

## MIT Open Access Articles

*Discrete Equidecomposability and Ehrhart Theory of Polygons*

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

**Citation:** Turner, Paxton and Yuhuai Wu. "Discrete Equidecomposability and Ehrhart Theory of Polygons." *Discrete & Computational Geometry* 65, 1 (June 2020): 90–115 © 2020 Springer Science Business Media

**As Published:** <http://dx.doi.org/10.1007/s00454-020-00211-1>

**Publisher:** Springer Science and Business Media LLC

**Persistent URL:** <https://hdl.handle.net/1721.1/129408>

**Version:** Author's final manuscript: final author's manuscript post peer review, without publisher's formatting or copy editing

**Terms of Use:** Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use.



## Discrete Equidecomposability and Ehrhart Theory of Polygons

**Cite this article as:** Paxton Turner and Yuhuai Wu, Discrete Equidecomposability and Ehrhart Theory of Polygons, Discrete & Computational Geometry <https://doi.org/10.1007/s00454-020-00211-1>

This Author Accepted Manuscript is a PDF file of an unedited peer-reviewed manuscript that has been accepted for publication but has not been copyedited or corrected. The official version of record that is published in the journal is kept up to date and so may therefore differ from this version.

Terms of use and reuse: academic research for non-commercial purposes, see here for full terms. <https://www.springer.com/aam-terms-v1>

Author accepted manuscript

;

Author accepted manuscript

Noname manuscript No.

(will be inserted by the editor)

Paxton Turner, Yuhuai Wu

## Discrete Equidecomposability and Ehrhart Theory of Polygons

Paxton Turner · Yuhuai Wu

Received: date / Accepted: date

**Abstract** Motivated by questions from Ehrhart theory, we present new results on discrete equidecomposability. Two rational polygons  $P$  and  $Q$  are said to be *discretely equidecomposable* if there exists a piecewise affine-unimodular bijection (equivalently, a piecewise affine-linear bijection that preserves the integer lattice  $\mathbb{Z}^2$ ) from  $P$  to  $Q$ . We develop an invariant for a particular version of this notion called *rational finite discrete equidecomposability*. We construct triangles that are Ehrhart equivalent but not rationally finitely discretely equidecomposable, thus providing a partial negative answer to a question of Haase–McAllister on whether Ehrhart equivalence implies discrete equidecomposability. Surprisingly, if we delete an edge from each of these triangles, there exists an *infinite* rational discrete equidecomposability relation between them. Our final section addresses the topic of infinite equidecomposability with concrete examples and a potential setting for further investigation of this phenomenon.

**Keywords** combinatorics, discrete geometry, dissections and valuations, lattice polytopes

**Mathematics Subject Classification (2000)** 52B20, 52B45

This material is based upon work supported by the National Science Foundation under Grant No. DMS-0931908 while the authors were in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI during the Summer@ICERM 2014 program.

P. Turner

Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA, 02139  
E-mail: pax@mit.edu

Y. Wu

Computer Science, University of Toronto, Toronto, Ontario M5S2E4  
E-mail: ywu@cs.toronto.edu

## 1 Introduction

### 1.1 Ehrhart Theory

Ehrhart theory is the study of counting the number of integer lattice points in integral dilates of polytopes [3]. Let  $P$  denote a polytope in  $\mathbb{R}^k$ . Given  $t \in \mathbb{N}$ , the  $t^{\text{th}}$  dilate of  $P$  is the set  $tP = \{tp \mid p \in P\}$ . Here  $tp$  denotes scalar multiplication of the point  $p$  by  $t$ . With this set-up, we can define the central object of Ehrhart theory, the function  $\text{ehr}_P(t)$  that counts the number of lattice points in  $tP$ :

$$\text{ehr}_P(t) := |\{tP \cap \mathbb{Z}^k\}|. \tag{1}$$

We say that  $P$  is an *integral polytope*, or simply,  $P$  is *integral* if its vertices lie in the lattice  $\mathbb{Z}^k$ . Similarly, a *rational polytope* has all of its vertices given by points whose coordinates are rational. If  $P$  is rational, the *denominator* of  $P$  is defined to be the least positive integer  $d$  such that  $dP$  is an integral polytope.

A fundamental theorem due to Ehrhart states that if  $P$  is a rational polytope with denominator  $d$ , then  $\text{ehr}_P(t)$  is a *quasi-polynomial* of period  $d$  [2, 3, 11, 12]. Polygons  $P$  and  $Q$  are defined to be *Ehrhart equivalent* if  $\text{ehr}_P(t) = \text{ehr}_Q(t)$ .

### 1.2 Discrete Equidecomposability

Unless explicitly stated otherwise, we restrict our attention to (not necessarily convex) closed polygons in  $\mathbb{R}^2$ . However, the concept of discrete equidecomposability and all the questions posed in this section make sense in higher dimensions. The main goal of this paper is to explore and more precisely characterize the relationship between Ehrhart equivalence and discrete equidecomposability for rational polygons in dimension 2.

The notion of discrete equidecomposability captures two sorts of symmetries: (1) translation by an integer vector and (2) lattice-preserving linear transformations. Note that both (1) and (2) preserve the number of lattice points in a region and, hence, Ehrhart quasi-polynomials. The *affine unimodular group*  $G := \text{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$  with the following action on  $\mathbb{R}^2$  captures both properties.

$$x \mapsto gx := Ux + v, \\ x \in \mathbb{R}^2, \quad g = U \ltimes v \in G = \text{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2.$$

Note that  $\text{GL}_2(\mathbb{Z})$  is precisely the set of integer  $2 \times 2$  matrices with determinant  $\pm 1$ . Two regions  $R_1$  and  $R_2$  are said to be *G-equivalent* if they are in the same  $G$ -orbit, that is,  $GR_1 = GR_2$ . Also, a *G-map* is a transformation on  $\mathbb{R}^2$  induced by an element  $g \in G$ . We slightly abuse notation and refer to both the element and the map induced by the element as  $g$ .

In the same manner as in [7], we define the notion of discrete equidecomposability in  $\mathbb{R}^2$  (labeled  $\text{GL}_2(\mathbb{Z})$ -*equidecomposability* in [7], see also [6] and

[10]), though in this definition we allow for a potentially infinite number of simplices.

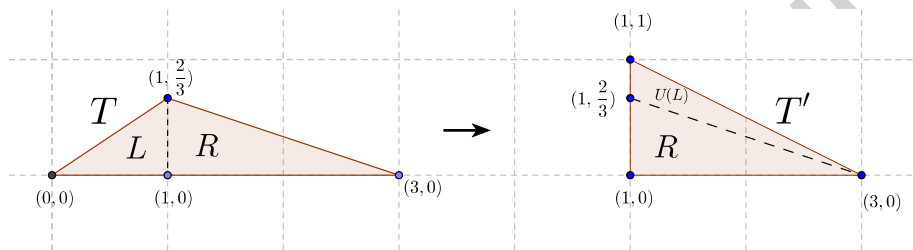
**Definition 1 (discrete equidecomposability)** Let  $P, Q \subset \mathbb{R}^2$ . Then  $P$  and  $Q$  are *discretely equidecomposable* if there exist relatively open simplices  $\{T_\alpha\}_{\alpha \in A}$  and  $G$ -maps  $\{g_\alpha\}_{\alpha \in A}$  such that

$$P = \bigsqcup_{\alpha \in A} T_\alpha \quad \text{and} \quad Q = \bigsqcup_{\alpha \in A} g_\alpha(T_\alpha).$$

Here,  $\bigsqcup$  indicates disjoint union.

A classic example of discrete equidecomposability shown in Figure 1 first appeared in [12]. Here

$$U(x) = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



**Fig. 1** This is an equidecomposability relation between the triangles  $T$  and  $T'$ . The identity map is applied to the closed triangle  $R$ , while the half-open triangle  $L$ , which is missing only the vertical edge, is mapped to  $U(L)$ .

In the case where the simplices  $\{T_\alpha\}_{\alpha \in A}$  are all rational, we refer to  $P$  and  $Q$  as in Definition 1 as *rationally* discretely equidecomposable. Moreover, if the index set  $A$  is finite, then we say that  $P$  and  $Q$  are *finitely* discretely equidecomposable. When the context is clear, we will occasionally omit the modifiers ‘rational’, ‘finite’, and ‘infinite’.

If  $P$  is a polygon, we refer to the collection of relatively open simplices  $\{T_\alpha\}_{\alpha \in A}$  as a *simplicial decomposition* or *triangulation*. If  $P$  and  $Q$  are discretely equidecomposable, then there exists a map  $\mathcal{F}$  which we call the *equidecomposability relation* that restricts to the specified  $G$ -map on each relatively open piece of  $P$ . Precisely, that is,  $\mathcal{F}|_{T_\alpha} = g_\alpha|_{T_\alpha}$  for all  $\alpha \in A$ . The map  $\mathcal{F}$  is thus a piecewise  $G$ -bijection. In the finite case, observe from the definition that the map  $\mathcal{F}$  preserves the Ehrhart quasi-polynomial; hence  $P$  and  $Q$  are Ehrhart equivalent if they are discretely equidecomposable<sup>1</sup>.

<sup>1</sup> The same is true in the infinite case as well, see Proposition 6.

## 1.3 Results

Our results primarily concern rational discrete equidecomposability of rational polygons in  $\mathbb{R}^2$ . Sections 2 and 3 study the finite case, while Section 4.1 focuses on the infinite case. In Section 2, we develop the necessary definitions (namely,  $d$ -minimal edges and triangles and their respective weights) and properties required for Section 3. In that section, we construct invariants for finite rational discrete equidecomposability and as a corollary demonstrate two triangles that are Ehrhart equivalent but not finitely rationally discretely equidecomposable. Finally, in Section 4.1, we produce an *infinite* equidecomposability relation between modified versions of these two triangle. We then generalize this example to give an infinite family of pairs of triangles that are Ehrhart equivalent, not finitely rationally equidecomposable, but *are* infinitely rationally equidecomposable after modification by removing an edge from each triangle. This motivates our conjecture that Ehrhart equivalence implies infinite rational discrete equidecomposability for full-dimensional polytopes in  $\mathbb{R}^2$ , a natural extension of a prior conjecture of Haase and McAllister [7]. We conclude with some limitations of infinite equidecomposability in terms of a specific example as well as further discussion of this conjecture.

Specifically, in Section 3, we introduce the *weight*  $W(P)$  of a rational polytope  $P$ . Our first main result is that weight is an invariant for finite rational discrete equidecomposability.

**Theorem 1** *The weight  $W(P)$  of a rational polygon  $P$  is preserved under equidecomposability relations. That is, if  $P$  and  $Q$  are finitely rationally discretely equidecomposable, then  $W(P) = W(Q)$ .*

We use this first result to find polygons in  $\mathbb{R}^2$  that are Ehrhart equivalent but not finitely rationally discretely equidecomposable, providing a partial negative answer to [7, Question 4.1]<sup>2</sup>. A striking aspect of the polygons we produce is that they are ‘almost’ *infinitely* equidecomposable.

**Theorem 2** *There exists an infinite family of pairs of triangles  $\{S_i, T_i\}$  with the following properties:*

1.  $S_i$  and  $T_i$  are Ehrhart equivalent but not finitely rationally discretely equidecomposable.
2. If a particular closed edge is deleted from both  $S_i$  and  $T_i$ , then the modified triangles are *infinitely* rationally discretely equidecomposable.

As explained in [4, Example 4], it is easy to construct rational line segments in  $\mathbb{R}^1$  that are Ehrhart equivalent but not finitely equidecomposable, taking for example  $L = [1/5, 6/5]$  and  $R = [2/5, 7/5]$ . In fact, they are not even infinitely

<sup>2</sup> Citing [6, Theorem 1.3], the authors of [7] claimed that Ehrhart equivalence implies finite rational discrete equidecomposability for rational polygons. This appears to be a slight misinterpretation, as the analogous notion of discrete equidecomposability in [6], referred to as a  $p\mathbb{Z}_0$  homeomorphism (see [6, Definition 1.1]), also allows for translation by a *rational* vector as opposed to a strictly integral one. Note, however, that [6, Theorem 1.3] does show that Ehrhart equivalence implies discrete equidecomposability for *integral* polygons.

decomposable, as can be seen by looking at the orbits under  $\mathrm{GL}_1(\mathbb{Z})$  of the denominator-5 points in  $L$  and  $R$  [4]. However, it is not obvious how to extend that example to polygons in  $\mathbb{R}^2$  with the same property. In particular, when the segments above are naturally embedded into the horizontal axis in  $\mathbb{R}^2$ ,  $S, T \subset \mathbb{R}^2$  become equidecomposable. Hence, if we take a point  $p \in \mathbb{Z}^2$ , it does not necessarily hold that the triangles  $P = \mathrm{Conv}(p, L)$  and  $Q = \mathrm{Conv}(p, R)$  are not discretely equidecomposable, even though they will have the same Ehrhart quasi-polynomial. It can be shown, e.g., that if  $p = (0, 1)$ , then  $P$  and  $Q$  are also equidecomposable.

Moreover, if we remove the left-endpoints from the one-dimensional examples  $L$  and  $R$ , then the resulting half-open intervals  $(1/5, 6/5]$  and  $(2/5, 7/5]$  are equidecomposable by fixing the interval  $(2/5, 6/5]$  and taking an integer translation of the interval  $(1/5, 2/5]$ . On the other hand, it is an open question as to whether or not there exists a *finite* rational equidecomposability relation between  $S_i$  and  $T_i$  from Theorem 2 after the appropriate closed edges are removed—we expect that the answer to this is negative.

Interestingly enough, we can also construct rational relatively closed line segments  $e, e' \subset \mathbb{R}^2$  that are Ehrhart equivalent but have very strong restrictions on potential (not necessarily rational or finite) equidecomposability relations between them. In particular, we show that if there exists an equidecomposability relation  $\mathcal{F}: e \rightarrow e'$ , then for all  $G$ -maps  $g$  involved in  $\mathcal{F}$ , the set of points that are mapped by  $g$  in  $\mathcal{F}$  cannot contain an interval (see Proposition 7). In particular, this rules out any finite, not necessarily rational, equidecomposability relation.

Theorem 2 and the previous discussion thus motivate the following conjectures. The first will be discussed in more detail in Section 4.1.

*Conjecture 1* Ehrhart equivalence is a sufficient condition for (not necessarily finite or rational) discrete equidecomposability of full-dimensional rational polytopes in dimension  $k \geq 2$ .

*Conjecture 2* Let  $\mathbf{0}_j$  denote the zero-vector in  $\mathbb{R}^j$ . Suppose that rational polytopes  $P, Q \subset \mathbb{R}^k$  are Ehrhart equivalent. Then there exists  $k' \geq k$  such that  $P \times \mathbf{0}_{k'-k}$  and  $Q \times \mathbf{0}_{k'-k}$  are rationally discretely equidecomposable.

A better understanding of the connection between Ehrhart equivalence and rational discrete equidecomposability for various classes of polytopes is a fascinating avenue for future research. In particular, the recent work [4] showed that the two notions are equivalent for integral polytopes in  $\mathbb{R}^3$ , and the recent work [5] showed the same is true for polytopes associated to cubic graphs.

## 2 $d$ -Minimal Triangles

In this section and Section 3, we only consider *finite rational* discrete equidecomposability. Hence in these sections, for the sake of brevity, we take the term ‘equidecomposability’ to mean ‘finite rational discrete equidecomposability.’



## 2.1 Definitions and Motivation

**Definition 2** Let  $d \in \mathbb{Z}_{\geq 1}$  and define  $\mathcal{L}_d = (1/d)\mathbb{Z} \times (1/d)\mathbb{Z}$ . A triangle  $T$  is said to be  $d$ -minimal if  $T \cap \mathcal{L}_d$  consists precisely of the vertices of  $T$ .

**Definition 3** A polygon  $P$  is said to be an  $\mathcal{L}_d$ -polygon if all of its vertices lie in  $\mathcal{L}_d$ .

It is well known that  $d$ -minimal triangles have area  $1/(2d^2)$  and that any  $\mathcal{L}_d$ -polygon can be triangulated by  $d$ -minimal triangles. See Section 3 of the thesis [9] for proofs of both of these results. Henceforth, we refer to a triangulation consisting entirely of  $d$ -minimal triangles as a  $d$ -minimal triangulation. Observe that any triangulation  $\mathcal{T}$  of  $P$  can be refined into a  $d'$ -minimal triangulation  $\mathcal{T}'$  for some positive integer  $d'$ . It is also useful to note that  $G$ -maps send  $d$ -minimal triangles to  $d$ -minimal triangles.

Now, in light of Definition 1 and the following discussion, a (finite, rational) equidecomposability relation  $\mathcal{F}: P \rightarrow Q$  can be viewed as bijecting a simplicial decomposition (that is, a triangulation)  $\mathcal{T}_1$  of  $P$  to a simplicial decomposition (triangulation)  $\mathcal{T}_2$  of some polygon  $Q$ . That is, to each relatively open simplex (face) in  $\mathcal{T}_1$ , we assign a  $G$ -map such that the overall map is a bijection. In this case we write  $\mathcal{F}: (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$ .

Define the denominator of a triangulation  $\mathcal{T}$  to be the least integer  $d$  such that for all faces  $F \in \mathcal{T}$ , the dilate  $dF$  is an integral simplex. The following remark follows from the definitions.

*Remark 1* If  $P$  and  $Q$  are  $\mathcal{L}_d$ -polygons and  $\mathcal{F}: (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$ , then  $\mathcal{T}_1$  and  $\mathcal{T}_2$  have the same denominator, call it  $d'$ . In this case, we write  $\mathcal{F}_{d'}: (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$  and say that  $\mathcal{F}$  has denominator  $d'$ . Moreover, this implies that  $d'$  is divisible by  $d$ .

Without loss of generality, we can refine  $\mathcal{T}_1$  to a  $d'$ -minimal triangulation  $\mathcal{T}'_1$  for some denominator  $d'$  and let  $\mathcal{F}$  instead act on  $\mathcal{T}'_1$ . Pointwise, the definition of  $\mathcal{F}$  has not changed, we are simply changing the triangulation upon which we view  $\mathcal{F}$  as acting. This yields the following useful remark.

*Remark 2* If  $P$  and  $Q$  are  $\mathcal{L}_d$ -polygons, any equidecomposability relation  $\mathcal{F}: P \rightarrow Q$  can be viewed as fixing a  $d'$ -minimal triangulation  $\mathcal{T}_1$  (in some denominator  $d'$  divisible by  $d$ ) of  $P$  and assigning a  $G$ -map  $g_F$  to each face  $F$  (vertex, edge, or facet, respectively) of  $\mathcal{T}_1$  such that  $g_F(F) = \mathcal{F}(F)$  is a face (vertex, edge, or facet, respectively) of some  $d'$ -minimal triangulation  $\mathcal{T}_2$  of  $Q$ . Hence, when analyzing equidecomposability relations with domain  $P$ , it suffices to consider, for all  $d'$  divisible by  $d$ , those  $\mathcal{F}$  that assign  $G$ -maps to the faces of a  $d'$ -minimal triangulation of  $P$ .

Therefore, it makes sense to study the  $d$ -minimal triangles in general, especially their  $G$ -orbits. In this section, we will first classify the  $G$ -orbits of  $d$ -minimal triangles according to an action of the dihedral group on 3 elements. Then we will introduce a  $G$ -invariant weighting system on edges of  $d$ -minimal

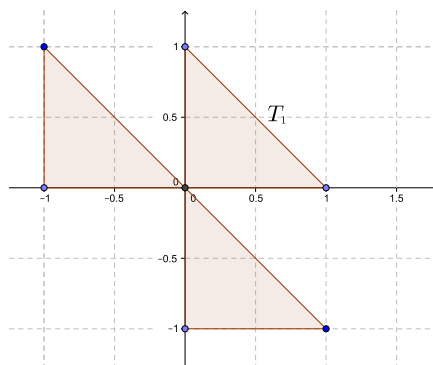


Fig. 2 The possible images of  $T_1$  under  $U$ .

triangles (known as  $d$ -minimal edges) which is crucial to the main results in Section 3. Finally, we show that the  $G$ -orbit of a  $d$ -minimal triangle is classified by the weights of its  $d$ -minimal edges. This last result provides an explicit parametrization of the  $G$ -equivalence classes of  $d$ -minimal triangles.

### 2.2 Classifying $G$ -orbits of $d$ -Minimal Triangles According to an Action of the Dihedral Group $D_3$

The following properties of  $G$ -maps are pivotal for all of the results of this section.

*Remark 3* Suppose we have a  $G$ -map  $g: P \rightarrow Q$ . Since  $g$  is an invertible affine linear map, it is a (topological) homeomorphism. Therefore  $g: \partial P \rightarrow \partial Q$  is also a homeomorphism. Moreover, if  $P$  and  $Q$  are polygons, linearity and invertibility guarantee that edges are sent to edges and vertices are sent to vertices.

The next proposition shows that it suffices to consider right triangles occurring in the unit square  $[0, 1] \times [0, 1]$ . It is a direct consequence of the fact that any two bases of the lattice  $\mathcal{L}_d$  are  $G$ -equivalent. Consider the  $d$ -minimal triangle  $T_1 := \text{Conv}((0, 0), (1/d, 0), (0, 1/d))$ .

**Proposition 1** *Every  $d$ -minimal triangle is  $G$ -equivalent to some translation  $T_1 + v$  of  $T_1$  with  $v \in \mathcal{L}_d \cap [0, 1]^2$ .*

Thus, the question of classifying  $G$ -orbits reduces to classifying the  $G$ -orbits of the  $\mathcal{L}_d$ -translations of  $T_1$  lying in the unit square. The next proposition classifies the types of transformations that send a triangle of the form  $T_1 + v$  to some  $T_1 + w$ .

**Proposition 2** *Suppose  $U(T_1 + v) + u = T_1 + w$  where  $U \in \text{GL}_2(\mathbb{Z})$  and  $u \in \mathbb{Z}^2$ . Then  $U$  belongs to the following set of matrices:*

$$D := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \right\}.$$

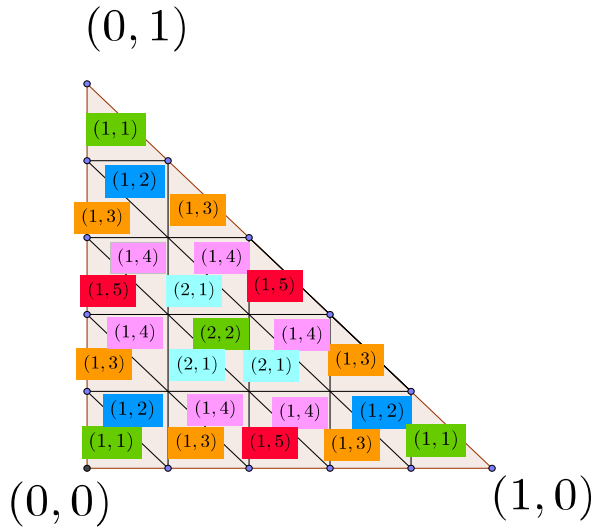


Fig. 3 The triangulation  $T$  on  $P$  in the case  $d = 5$ .

Note that  $D$  defines a group isomorphic to the dihedral group  $D_3 = \langle A, B \mid A^3 = B^2 = ABAB = I \rangle$ . Simply set

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is the crucial observation required for the proof of the main theorem later in this section.

*Proof* If  $U(T_1 + v) + u = T_1 + w$ , then  $UT_1 + Uv + u = T_1 + w$ . Thus  $UT_1 + Uv + u - w = T_1$ . Note that invertible linear transformations preserve the vertices of a triangle by Remark 3. Precisely, the vertices of the original triangle map to the vertices of its image. Hence,  $Uv + u - w$  is a vertex of  $T_1$ . This implies that  $UT_1$  is either  $T_1$ ,  $T_1 - (1, 0)^T$ , or  $T_1 - (0, 1)^T$ . These triangles are shown in Figure 2.

Since  $U$  is linear and sends vertices to vertices, we just need to compute the number of ways to send the ordered basis (and vertices of  $T_1$ )  $\{(0, 1), (1, 0)\}$  to other non-zero vertices of the previous triangles listed. This amounts to computing six changes of basis matrices, which are precisely given by the matrices in the set  $D$ .  $\square$

The next proposition will accomplish our classification of  $G$ -orbits, as mentioned before, via a relationship with the dihedral group. Consider the half

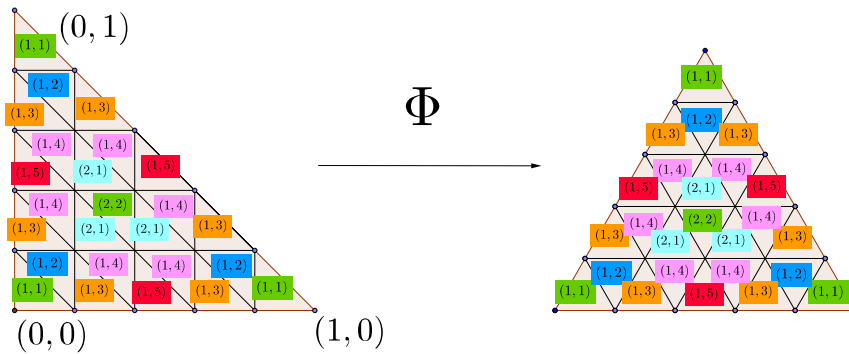


Fig. 4 The map  $\Phi$  in the case  $d = 5$ .

of the unit square given by  $P = \text{Conv}((0, 0), (1, 0), (0, 1))$  and the  $d$ -minimal triangulation  $\mathcal{T}$  of  $P$  given by cutting grid-squares in half by lines of slope  $-1$ . An example is shown in Figure 3.

This triangulation involves translations of  $T_1$  and translations of reflections of  $T_1$  about the line  $y = -x$ . These reflected triangles are, however, equivalent under the lattice-preserving transformation given by the flip about the line  $y = -x$  to translations of  $T_1$  lying in the unit square. Hence, it suffices to classify the  $G$ -orbits of the triangles lying in  $P$ . To simplify things, we can consider instead the quotient  $\bar{P} := P + \mathbb{Z}^2 \subset \mathcal{L}_d/\mathbb{Z}^2$ .

It is straightforward to check that we get an action of  $D_3 \cong D$  on  $\bar{P}$  and its triangulation by checking that each matrix in  $D$  preserves the triangulation. Hence, this action defines a permutation on  $\mathcal{T}$ . Let  $\Phi$  be the bijection from  $\bar{P}$  (and its constituent triangles) to a triangulated equilateral triangle  $T$  as shown in Figure 4 for the case  $n = 5$ .

Now, we also have an action of  $D$  on  $T$  by letting  $A$  act as a counterclockwise  $60^\circ$  rotation and  $B$  as a reflection about the angle bisector of the leftmost vertex, where  $A$  and  $B$  are defined in Proposition 2. We regard this action as a permutation on the set of triangles in the given triangulation of  $T$ . We claim that these two actions are compatible. This gives us an explicit understanding of the distribution of  $d$ -minimal triangles in  $\bar{P}$ .

**Proposition 3** *The actions of  $D$  on the constituent triangles of  $\bar{P}$  and on the constituent triangles of  $T$  are compatible in the sense that given  $\alpha \in D$ ,  $\alpha\Phi = \Phi\alpha$ .*

### 2.3 A $G$ -invariant $d$ -Minimal Edge Weighting System

It is convenient to work with  $d$ -minimal segments, which are the 1-dimensional counterparts of  $d$ -minimal triangles.

**Definition 4 (minimal edge)** A line segment  $E$  with endpoints in  $\mathcal{L}_d$  is said to be a  $d$ -minimal segment if  $E \cap \mathcal{L}_d$  consists precisely of the endpoints of  $E$ .

In particular, observe that the edges of  $d$ -minimal triangles are  $d$ -minimal segments. Our goal in the next two subsections is to develop a  $G$ -invariant weighting system on  $d$ -minimal edges that we will extend to a weighting system on  $d$ -minimal triangles. The existence of this invariant is the key to all of our main results. Note that this weight is defined on *oriented*  $d$ -minimal edges. The notation  $E_{p \rightarrow q}$  denotes that the edge  $E$  is oriented from the source  $p$  to the sink  $q$ .

**Definition 5 (weight of an edge)** Let  $E_{p \rightarrow q}$  be an oriented  $d$ -minimal edge from endpoint  $p = (w/d, x/d)$  to  $q = (y/d, z/d)$ . Then define the weight  $W(E_{p \rightarrow q})$  to be

$$W(E_{p \rightarrow q}) = \det \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \begin{pmatrix} w/d & y/d \\ x/d & z/d \end{pmatrix} = \det \begin{pmatrix} w & y \\ x & z \end{pmatrix} \pmod{d}.$$

As mentioned,  $W$  is invariant (up to sign) under the action of  $G$ . This is a straightforward computation.

**Proposition 4 ( $W$  is  $G$ -invariant)** Let  $E_{p \rightarrow q}$  be an oriented  $d$ -minimal edge and let  $g \in G$ . Then  $W(E_{p \rightarrow q}) = \pm W(g(E)_{g(p) \rightarrow g(q)})$ . Precisely, if  $g$  is orientation preserving (i.e.,  $\det g = 1$ ),  $W(E_{p \rightarrow q}) = W(g(E)_{g(p) \rightarrow g(q)})$ , and if  $g$  is orientation reversing (i.e.,  $\det g = -1$ ),  $W(E_{p \rightarrow q}) = -W(g(E)_{g(p) \rightarrow g(q)})$ .

We state a nice geometric interpretation of the weight which is critical to the proofs in Section 2.4. Doing so requires the following definition.

**Definition 6** Let  $E$  be an oriented  $d$ -minimal edge with endpoints  $p = (w/d, x/d)$  and  $q = (y/d, z/d)$ . The edge  $E$  is said to be *oriented counterclockwise* if

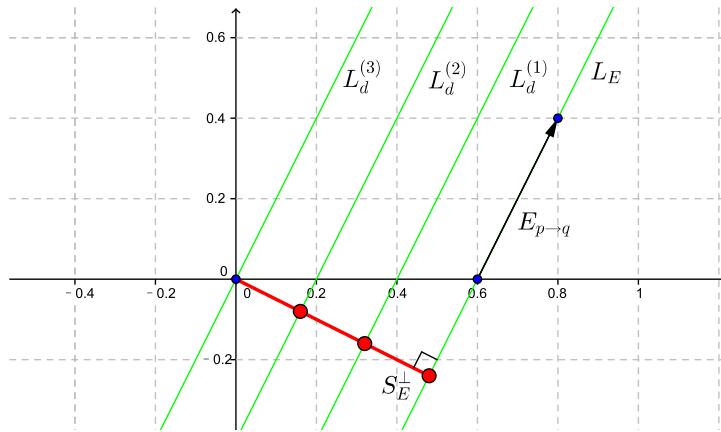
$$\det \begin{pmatrix} w & y \\ x & z \end{pmatrix} \geq 0$$

and *oriented clockwise* if

$$\det \begin{pmatrix} w & y \\ x & z \end{pmatrix} \leq 0.$$

In particular, if the determinant above is 0, then  $E$  is said to be both *oriented counterclockwise* and *clockwise*.

Now we define  $\text{dis}(E)$ , the *lattice-distance* (with respect to the lattice  $\mathcal{L}_d$ ) of a  $d$ -minimal segment  $E$  from the origin. A line in  $\mathbb{R}^2$  is said to be an  $\mathcal{L}_d$ -line if its intersection with  $\mathcal{L}_d$  is non-empty. Let  $L_E$  be the line extending the segment  $E$  and call  $L_E^\parallel$  the set of all  $\mathcal{L}_d$ -lines parallel to  $L_E$ . Construct  $S_E^\perp$ ,



**Fig. 5**  $\text{dis}(E_{p \rightarrow q}) = 4 - 1 = 3$  in this case, where  $E$  is the edge with end points  $p$ ,  $(0.6, 0)$  and  $q$ ,  $(0.8, 0.4)$  (both lie in  $\mathcal{L}_5$ ). The  $L_E^{(i)}$  indicate the lines between  $L_E$  and the origin.

the (closed) line segment perpendicular to  $L_E$  from the origin to  $L_E$ . Finally, we may define  $\text{dis}(E)$  formally as follows:

$$\text{dis}(E) := |\{L_E^\parallel \cap S_E^\perp\}| - 1.$$

In words,  $\text{dis}(E)$  is the number of  $\mathcal{L}_d$  lines parallel to  $E$  between the origin and the line  $L_E$  containing  $E$ , inclusive, minus one for the line through the origin. Alternatively,  $\text{dis}(E)$  is the relative length of the segment  $S_E^\perp$  in  $\mathcal{L}_d$ .

It is a known fact that we attribute to folklore, told to us by Tyrrell McAllister, that the determinant used to define our weight before taking residues computes  $\text{dis}(E)$ . We thank an anonymous referee for suggesting the following simple proof.

**Proposition 5 (lattice-distance interpretation of the weight  $W$ )** *Let  $E_{p \rightarrow q}$  be a counterclockwise oriented  $d$ -minimal segment. Then*

$$W(E_{p \rightarrow q}) = \text{dis}(E_{p \rightarrow q}) \pmod d.$$

*Proof* Let  $p = (p_1/d, p_2/d)$  and  $q = (q_1/d, q_2/d)$ . We show that the statement holds before taking residues; that is

$$\det \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} = \text{dis}(E_{p \rightarrow q}).$$

A lattice line parallel to  $E_{p \rightarrow q}$  has an equation of the form

$$(q_2 - p_2)x - (q_1 - p_1)y = \frac{c}{d} \tag{2}$$

in the  $(x, y)$ -plane, for some  $c \in \mathbb{Z}$ . In particular, the line  $\mathcal{L}_{E_{p \rightarrow q}}$  has the equation

$$(q_2 - p_2)x - (q_1 - p_1)y = \frac{1}{d} \det \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}.$$

Since  $E_{p \rightarrow q}$  is oriented counterclockwise, by definition

$$w := \det \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \geq 0.$$

As the nonnegative integer  $c$  ranges from 0 to  $w$ , Eq. (2) parametrizes all of the lines parallel to  $\mathcal{L}_{E_{p \rightarrow q}}$  that lie between the origin and  $\mathcal{L}_{E_{p \rightarrow q}}$ , inclusive. Therefore,

$$w = |\{L_{E_{p \rightarrow q}} \cap S_{E_{p \rightarrow q}}^\perp\}| - 1 = \text{dis}(E_{p \rightarrow q}),$$

so taking residues yields the desired result.  $\square$

### 2.4 Classifying $G$ -orbits of $d$ -Minimal Triangles via Weights

Using the geometric properties described in Propositions 2 and 5, we can classify  $d$ -minimal triangles by the weights of their  $d$ -minimal edges. We begin with the definition of this weight.

**Definition 7** Let  $T$  be a  $d$ -minimal triangle with vertices  $p, q$ , and  $r$ . Orient  $T$  counterclockwise, and suppose, without loss of generality, this orients the edges of  $T$  as follows:  $E_{p \rightarrow q}$ ,  $E_{q \rightarrow r}$ , and  $E_{r \rightarrow p}$ . Then we define the weight  $W(T)$  of  $T$  to be the (unordered) multiset as follows:

$$W(T) := \{W(E_{p \rightarrow q}), W(E_{q \rightarrow r}), W(E_{r \rightarrow p})\}.$$

The weight of a  $d$ -minimal triangle determines its  $G$ -orbit.

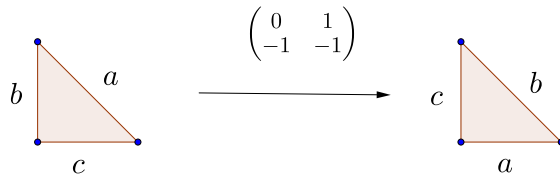
**Theorem 3** Two  $d$ -minimal triangles  $S$  and  $T$  are  $G$ -equivalent if and only if  $W(S) = W(T)$ .

*Proof* The left-to-right direction is a direct consequence of Proposition 4.

Now, suppose  $W(S) = W(T)$ . Use Proposition 1 to map  $S$  and  $T$  to translations  $S'$  and  $T'$ , respectively, of the triangle  $T_1 = \text{Conv}((0, 0), (1/d, 0), (0, 1/d))$  by a vector in  $\mathcal{L}_d$ . Let's temporarily order the sets  $W(S')$  and  $W(T')$ , starting from the hypotenuse and reading off weights counterclockwise.

By Propositions 2 and 3, we may act on  $S'$  by a  $G$ -map  $g$  such that (1)  $g(S')$  is still a translation of  $T_1$  by a denominator  $d$  vector and (2) the ordered weight of  $g(S')$  is a permutation of the ordered weight of  $S'$ . In fact, all permutations of orderings are possible because the dihedral group on 3 elements is precisely the symmetric group on 3 elements. See Figure 6 for a particular permutation.

Thus, we may choose a map  $g$  so that the ordered weights of  $T'$  and  $S'' = g(S')$  agree. By Proposition 5, the lattice distance of the vertical (respectively, horizontal) edges of  $T'$  and  $S''$  agree modulo  $d$ . Therefore, the coordinates of the vertex of  $T'$  opposite the hypotenuse must agree with the coordinates of the vertex of  $S''$  opposite its hypotenuse modulo integer translation. We conclude that  $S''$  is an integer translate of  $T'$ , so indeed  $S$  and  $T$  are  $G$ -equivalent.  $\square$



**Fig. 6** Each triangle in this figure is a translate of  $T_1$ . The matrix shown sends the ordered weight  $\{a, b, c\}$  to  $\{b, a, c\}$ . In general, each matrix from Proposition 2 acts as one of the permutations in the symmetric group of three letters on  $\{a, b, c\}$ .

### 3 Weight is an Invariant for Equidecomposability

Our goal in this section is to generalize the weight  $W$  to arbitrary rational polygons (not just  $d$ -minimal triangles) and show that it serves as an invariant for finite rational discrete equidecomposability. As in the previous section, we only consider finite rational discrete equidecomposability, which we also refer to in this section as ‘equidecomposability’ for brevity.

#### 3.1 Weight of a Rational Polygon

**Definition 8 (weight of a rational polygon)** Let  $P$  be a counterclockwise oriented  $\mathcal{L}_d$  polygon, and let  $d'$  denote a positive integer divisible by  $d$ . Then we may uniquely regard the boundary of  $P$  as a finite union  $\bigcup E^i$  of oriented  $d'$ -minimal segments  $\{E^i\}$ . We define the  $d'$ -weight  $W_{d'}(P)$  of the polygon  $P$  to be the unordered multiset

$$W_{d'}(P) := \bigcup \{W(E_i)\}.$$

Observe that for a  $d$ -minimal triangle  $T$ ,  $W_d(T)$  agrees with  $W(T)$  as described by Definition 7. We will work an example for clarity.

*Example 1* Let  $T_{(1,2)} = \text{Conv}((1/5, 0), (0, 1/5), (1/5, 1/5))$  and  $T_{(1,4)} = \text{Conv}((2/5, 0), (1/5, 1/5), (2/5, 1/5))$ . The triangles  $T_{(1,2)}$  and  $T_{(1,4)}$  are the denominator 5 triangles labeled (1, 2) and (1, 4), respectively, on the bottom row of the right hand side of Figure 4.

We compute the edge-weights in the multiset  $W_5(T_{(1,2)})$  below. To do so, we orient  $T_{(1,2)}$  counterclockwise. Also, in such computations modulo  $d$ , for our purposes, it is convenient to select our set of residues to be centered around



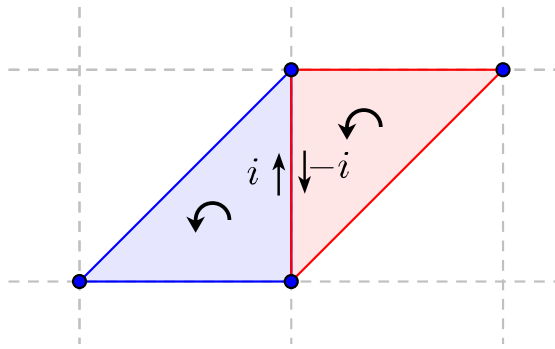


Fig. 7 The two-sided edge-weighting system for interior edges.

0. For example, if  $d = 5$ , we choose our residues from the set  $\{-2, -1, 0, 1, 2\}$ .

$$\begin{aligned}
 W(E_{(0,1/5) \rightarrow (1/5,0)}) &= \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod d = -1 \\
 W(E_{(1/5,1/5) \rightarrow (0,1/5)}) &= \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod d = 1 \\
 W(E_{(1/5,0) \rightarrow (1/5,1/5)}) &= \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \pmod d = 1
 \end{aligned}$$

So  $W_5(T_{(1,2)}) = \{1, 1, -1\}$ . In the same fashion, we can compute  $W_5(T_{(1,4)}) = \{2, -2, 1\}$ . Observe that  $W_5(T_{(1,2)}) \neq W_5(T_{(1,4)})$ .

### 3.2 $W_d$ is an Invariant for Equidecomposability

Recall from Remark 2 that a (finite, rational) equidecomposability relation  $\mathcal{F}: P \rightarrow Q$  between  $\mathcal{L}_d$ -polygons may be regarded as an assignment of  $G$ -maps to a  $d'$ -minimal triangulation  $\mathcal{T}_1$  of  $P$ . Since  $\mathcal{F}$  is a bijection and  $G$ -maps preserve the lattice  $\mathcal{L}_{d'}$ ,  $\mathcal{T}_1$  is sent to a  $d'$ -minimal triangulation  $\mathcal{T}_2$  of  $Q$ . From this point on, we will write  $\mathcal{F}_{d'}: (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$  to indicate the underlying triangulations and their denominator.

Since weights of facets (that is,  $d'$ -minimal triangles) in  $\mathcal{T}_1$  are preserved by  $G$ -maps, we see the multiset of weights of facets comprising  $\mathcal{T}_1$  must be in bijection with the multiset of weights of facets comprising  $\mathcal{T}_2$ . Concretely, if there is a triangle with weight  $\omega$  in the triangulation  $\mathcal{T}_1$ , there must be a triangle of weight  $\omega$  in  $\mathcal{T}_2$  and vice versa.

The edges in  $\mathcal{T}_1$  have a weighting (up to sign) induced by the weights of the facets. An edge  $E$  in the interior of  $\mathcal{T}_1$  is bordered by two facets  $F_1$  and  $F_2$  of  $\mathcal{T}_1$ . If the facet  $F_1$  induces the weight  $i$  on  $E$ , then  $F_2$  induces the weight  $-i$  on  $E$ , since both  $F_1$  and  $F_2$  are oriented counterclockwise. Hence, the induced

weighting on the edges of  $\mathcal{T}_1$  by the facets of  $\mathcal{T}_1$  is properly regarded as a *two-sided* edge-weighting system on interior edges. See Figure 7 for illustration. However, note that boundary edges in  $\mathcal{T}_1$  only border a single facet in  $\mathcal{T}_1$ , and hence have a well-defined orientation and weight induced by the orientation on the facet weights. In particular, the multiset of weights of boundary edges induced by the neighboring facets agrees with  $W_{d'}(P)$ . To proceed further, we need the following definition.

**Definition 9 ( $\pm i$   $d$ -minimal edges)** A  $d$ -minimal edge  $E$  is said to be a  $\pm i$   $d$ -minimal edge (or simply a  $\pm i$  edge when the denominator is clear) if there is an orientation on  $E$  such that  $W(E) = i$ .

If  $\mathcal{F}_{d'}: (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$  is an equidecomposability relation of denominator  $d'$  between  $\mathcal{L}_d$ -polygons  $P$  and  $Q$ , we see by Proposition 4 the number of  $\pm i$   $d'$ -minimal edges in  $\mathcal{T}_1$  is the same as the number of  $\pm i$  weighted  $d'$ -minimal edges in  $\mathcal{T}_2$ .

*Remark 4* Suppose  $\mathcal{F}_{d'}: (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$  is an equidecomposability relation. Then:

1. The multiset  $\{W_{d'}(F) \mid F \text{ a facet in } \mathcal{T}_1\}$  is in bijection with the multiset  $\{W_{d'}(F) \mid F \text{ a facet in } \mathcal{T}_2\}$ .
2. The set  $\{E \mid E \text{ an edge in } \mathcal{T}_1, W(E) = \pm i\}$  is in bijection with the set  $\{E \mid E \text{ an edge in } \mathcal{T}_2, W(E) = \pm i\}$  for all residues  $i$  modulo  $d'$ .

The following two definitions, the signed and unsigned weights of a polygon  $P$ , can be used to compute  $W_d(P)$ . These definitions are convenient because they can each be shown with some simple combinatorial arguments to be invariant under equidecomposability relations.

*Remark 5* In all statements and definitions that follow in this section,  $P$  and  $Q$  denote  $\mathcal{L}_d$ -polygons,  $d'$  is a positive integer divisible by  $d$ , and  $\mathcal{F}_{d'}$  is an equidecomposability relation from  $(P, \mathcal{T}_1)$  to  $(Q, \mathcal{T}_2)$  where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are both  $d'$ -minimal triangulations.

**Definition 10 (signed  $d'$ -weight  $SW_{d'}$ )** Fix a residue  $i \pmod{d'}$  and let  $\mathbb{1}_i$  denote the indicator function of  $i$  on the multiset  $W_{d'}(P)$ . That is,

$$\mathbb{1}_i(j) = \begin{cases} 1 & \text{if } j \equiv i \pmod{d'}, \\ 0 & \text{else.} \end{cases}$$

Then  $SW_{d'}(P)$  is a vector indexed by  $\mathbb{Z}/d'\mathbb{Z}$  as follows:

$$(SW_{d'}(P))_i = \sum_{j \in W_{d'}(P)} \mathbb{1}_i(j) - \sum_{j \in W_{d'}(P)} \mathbb{1}_{-i}(j).$$

That is,  $(SW_{d'}(P))_i$  is the difference between the number of appearances of  $i$  and  $-i$  in the multiset  $W_{d'}(P)$ .

*Example 2* Recall triangles  $T_{(1,2)}$  and  $T_{(1,4)}$  from Example 1. Let's compute  $SW_5$  of these two triangles. Recall that  $W_5(T_{(1,2)}) = \{1, 1, -1\}$  and  $W_5(T_{(1,4)}) = \{2, -2, 1\}$ . Represent  $SW_5$  by a five-entry vector, starting with the  $i = -2$  index and ending at the  $i = 2$  index. Then  $SW_5(T_{(1,2)}) = SW_5(T_{(1,4)}) = (0, -1, 0, 1, 0)$ . As an example, let's show  $SW_5(T_{(1,4)})_1 = 1$ . By definition,

$$SW_5(T_{(1,4)})_1 = \sum_{j \in W_5(T_{(1,4)})} \mathbb{1}_1(j) - \sum_{j \in W_5(T_{(1,4)})} \mathbb{1}_{-1}(j) = 1 - 0 = 1.$$

**Definition 11 (unsigned  $d'$ -weight  $UW_{d'}$ )** Fix a residue  $i \pmod{d'}$  and let  $\mathbb{1}_{\pm i}$  denote the indicator function of  $\{i, -i\}$  on the multiset  $W_{d'}(P)$ . That is,

$$\mathbb{1}_{\pm i}(j) = \begin{cases} 1 & \text{if } j \equiv i \pmod{d'} \text{ or } j \equiv -i \pmod{d'}, \\ 0 & \text{else.} \end{cases}$$

Then  $UW_{d'}(P)$  is a vector indexed by  $\mathbb{Z}/d'\mathbb{Z}$  as follows:

$$(UW_{d'}(P))_i = \sum_{j \in W_{d'}(P)} \mathbb{1}_{\pm i}(j).$$

That is,  $(UW_{d'}(P))_i$  is the total number of edges in  $W_{d'}(P)$  with weight  $\pm i$ .

*Example 3* Let's compute  $UW_5$  of the triangles  $T_{(1,2)}$  and  $T_{(1,4)}$ , recalling again that  $W_5(T_{(1,2)}) = \{1, 1, -1\}$  and  $W_5(T_{(1,4)}) = \{2, -2, 1\}$ . As in Example 2, let's index the 5-entry vector  $UW_5$  so that the entries run starting with the index  $i = -2$  and ending at the index  $i = 2$ . We see that  $UW_5(T_{(1,2)}) = (0, 3, 0, 3, 0)$  but  $UW_5(T_{(1,4)}) = (2, 1, 0, 1, 2)$ . For example, using the definition,

$$\begin{aligned} UW_5(T_{(1,2)})_{-1} &= \sum_{j \in W_{d'}(T_{(1,2)})} \mathbb{1}_{\pm 1}(j) = 3, \\ UW_5(T_{(1,4)})_2 &= \sum_{j \in W_{d'}(T_{(1,4)})} \mathbb{1}_{\pm 2}(j) = 2. \end{aligned}$$

Now we prove the invariance under equidecomposability of the signed and unsigned weight.

**Lemma 1 (signed  $d'$ -weight invariance)** Let  $\mathcal{F}_{d'} : (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$ . Then  $SW_{d'}(P) = SW_{d'}(Q)$ .

*Proof* The key observation is that

$$\sum_{F \text{ a facet in } \mathcal{T}_1} SW_{d'}(F) = SW_{d'}(P), \tag{3}$$

where we sum up the vectors  $SW_{d'}(F)$  componentwise<sup>3</sup>. Eq. (3) is justified as follows. Let  $i \pmod{d'}$  be a residue modulo  $d'$ . We show

$$\sum_{F \text{ a facet in } \mathcal{T}_1} SW_{d'}(F)_i = SW_{d'}(P)_i.$$

<sup>3</sup> This implies that  $SW_{d'}(P)$  is a *discrete valuation* in the sense of [13].

The LHS adds 1 for every weight  $i$  edge among the facets in  $\mathcal{T}_1$  and adds  $-1$  for every weight  $-i$  edge among the facets in  $\mathcal{T}_1$ . Formally, we have:

$$\begin{aligned} \sum_{F \text{ a facet in } \mathcal{T}_1} SW_{d'}(F)_i &= \sum_{F \text{ a facet in } \mathcal{T}_1} \sum_{E \text{ an edge of } F} \{\mathbb{1}_i(W(E)) - \mathbb{1}_{-i}(W(E))\} \\ &= \sum_{F \text{ a facet in } \mathcal{T}_1} \sum_{E \text{ an edge of } F} \mathbb{1}_i(W(E)) - \sum_{F \text{ a facet in } \mathcal{T}_1} \sum_{E \text{ an edge of } F} \mathbb{1}_{-i}(W(E)) \end{aligned} \quad (4)$$

Recall from Figure 7 and the neighboring discussion that interior weight  $i$  edges would appear once in the first summand of the RHS of Eq. (4) with sign  $+1$  and once in the second summand of the RHS of Eq. (4) with sign  $-1$ . Therefore, the contribution of any interior edge to the sum in Eq. (4) is 0. Only the boundary edges with weight  $i$  (when given the orientation induced by the counterclockwise orientation on  $P$ ) need to be taken into account. Thus,

$$\sum_{F \text{ a facet in } \mathcal{T}_1} SW_{d'}(F)_i = \sum_{\substack{E \in \mathcal{T}_1 \\ E \subset \partial P}} \{\mathbb{1}_i(W(E)) - \mathbb{1}_{-i}(W(E))\} = SW_{d'}(P)_i,$$

where the last equality follows from unraveling Definitions 8 and 10. Thus, Eq. (3) holds.

Finally, the LHS of Eq. (3) is invariant under the equidecomposability relation  $\mathcal{F}$ , because  $\mathcal{F}$  restricts to a  $G$ -map on facets of  $\mathcal{T}_1$ , and the weights of these facets are invariant under  $G$ -maps. That is,

$$\begin{aligned} \sum_{F \text{ a facet in } \mathcal{T}_1} SW_{d'}(F) &= \sum_{\substack{\mathcal{F}(F) \text{ s.t.} \\ F \text{ a facet in } \mathcal{T}_1}} SW_{d'}(\mathcal{F}(F)) \\ &= \sum_{F' \text{ a facet in } \mathcal{T}_2} SW_{d'}(F') = SW_{d'}(Q). \end{aligned}$$

The proof is complete.  $\square$

**Lemma 2 (unsigned  $d'$ -weight invariance)** *Let  $\mathcal{F}_{d'} : (P, \mathcal{T}_1) \rightarrow (Q, \mathcal{T}_2)$ . Then  $UW_{d'}(P) = UW_{d'}(Q)$ .*

*Proof* Let  $i \bmod d'$  denote a residue modulo  $d'$ . Let  $k \in \{1, 2, 3\}$  and define

$$\Delta_k^{\pm i}(\mathcal{T}_1) = \left\{ F \mid F \text{ is a facet in } \mathcal{T}_1 \text{ and } \sum_{E \text{ an edge in } F} \mathbb{1}_{\pm i}(W(E)) = k \right\}. \quad (5)$$

That is,  $\Delta_k^{\pm i}(\mathcal{T}_1)$  is the set of facets in  $\mathcal{T}_1$  having precisely  $k$  edges of weight  $\pm i$ . For the next part of this proof until Eq. (7), we will take the underlying triangulation  $\mathcal{T}_1$  as implicit and simply write  $\Delta_k^{\pm i}$  to represent  $\Delta_k^{\pm i}(\mathcal{T}_1)$ . We make the following claim:

$$\begin{aligned} \sum_{E \text{ an edge in } \mathcal{T}_1} \mathbb{1}_{\pm i}(W(E)) &= \frac{1}{2} (|\Delta_1^{\pm i}| + 2|\Delta_2^{\pm i}| + 3|\Delta_3^{\pm i}|) \\ &\quad + \frac{1}{2} \sum_{\substack{E \text{ an edge in } \mathcal{T}_1 \\ E \subset \partial P}} \mathbb{1}_{\pm i}(W(E)). \end{aligned} \quad (6)$$

To see this, note first that the LHS of Eq. (6) counts the total number of edges in  $\mathcal{T}_1$  with weight  $\pm i$ . To understand the RHS, let  $\mathcal{E}_{\pm i}^\circ$  be the set of all interior edges in  $\mathcal{T}_1$  with weight  $\pm i$ . Each edge  $E \in \mathcal{E}_{\pm i}^\circ$  is bordered by precisely two facets, say  $F_{E,1}$  and  $F_{E,2}$ . Observe that  $F_{E,1}$  (respectively,  $F_{E,2}$ ) is a member of exactly one of the sets  $\Delta_1^{\pm i}$ ,  $\Delta_2^{\pm i}$ , and  $\Delta_3^{\pm i}$ . Thus the expression  $|\Delta_1^{\pm i}| + 2|\Delta_2^{\pm i}| + 3|\Delta_3^{\pm i}|$  counts each edge in  $\mathcal{E}_{\pm i}^\circ$  exactly twice. Therefore,  $E$  is counted precisely once by the RHS of Eq. (6) (note the normalizing factor  $1/2$ ), since  $E$  is not a boundary edge.

Similarly, if  $E$  is a boundary edge of  $\mathcal{T}_1$ , then  $E$  only borders one facet in  $\mathcal{T}_1$ . Therefore,  $E$  is counted once by the expression  $|\Delta_1^{\pm i}| + 2|\Delta_2^{\pm i}| + 3|\Delta_3^{\pm i}|$ . Furthermore,  $E$  is counted precisely once by the term

$$\sum_{\substack{E \text{ an edge in } \mathcal{T}_1 \\ E \subset \partial P}} \mathbb{1}_{\pm i}(W(E)).$$

Since we normalize by  $1/2$ ,  $E$  is counted precisely once on the RHS of Eq. (6), as desired.

Observe by Definitions 8 and 11 and rearranging Eq. (6) that

$$\begin{aligned} UW_{d'}(P)_i &= \sum_{\substack{E \text{ an edge in } \mathcal{T}_1 \\ E \subset \partial P}} \mathbb{1}_{\pm i}(W(E)) = 2 \sum_{E \text{ an edge in } \mathcal{T}_1} \mathbb{1}_{\pm i}(W(E)) \\ &\quad - (|\Delta_1^{\pm i}(\mathcal{T}_1)| + 2|\Delta_2^{\pm i}(\mathcal{T}_1)| + 3|\Delta_3^{\pm i}(\mathcal{T}_1)|). \end{aligned} \tag{7}$$

We observe using Remark 4 that the total number of  $\pm i$  weighted edges in  $\mathcal{T}_1$  is preserved by  $\mathcal{F}$ . That is,  $\mathcal{T}_2$  has the same amount of  $\pm i$  weighted edges as  $\mathcal{T}_1$ . Also by Remark 4,  $|\Delta_k^{\pm i}(\mathcal{T}_1)| = |\Delta_k^{\pm i}(\mathcal{T}_2)|$  for all  $k \in \{1, 2, 3\}$ . Therefore, the RHS of Eq. (7) is preserved by  $\mathcal{F}$ , which implies  $UW_{d'}(P) = UW_{d'}(\mathcal{F}(P))$ , as desired.  $\square$

**Lemma 3** *The weight  $W_{d'}(P)$  is uniquely determined by the signed weight  $SW_{d'}(P)$  and unsigned weight  $UW_{d'}(P)$ .*

*Proof* Let  $n_i$  denote the number of times the residue  $i$  occurs in the multiset  $W_{d'}(P)$ . If  $i = -i \pmod{d}$ , then we have by Definitions 8 and 11 that  $(UW_{d'}(P))_i = n_i$ . Now suppose that  $i \neq -i \pmod{d}$ . In this case,

$$\mathbb{1}_{\pm i} = \mathbb{1}_i + \mathbb{1}_{-i}.$$

From Definitions 10 and 11 we see that  $n_i - n_{-i} = SW_{d'}(P)_i$  and  $n_i + n_{-i} = UW_{d'}(P)_i$ . Therefore,  $n_i = (SW_{d'}(P) + UW_{d'}(P))/2$ . Thus, the multiset  $W_{d'}(P)$  is uniquely determined by  $SW_{d'}(P)$  and  $UW_{d'}(P)$ .  $\square$

**Lemma 4** *Recall that  $d'$  is a fixed integer divisible by  $d$  and that  $P$  and  $Q$  are  $\mathcal{L}_d$ -polygons. If  $W_{d'}(P) = W_{d'}(Q)$ , then  $W_d(P) = W_d(Q)$ .*

*Proof* Our proof strategy is that given  $W_{d'}(P)$ , we can reconstruct  $W_d(P)$  uniquely. If we can show this, then Lemma 4 is justified.

Let  $n = d'/d$ . Label the counterclockwise oriented boundary  $d$ -minimal segments of  $P$  as  $\{E^i\}_{i=1}^N$ . Each  $E^j$  is subdivided into  $n$   $d'$ -minimal segments  $\{E^{j,k}\}_{k=1}^n$  when we regard  $P$  as an  $\mathcal{L}_{d'}$ -polygon for the purposes of computing  $W_{d'}(P)$ . For fixed  $j$ , each segment  $E^{j,k}$  has the same  $d'$ -weight since each lies in the same line (apply Proposition 5 to see this). Therefore,

$$W_{d'}(P) = \bigcup_{j=1}^N \bigcup_{k=1}^n \{W_{d'}(E^{j,k})\}.$$

Suppose  $E^j$  goes from  $p = (w/d, x/d)$  to  $q = (y/d, z/d)$ . Then the edge from  $p = (w/d, x/d)$  to  $q' = p + (y - w, z - x)/d'$  is in the set  $\{E^{j,k}\}_{k=1}^n$ . Without loss of generality, say this  $d'$ -minimal edge is  $E^{j,1}$ . Then,

$$\begin{aligned} W_{d'}(E^{j,1}) &= \det \begin{pmatrix} nw & nw + (y - w) \\ nx & nx + (z - x) \end{pmatrix} = \det \begin{pmatrix} nw & y - w \\ nx & z - x \end{pmatrix} \\ &= nwz - nwx - nxy + nxw = n(wz - xy) = n \det \begin{pmatrix} w & y \\ x & z \end{pmatrix} \pmod{d'}. \end{aligned}$$

We claim that there exists a unique choice  $r$  of residue modulo  $d$  such that

$$W_{d'}(E^{j,1}) = nr \pmod{d'}. \tag{8}$$

Eq. (8) says that for some integer  $t$ ,

$$W_{d'}(E^{j,1}) = nr + td' = nr + tnd = n(r + td).$$

Therefore, the residue  $n$  divides  $W_{d'}(E^{j,1})$  and we get

$$(W_{d'}(E^{j,1})/n) = r + td \iff r \equiv W_{d'}(E^{j,1})/n \pmod{d}.$$

We conclude that

$$W_d(P) = \bigcup_{j=1}^N \{(W_{d'}(E^{j,1})/n)\}.$$

Indeed,  $W_d(P)$  is uniquely determined by  $W_{d'}(P)$ . □

**Theorem 4 (weight  $W_d$  is an invariant for equidecomposability)** *Let  $P$  and  $Q$  be  $\mathcal{L}_d$ -polygons, and suppose  $\mathcal{F}: P \rightarrow Q$  is an equidecomposability relation. Then  $W_d(P) = W_d(Q)$ .*

*Proof* If  $\mathcal{F}_{d'}: P \rightarrow Q$ , then by Lemmas 1 and 2,  $SW_{d'}(P) = SW_{d'}(Q)$  and  $UW_{d'}(P) = UW_{d'}(Q)$ . By Lemmas 3 and 4, this implies  $W_d(P) = W_d(Q)$ , as desired. □

*Remark 6* Note that all our definitions and results immediately generalize to the case where  $P$  or  $Q$  is a finite union of rational polygons. Moreover, nowhere have we used any assumptions about convexity, and in general, we make no assumptions about convexity for this paper.

Theorem 4 comes with the following interesting corollary regarding  $d$ -minimal triangles.

**Corollary 1** *If  $S$  and  $T$  are  $d$ -minimal triangles, then the following are equivalent.*

- (1)  $S$  and  $T$  are  $G$ -equivalent.
- (2)  $W(S) = W(T)$ .
- (3)  $S$  and  $T$  are finitely rationally discretely equidecomposable.

*Proof* (1)  $\Leftrightarrow$  (2) is the content of Theorem 3. (3)  $\Rightarrow$  (2) is the content of Theorem 4. (1)  $\Rightarrow$  (3) is true because a  $G$ -map from  $S$  to  $T$  is automatically an equidecomposability relation.  $\square$

### 3.3 Ehrhart Equivalence Does not Imply Rational Finite Discrete Equidecomposability

The computational software **LattE** [1] can be used to show  $\text{ehr}_{T_{(1,2)}}(t) = \text{ehr}_{T_{(1,4)}}(t)$ . The explicit formulas are below.

$$\text{ehr}_{T_{(1,2)}}(t) = \text{ehr}_{T_{(1,4)}}(t) = \begin{cases} \frac{x^2}{50} + \frac{3x}{50} - \frac{2}{25} & \text{if } t \equiv 1 \pmod{5}, \\ \frac{x^2}{50} + \frac{x}{50} - \frac{3}{25} & \text{if } t \equiv 2 \pmod{5}, \\ \frac{x^2}{50} - \frac{x}{50} - \frac{3}{25} & \text{if } t \equiv 3 \pmod{5}, \\ \frac{x^2}{50} - \frac{3x}{50} - \frac{2}{25} & \text{if } t \equiv 4 \pmod{5}, \\ \frac{x^2}{50} + \frac{3x}{10} + 1 & \text{if } t \equiv 5 \pmod{5}. \end{cases}$$

However, by Theorem 4, we see that  $T_{(1,2)}$  and  $T_{(1,4)}$  are NOT rationally, finitely equidecomposable because  $W_5(T_{(1,2)}) = \{1, 1, -1\}$  and  $W_5(T_{(1,4)}) = \{2, -2, 1\}$ . This provides the partial negative answer in dimension 2 to Haase–McAllister’s [7] question of whether Ehrhart equivalence implies equidecomposability.

## 4 An Infinite Equidecomposability Relation

In this section—unlike in Sections 2 and 3—we consider discrete equidecomposability in full generality as stated in Definition 1. Hence, equidecomposability relations need not be finite or consist entirely of rational simplices.

### 4.1 Construction of the Infinite Equidecomposability Relation

Previously, by Section 3.3, we have shown that the two special triangles  $T_{(1,2)}$  and  $T_{(1,4)}$  with the same Ehrhart quasi-polynomial are not finitely rationally equidecomposable. However, if we delete an edge from each triangle, there does exist an infinite rational equidecomposability relation mapping one to the other. The existence of this infinite construction also explains why these two triangles share the same Ehrhart quasi-polynomial. It also raises many interesting problems discussed at the end of this section.

Label two edges of each of these triangles as follows. Let  $e_1$  denote the closed edge of  $T_{(1,2)}$  with endpoints  $(1/5, 0)$ ,  $(0, 1/5)$ , let  $e_2$  denote the closed edge with endpoints  $(1/5, 0)$ ,  $(1/5, 1/5)$ , let  $e_3$  denote the closed edge of  $T_{(1,4)}$  with endpoints  $(2/5, 0)$ ,  $(1/5, 1/5)$ , and let  $e_4$  denote the closed edge with endpoints  $(2/5, 0)$ ,  $(2/5, 1/5)$ .

**Theorem 5** *Denote by  $T_{(1,2)}$  the triangle with vertices  $(1/5, 0)$ ,  $(0, 1/5)$ ,  $(1/5, 1/5)$ , and by  $T_{(1,4)}$  the triangle with vertices  $(2/5, 0)$ ,  $(1/5, 1/5)$ ,  $(2/5, 1/5)$ . If we delete either one of  $e_1, e_2$  from  $T_{(1,2)}$  and either one of  $e_3, e_4$  from  $T_{(1,4)}$ , the remaining half-open polygons are infinitely equidecomposable.*

*Proof* Since the unimodular transformation

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

maps  $e_1$  to  $e_2$  and  $e_3$  to  $e_4$ , respectively, we delete either  $e_1$  or  $e_2$  from  $T_{(1,2)}$  and either  $e_3$  or  $e_4$  from  $T_{(1,4)}$ , and claim that the remaining half-open  $\Delta$ -complexes<sup>4</sup> are infinitely equidecomposable. Therefore, without loss of generality, we delete  $e_1$  from  $T_{(1,2)}$  and  $e_3$  from  $T_{(1,4)}$ .

*Part I: Choosing a  $\Delta$ -complex decomposition of  $T_{(1,2)}$  and  $T_{(1,4)}$  (cutting).* Let  $I = (1/5, 0)$ . Construct all the lines  $\{l_{4,i}\}$  connecting  $I$  and the lattice points of  $(1/5)\mathbb{Z} \times (1/5)\mathbb{Z}$  contained in the line  $y = 1/5$ , starting from  $(2/5, 1/5)$  and going to the right. The  $i^{\text{th}}$  line  $l_{4,i}$  will divide  $T_{(1,4)}$  into one more region. Denote the upper new region resulting from constructing  $l_{4,i}$  as  $R_i$ . We restrict  $R_i$  to be open. Let  $r_{i,i+1}$  denote the relatively open edge between  $R_i$  and  $R_{i+1}$ . Also,  $R_i$  has one side of its boundary bordering one of the two non-deleted edges of  $T_{(1,4)}$ . We let  $n_i$  denote this relatively open edge of  $R_i$ . Finally, let  $N_i$  be the point of intersection of the line  $l_{4,i}$  and the edge  $e_4$ . See Figure 8. Thus we have

$$T_{(1,4)} - e_3 = \bigsqcup_i \{R_i\} \bigsqcup_i \{r_{i,(i+1)}\} \bigsqcup_i \{n_i\} \bigsqcup_i \{N_i\}.$$

<sup>4</sup> For our purposes, a  $\Delta$ -complex can be thought of as a disjoint union of relatively open simplices. This is looser than the notion of simplicial complex because it is not required that the boundary of a face of a  $\Delta$ -complex be a part of the  $\Delta$ -complex, *i.e.*, relatively half-open structures are allowed. See Chapter 2 of [8] for a more general topological definition of  $\Delta$ -complexes.



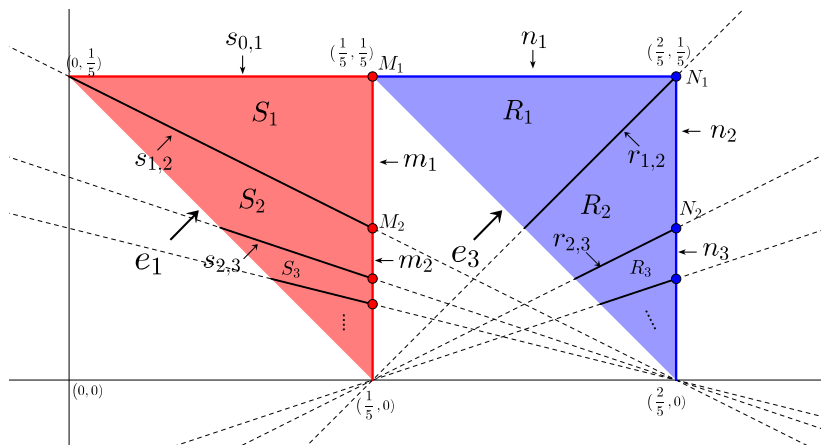


Fig. 8 An Infinite Equidecomposability Relation.

Next, choose another point  $J = (2/5, 0)$ . Construct all the lines  $\{l_{2,i}\}$  connecting  $J$  and the lattice points of  $(1/5)\mathbb{Z} \times (1/5)\mathbb{Z}$  contained in the line  $y = 1/5$ , starting from  $(1/5, 1/5)$  and going to the left. The  $i^{\text{th}}$  line  $l_{2,i}$  will divide  $T_{(1,2)}$  into one more region. Denote the upper new open region resulted from cutting by  $l_{2,i}$  as  $S_i$ . Let  $s_{0,1}$  denote the relatively open edge bordering  $S_1$  that lies on the line  $y = 1/5$ . Denote by  $s_{i,i+1}$  the relatively open edge lying between the regions  $S_i$  and  $S_{i+1}$ . For each  $S_i$ , denote the relatively open edge bordering  $S_i$  that lies on  $e_2$  of  $T_{(1,2)}$  by  $m_i$ . Finally, let  $M_i$  be the point of intersection of the line  $l_{2,i}$  and the edge  $e_2$ . See Figure 8. Thus,

$$T_{(1,2)} - e_1 = \bigsqcup_i \{S_i\} \bigsqcup_i \{s_{(i-1),i}\} \bigsqcup_i \{m_i\} \bigsqcup_i \{M_i\}.$$

Part II: Mapping the selected  $\Delta$ -complex decompositions (pasting). Let

$$U_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}.$$

$U_i$  sends each  $S_i$  to  $R_i$ ,  $s_{i-1,i}$  to  $n_i$ ,  $m_i$  to  $r_{i,i+1}$ , and  $M_i$  to  $N_i$ . The details can be verified by writing down explicit expression of the coordinates of each piece. Intuitively, we can think of the transformation  $U$  as mapping the triangle  $T_{(1,2)}$  to cover  $T_{(1,4)}$ , step by step. Since  $U_i$  fixes the points on  $y = 0$  but moves the lattice points on  $y = 1/5$  to the right by  $i$  units of the lattice,  $T_{(1,4)}$  is covered by  $T_{(1,2)}$  completely.  $\square$

We observe that the construction from Theorem 5 can be applied to a much larger family of pairs of triangles.

**Definition 12** Let  $T_l$  and  $T_r$  be two  $d$ -minimal triangles. The pair of triangles  $T_l$  and  $T_r$  are called *similar neighbors* if  $T_l$  is  $G$ -equivalent to  $T'_l$  and  $T_r$  is  $G$ -equivalent to  $T'_r$ , where

$$\begin{aligned} T'_l &= \text{Conv} \left( \left( \frac{1+t}{d}, \frac{1}{d} \right), \left( \frac{1+t}{d}, 0 \right), \left( \frac{t}{d}, \frac{1}{d} \right) \right), \\ T'_r &= \text{Conv} \left( \left( \frac{2+t}{d}, \frac{1}{d} \right), \left( \frac{2+t}{d}, 0 \right), \left( \frac{1+t}{d}, \frac{1}{d} \right) \right), \end{aligned} \tag{9}$$

for some fixed integer  $t$ , where  $1+t$ ,  $2+t$ , and  $d$  are pairwise relatively prime, and  $3+2t \not\equiv 0 \pmod{d}$ . When  $3+2t \equiv 0 \pmod{d}$ ,  $T_l$  and  $T_r$  are actually  $G$ -equivalent, so we exclude this case.

*Remark 7* Observe that when  $d < 5$  there are no similar neighbors. One example of similar neighbors is the pair of our favorite two triangles  $T_{(1,2)}$  and  $T_{(1,4)}$  (see Figure 3).

**Theorem 6** *If two triangles  $T_l$  and  $T_r$  are similar neighbors, then they share the same Ehrhart quasi-polynomials but are not finitely rationally equidecomposable.*

*Proof* Without loss of generality, let  $T_l$  and  $T_r$  be as  $T'_l$  and  $T'_r$  in (9), respectively. First let's show they share the same Ehrhart quasi-polynomials. Let  $e_{l,1}$  be the edge with endpoints  $((1+t)/d, 0)$  to  $(t/d, 1/d)$ , and let  $e_{l,2}$  be the edge with endpoints  $((1+t)/d, 0)$ ,  $((1+t)/d, 1/d)$ . Let  $e_{r,1}$  be the edge with endpoints  $((2+t)/d, 0)$  to  $((1+t)/d, 1/d)$ , and let  $e_{r,2}$  be the edge with endpoints  $((2+t)/d, 0)$ ,  $((2+t)/d, 1/d)$ . By the same construction as in the proof of Theorem 5, we can map the  $\Delta$ -complex  $(T_l - e_{l,1})$  to  $(T_r - e_{r,1})$ . By Proposition 6,  $(T_l - e_{l,1})$  and  $(T_r - e_{r,1})$  share the same Ehrhart quasi-polynomial. It remains to show the edges  $e_{l,1}$  and  $e_{r,1}$  have the same Ehrhart quasi-polynomial.

Observe that the matrix

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

maps  $e_{l,1}$  to  $e_{l,2}$  (respectively,  $e_{r,1}$  to  $e_{r,2}$ ). Thus it suffices to show that  $e_{l,2}$  and  $e_{r,2}$  share the same Ehrhart quasi-polynomial. Because  $1+t$ ,  $2+t$ , and  $d$  are pairwise relatively prime, we have a bijection between the denominator  $d'$  points in  $e_{l,2}$  and the denominator  $d'$  points in  $e_{r,2}$  by projecting the segment  $e_{l,2}$  onto  $e_{r,2}$ . Thus  $T_l$  and  $T_r$  share the same Ehrhart quasi-polynomial. To show they are not finitely rationally equidecomposable, we compute  $W_d(T_l) = \{1+t, 1, -1-t\}$  and  $W_d(T_{(1,2)}) = \{2+t, 1, -2-t\}$ . Since  $3+2t \not\equiv 0 \pmod{d}$ ,  $W_d(T_l) \neq W_d(T_r)$ . This completes the proof.  $\square$

#### 4.2 General Discrete Equidecomposability

It is natural to ask if the Ehrhart quasi-polynomial is even preserved in general by equidecomposability relations consisting of an infinite amount of (not necessarily rational) simplices. To do so, let's first provide the general definition

of an Ehrhart function of a bounded subset of  $\mathbb{R}^2$ . In this section, when we refer to equidecomposability, we mean it in the most general sense as allowed by Definition 1.

Here we denote as  $\text{ehr}(t)$ , the Ehrhart function (also referred to as the Ehrhart counting function), the number of lattice points in the  $t^{\text{th}}$  dilate of a general subset  $S \subset \mathbb{R}^2$ . As opposed to the case of rational polytopes, this function may not necessarily be a quasipolynomial. Observe that the definition provided below agrees with the Ehrhart function of a polygon provided in Eq. (1) of Section 1.

**Definition 13 (Ehrhart function of a subset of  $\mathbb{R}^2$ )** Let  $S$  be a bounded subset of  $\mathbb{R}^2$ . Then the *Ehrhart function* of  $S$  is defined to be

$$\text{ehr}_S(t) := |tS \cap \mathbb{Z}^2|$$

where  $t \in \mathbb{Z}_{\geq 1}$  and  $tS$  denotes the  $t^{\text{th}}$  dilate of  $S$ .

We require  $S$  to be bounded so that  $\text{ehr}_S(t)$  is always finite. The fact that Ehrhart functions are preserved by potentially infinite equidecomposability relations follows fairly quickly from the definitions.

**Proposition 6** *Let  $S$  and  $S'$  be bounded subsets of  $\mathbb{R}^2$  with (not necessarily finite or rational)  $\Delta$ -complex decompositions  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively. If  $\mathcal{F}: (S, \mathcal{T}) \rightarrow (S', \mathcal{T}')$  is an equidecomposability relation, then*

$$\text{ehr}_S(t) = \text{ehr}_{S'}(t).$$

*Proof* Let  $t$  be a positive integer. We want to construct a bijection  $\Phi$  from  $\mathbb{Z}^2 \cap tS$  to  $\mathbb{Z}^2 \cap tS'$ . Given  $p \in \mathbb{Z}^2 \cap tS$ , define  $\Phi(p) = t\mathcal{F}(p/t)$ . Injectivity of  $\Phi$  is clear because both the scaling map  $p \mapsto p/t$ , equidecomposability relation, and dilation map  $\mathcal{F}(p/t) \mapsto t\mathcal{F}(p/t)$  are all injective. Also,  $t\mathcal{F}(p/t)$  is an integer point because  $p$  is an integer point and  $\mathcal{F}$  preserves denominators.

Now we show surjectivity. Suppose  $tq \in tS' \cap \mathbb{Z}^2$ . Then  $q \in S'$ . Since  $\mathcal{F}$  is surjective, there exists  $p \in S$  such that  $\mathcal{F}(p) = q$ . Since  $tp \in tP$ , we observe that  $\Phi(tp) = t\mathcal{F}(p) = tq$ . Again, since  $\mathcal{F}$  preserves denominators, if  $tq$  is an integer point, it follows that  $tp$  is an integer point, as desired.

Indeed,  $\mathbb{Z}^2 \cap tS$  is in bijection with  $\mathbb{Z}^2 \cap tS'$ . Moreover, both sets are finite by the boundedness of  $S$  and  $S'$ . □

We restate Conjecture 1 from the introduction.

*Conjecture (formal statement of Conjecture 1 in dimension two)* Let  $P$  and  $Q$  be full-dimensional (closed, open, or half-open) rational polygons in  $\mathbb{R}^2$ . If

$$\text{ehr}_P(t) = \text{ehr}_Q(t),$$

then there exists an equidecomposability relation  $\mathcal{F}: (P, \mathcal{T}) \rightarrow (Q, \mathcal{T}')$ , where  $\mathcal{T}$  and  $\mathcal{T}'$  are  $\Delta$ -complex decompositions of  $P$  and  $Q$ , respectively.

Note that we just proved the forward direction. The key difficulty of the backward direction is that it is hard to keep track of the irrational points in an arbitrary decomposition, although their Ehrhart function is trivially 0.

Referring back to Theorem 5, the question remains as to why we must delete an edge from each triangle. Since we allow decompositions consisting of infinitely many pieces, can we also find an infinite equidecomposable relation between  $e_1$  and  $e_3$ ? This depends on the restrictions on decompositions, leading us to consider special types of decompositions.

**Definition 14** Let  $S$  and  $S'$  be bounded subsets of  $\mathbb{R}^2$ . Consider a  $\Delta$ -complex decomposition of  $S$  consisting entirely of 0-simplices:  $S = \bigsqcup_{\alpha \in A} S_\alpha$ , where the  $S_\alpha$  are vertices. Suppose that  $S$  and  $S'$  are equidecomposable, and let  $\mathcal{G} = \{g_\alpha\}_{\alpha \in A}$  denote a set of  $G$ -maps such that

$$S' = \bigsqcup_{\alpha \in A} g_\alpha(S_\alpha).$$

Let  $\mathcal{A}: S \rightarrow \mathcal{G}$  denote the naturally resulting assignment  $\alpha \mapsto g_\alpha$ . Given  $g \in \mathcal{G}$ , we refer to the pre-image  $\mathcal{A}^{-1}(g)$  as the *maximal subcomplex* associated to  $g$ . Furthermore, we define the *maximal decomposition*  $\mathcal{T}(\mathcal{A})$  associated to  $\mathcal{A}$  as follows:

$$\mathcal{T}(\mathcal{A}) := \{\mathcal{A}^{-1}(g) \mid g \in \mathcal{G}\}.$$

Since the group  $G$  is countable, it holds that  $\mathcal{G}$  and  $\mathcal{T}(\mathcal{A})$  are countable as well. Therefore, for typical equidecomposable sets  $S, S' \subset \mathbb{R}^2$ , some of the maximal subcomplexes  $\mathcal{A}^{-1}(g)$  are uncountable.

An assignment  $\mathcal{A}$  as in the previous definition yields an equidecomposability relation  $\mathcal{F}_\mathcal{A}: S \rightarrow S'$ . Note that an equidecomposability relation  $\mathcal{F}: S \rightarrow S'$  may have two distinct assignments  $\mathcal{A}$  and  $\mathcal{A}'$  such that  $\mathcal{F}_\mathcal{A} \equiv \mathcal{F}_{\mathcal{A}'} \equiv \mathcal{F}$ . To avoid any ambiguity in what follows, we take the assignment  $\mathcal{A}$  to be part of the data associated with the resulting equidecomposability relation denoted by  $\mathcal{F}_\mathcal{A}$ .

**Definition 15** Let  $S$  and  $S'$  be bounded subsets in  $\mathbb{R}^2$  of infinite cardinality that are equidecomposable, and let  $\mathcal{A}$  denote an assignment of points in  $S$  to a subset  $\mathcal{G} \subset G$  as in Definition 14 that yields an equidecomposability relation  $\mathcal{F}_\mathcal{A}: S \rightarrow S'$ . If for every  $g \in \mathcal{G}$ , the maximal subcomplex  $\mathcal{A}^{-1}(g)$  is totally disconnected, then  $\mathcal{F}_\mathcal{A}$  is called an *intractable* equidecomposability relation. Otherwise, it is referred to as *tractable*.

The Ehrhart function turns out to not be a sufficient condition for the existence of a tractable equidecomposability relation. It can easily be computed by hand that  $\text{ehr}_{e_1}(t) = \text{ehr}_{e_3}(t)$ , and the following shows that the two segments are not tractably equidecomposable.

**Proposition 7** *There does not exist a tractable equidecomposability relation between edges  $e_1$  with endpoints  $(1/5, 0)$ ,  $(0, 1/5)$  and  $e_3$  with endpoints  $(2/5, 0)$ ,  $(1/5, 1/5)$ .*

*Proof* Suppose there exists an assignment  $\mathcal{A}$  as in Definition 14 so that the resulting equidecomposability relation  $\mathcal{F}_{\mathcal{A}}: e_1 \rightarrow e_3$  is tractable. By tractability, there exists  $g \in G$  such that the maximal subcomplex  $\mathcal{A}^{-1}(g)$  is not totally disconnected. Hence, this maximal subcomplex contains some relatively open segment  $e' \subset e_1$ . Moreover,  $\mathcal{F}_{\mathcal{A}}(e') = g(e')$  is a relatively open segment contained in  $e_3$ . Recall that  $G$ -maps send 5-minimal segments to 5-minimal segments. Therefore, since  $e' \subset e_1$ ,  $g(e_1)$  is a 5-minimal segment containing  $g(e')$ . However, there can be at most one 5-minimal segment containing a given relatively open segment. It follows that  $g(e_1) = e_3$ . Yet this is a contradiction because  $\pm 1 = W(e_1) \neq W(e_3) = \pm 2 \pmod{5}$ , and Proposition 4 says that the weight of a  $d$ -minimal edge is preserved up to sign by  $G$ -maps.  $\square$

We recall Conjecture 1. Now to be more precise, we conjecture the Ehrhart function is a necessary and sufficient criterion for equidecomposability of full-dimensional polygons if we allow intractable equidecomposability relations. An example of a possible maximal subcomplex in an intractable equidecomposability relation is a fat Cantor set, which is a nowhere dense set of points that has uncountable cardinality and positive measure. We also wonder if there exists a tractable equidecomposability relation between a pair of similar neighbors.

**Acknowledgements** The authors express their deepest gratitude to Sinai Robins for introducing us to the area and mentoring us throughout the project. We also thank Tarik Aougab, Quang Nhat Le, Tyrrell McAllister, Jim Propp, Sanya Pushkar, Hugh Thomas, and Emmanuel Tsukerman for beneficial conversations and suggestions. We thank ICERM for providing an excellent research environment and thank Michael Mossinghoff and Sinai Robins for coordinating the 2014 Summer@ICERM program. We thank anonymous referees for pointing out the one-dimensional counterexample in the reference [4] and more generally for their helpful comments and suggestions.

## References

1. Baldoni, V., Berline, N., Loera, J.D., Dutra, B., Köppe, M., Moreinis, S., Pinto, G., Vergne, M., Wu, J.: Software and user's guide for LattE integrale (2014). <https://www.math.ucdavis.edu/~latte>
2. Beck, M., Robins, S.: Computing the Continuous Discretely. Undergraduate Texts in Mathematics. Springer, New York (2007)
3. Ehrhart, E.: Sur les polyèdres rationnels homothétiques a  $n$  dimensions. C. R. Acad. Sci. Paris **254**, 616–618 (1962)
4. Erbe, J., Haase, Ch., Santos, F.: Ehrhart-equivalent 3-polytopes are equidecomposable. Proc. Amer. Math. Soc. **147**(12), 5373–5383 (2019)
5. Fernandes, C.G., de Pina, J.C., Ramírez Alfonsín, J.L., Robins, S.: Cubic graphs, their Ehrhart quasi-polynomials, and a scissors congruence phenomenon. Discrete Comput. Geom. (2020). DOI: 10.1007/s00454-020-00192-1
6. Greenberg, P.: Piecewise  $SL_2\mathbf{Z}$  geometry. Trans. Amer. Math. Soc. **335**(2), 705–720 (1993)

7. Haase, Ch., McAllister, T.B.: Quasi-period collapse and  $GL_n(\mathbb{Z})$ -scissors congruence in rational polytopes. In: Integer Points in Polyhedra Geometry, Number Theory, Representation Theory, Algebra, Optimization, Statistics. Contemp. Math., vol. 452, pp. 115–122. Amer. Math. Soc., Providence (2008)
8. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
9. Hermann, A.: Classification of Ehrhart quasi-polynomials of Half-Integral Polygons. MSc thesis, San Francisco State University (2010). <http://math.sfsu.edu/beck/teach/masters/andrewh.pdf>
10. Kantor, J.-M.: Triangulations of integral polytopes and Ehrhart polynomials. *Beiträge zur Algebra und Geometrie* **39**(1), 205–218 (1998)
11. McAllister, T.B.: Coefficient functions of the Ehrhart quasi-polynomial of rational polygons. In: International Conference on Information Theory and Statistical Learning (Las Vegas 2008), pp. 114–118. CSREA Press (2008)
12. McAllister, T.B., Woods, K.M.: The minimum period of the Ehrhart quasi-polynomial of a rational polytope. *J. Combin. Theory Ser. A* **109**(2), 345–352 (2005)
13. McMullen, P.: Lattice invariant valuations on rational polytopes. *Arch. Math. (Basel)* **31**(5), 509–516 (1978)

Author accepted manuscript