Equivalence of formulations of the MKP hierarchy and its polynomial tau-functions

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1 Introduction

The KP hierarchy was introduced by Sato in his seminal paper [15] as the hierarchy of evolution equations of Lax type

$$\frac{dL}{dt_n} = [(L^n)_+, L],\ n = 1, 2, \ldots,$$

on the pseudodifferential operator $L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \ldots$, where $\partial = \frac{\partial}{\partial t_1}$ and $+$ stands for the differential part. He also introduced the associated wave functions and the tau-function, and discussed reductions of the KP hierarchy. His ideas have been subsequently developed by his school in a series of papers, which were reviewed in [10] and [7].

In the review [10] Jimbo and Miwa also introduced the modified KP hierarchy (MKP hierarchy), as a set of bilinear equations on the tau-functions τ_ℓ, $\ell \in \mathbb{Z}$, see [10], eq.$(2.4)_{\ell,\ell'}$, each τ_ℓ being a tau-function of the KP hierarchy. It was subsequently shown in [13] that these equations arise naturally from the fermionic formulation of the MKP hierarchy and the boson-fermion correspondence. This implies that the MKP tau-functions $(\ldots, \tau_{\ell-1}, \tau_\ell, \tau_{\ell+1}, \ldots)$ are naturally parameterized by the infinite-dimensional flag manifold ([13], Corollary 8.1), in analogy with the famous observation of Sato [15] that tau-functions of the KP hierarchy are parametrized by the infinite-dimensional Grassmann manifold. Note that the tau-functions of the discrete KP hierarchy, studied in [2], are precisely those, satisfying the Jimbo-Miwa equations from [10].

On the other hand, Dickey proposed a Lax type formulation of the MKP hierarchy in [5] (see also [6]), which is an extension of the Sato formulation of KP. The first result of the present paper is the equivalence of Jimbo-Miwa's tau-function formulation and Dickey's Lax type formulation of the MKP hierarchy (Theorem 3 in Section 4), in analogy with the well developed theory of the KP hierarchy (see e.g. [10], [7]). Similar equivalences are established for the discrete KP hierarchy in [2]. The vertex operator construction of the Lie agebra $g\ell_\infty$ provides solutions to the tau-function formulation of the MKP hierarchy [13], hence to the Lax type formulation of it. Similar solutions have been constructed in [2] for the discrete KP hierarchy.

In Section 5 we give eigenfunction formulations of the MKP hierarchy, closely related to the work [9]. As a byproduct, we find in Section 6 an astonishingly simple explicit description of all polynomial tau-functions of the KP and the MKP hierarchies (Theorem 16). Of course, it is a well-known result of Sato [15] that all Schur polynomials are tau-functions of the KP hierarchy. We show that, moreover, all polynomial tau-functions of the KP hierarchy can be obtained from Schur polynomials by certain shifts of arguments.

We discuss in Section 7 the reductions of the MKP hierarchy to the modified n-KdV hierarchies for each integer $n \geq 2$, the $n = 2$ case being the classical modified KdV hierarchy (cf. [5]). Finally, in Section 8 we find all polynomial tau-functions for the n-KdV hierarchy, and (implicitly) for the modified n-KdV hierarchy. This was known only for $n = 2$ [13].

2 The fermionic formulation of MKP

Recall the semi-infinite wedge representation [13], [12]. Consider the infinite matrix group GL_∞, consisting of all complex matrices $G = (g_{ij})_{i,j\in\mathbb{Z}}$ which are invertible and all but a finite number of $g_{ij} - \delta_{ij}$ are 0. It acts naturally on the vector space $\mathbb{C}^\infty = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}e_j$ (via the usual formula $E_{ij}(e_k) = \delta_{jk}e_i$).

The semi-infinite wedge space $F = \Lambda^{\frac{1}{2}\infty}\mathbb{C}^\infty$ is the vector space with a basis consisting of all semi-infinite monomials of the form $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \ldots$, where $i_1 > i_2 > i_3 > \ldots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell >> 0$. One defines the representation R of GL_∞ on F by

$$R(G)(e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots) = Ge_{i_1} \wedge Ge_{i_2} \wedge Ge_{i_3} \wedge \cdots ,$$

and apply linearity and anticommutativity of the wedge product $\wedge$.

The corresponding representation r of the Lie algebra $g\ell_\infty$ of GL_∞ can be described in terms of a Clifford algebra. Define the wedging and contracting operators ψ_j^+ and ψ_j^- $(j \in \mathbb{Z} + \frac{1}{2})$ on F by

$$\psi_j^+(e_{i_1} \wedge e_{i_2} \wedge \cdots) = e_{-j+\frac{1}{2}} \wedge e_{i_1} \wedge e_{i_2} \cdots ,$$

$$\psi_j^-(e_{i_1} \wedge e_{i_2} \wedge \cdots) = \begin{cases} 0 & \text{if } j - \frac{1}{2} \neq i_s \text{ for all } s \\ (-1)^{s+1} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \cdots & \text{if } j = i_s - \frac{1}{2}. \end{cases}$$

These operators satisfy the relations $(i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)$:

$$\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda,-\mu}\delta_{i,-j},$$

hence they generate a Clifford algebra, which we denote by $\mathcal{C}\ell$. Introduce the following elements of F $(m \in \mathbb{Z})$:

$$|m\rangle = e_m \wedge e_{m-1} \wedge e_{m-2} \wedge \cdots . \tag{1}$$

It is clear that F is an irreducible $\mathcal{C}\ell$-module such that

$$\psi_j^\pm|0\rangle = 0 \text{ for } j > 0.$$

The representation r of $g\ell_\infty$ in F, corresponding to the representation R of GL_∞, is given by the formula $r(E_{ij}) = \psi^+_{-i+\frac{1}{2}}\psi^-_{j-\frac{1}{2}}$. Define the *charge decomposition*

$$F = \bigoplus_{m\in\mathbb{Z}} F^{(m)}, \qquad \text{where charge}(|m\rangle) = m \text{ and charge}(\psi_j^\pm) = \pm 1.$$

The space $F^{(m)}$ is an irreducible highest weight $g\ell_\infty$-module, with highest weight vector $|m\rangle$:

$$r(E_{ij})|m\rangle = 0 \text{ for } i < j, \quad r(E_{ii})|m\rangle = 0 \text{ (resp. } = |m\rangle) \text{ if } i > m \text{ (resp. if } i \le m).$$

Let

$$\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}$$

be the GL_∞-orbit of the highest weight vector $|m\rangle$.

Theorem 1 ([13], Theorem 5.1) *Let I be a non-empty finite subset of $\mathbb{Z}$ and let $f = \oplus_{m\in I} f_m \in \oplus_{m\in I} F^{(m)}$ be such that all $f_m \neq 0$. Then $f \in \oplus_{m\in I}\mathcal{O}_m$ if and only if for all $k, \ell \in I$, such that $k \ge \ell$, one has*

$$\sum_{i\in\mathbb{Z}+\frac{1}{2}} \psi_i^+ f_k \otimes \psi_{-i}^- f_\ell = 0\,. \tag{2}$$

Equation (2) is called the $(k-\ell)$-th modified KP hierarchy in the fermionic picture. The 0-th modified KP is the KP hierarchy. The collection of all such equations $k, \ell \in \mathbb{Z}$ with $k \ge \ell$ is called the (full) MKP hierachy in the fermionic picture.

3 The bosonic formulation of MKP

Define the fermionic fields by $\psi^\pm(z) = \sum_{i\in\mathbb{Z}+\frac{1}{2}} \psi_i^\pm z^{-i-\frac{1}{2}}$ and the bosonic field $\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} =: \psi^+(z)\psi^-(z):$. Then there exists a unique vector space isomorphism, called the boson-fermion correspondence, $\sigma : F \to B = \mathbb{C}[q, q^{-1}] \otimes \mathbb{C}[t_1, t_2, \ldots]$ such that $\sigma(|m\rangle) = q^m$, $\sigma\alpha_n\sigma^{-1} = \frac{\partial}{\partial t_n}$, $\sigma\alpha_{-n}\sigma^{-1} = nt_n$, for $n > 0$ and $\sigma\alpha_0\sigma^{-1} = q\frac{\partial}{\partial q}$. Moreover, one has

$$\sigma\psi^\pm(z)\sigma^{-1} = q^{\pm 1} z^{\pm q\frac{\partial}{\partial q}} \exp\left(\pm\sum_{k=1}^\infty t_k z^k\right) \exp\left(\mp\sum_{k=1}^\infty \frac{\partial}{\partial t_k}\frac{z^{-k}}{k}\right). \tag{3}$$

For $f_m \in \mathcal{O}_m \cup \{0\}$ we write: $\sigma(f_m) = \tau_m(t)q^m$, where $t = (t_1, t_2, \ldots)$. Such a τ_m is called a tau-function. Under the isomorphism σ we can rewrite (2), using (3), to obtain a Hirota bilinear identity for tau-functions

The first formulation of the MKP hierarchy:
Let $[z]=(z,\frac{z^2}{2},\frac{z^3}{3},\ldots)$, $y=(y_1,y_2,\ldots)$, *and* $\operatorname{Res}\sum_i f_i z^i dz = f_{-1}$, *then*

$$\operatorname{Res} z^{k-\ell}\tau_k(t-[z^{-1}])\tau_\ell(y+[z^{-1}])\exp\left(\sum_{i=1}^{\infty}(t_i-y_i)z^i\right)dz=0, \qquad k\ge\ell. \tag{4}$$

The equations (4) first appeared in [10], $(2.4)_{l,l'}$.

Divide (4) by $\tau_k(t)\tau_\ell(y)$ and introduce the wave functions w_m^+ and adjoint wave function w_m^- ($m\in\mathbb{Z}$) by

$$\begin{aligned} w_m^{\pm}(t,z)=&q^{\mp1}\frac{\sigma\left(\psi^{\pm}(z)f_m\right)}{\sigma(f_m)}\\ =&z^{\pm m}\frac{\tau_m(t\mp[z^{-1}])}{\tau_m(t)}e^{\pm t\cdot z}. \end{aligned} \tag{5}$$

Here and thereafter we use the shorthand notation

$$t\cdot z=\sum_{i=1}^{\infty}t_i z^i .$$

Then (4) becomes

$$\operatorname{Res} w_k^+(t,z)w_\ell^-(y,z)dz=0, \qquad k\ge\ell. \tag{6}$$

4 The Lax type formulation of MKP

We now want to express the wave functions in terms of formal pseudodifferential operators in $\partial=\frac{\partial}{\partial t_1}$. A formal pseudodifferential operator is an expression of the form

$$P(t,\partial)=\sum_{j\le N}P_j(t)\partial^j ,$$

where the $P_j(t)$ are functions in t, infinitely differentiable in t_1. The differential part of $P(t,\partial)$ is $P_+(t,\partial):=\sum_{j=0}^N P_j(t)\partial^j$, and $P_-:=P-P_+$. These operators form an associative algebra with multiplication $\circ$, defined by ($k,\ell\in\mathbb{Z}$)

$$A(t)\partial^k\circ B(t)\partial^\ell=\sum_{i=0}^{\infty}\binom{k}{i}\frac{\partial^i A(t)}{\partial t_1^i}B(t)\partial^{k+\ell-i}.$$

The formal adjoint of $P(t,\partial)$ is defined by the following formula:

$$(\sum_j P_j(t)\partial^j)^*=\sum_j(-\partial)^j\circ P_j(t).$$

The residue of $P(t,\partial)$ is $\operatorname{Res}_\partial P(t,\partial):=P_{-1}(t)$.

Let

$$P_m^{\pm}(t,\pm z) = \frac{\tau_m(t \mp [z^{-1}])}{\tau_m(t)} = 1 \pm p_1^{\pm}(t) z^{-1} + p_2^{\pm}(t) z^{-2} \pm \cdots , \tag{7}$$

so that

$$\begin{aligned} w_m^{\pm}(t,z) &= P_m^{\pm}(t,\pm z) z^{\pm m} e^{\pm t\cdot z} = P_m^{\pm}(t,\partial) \circ (\pm\partial)^{\pm m}(e^{\pm t\cdot z}) \\ &= P_m^{\pm}(t,\partial) \circ \partial^{\pm m} \circ \exp\left(\pm \sum_{i=2}^{\infty} t_i (\pm\partial)^i \right) (e^{\pm t_1 z}) . \end{aligned} \tag{8}$$

Then (4) is equivalent to

$$\operatorname{Res} P_k^{+}(t,z) z^k e^{t\cdot z} P_\ell^{-}(y,-z) z^{-\ell} e^{-y\cdot z} dz = 0. \tag{9}$$

The following lemma is crucial. It involves only the first variable t_1. When we use it, the variables $t_2, t_3, \ldots$ are seen as extra parameters.

Lemma 2 ([12], Lemma 4.1) *Let $P(t_1,\partial)$ and $Q(t_1,\partial)$ be two formal pseudo-differential operators, then*

$$\operatorname{Res} P(t_1,z) e^{t_1 z} Q(y_1,-z) e^{-y_1 z} dz = \operatorname{Res}_{\partial} P(t_1,\partial) \circ Q(t_1,\partial)^* \circ e^{u\partial}|_{u=t_1-y_1} .$$

Applying the lemma to the bilinear identity (6), while using the expression (8) for the wave functions, one deduces

$$P_k^{-}(t,\partial)^* = P_k^{+}(t,\partial)^{-1}, \quad \left(P_k^{+}(t,\partial) \circ \partial^{(k-\ell)} \circ P_\ell^{+}(t,\partial)^{-1}\right)_- = 0. \tag{10}$$

We obtain the Sato-Wilson equation

$$\frac{\partial P_k^{+}(t,\partial)}{\partial t_j} = \left(P_k^{+}(t,\partial) \circ \partial^j \circ P_k^{+}(t,\partial)^{-1}\right)_- \circ P_k^{+}(t,\partial) , \tag{11}$$

by differentiating (6) by t_j, using the first equation of (10) and then applying Lemma 2 (see e.g. [12], proof of Lemma 4.2) .

Introduce the Lax operator L_k by dressing ∂ by (the dressing operator) P_k^+:

$$L_k = L_k(t,\partial) = P_k^{+}(t,\partial) \circ \partial \circ P_k^{+}(t,\partial)^{-1}. \tag{12}$$

Differentiate (8) by t_j and apply the Sato-Wilson equation (11). This gives the following linear equation (= linear problem) for the wave function w_k^+ ($k \in \mathbb{Z}$):

$$L_k w_k^{+}(t,z) = z w_k^{+}(t,z), \quad \frac{\partial w_k^{+}(t,z)}{\partial t_j} = \left(L_k^j\right)_+ w_k^{+}(t,z) \tag{13}$$

and the adjoint wave function w_k^-:

$$L_k^* w_k^{-}(t,z) = z w_k^{-}(t,z), \quad \frac{\partial w_k^{-}(t,z)}{\partial t_j} = -\left(L_k^j\right)_+^* w_k^{-}(t,z) . \tag{14}$$

From (11) it is easy to deduce the Lax equations on L_k (see e.g. [12], Lemma 4.3):

$$\frac{\partial L_k}{\partial t_j} = \left[(L_k^j)_+, L_k\right]\ , \quad j = 1, 2, \ldots, \tag{15}$$

which are the compatibility conditions of the linear problem (13). From (7) we find that

$$P_k^+(t, \partial) = 1 - \partial(\log \tau_k(t))\partial^{-1} + \cdots ,$$

hence the second equation of (10) for $k = \ell + 1$ gives that

$$P_{\ell+1}^+(t,\partial)\circ\partial\circ P_\ell^+(t,\partial)^{-1} = \left(P_{\ell+1}^+(t,\partial)\circ\partial\circ P_\ell^+(t,\partial)^{-1}\right)_+ = \partial + \partial(\log(\tau_\ell(t)) - \partial(\log(\tau_{\ell+1}(t))\,,$$

and hence

$$P_{\ell+1}^+(t,\partial)\partial = (\partial + v_\ell(t))\circ P_\ell^+(t,\partial), \quad \text{where } v_\ell(t) = \partial\left(\log\frac{\tau_\ell(t)}{\tau_{\ell+1}(t)}\right). \tag{16}$$

This leads to another formulation of MKP, which was suggested by Dickey [5], [6]:

The second formulation of the MKP hierarchy:

Let $U = \mathbb{C}[u_i^{(n)}, v_j^{(n)} | i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}]$ be the algebra of differential polynomials in u_i and v_j, where $\partial u_i^{(n)} = u_i^{(n+1)}$, $\partial v_j^{(n)} = v_j^{(n+1)}$. Let $L_0(\partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2}\ldots \in U((\partial^{-1}))$ be a pseudo-differential operator. Then the MKP hierarchy is the following system of evolution equations in U ($j \in \mathbb{Z}_{\geq 1}$, $i \in \mathbb{Z}$):

$$\frac{\partial L_0(\partial)}{\partial t_j} = [(L_0(\partial)^j)_+, L_0(\partial)], \quad \frac{\partial v_i}{\partial t_j} = \left(L_{i+1}(\partial)^j\right)_+ \circ(\partial + v_i) - (\partial + v_i)\circ\left(L_i(\partial)^j\right)_+\,, \tag{17}$$

where $L_i(\partial)$ and $L_{-i}(\partial)$, for $i > 0$, are defined by

$$L_i(\partial) = (\partial + v_{i-1})\circ L_{i-1}(\partial)\circ(\partial + v_{i-1})^{-1}, \qquad L_{-i}(\partial) = (\partial + v_{-i})^{-1}\circ L_{1-i}(\partial)\circ(\partial + v_{-i})\,. \tag{18}$$

Theorem 3 *The first and the second formulation of MKP are equivalent.*

Proof. To prove that the first formulation implies the second, first note that, using the first formula of (16), one indeed gets that for $\ell > 0$:

$$\begin{aligned} L_\ell &= P_\ell^+ \circ\partial\circ(P_\ell^+)^{-1} = (\partial + v_{\ell-1})\circ P_{\ell-1}^+\circ\partial(P_{\ell-1}^+)^{-1}\circ(\partial + v_{\ell-1})^{-1} \\ &= (\partial + v_{\ell-1})\circ L_{\ell-1}\circ(\partial + v_{\ell-1})^{-1} \quad \text{and} \\ L_{-\ell} &= P_{-\ell}^+\circ\partial\circ(P_{-\ell}^+)^{-1} = (\partial + v_{-\ell})^{-1}\circ P_{1-\ell}^+\circ\partial\circ(P_{1-\ell}^+)^{-1}\circ(\partial + v_{-\ell}) \\ &= (\partial + v_{-\ell})^{-1}\circ L_{1-\ell}\circ(\partial + v_{-\ell})\,. \end{aligned} \tag{19}$$

Secondly, we show that the second equation of (17) holds. This follows from the Sato-Wilson equation (11). Indeed,

$$\begin{aligned} \frac{\partial P_{\ell+1}^+(t,\partial)}{\partial t_j} &= -\left(L_{\ell+1}(t,\partial)^j\right)_- \circ(\partial + v_\ell(t))\circ P_\ell^+(t,\partial) \\ &= \frac{\partial v_\ell(t)}{\partial t_j}P_\ell^+(t,\partial) - (\partial + v_\ell(t))\circ\left(L_\ell(t,\partial)^j\right)_- \circ P_\ell^+(t,\partial)\,, \end{aligned}$$

we deduce that

$$\begin{aligned}\frac{\partial v_\ell(t)}{\partial t_j} &= -\left(L_{\ell+1}(t,\partial)^j\right)_- \circ (\partial + v_\ell(t)) + (\partial + v_\ell(t)) \circ \left(L_\ell(t,\partial)^j\right)_- \\ &= -L_{\ell+1}(t,\partial)^j \circ (\partial + v_\ell(t)) + \left(L_{\ell+1}(t,\partial)^j\right)_+ \circ (\partial + v_\ell(t)) \\ &\qquad + (\partial + v_\ell(t)) \circ L_\ell(t,\partial)^j - (\partial + v_\ell(t)) \circ \left(L_\ell(t,\partial)^j\right)_+ \\ &= \left(L_{\ell+1}(t,\partial)^j\right)_+ \circ (\partial + v_\ell(t)) - (\partial + v_\ell(t)) \circ \left(L_\ell(t,\partial)^j\right)_+ .\end{aligned}$$

Here we have used that $L_{\ell+1}(t,\partial)^j \circ (\partial + v_\ell(t)) = (\partial + v_\ell(t)) \circ L_\ell(t,\partial)^j$.

To prove the converse, we use a result of Shiota [16], the Claim of Section 1.2. He shows that if L_0 satisfies the Lax equation (15), then $w_0^+(t,z)$ is uniquely determined by the linear problem (13), up to multiplication by elements of the form $1 + \sum_{i>0} a_i z^{-i}$, with $a_i \in \mathbb{C}$ or rather $P_0(t,\partial) = 1 + \sum_{i>0} w_i(t)\partial^{-i}$ is a unique solution up to multiplication from the right by elements of the form $1 + \sum_{i>0} a_i \partial^{-i}$, with $a_i \in \mathbb{C}$, of the equations

$$L_0 \circ P_0^+(t,\partial) = P_0^+(t,\partial) \circ \partial, \quad \frac{\partial P_0^+(t,\partial)}{\partial t_j} = P_0^+(t,\partial) \circ \partial^j - \left(L_0^j\right)_+ \circ P_0^+(t,\partial) .$$

Hence, $w_0^+(t) = P_0^+(t,\partial)e^{t\cdot z}$ satisfies (13) and thus is a wave function for L_0, so that $w_0^-(t) = (P_0^+(t,\partial))^{*-1}e^{-t\cdot z}$ is the adjoint wave function. For $i > 0$ let

$$\begin{aligned}P_i^+ &= (\partial + v_{i-1}) \circ (\partial + v_{i-2}) \circ \cdots \circ (\partial + v_0) \circ P_0^+, \\ P_{-i}^+ &= (\partial + v_{-i})^{-1} \circ (\partial + v_{1-i})^{-1} \circ \cdots \circ (\partial + v_{-1})^{-1} \circ P_0^+ ,\end{aligned}$$

and construct all other (adjoint) wave functions via

$$\begin{aligned} w_i^+ &= (\partial + v_{i-1})(w_{i-1}^+), & w_i^- &= (\partial + v_{i-1})^{*-1}(w_{i-1}^-), \\ w_{-i}^+ &= (\partial + v_{-i})^{-1}(w_{1-i}^+), & w_{-i}^- &= (\partial + v_{-i})^{*}(w_{1-i}^-) .\end{aligned} \tag{20}$$

By(17) and (18) these (adjoint) wave functions satisfy the linear problem (13). In order to show that the bilinear identity holds for the wave functions, we first prove that

$$\left(\partial^j P_k^+(t,\partial) P_\ell^{-*}(t,\partial)\right)_- = 0 \quad \text{for all} \quad k \geq \ell,\ j \geq 0. \tag{21}$$

We show this for $k \geq 0$ and $\ell < 0$ (all other cases are obvious):

$$\begin{aligned}\partial^j \circ P_k^+ P_\ell^{-*} &= \partial^j \circ (\partial + v_{k-1}) \circ \cdots \circ (\partial + v_0) \circ P_0^+ \circ (P_0^+)^{-1} \circ (\partial + v_{-1}) \circ \cdots \circ (\partial + v_\ell) \\ &= \partial^j \circ (\partial + v_{k-1}) \circ (\partial + v_{k-2}) \circ \cdots \circ (\partial + v_\ell).\end{aligned}$$

Using Lemma 2, we deduce from (21) that

$$\operatorname{Res} \frac{\partial^j w_k^+(s_1,t_2,t_3\cdots,z)}{\partial s_1^j} w_\ell^-(t_1,t_2,t_3,\ldots,z)dz = 0.$$

The second formula of (13) implies that

$$\operatorname{Res} \frac{\partial^{j_1+j_2+\cdots+j_n} w_k^+(s_1,t_2,t_3\cdots,z)}{\partial s_1^{j_1}\partial t_2^{j_2}\cdots\partial t_n^{j_n}} w_\ell^-(t_1,t_2,t_3,\ldots,z)dz = 0.$$

Using Taylor's formula we obtain the bilinear identity (6) for the wave function. The tau-functions τ_i are then obtained up to a scalar factor by the formula (see e.g. [12] eq. (111), which is a direct consequence of (7)):

$$\frac{\partial \log \tau_i(t)}{\partial t_j} = \operatorname{Res} z^j \left(\frac{\partial}{\partial z} - \sum_{k>0} z^{-k-1} \frac{\partial}{\partial t_k} \right) P_i^+(t, z) \,.$$

Hence, multiplying (6) by $\tau_k(t)\tau_\ell(y)$, we obtain the bilinear identities (4) for the tau-functions, which is the first formulation of MKP. Thus the two formulations are equivalent. □

The v_j are expressed in terms of the tau-functions via the second formula of (16). Using (7), we see that

$$P_0^\pm(t, \partial) = \sum_{i,j=0}^{\infty} \frac{S_i(\mp D)\tau_0}{\tau_0} \partial^{-i}, \text{ where } \sum_{i=0}^{\infty} S_i(D) z^i = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \frac{\partial}{\partial t_k} \right).$$

This and the fact that L_0 is given by (12), gives that the u_i can be calculated by the following formula

$$L_0(t, \partial) = \sum_{i,j=0}^{\infty} \frac{S_i(-D)\tau_0}{\tau_0} \partial^{1-i-j} \circ \frac{S_j(D)\tau_0}{\tau_0} \,.$$

Remark 4 *Dickey shows that all flows $\frac{\partial}{\partial t_k}$, defined by (17), commute ([5], Proposition 2.3). Hence (17) is an integrable system of compatible evolution equations in U.*

Remark 5 *The differential algebra U carries an automorphism S (commuting with ∂), defined by*

$$S(v_j) = v_{j+1}, \quad S(L) = (\partial + v) \circ L \circ (\partial + v)^{-1}.$$

The MKP hierarchy can be understood as the following system of partial differential-difference equations ($j = 1, 2, \ldots$)

$$\begin{cases} \frac{dL}{dt_j} = [(L^j)_+, L] \\ \frac{dv}{dt_j} = (S(L)^j)_+ \circ (\partial + v) - (\partial + v) \circ (L^j)_+ \,. \end{cases}$$

Here $L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \ldots$ and $v = v_0$.

5 Eigenfunction formulation of MKP

There is yet another formulation of MKP. It is given in terms of eigenfunctions and adjoint eigenfunctions of the Lax operators L_k.

Definition 6 *Let $L = L(t, \partial)$ be a pseudodifferential operator with coefficients in $\mathbb{C}(t_1, t_2, \ldots)$, where $\partial = \frac{\partial}{\partial t_1}$. An element $\phi \in \mathbb{C}(t_1, t_2, \ldots)$ is called an eigenfunction (resp. adjoint eigenfunction) for L if*

$$\frac{\partial \phi(t)}{\partial t_n} = (L^n)_+ (\phi(t)) \quad \left(resp.\ \frac{\partial \phi(t)}{\partial t_n} = -(L^n)^*_+ (\phi(t)) \right), \ n = 1, 2, \cdots . \tag{22}$$

Example 7 *Let $L = L(t, \partial)$ be a pseudodifferential operator and $w^+(t, z)$ (resp $w^-(t, z)$) satisfy*

$$\frac{\partial w^+(t,z)}{\partial t_j} = \left(L^j\right)_+ w^+(t,z), \ \left(resp.\ \frac{\partial w^-(t,z)}{\partial t_j} = -\left(L^j\right)^*_+ w^-(t,z) \right),$$

cf. (13) and (14). Then for each $f(z) \in \mathbb{C}((z^{-1}))$ the functions

$$q_f^{\pm}(t) = \mathrm{Res} f(z) w^{\pm}(t, z) dz \,, \tag{23}$$

are eigenfunctions (taking $+$) and adjoint eigenfunctions (taking $-$) for L. In particular if $L = \partial$, then

$$q_f^{\pm}(t) = \mathrm{Res} f(z) e^{\pm t \cdot z} dz \,,$$

are its (adjoint) eigenfunctions.

These (adjoint) eigenfunctions were used by Matveev and Salle [14] to construct new solutions of the KP equation from old ones. In fact we will prove later the following

Proposition 8 *If $\tau(t)$ is a tau-function, satisfying (4) for $k = \ell$, and $L = P^+ \circ \partial \circ (P^+)^{-1}$ is the corresponding Lax operator, where P^+ is given by (7), then $\phi^{\pm}(t)\tau(t)$ is again a tau-function, provided that $\phi^{\pm}(t)$ is an (adjoint) eigenfunction for L.*

We will show (see also [9]) that $\tau(t)$ and $\phi^{\pm}(t)\tau(t)$ satisfy the 1st modified KP hierarchy (4) for $k - \ell = 1$. The converse of this statement also holds, namely we have

Proposition 9 *Let $\tau_k(t)$ and $\tau_{k+1}(t)$ be KP tau-functions that satisfy (4) for $k - \ell = 1$. Then their ratio $\phi_k(t) = \frac{\tau_{k+1}(t)}{\tau_k(t)}$ is an eigenfunction for $L_k = P_k^+ \partial P_k^{+-1}$ and $\frac{1}{\phi_k(t)}$ is an adjoint eigenfunction for $L_{k+1} = P_{k+1}^+ \partial P_{k+1}^{+-1}$, where P_m^+ is given by (7).*

Proof. The tau-function formulation of the 1-st MKP hierarchy, i.e. (4) for $k - \ell = 1$ is equivalent to (see e.g. [11], Theorem 2.3 (c), for $l = 1$).

$$\mathrm{Res}\, z^{-1} \tau_k(t - [z^{-1}]) \tau_{k+1}(y + [z^{-1}]) \exp\left(\sum_{i=1}^{\infty} (t_i - y_i) z^i \right) dz = \tau_{k+1}(t) \tau_k(y). \tag{24}$$

Divide equation (24) by $\tau_{k+1}(t)\tau_k(y)$, to obtain:

$$\mathrm{Res}\, \phi_k(t)^{-1} w_k^+(t, z) \phi_k(y) w_{k+1}^-(y, z) dz = 1 \,. \tag{25}$$

Differentiate this equation by t_n and then multiply by $\phi_k(t)$, to obtain

$$\mathrm{Res}\left(-\frac{\partial\phi_k(t)}{\partial t_n}\phi_k(t)^{-1}w_k^+(t,z)+(L_k^n)_+\left(w_k^+(t,z)\right)\right)\phi_k(y)w_{k+1}^-(y,z)dz=0.$$

Using Lemma 2, (7), (8) and the fact that

$$w_{k+1}^-(y,z)=\frac{1}{\phi_k(y)}(-\partial)^{-1}\circ\phi_k(y)\left(P_k^+(y,\partial)\right)^{*-1}e^{-\sum_i y_iz^i},$$

we obtain

$$\left(-\frac{\partial\phi_k(t)}{\partial t_n}\phi_k(t)^{-1}P_k^+(t)\circ P_k^+(t)^{-1}\circ\phi_k(t)\partial^{-1}+(L_k^n)_+\circ P_k^+(t)\circ P_k^+(t)^{-1}\circ\phi_k(t)\partial^{-1}\right)_-=0\,.$$

Taking the residue of this expression (i.e. the coefficient of ∂^{-1}) gives equation (22). The second formula can be also obtained from (25) in almost the same way, but now one has to differentiate this equation by y_1 and continue in a similar manner. □

One also has

Proposition 10 *Let $\phi_k(t)$ be as in the previous Proposition and let $w_k^+(t,z)=P_k^+(t,z)z^ke^{t\cdot z}$ and $w_{k+1}^-(t,z)=P_{k+1}^-(t,-z)z^{-k-1}e^{-t\cdot z}$ be the (adjoint) wave function, corresponding to τ_k and τ_{k+1}, i.e., given by (7) and (8) satisfying (6) for $\ell=k+1$. Then*

$$P_{k+1}^+(t,\partial)\circ\partial=\phi_k(t)\partial\circ\frac{1}{\phi_k}(t)P_k^+(t,\partial) \tag{26}$$

and

$$L_{k+1}=\phi_k(t)\partial\circ\frac{1}{\phi_k(t)}L_k\circ\phi_k(t)\partial^{-1}\circ\frac{1}{\phi_k(t)}. \tag{27}$$

Proof. If we divide equation (24) by $\tau_k(t)\tau_{k+1}(y)$, we obtain

$$\mathrm{Res}\,w_k^+(t,z)\phi_k(y)w_{k+1}^-(y,z)dz=\phi_k(t)\frac{1}{\phi_k(y)}\,. \tag{28}$$

which is equivalent to (6). Using Lemma 2 and (10), we deduce that

$$P_k^+(t,\partial)\circ\partial^{-1}\circ P_{k+1}^+(t,\partial)^{-1}=\phi_k(t)\partial^{-1}\circ\frac{1}{\phi_k(t)},$$

which gives (26). Then (27) follows from (12). □

The converse also holds:

Proposition 11 *Let $\phi^+(t)$ be an eigenfunction and $\phi^-(t)$ be an adjoint eigenfunction for the Lax operator $L=P\partial P^{-1}$, i.e. L satisfies (15), where P is a dressing operator, satisfying the Sato-Wilson equation (11), then*

$$Q=\phi^+(t)\partial\circ\frac{1}{\phi^+(t)}P\quad \textit{and } R=\frac{1}{\phi^-(t)}\partial^{-1}\circ\phi^-(t)P$$

also satisfy (11) and both

$$Q \circ \partial \circ Q^{-1} \quad \text{and } R \circ \partial \circ R^{-1}$$

are Lax operators.

For a proof of this proposition, see pages 499 and 500 of [9].

Proof of Proposition 8. We will only consider the case of eigenfunctions. The proof for adjoint eigenfunctions is similar. Use the previous Proposition, then

$$\operatorname{Res} Q e^{t\cdot z}(P^*)^{-1} e^{y\cdot z} dz = \phi^+(t)\partial_{t_1} \circ \frac{1}{\phi^+(t)} \operatorname{Res} P e^{t\cdot z}(P^*)^{-1} e^{-y\cdot z} dz = 0\,.$$

Hence the wave function $Qe^{t\cdot z}$ and the adjoint wave function $(P^*)^{-1}e^{-y\cdot z}$ satisfy the 1-st modified KP hierarchy, (6) for $k = \ell + 1$. Therefore, $Pe^{t\cdot z}$ and $(Q^*)^{-1}e^{-y\cdot z}$ satisfy (25), i.e.,

$$\operatorname{Res} P e^{t\cdot z} Q^{*-1} e^{y\cdot z} dz = \frac{\phi^+(t)}{\phi^+(y)}\,.$$

Let τ be the tau-function which corresponds to P and τ_1 be the tau-function that corresponds to Q, then

$$\operatorname{Res} z^{-1}\tau(t - [z^{-1}])\tau_1(y + [z^{-1}]) \exp\left(\sum_{i=1}^{\infty}(t_i - y_i)z^i\right) dz = \tau(t)\phi^+(t)\frac{\tau_1(y)}{\phi^+(y)}\,,$$

which must be equation (24). Thus $\tau_1(t) = \phi^+(t)\tau(t)$. □

Define

$$\phi_k^+(t) = \phi_k(t) \quad \left(\text{resp. } \phi_k^-(t) = \frac{1}{\phi_{-k-1}}\right) \text{ for } k \geq 0\,,$$

which are eigenfunctions for L_k (resp. adjoint eigenfunctions for L_{-k}). Then by Proposition 9,

$$\phi_k^+(t) = \frac{1}{\phi_{-k-1}^-(t)} = \frac{\tau_{k+1}(t)}{\tau_k(t)}, \tag{29}$$

and (by (16) and Proposition 9)

$$\partial + v_k(t) = \begin{cases} \partial - \partial(\log \phi_k^+(t)) = \phi_k^+(t)\partial \circ \frac{1}{\phi_k^+(t)} & \text{for } k \geq 0, \\ \partial + \partial(\log \phi_{-k-1}^-(t)) = \frac{1}{\phi_{-k-1}^-(t)}\partial \circ \phi_{-k-1}^-(t) & \text{for } k < 0, \end{cases} \tag{30}$$

and

$$\begin{aligned} w_{k+1}^{\pm}(t,z) &= \pm(\phi_k^+(t)^{\pm 1}\partial^{\pm 1} \circ \phi_k^+(t)^{\mp 1})w_k^{\pm}(t,z), \\ w_{-k-1}^{\pm}(t,z) &= \pm(\phi_k^-(t)^{\mp 1}\partial^{\mp 1} \circ \phi_k^-(t)^{\pm 1})w_{-k}^{\pm}(t,z). \end{aligned} \tag{31}$$

It is clear that the first and the second formulation of MKP imply

The third formulation of the MKP hierarchy:

Let $W = \mathbb{C}[u_i^{(n)}, \phi_j^{\pm(n)} \mid i \in \mathbb{Z}_{\geq 1},\ j,n \in \mathbb{Z}_{\geq 0}]$ *be the algebra of differential polynomials in* u_i *and* $\phi_j^{\pm}$*, where* $\partial u_i^{(n)} = u_i^{(n+1)}$*,* $\partial \phi_j^{\pm(n)} = \phi_j^{\pm(n+1)}$*. Let* $L_0(\partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2}\ldots \in W((\partial^{-1}))$ *be a pseudo-differential operator. Then the MKP hierarchy is the following system of evolution equations in* W*:*

$$\frac{\partial L_0(\partial)}{\partial t_j} = [(L_0(\partial)^j)_+, L_0(\partial)], \quad \frac{\partial \phi_i^+}{\partial t_j} = \left(L_i(\partial)^j\right)_+ (\phi_i^+), \quad \frac{\partial \phi_i^-}{\partial t_j} = -\left(L_{-i}(\partial)^j\right)_+^* (\phi_i^-) \tag{32}$$

for $j \in \mathbb{Z}_{\geq 1}$ *and* $i \in \mathbb{Z}_{\geq 0}$*, where the* L_i *and* L_{-i}*, for* $i > 0$*, are defined by*

$$L_i = \phi_{i-1}^+ \partial \circ \frac{1}{\phi_{i-1}^+} L_{i-1} \circ \phi_{i-1}^+ \partial^{-1} \circ \frac{1}{\phi_{i-1}^+}, \quad L_{-i} = \frac{1}{\phi_{i-1}^-}\partial^{-1} \circ \phi_{i-1}^- L_{1-i} \circ \frac{1}{\phi_{i-1}^-}\partial \circ \phi_{i-1}^- .$$

Theorem 12 *All three formulations of the MKP are equivalent.*

Proof Assume the third formulation of MKP holds. Define for $i \geq 0$ the function $v_i = -\partial \log \phi_i^+$ and $v_{-i-1} = \partial \log \phi_i^-$. Then

$$w_{i+1}^+(t,z) = \phi_i^+(t)\partial \circ \frac{1}{\phi_i^+(t)}(w_i^+(t,z)) = (\partial + v_i(t))(w_i^+(t,z))$$

is a wave function for $L_{i+1} = (\partial + v_i(t))L_i(\partial + v_i(t))^{-1}$. One finds similar wave functions and relations between these wave functions if $i < 0$. Hence, the same proof as the proof of Theorem 3 gives the second equation of (17). Equation (18) is obvious.□

Now, for $i > 0$, the tau-function is equal to (by (29))

$$\tau_{\pm i} = \phi_{i-1}^{\pm}\tau_{\pm(i-1)} = \phi_{i-1}^{\pm}\phi_{i-2}^{\pm}\tau_{\pm(i-2)} = \cdots = \phi_{i-1}^{\pm}\phi_{i-2}^{\pm}\cdots\phi_0^{\pm}\tau_0 , \tag{33}$$

and the (adjoint) wave function $w_{\pm i}^{\pm}(t,z) = M_{\pm i}(t,\partial)\left(w_0^{\pm}(t,z)\right)$, where $M_0 = 1$ and by (31) and (30):

$$\begin{aligned} M_{\pm i}(t,\partial) &= (\pm\partial + v_{\pm(i-\frac{1}{2}\mp\frac{1}{2})}) \circ M_{\pm(i-1)}(t,\partial) \\ &= \pm\phi_{i-1}^{\pm}\partial \circ \frac{1}{\phi_{i-1}^{\pm}} M_{\pm(i-1)}(t,\partial) \\ &= \phi_{i-1}^{\pm}\partial \circ \frac{1}{\phi_{i-1}^{\pm}}\phi_{i-2}^{\pm}\partial \circ \frac{1}{\phi_{i-2}^{\pm}} M_{\pm(i-2)}(t,\partial) \\ &= \cdots \\ &= (\pm 1)^i \phi_{i-1}^{\pm}\partial \circ \frac{\phi_{i-2}^{\pm}}{\phi_{i-1}^{\pm}}\partial \circ \frac{\phi_{i-3}^{\pm}}{\phi_{i-2}^{\pm}}\partial \circ \cdots \circ \frac{\phi_0^{\pm}}{\phi_1^{\pm}}\partial \circ \frac{1}{\phi_0^{\pm}} , \end{aligned} \tag{34}$$

is an i-th order differential operator. Using the connection between the wave function and adjoint wave function we have , $w_{-i}^+(t,z) = M_{-i}^{*-1}(t,\partial)\left(w_0^+(t,z)\right)$ and using the relation between the wave function and the Lax operator (12), we find

$$L_i = M_i \circ L_0 \circ M_i^{-1} \text{ and } L_{-i} = (M_{-i}^*)^{-1} \circ L_0 \circ M_{-i}^* . \tag{35}$$

In the polynomial case, using the boson-fermion correspondence σ, it is not difficult to find these (adjoint) eigenfunctions. We know from the results of [13] that if $\sigma^{-1}(\tau_n q^n) = f_n \in \mathcal{O}_n$, then $\sigma^{-1}(\tau_{n+1} q^{n+1}) = w \wedge f_n$ for some $w = \sum_i a_i e_i \in \mathbb{C}^\infty$. We have

$$f_{n+1} = w \wedge f_n = \left(\sum_i a_i e_i \right) \wedge f_n = \sum_i a_i \psi^+_{-i+\frac{1}{2}}(f_n) = \text{Res} \sum_i a_i z^{-i} \psi^+(z)(f_n) dz \,,$$

since this holds for $f_n = |0\rangle$ and $f_{n+1} = |n+1\rangle$. Thus if we define $\phi_n^+(t) = \text{Res} \sum_i a_i z^{-i} w_n^+(t,z) dz$, then by (5) we find that

$$\begin{aligned} \tau_{n+1} q^{n+1} =& \sigma \left(\text{Res} \sum_i a_i z^{-i} \psi^+(z)(f_n) dz \right) \\ =& \text{Res} \sum_i a_i z^{-i} \sigma \psi^+(z) \sigma^{-1} dz \, \tau_n q^n \\ =& \text{Res} \sum_i a_i z^{-i} w_n^+(t,z) dz \, \tau_n q^{n+1} \\ =& \phi_n^+ \tau_n q^{n+1} \,, \end{aligned} \tag{36}$$

hence

$$\tau_{n+1} = \phi_n^+ \tau_n, \text{ where } \phi_n^+ = \text{Res} \sum_i a_i z^{-i} w_n^+(t,z) dz \,. \tag{37}$$

Since $f_{n-1} = \sum_i b_i \psi^-_{i+\frac{1}{2}}(f_n)$, with $b_i \in \mathbb{C}$, in a similar way we find

$$\tau_{-n-1} = \phi_n^- \tau_{-n}, \text{ where } \phi_n^-(t) = \text{Res} \sum_i b_i z^i w_{-n}^-(t,z) dz \,. \tag{38}$$

Thus we have the following

Lemma 13 *In the polynomial setting every (adjoint) eigenfunction is of the form (23).*

Observe that since $\phi_1^\pm = \text{Res}\, f(z) w_{\pm 1}^\pm(z) dz$ for some $f(z)$, we find that if we define $q_0^\pm = \phi_0^\pm$ and $q_1^\pm = \text{Res}\, f(z) w_0^\pm(z) dz$, which are both (adjoint) eigenfunctions of L_0, then using (31) we deduce that

$$\begin{aligned} \phi_1^\pm =& \text{Res}\, f(z) w_{\pm 1}^\pm(z) dz \\ =& \pm \text{Res}\, f(z) \phi_0^\pm \partial \left(\frac{w_0^\pm(z)}{\phi_0^+} \right) dz \\ =& \pm q_0^\pm \partial \left(\frac{q_1^\pm}{q_0^\pm} \right) \\ =& \pm \left(\partial(q_1^\pm) - \frac{q_1^\pm}{q_0^\pm} \partial(q_0^\pm) \right) . \end{aligned}$$

Thus

$$\tau_{\pm 2} = \phi_0^\pm \phi_1^\pm \tau_0 = \pm q_0^\pm \left(\partial(q_1^\pm) - \frac{q_1^\pm}{q_0^\pm} \partial(q_0^\pm) \right) \tau_0 = \pm \det \begin{pmatrix} q_0^\pm & q_1^\pm \\ \partial(q_0^\pm) & \partial(q_1^\pm) \end{pmatrix} \tau_0 \,.$$

Note that we can remove the possible minus sign in front of the determinant. If τ_2 is a tau-function, then a multiple of τ_2 is also a tau-function. From now on we will always do so, i.e. forget about the sign of the tau-function.

Using formula (34), we deduce that

$$M_{\pm 1} = \pm \phi_0^\pm \partial \circ \frac{1}{\phi_0^\pm}$$

and

$$\begin{aligned} M_{\pm 2} =& \phi_1^\pm \partial \circ \frac{\phi_0^\pm}{\phi_1^\pm} \partial \circ \frac{1}{\phi_0^\pm} \\ =& \frac{1}{q_0^\pm} \left(q_0^\pm \partial(q_1^\pm) - q_1^\pm \partial(q_0^\pm) \right) \partial \circ \frac{(q_0^\pm)^2}{q_0^\pm \partial(q_1^\pm) - q_1^+ \partial(q_0^\pm)} \partial \circ \frac{1}{q_0^\pm} \\ =& \left(\det \begin{pmatrix} q_0^\pm & q_1^\pm \\ \partial(q_0^\pm) & \partial(q_1^\pm) \end{pmatrix} \right)^{-1} \det \begin{pmatrix} q_0^\pm & q_1^\pm & 1 \\ \partial(q_0^\pm) & \partial(q_1^\pm) & \partial \\ \partial^2(q_0^\pm) & \partial^2(q_1^\pm) & \partial^2 \end{pmatrix} . \end{aligned}$$

Continuing in this way, see e.g. Theorem 5.1 of [9] for more details, it is possible to express $M_{\pm i}$ in terms of certain (adjoint) eigenfunctions $q_k^\pm(t)$ of the operator L_0, i.e. if

$$\phi_k^\pm = \mathrm{Res} f_k(z) w_{\pm k}^\pm dz \,,$$

for some $f_k(z) \in \mathbb{C}[z, z^{-1}]$, then we define

$$q_k^\pm = \mathrm{Res} f_k(z) w_0^\pm dz \,.$$

These $q_k^\pm(t)$ are (adjoint) eigenfumctions for $L_0(\partial)$ by (23). We have the following formulas:

$$\tau_{\pm i} = W_{\pm i} \tau_0 \text{ and } w_{\pm i}^\pm = M_{\pm i}(w_0^\pm) \text{ and } w_{-i}^+ = (M_{-i}^*)^{-1}(w_0^+), \tag{39}$$

where $M_{\pm i} = (\pm 1)^i W_{\pm i}(\partial)/W_{\pm i}$, and

$$W_{\pm i}(\partial) = \det \begin{pmatrix} q_0^\pm & \cdots & q_{i-1}^\pm & 1 \\ \partial(q_0^\pm) & \cdots & \partial(q_{i-1}^\pm) & \partial \\ \vdots & \ddots & \vdots & \vdots \\ \partial^i(q_0^\pm) & \cdots & \partial^i(q_{i-1}^\pm) & \partial^i \end{pmatrix} \text{ and } W_{\pm i} = \det \begin{pmatrix} q_0^\pm & \cdots & q_{i-1}^\pm \\ \partial(q_0^\pm) & \cdots & \partial(q_{i-1}^\pm) \\ \vdots & \ddots & \vdots \\ \partial^{i-1}(q_0^\pm) & \cdots & \partial^{i-1}(q_{i-1}^\pm) \end{pmatrix} \tag{40}$$

are Wronskian determinants. The determinants $W_{\pm i}(\partial)$ are computed by expanding along the last column, putting the cofactors to the left of the ∂^j's.

Let us prove the formulas of (39). If $\tau_{\pm i} = W_{\pm i}\tau_0$, then

$$\begin{aligned}
\tau_{\pm i \pm 1} &= \phi_i^{\pm}\tau_{\pm i} \\
&= \mathrm{Res}\, f_i(z) w_{\pm i}^{\pm} dz\, W_{\pm i}\tau_0 \\
&= \mathrm{Res}\, f_i(z) M_{\pm i}(w_0^{\pm}) dz\, W_{\pm i}\tau_0 \\
&= \mathrm{Res}\, f_i(z) W_{\pm i}(\partial)(w_0^{\pm}) dz\, \tau_0 \\
&= W_{\pm i}(\partial)(\mathrm{Res}\, f_i(z) w_0^{\pm} dz)\tau_0 \\
&= W_{\pm i}(\partial)(q_i^{\pm})\tau_0 \\
&= W_{\pm(i+1)}\tau_0\, .
\end{aligned}$$

Thus

$$\phi_i^{\pm} = \frac{W_{\pm(i+1)}}{W_{\pm i}},$$

and using this, we find that

$$\begin{aligned}
w_{\pm(i+1)}^{\pm} &= \pm\phi_i^{\pm}\partial \circ \frac{1}{\phi_i^{\pm}}(w_{\pm i}^{\pm}) \\
&= (\pm 1)^{i+1}\frac{W_{\pm(i+1)}}{W_{\pm i}}\partial \circ \frac{W_{\pm i}}{W_{\pm(i+1)}} \circ M_{\pm i}(w_0^{\pm}) \\
&= (\pm 1)^{i+1}\frac{W_{\pm(i+1)}}{W_{\pm i}}\partial \circ \frac{W_{\pm i}}{W_{\pm(i+1)}} \left(\frac{W_{\pm i}(\partial)(w_0^{\pm})}{W_{\pm i}}\right) \\
&= (\pm 1)^{i+1}\frac{W_{\pm(i+1)}(\partial)(w_0^{\pm})}{W_{\pm(i+1)}} \\
&= M_{\pm(i+1)}(w_0^{\pm}).
\end{aligned}$$

The next to the last equality follows from Crum's Identity for Wronskian determinants (which is in fact the Desnanot-Jacobi identity for Wronskians, see [4], section 3):

$$W_{\pm(i+1)}\partial \circ W_{\pm i}(\partial) - \partial(W_{\pm(i+1)})W_{\pm i}(\partial) = W_{\pm i}W_{\pm(i+1)}(\partial)\, . \tag{41}$$

Thus $w_{-i}^{+} = (M_{-i}^{*})^{-1}(w_0^{+})$. Now by (35) we find that

$$\begin{aligned}
L_i &= M_i \circ L_0 \circ M_i^{-1} = W_i(\partial)/W_i \circ L_0 \circ (W_i(\partial)/W_i)^{-1} \\
L_{-i} &= M_{-i}^{*-1} \circ L_0 \circ M_{-i}^{*} = (W_{-i}(\partial)/W_{-i})^{*-1} \circ L_0 \circ (W_{-i}(\partial)/W_{-i})^{*}\, .
\end{aligned} \tag{42}$$

Remark 14 *Let $i \geq 0$ and let $f_i = \sigma^{-1}(\tau_i(t)q^i)$. Then $f_i \in \mathcal{O}_i$, which means that*

$$f_i = v_i \wedge v_{i-1} \wedge \cdots \wedge v_2 \wedge v_1 \wedge f_0, \;\; \textit{where } v_j = \sum_s a_{sj}e_s,\; f_0 \in \mathcal{O}_0, \tag{43}$$

and the eigenfunctions of L_j are of the form

$$\phi_j^{+}(t) = \mathrm{Res}\, w_j^{+}(t,z)\sum_i a_{i,j+1}z^{-i}dz.$$

Hence, this eigenfunction is determined by $w_j^+(t,z)$ *and by* v_{j+1}. *Define*

$$q_j^+(t) = \operatorname{Res} w_0^+(t,z) \sum_i a_{i,j+1} z^{-i} dz.$$

Since M_i *is of the form (34),* $\phi_0^+(t) = q_0^+(t)$ *is in the kernel of* M_i. *However, if we reorder the* v_j*'s in (43) we get the same element up to a sign. This gives different eigenfunctions* ϕ_j^+ *and different* L_j *for* $j = 1, 2, \ldots, i-1$*, but* M_i *is the same and* L_i *is the same. Hence we can put every* v_j *in (43) just before* f_0*, which means that the new* $f_1 = v_j \wedge f_0$*, thus we get a new eigenfunction* ϕ_0^+ *which is now equal to* $q_j^+(t)$. *Moreover, if* $f_i \neq 0$*, then* $q_j^+(t) \neq 0$. *Thus* $q_0^+(t), q_1^+(t), \ldots, q_{i-1}^+(t)$ *are nonzero eigenfunctions for* L_0 *which are all in the kernel of* M_i*, and clearly must be linearly independent otherwise the element* f_i *would be 0. Similarly*

$$f_{-i} = v_{-i}(v_{1-i}(\cdots(v_{-2}(v_{-1}(f_0)\cdots),$$

where $v_j = \sum_i b_{ij} \psi^-_{i+\frac{1}{2}}$. *Then*

$$\phi^-_{j-1}(t) = \operatorname{Res} w^-_{1-j}(t,z) \sum_i b_{i,-j} z^i dz, \text{ and } \quad q^-_{j-1}(t) = \operatorname{Res} w_0^-(t,z) \sum_i b_{i,-j} z^i dz,$$

and all $q_j^-(t)$ *for* $0 \leq j < i$ *are in the kernel of* M_{-i}.

The fourth formulation of the MKP hierarchy:
Let $V = \mathbb{C}[u_i^{(n)}, q_j^{\pm(n)} | i \in \mathbb{Z}_{\geq 1},\ j,n \in \mathbb{Z}_{\geq 0}]$ *be the algebra of differential polynomials in* u_i *and* $q_j^{\pm}$. *Let* $L_0 = \partial + u_1(t)\partial^{-1} + \ldots \in V((\partial^{-1}))$ *be a pseudo-differential operator. Then the MKP hierarchy is the following system of evolution equations in* V*:*

$$\frac{\partial L_0(\partial)}{\partial t_j} = [(L_0(\partial)^j)_+, L_0(\partial)], \quad \frac{\partial q_i^+}{\partial t_j} = \left(L_0(\partial)^j\right)_+ (q_i^+), \quad \frac{\partial q_i^-}{\partial t_j} = -\left(L_0(\partial)^j\right)^*_+ (q_i^-). \tag{44}$$

Now we are able to prove the following

Theorem 15 *In the polynomial setting, all four formulations of MKP are equivalent.*

Proof. It suffices to establish the equivalence between the third and fourth formulation. To obtain the fourth formulation from the third, we use the fact that if $\phi_i^{\pm}(t)$ is given, then by Lemma 13 this (adjoint) eigenfunction for $L_{\pm i}$ is equal to

$$\phi_i^{\pm}(t) = \operatorname{Res} f^{\pm}(z) w^{\pm}_{\pm i}(z) dz \text{ for some } f(z) \in \mathbb{C}((z^{-1})).$$

Then we define the $q_i^{\pm}(t)$ of the fourth formulation by

$$q_i^{\pm}(t) = \operatorname{Res} f^{\pm}(z) w_0^{\pm}(z) dz \text{ for the same } f(z) \in \mathbb{C}((z^{-1})),$$

which now is an (adjoint) eigenfunction for L_0. This $q_i^{\pm}(t)$ for $i \geq 0$ is (by Remark 14) in the kernel of $M_{\pm j}$ (defined in (34)) for $j \geq i$, and since it is an (adjoint) eigenfunction for L_0, it satisfies the second (third) formula of (44). Hence this establishes the fourth formulation of MKP.

Assume the fourth formulation holds. Define $\phi^{\pm}_{\pm n} = (-1)^n \frac{W_{\pm n \pm 1}}{W_{\pm n}}$; together with L_0 they form the data of the third formulation. Since $q_i^{\pm}$ is an (adjoint) eigenfunction of L_0, then by Lemma 13 there exist functions $f_i^{\pm}(z) \in \mathbb{C}((z^{-1}))$, such that

$$q_i^{\pm}(t) = \operatorname{Res} f_i^{\pm}(z) w_0^{\pm}(t,z) dz \,. \tag{45}$$

Let τ_0 be the tau-function for L_0. Since $q_0^{\pm} = \phi_0^{\pm}$ is an (adjoint) eigenfunction of L_0, by Proposition 8, the tau-functions for $L_{\pm}$ are

$$\tau_{\pm 1} = W_{\pm 1}\tau_0 = \phi_0^{\pm}\tau_0 \,.$$

The corresponding (adjoint) wave functions are (by Propositions 10 and 11)

$$\begin{aligned} w_1^+(t,z) &= M_1(w_0^+(t,z)) = \phi_0^+(t)\partial \circ \frac{1}{\phi_0^+(t)}(w_0^+(t,z))\,, \\ w_{-1}^-(t,z) &= M_{-1}(w_0^-(t,z)) = -\phi_0^-(t)\partial \circ \frac{1}{\phi_0^+(t)}(w_0^-(t,z))\,, \end{aligned} \tag{46}$$

where $M_{\pm 1}$ is given by (39). The corresponding Lax operator $L_{\pm 1}$ is defined by (42), which is the same as $L_{\pm 1}$ in the third formulation, because of (46). Let

$$\begin{aligned} \phi_1^{\pm}(t) &= \operatorname{Res} f_1^{\pm}(z) w_{\pm 1}^{\pm}(t,z) dz \\ &= \pm \operatorname{Res} f_1^{\pm}(z) \frac{W_{\pm 1}(\partial)(w_0^{\pm}(t,z))}{W_{\pm 1}} \\ &= \pm \frac{W_{\pm 1}(\partial)(q_1^{\pm}(t))}{W_{\pm 1}} \\ &= \pm \frac{W_{\pm 2}}{W_{\pm 1}}\,, \end{aligned}$$

where $f_1^{\pm}(z)$ is given by (45), which are non-zero by Remark 14. Now, $\phi_1^{\pm}(t)$ is an (adjoint) eigenfunction for $L_{\pm 1}$, hence the second equation of (32) holds for $L_{\pm 1}$ and $\phi_1^{\pm}$. Thus (by (39)) we obtain that the tau-functions for $L_{\pm 2}$ are equal to

$$\tau_{\pm 2} = W_{\pm 2}\tau_0 = \frac{W_{\pm 2}}{W_{\pm 1}} W_{\pm 1}\tau_0 = \pm \phi_1^{\pm}\tau_{\pm 1}\,.$$

The corresponding (adjoint) wave functions are given by (39) and (40), and we have

$$w_2^+(t,z) = M_2(w_0^+(t,z)) = \frac{W_2(\partial)(w_0^+(t,z))}{W_2}\,.$$

By Crum's identity (41) we find that

$$\begin{aligned}
w_2^+(t,z) =& \frac{W_2}{W_1}\partial\circ\frac{W_1}{W_2}\left(\frac{W_1(\partial)(w_0^+(t,z)}{W_1}\right)\\
&= \phi_1^+(t)\partial\circ\frac{1}{\phi_1^+(t)}\circ M_1(w_0^+(t,z))\\
&= \phi_1^+(t)\partial\circ\frac{1}{\phi_1^+(t)}(w_1^+(t,z)),\\
w_{-2}^-(t,z) &= M_{-2}(w_0^-(t,z)) = -\phi_1^-(t))\partial\circ\frac{1}{\phi_1^-(t)}(w_{-1}^-(t,z))\,.
\end{aligned} \tag{47}$$

The corresponding Lax operator $L_{\pm 2}$ is defined by (42), which is the same as the one in the third formulation, because of (47). Let

$$\phi_2^\pm(t) = \mathrm{Res}\, f_2^\pm(z) w_{\pm 2}^\pm(t,z)dz = \mathrm{Res}\, f_2^\pm(z)\frac{W_{\pm 2}(\partial)(w_0^\pm(t,z))}{W_{\pm 2}}dz = \frac{W_{\pm 3}}{W_{\pm 2}}\,,$$

where again $f_2^\pm(z)$ is given by (45). This is again an (adjoint) eigenfunction for $L_{\pm 2}$ and hence the second equation of (32) holds for $L_{\pm 2}$ and $\phi_2^\pm$. Continuing along these lines gives the third formulation and hence we have proved that all four formulations are equivalent. □

6 Polynomial solutions of MKP

We are now going to construct polynomial tau-functions for MKP. We assume that $f_0 = |0\rangle$ which means that $\tau_0(t) = 1$, $w^\pm(t,z) = e^{\pm t\cdot z}$ and $L_0 = \partial$. We construct a $L_0 = \partial$ eigenfunction by the procedure described in Example 7 at the beginning of Section 5. Since $f_1 = w \wedge f_0 = w \wedge |0\rangle$ and the vacuum is given by (1), such a w can be chosen of the form $w = \sum_{j=0}^\infty a_j e_{j+1}$, thus the corresponding eigenfunction $q^+(t) = \tau_1(t)$ is of the form (see Example 7)

$$q^+(t) = \mathrm{Res} \sum_{j=0}^\infty a_j z^{-j-1} e^{t\cdot z} dz\,.$$

A similar construction is possible for the adjoint eigenfunction, in fact we have that all (adjoint) eigenfunctions are of the form

$$q_i^\pm(t) = \mathrm{Res}\, f_i^\pm(z) e^{\pm t\cdot z}dz,\ \text{for some}\ f_i^\pm(z) = \sum_{j=0}^\infty a_{ji}^\pm z^{-j-1}\,. \tag{48}$$

Since $\tau_0 = 1$ and $\tau_n = W_n\tau_0$ (see (39)), the corresponding tau-function is equal to $\tau_n = W_n$, for $n \in \mathbb{Z}$, the Wronskian determinant of the (adjoint) eigenfunctions. Now using the elementary Schur polynomials, which are defined by

$$e^{t\cdot z} = \sum_{j=0}^\infty s_j(t) z^j\,, \tag{49}$$

we find (see (48)) that

$$q_i^{\pm}(t) = \operatorname{Res} f_i^{\pm}(z)e^{\pm t\cdot z}dz = \sum_{j=0}^{\infty} a_{ji}^{\pm} s_j(\pm t)\,.$$

One obtains polynomial tau-functions by taking $f_i^{\pm}(z) = \sum_{j=0}^{M_i^{\pm}} a_{ji}^{\pm} z^{-j-1}$. To simplify notation we shall sometimes drop the superscrips $\pm$. Without loss of generality we may assume that $a_{M_i,i} = 1$, then

$$q_i^{\pm}(t) = s_{M_i}(\pm t) + \sum_{j=0}^{M_i-1} a_{ji}s_j(\pm t).$$

One can find recursively constants $c_i = (c_{1i}, c_{2i}, \dots, c_{M_i i})$, such that

$$q_i^{\pm}(t) = s_{M_i}(\pm t) + \sum_{j=0}^{M_i-1} a_{ji}s_j(\pm t) = s_{M_i}(\pm(t + c_i))\,. \tag{50}$$

Indeed, since, $s_{M_i}(t + c_i) = \sum_{j=0}^{M_i} s_j(c_i)s_{M_i-j}(t)$, which follows immediately from (49), one has to solve equations of the form $s_j(c_i) = a_{M_i-j,i}$ and this can be done recursively since $s_j(c_i) = c_{ji} + p_j(c_{1i}, \dots, c_{j-1,i})$, where p_j is some polynomial. First, determine $c_{1,i}$, which is determined by $a_{M_i-1,i}$, then $c_{2,i}$, which is determined by $a_{M_i-2,i}$ and c_{1i}, then c_{3i}, which is determined by $a_{M_i-3,i}$, c_{1i} and c_{2i}, etc. In fact there is an explicit formula for these constants. Since

$$1 + \sum_{j=1}^{M_i} a_{M_i-j,i}z^j = \sum_{j=0}^{M} s_j(c_i)z^j\,,$$

which is equal to the first $M_i + 1$ terms of $\exp(\sum_{j=1}^{M_i} c_{ji}z^j)$, the logarithm of this gives that

$$\sum_{\ell=1}^{M_i} c_{\ell i}z^{\ell} i + \text{ higher order terms} = \log\left(1 + \sum_{k=1}^{M_i} a_{M_i-k,i}z^k\right).$$

Hence

$$c_{ki} = -\sum_{\substack{m_1+2m_2+\cdots+km_k=k \\ m_1\ge 0, m_2\ge 0,\dots,m_k\ge 0}} \prod_{j=1}^{k} \frac{(-a_{M_i-j,i})^{m_j}}{m_j}\,.$$

Since $\tau_0 = 1$ and $\tau_{\pm n} = W_{\pm n}\tau_0$, we have (see (40) and (50)) that

$$\begin{aligned} \tau_{\pm n}(t) =& W(q_0^{\pm}(t), q_1^{\pm}(t), \dots, q_{n-1}^{\pm}(t)) \\ =& W\left(s_{M_0^{\pm}}(\pm t + c_0^{\pm}), s_{M_1^{\pm}}(\pm t + c_1^{\pm}), \dots, s_{M_{n-1}^{\pm}}(\pm t + c_{n-1}^{\pm})\right), \end{aligned} \tag{51}$$

where $W(\ )$ stands for the Wronskian determinant of those (adjoint) eigenfunctions, satisfies the KP hierarchy. This shows that every function of the form (51) is a polynomial tau-function. Moreover, one has the following remarkable

Theorem 16 *(a) All polynomial tau-functions of the KP hierarchy are, up to a constant factor, of the form*

$$\tau_{\lambda_1,\lambda_2,\ldots,\lambda_k}(t;c_1,c_2,\ldots,c_k)=\det\left(s_{\lambda_i+j-i}(t_1+c_{1,i},t_2+c_{2i},t_3+c_{3i},\ldots)\right)_{1\le i,j\le k}\ , \quad (52)$$

where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ *is a partition and* $c_i=(c_{1i},c_{2i},\ldots)\in\mathbb{C}^k$ *are arbitrary.*
(b) All polynomial tau-functions of the MKP hierarchy are the sequences $(\ldots,\tau_n,\tau_{n+1},\ldots)$*, where each* τ_n *is, up to a constant factor, of the form (52)), and* τ_{n+1} *is obtained from* τ_n*, up to a constant factor, in one of the following three possible ways:*

- $\tau_{\mu,\lambda_1,\lambda_2,\ldots,\lambda_k}(t;d,c_1,c_2,\ldots,c_k)$*, with* $\mu\ge\lambda_1$*;*
- $\tau_{\lambda_1-1,\lambda_2-1,\ldots,\lambda_i-1,\mu,\lambda_{i+1},\ldots\lambda_k}(t;c_1,c_2,\ldots,c_i,d,c_{i+1},\ldots c_k)$*, for* $i=1,2,\ldots k$*, with* $\lambda_i>\mu\ge\lambda_{i+1}$*;*
- $\tau_{\lambda_1-1,\lambda_2-1,\ldots,\lambda_k-1}(t;c_1,c_2,\ldots,c_k)$.

Here $d=(d_1,d_2,\ldots)$ *is a set of constants connected to the part* μ *of the partition, that appears in* τ_{n+1}*, in the first two cases. In the third case one has to delete* λ_j-1*'s and the corresponding* c_j*'s, whenever* λ_j-1 *is equal to* 0.

Proof. (a) First reorder the functions in (51) such that $M_0>M_1>M_2>\cdots>M_{k-1}$, which leaves the tau-function unchanged up to a sign. If one writes out (51), (cf. (40)), where q_i^+ is an elementary Schur function s_{M_i}, using that $\frac{\partial^\ell s_{M_i}}{\partial t_1^\ell}=s_{M_i-\ell}$, it is immediate to check that the the Wronskian matrix of (51) is the transposed of the matrix in:

$$\tau_k(t)=\det\left(s_{M_{i-1}+j-k}(t_1+c_{1,i},t_2+c_{2i},t_3+c_{3i},\ldots)\right)_{ij}\ . \quad (53)$$

Now, $\tau_n(t)$ is the image under the map σ in B of the following element of $F^{(0)}$ (cf. (50), where we remove the upper index $+$, to simplify notation):

$$\left(e_{M_0+1-k}+\sum_{j=1}^{M_0}a_{j-1,0}e_{j-k}\right)\wedge\cdots\wedge\left(e_{M_{k-1}+1-k}+\sum_{j=1}^{M_{k-1}}a_{j-1,k-1}e_{j-k}\right)\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots$$
$$=R\left(I+\sum_{\ell=0}^{k-1}\sum_{j=1}^{M_\ell}a_{j-1,\ell}E_{j-k,M_\ell+1-k}\right)(e_{M_0+1-k}\wedge\cdots\wedge e_{M_{k-1}+1-k}\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots).$$

Recall that (see [13])

$$\sigma(e_{M_0+1-k}\wedge e_{M_1+1-k}\wedge\cdots\wedge e_{M_{k-1}+1-k}\wedge e_{-k}\wedge e_{-k-1}\wedge e_{-k-2}\wedge\cdots))=s_\lambda(t)\,,$$

where

$$s_\lambda(t)=\det(s_{\lambda_i+j-i}(t))_{1\le i,j\le k}$$

is the Schur polynomial, corresponding to the partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$, with $\lambda_i=M_{i-1}+i-k$. Thus (53) lies in $\sigma R(U)\sigma^{-1}\cdot s_\lambda(t)$, where R is the representation of GL_∞ in F (see Section 2), so that $\sigma R\sigma^{-1}$ is the corresponding representation in

B, and U is the subgroup of GL_∞, consisting of upper triangular matrices with 1's on the diagonal.

We will next show that the dimension of the space of all polynomials of the form (53) is $-\frac{1}{2}k(k-1)+\sum_{i=0}^{k-1} M_i$, or in terms of the corresponding partition λ, it is $|\lambda| = \lambda_1+\lambda_2+\cdots+\lambda_k$. To show this, we first calculate the degrees of freedom of such a solution. Since it is difficult to determine this in terms of the degrees of freedom of the constants c_{ij}, we calculate this for the constants $a_{j\ell}$ which appear in (50), or rather in $f_i(z) = z^{-M_i-1}+\sum_{j=0}^{M_i-1} a_{ji}z^{-j-1}$. Note that, the corresponding tau-function does not change if we use Gauss elimination, i.e., if we add a multiple of the function $f_i(z)$ to the function $f_j(z)$ With this we can eliminate with $f_i(z)$ the constant $a_{M_i,j}$ in $f_j(z)$ for all $j<i$. This eliminates all dependence in the constants $a_{j\ell}$ and no more constants can be set to zero. Hence, the degrees of freedom that remain are $\lambda_k = M_{k-1}$ for $f_{k-1}(z)$, $\lambda_{k-1} = M_{k-2}-1$ for $f_{k-1}(z)$, ..., $\lambda_1 = M_0-k+1$ for f_0. If we add this all up, we obtain $|\lambda| = -\frac{1}{2}k(k-1)+\sum_{i=0}^{k-1} M_i$, the desired result.

Now recall that the set of all polynomial tau-functions of the KP hierarchy is the orbit $\mathcal{O}_0$ of $\mathbb{C}1 \in B$ under the representation $\sigma R \sigma^{-1}$ of the group GL_∞. Let P be the stabilizer of the line $\mathbb{C}1$, let W be the subgroup of permutations of basis vectors of $\mathbb{C}^\infty$ and let W_0 be its subgroup, consisting of permutations, permuting vectors with non-positive indices between themselves. Then one has the Bruhat decomposition:

$$GL_\infty = \bigcup_{w\in W/W_0} UwP \ \text{ (disjoint union)}.$$

Applying this to $\mathbb{C}1$, we obtain that the projectivised orbit $\mathbb{P}\mathcal{O}_0$ is a disjoint union of Schubert cells $C_w = Uw\cdot 1$, $w \in W/W_0$. It is well known (see, e.g. [13]) that each $w\cdot 1$ is a Schur polynomial s_λ for some partition $\lambda = \lambda(w)$, and the corresponding Schubert cell $C_\lambda = U\cdot s_{\lambda(w)}$ is an affine algebraic variety isomorphic to $\mathbb{C}^{|\lambda|}$.

On the other hand, by the previous discussion, we have constructed an injective polynomial map from the space $\mathbb{C}^{|\lambda|}$ to the Schubert cell C_λ. But, by Nagata's lemma, if an affine variety X is embedded in an irreducible affine variety Y of the same dimension, then either $X=Y$, or the complement Z of X in Y is a closed subvariety of Y of codimension 1. Since in our situation Y is an affine space, there exists a polynomial F on Y, whose set of zeros is Z. But then the restriction of F to X is a non-constant invertible polynomial function on X, which in our situation is an affine space as well. This is a contradiction.

(b) By part (a), every τ_n must be of the form (52). Since we can shift the index n of τ_n, we may assume, without loss of generality, that $n=k$ and that $\tau_k(t) = \tau_{\lambda_1,\lambda_2,\ldots,\lambda_k}(t;c_1,c_2,\ldots,c_k)$. Since (51) and (52) give the same tau-function, we find that

$$\tau_k(t) = W(s_{\lambda_1+k-1}(t+c_1), s_{\lambda_2+k-2}(t+c_2),\ldots,s_{\lambda_k}(t+c_k)).$$

Using the relation between MKP tau-functions and the infinite flag manifold, as used in [13] and [9], see also Remark 14, we have

$$\sigma^{-1}(\tau_k) = w_k\wedge w_{k-1}\wedge\cdots\wedge w_1\wedge|0\rangle$$

and

$$\sigma^{-1}(\tau_{k+1}) = w_{k+1} \wedge w_k \wedge w_{k-1} \wedge \cdots \wedge w_1 \wedge |0\rangle,$$

hence the non-zero polynomial tau-function $\tau_{k+1}(t)$ must be the Wronskian determinant of the same functions, but now with one eigenfunction of $L = \partial$ added. Such an eigenfunction is of the form (50), thus

$$\tau_{k+1}(t) = W(s_M(t+d), s_{\lambda_1+k-1}(t+c_1), s_{\lambda_2+k-2}(t+c_2), \ldots, s_{\lambda_k}(t+c_k)).$$

Moreover, we may assume that $M \neq \lambda_i + k - i$, otherwise we can use Gauss elimination to get a smaller M. Now reorder $M,\ \lambda_1 + k - 1,\ \lambda_2 + k - 2, \ldots,\ \lambda_k$ to a decreasing order. If $M > \lambda_1 + k - 1$, then the Wronskian determinant is equal to the first possibility, where $\mu = M - k$. If $\lambda_i + k - i > M > \lambda_{i+1} + k - i - 1$ or $\lambda_k > M \neq 0$, we get the second possibility with $\mu = M + i - k$. And finally, when $M = 0$, we obtain the last possibility. □

7 Reduction of MKP to n-MKdV

Let n be an integer, $n \geq 2$. The n-th Gelfand-Dickey hierarchy, or n-KdV, describes the group orbit in a projective representation of the loop group of SL_n. This is not a subgroup of Gl_∞, one has to take a bigger group, containing it, as, e.g in [13]. Then the representation R of GL_∞ extends to a projective representation, denoted by $\hat{R}$, of this bigger group. An element of the loop group of SL_n commutes with the operator q^n (in the space B), which means that $\tau_{k+n}(t) = \tau_k(t)$ and hence $v_{k+n}(t) = v_k(t)$ and $P^{\pm}_{n+k}(t,\partial) = P^{\pm}_k(t,\partial)$. This gives that $L_{k+n} = L_k$ and that

$$(L_k^n)_- = \left(P_k^+ \circ \partial^n \circ P_k^{+-1}\right)_- = \left(P_{n+k}^+ \circ \partial^n \circ P_k^{+-1}\right)_- = 0\,,$$

which means that L_k^n is a differential operator. Using the Sato-Wilson equations (11), we deduce that $\frac{\partial P_k^+}{\partial t_{jn}} = 0$, for $j = 1, 2, \ldots$, and hence, since $L_k = P_k^+ \circ \partial^n \circ P_k^{+-1}$, that also $\frac{\partial L_k}{\partial t_{jn}} = 0$. The corresponding tau-function then satisfies $\frac{\partial \tau_k}{\partial t_{jn}} = a_j \tau_k$ for some constants a_j, and hence is of the form

$$\tau_k(t) = T_k(t) \exp\left(\sum_{j=1}^{\infty} a_j t_{jn}\right), \quad \text{where } \frac{\partial T_k(t)}{\partial t_{jn}} = 0 \text{ for } j = 1, 2, \ldots. \tag{54}$$

Differentiating (6) by t_{jn} and using that $\frac{\partial P_k^+}{\partial t_{jn}} = 0$, we obtain

The first formulation of the n-MKdV:

$$\operatorname{Res} z^{jn+k-\ell} \tau_k(t - [z^{-1}]) \tau_\ell(y + [z^{-1}]) \exp\left(\sum_{i=1}^{\infty} (t_i - y_i) z^i\right) dz = 0, \tag{55}$$

for all $0 \leq k, \ell \leq n-1$ *and* $j \geq 0$ *, provided that* $jn + k - \ell \geq 0$.

Let $\epsilon = \exp\frac{2\pi i}{n}$. One can reformulate (55) to one identity for each pair k and ℓ as in [8], equation (8):

$$z^{-1}\sum_{a=1}^{n}(\epsilon^a z)^{k-\ell+1+\delta n}\tau_k(t-[(\epsilon^a z)^{-1}])\tau_\ell(y+[(\epsilon^a z)^{-1}])\exp\left(\sum_{i=1}^{\infty}(t_i-y_i)(\epsilon^a z)^i\right)$$

has no negative powers of z, for $0\le k,\ell\le n-1$, and $\delta=0$ if $k-\ell\ge 0$ and $=1$ if $k-\ell<0$.

The fact $P_n^+ = P_0^+$ and that L_0^n is a differential operator, gives that L_0 is the n-th root of a differential operator [5], [6]

$$\begin{aligned}\mathcal{L}_0 &= \partial^n + w_{n-2}(t)\partial^{n-2}+\cdots+w_1(t)\partial+w_0(t) = L_0^n = P_n^+(t)\circ\partial^n\circ P_0^+(t)^{-1}\\ &= (\partial+v_{n-1}(t))\circ(\partial+v_{n-2}(t))\circ\cdots\circ(\partial+v_0(t))P_0^+(t)P_0^+(t)^{-1}\\ &= (\partial+v_{n-1}(t))\circ(\partial+v_{n-2}(t))\circ\cdots\circ(\partial+v_0(t))\,.\end{aligned}$$

The explicit form (16) of the $v_j(t)$ expressed in terms of the tau-functions (54), gives that

$$v_0(t)+v_1(t)+\cdots v_{n-1}(t)=0,\quad \text{and that } \frac{\partial v_k(t)}{\partial t_{jn}}=0, \text{ for all } j=1,2,\ldots.$$

Note that by (18):

$$\mathcal{L}_j := L_j^n = (\partial+v_{j-1}(t))\circ(\partial+v_{j-2}(t))\circ\cdots\circ(\partial+v_0(t))\circ(\partial+v_{n-1}(t))\circ(\partial+v_{n-2}(t))\circ\cdots\circ(\partial+v_j(t))\,,$$

which is a Darboux transformation of $\mathcal{L}_0$, i.e. a cyclic permutation of the factors $\partial+v_j$ of $\mathcal{L}_0$.

Since now $L_i=\mathcal{L}_i^{\frac{1}{n}}$ is only expressed in the v_j, the second set of equations of (17), which now have the form

$$\frac{\partial v_i}{\partial t_j} = \left(\mathcal{L}_{i+1}^{\frac{j}{n}}\right)_+\circ(\partial+v_i)-(\partial+v_i)\circ\left(\mathcal{L}_i^{\frac{j}{n}}\right)_+,\quad \text{where } \mathcal{L}_{n+i}=\mathcal{L}_i, \tag{56}$$

imply the first ones, the Lax equations, of (17).

We can reformulate the equations (56) by one compact formula (see e.g. [17]). Let

$$\mathcal{L} = \text{diag }(\mathcal{L}_0,\mathcal{L}_1,\ldots,\mathcal{L}_{n-1}) \tag{57}$$

and

$$M = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \partial+v_{n-1}(t)\\ \partial+v_0(t) & 0 & & & 0\\ 0 & \partial+v_1(t) & 0 & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & \partial+v_{n-2}(t) & 0\end{pmatrix} \tag{58}$$

Then $\mathcal{L}=M^n$, and the equation (56) is exactly the $(i+2)$ mod n-th row of the equation

$$\frac{\partial M}{\partial t_j} = \left[\left(\mathcal{L}^{\frac{j}{n}}\right)_+, M\right],\ j=1,2,\cdots. \tag{59}$$

Hence we obtain:

The second formulation of the n-MKdV:
Let $U_n = \mathbb{C}[v_i^{(m)} | i = 0, 1, 2, \cdots, n-1,\ m \in \mathbb{Z}_{\geq 0}]/(v_0 + v_1 + \cdots v_{n-1})$ be the quotient of the algebra of differential polynomials in v_j by the differential ideal, generated by $v_0 + v_1 + \cdots + v_{n-1}$. Then the n-MKdV hierarchy is the system of evolution equations (59) in U_n, where $\mathcal{L}$ and M are given by (57) and (58).

Example. For $n = 2$, we get the modified KdV equation in $v = v_0 = -v_1$. Indeed:

$$\mathcal{L}_0 = \partial^2 + u_0 = (\partial - v) \circ (\partial + v) = \partial^2 + \frac{\partial v}{\partial t_1} - v^2 ,$$

$$\mathcal{L}_1 = \partial^2 + u_1 = (\partial + v) \circ (\partial - v) = \partial^2 - \frac{\partial v}{\partial t_1} - v^2 ,$$

and

$$\frac{\partial v}{\partial t_j} = \left(\mathcal{L}_1^{\frac{j}{2}}\right)_+ \circ (\partial + v) - (\partial + v) \circ \left(\mathcal{L}_0^{\frac{j}{2}}\right)_+ , \quad j = 1, 3, 5, \ldots .$$

For $j = 3$ this gives the classical modified KdV equation:

$$\frac{\partial v}{\partial t_3} = -\frac{3}{2} v^2 \frac{\partial v}{\partial t_1} + \frac{\partial^3 v}{\partial t_1^3} .$$

8 Polynomial solutions of n-KdV and n-MKdV

We can use the ideas of Section 6 to obtain polynomial tau-functions of n-MKdV. We will first construct a polynomial tau-function for the n-KdV hierarchy. Let π be a permutation of $1, 2, \ldots, n$, such that $\pi(i) = j_i$, and choose n formal power series

$$f_i(z) = z^{j_i - 1} + \sum_{k=j_i}^{\infty} a_{ki} z^k, \quad i = 1, 2, \ldots, n.$$

Choose non-negative integers $m_1, m_2, \ldots, m_n$, such that at least one $m_i = 0$ and one m_i non-zero (all $m_i = 0$ would lead to the trivial solution $\tau_0 = 1$). We construct $L_0 = \partial$ eigenfunctions from these data. For $\ell = 1, 2, \ldots m_i$, define

$$q_{\ell,i}(t) = \mathrm{Res} z^{-\ell n} f_i(z) e^{t \cdot z} dz = s_{\ell n - j_i}(t) + \sum_{k \geq j_i} a_{ki} s_{\ell n - k - 1}(t) = s_{\ell n - j_i}(t + c_i) , \quad (60)$$

for certain constants $c_i = (c_{1i}, c_{2i}, \ldots)$. Then $\tau_0(t)$ is the Wronskian determinant of all functions

$$s_{\ell n - j_i}(t + c_i), \quad \text{for } 1 \leq i \leq n,\ 1 \leq \ell \leq m_i \text{ and } \ell n - j_i \geq 0 .$$

This determinant clearly becomes zero after differentiating by t_{pn} since differentiating the function $s_{\ell n - j_i}(t + c_i)$ by t_{pn} gives $s_{(\ell - p)n - j_i}(t + c_i)$, which is either zero if $(\ell - p)n - j_i < 0$ or it already appears as an eigenfunction in the Wronskian determinant. Hence $\tau_0(t)$ is an n-KdV tau-function.

We obtain τ_1 by adding the eigenfunction $s_{(m_1+1)n-j_1}(t+c_1)$ to the Wronskian determinant. We obtain τ_2 by adding this function and also $s_{(m_2+1)n-j_2}(t+c_2)$. For τ_3 we add besides these two also $s_{(m_3+1)n-j_3}(t+c_3)$, etc. For τ_n we add the functions

$$s_{(m_1+1)n-j_1}(t+c_1), s_{(m_2+1)n-j_2}(t+c_2), \dots, s_{(m_n+1)n-j_n}(t+c_n)\,.$$

This however gives no new tau-function: it is straightforward to check, but rather tedious, that τ_n is a scalar multiple of τ_0. In fact the theorem, that we shall prove later on in this section, then implies that this construction gives all possible polynomial tau-functions for n-MKdV.

Example 17 *Let us inspect the case $n = 2$. In this case either $m_1 = 0$ or $m_2 = 0$ and π is the identity or the transposition* (12)*. This gives two possible solutions, viz*

$$\tau_0(t) = s_{k,k-1,\dots,2,1}(t+c) \quad \textit{and } \tau_1(t) = s_{k+1,k,k-1,\dots,2,1}(t+c), \ \textit{or}$$
$$\tau_0(t) = s_{k,k-1,\dots,2,1}(t+c) \quad \textit{and } \tau_1(t) = s_{k-1,k-2,\dots,2,1}(t+c)\,,$$

where $c = (c_1, c_2, \ldots)$, which are all polynomial tau-functions of the KdV and the modified KdV hierarchies. This is a result of [13], Theorem 9.1(b). Note that these tau-functions are independent of the even times t_{2k}.

For general n to describe all tau-functions that satisfy the n-MKdV hierarchy in terms of a formula like (52) is rather complicated. Not only are there special partitions λ connected to the case of n-KdV. But also instead of arbitrary constants $c_i = (c_{1i}, c_{2i}, \ldots)$ connected to part λ_i of the partition λ, there are certain restrictions. This time there are series of constants that depend on the shifted parts $\lambda_i - i + 1$, but then calculated modulo n. Hence, there are n of such series $c_{\bar{i}} = (c_{1\bar{i}}, c_{2\bar{i}}, \ldots)$ of which at most $n-1$ appear in the tau-function. Here and thereafter $\bar{s}$ stands for remander of the division of s by n.

We claim that the Wronskian determinant

$$W(s_{\lambda_1+k-1}(t+c_{\overline{\lambda_1}}), s_{\lambda_2+k-2}(t+c_{\overline{\lambda_2-1}}), \dots, s_{\lambda_k}(t+c_{\overline{\lambda_k-k+1}}))\,, \tag{61}$$

is a polynomial tau-function of the n-KdV if and only if the set of shifted parts

$$V_\lambda = \{\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots, \lambda_k - k + 1, -k, -k-1, -k-2, \dots\}$$

satisfies the condition that

$$\text{if } j \in V_\lambda, \quad \text{then also } j - n \in V_\lambda\,.$$

This condition reflects the condition that if the eigenfunction $q_{\ell,i}(t)$, defined in (60) appears in the Wronskian determinant, then also $q_{\ell-n,i}(t)$, if it is non-zero, must appear in this determinant as well. Or stated differently, if $s_{\lambda_i+k-i}(t+c_{\overline{\lambda_i-i+1}})$ appears in the Wronskian determinant of (61), then either $\frac{\partial s_{\lambda_i+k-i}(t+c_{\overline{\lambda_i-i+1}})}{\partial t_n} = 0$ or $s_{\lambda_i+k-i-n}(t+c_{\overline{\lambda_i-i+1}})$ also appears in this determinant as well. This leads us to the following notion.

Definition 18 *A partition λ is called n-periodic if the corresponding infinite sequence V_λ is mapped to itself when subtracting n from each term.*

Theorem 19 *All polynomial tau-functions of the n-KdV hierarchy are, up to a constant factor, of the form*

$$\tau^n_{\lambda_1,\lambda_2,\ldots,\lambda_k}(t;c_{\overline{\lambda_1}},c_{\overline{\lambda_2-1}},\ldots,c_{\overline{\lambda_k-k+1}}) = \det\left(s_{\lambda_i+j-i}(t_1+c_{1,\overline{\lambda_i-i+1}},t_2+c_{2,\overline{\lambda_i-i+1}}\ldots)\right)_{1\le i,j\le k}\,, \tag{62}$$

where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ is an n-periodic partition. Here the $c_{\overline{i}}=(c_{1\overline{i}},c_{2\overline{i}},\ldots)$ for $i=1,2,\ldots n$ (where at most $n-1$ of such $\overline{i}$'s appear) are arbitrary constants.

Before we give the proof, let us make calculations in an explicit example. Let $n=4$ and $\lambda=(6,3,2,1)$. Then

$$V_\lambda=\{6,2,0,-2,-4,-5,-6,\ldots\}\,,$$

hence λ is 4-periodic, and the corresponding tau-function is

$$\begin{aligned}\tau^4_{(6,3,2,1)}(t;c_{\overline{2}},c_{\overline{2}},c_{\overline{4}},c_{\overline{2}}) =& W(s_9(t+c_{\overline{2}}),s_5(t+c_{\overline{2}}),s_3(t+c_{\overline{2}}),s_1(t+c_{\overline{2}}))\\ =& \begin{vmatrix} s_6(t+c_{\overline{2}}) & s_7(t+c_{\overline{2}}) & s_8(t+c_{\overline{2}}) & s_9(t+c_{\overline{2}})\\ s_2(t+c_{\overline{2}}) & s_3(t+c_{\overline{2}}) & s_4(t+c_{\overline{2}}) & s_5(t+c_{\overline{2}})\\ s_0(t+c_{\overline{4}}) & s_1(t+c_{\overline{4}}) & s_2(t+c_{\overline{4}}) & s_3(t+c_{\overline{4}})\\ 0 & 0 & s_0(t+c_{\overline{2}}) & s_1(t+c_{\overline{2}})\end{vmatrix}\,,\end{aligned} \tag{63}$$

which depends on two series of constants, viz. $c_{\overline{6}}=c_{\overline{2}}$ and $c_{\overline{0}}=c_{\overline{4}}$. The $\overline{6}$ and $\overline{0}$ are the elements of the following set

$$U^{(4)}_\lambda=\{6,2,0,-2\}\backslash\{2,-2,-4,-6\}=\{6,0\},$$

which are all the elements j of V_λ where one removes all elements $j-4$.

Now

$$\begin{aligned} s_9(t+c_{\overline{2}}) &= s_9(t)+\sum_{j=0}^{8}a_{9-j,\overline{2}}s_j(t) \text{ and } & f_6(z) &= z^{-10}+\sum_{j=0}^{8}a_{9-j,\overline{2}}z^{-j-1},\\ s_5(t+c_{\overline{2}}) &= s_5(t)+\sum_{j=0}^{4}a_{5-j,\overline{2}}s_j(t), & f_2(z) &= z^{-6}+\sum_{j=0}^{4}a_{5-j,\overline{2}}z^{-j-1},\\ s_3(t+c_{\overline{4}}) &= s_3(t)+\sum_{j=0}^{2}a_{3-j,\overline{4}}s_j(t), & f_0(z) &= z^{-4}+\sum_{j=0}^{2}a_{3-j,\overline{4}}z^{-j-1},\\ s_1(t+c_{\overline{2}}) &= s_1(t)+a_{1,\overline{2}}s_0(t), & f_{-2}(z) &= z^{-2}+a_{1,\overline{2}}z^{-1},\end{aligned}$$

where $a_{k,\overline{j}}=s_k(c_{\overline{j}})$. And as in the proof of Theorem 16 we can eliminate the coefficients of z^{-2}, in $f_0(z)$, $f_2(z)$ and $f_6(z)$, and the coefficient of z^{-4} in $f_2(z)$ and $f_6(z)$ and the coefficient of z^{-6} in $f_6(z)$, leaving a freedom of $9-3=6=\lambda_1$ in $f_6(z)$

and similarly a freedom of $3-1=2=\lambda_3$ in $f_0(z)$. Hence the dimension of the space of polynomials (63) is

$$8 = 6+2 = \lambda_1+\lambda_3 = \sum_{\lambda_i\in\Lambda^{(4)}(\lambda)} \lambda_i\,,$$

where

$$\Lambda^{(4)}((6,3,2,1) = \{\lambda_1,\lambda_3\} = \{6,2\}\,.$$

Let us next investigate the element $s_{(6,3,2,1)}(t)$ the corresponding element under σ^{-1} is

$$\begin{aligned}\sigma^{-1}(s_{(6,3,2,1)}(t)) =& e_6\wedge e_2\wedge e_0\wedge e_{-2}\wedge e_{-4}\wedge e_{-5}\wedge\cdots\\ =& t^{-1}u_2\wedge u_2\wedge tu_4\wedge tu_2\wedge t^2u_4\wedge t^2u_3\wedge t^2u_2\wedge t^2u_1\wedge t^3u_4\wedge\cdots\,.\end{aligned}$$

Here we make the identification $t^{-k}u_j = e_{4k+j}$ and $t^k e_{ij} = \sum_{s\in\mathbb{Z}} E_{4(s-k)+i,4s+j}$ as in [13] , eq. (9.1-2). And this is up to some infinite reordering "equal to"

$$\begin{aligned}&\hat{R}(t^1e_{11}+t^{-2}e_{22}+t^1e_{33}+e_{44})(tu_4\wedge tu_3\wedge tu_2\wedge tu_1\wedge t^2u_4\wedge\cdots)\\ &\quad = \hat{R}(t^1e_{11}+t^{-2}e_{22}+t^1e_{33}+e_{44})|0\rangle\,.\end{aligned}$$

We now reconstruct our λ from the element $t^1e_{11}+t^{-2}e_{22}+t^1e_{33}+e_{44}$. For this we invert the process above. We first calculate the corresponding infinite wedge product and need to find the place of $e_6 = t^{-1}u_2 = t^{-2}e_{22}tu_2$ and $e_0 = tu_4 = e_{44}tu_4$ in this product. It is the place 0 and the place -2, which gives the elements $\lambda_1 = 6-0$ and $\lambda_3 = 0-(-2) = 2$ of λ .

We now want to use some of the above features of the example in the

Proof of Theorem 19. First observe that (61) is equal to (62).

As in the proof of Theorem 16, we can calculate the degrees of freedom of the constants in a similar way. Let $\lambda = (\lambda_1,...,\lambda_k)$ be a partition. As before,

$$s_{\lambda_i+k-i}(t+c_{\overline{\lambda_i-i+1}}) = s_{\lambda_i+k-i}(t)+\sum_{j=0}^{\lambda_i+k-i-1} a_{\lambda_i+k-i-j,\overline{\lambda_i-i+1}}s_j(t) = \operatorname{Res} f_{\lambda_i-i+1}(z)e^{t\cdot z}dz\,,$$

for $a_{j,\overline{\lambda_i-i+1}} = s_j(c_{\overline{\lambda_i-i+1}})$, and

$$f_{\lambda_i-i+1}(z) = z^{-(\lambda_i-i+1)-k}+\sum_{j=0}^{\lambda_i+k-i-1} a_{j,\overline{\lambda_i-i+1}}z^{-(\lambda_i+k-i)+j-1}\,.$$

Note that $f_{\lambda_i-i+1-n}(z)$ also appears as some $z^nf_{\lambda_j-j+1}(z)$, for some $j>i$ and it has the form

$$f_{\lambda_i-i+1-n}(z) = (z^nf_{\lambda_i-i+1}(z))_- = z^{-(\lambda_i-i+1)-k-n}+\sum_{j=0}^{\lambda_i+k-i-1-n} a_{j,\overline{\lambda_i-i+1}}z^{-(\lambda_i+k-i)+j+n-1}\,.$$

Hence, proceeding in a similar way as in the proof of Theorem 16, we can use $f_{\lambda_i-i+1}(z)$ to eliminate the constant $a_{\lambda_\ell-\lambda_i+i-\ell-1,\overline{\lambda_\ell-\ell+1}}$, in front of $z^{-(\lambda_i-i+1)-k}$ in $f_{\lambda_\ell-\ell+1}(z)$ for all $\ell<i$. Note that we cannot eliminate more constants. Hence we have λ_k degrees of freedom for $f_{\lambda_k-k+1}(z)$, λ_{k-1} for $f_{\lambda_{k-1}-k+2}(z)$, λ_{k-2} for $f_{\lambda_{k-2}-k+3}(z)$, $\dots$, λ_1 for $f_{\lambda_1}(z)$. This is similar to the KP case, except that some of the $f_{\lambda_i-i+1}(z)$ are related, as described above. Hence we have to find those $f_{\lambda_i-i+1}(z)$ with the highest possible index that are not related to the one with a higher index. These are all the $f_j(z)$'s, with j from the following set:

$$U_\lambda^{(n)}=\{\lambda_1,\lambda_2-1,\dots,\lambda_k-k+1\}\backslash\{\lambda_1-n,\lambda_2-n+1,\dots,\lambda_k-n-k+1\}\,.$$

If $j\in U_\lambda$, then $j=\lambda_i-i+1$ for some i and $f_j(z)=f_{\lambda_i-i+1}(z)$ has λ_i degrees of freedom. Hence, defining

$$\Lambda^{(n)}(\lambda)=\{\lambda_i|\lambda_i-i+1\in U_\lambda^{(n)}\}\,,$$

the freedom of choosing constants (or the dimension of this subspace of polynomials) is equal to

$$\sum_{\lambda_i\in\Lambda^{(n)}(\lambda)}\lambda_i\,.$$

As before, the the tau-function (62) is the image under σ in B of the following element of $F^{(0)}$:

$$\begin{aligned}\left(e_{\lambda_1}+\sum_{j=1}^{\lambda_1+k-1}a_{j-1,\overline{\lambda_1}}e_{\lambda_1-j}\right)\wedge\left(e_{\lambda_2-1}+\sum_{j=1}^{\lambda_2+k-2}a_{j-1,\overline{\lambda_2-1}}e_{\lambda_2-1-j}\right)\wedge\cdots\\ \cdots\wedge\left(e_{\lambda_k-k+1}+\sum_{j=1}^{\lambda_k}a_{j-1,\overline{\lambda_k-k+1}}e_{\lambda_k-k+1-j}\right)\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots\,,\end{aligned}\tag{64}$$

which is equal to

$$R\left(I+\sum_{i=1}^{k}\sum_{j=1}^{\lambda_i+k-i}a_{j-1,\overline{\lambda_i-i+1}}E_{\lambda_i-i+1-j,\lambda_i-i+1}\right)(e_{\lambda_1}\wedge e_{\lambda_2-1}\wedge\cdots\wedge e_{\lambda_k-k+1}\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots)\,,$$

where

$$\sigma(e_{\lambda_1}\wedge e_{\lambda_2-1}\wedge\cdots\wedge e_{\lambda_k-k+1}\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots)=s_\lambda(t)\,.$$

We can rewrite (64) as follows:

$$R\left(I+\sum_{p\in U_\lambda^{(n)}}\sum_{0\le s<\frac{k+p}{n}}\sum_{j=1}^{p+k-sn-1}a_{j-1,\overline{p}}E_{p-j-sn,p-sn}\right)(e_{\lambda_1}\wedge e_{\lambda_2-1}\wedge\cdots\wedge e_{\lambda_k-k+1}\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots)\,.$$

Note that replacing the upper bound $p+k-sn-1$ of j by $p+k-1$ does not change the element. We can also drop the lower bound of s because this will give a matrix element that acts as zero on every vector of the wedge product $\sigma^{-1}(s_\lambda(t))$. We can also drop the upper bound of s. Indeed, if we do that, the new element transforms the element e_ℓ for $\ell \le -k$ into an element of the form $v_\ell = e_\ell + \sum_{-\infty<<i<\ell} b_i e_i$. We can then use the v_j for $j < \ell$ to eliminate all the coefficients of b_i (we have to do this procedure infinitely many times). In this way we get that (64) is equal to

$$\hat{R}\left(I + \sum_{p\in U_\lambda^{(n)}} \sum_{j=1}^{p+k-1} a_{j-1,\overline{p}} \sum_{s\in\mathbb{Z}} E_{p-j+sn,p+sn}\right)(e_{\lambda_1}\wedge e_{\lambda_2-1}\wedge\cdots\wedge e_{\lambda_k-k+1}\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots).$$

Now we relate the above element of the completed GL_∞ to an element of of the loop group $SL_n(\mathbb{C}[t,t^{-1}])$ by making the identification $t^{-k}u_j = e_{kn+j}$ and $t^k e_{ij} = \sum_{s\in\mathbb{Z}} E_{(s-k)n+i,sn+j}$ as in [13], eq. (9.1-2). Let

$$U = \{A(t) \in SL_n(\mathbb{C}[t])|\, A(0) \text{ is upper triangular with 1's on the diagonal}\}.$$

Then, under the above identification we have

$$I + \sum_{p\in U_\lambda^{(n)}} \sum_{j=1}^{p+k-1} a_{j-1,\overline{p}} \sum_{s\in\mathbb{Z}} E_{p-j+sn,p+sn} \in U.$$

Let $T = \{\sum_{i=1}^n t^{k_i} e_{ii}|\, k_i \in \mathbb{Z},\, \sum_{i=1}^n k_i = 0\} \subset SL_n(\mathbb{C}[t,t^{-1}])\}$. Fix $w = \sum_{i=1}^n t^{k_i} e_{ii} \in T$. We want to find the partition that corresponds to $\hat{R}(w)|0\rangle$, i.e., to find λ such that $\sigma(\hat{R}(w)|0\rangle) = s_\lambda(t)$. In fact, if $\lambda = (\lambda_1, \lambda_2, \ldots)$, we want to find its parts λ_i that are in $\Lambda^{(n)}(\lambda)$. We will denote these elements by $\hat\lambda_1, \hat\lambda_2, \ldots, \hat\lambda_p$. Now, $\hat{R}(w)|0\rangle$ is a semi-infinte wedge product of the elements $t^{k_i+j}u_i = e_{-(k_i+j)n+i}$, for $j > 0$ and all $1 \le i \le n$. We have to order these e_ℓ in a decreasing order in this wedge product, from which we then can determine the corresponding partition λ. For this, first reorder the elements k_i to the decreasing order without interchanging k_i's, if they are the same. Then p is the same as the number of k_i's which are smaller than the maximum of this set. Let π be the permutation that assigns to i the number j if k_j is in the i-th place in the decreasing order. The corresponding $\Lambda^{(n)}(\lambda)$ has p elements $\hat\lambda_i$, which we put in decreasing order: $\hat\lambda_1 \ge \hat\lambda_2 \ge \cdots \ge \hat\lambda_p$. The part λ_1, which is always an element of $\Lambda^n(\lambda)$, corresponds to the place of $t^{k_{\pi(n)}+1}u_{\pi(n)} = e_{-k_{\pi(n)}n+\pi(n)-n}$ in the semi-infinite wedge product, which is always on the 0-th place. Hence

$$\hat\lambda_1 = \lambda_1 = -k_{\pi(n)}n + \pi(n) - n$$

and $\lambda_2 = -k_{\pi(n)}n+\pi(n)-2n+1$, since it corresponds to $t^{k_{\pi(n)}+2}u_{\pi(n)} = e_{-k_{\pi(n)}n+\pi(n)-2n}$, then $\lambda_3 = -k_{\pi(n)}n+\pi(n)-3n+2$ and we continue as long as $k_{\pi(n)}+1$, $k_{\pi(n)}+2, \ldots$ is smaller than $k_{\pi(n-1)}$. To determine $\hat\lambda_2$ of $\Lambda^{(n)}(\lambda)$, is already a bit more complicated. One has to consider two cases. It is $\lambda_{k_{\pi(n-1)}-k_{\pi(n)}+2}$, if $k_{\pi(n-1)} = k_{\pi(n)}$ or if $k_{\pi(n-1)} > k_{\pi(n)}$ and $\pi(n-1) < \pi(n)$. Then the element $t^{k_{\pi(n-1)}+1}u_{\pi(n-1)}$, which is equal to $e_{-k_{\pi(n-1)}n-+\pi(n-1)-n}$ is in the $-k_{\pi(n-1)} + k_{\pi(n)} - 1$-th place in the semi-infinite wedge product. Hence $\hat\lambda_2 = -k_{\pi(n-1)}(n-1) - k_{\pi(n)} + \pi(n-1) - (n-1)$. However, if $k_{\pi(n-1)} > k_{\pi(n)}$ and $\pi(n-1) > \pi(n)$, then $\hat\lambda_2 = \lambda_{k_{\pi(n-1)}-k_{\pi(n)}+1}$ and this corresponds to the same element $e_{-k_{\pi(n-1)}n+\pi(n-1)-n}$, hence $\hat{\lambda}_2 = -k_{\pi(n-1)}(n-1) - k_{\pi(n)} + \pi(n-1) - (n-1) - 1$. The extra -1 at the end comes from the inversion of π between the elements $n-1$ and n, viz. in this case $\pi(n-1) > \pi(n)$. The number of inversions will turn out to be important, so let us introduce some notation. Let

$$J_j = |\{i > j | \pi(i) < \pi(j)\}|,$$

then

$$\hat{\lambda}_2 = -k_{\pi(n-1)}(n-1) - k_{\pi(n)} + \pi(n-1) - (n-1) - J_{n-1}\,.$$

For the next one we have $\hat{\lambda}_3 = \lambda_{2k_{\pi(n-2)}-k_{\pi(n-1)}-k_{\pi(n)}} + 2 - J_{n-2}$ and the corresponding element is $t^{k_{\pi(n-2)}+1}u_{\pi(n-2)} = e_{-k_{\pi(n-2)}n+\pi(n-2)-n}$, which gives

$$\hat{\lambda}_3 = -k_{\pi(n-2)}(n-2) - k_{\pi(n-1)} - k_{\pi(n)} + \pi(n-2) - (n-2) - J_{n-2}\,.$$

Continuing in this way we find

$$\hat{\lambda}_j = -k_{\pi(n-j+1)}(n-j+1) - k_{\pi(n-j+2)} - \cdots k_{\pi(n-1)} - k_{\pi(n)} + \pi(n-j+1) - (n-j+i) - J_{n-j+1}\,,$$

where the last one is $\hat{\lambda}_p$. The dimension of this space is $\hat{\lambda}_1 + \hat{\lambda}_2 + \cdots + \hat{\lambda}_p$, which is equal to

$$\sum_{j=n-p+1}^{n} (n-p-2j+1)k_{\pi(j)} + \pi(j) - j - J_j\,. \tag{65}$$

Since $\sum_i k_i = 0$, we can add a multiple of this sum, thus equation (65) is equal to

$$p\sum_{i=1}^{n-p} k_{\pi(i)} + \sum_{j=n-p+1}^{n} (n-2j+1)k_{\pi(j)} + \pi(j) - j - J_j\,. \tag{66}$$

Now, $k_{\pi(1)} = k_{\pi(2)} = \cdots = k_{\pi(n-p)}$, hence

$$p\sum_{i=1}^{n-p} k_{\pi(i)} = p(n-p)k_{\pi(1)} = \sum_{i=1}^{n-p}(n-2i+1)k_{\pi(i)}.$$

Thus (66) is equal to

$$\sum_{i=1}^{n}(n-2i+1)k_{\pi(i)} - \sum_{j=n-p+1}^{n} j - \pi(j) + J_j\,. \tag{67}$$

Note that $\pi(1) < \pi(2) < \cdots < \pi(n-p)$ and $j - \pi(j) + J_j$ are the number of inversions between j and all elements i with $i < j$, thus

$$\sum_{j=n-p+1}^{n} j - \pi(j) + J_j = \text{ number of inversions of } \pi\,,$$

hence, the dimension of the space which corresponds to w is

$$\sum_{\lambda_i \in \Lambda^{(n)}(\lambda)} \lambda_i = \sum_{i=1}^{n}(n-2i+1)k_{\pi(i)} - (\text{number of inversions of } \pi) \tag{68}$$

We now have to prove that this is indeed the right dimension to obtain all possible polynomial tau-functions. Recall that the set of all polynomial tau-functions of the n-KdV hierarchy is the orbit $\mathcal{O}_0^n$ of $\mathbb{C}1 \in B$ under the projecive representation $\hat{R}$ of the group $SL_n(\mathbb{C}[t, t^{-1}])$. Let $P = SL_n(\mathbb{C}[t])$. Then one has the Bruhat decomposition:

$$SL_n(\mathbb{C}[t, t^{-1}]) = \bigcup_{w \in T} UwP \text{ (disjoint union).}$$

Applying this to $\mathbb{C}1$, we obtain that the projectivisation of the orbit $\mathcal{O}_0^n$ is a disjoint union of Schubert cells $C_w = Uw \cdot 1$, for all possible $w = \mathrm{diag}(t^{k_1}, ..., t^{k_n}) \in T$. Now, $UwP = ww^{-1}UwP$, hence elements of U that w conjugates to elements in P get absorbed in P , and the elements $t^c e_{ij} \in U$ that get mapped under conjugation by w to elements $t^d e_{ij}$ with $d < 0$ give the cell. Hence we have to count the possible values of c such that $c - k_i + k_j < 0$. This is straigtforward, for $i < j$ it is $|k_i - k_j|$ if $k_i > k_j$ and 0 otherwise. For $j < i$ we find $|k_i - k_j| - 1$ if $k_i > k_j$ and 0 otherwise. Hence, we obtain as dimension the sum of all values $|k_i - k_j|$ for $1 \le i < j \le n$, where we have to subtract 1 if $k_i > k_j$. We find that the dimension of this Schubert cell is

$$\sum_{1 \le i < j \le n} \left(|k_i - k_j| - \begin{cases} 1 & \text{if } k_i > k_j, \\ 0, & \text{otherwise.} \end{cases} \right)$$

Now ordering the k_i's in decreasing order (where π is the permutation as before), we can remove the absolute value and obtain that the dimension is equal to

$$\sum_{1 \le i < j \le n} \left(k_{\pi(i)} - k_{\pi(j)} - \begin{cases} 1 & \text{if } \pi(i) > \pi(j) \\ 0, & \text{otherwise.} \end{cases} \right)$$

In this sum $k_{\pi(i)}$ appears $n - 1$ times, with $n - i$ plus signs and $i - 1$ minus signs, hence we obtain that the dimension of the Schubert cell C_w is equal to

$$\sum_{i=1}^{n} (n - 2i + 1) k_{\pi(i)} - (\text{number of inversions of } \pi) = \sum_{\lambda_i \in \Lambda^{(n)}(\lambda)} \lambda_i \,,$$

which is the dimension of the space of polynomials of the form (62). The same algebro-geometric argument as in the KP case completes the proof of the theorem. □

Example 20 *For $n = 3$ we have the following possible polynomial tau-functions of the 3-KdV hierarchy. Let $k, \ell = 0, 1, 2, \ldots$, then we find two series (see (62)):*

$$\tau^3_{k+2\ell, k+2\ell-2, \ldots \ell+2, \ell, \underline{\ell}, \ell-1, \underline{\ell-1}, \cdots, 1, \underline{1}}(t; c, c, \ldots c, c, \underline{c}, c, \underline{c}, \ldots, c, \underline{c})$$

and

$$\tau^3_{k+2\ell+1, k+2\ell-1, \ldots \ell+3, \ell+1, \underline{\ell}, \ell, \underline{\ell-1}, \ell-1, \cdots, \underline{1}, 1}(t; c, c, \ldots, c, c, \underline{c}, c, \underline{c}, c, \ldots, \underline{c}, c)$$

We have at most two series of constants that appear, viz. $c = (c_1, c_2, c_3, \ldots)$ and $\underline{c} = (\underline{c}_1, \underline{c}_2, \underline{c}_3, \ldots)$, and c is coupled to the parts of the partition which are not underlined and $\underline{c}$ to all underlined parts of the partition. In both cases the tau-functions are independent of all times t_{3k}.

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