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A Reduced Basis Method for the Wave Equation

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ABSTRACT
In this contribution, we derive an a posteriori error estimator for the second order wave equation motivated by energy-based a priori estimates by Bernardi and Süli (2005). This estimate (which is valid for general discretizations) is then used to derive a POD-Greedy reduced basis approach for the parameterized wave equation. The quantitative performance of the online-efficient error estimator is shown for an illustrative example, keeping in mind that model reduction of parametrized hyperbolic problems is a challenge.

KEYWORDS
Wave equation, error estimates, reduced basis methods.

1. Introduction

The reduced basis method (RBM) is a widely accepted and intensively studied model reduction technique for parameterized partial differential equations (PPDEs), see e.g. Haasdonk (2017); Hesthaven, Rozza and Stamm (2016); Quarteroni, Manzoni and Negri (2016). Significant progress has been made in particular for elliptic and parabolic problems, Haasdonk and Ohlberger (2008); Urban and Patera (2012, 2014). Hyperbolic problems still remain a challenge. In fact, even for linear transport problems, it may happen that such problems cannot be well reduced in the sense that many degrees of freedom are required for a reduced system of desired accuracy, Ohlberger and Rave (2016).

In this paper, we consider the hyperbolic wave equation. Our point of departure is an a priori estimate of energy type introduced in Bernadi and Süli (2005), see §3. We put this estimate in the framework of a variational formulation of the wave equation (§2), which then allows us to extend it to an a posteriori error estimate involving the residual in each step of a time-marching scheme, §4. This estimate can then be used within a POD-Greedy framework (which has been derived for parametrized evolution problems in Haasdonk and Ohlberger (2008)) to derive a certified reduced basis approximation for the parameterized wave equation, §5. We illustrate the quantitative performance of the derived model reduction by a numerical example, namely a traveling wave problem, which is particularly challenging for model reduction, §6.

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2. Variational Formulation of the Wave Equation

We start by formulating a general linear equation of wave type. To this end, consider a Gelfand triple of Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$ and a symmetric, bounded and positive operator $A \in \mathcal{L}(V, V')$ given by $\langle A\phi, \psi\rangle_{V'\times V} = a(\phi, \psi)$, $\phi, \psi \in V$, induced by a symmetric, bounded, coercive bilinear form $a : V \times V \to \mathbb{R}$, such that

$$\alpha_a \|\psi\|_V \le \|A\psi\|_{V'} \le \gamma_a \|\psi\|_V, \qquad \psi \in V, \tag{1}$$

i.e., α_a and γ_a are coercivity and continuity constants, respectively. Setting $I := (0, T)$, $T > 0$, given $g \in L_2(I; V')$, $u_0 \in H$, $u_1 \in V'$, we look for $u(t) \in H$, $t \in \bar{I}$, such that

$$\ddot{u}(t) + A\,u(t) = g(t) \quad \text{in}\, V',\ \ t \in I \text{ a.e.}, \tag{2a}$$

$$u(0) = u_0 \in H, \quad \dot{u}(0) = u_1 \in V'. \tag{2b}$$

We restrict ourselves to LTI systems even though some of our results can be extended to the more general situation of a time-dependent $A(t)$.

The maybe most standard approach for deriving a weak formulation of (2) is to multiply (2a) with a test function $v \in V$ (i.e., in space only), integrate over space and perform integration by parts, i.e.,

$$\langle \ddot{u}(t), v\rangle_{V'\times V} + a(u(t), v) = \langle g(t), v\rangle_{V'\times V} \quad \forall v \in V,\, t \in I \text{ a.e.}, \tag{3a}$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1. \tag{3b}$$

It is well-known (see e.g. (Grossmann, Roos and Stynes 2007, Thm. 5.30), (Wloka 1987, Thm. 29.1)) that (3) admits a unique solution $u \in H^2(I; V') \cap H^1(I; H) \cap L_2(I; V)$ for all $g \in L_2(I; H)$, $u_0 \in V$ and $u_1 \in H$. Moreover, $u(t)$ stays in the same space as the initial data u_0. Hence $u_0 \in V$ implies $u \in L_2(I; V)$, which is often too restrictive in real wave phenomena, where one may only have $u_0 \in H$. However, the above-mentioned well-posedness result has been extended in (Lions and Magenes 1972, Thm. 9.4) to the situation in which the data only satisfies $g \in L_2(I; V')$, $u_0 \in H$ and $u_1 \in V'$ yielding a unique solution $u \in H^2(I; W') \cap H^1(I; V') \cap L_2(I; H)$, where W' is the dual space of the domain of A typically defined as $W := \text{Dom}(A) := \{u \in H : Au \in H\}$. This latter result is also aligned with d'Alembert's formula for the 1D wave equation which shows that the solution stays (only) in H if the initial data are only that regular. Finally, it is also known that $u \in C^0(\bar{I}; H) \cap C^1(\bar{I}; V')$, so that the initial conditions are in fact well-defined as well as terms like $\|u(t)\|_H$, $\|\dot{u}(t)\|_{V'}$, $t \in \bar{I}$, which we will need later.

Finally, we denote the *Riesz operator* $R : V' \to V$ defined by $(R\tilde{\psi}, \psi)_V = \langle \tilde{\psi}, \psi\rangle_{V'\times V}$, $\tilde{\psi} \in V'$, $\psi \in V$. We recall some well-known quantities, namely $\|R\tilde{\psi}\|_V = \|\tilde{\psi}\|_{V'}$ and $(\tilde{\psi}, \tilde{\phi})_{V'} = \langle R\tilde{\psi}, \tilde{\phi}\rangle_{V\times V'} = (R\tilde{\psi}, R\tilde{\phi})_V$, $\tilde{\psi}, \tilde{\phi} \in V'$, with the inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_{V'}$ on V and V', respectively as well as the duality paring $\langle\cdot,\cdot\rangle_{V'\times V}$ of V' and V induced by H.

3. A Priori Estimates

As already mentioned, we follow Bernadi and Süli (2005); Georgoulis, Lakis and Stynes (2013) to derive a priori energy-type estimates which we will later also use in an a posteriori manner.

Lemma 3.1. *Let $g \in L_2(I;V')$, $u_0 \in H$, $u_1 \in V'$ and denote the unique solution of (3) by $u \in H^2(I;W') \cap H^1(I;V') \cap L_2(I;H)$. Then, we get for all $t \in I$ the estimate*

$$\sqrt{\|\dot u(t)\|_{V'}^2 + \alpha_a \|u(t)\|_H^2} \le \sqrt{\|\dot u(0)\|_{V'}^2 + \gamma_a \|u(0)\|_H^2} + \int_0^t \|g(s)\|_{V'} ds. \qquad (4)$$

Proof. We start by (2) for some $s \in I$ and test it with $R\dot u(s) \in V$, which is the same as taking the inner V'-inner product with $\dot u(s) \in V'$. Then, we get

$$(\ddot u(s), \dot u(s))_{V'} + a(u(s), \dot u(s)) = (g(s), \dot u(s))_{V'}.$$

Next, we use the well-known equalities $(\ddot u(s), \dot u(s))_{V'} = \frac{1}{2}\frac{d}{dt}\|\dot u(s)\|_{V'}^2$ and $a(u(s), \dot u(s)) = \frac{1}{2}\frac{d}{dt}\|A^{1/2}u(s)\|_{V'}^2$. We define $Z(s) := \left(\|\dot u(s)\|_{V'}^2 + \|A^{1/2}u(s)\|_{V'}^2\right)^{1/2}$, where $Z(s) \ge 0$ (and also $Z(s) > 0$ for $u \ne 0$) and arrive at

$$\begin{aligned} Z(s)\dot Z(s) &= \frac{1}{2}\frac{d}{ds}Z(s)^2 = \frac{1}{2}\frac{d}{ds}\|\dot u(s)\|_{V'}^2 + \frac{1}{2}\frac{d}{ds}\|A^{1/2}u(s)\|_{V'}^2 \\ &= (g(s), \dot u(s))_{V'} \le \|g(s)\|_{V'}\|\dot u(s)\|_{V'} \\ &\le \|g(s)\|_{V'} \left(\|\dot u(s)\|_{V'}^2 + \|A^{1/2}u(s)\|_{V'}^2\right)^{1/2} = \|g(s)\|_{V'} Z(s), \end{aligned}$$

thus $\dot Z(s) \le \|g(s)\|_{V'}$. Integrating over $[0,t]$ yields $Z(t) - Z(0) \le \int_0^t \|g(s)\|_{V'} ds$. By inserting the definition of $Z(s)$, we get

$$\sqrt{\|\dot u(t)\|_{V'}^2 + \|A^{1/2}u(t)\|_{V'}^2} \le \sqrt{\|\dot u(0)\|_{V'}^2 + \|A^{1/2}u(0)\|_{V'}^2} + \int_0^t \|g(s)\|_{V'} ds.$$

Finally, note, that (1) immediately yields $\sqrt{\alpha_a}\,\|\phi\|_H \le \|A^{1/2}\phi\|_{V'} \le \sqrt{\gamma_a}\,\|\phi\|_H$, $\phi \in H$. Thus, it follows

$$(\|\dot u(t)\|_{V'}^2 + \alpha_a\|u(t)\|_H^2)^{1/2} \le (\|\dot u(0)\|_{V'}^2 + \gamma_a\|u(0)\|_H^2)^{1/2} + \int_0^t \|g(s)\|_{V'} ds\,,$$

which proves the claim. □

4. An A Posteriori Error Estimate

We can now easily obtain error estimates from the above Lemma 3.1 for a semi-discretization in space. Let us stress the fact that the derived estimates are valid of any suitable discretization and are not restricted to the RBM.

To this end, let $H_h = \operatorname{span}\{\phi_1, \dots, \phi_{\mathcal N}\} \subset H$ and $V_h := \operatorname{span}\{\psi_1, \dots, \psi_{\mathcal N}\} \subset V$, be linear spaces of (possibly large) dimension $\mathcal N \equiv \mathcal N_h \in \mathbb N$. In this case, the discrete problem is to find $u_h(t) = \sum_{i=1}^{\mathcal N} u_{h,i}(t)\,\phi_i$, $u_{h,i}(t) \in \mathbb R$ ($\boldsymbol u_h(t) := (u_{h,i}(t))_{i=1,\dots,\mathcal N}$ being the vector of the expansion coefficients of the approximation u_h at time t), such that

$$\langle \ddot u_h(t), v_h\rangle_{V'\times V} + a(u_h(t), v_h) = \langle g(t), v_h\rangle_{V'\times V} \quad \forall v_h \in V_h, \qquad (5a)$$
$$u_h(0) = u_{0,h}, \quad \dot u_h(0) = u_{1,h}, \qquad (5b)$$

where $u_{0,h}$ and $u_{1,h}$ are finite approximations of u_0 and u_1, respectively. We shall assume that (5) is well-posed. Then, we denote the *error* by $e_h(t) := u(t) - u_h(t)$ and the *residual* as usual by $r_h(t) := g(t) - \ddot{u}_h(t) - Au_h(t)$. These quantities are known to be related by an initial-value problem very similar to (5), namely

$$\langle \ddot{e}_h(t), v_h\rangle_{V'\times V} + a(e_h(t), v_h) = \langle r_h(t), v_h\rangle_{V'\times V} \quad \forall v_h \in V_h, \tag{6a}$$

$$e_h(0) = e_{0,h}, \quad \dot{e_h}(0) = e_{1,h}, \tag{6b}$$

where $e_{0,h} := u_0 - u_{0,h}$, $e_{1,h} := u_1 - u_{1,h}$. As a consequence of Lemma 3.1, we get

Corollary 4.1. *Under the assumptions of Lemma 3.1 and with H_h, V_h as above, we get for all $t \in I$ the estimate*

$$\|u(t) - u_h(t)\|_H \leq \sqrt{\frac{\gamma_a}{\alpha_a}\|e_{0,h}\|_H^2 + \frac{1}{\alpha_a}\|e_{1,h}\|_{V'}^2} + \frac{1}{\sqrt{\alpha_a}}\int_0^t \|r_h(s)\|_{V'}\, ds. \tag{7}$$

5. Reduced Basis Method (RBM)

As already mentioned earlier, the RBM is a model reduction technique for *parameterized* partial differential equations[1]. Thus, we introduce a general notion of a parametric version of the weak formulation of the wave equation (3). Let $\mu \in \mathcal{P} \subset \mathbb{R}^P$ be a parameter and consider the parametric form $a : V \times V \times \mathcal{P} \to \mathbb{R}$. We arrive at the (already discretized in space) parametric semi-discrete variational formulation

$$\langle \ddot{u}_h(t;\mu), v_h\rangle_{V'\times V} + a(u_h(t;\mu), v_h) = \langle g(t;\mu), v_h\rangle_{V'\times V} \quad \forall v_h \in V_h, \tag{8a}$$

$$u_h(0;\mu) = u_{0,h}, \quad \dot{u}_h(0;\mu) = u_{1,h}. \tag{8b}$$

We detail the reduced basis method for this kind of parametric wave equation.

5.1. *Detailed Solution, the "Truth"*

Any reduced basis approximation is based upon an approximation of the underlying problem. This discretization is assumed to be sufficiently fine in order to be able to represent the solution for any parameter with the desired accuracy. Such a detailed discretization is also called the "truth". Thus, we assume that V_h is sufficiently refined and for the remaining temporal discretization we apply a well-known θ-scheme with $\tau := T/K$ for some $K > 1$ and $t^k := k\,\tau$. Then, we seek an approximation $u_h^k(\mu) \approx u_h(t^k;\mu)$, where we will often omit the μ-dependency in order to shorten the notation. Then, the fully discrete problem for a given parameter $\mu \in \mathcal{P}$ amounts to find functions $u_h^k = u_h^k(\mu) \in V_h$, $k = 2, \ldots, K$ (given the initial value $u_h^0 := u_{0,h}$ and a second-order accurate approximation u_h^1 of $u_h(t^1;\mu)$), such that for all $v_h \in V_h$

$$\begin{aligned}\frac{1}{\tau^2}(u_h^{k+1} - 2u_h^k + u_h^{k-1}, v_h)_{L_2(\Omega)} + a(\theta u_h^{k+1} + (1-2\theta)u_h^k + \theta u_h^{k-1}, v_h;\mu) =\\ = \theta\langle g(t^{k+1};\mu), v_h\rangle_{V'\times V} + (1-2\theta)\langle g(t^k;\mu), v_h\rangle_{V'\times V} + \theta\langle g(t^{k-1};\mu), v_h\rangle_{V'\times V}\\ =: \langle g_h^k(\mu), v_h\rangle_{V'\times V}, \qquad k = 1, \ldots, K-1.\end{aligned} \tag{9}$$

[1]For recent introductions to the RBM, we refer to Haasdonk (2017); Hesthaven, Rozza and Stamm (2016); Quarteroni, Manzoni and Negri (2016).

For $\theta \geq 1/4$, the above scheme is unconditionally stable, we use $\theta = 1/4$. The above iteration (9) can be rewritten in operator form for fixed θ as

$$\mathcal{L}_I u_h^{k+1} = \mathcal{L}_{E_1} u_h^k + \mathcal{L}_{E_2} u_h^{k-1} + b_h^k, \quad k = 1, \ldots, K-1, \tag{10}$$

where $\mathcal{L}_I = \mathcal{L}_I(\mu) := I_h + \theta\tau^2 A_h(\mu) : V_h \to V_h$ is the implicit part, $\mathcal{L}_{E_1} = \mathcal{L}_{E_1}(\mu) := 2I_h - (1-2\theta)\tau^2 A_h(\mu)$, $\mathcal{L}_{E_2} = \mathcal{L}_{E_2}(\mu) := -I_h - \theta\tau^2 A_h(\mu) : V_h \to V_h$ the two explicit ones as well as $A_h(\mu) : V_h \to V_h$ is the operator induced by $a(\cdot,\cdot;\mu)$ on $V_h \times V_h$, I_h is the identity on V_h and $b_h^k := \tau^2\, g_h^k(\mu)$.

5.2. *Reduced Basis Error Estimate*

Both for the offline determination of a reduced model as well as for the online certification of a reduced basis approximation, we need an error estimator, which is *online-efficient*, i.e., whose computational work is independent of the detailed dimension $\mathcal{N}$. Of course, for the considered wave equation, such an estimator also needs to incorporate the temporal evolution of a reduced system.

In order to obtain a reduced temporal evolution, we replace V_h in the θ-scheme for the detailed approximation by some subspace $V_N \subset V_h$ of dimension $N \ll \mathcal{N}$ (the construction of V_N will be detailed below). Hence, we compute $(u_N^k)_{k=1,\ldots,K}$ such that

$$\begin{aligned}\frac{1}{\tau^2}(u_N^{k+1} - 2u_N^k + u_N^{k-1}, v_N)_{L_2(\Omega)} + a(\theta u_N^{k+1} + (1-2\theta)u_N^k + \theta u_N^{k-1}, v_N; \mu) = \\ = \langle g_h^k(\mu), v_N\rangle_{V' \times V}, \qquad k = 1, \ldots, K-1,\end{aligned}$$

for all $v_N \in V_N$, given some sufficiently accurate approximations of the initial data, namely u_N^0 and u_N^1. Recalling that we have an implicit scheme, the above iteration requires to solve K linear problems of reduced dimension N, which can be written as

$$\mathcal{L}_{I,N} u_N^{k+1} = \mathcal{L}_{E_1,N} u_N^k + \mathcal{L}_{E_2,N} u_N^{k-1} + b_N^k, \quad k = 1, \ldots, K-1,$$

where $\mathcal{L}_{I,N} := P_N \circ \mathcal{L}_I$, $\mathcal{L}_{E_i,N} := P_N \circ \mathcal{L}_{E_i}$, $i = 1, 2$, and $b_N^k := P_N b_h^k$, and $P_N : V_h \to V_N$ denotes the orthogonal projection. Due to the assumed separability w.r.t. the parameter, this iteration can be performed online-efficient.

Next, for $\mu \in \mathcal{P}$, we want to estimate the (truth) *error* at time-step k defined by $e_N^k(\mu) := u_h^k(\mu) - u_N^k(\mu)$ in terms of the *residual* $\mathcal{R}_N^k(\mu)$, i.e.,

$$\mathcal{R}_N^{k+1}(\mu) := \frac{1}{\tau^2}\left(\mathcal{L}_{E_2} u_N^{k-1} + \mathcal{L}_{E_1} u_N^k - \mathcal{L}_I u_N^{k+1} + b_h^k\right), \quad k = 1, \ldots, K-1. \tag{11}$$

We look for an *error-residual relation.* A standard key observation is the fact that the error satisfies an evolution problem, namely (for $k = 1, \ldots, K-1$)

$$\mathcal{L}_I e_N^{k+1} = \mathcal{L}_I u_h^{k+1} - \mathcal{L}_I u_N^{k+1} = \mathcal{L}_{E_2} e_N^{k-1} + \mathcal{L}_{E_1} e^k + \tau^2\, \mathcal{R}_N^{k+1}. \tag{12}$$

Then, the desired a-posteriori error bound follows from Corollary 4.1 applied to (12),

$$\|u_h^k(\mu) - u_N^k(\mu)\|_H \leq \Delta_N^k(\mu), \quad k = 0, \ldots, K,$$

with the reduced basis error estimator given by

$$\Delta_N^k(\mu) := \sqrt{\tfrac{\gamma_a}{\alpha_a}\|e_N^0(\mu)\|_H^2 + \tfrac{1}{\alpha_a}\|\dot{e}_N^0(\mu)\|_{V'}^2} \\ + \frac{1}{\sqrt{\alpha_a}}\Big(\frac{\|\mathcal{R}_N^2\|_{V'}}{2} + \sum_{i=3}^{k-1}\|\mathcal{R}_N^i\|_{V'} + \frac{\|\mathcal{R}_N^k\|_{V'}}{2}\Big).$$

Here $\dot{e}_N^k(\mu) := \dot{u}_h^k(\mu) - \dot{u}_N^k(\mu)$ is the error of the temporal derivative. Moreover, the integral on the right-hand side in (7) is approximated by the trapezoidal rule. We will always use $\Delta_N^K(\mu)$, i.e., the error bound at the final time T, in order to control the full evolution.

Note, that the residual $\mathcal{R}_N^k(\mu)$ in (11) is separable w.r.t. the parameter, so that $\Delta_N^k(\mu)$ can be computed online-efficient by an offline/online computational procedure.

5.3. *Reduced Basis Method via POD-Greedy*

The POD-Greedy method has been introduced in Haasdonk and Ohlberger (2008) as a model reduction method for linear evolution equations. It is a combination of the standard Greedy algorithm for the parameter search and a Proper Orthogonal Decomposition (POD) in time in order to select the time step containing the maximal information of the trajectory for the given parameter. We use this framework in combination with our a posteriori error estimator $\Delta_N^k(\mu)$ introduced above. The resulting scheme is shown in Algorithm 1.

Algorithm 1 POD-Greedy algorithm, see Haasdonk and Ohlberger (2008)

Require: Given $N_{\max} > 0$, finite training set $\mathcal{P}_{\text{train}} \subset \mathcal{P}$, accuracy ϵ_{tol}

1: choose $\mu^1 \in \mathcal{P}_{\text{train}}$, k arbitrarily; set $\Psi_1 := \left\{\frac{u_h^k(\mu^1)}{\|u_h^k(\mu^1)\|_V}\right\}$, $V_1 := \text{span}(\Psi_1)$; set $\ell := 1$
2: **while** $\max_{\mu\in\mathcal{P}_{\text{train}}} \Delta_\ell^K(\mu) > \epsilon_{\text{tol}}$ **do**
3: define $\mu^{\ell+1} := \text{argmax}_{\mu\in\mathcal{P}_{\text{train}}} \Delta_\ell^K(\mu)$ ▷ Greedy
4: define $\tilde{\psi}_{\ell+1} := \text{POD}\big\{u_h^k(\mu^{\ell+1}) - \text{Proj}_{V_\ell}(u_h^k(\mu^{\ell+1}))\big\}_{k=0,\ldots,K}$ ▷ POD
5: define $\Psi_{\ell+1} := \text{orthonormalize}\big(\Psi_\ell \cup \{\tilde{\psi}_{\ell+1}\}\big)$, $V_{\ell+1} := \text{span}(\Psi_{\ell+1})$, $\ell := \ell + 1$
6: **end while**
7: define $V_N := V_\ell$, $N :=\dim(V_N)$ **return** V_N, N;

Let us comment on line 1: Often, the initial values are taken as the first basis functions. However, since we consider homogeneous initial conditions, we use some arbitrary snapshot as initialization for the POD-Greedy scheme. Algorithm 1 yields an N-dimensional reduced space $V_N = \text{span}\{\psi_1, \ldots, \psi_N\}$.

6. A Numerical Experiment

We present some results of one of our numerical experiments.

Model data. Choose $\Omega = (0,1)$ and $I := (0,1)$. We set $V := H_0^1(\Omega) \hookrightarrow L_2(\Omega) =: H$, hence $V' = H^{-1}(\Omega)$. The parameter $\mu \in \mathcal{P} := [0.3, 2] \subset \mathbb{R}$ is chosen to be the wave speed. Then, we aim at determining $u(\cdot,\cdot;\mu) : \overline{I \times \Omega} \to \mathbb{R}$ that solves (in weak form)

$$\ddot{u}(t,x;\mu) - \mu^2\, u_{xx}(t,x;\mu) = 0 \qquad \text{on } I \times \Omega, \tag{13a}$$
$$u(t,0;\mu) = \tanh(5t)^3 \qquad \forall\, t \in I, \tag{13b}$$
$$u(t,1;\mu) = 0 \qquad \forall\, t \in I, \tag{13c}$$
$$u(0,x;\mu) = 0 \qquad \forall\, x \in \Omega, \tag{13d}$$
$$\dot{u}(0,x;\mu) = 0 \qquad \forall\, x \in \Omega. \tag{13e}$$

The separability w.r.t. the parameter is obvious. The wave speed μ is normalized in such a way that the wave front reaches the right endpoint $x = 1$ of Ω exactly at the terminal time $T = 1$ for the choice $\mu = 1$, see Figure 1. For $\mu < 1$, the wave front does

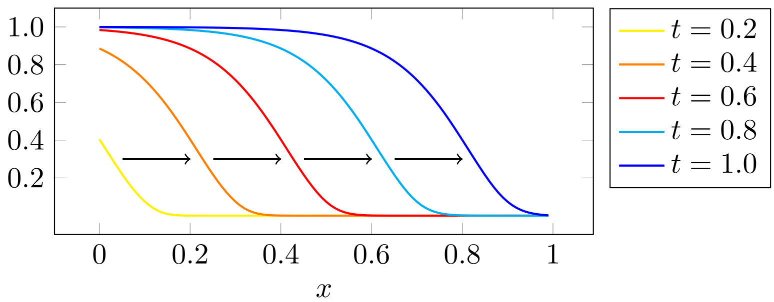


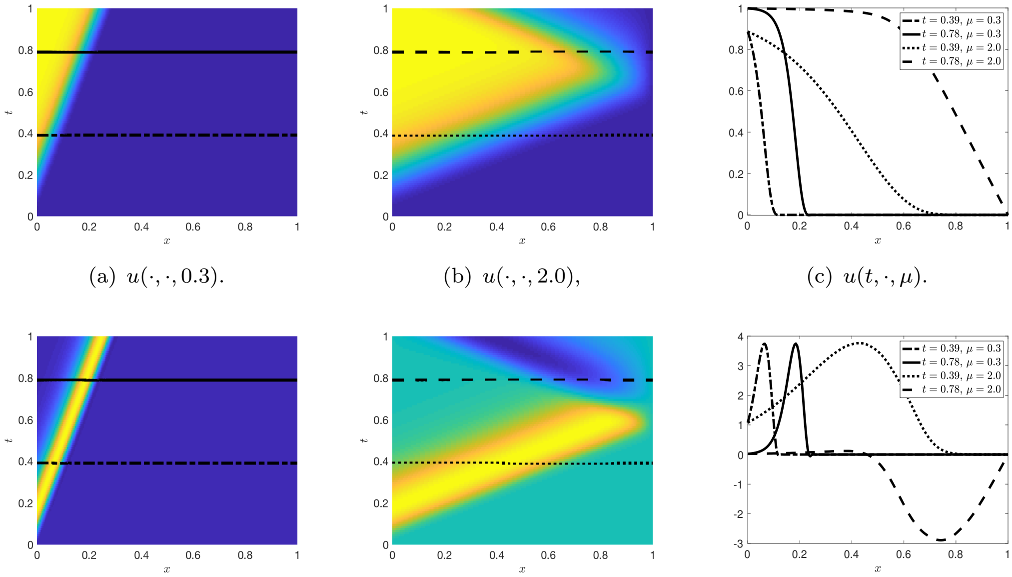
Figure 1. Solution $u(t,\cdot;1)$ of (13) for $\mu = 1$ and different times $t = 0.2, 0.4, 0.6, 0.8, 1.0$ (yellow, orange, red, cyan, blue): The wave evolves in time from left to right from $t_0 = 0$ to $T = 1$ due to the left-hand side boundary condition (13b).

not reach $x = 1$ within the period of time I, whereas for $\mu > 1$, the wave is reflected at the right end point due to the homogeneous Dirichlet boundary conditions in (13c), see also the upper row in Figure 2, where we depict the solutions for $\mu = 0.3$ and $\mu = 2.0$. The reflection is clearly visible for $\mu = 2.0$ in Figure 2(b).

The strongly parameter-dependent shape of the solutions leads us to the expectation that this is a tough problem for model reduction since a linear combination of $u(\cdot,\cdot;\mu^i)$, $i = 1,\dots,N$ cannot be expected to yield a good approximation for $u(\cdot,\cdot;\mu)$, $\mu \neq \mu^i$. In order to make that point even clearer, we also take the spatial derivatives $u_x(\cdot,\cdot;\mu)$ into consideration, see the bottom row of Figure 2. As we see, the dependency of the derivative w.r.t. the parameter is even more pronounced – another hint that we are facing a challenging problem for model reduction.

We divide Ω into 400 and I into 100 subintervals, i.e., $\mathcal{N} = 400$ and $K = 101$. The training set $\mathcal{P}_{\text{train}}$ is chosen as 60 equidistantly distributed points in $\mathcal{P}$.

Results. We start by investigating the rate of the decay of the truth RB error $\|u_h^K - u_N^K\|_{L_2}$ at the final time in the L_2-norm. As we consider a tough problem for model reduction and having the slow decay of $N^{-1/2}$ for the transport equation in mind (Ohlberger and Rave (2016),Brunken, Smetana and Urban (2019)), we expect some polynomial decay. In fact, as can be seen on the left of Figure 3, we obtain an average rate of $N^{-7/2}$ – as opposed to exponential rates observed e.g. for certain elliptic and parabolic problems. This was to be expected since the reachable rate is limited by the decay of the Kolmogorov N-width $d_N(\mathcal{P})$. However, using similar arguments as in Ohlberger and Rave (2016), one can in fact show that the decay of $d_N(\mathcal{P})$ for the



(a) $u(\cdot, \cdot, 0.3)$. (b) $u(\cdot, \cdot, 2.0)$, (c) $u(t, \cdot, \mu)$.

(d) $u_x(\cdot, \cdot, 0.3)$. (e) $u_x(\cdot, \cdot, 2.0)$. (f) $u_x(t, \cdot, \mu)$.

Figure 2. Detailed solutions and spatial derivatives for different parameter values. Right column: slice at fixed times $t \in \{0.39, 0.78\}$.

wave equation is (at least in the worst case) not better than for transport equations.

Moreover, the comparison of the exact error (in blue) and the error estimator (in red) shows that our error estimator introduced above in fact realizes the same decay rate as the exact error – in the RB language this means that the weak Greedy decay is comparable to the strong Greedy decay. This shows that $\Delta_N^K(\mu)$ properly reflects the behavior of the problem (on the chosen training set $\mathcal{P}_{\text{train}}$).

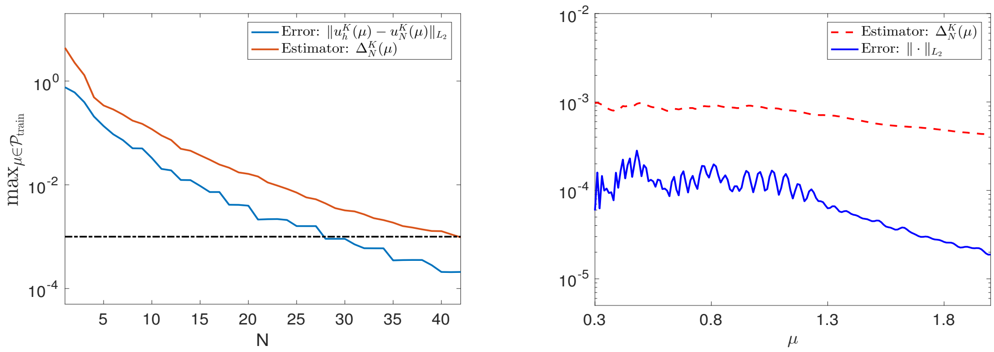


Figure 3. Left: RB-Greedy approximation error (red: error estimator $\Delta_N^K(\mu)$; blue: true L_2-error). Right: L_2-Error (blue, solid) and error estimator (red, dashed) over a test set $\mathcal{P}_{\text{test}} \subset [0.3, 2] = \mathcal{P}$.

Next, we consider a test set $\mathcal{P}_{\text{test}}$ (disjoint from $\mathcal{P}_{\text{train}}$), which is chosen as 171 equidistantly distributed points in $\mathcal{P}$. For those parameters, we compare the true RB error (in solid blue) with the error estimator $\Delta_N^K(\mu)$. Here $N := 42$ is chosen in such a way, that the POD-Greedy error is below $\epsilon_{\text{tol}} := 10^{-3}$ (indicated by the horizontal line on the left of Figure 3). First, we see that this tolerance is in fact realized, the red dashed line is always below ϵ_{tol}. Second, we see that the slope of $\Delta_N^K(\mu)$ mainly follows the true error with slightly increasing distance for growing wave numbers μ.

Defining the effectivity ratio in the usual way as $\eta_N(\mu) := \frac{\Delta_N^K(\mu)}{\|u_h^K(\mu)-u_N^K(\mu)\|_{L_2}}$, we obtain

$$\frac{1}{|\mathcal{P}_{\text{test}}|}\sum_{\mu\in\mathcal{P}_{\text{test}}}\eta_N(\mu) = 10.72, \qquad \frac{1}{|\mathcal{P}_{\text{test}}\cap[0.3,1]|}\sum_{\mu\in\mathcal{P}_{\text{test}}\cap[0.3,1]}\eta_N(\mu) = 6.92,$$

which also shows that the problem is increasingly tough for increasing wave speed.

7. Conclusions and Outlook

We have presented an energy-based error estimate for the wave equation and have used it for model reduction in a wave speed-parameterized problem. We have proven the a posteriori error bound and have quantitatively investigated the performance of the arising POD-Greedy reduced basis approximation for a traveling wave problem. We have observed polynomial decay $N^{-7/2}$ of the reduced basis error and average effectivities of about 10.72.

The presented results seem to justify further investigations. We will extend this work to first-order system formulations of the wave equations also including additional dissipation for numerical stabilization. Moreover, we will also consider space-time variational formulations of the wave equation in the spirit of Brunken, Smetana and Urban (2019); Urban and Patera (2012, 2014).

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