipses refer to lower power of τ. Because we are interested in the late time asymptotic form of the light cone we only need to focus on the largest power of τ. This exponent $\gamma_i^{(n)}$ is given by

$$\gamma_1^{(n)} = \begin{cases} d+n & n \leqslant n^* \\ \dfrac{\sigma\alpha + n[1+d-\sigma(1+\alpha)]}{1-\sigma} & n > n^* \end{cases}, \tag{D4}$$

$$\gamma_2^{(n)} = 1 + \frac{d}{1-\sigma} + \max\left[0, \frac{n-2}{1-\sigma}\{1+d-\sigma(1+\alpha)\}\right]. \tag{D5}$$

An important remark is that if $\sigma > (d+1)/(\alpha+1)$ then $1 + d - \sigma(1+\alpha) < 0$. In this regime, increasing n reduces $\gamma_1^{(n)}$ for $n > n^*$ and does not change γ_2.

The spatial decay is then given by

$$\mu_1^{(n)} = \begin{cases} -n\sigma\alpha + d(\sigma+n-1) & n \leqslant n^* \\ -\sigma\alpha & n > n^* \end{cases}, \tag{D6}$$

$$\mu_2^{(n)} = -\sigma\alpha. \tag{D7}$$

Each of the three terms (the exponential and the two polynomials) will lead to a LC1 exponent. The final exponent is the largest of the three for some n and σ. Optimizing over these two parameters yields the best β^{LC1}:

$$\beta^{\mathrm{LC1}}_{n;\exp} = \frac{1}{1-\sigma}, \tag{D8}$$

$$\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}} = \frac{\gamma_1^{(n)}}{-\mu_1^{(n)}}, \tag{D9}$$

$$\beta^{\mathrm{LC1}}_{n;\mathrm{poly2}} = \frac{\gamma_2^{(n)}}{-\mu_2^{(n)}}. \tag{D10}$$

Immediately, one can see that $\beta^{\mathrm{LC1}}_{n;\exp}$ is an increasing function of σ. At the same time $\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}}$ is a decreasing function of σ for any fixed n, as shown below. The intersection of the two curves provides the best LC1 exponent from the exponential and first polynomial term alone. This intersection occurs at $\sigma = \sigma_{1;\exp}$. If $\beta^{\mathrm{LC1}}_{n;\mathrm{poly2}}(\sigma_{1;\exp})$ is less than or equal to the other two curves at this point, it corresponds to the correct LC1 exponent.

We begin by showing that $\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}}$ is a decreasing function. If $n \leqslant n^*$,

$$\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}} = \frac{d+n}{\sigma(n\alpha - d) - (n-1)d}, \tag{D11}$$

which is a decreasing function of σ.

We now focus on the case $n > n^*$:

$$\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}} = \frac{\sigma\alpha + n[1+d-\sigma(1+\alpha)]}{(1-\sigma)\sigma\alpha}. \tag{D12}$$

First, let us note that n^* is a decreasing function of σ. Moreover, for this calculation to be meaningful we need

$$n > n^*(\sigma = 1) \Rightarrow n \geqslant n^*(\sigma = 1) + 1 \tag{D13}$$

$$= \left\lceil \frac{d}{\alpha - d} \right\rceil + 1 = \left\lceil \frac{\alpha}{\alpha - d} \right\rceil \geqslant \frac{\alpha}{\alpha - d}. \tag{D14}$$

We can now compute the derivative of $\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}}$ with respect to σ:

$$\frac{d}{d\sigma}\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}} = \frac{\alpha\sigma^2 - n[(1+d)(1-2\sigma) + \sigma^2(1+\alpha)]}{\alpha\sigma^2(1-\sigma)^2}.$$

Parametrizing $\alpha = d + \epsilon$ and $n = \alpha/(\alpha - d) + \delta = (d + \epsilon)/\epsilon + \delta$, with $\epsilon > 0$ and $\delta \geqslant 0$, we obtain

$$\frac{d}{d\sigma}\beta^{\mathrm{LC1}}_{n;\mathrm{poly1}} = -\frac{1}{\epsilon}\frac{1+d}{\sigma^2} - \delta\frac{(1+d)(1-\sigma)^2 + \sigma^2\epsilon}{(d+\epsilon)\sigma^2(1-\sigma)^2},$$

which is always negative. Since the function is continuous, and in both cases it is decreasing, it is always decreasing.

The intersection of the two curves then occurs as follows.

(1) If $n \leqslant n^*$,

$$\frac{d+n}{\sigma(n\alpha - d) - (n-1)d} = \frac{1}{1-\sigma}$$
$$\Rightarrow \sigma_{1;\exp} = \frac{d+1}{\alpha+1}. \tag{D15}$$

(2) If $n > n^*$,

$$\frac{\sigma\alpha + n[1+d-\sigma(1+\alpha)]}{\sigma\alpha(1-\sigma)} = \frac{1}{1-\sigma}$$
$$\Rightarrow \sigma_{1;\exp} = \frac{d+1}{\alpha+1}, \tag{D16}$$

which regardless of the regime occurs at the same value of σ, leading to

$$\beta^{\mathrm{LC1}}_{\exp;\mathrm{poly1}} = \frac{\alpha+1}{\alpha-d}. \tag{D17}$$

At the same time,

$$\beta^{\mathrm{LC1}}_{n;\mathrm{poly2}}(\sigma = \sigma_{1;\exp}) = \frac{\alpha+1}{\alpha-d} = \beta^{\mathrm{LC1}}_{\exp;\mathrm{poly1}} = \beta^{\mathrm{LC1}}, \tag{D18}$$

which corresponds to the best LC1 light cone for this bound (equal for any number of iterations), in agreement with the previous works [21,24].

2. Light cone 2 for power-law interactions

a. *Foss-Feig et al. [21]*

We can summarize the bound obtained in Ref. [21] as [53]

$$\mathcal{C}_{\mathrm{FF}}(r,t) = \exp\left[vt - \frac{r}{t^{\gamma}}\right] + \frac{t^{(\alpha+d)(1+\gamma)}}{r^{\alpha+d}}$$

$$\text{where} \quad \gamma = \frac{1+d}{\alpha-d}. \tag{D19}$$

One can immediately extract the light cone associated with LC2 for the two terms:

$$\beta^{\mathrm{LC2}}_{\mathrm{FF};\exp} = \frac{\alpha+1}{\alpha-d}, \tag{D20}$$

$$\beta^{\mathrm{LC2}}_{\mathrm{FF};\mathrm{poly}} = \frac{\alpha+d}{\alpha}\frac{\alpha+1}{\alpha-d} + \frac{1}{\alpha}. \tag{D21}$$

Since the latter is larger, it sets $\beta^{\mathrm{LC2}}_{\mathrm{FF}}$, which is valid for $\alpha > d$.

Let us note that because the Lieb-Robinson bound in Ref. [21] holds only for two-body interactions the calculation of LC2 also only holds for such Hamiltonians H_1. Moreover, because the bound is only valid for operators A and B which lie at a single site, the derivation in Sec. III needs to consider the size of each term H_z, leading to an overall extra factor of 2 (which can be absorbed into the constant C).

b. *Matsuta et al. [24]*

In analyzing Ref. [24], we can make use of the results obtained in our iterative procedure after a single iteration. Using Eq. (28) of the main text, we can immediately compute the exponent of the LC2 power-law light cone arising from each term of the bound:

$$\beta^{\mathrm{LC2}}_{n=1;\exp}(\sigma) = \frac{1}{1-\sigma}, \tag{D22}$$

$$\beta^{\mathrm{LC2}}_{n=1;\mathrm{poly1}}(\sigma) = \frac{d+2}{\sigma(\alpha-d)-d}, \tag{D23}$$

$$\beta^{\mathrm{LC2}}_{n=1;\mathrm{poly2}}(\sigma) = \frac{2}{\sigma\alpha - d} \tag{D24}$$

for $d/(\alpha - d) < \sigma < 1$. This condition immediately requires $\alpha > 2d$ for there to exist a power-law LC2. Having the exponents as a function of σ, $\beta^{\mathrm{LC2}}_{\mathrm{M}}$ is given by the optimized exponent with respect to σ:

$$\beta^{\mathrm{LC2}}_{\mathrm{M}} = \inf_{d/(\alpha-d)<\sigma<1}\Big[\max\Big(\beta^{\mathrm{LC2}}_{n=1;\exp}(\sigma),$$
$$\times\ \beta^{\mathrm{LC2}}_{n=1;\mathrm{poly1}}(\sigma), \beta^{\mathrm{LC2}}_{n=1;\mathrm{poly2}}(\sigma)\Big)\Big]. \tag{D25}$$

For all σ, $\beta^{\rm LC2}_{n=1;{\rm poly1}}(\sigma) > \beta^{\rm LC2}_{n=1;{\rm poly2}}(\sigma)$ and both are decreasing functions, while $\beta^{\rm LC2}_{\rm exp}$ is an increasing function. As a result, the minimum occurs at the intersection between $\beta^{\rm LC2}_{n=1;{\rm poly1}}$ and $\beta^{\rm LC2}_{\rm exp}$, which occurs at $\sigma = (2d+2)/(\alpha+2)$, leading to the light cone exponent:

$$\beta^{\rm LC2}_{n=1} = \frac{\alpha+2}{\alpha-2d} = \beta^{\rm LC2}_{\rm M}, \tag{D26}$$

for $\alpha > 2d$.

c. Our paper

Since we have considered the case of $n = 1$ in Appendix D 2 b, we now restrict our attention to $n > 1$. We will begin our calculation by focusing on the contribution from $\beta^{\rm LC2}_{n;{\rm exp}}$ and $\beta^{\rm LC2}_{n;{\rm poly2}}$ first and then confirming that the other polynomial term will not change the obtained exponent.

Based on the exponential term and the polynomial exponents in Eqs. (D5) and (D7) we obtain

$$\beta^{\rm LC2}_{n;{\rm poly2}} = \frac{1+\gamma_2^{(n)}}{-\mu_2^{(n)}-d} \tag{D27}$$

$$= \begin{cases} \frac{-d+2\sigma\alpha+n[1+d-\sigma(1+\alpha)]}{(1-\sigma)(\sigma\alpha-d)} & \sigma < \frac{d+1}{\alpha+1} \\ \frac{2(1-\sigma)+d}{(1-\sigma)(\sigma\alpha-d)} & \sigma \geqslant \frac{d+1}{\alpha+1} \end{cases}, \tag{D28}$$

$$\beta^{\rm LC2}_{n;{\rm exp}} = \frac{1}{1-\sigma}. \tag{D29}$$

Because $\beta^{\rm LC2}_{n;{\rm poly2}}$ is a convex function, the correct LC2 exponent $\beta^{\rm LC2}$ will occur in one of two regimes: at the minimum of $\beta^{\rm LC2}_{n;{\rm poly2}}$ or at the intersection of $\beta^{\rm LC2}_{n;{\rm poly2}}$ and $\beta^{\rm LC2}_{n;{\rm exp}}$ (i.e., at the first intersection of $\beta^{\rm LC2}_{n;{\rm poly2}}$ and $\beta^{\rm LC2}_{n;{\rm exp}}$; the second intersection occurs as $\sigma \to 1$, where both exponents become infinite).

The location of the minimum occurs at

$$\sigma_{2;\min} = 1 + \frac{d}{2} - \frac{d}{2}\sqrt{1+\frac{2}{d}-\frac{2}{\alpha}} \tag{D30}$$

$$\Rightarrow \quad \beta^{\rm LC2}_{n;{\rm poly2}}(\sigma_{2;\min}) = 2\,\frac{\alpha-d+d\alpha[1+\sqrt{1+2/d-2/\alpha}]}{(\alpha-d)^2}. \tag{D31}$$

The intersection, on the other hand, occurs at

$$\sigma_{2;\exp} = \frac{2d+2}{\alpha+2}$$

$$\Rightarrow \quad \beta^{\rm LC2}_{n;{\rm poly2}}(\sigma = \sigma_{2;\exp}) = \frac{\alpha+2}{\alpha-2d}, \tag{D32}$$

which requires, for consistency, $\alpha > 2d$. This exponent matches that of $n = 1$ and Ref. [24].

Because $\sigma_{2;\min}, \sigma_{2;\exp} > (d+1)/(\alpha+1)$, only the second branch of Eq. (D28) is relevant for this minimization procedure. This branch is independent of the number of iterations performed; the above results are valid for all $n > 1$.

If we now consider $\beta^{\rm LC2}_{n;{\rm poly1}}$ it can never improve on this minimization; it only worsens it. Moreover, by choosing $\sigma > (d+1)/(\alpha+1)$ and $n \geqslant n^* + 1$ iterations, one ensures that $\mathfrak{F}_2^{(n*+1)}$ contains the dominant asymptotic time dependence of the polynomial terms, ensuring that $\beta^{\rm LC2}_{n;{\rm poly2}} \geqslant \beta^{\rm LC2}_{n;{\rm poly1}}$. As a result, considering $\beta^{\rm LC2}_{n;{\rm poly1}}$ does not change our analysis of the LC2 exponent; it only imposes that $n \geqslant n^* + 1$.

Then, by choosing $n \geqslant n^* + 1$, we can immediately compute the LC2 exponent $\beta^{\rm LC2}$ by just considering $\beta^{\rm LC2}_{n;{\rm poly2}}$ and $\beta^{\rm LC2}_{n;{\rm exp}}$. In this regime, the exponents are independent of n, as shown above. There are two regimes that can determine $\beta^{\rm LC2}$.

(1) $\beta^{\rm LC2}$ occurs at the intersection of the curves $\beta^{\rm LC2}_{n;{\rm poly2}}(\sigma)$ and $\beta^{\rm LC2}_{n;{\rm exp}}$, which occurs at $\sigma = \sigma_{2;\exp}$. This requires that $\sigma_{2;\exp} \leqslant \sigma_{2;\min}$, which gives us a condition for α:

$$\alpha \geqslant \alpha_M \equiv \frac{3d}{2}\left(1+\sqrt{1+\frac{8}{9d}}\right), \tag{D33}$$

which is consistent with the requirement $2d < \alpha_M$ for the intersection solution to be meaningful.

(2) $\beta^{\rm LC2}$ occurs at the minimum of $\beta^{\rm LC2}_{n;{\rm poly2}}$, which occurs for $\alpha < \alpha_M$.

Thus, we can summarize our result as follows.

(1) If $d < \alpha < \alpha_M$,

$$\beta^{\rm LC2} = \frac{2}{(\alpha-d)^2}\{\alpha-d+d\alpha[1+\sqrt{1+2/d-2/\alpha}]\}. \tag{D34}$$

(2) If $\alpha_M < \alpha$,

$$\beta^{\rm LC2} = \frac{\alpha+2}{\alpha-2d}. \tag{D35}$$

Then, for $\alpha > \alpha_M$, our LC2 light cone matches that of Matsuta *et al.* [24], while for $\alpha < \alpha_M$ our iterative procedure ensures a better LC2 under similar assumptions. In fact, for $\alpha < 2d$, our LC2 is well defined while the LC2 of Matsuta *et al.* [24] diverges. However, our LC2 is bigger than that of a purely two-body power-law interacting system [21]. This situation is summarized in Table II of the main text, and the α dependencies of $\beta^{\rm LC2}$ are plotted for the Lieb-Robinson bounds of the present paper and Refs. [21,24] for $d = 1$ in Fig. 2 of the main text.

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