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As Published: <https://doi.org/10.1007/s00440-019-00918-0>

Publisher: Springer Berlin Heidelberg

Persistent URL: <https://hdl.handle.net/1721.1/131296>

Version: Author's final manuscript: final author's manuscript post peer review, without publisher's formatting or copy editing

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Cite this article as: Nina Holden and Scott Sheffield, Scaling limits of the Schelling model, Probability Theory and Related Fields <https://doi.org/10.1007/s00440-019-00918-0>

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Author accepted manuscript

Scaling limits of the Schelling model

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May 3, 2019

Abstract

The Schelling model of segregation, introduced by Schelling in 1969 as a model for residential segregation in cities, describes how populations of multiple types self-organize to form homogeneous clusters of one type. In this model, vertices in an N -dimensional lattice are initially assigned types randomly. As time evolves, the type at a vertex v has a tendency to be replaced with the most common type within distance w of v . We present the first mathematical description of the dynamical scaling limit of this model as w tends to infinity and the lattice is correspondingly rescaled. We do this by deriving an integro-differential equation for the limiting Schelling dynamics and proving almost sure existence and uniqueness of the solutions when the initial conditions are described by white noise. The evolving fields are in some sense very “rough” but we are able to make rigorous sense of the evolution. In a key lemma, we show that for certain Gaussian fields h , the supremum of the occupation density of $h - \phi$ at zero (taken over all 1-Lipschitz functions ϕ) is almost surely finite, thereby extending a result of Bass and Burdzy. In the one dimensional case, we also describe the scaling limit of the limiting clusters obtained at time infinity, thereby resolving a conjecture of Brandt, Immorlica, Kamath, and Kleinberg.

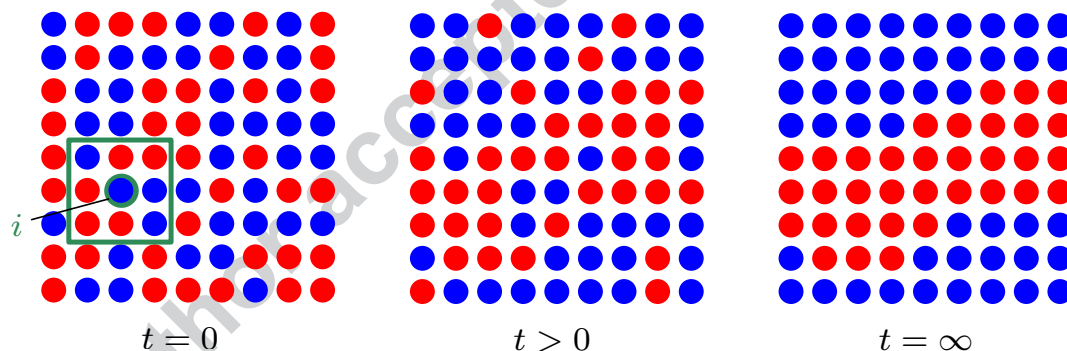


Figure 1: The Schelling model on the two-dimensional torus with two types (red and blue). At time $t = 0$ (left) the types of the nodes are chosen uniformly and independently at random. Each node i is associated with a neighborhood (shown in green) and an independent rate one Poisson clock. Every time the clock of a node rings, it updates its type to the most common type in its neighborhood. Eventually we reach a stable configuration (right).

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1 Introduction

The Schelling model [Sch69, Sch71, Sch78] was initially introduced to explain residential segregation in cities, and is one of the earliest and most influential agent-based models studied by economists. Variants of the model have been studied by thousands of researchers within a number of disciplines, e.g. social sciences, statistical mechanics, evolutionary game theory, and computer science, see the works referenced below and [Cla91, PW01, LJ03, VK07, PV07, SS07, DCM08, Odo08, GGS⁺08, GVN09, GBLJ09, SVW09] for a small and incomplete selection of these works. Until recently [BIKK12, BEL14, IKLZ17, BEL15b, BEL16, BEL15a, OF18] most analysis of the model was based either on simulation, non-rigorous analysis or so called “perturbed” versions of the model (discussed below). We will discuss the Schelling model history and give an informal overview of the paper in Sections 1.1 and 1.2, and we provide a precise definition of the model in Section 1.3.

1.1 History

In the original formulation of the model, individuals of two “types” occupy a subset of the nodes of a graph, and at random times an individual moves to a free (i.e., unoccupied) position in the graph. Individuals move to locations at which they will have more neighbors of their own type. Schelling showed, using simulations he implemented manually with pennies and dimes on a ruled sheet of paper [Sch78], that segregation occurs even if the agents have only a weak preference for being in regions with a high density of their own type. His findings have been confirmed later by a huge number of simulations of other researchers, and his findings have strongly influenced debates about the causes of residential segregation [CF08]. The introduction of the model also contributed to Schelling winning the Nobel Memorial Prize in Economics in 2005 [N05].

The first mathematically rigorous results on the model considered a variant where the dynamics describing the transition between states were “perturbed” in the sense that agents have a small probability $p > 0$ of acting against their preference [You01, Zha04]. The perturbed model was analyzed by studying the stationary distribution of the associated Markov chain. In particular, the stochastically stable states, which are states whose stationary probability is bounded away from zero when $p \rightarrow 0$, were studied. The stochastically stable states are proved to be those which minimize the length of the interface between the two types of individuals (i.e., the number of neighboring pairs containing one individual of each type) so that using the terminology of statistical physics the stochastically stable states correspond to Ising model ground states.

One interesting property of the unperturbed model is that it can be shown to *stabilize* a.s. in finite time, i.e., after some point in time the agents stop moving. It has been argued (see e.g. [IKLZ17]) that these limiting stable configurations have at least some properties in common with the segregation patterns observed in real cities, e.g. since they tend to appear more irregular than the stochastically stable states of the perturbed model. We will not address real world segregation patterns in this paper.

The first mathematically rigorous analysis of the unperturbed model by Brandt, Immorlica, Kamath, and Kleinberg [BIKK12] concerns a version of the model on the one-dimensional torus where the neighborhood of a node is given by its nearest $2w + 1$ neighbors (including itself) for some $w \in \mathbb{N}$. They prove that the configuration of types in the stable limiting configuration consists of intervals of length at most polynomial in w . In [IKLZ17] the authors consider a two-dimensional Schelling model where an agent only changes location if the fraction of his neighbors having the same type as himself is $< 1/2 - \epsilon$ for some $\epsilon \ll 1$. They prove that the expected diameter of the segregated region containing the origin in the final configuration grows at least exponentially in w^2 . In [OF18] an upper bound on the size of this region of the same order is established, and the authors prove that their result holds for $\epsilon < 0.134$. See also [BEL14, BEL15b, BEL16, BEL15a] for recent rigorous results on the unperturbed Schelling model in one, two, and three dimensions. The models studied in these papers have a more general initial configuration of types and/or more general tolerance parameters than the models in [BIKK12, IKLZ17, OF18] and the current paper. The authors are particularly interested in parameter values which lead to either very high degree of segregation, total takeover of one type, or almost no changes relative to the initial configuration. The Schelling model is a variant of the so-called *threshold voter model* [Lig13]; see the end of Section 1.3 for further details.

We will focus on a variant of the model which we call the *single-site-update* Schelling model, briefly explained later in this paragraph, in which individual vertices are updated one at a time. This variant of the model is also the one considered in [BEL15b, BEL16, IKLZ17]. Some of the other papers mentioned above consider the *pair-swapping* Schelling model, where two individuals of different types will swap positions with each other if this leads to both nodes having more neighbors of their own type. In both variants of the model all the nodes of the considered graph are occupied, i.e., there are no free or unoccupied nodes. In the single-site-update Schelling model unsatisfied individuals *change* types, instead of swapping with each other. That is, one picks a random *individual* and allows that individual to change type if desired, instead of picking a *pair* of individuals and asking them to swap locations if desired. In the single-site-update version, the number of vertices of a given type is not constant. Instead, one imagines that there is a larger “outside world” beyond the graph being considered, and that when a vertex changes type, it corresponds to an individual within the configuration swapping location with someone from the “outside world.” The single-site-update evolution is essentially equivalent to the pair-swapping model evolution in a setting with an “outside world” region (disconnected from the main lattice graph under consideration) that contains a large number of unsatisfied individuals of each type. We will focus on the single-site-update variant in this paper because it is cleaner mathematically (one only has to deal with one individual at a time when making updates), but we will explain at the end of Section 1.4 that our first main result (Theorem 1.2) also holds in the pair-swapping setting.

1.2 Overview

We study an unperturbed, single-site-update version of the Schelling model on an N -dimensional lattice with $M \geq 2$ different types. A node is unsatisfied if the most common type in its neighborhood differs from its current type, and the size of the neighborhood is described by a constant $w \in \mathbb{N}$. Adapting the vocabulary of majority dynamics (see e.g. the survey [MT17] and references therein), we call the type of the node the *opinion* of the node. At time zero, each node is assigned an opinion uniformly and independently at random. Each node is associated with an independent Poisson clock, and every time the clock of a node rings it updates its opinion to the most common opinion in its neighborhood. In other words, a node changes its opinion when its Poisson clock rings if and only if the node is currently not satisfied.

We prove a dynamical scaling limit result for the early phase of the Schelling dynamics for any $N \in \mathbb{N}$ and $M \in \{2, 3, \dots\}$. We define a vector-valued function Y^w called the normalized bias function; as explained below, $Y^w(z)$ is the vector whose components are (a normalizing constant times) the M opinion densities

(minus their expectations) in the radius w box centered at z . We prove that Y^w converges in the scaling limit to the solution Y of a differential equation (more precisely, an *integro-differential equation*) with Gaussian initial data. We call the function Y the continuum bias function, and we call the associated initial value problem the continuum Schelling model. See Theorem 1.2 and Proposition 3.1. Solutions of the differential equation are not unique for all choices of initial data. However, we prove existence and uniqueness of solutions for Gaussian initial data.

The basic idea of the argument is to note that even though the initial Gaussian normalized bias function is very rough, the *change* in its value from the initial time to a finite later time is a.s. a (random) Lipschitz function. Focusing on this difference, we are left with a random ODE in the (more regular) space of Lipschitz functions. To establish existence and uniqueness of the evolution within this space, we wish to apply a variant of the Picard-Lindelöf theorem, but doing so requires some sort of continuity in the corresponding ODE, which requires us to understand whether there are situations where two coordinates of the normalized bias function are *very* close on a large set, so that even small perturbations lead to big changes in the direction the functions are evolving. It turns out that one can show these situations are unlikely by establishing control on the maximal *occupation kernel* corresponding to the intersection of the initial Gaussian function with a Lipschitz function (where the maximum is taken over all functions with Lipschitz norm bounded by a fixed constant). Analogous results for Brownian local time, established by Bass and Burdzy in [BB01], turn out to be close to what we need, and we are able to adapt the techniques of [BB01] to our higher dimensional setting with a few modifications.

In the special case when $N = 1$ and $M = 2$ we also prove a scaling limit result for the final configuration of opinions. This confirms a variant of a conjecture in [BIKK12].¹ More precisely, we show in Theorem 1.3 below that the law on subsets of \mathbb{Z} describing the limiting opinion of each node converges upon rescaling by w . The theorem says that if we study the model on the rescaled lattice $w^{-1}\mathbb{Z}$ and let $A \subset w^{-1}\mathbb{Z}$ be the set of nodes for which the limiting opinion is 1, then A converges in law as $w \rightarrow \infty$, viewed as an element in the space of closed subsets of \mathbb{R} equipped with the Hausdorff distance. The scaling limit result is proved by studying the long-time behavior of the continuum bias function Y . This is one of the more technically interesting parts of the paper, as a number of tricks are used to rule out anomalous limiting behavior. Our conclusion is that in the limit one obtains a random collection of homogeneous neighborhoods, each of width strictly greater than one. The idea of the proof is to show that if this does not occur, then it will occur if we make a slight perturbation to the initial data, and a delicate analysis of the differential equation is required to show that this is indeed the case.

After we describe the continuum dynamics and (for $N = 1, M = 2$) its limiting behavior, we will need to do some additional work to make the connection with the discrete model. We consider two phases separately: First we study the model up to time $Cw^{-N/2}$ for $C \gg 1$, and then (for $N = 1, M = 2$) we consider times larger than $Cw^{-1/2}$. The first phase of the evolution is governed by the differential equation, and we prove that the differential equation predicts the evolution of the discrete model well by bounding the error which accumulates during a short interval of length Δt . The second phase starts when the solution of the differential equation has almost reached its limiting state with homogeneous intervals. We show that with high probability the homogeneous intervals observed at time $Cw^{-1/2}$ will continue to exist until all nodes have reached their final opinion. Nodes near the boundary between two intervals at time $Cw^{-1/2}$ have approximately half of their neighbors of each opinion, which makes it hard to control the evolution of the bias for these nodes; however, we do manage to show that with high probability each interval does not shrink too much before all nodes have reached their final opinion.

In the remainder of the introduction we will give a precise definition of the Schelling model and state our main results. In Section 2.3 we show existence and uniqueness of solutions of the continuum Schelling model by using results from Section 2.2, and in Section 2.4 we prove that for the one-dimensional model with $M = 2$ the sign of the solution converges a.s. at almost every point. In Section 3.1 we prove that the continuum Schelling model describes the discrete Schelling model well for small times and large w . In Section

¹The original version of the conjecture in [BIKK12] is for the pair-swapping variant of the model, while we mainly consider the single-site-update variant in this paper. As we explain in Sections 1.4 and 2, most of our arguments can be adapted to the pair-swapping setting.

3.2 we conclude the proof of the scaling limit result for the one-dimensional Schelling model. We also prove (proceeding similarly as in [TT15]) that the opinion of each node converges a.s. for any $N \in \mathbb{N}$ and $M \geq 2$, and we include a lemma which might be related to the typical cluster size for the limiting opinions in higher dimensions. We conclude the paper with a list of open questions in Section 4.

1.3 The Schelling model

We will start by defining the Schelling model on a general simple graph G with vertex set $V(G)$ and edge set $E(G)$. The vertex set $V(G)$ may be infinite, but we assume G is locally finite. Let $M \in \{2, 3, 4, \dots\}$. Each $i \in V(G)$ is associated with an opinion in $[M] := \{1, \dots, M\}$ and an independent unit rate Poisson clock, i.e., each node is associated with a clock such that the times between two consecutive rings of the clock are distributed as i.i.d. unit rate exponential random variables. Let $X(i, t) \in [M]$ denote the opinion of node i at time $t \geq 0$, and let $\mathfrak{N}(i) := \{j \in V(G) : (i, j) \in E(G)\} \cup \{i\}$ be the neighborhood of i . Every time the Poisson clock of a node rings, the node updates its opinion according to the following rules:

- (i) The node chooses the most common opinion in its neighborhood if this is unique. In other words, we set $X(i, t) = m$ for $m \in [M]$ if for all $m' \in [M] \setminus \{m\}$ we have $|\{j \in \mathfrak{N}(i) : X(j, t) = m\}| > |\{j \in \mathfrak{N}(i) : X(j, t) = m'\}|$.
- (ii) If there is a draw between different opinions, and the current opinion of the node is one of these opinions, the node keeps its current opinion.
- (iii) If there is a draw between different opinions, and none of these opinions are equal to the current opinion of the node, the new opinion of the node will be chosen uniformly at random from the set of most common opinions in its neighborhood.

Note that a.s. no two Poisson clocks will ring simultaneously, even when $|V(G)|$ is infinite. Furthermore, one can show that for graphs with bounded degree, and for any fixed vertex $i \in V(G)$ and a fixed time $t \in \mathbb{R}_+$, the set of vertices whose initial opinion may have influenced the opinion of i at time t , given the times at which the various Poisson clocks were ringing, is a.s. finite, see e.g. [TT15, Claim 3.5]. These two observations imply that the configuration at each time t is a.s. determined by the initial configuration and the ring times, along with knowledge about how the draws described in (iii) are resolved.

In this paper, we will consider the Schelling model on a lattice, and we consider scaling limits as the neighborhood size tends to infinity. Let $N \in \mathbb{N}$ be a parameter describing the dimension of the graph, and let $w \in \mathbb{N}$ be a parameter we call the window size. Let $\mathcal{N} = (-1, 1)^N$ (or, alternatively, let $\mathcal{N} \subset (-1, 1)^N$ be a sphere or some other shape; precise conditions on \mathcal{N} appear below). Define the neighborhood $\mathfrak{N}(i)$ of an element i of \mathbb{Z}^N by

$$\mathfrak{N}(i) = \{j \in \mathbb{Z}^N : w^{-1}(j - i) \in \mathcal{N}\}. \tag{1}$$

In most of the paper (at least for $N > 1$) we will work on a torus whose size is a large constant times the neighborhood size; precisely, for a fixed constant $R \in \{3, 4, \dots\}$ and defining the one-dimensional torus $\mathfrak{S} = \mathbb{Z}/(Rw\mathbb{Z})$ we will work on the torus \mathfrak{S}^N . Some places we also consider the model on \mathbb{Z}^N . In the remainder of this section we describe the Schelling model in terms of \mathfrak{S}^N rather than \mathbb{Z}^N , but we obtain the model on \mathbb{Z}^N by repeating the description with \mathbb{Z}^N instead of \mathfrak{S}^N . To simplify notation when considering the Schelling model on \mathfrak{S}^N , we identify an element $i \in \mathbb{Z}^N$ with its equivalence class in \mathfrak{S}^N .

Assume the initial opinions of the nodes are i.i.d. random variables satisfying $\mathbb{P}[X(i, 0) = m] = M^{-1}$ for all $i \in \mathfrak{S}^N$ and $m \in [M]$. Throughout the paper we let \mathcal{R} denote the set of rings of the Poisson clocks

$$\mathcal{R} = \{(i, t) \in \mathfrak{S}^N \times \mathbb{R}_+ : \text{the clock of node } i \in \mathfrak{S}^N \text{ rings at time } t\}. \tag{2}$$

We define the *bias* of node $i \in \mathfrak{S}^N$ towards opinion $m \in [M]$ at time $t > 0$, to be the sum $\sum_{j \in \mathfrak{N}(i)} \mathbf{1}_{X(j, t) = m}$. When the clock of a node rings, the node updates its opinion to the opinion towards which it has the strongest bias, with draws resolved as described in (ii)-(iii) above. We say that $i \in \mathfrak{S}^N$ agrees with the most common opinion in its neighborhood at time t if $\sum_{j \in \mathfrak{N}(i)} \mathbf{1}_{X(j, t) = X(i, t)} \geq \sum_{j \in \mathfrak{N}(i)} \mathbf{1}_{X(j, t) = m}$ for any $m \in [M]$. In

other words, i agrees with the most common opinion in its neighborhood if and only if it would not update its opinion if its Poisson clock were ringing.

In the description of the Schelling model above we considered (for simplicity of the description) the neighborhood $\mathcal{N} = (-1, 1)^N$, but our results are proved for more general neighborhoods, since the more general case is not significantly more difficult to analyze. Unless otherwise stated, we will assume that $\mathcal{N} \subset (-1, 1)^N$ is an arbitrary open set containing 0, such that $\partial\mathcal{N}$ has upper Minkowski dimension strictly smaller than N , and such that the following technical condition is satisfied. If x_0 is a unit vector in an arbitrary direction, λ denotes Lebesgue measure, and we let $\mathcal{N} + sx_0$ denote the set $\{x + sx_0 : x \in \mathcal{N}\}$ for some $s \in \mathbb{R}$, then

$$\liminf_{t \rightarrow 0^+} \frac{1}{t} \lambda \left((\mathcal{N} + tx_0) \setminus \bigcup_{s \leq 0} (\mathcal{N} + sx_0) \right) > 0. \tag{3}$$

The condition (3) will be used in the proof of Lemmas 2.5 and 2.7. It is obviously satisfied by most of the smooth-boundary regions one would be inclined to consider. We may for example let $\mathcal{N} = \mathcal{N}_p$, where $\mathcal{N}_p := \{x \in \mathbb{R}^N : \|x\|_p < 1\}$ is a metric ball for the ℓ^p norm for some $p \in [1, \infty]$.

We remark that the first main result of the next section holds for the general N -dimensional Schelling model on the torus with $M \in \{2, 3, \dots\}$ opinions, while the second main result holds for the torus or the real line for $N = 1$, $M = 2$, and $\mathcal{N} = \mathcal{N}_\infty$.

Remark 1.1. For the reader who prefers to focus on a single variant of the Schelling model we advise to assume the following throughout the paper: dimension $N = 1$, $M = 2$ opinions, torus $\mathfrak{S}^N = \mathfrak{S}$ (rather than $\mathbb{Z}^N = \mathbb{Z}$), and neighborhood $\mathcal{N} = (-1, 1)^N = (-1, 1)$. Most of the phenomena and mathematical challenges of our analysis are present already with these parameters. The model for $N > 1$ does exhibit different qualitative properties than the case $N = 1$, but, besides Section 2.2, most techniques we develop to treat this case are not specific for higher dimensions.

Finally, let us mention that the Schelling model with $M = 2$ is a special case of the *threshold voter model*. The threshold voter model with parameter T is defined just as the Schelling model on \mathbb{Z}^N with $M = 2$, \mathcal{N} symmetric, and w fixed, except that the initial opinions may be sampled according to some other distribution, and that a node changes its opinion exactly when its clock rings and when at least T of its neighbors are of a different opinion. Note that the Schelling model corresponds to setting $T = \lfloor |\mathfrak{N}(0)|/2 + 1 \rfloor$. The threshold voter model has been proved to fixate for any $T > (|\mathfrak{N}(0)| - 1)/2$ [DS93], i.e., for any i the limit $\lim_{t \rightarrow \infty} X(i, t)$ exists a.s. For $T = (|\mathfrak{N}(0)| - 1)/2$ and dimension $N = 1$ the model clusters, which means that for any fixed i, j we have $\lim_{t \rightarrow \infty} \mathbb{P}(X(i, t) \neq X(j, t)) = 0$ [ALM92] (Note that this value of T is slightly larger than the one relevant for the Schelling model). For $T < \theta|\mathcal{N}|$ with $\theta < 1/4$ and \mathcal{N} sufficiently large, the process coexists, meaning that there are invariant measures which are not a mixture of the extremal measures with only 1 or only 2 [Lig94].

1.4 Main results

Our first main result is a dynamical scaling limit result for the Schelling model. We prove that a function Y^w describing the opinions of the nodes in the early phase (times up to order $w^{-N/2}$) of the Schelling model on the torus, converges in law as $w \rightarrow \infty$. For $R \in \{3, 4, \dots\}$ let \mathcal{S} be the torus of width R , i.e., $\mathcal{S} = \mathbb{R}/\sim$, where \sim is the equivalence relation on \mathbb{R} defined by $x \sim y$ iff $x - y$ is an integer multiple of R . Denote the N -dimensional torus of side length R by $\mathcal{S}^N = \mathcal{S} \times \dots \times \mathcal{S}$. Let $\mathcal{C}(\mathcal{S}^N \times \mathbb{R}_+)$ denote the set of continuous real-valued functions on $\mathcal{S}^N \times \mathbb{R}_+$, and let $\mathcal{C}_M(\mathcal{S}^N \times \mathbb{R}_+)$ denote the set of functions which can be written in the form $f = (f_1, \dots, f_M)$ for $f_m \in \mathcal{C}(\mathcal{S}^N \times \mathbb{R}_+)$ and $m = 1, \dots, M$. Equip $\mathcal{C}_M(\mathcal{S}^N \times \mathbb{R}_+)$ with the topology of uniform convergence on compact sets. Define the unscaled bias function \mathcal{Y}_m by

$$\mathcal{Y}_m(i, t) = \sum_{j \in \mathfrak{N}(i)} \left(\mathbb{1}_{X(j,t)=m} - \frac{1}{M} \right).$$

Then define the (normalized) bias function $Y^w = (Y_1^w, \dots, Y_M^w) \in \mathcal{C}_M(\mathcal{S}^N \times \mathbb{R}_+)$ by

$$Y_m^w(x, t) := \frac{1}{w^{N/2}} \mathcal{Y}_m(xw, tw^{-N/2}) \quad m \in [M], \quad x \in w^{-1}\mathbb{Z}^N, \quad t \geq 0, \quad (4)$$

and for $x \notin w^{-1}\mathbb{Z}^N$ let $Y_m^w(x, t)$ be a weighted average of the $\leq 2^N$ points $\tilde{x} \in w^{-1}\mathbb{Z}^N$ which satisfy $\|x - \tilde{x}\|_\infty < w^{-1}$

$$Y_m^w(x, t) = \sum_{\tilde{x} \in w^{-1}\mathbb{Z}^N : \|x - \tilde{x}\|_\infty < w^{-1}} \left(\prod_{k=1}^N (1 - w|x_k - \tilde{x}_k|) \right) Y_m^w(\tilde{x}, t). \quad (5)$$

See Figure 2. Note that Y^w encodes the bias of each node towards each opinion $1, \dots, M$. Also note that when defining Y^w we do not only scale space; we also scale time by $w^{-N/2}$. The rescaling in space guarantees that $Y^w(\cdot, 0)$ converges in law to a Gaussian field as $w \rightarrow \infty$. The rescaling in time guarantees that for fixed $x \in \mathcal{S}$ the change in $Y^w(x, \cdot)$ during a time interval of order 1 is of order 1, since during an unscaled time interval of length $w^{-N/2}$ there are typically order $w^{-N/2} \cdot w^N = w^{N/2}$ nodes in the neighborhood of any fixed node which change opinion, which implies that the change to $Y^w(x, \cdot)$ for any fixed x is of order 1. Due to the rescaling of time, the convergence result of the following theorem only describes the evolution of the bias in the very beginning (more precisely, up to times of order $w^{-N/2}$) of the Schelling model. However, as we describe below, for the one-dimensional model one can use the theorem to also obtain information about the final configuration of opinions, since the sign of the bias function can be shown to stabilize after times of order $w^{-N/2} = w^{-1/2}$ for $N = 1$.

Theorem 1.2. *In the setting described above, Y^w converges in law in $\mathcal{C}_M(\mathcal{S}^N \times \mathbb{R}_+)$ to a random function Y as $w \rightarrow \infty$.*

The theorem is an immediate consequence of Proposition 3.1 in Section 3.1, which identifies Y as the solution of a particular differential equation with Gaussian initial data. Note that this theorem holds in rather large generality: For all dimensions $N \in \mathbb{N}$, number of opinions $M \in \{2, 3, \dots\}$, and general neighborhoods \mathcal{N} . However, we have not proved the theorem for the model on \mathbb{Z}^N , only for the model on the torus; the reason for this is that we do not establish that the differential equation describing the evolution of Y^w is well-defined on \mathbb{R}^N for $N > 1$.

As we discuss right below, Theorem 1.2 is an important input to the proof of our second main result, Theorem 1.3, which is a scaling limit result for the final configuration of opinions in the one-dimensional Schelling model. In Theorem 1.3 we consider the model on either the torus \mathfrak{S} or on \mathbb{Z} , and have $M = 2$ opinions and neighborhood $\mathcal{N} = \mathcal{N}_\infty$. In other words, we consider the base case in Remark 1.1, except that we allow both \mathfrak{S} and \mathbb{Z} . By [DS93] (see Proposition 3.10 below for the analogous result for general M) the opinion of each node converges a.s. as time goes to infinity, hence each node is associated with a unique opinion in $\{1, 2\}$ describing its limiting opinion.

For $V = \mathcal{S}$ and $V = \mathbb{R}$ define

$$\mathcal{D}(V) := \{\mathcal{A} = (A_1, A_2) : A_m \subset V \text{ is closed for } m = 1, 2\}.$$

Equip $\mathcal{D}(V)$ with the topology of convergence of A_1 and A_2 for the Hausdorff distance on compact sets. The appropriately normalized limiting distribution of opinions in the Schelling model, is a random variable in $\mathcal{D}(V)$. The following theorem says that this random variable converges in law in $\mathcal{D}(V)$ as the window size w converges to ∞ . In the theorem below we identify \mathfrak{S} and \mathcal{S} with $\{0, \dots, Rw - 1\}$ and $[0, R)$, respectively.

Theorem 1.3. *Let $\mathcal{V} = \mathbb{Z}$ and $V = \mathbb{R}$, or let $\mathcal{V} = \mathfrak{S}$ and $V = \mathcal{S}$. Consider the one-dimensional Schelling model on \mathcal{V} as described in Section 1.3 with window size $w \in \mathbb{N}$, $M = 2$ opinions, and $\mathcal{N} = \mathcal{N}_\infty$. Define $\mathcal{A}^w \in \mathcal{D}(V)$ by*

$$\mathcal{A}^w := (A_1^w, A_2^w), \quad A_m^w := \{jw^{-1} \in V : j \in \mathcal{V}, \lim_{t \rightarrow \infty} X(j, t) = m\} \text{ for } m = 1, 2.$$

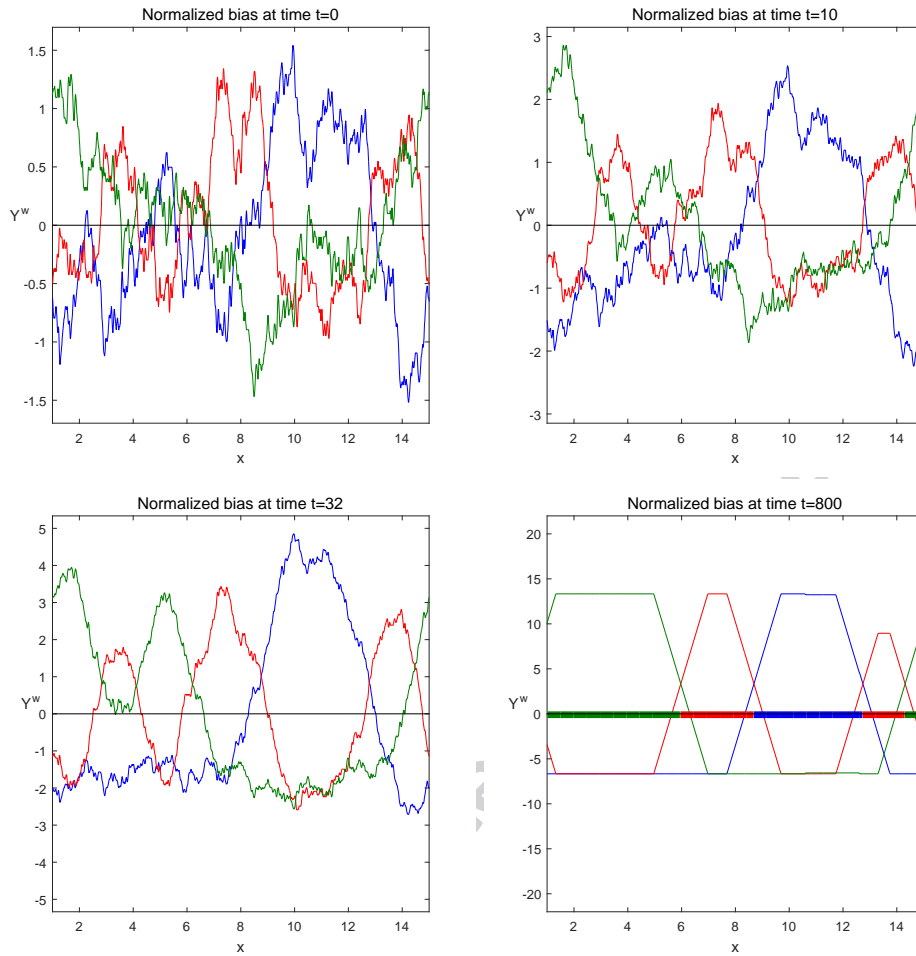


Figure 2: The graphs show the bias function Y^w at four different times for the Schelling model on the torus of dimension $N = 1$ with $M = 3$ opinions. Each color in the figure corresponds to one of the M opinions. The initial data of Y^w (upper left figure) converges in law to the Gaussian field B defined in Section 2 when $w \rightarrow \infty$. The evolution of Y^w can be well approximated by the differential equation (6) on compact time intervals, see Theorem 1.2 and Proposition 3.1. When we have reached the final configuration of opinions, each function Y_m^w , $m = 1, 2, 3$, is piecewise linear with slopes $\pm 2w^{1/2}$ and 0, and values in the interval $[-2w^{1/2}\frac{1}{M}, 2w^{1/2}(1 - \frac{1}{M})]$. Regions in which Y_m^w equals $2w^{1/2}(1 - \frac{1}{M})$ correspond to long intervals where the nodes have limiting opinion m . We see that the natural rescaling of the discrete bias function to get a non-trivial limit for the final configuration is to replace the multiplicative factor $w^{-1/2}$ in (4) by w^{-1} . For the continuum analog Y of Y^w , each function Y_m converges to $\pm\infty$ at almost every point, such that for each fixed x the function $t \mapsto |Y_m(x, t)|$ is approximately linear in t . The thick line at the x axis in the lower right figure represents the limiting opinion of the nodes. The plots are made with window size $w = 100$ and torus width $R = 14$.

Then $A_1^w \cup A_2^w = \{jw^{-1} : j \in \mathcal{V}\}$ a.s., and \mathcal{A}^w converges in law as a random variable in $\mathcal{D}(V)$ to a limiting random variable $\mathcal{A} = (A_1, A_2)$. The sets A_1 and A_2 have disjoint interior and union V a.s., and each set A_m for $m = 1, 2$ is a.s. the union of at most countably many closed intervals each of length strictly larger than 1.

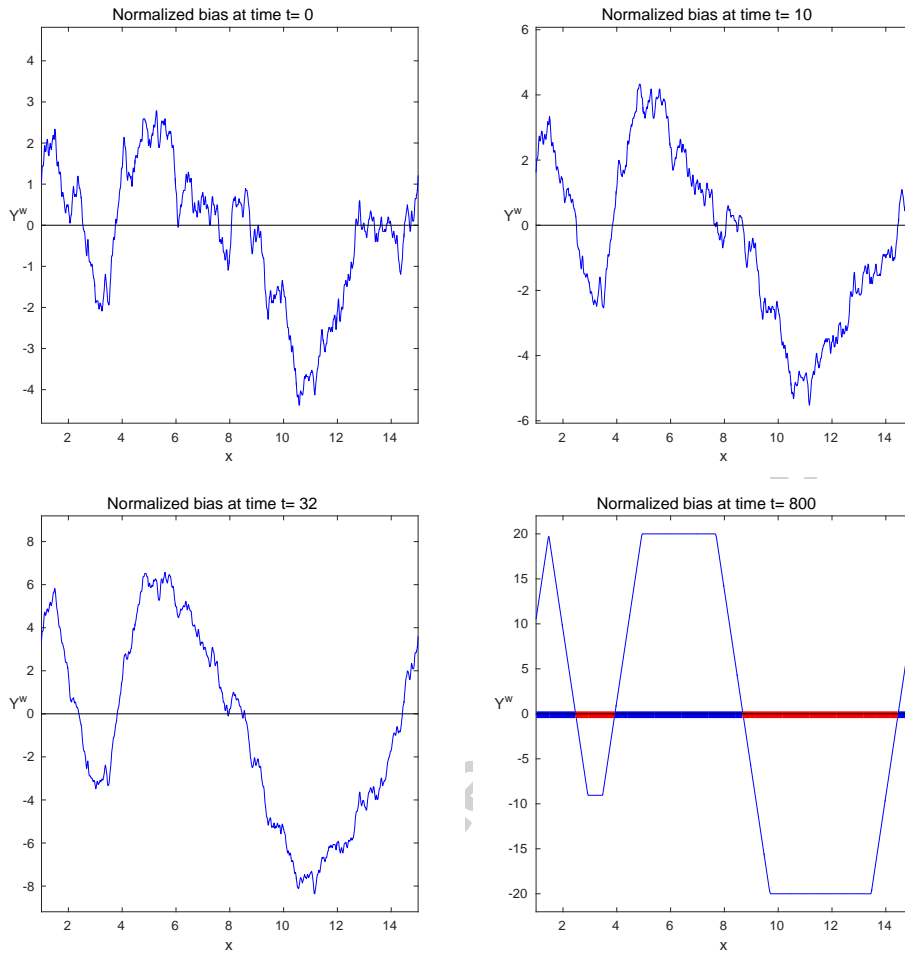


Figure 3: The graphs show the function $\widehat{Y}^w := Y_1^w - Y_2^w$ for the Schelling model on the torus of dimension $N = 1$ with $M = 2$ opinions, which evolves approximately as described by (11). The plots are made with window size $w = 100$ and torus width $R = 14$.

When we prove the theorem in Section 3.2 we will describe the limiting random variable \mathcal{A} in terms of the solution Y of the initial value problem mentioned above. We will not describe the law of this random variable further, but we remark that if I is defined to be the maximal interval satisfying either $0 \in I \subset A_1$ or $0 \in I \subset A_2$, then the length of I decays at least exponentially; this holds by Lemma 3.9, and since the event considered in this lemma holds independently and with uniformly positive probability on each interval $[10k, 10k + 5]$, $k \in \mathbb{Z}$ (see the proof of [BIKK12, Theorem 1] for a similar argument).

The solution Y of the differential equation evolves deterministically once the random initial data are fixed. Therefore the final configuration in the discrete one-dimensional model is essentially determined at time $t = 0$, i.e., the randomness of the Poisson clocks has only minor impact on the final configuration of opinions. Furthermore, due to the rescaling in time in (4), one can see already at time $O(w^{-1/2})$ approximately what the final configuration of opinions will be, since the signs of Y eventually stabilize in time. More precisely, for each fixed x the limit $\lim_{t \rightarrow \infty} \text{sign } Y(x, t)$ exists a.s., and \mathcal{S} can be divided into intervals of length strictly greater than one on which the limiting sign of Y is constant. By a coupon collector argument, the time it takes until the final configuration is reached in some interval of length $\Theta(w)$ is $\Omega(\log w)$, since the clock of

a constant fraction of the nodes must ring in order for this to happen. We note that the *bias* of most nodes stabilizes already at time $O(w^{-1/2})$, while the *opinion* of a node does not stabilize until a time of order 1 with constant order probability. We remark that the strategy outlined here for the one-dimensional Schelling model does not immediately apply to the higher-dimensional Schelling model, since we have not proved that $\lim_{t \rightarrow \infty} \text{sign } Y(x, t)$ exists a.s. in higher dimensions.

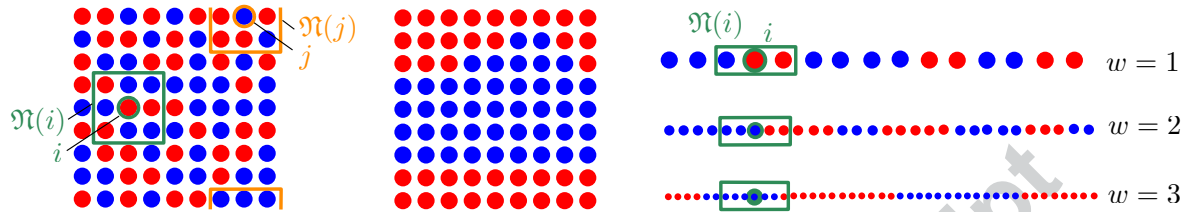


Figure 4: Left: initial configuration of opinions in the Schelling model on the torus for $N = 2$, $M = 2$, $R = 9$, and $w = 1$. Middle: one possible final configuration of opinions with initial data as on the left figure. Observe that all nodes agree with the most common opinion in their neighborhood. Right: final configuration of opinions in the Schelling model on the torus for $N = 1$, $M = 2$, $R = 15$, and $w = 1, 2, 3$.

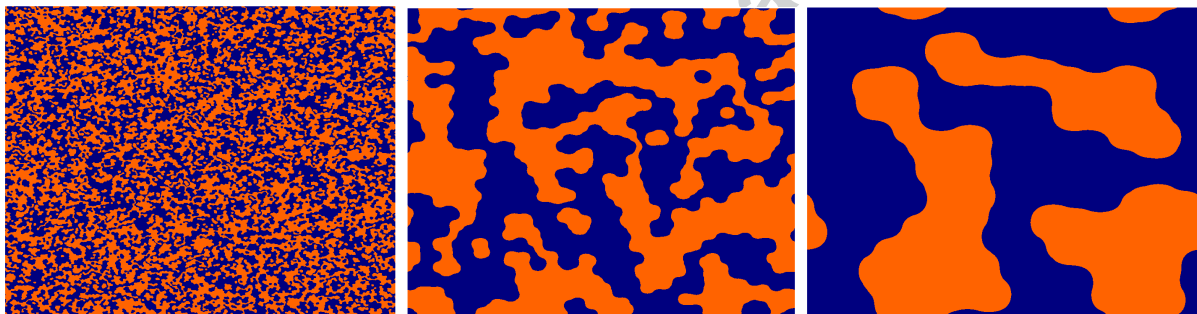


Figure 5: Final configuration of opinions in the Schelling model on the torus for $N = 2$, $M = 2$, and a torus width of 4000 nodes. Left: $w = 4$, middle: $w = 8$, and right: $w = 12$. Simulations by Omer Tamuz.

Other results in the paper of independent interest include Theorem 2.1, which establishes existence and uniqueness of the solution of the differential equation mentioned above, and Theorem 2.4, which says that for a certain family of Gaussian fields the supremum of its occupation kernel on Lipschitz functions is finite.

Finally we remark that our methods can be adapted easily to certain other variants of the Schelling model. For example, we may consider a perturbed variant of the model, where each node acts against its own preference with probability $p \in (0, 1)$ every time its Poisson clocks rings, e.g. it chooses some opinion uniformly at random from $[M]$ instead of changing its opinion to the most common opinion in its neighborhood. In this perturbed model a variant of Theorem 1.2 still holds, but the continuum bias function would evolve slower than with the unperturbed dynamics.

The results above are stated for the single-site-update variant of the Schelling model. In the version of the Schelling model studied in certain other papers, however, the nodes swap opinions rather than changing opinions; equivalently, the nodes have a fixed opinion and they change locations in order to be surrounded by nodes of a similar opinion to themselves. In this formulation of the model we would consider a finite grid (e.g. the torus), and each time step could consist of choosing two nodes i, j uniformly at random and swapping their opinions if the opinion of node i (resp. j) equals the most common opinion in the neighborhood of node j (resp. i). Defining Y^w using (4) and (5), the continuum approximation Y to Y^w would evolve as

described by a particular differential equation with random initial data, i.e., a variant of Theorem 1.2 still holds in this setting. This differential equation also describes certain variants closely related to Schelling’s original model, where some nodes are unoccupied and individuals may move to unoccupied sites if they are not satisfied. See the introduction of Section 2 for more details.

A variant of Theorem 1.3 also holds for the pair-swapping variant of the Schelling model if the torus width $R \rightarrow \infty$ sufficiently fast as $w \rightarrow \infty$. Restricted to some bounded neighborhood J of the origin, the main difference between the pair-swapping Schelling model and the single-site-update Schelling model, is that in the former variant of the model unsatisfied nodes of one type will change type at a higher rate than unsatisfied nodes of the other type. For example, if $M = 2$ and there are more unsatisfied nodes of type 1 than of type 2 in the system overall, unsatisfied nodes of type 1 will tend to get replaced more slowly than unsatisfied nodes of type 2. When $R \rightarrow \infty$, the total variation distance between the two variants of the model restricted to J and some bounded time interval goes to zero. Since the system stabilizes in finite time, for any given w we can find R such that the two models restricted to J evolve very similarly in total variation distance. In particular, if $R \rightarrow \infty$ sufficiently fast, then the limiting configuration of opinions in the pair-swapping Schelling model restricted to J converges in law to the same limit as the single-site-update Schelling model on \mathbb{Z} . The variant of the problem considered here (pair-swapping model; $R \rightarrow \infty$ sufficiently fast as $w \rightarrow \infty$) is exactly the variant studied in [BIKK12], hence the discussion in this paragraph addresses the precise form of conjecture in [BIKK12] about scaling limits for the one-dimensional Schelling model.

Our results above also extend easily to other lattices than \mathbb{Z}^N and \mathfrak{S}^N (assuming $\mathfrak{N}(i)$ is still defined by (1) for each node i).

1.5 Notation

We will use the following notation:

- If a and b are two quantities whose values depend on some parameters, we write $a \preceq b$ (resp. $a \succeq b$) if there is a constant C independent of the parameters such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \asymp b$ if $a \preceq b$ and $a \succeq b$. (We will sometimes abuse notation and use the same terminology when C depends on some parameters but not others, but this will be made clear in context.)
- For any $N \in \mathbb{N}$ let λ denote the Lebesgue measure on \mathbb{R}^N or the torus \mathfrak{S}^N .
- For $N \in \mathbb{N}$ and a topological space V let $\mathcal{C}(V^N)$ denote the space of continuous real-valued functions on V^N , equipped with the topology of uniform convergence on compact sets.
- For $N \in \mathbb{N}$ and either $V = \mathfrak{S}$ or $V = \mathbb{R}$ let $\mathcal{L}(V^N)$ denote the space of real-valued Lipschitz continuous functions on V^N equipped with the topology of uniform convergence on compact sets. For $K > 0$ define $\mathcal{L}^K(V^N) \subset \mathcal{L}(V^N)$ by

$$\mathcal{L}^K(V^N) = \left\{ y \in \mathcal{L}(V^N) : \|y\|_\infty \leq K2^N \text{ and } \forall k \in \{1, \dots, N\}, \sup_{\substack{x, x' \in V^N, x_k \neq x'_k, \\ x_j = x'_j \forall j \neq k}} \frac{|y(x) - y(x')|}{|x_k - x'_k|} \leq K2^{N-1} \right\}.$$

- For $M, N \in \mathbb{N}$ and either $V = \mathfrak{S}$ or $V = \mathbb{R}$ define $\mathcal{C}_M(V^N)$ (resp. $\mathcal{L}_M(V^N)$, $\mathcal{L}_M^K(V^N)$) to be the space of functions $f = (f_1, \dots, f_M)$ taking values in \mathbb{R}^M , such that for each $m \in [M]$ we have $f_m \in \mathcal{C}(V^N)$ (resp. $f_m \in \mathcal{L}(V^N)$, $f_m \in \mathcal{L}^K(V^N)$).
- For any topological space V let $\mathcal{B}(V)$ denote the Borel σ -algebra.

See Sections 1.3, 1.4, and 2 for additional notation.

Acknowledgements

We thank Omer Tamuz for introducing us to the Schelling model, for numerous helpful discussions, for comments on an earlier draft of this paper, and for allowing us to use his simulations in Figure 5. We thank Chris Burdzy for our discussion about his paper [BB01], and we thank Matan Harel for our discussions about Proposition 3.11. We thank Nicole Immorlica, Robert Kleinberg, Brendan Lucier, and Rad Niazadeh for discussions about this paper and some of their related work [BIKK12, IKLZ17]. The first author was supported by a fellowship from the Norwegian Research Council. The second author was supported by NSF grants DMS 1209044 and DMS 1712862.

2 The continuum Schelling model

In this section we will introduce a differential equation which describes the early phase of the Schelling dynamics when the window size w is large. We call this differential equation with appropriate initial data the continuum Schelling model.

The main result of this section is the existence and uniqueness of solutions of the differential equation (Theorem 2.1), along with a result on the occupation kernel of Gaussian fields (Theorem 2.4) and some properties of the continuum one-dimensional dynamics (Proposition 2.14).

Let $N \in \mathbb{N}$, $M \in \{2, 3, \dots\}$, and $R \in \{3, 4, \dots\}$, and let $V = \mathcal{S}$ or $V = \mathbb{R}$. The solution of the differential equation we will define just below is a function $Y = (Y_1, \dots, Y_M)$, $Y_m : V^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$, which is the continuum analog of the function Y^w defined by (4) and (5). Define the plurality function $p : \mathbb{R}^M \mapsto \{0, 1, \dots, M\}$ by

$$p(y) = \begin{cases} m & \text{if } y_m > \max_{m' \in [M] \setminus \{m\}} y_{m'}, \\ 0 & \text{if there is no } m \text{ for which } y_m > \max_{m' \in [M] \setminus \{m\}} y_{m'}. \end{cases}$$

Letting $\mathcal{N} \subset (-1, 1)^N$ be as in Section 1.3 define the neighborhood of $x \in V^N$ by

$$\mathcal{N}(x) = \{x' \in V^N : x' - x \in \mathcal{N}\},$$

where we view $x' - x$ modulo addition of an element in $R\mathbb{Z}^N$ if $V = \mathcal{S}$. Consider the following differential equation

$$\frac{\partial Y_m}{\partial t}(x, t) = \int_{x' \in \mathcal{N}(x)} (1 - M^{-1}) \mathbf{1}_{p(Y(x', t))=m} - M^{-1} \mathbf{1}_{p(Y(x', t)) \neq m} dx' \quad m \in [M], x \in V^N, t \geq 0. \quad (6)$$

We will prove in Proposition 3.1 that this differential equation approximates the early phase of the Schelling dynamics well for large window size.

An (N, M) -random field is a random map from V^N to \mathbb{R}^M . Let B be a multivariate Gaussian (N, M) -random field (see Section 2.1) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with mean and covariance functions given by $\mathbf{m}_m(x) = 0$ and

$$\mathfrak{C}_{m, m'}(x, x') = \begin{cases} \frac{M-1}{M^2} \lambda(\mathcal{N}(x) \cap \mathcal{N}(x')) & \text{for } m = m', \\ \frac{-1}{M^2} \lambda(\mathcal{N}(x) \cap \mathcal{N}(x')) & \text{for } m \neq m'. \end{cases} \quad (7)$$

We prove in Lemma 3.2 that B is well-defined as a continuous field. Although we will not need this formulation, we remark that one way to construct B involves starting with $W = (W_1, W_2, \dots, W_M)$, where the W_i are i.i.d. instances of white noise (each rescaled by $1/\sqrt{M}$) on N -dimensional space, and then writing

$$\widetilde{W}_i = W_i - \frac{1}{M} \sum_{i=1}^M W_i,$$

so that the \widetilde{W}_i sum up to zero a.s. and each \widetilde{W}_i describes (in a limiting sense) the ‘‘surplus’’ of individuals with opinion i . We can then let $B(x)$ denote the integral of $\widetilde{W}/\sqrt{2}$ over the set $\mathcal{N}(x)$.

Let the initial data of (6) be given by B

$$Y(x, 0) = B(x), \quad \forall x \in V^N. \quad (8)$$

The following theorem will be proved in Section 2.3.

Theorem 2.1 (Existence and uniqueness for (6), (8)). *Let $V = \mathcal{S}$ and $N \in \mathbb{N}$, or let $V = \mathbb{R}$ and $N = 1$. Let $M \in \{2, 3, \dots\}$ and $R \in \{3, 4, \dots\}$, and let $\mathcal{N} \subset (-1, 1)^N$ be as defined in Section 1.3. Then the initial value problem (6), (8) a.s. has a solution $Y : V^N \times [0, \infty) \rightarrow \mathbb{R}^M$. This solution can be written as the sum of the function $(x, t) \mapsto B(x)$ and a function $y : V^N \times [0, \infty) \rightarrow \mathbb{R}^M$ satisfying the following properties. For any $t > 0$, we have $y(\cdot, t) \in \mathcal{L}_M^t(V^N)$, so that in particular $y(\cdot, 0) = 0$, and y is continuously differentiable in t . The solution Y just described is unique in the space of functions satisfying these properties.*

Note that we have not proved that the initial value problem (6), (8) is well-defined for $N \geq 2$ and $V = \mathbb{R}$, but we believe that the above theorem also holds in this case. As we will discuss in Section 2.3 there exist initial data for which (6) does not have a unique solution (also when $N = 1$ and/or $V = \mathcal{S}$), and solutions of (6) do not in general vary continuously with the initial data.

For any $\omega \in \Omega$ and $m \in [M]$ define the random function $\phi_m^\omega : \mathcal{C}_M(V^N) \rightarrow \mathcal{L}^1(V^N)$ by

$$\phi_m^\omega(y) : x \mapsto \int_{x' \in \mathcal{N}(x)} (1 - M^{-1}) \mathbf{1}_{p(B(x') + y(x')) = m} - M^{-1} \mathbf{1}_{p(B(x') + y(x')) \neq m} dx', \quad \forall x \in V^N, y \in \mathcal{C}_M(V^N). \quad (9)$$

Also define $\phi^\omega : \mathcal{C}_M(V^N) \rightarrow \mathcal{L}_M^1(V^N)$ by $\phi^\omega = (\phi_1^\omega, \dots, \phi_M^\omega)$. Note that solving (6), (8) is equivalent to solving the following initial value problem for a.e. $\omega \in \Omega$

$$\begin{aligned} \frac{\partial y}{\partial t}(\cdot, t) &= \phi^\omega(y(\cdot, t)), & t \geq 0, \\ y(x, 0) &= 0, & x \in V^N. \end{aligned} \quad (10)$$

We obtain a solution to (6), (8) by defining $Y(x, t) = y(x, t) + B(x)$ for all $t \geq 0$ and $x \in V^N$.

We also observe that (6) is equivalent to a single differential equation when $M = 2$. Let $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ denote the sign function, which is defined to be 0 at 0. Defining $\widehat{Y} = Y^1 - Y^2$, the initial value problem (6), (8) is equivalent to

$$\begin{aligned} \frac{\partial \widehat{Y}}{\partial t}(x, t) &= \int_{x' \in \mathcal{N}(x)} \text{sign} \widehat{Y}(x', t) dx', & \forall x \in V^N, t \geq 0, \\ \widehat{Y}(x, 0) &= \widehat{B}(x), \end{aligned} \quad (11)$$

where \widehat{B} is the centered Gaussian field with covariances $\text{Cov}(\widehat{B}(x), \widehat{B}(x')) = \lambda(\mathcal{N}(x) \cap \mathcal{N}(x'))$.

Finally, we will briefly state the analog of (6), (8) for the setting of the pair-swapping variant of the Schelling model. See the end of Section 1.4 for the definition of this model. In this variant of the model the initial data for the continuum approximation $Y : \mathcal{S}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}^M$ to $Y^w : \mathcal{S}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}^M$ is still given by (8), while the differential equation describing the evolution of Y is given by

$$\begin{aligned} \frac{\partial Y_m}{\partial t}(x, t) &= \int_{x' \in \mathcal{N}(x)} (\lambda(\mathcal{S}^N) - \Lambda_m(t)) \mathbf{1}_{p(Y(x', t)) = m} - \Lambda_m(t) \mathbf{1}_{p(Y(x', t)) \neq m} dx', \\ m \in [M], \quad x \in \mathcal{S}^N, \quad t \geq 0, \quad \Lambda_m(t) &:= \lambda(\{x \in \mathcal{S}^N : p(Y(x, t)) = m\}). \end{aligned} \quad (12)$$

The main difference between (6) and (12) is that the rate at which Y changes in (12) depends on the *overall* fraction $\Lambda_m(t)$ of points with the various biases. Also observe that the integral of $Y_m(\cdot, t)$ is constant in time, which is consistent with the fact that the number of nodes with opinion m is constant. Theorems 1.2 and 2.1 (for the model on the torus) also hold for the pair-swapping Schelling model, and are proved exactly as before. Notice in particular that any solution of (12), (8) can be written on the same form as the

function Y in Theorem 2.1 (except that $\mathcal{L}_M^t(V^N)$ is replaced by $\mathcal{L}_M^{Ct}(V^N)$ for some constant $C > 1$), and since $y(\cdot, t) \in \mathcal{L}_M^{Ct}(V^N)$ we can still apply Theorem 2.4 in this setting. The initial value problem (12), (8) also describes the situation where each node is unoccupied with constant probability in the initial configuration, and individuals (who have a fixed type or opinion) may move to an unoccupied node if this would make them satisfied.

2.1 Occupation measures of random fields: basic definitions

We now give a short introduction to the theory of occupation measures of random fields, which is used frequently in our study of the continuum Schelling model. We refer to [GH80] for further information.

Let $N, M \in \mathbb{N}$. An (N, M) -random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection B of random variables with values in \mathbb{R}^M , which are indexed by the N -dimensional vector space V^N for $V = \mathbb{R}$ or $V = \mathcal{S}$, i.e., $B = \{B(x) : x \in V^N\}$ or $B = (B(x))_{x \in V^N}$. If $M = 1$ we say that the field is an N -random field.

The field B is Gaussian if $M = 1$ and if, for every $k \in \mathbb{N}$ and $x^1, \dots, x^k \in V^N$, the random variable $(B(x^1), \dots, B(x^k))$ is multivariate Gaussian. We say that $B = (B_1, \dots, B_M)$ is a multivariate (N, M) -Gaussian field if, for every $\alpha \in \mathbb{R}^M$, the weighted sum $\sum_{m=1}^M \alpha_m B_m$ is a real-valued Gaussian field. By e.g. [Adl10], a multivariate Gaussian field is uniquely determined by its mean $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_M)$ and covariance matrix $\mathfrak{C} = (\mathfrak{C}_{m,m'})_{m,m' \in [M]}$, which satisfy the following relations with t denoting the transpose of a matrix

$$\mathbf{m}(x) = \mathbb{E}[B(x)], \quad \mathfrak{C}(x, x') = \mathbb{E}[(B(x) - \mathbf{m}(x))^t (B(x') - \mathbf{m}(x'))], \quad x, x' \in \widehat{V}.$$

White noise on V^N for $V = \mathcal{S}$ or $V = \mathbb{R}$ is a collection of random variables $W = (W(X) : X \in \mathcal{B}(V^N))$ such that (i) $W(X) \sim \mathcal{N}(0, \lambda(X))$ for any $X \in \mathcal{B}(V^N)$, (ii) $W(X_1 \cup X_2) = W(X_1) + W(X_2)$ if $X_1, X_2 \in \mathcal{B}(V^N)$ and $X_1 \cap X_2 = \emptyset$, and (iii) $W(X_1)$ and $W(X_2)$ are independent if $X_1, X_2 \in \mathcal{B}(V^N)$ and $X_1 \cap X_2 = \emptyset$. Note that for a fixed set $X \in \mathcal{B}(V^N)$ we can define an N -random field $\widehat{B} = \{\widehat{B}(x) : x \in V^N\}$ by defining $\widehat{B}(x) := W(x + X)$, where $x + X = \{x + x' : x' \in X\}$ for any $x \in V^N$. We call a field that can be written on this form a moving average Gaussian field.

The following definition is from [GH80, Section 21]. See [GH80, Theorem 6.3] for a proof that the occupation kernel α described below is well-defined when μ_X is a.s. absolutely continuous with respect to Lebesgue measure λ .

Definition 2.2 (Occupation measure and occupation kernel). Let $N, M \in \mathbb{N}$, let $V = \mathcal{S}$ or $V = \mathbb{R}$, and consider an (N, M) -random field $B = (B_1(x), \dots, B_M(x))_{x \in V^N}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- The *occupation measure* $\mu = (\mu_X)_{X \in \mathcal{B}(V^N)}$ is defined by

$$\mu_X(A) := \lambda(X \cap B^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}^M).$$

- If μ_X is a.s. absolutely continuous with respect to Lebesgue measure λ for all $X \in \mathcal{B}(V^N)$, let $\alpha(a, X)$ denote the Radon-Nikodym derivative of μ_X with respect to λ , i.e.,

$$\lambda(X \cap B^{-1}(A)) = \int_A \alpha(a, X) da, \quad A \in \mathcal{B}(\mathbb{R}^M). \tag{13}$$

Let α be chosen such that $\alpha(\cdot, X)$ is measurable for each fixed $X \in \mathcal{B}(V^N)$, and $\alpha(a, \cdot)$ is a σ -finite measure on $(V^N, \mathcal{B}(V^N))$ for each $a \in \mathbb{R}^M$. We call α the *occupation kernel* of B .

- For $f : V^N \rightarrow \mathbb{R}^M$ let $\alpha(f, \cdot, \cdot)$ denote the occupation kernel (provided it exists) of the field $B - f$.
- More generally, for $k \in \{1, \dots, N - 1\}$, $x' = (x'_1, \dots, x'_k) \in V^k$, and $f : V^N \rightarrow \mathbb{R}^M$, let $\alpha(x', f, \cdot, \cdot)$ denote the occupation kernel (provided it exists) of the $(N - k, M)$ -random field

$$(B(x_1, \dots, x_{N-k}, x'_1, \dots, x'_k) - f(x_1, \dots, x_{N-k}, x'_1, \dots, x'_k))_{(x_1, \dots, x_{N-k}) \in V^{N-k}}.$$

2.2 Supremum of the occupation kernel on Lipschitz functions for moving average Gaussian fields

In [BB01, BB02] the authors prove that the supremum on Lipschitz curves of Brownian local time is finite. In this section we will prove a higher-dimensional analog of this result, stated in Theorem 2.4 below. We consider the supremum on Lipschitz functions of the occupation kernel of a particular centered moving average Gaussian field B .

The idea of the proof is to define various $(N + 1)$ -dimensional boxes on different scales, and bound the number of such boxes intersected by both the graph of B and a Lipschitz function f , uniformly over all choices of f . On each scale we proceed by using that f is approximately constant, while B fluctuates rapidly. We also prove that if f and g are uniformly close then the occupation kernel of B on f and g , respectively, are close with high probability. This allows us to show that the occupation kernel is continuous on a countable dense set of functions. Since the set of Lipschitz continuous functions with bounded Lipschitz constant is compact for the supremum norm, we can conclude that the continuous extension of the occupation kernel on this countable dense set has a finite supremum. We start the section by proving existence and basic properties of the occupation kernel of B on any fixed Lipschitz function.

Theorem 2.4 will imply that for any solution Y of (6), (8) on \mathcal{S}^N and any $m, m' \in [M]$, $m \neq m'$, the field $Y_m - Y_{m'}$ is close to 0 only for a small subset of \mathcal{S}^N simultaneously. This will help us to prove existence, uniqueness and other properties of solutions to (6), (8). Since $B_m - B_{m'}$ has the law of a constant multiple of B_m , it will be sufficient to obtain our result for the following real-valued field B .

Remark 2.3. We will assume throughout the section that $B = (B(x))_{x \in \mathcal{S}^N}$ is the centered moving average Gaussian field with covariances given by

$$\mathfrak{C}(x, x') = \lambda(\mathcal{N}(x) \cap \mathcal{N}(x')), \quad x, x' \in \mathcal{S}^N. \quad (14)$$

Observe that B is a moving average Gaussian field as defined in Section 2.1. The proof of the following theorem uses ideas from [BB01, BB02]. See Section 2.1 for the definition and basic properties of the occupation measure of random fields.

Theorem 2.4. *Let $B = (B(x))_{x \in \mathcal{S}^N}$ be the centered Gaussian field with covariances given by (14), and let $K > 0$. For each fixed $f \in \mathcal{L}^K(\mathcal{S}^N)$ the occupation kernel $\alpha(f, \cdot, \cdot)$ of $B - f$ exists a.s. Furthermore, a.s. there exists a random field $\tilde{\alpha} = \{\tilde{\alpha}(f) \in \mathbb{R} : f \in \mathcal{L}^K(\mathcal{S}^N)\}$ satisfying the following properties.*

- (I) *For each fixed $f \in \mathcal{L}^K(\mathcal{S}^N)$, we have $\tilde{\alpha}(f) = \alpha(f, 0, \mathcal{S}^N)$ a.s.*
- (II) *Almost surely, $f \mapsto \tilde{\alpha}(f)$ is continuous on $\mathcal{L}^K(\mathcal{S}^N)$ equipped with the supremum norm.*
- (III) *Almost surely, $\sup_{f \in \mathcal{L}^K(\mathcal{S}^N)} \tilde{\alpha}(f) < \infty$.*

First we will prove the existence and various properties of the occupation kernel for a certain class of Gaussian random fields. In particular, we study fields given by B plus a Lipschitz continuous function, and fields obtained by fixing some coordinates of this field.

Lemma 2.5. *Let $K > 0$ and $f \in \mathcal{L}^K(\mathcal{S}^N)$. Define $\tilde{B} = B - f$, where B is the centered Gaussian field on \mathcal{S}^N with covariances given by (14). Then the following holds a.s.*

- (I) *\tilde{B} has an occupation kernel $\alpha(f, \cdot, \cdot)$.*
- (II) *For all $k \in \{1, \dots, N - 1\}$ and almost all $x' = (x'_1, \dots, x'_k) \in \mathcal{S}^k$ the $(N - k)$ -field*

$$x'' \mapsto \tilde{B}(x''_1, \dots, x''_{N-k}, x'_1, \dots, x'_k)$$

for $x'' = (x''_1, \dots, x''_{N-k}) \in \mathcal{S}^{N-k}$, has an occupation kernel $\alpha(x', f, \cdot, \cdot)$. For almost all $a \in \mathbb{R}$ the following holds for all $X' \in \mathcal{B}(\mathcal{S}^k)$ and $X'' \in \mathcal{B}(\mathcal{S}^{N-k})$

$$\alpha(f, a, X'' \times X') = \int_{X'} \alpha(x', f, a, X'') dx'. \quad (15)$$

Proof of Lemma 2.5. By (14), we have $\text{Var}(|\tilde{B}(x') - \tilde{B}(x)|) \asymp \lambda(\mathcal{N}(x)\Delta\mathcal{N}(x'))$, where the implicit constant is independent of $x, x' \in \mathcal{S}^N$ and Δ denotes symmetric difference. Since $x, x' \in \mathcal{S}^N$, the difference $x - x'$ is only defined as an element in \mathbb{R}^N modulo $R\mathbb{Z}^N$, but we will view $x - x'$ as an element in \mathbb{R}^N by choosing the equivalence class such that $\|x - x'\|_1$ is minimized (with some arbitrary choice of equivalence class in case of draws). By (3) it follows that $\text{Var}(|\tilde{B}(x') - \tilde{B}(x)|) \succeq \|x' - x\|_1$, where the implicit constant is again independent of $x, x' \in \mathcal{S}^N$, but may depend on all other parameters. Since the probability density function of a standard normal random variable is bounded, for any $x \in \mathcal{S}^N$,

$$\liminf_{\epsilon \rightarrow 0} \epsilon^{-1} \int_{\mathcal{S}^N} \mathbb{P}[|\tilde{B}(x') - \tilde{B}(x)| \leq \epsilon] dx' \preceq \int_{\mathcal{S}^N} \|x' - x\|_1^{-1/2} dx' < \infty. \quad (16)$$

By [GH80, Theorem 21.12] we know that if \tilde{B} is a random field for which the left side of (16) is finite, then \tilde{B} has an occupation kernel a.s. Applying this result concludes our proof of (I).

The same theorem also implies the existence of the occupation kernel $\alpha(x', f, \cdot, \cdot)$ for a.e. fixed $x' = (x'_1, \dots, x'_k)$. The identity (15) follows by [GH80, Theorem 23.5], since for any $k \in \{1, \dots, N-1\}$ and $x = (x''_1, \dots, x''_{N-k}, x'_1, \dots, x'_k) \in \mathcal{S}^N$, $x'' = (x''_1, \dots, x''_{N-k}) \in \mathcal{S}^{N-k}$, $x' = (x'_1, \dots, x'_k) \in \mathcal{S}^k$,

$$\begin{aligned} \int_{\mathcal{S}^{N-k}} \sup_{\epsilon > 0} \epsilon^{-1} \mathbb{P}[|\tilde{B}((\tilde{x}_1, \dots, \tilde{x}_{N-k}, x'_1, \dots, x'_k)) - \tilde{B}((x''_1, \dots, x''_{N-k}, x'_1, \dots, x'_k))| \leq \epsilon] d\tilde{x} \\ \preceq \int_{\mathcal{S}^{N-k}} \|\tilde{x} - x''\|_1^{-1/2} d\tilde{x} < \infty. \end{aligned}$$

□

Now we will construct a countable set of functions which is dense in $\mathcal{L}^K(\mathcal{S}^N)$ for the supremum norm. For $K > 1$ let $\mathcal{L}^{K,k}(\mathcal{S}^N)$ denote the space of functions $f \in \mathcal{L}(\mathcal{S}^N)$ that satisfy the following properties: (i) for each $x \in 2^{-k}\mathbb{Z}^N$, $f(x) \in 2^{-k}\mathbb{Z}$, (ii) for each $x \notin 2^{-k}\mathbb{Z}^N$, $f(x)$ is a weighted average of f at the $\leq 2^N$ points $\tilde{x} \in 2^{-k}\mathbb{Z}^N$ which satisfy $\|x - \tilde{x}\|_\infty < 2^{-k}$, where the weights are defined as in (5), (iii) $\|f\|_{L^\infty(\mathcal{S}^N)} \leq K2^N + 2^{-k}$, and (iv) if $x, x' \in 2^{-k}\mathbb{Z}^N$ satisfy $\|x - x'\|_1 = 2^{-k}$ then $|f(x) - f(x')| \leq K2^{N-k-1} + 2^{2-k}$. Define

$$\tilde{\mathcal{L}}^K(\mathcal{S}^N) = \bigcup_{k \in \mathbb{N}} \mathcal{L}^{K,k}(\mathcal{S}^N). \quad (17)$$

The following lemma implies that $\tilde{\mathcal{L}}^K(\mathcal{S}^N)$ is dense in $\mathcal{L}^K(\mathcal{S}^N) \cup \tilde{\mathcal{L}}^K(\mathcal{S}^N)$ for the supremum norm.

Lemma 2.6. *For any $f \in \mathcal{L}^K(\mathcal{S}^N)$ and $k \in \mathbb{N}$ there is a canonically defined element $f^k \in \mathcal{L}^{K,k}(\mathcal{S}^N)$ such that $\|f - f^k\|_{L^\infty} \leq 2^{-k}$, where the implicit constant is independent of k and f , but may depend on K and k .*

Proof. For each $x \in 2^{-k}\mathbb{Z}^N$ let $f^k(x)$ be the multiple of 2^{-k} which is closest to $f(x)$. For $x \notin 2^{-k}\mathbb{Z}^N$ let $f^k(0)$ be defined such that condition (ii) in the definition of $\mathcal{L}^{K,k}(\mathcal{S}^N)$ is satisfied. It is immediate that the properties (i)-(iv) in the definition of $\mathcal{L}^{K,k}(\mathcal{S}^N)$ are satisfied. □

Our next lemma proves some basic properties of the field B . From (20) one can deduce that for two close points $x, x' \in \mathcal{S}$ the random variable $B(x) - B(x')$ is typically not too small, which will help to lower bound the occupation kernel at any fixed level. From (21) we will deduce that if f, g are close then the occupation kernel of B on f and on g are typically not too different. This will help us later to establish continuity of the occupation kernel in f . By (14) we may couple B with an instance of white noise W on \mathcal{S}^N such that for any $x \in \mathcal{S}$, we have $B(x) = W(\mathcal{N}(x))$. Define a filtration $(\mathcal{F}_t)_{t \in [0, R-2]}$ by

$$\mathcal{F}_t = \sigma(W|_{\mathcal{B}(D_t)}), \quad D_t := \bigcup_{s \in [0, t], (x_2, \dots, x_N) \in \mathcal{S}^{N-1}} \mathcal{N}((s, x_2, \dots, x_N)). \quad (18)$$

Let $0 \leq s < t < R-2$ and $(x_2, \dots, x_N) \in \mathbb{R}^{N-1}$. Conditioned on \mathcal{F}_s the random variable $B(t, x_2, \dots, x_N)$ is a Gaussian random variable with expectation $W(\mathcal{N}((t, x_2, \dots, x_N)) \cap D_s)$ and variance depending only on

$t - s$. Therefore there exists a function $p : (0, R - 2) \times \mathbb{R} \times \mathbb{R}$ such that $p(t - s, W(\mathcal{N}((t, x_2, \dots, x_N)) \cap D_s), \cdot)$ is the probability density function of $B(t, x_2, \dots, x_N)$ conditioned on \mathcal{F}_s . By (14), and with $x = (t, 0, \dots, 0)$ and $\tilde{p}(s, a, y) := (2\pi s)^{-1/2} \exp(-|y - a|^2/(2s))$,

$$p(t, a, y) = \tilde{p}(\lambda(\mathcal{N}(x) \setminus D_0), a, y). \tag{19}$$

Lemma 2.7. *Let p be given by (19). For $t \in (0, R - 2)$ and $a, y \in \mathbb{R}$,*

$$p(t, a, y) \leq t^{-1/2}, \tag{20}$$

where the implicit constant is independent of t, a, y , but may depend on \mathcal{N} . Let $\delta \in (0, 1/2)$, $a \in \mathbb{R}$, and assume $f, g : [0, R - 2] \rightarrow \mathbb{R}$ satisfy $\|f - g\|_{L^\infty([0, R-2])} < \delta$. Then

$$\int_0^{R-2} |p(t, a, f(t)) - p(t, a, g(t))| dt \leq \delta \log(1/\delta), \tag{21}$$

where the implicit constant may depend on \mathcal{N} .

Proof. Since p is continuous in t by (19) and since $p(t, a, y) \leq 1$ for $t > 2$ and any $a, y \in \mathbb{R}$, in order to prove (20) it is sufficient to prove $\liminf_{t \rightarrow 0} t^{1/2} p(t, a, y) \leq 1$. By the explicit formula for p this follows from (3).

The estimate (21) follows by the exact same argument as for the case $p = \tilde{p}$, which is considered in [BB01, Lemma 3.4], except that we use (20) instead of the corresponding estimate for \tilde{p} . \square

The next lemma will be used to show that if two functions f, g are close for the supremum norm, then the occupation kernel of B on f is typically not too different from then the occupation kernel of B on g .

Lemma 2.8. *Let $d \in (0, 1 \wedge (R - 2)]$, and let f, g be two functions defined on some N -dimensional cube $I \subset \mathcal{S}^N$ with side lengths d . Assume that $\|f - g\|_{L^\infty(I)} \leq \delta$ for some $\delta > 0$. Then for all $b \geq 1$ and with $\alpha(f, I) = \alpha(f, 0, I)$ and $\alpha(g, I) = \alpha(g, 0, I)$ as in Definition 2.2,*

$$\log \mathbb{P} \left[\alpha(f, I) - \alpha(g, I) \geq bd^{(N-3/4)} (\delta \log(\delta^{-1}))^{1/2} \right] \leq -b,$$

where the implicit constant is independent of δ, f and g .

Proof. Assume without loss of generality that $I = [0, d]^N$. Couple B with an instance of white noise W as described above the statement of Lemma 2.7, and recall the filtration $(\mathcal{F}_t)_t$ and the sets $D_t \subset \mathbb{R}^N$ defined by (18). For any $t \in [0, d]$ define $I_t = [0, t] \times [0, d]^{N-1} \subset \mathcal{S}^N$, and let $A_t^f = \alpha(f, I_t)$, $A_t^g = \alpha(g, I_t)$, and $\tilde{A}_t = A_t^f - A_t^g$. Then $(A_t^f)_{t \in [0, d]}$, $(A_t^g)_{t \in [0, d]}$, and $(\tilde{A}_t)_{t \in [0, d]}$ are stochastic processes adapted to the filtration $(\mathcal{F}_t)_{t \in [0, d]}$. To simplify notation we write $f(t, x')$ instead of $f(t, x'_1, \dots, x'_{N-1})$ for $t \in \mathbb{R}$ and $x' = (x'_1, \dots, x'_{N-1}) \in \mathbb{R}^{N-1}$. Let $t_1, t_2 \in [0, d]$ satisfy $t_1 < t_2$. By Lemma 2.5 (II) and (20) of Lemma 2.7,

$$\begin{aligned} \mathbb{E} \left[A_{t_2}^f - A_{t_1}^f \mid \mathcal{F}_{t_1} \right] &= \mathbb{E} \left[\int_{[0, d]^{N-1}} \alpha(x', f(\cdot, x'), [t_1, t_2]) dx' \mid \mathcal{F}_{t_1} \right] \\ &= \int_{[0, d]^{N-1}} \int_0^{t_2-t_1} p(s, \mathcal{N}((t_2, x'_1, \dots, x'_{N-1})) \cap D_{t_1}, f(t_1 + s, x')) ds dx' \\ &\leq d^{N-1} \int_0^{t_2-t_1} s^{-1/2} ds \\ &\leq d^{N-1/2}. \end{aligned}$$

By a similar argument $\mathbb{E}[A_{t_2}^g - A_{t_1}^g | \mathcal{F}_{t_1}] \leq d^{N-1/2}$. By (21) we get further

$$\begin{aligned} |\mathbb{E}[\tilde{A}_{t_2} - \tilde{A}_{t_1} | \mathcal{F}_{t_1}]| &= \left| \mathbb{E} \left[\int_{[0,d]^{N-1}} \alpha(x', \tilde{f}(\cdot, x'), [t_1, t_2]) - \alpha(x', \tilde{g}(\cdot, x'), [t_1, t_2]) dx' \mid \mathcal{F}_{t_1} \right] \right| \\ &\leq \int_{[0,d]^{N-1}} \int_0^{t_2-t_1} \left| p\left(s, \mathcal{N}((t_2, x'_1, \dots, x'_{N-1})) \cap D_{t_1}, f(t_1 + s, x')) \right. \right. \\ &\quad \left. \left. - p\left(s, \mathcal{N}((t_2, x'_1, \dots, x'_{N-1})) \cap D_{t_1}, g(t_1 + s, x')) \right) \right| ds dx' \\ &\leq d^{N-1} \delta \log \delta. \end{aligned}$$

The lemma now follows by [BB01, Lemma 2.1]. □

The next lemma bounds from above the number of rectangles of width L^{-1} and height $L^{-1/2}$ that intersect that graph of B .

Lemma 2.9. *Let $L > 1$, $a \in \mathbb{R}$, and define $\tilde{L} := \lceil L^{1/2} \rceil$. For $m \in \{1, \dots, \tilde{L}\}$ define the $(N+1)$ -dimensional rectangle J^m by*

$$J^m = [(m-1)L^{-1}, mL^{-1}] \times [0, L^{-1}]^{N-1} \times [a, a + \tilde{L}^{-1}].$$

Let A be the number of rectangles J^m for $m \in \{1, \dots, \tilde{L}\}$ which intersect the graph $\{(x, B(x)) \in \mathcal{S}^N \times \mathbb{R} : x \in \mathcal{S}^N\}$ of B . Then the following estimate holds for all $\xi > 1$ and $\epsilon \in (0, 1)$

$$\mathbb{P}[A \geq \tilde{L}^{1/2+\epsilon}] \leq L^{-\xi},$$

where the implicit constant is independent of a and L , but depends on ϵ and ξ .

Proof. Let E_m be the event that the graph of B intersects J^m , let $\tilde{J}^m = [(m-1)L^{-1}, mL^{-1}] \times [0, L^{-1}]^{N-1}$, let $x^m = (mL^{-1}, 0, \dots, 0) \in \tilde{J}^m$ and define the event E'_m by $E'_m = \{|B(x^m) - a| < \tilde{L}^{-1+\epsilon/10}\}$. Define the random variable A^m by $A^m = \sum_{m' \leq m} \mathbf{1}_{E'_{m'}}$. By (20), for $m_2 > m_1$,

$$\mathbb{P}[E'_{m_2} | \mathcal{F}_{m_1 L^{-1}}] \leq ((m_2 - m_1) \cdot L^{-1})^{-1/2} \tilde{L}^{-1+\epsilon/10} \asymp (m_2 - m_1)^{-1/2} \tilde{L}^{\epsilon/10}.$$

Therefore, for any $m \in \{1, \dots, \tilde{L}\}$,

$$\mathbb{E}[A^{\tilde{L}} - A^m | \mathcal{F}_{m L^{-1}}] \leq \sum_{d=1}^{\tilde{L}} d^{-1/2} \tilde{L}^{\epsilon/10} \leq \tilde{L}^{1/2+\epsilon/10}.$$

By applying [Bas95, Corollary I.6.12] to a constant multiple of the sequence $\{A^m / \tilde{L}^{1/2+\epsilon/10}\}_{1 \leq m \leq \tilde{L}}$, we get

$$\log \mathbb{P}(A^{\tilde{L}} > \tilde{L}^{1/2+\epsilon}) \leq -\tilde{L}^{\epsilon/10}.$$

For any $x, x' \in \mathcal{S}$ and $b > 1$ we have $\mathbb{E}[|B(x) - B(x')|^b] \leq \|x - x'\|^{b/2}$ for an implicit constant depending on b . By a quantitative version of the Kolmogorov-Chentsov theorem as in e.g. [MS16, Proposition 2.3], the function B is γ -Hölder continuous with (random) constant $C(\gamma)$ for any $\gamma < 1/2$, and $\mathbb{P}[C(\gamma) > C]$ decays faster than any power of C . In particular, $\mathbb{P}[C(1/2 - \epsilon/100) > L^{\epsilon/100}] \leq L^{-\xi}$ for any ξ . Observe that if $C(1/2 - \epsilon/100) \leq L^{\epsilon/100}$ and E_m occurs, and if L is sufficiently large, then E'_m also occurs since for some $x \in \tilde{J}^m$,

$$|B(x^m) - a| \leq |B(x^m) - B(x)| + |B(x) - a| \leq NL^{\epsilon/100} (L^{-1})^{1/2-\epsilon/100} + \tilde{L}^{-1} < \tilde{L}^{-1+\epsilon/10}.$$

Therefore

$$E_m \subset E'_m \cup \{C(1/2 - \epsilon/100) > L^{\epsilon/100}\},$$

so

$$\{A \geq \tilde{L}^{1/2+\epsilon}\} \subset \{A^{\tilde{L}} \geq \tilde{L}^{1/2+\epsilon}\} \cup \{C(1/2 - \epsilon/100) > L^{\epsilon/100}\},$$

and the lemma follows by a union bound. □

In the next lemma we define a number of high probability events G_k^1, G_k^2, G_k^3 which will help us conclude the proof of Theorem 2.4. On the event G_k^1 , for any $g \in \mathcal{L}^K(\mathcal{S}^N)$ we have an upper bound for the number of cubes of side length 2^{-k} that intersect the graph of both B and g . This will imply an upper bound for the occupation measure of B on g . Occurrence of the events G_k^2 and G_k^3 will guarantee continuity of the occupation kernel for the supremum norm.

Lemma 2.10. *For any $k \in \mathbb{N}$ divide \mathcal{S}^N into N -dimensional cubes I_j , $j = 1, \dots, (R2^k)^N$, of side length 2^{-k} , such that the cubes have pairwise disjoint interior. Also divide $\mathcal{S}^N \times \mathbb{R}$ into $(N + 1)$ -dimensional cubes I'_{ij} of side length 2^{-k} for $j = 1, \dots, (R2^k)^N$ and $i \in \mathbb{Z}$, such that the cubes have pairwise disjoint interior. Let $\epsilon \in (0, 1/100)$ and define the three events G_k^1, G_k^2, G_k^3 as follows*

- G_k^1 is the event that for any $g \in \mathcal{L}^K(\mathcal{S}^N)$ the number of cubes I'_{ij} intersecting the graph of both g and B , is bounded by $2^{k(N-1/2+\epsilon)}$.
- G_k^2 is the event that for any $f \in \mathcal{L}^K(\mathcal{S}^N)$ and $j \in \{1, \dots, 2^k\}$, the approximations f^k and f^{k+1} to f defined in Lemma 2.6 satisfy

$$|\alpha(f^k, I_j) - \alpha(f^{k+1}, I_j)| \leq 2^{k(-N+1/4+\epsilon)}.$$

- G_k^3 is the event that for any $j \in \{1, \dots, 2^k\}$ and any two $f, g \in \mathcal{L}^K(\mathcal{S}^N)$ satisfying $\|f - g\|_{L^\infty} \leq 2^{-k}$, we have

$$|\alpha(f^k, I_j) - \alpha(g^k, I_j)| \leq 2^{k(-N+1/4+\epsilon)}.$$

Finally define G_k by

$$G_k := \left(\bigcap_{k' \geq k} G_{k'}^1 \right) \cap \left(\bigcap_{k' \geq k} G_{k'}^2 \right) \cap \left(\bigcap_{k' \geq k} G_{k'}^3 \right).$$

Then $\mathbb{P}[G_k] \rightarrow 1$ as $k \rightarrow \infty$.

Proof. It is sufficient to prove that for $b = 1, 2, 3$ it holds that $\lim_{k \rightarrow \infty} \mathbb{P}[\bigcap_{k' \geq k} G_{k'}^b] = 1$. We consider the three cases $b = 1, 2, 3$ separately. All implicit constants may depend on R and K , but not on k .

Case $b = 1$: For any $L \in \mathbb{N}$ divide $\mathcal{S}^N \times \mathbb{R}$ into $(N + 1)$ -dimensional cubes $I'_{ij}(L)$ for $i \in \mathbb{Z}$ and $j \in \{1, \dots, (LR)^N\}$, such that the interior of the cubes are disjoint, and each cube has side length L^{-1} . Observe that $I'_{ij} = I'_{ij}(2^k)$ with I'_{ij} as in the statement of the lemma. Define $\tilde{L} := \lceil L^{1/2} \rceil \in \mathbb{N}$. Divide $\mathcal{S}^N \times \mathbb{R}$ into $(N + 1)$ -dimensional rectangles $J_{ij} = J_{ij}(L)$ for $i \in \mathbb{Z}$ and $j \in \{1, \dots, R^N L^{N-1} \tilde{L}\}$, such their interiors are pairwise disjoint, and such that each rectangle is a translation of $J_{i1} := [0, \tilde{L}^{-1}] \times [0, L^{-1}]^{N-1} \times [0, \tilde{L}^{-1}]$. Assume $I'_{1,1}(L) = [0, L^{-1}]^{N+1}$, and that the projection of $I'_{ij}(L)$ (resp. J_{ij}) onto the last coordinate is given by $L^{-1}[i - 1, i]$ (resp. $\tilde{L}^{-1}[i - 1, i]$).

Consider one of the rectangles J_{ij} . Find a cover $\{J_{ij}^m\}_{m=1}^{\tilde{L}}$ of J_{ij} of $(N + 1)$ -dimensional rectangles whose interiors are disjoint, and each of which is a translation of $[0, L^{-1}]^N \times [0, \tilde{L}^{-1}]$ (note that unless $\tilde{L}^2 = L$, a small fraction of the rectangles J_{ij}^m have a non-empty intersection with the complement of J_{ij}). Let $A_{ij} = A_{ij}(L)$ denote the number of such rectangles that contain a point of the graph of B . By Lemma 2.9 the following holds for all i, j and any $\xi > 1$

$$\mathbb{P}[A_{ij} \geq \tilde{L}^{1/2+\epsilon}] \preceq \tilde{L}^{-\xi}, \tag{22}$$

where the implicit constant depends on ϵ and ξ . We will now prove that on the event

$$\tilde{E}_L := \bigcap_{1 \leq j \leq R^N L^{N-1} \tilde{L}, -KL \leq i < KL} \{A_{ij} < \tilde{L}^{1/2+\epsilon}\}$$

the number of cubes $I'_{ij}(L)$ intersecting the graph of both $f \in \mathcal{L}^K(\mathcal{S}^N)$ and B is $\preceq L^{(N-1/4+\epsilon/2)}$. Since $f \in \mathcal{L}^K(\mathcal{S}^N)$ the number of rectangles $J_{i'j'}$ intersecting the graph of f is $\preceq L^{N-1/2}$. Assuming the event

\tilde{E}_L occurs, for each such i', j' there are $< \tilde{L}^{1/2+\epsilon}$ rectangles $J_{i'j'}^m$, intersecting both $J_{i'j'}$ and the graph of B . By using $f \in \mathcal{L}^K(\mathcal{S}^N)$ again it follows that, for each i', j' , the number of cubes $I'_{ij}(L)$ intersecting both $J_{i'j'}$ and the graph of B and f , is $\leq \tilde{L}^{1/2+\epsilon}$. Hence the total number of cubes $I'_{ij}(L)$ intersecting the graph of both B and f is $\leq L^{N-1/2} \times \tilde{L}^{1/2+\epsilon} \leq L^{N-1/4+\epsilon/2}$.

By (22) and a union bound, in order to complete the proof for the case $b = 1$ it is sufficient to prove that, conditioned on the event $\cap_{L \in \mathbb{N}: L \geq L_0} \tilde{E}_L$ for some $L_0 \in \mathbb{N}$, the number of rectangles $I'_{ij}(L)$ intersecting the graph of both f and B is bounded by $L^{N-1/2+\epsilon}$ for all sufficiently large L . We will proceed by iterations as in the proof of [BB01, Proposition 3.3].

Condition on the event $\cap_{L \in \mathbb{N}: L \geq L_0} \tilde{E}_L$, and choose $L \geq (2L_0)^2$. Since $\tilde{L} > L_0$ the number of cubes $I'_{ij}(\tilde{L})$ intersecting the graph of both f and B is $< \tilde{L}^{N-1/4+\epsilon/2} < \tilde{L}^{N-1/4+\epsilon}$. Each of these cubes $I'_{ij}(\tilde{L})$ is contained in the union of \tilde{L}^{N-1} rectangles $J_{i'j'}(L)$. By the definition of \tilde{E}_L and by $f \in \mathcal{L}^K(\mathcal{S}^N)$, for each fixed i', j' , the number of cubes $I'_{i'j'}(L)$ intersecting both $J_{i'j'}(L)$, the graph of f and the graph of B , is bounded by $\tilde{L}^{1/2+\epsilon}$. It follows that the number of cubes $I'_{i'j'}(L)$ intersecting both $I'_{ij}(\tilde{L})$, the graph of f and the graph of B , is $\leq \tilde{L}^{N-1/2+\epsilon}$. Further we get that the number of cubes $I'_{i'j'}(L)$ intersecting the graph of both f and B is $\leq \tilde{L}^{N-1/4+\epsilon} \times \tilde{L}^{N-1/2+\epsilon} \asymp L^{N-3/8+\epsilon}$.

Now choose $L \geq (4L_0)^4$. Since $\tilde{L} \geq (2L_0)^2$ the number of cubes $I'_{ij}(\tilde{L})$ intersecting the graph of both f and B is $\leq \tilde{L}^{N-3/8+\epsilon}$. For fixed i, j the number of cubes $I'_{i'j'}(L)$ intersecting both $I'_{ij}(\tilde{L})$, the graph of f and the graph of B , is $\leq \tilde{L}^{N-1/2+\epsilon}$, so the number of cubes $I'_{i'j'}(L)$ intersecting the graph of both f and B is $\leq \tilde{L}^{N-3/8+\epsilon} \times \tilde{L}^{N-1/2+\epsilon} \asymp L^{N-7/16+\epsilon}$. By continued iterations it follows that, for sufficiently large L , the number of cubes $I'_{i'j'}(L)$ intersecting the graph of both f and B is $\leq L^{N-1/2+2\epsilon}$. This completes the proof.

Case $b = 2$: By Lemma 2.6 we have $\|f^k - f^{k+1}\|_{L^\infty(\mathcal{S}^N)} \leq 2^{-k}$. Therefore, for any $f \in \mathcal{L}^K(\mathcal{S}^N)$ and $j \in \{1, \dots, 2^k\}$, Lemma 2.8 implies that

$$\log \mathbb{P} [|\alpha(f^k, I_j) - \alpha(f^{k+1}, I_j)| \geq 2^{k(-N+1/4+\epsilon)}] \leq -2^{k\epsilon/2}.$$

For each fixed $x \in I_{ij} \cap (2^k \mathbb{Z}^N)$, $f^k(x)$ can take $\leq 2^k$ different values. Conditioned on $f^k(x)$ the number of possible realizations of $f^k|_{I_j}$ and $f^{k+1}|_{I_j}$ is bounded by a constant. It follows that there are $\leq 2^k$ possibilities for $f^k|_{I_j}$ and $f^{k+1}|_{I_j}$. Since the number of cubes I_j is $\asymp 2^{kN}$, a union bound implies that

$$\log(1 - \mathbb{P}[G_k^2]) \leq -2^{k\epsilon/2}.$$

We conclude by a union bound.

Case $b = 3$: We proceed exactly as in the case $b = 2$. The result follows by a union bound, Lemma 2.8, and by observing that the number of possible realizations of $f^k|_{I_j}$ and $g^k|_{I_j}$ for each fixed j is $\asymp 2^k$. \square

Proof of Theorem 2.4. We start by proving uniform continuity of $g \mapsto \alpha(g)$ on $\tilde{\mathcal{L}}^K(\mathcal{S}^N)$, where $\tilde{\mathcal{L}}^K(\mathcal{S}^N)$ is defined by (17). Let $\eta, \beta > 0$, and choose $\tilde{k} \in \mathbb{N}$ sufficiently large such that the event $G_{\tilde{k}}^-$ of Lemma 2.10 holds with probability at least $1 - \eta$. Condition on the event $G_{\tilde{k}}^-$. Consider any $f, g \in \tilde{\mathcal{L}}^K(\mathcal{S}^N)$ such that $\|f - g\|_{L^\infty} \leq 2^{-\tilde{k}}$, and let $f^k, g^k \in \mathcal{L}^{K,k}(\mathcal{S}^N)$ denote the approximations to f, g , respectively, defined in Lemma 2.6. By the definition of the events G_k^1 and G_k^2 ,

$$|\alpha(f^{k+1}) - \alpha(f^k)| \leq 2^{k(N-1/2+\epsilon)} \cdot 2^{k(-N+1/4+\epsilon)} = 2^{k(-1/4+2\epsilon)}$$

for all $k \geq \tilde{k}$ and a universal implicit constant. Since $f = f^k$ for sufficiently large k the triangle inequality implies that, after we increase \tilde{k} if necessary, we have $|\alpha(f) - \alpha(f^{\tilde{k}})| \leq \beta/3$. By the same argument $|\alpha(g) - \alpha(g^{\tilde{k}})| \leq \beta/3$. By the definition of G_k^1 and G_k^3 ,

$$|\alpha(f^{\tilde{k}}) - \alpha(g^{\tilde{k}})| \leq 2^{\tilde{k}(N-1/2+\epsilon)} \cdot 2^{\tilde{k}(-N+1/4+\epsilon)} = 2^{\tilde{k}(-1/4+2\epsilon)}.$$

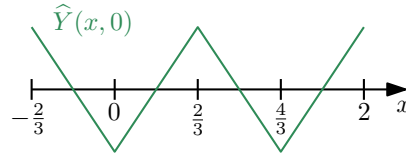


Figure 6: Illustration of the periodic initial data defined in (23).

Increasing \tilde{k} if necessary, it follows by the triangle inequality that $|\alpha(f) - \alpha(g)| \leq \beta$ with probability at least $1 - \eta$ for all $f, g \in \tilde{\mathcal{L}}(\mathcal{S}^N)$ satisfying $\|f - g\|_{L^\infty} \leq 2^{\tilde{k}}$. Note that \tilde{k} is a function of β and η , i.e., $\tilde{k} = \tilde{k}(\beta, \eta)$.

Fix some $\eta > 0$ and a sequence $(\beta_m)_{m \in \mathbb{N}}$ converging to 0. For any $m \in \mathbb{N}$ and any $f, g \in \tilde{\mathcal{L}}^K(\mathcal{S}^N)$ satisfying $\|f - g\|_{L^\infty} \leq 2^{-\tilde{k}(\beta_m, \eta 2^{-m})}$ we have $|\alpha(f) - \alpha(g)| \leq \beta_m$ with probability at least $1 - \eta 2^{-m}$. By a union bound it holds with probability at least $1 - \eta$ that $|\alpha(f) - \alpha(g)| \leq \beta_m$ for all $m \in \mathbb{N}$ and all $f, g \in \tilde{\mathcal{L}}^K(\mathcal{S}^N)$ satisfying $\|f - g\|_{L^\infty} \leq 2^{-\tilde{k}(\beta_m, \eta 2^{-m})}$. Since the choice of η was arbitrary this implies uniform continuity of $g \mapsto \alpha(g)$.

Define $\tilde{\alpha}$ to be the restriction to $\mathcal{L}^K(\mathcal{S}^N)$ of the continuous extension of α from $\tilde{\mathcal{L}}^K(\mathcal{S}^N)$ to $\mathcal{L}^K(\mathcal{S}^N) \cup \tilde{\mathcal{L}}^K(\mathcal{S}^N)$. Part (II) of the theorem follows by the definition of $\tilde{\alpha}$ and uniform continuity of α on $\tilde{\mathcal{L}}^K(\mathcal{S}^N)$. Part (III) of the theorem follows by using part (II) and that $\mathcal{L}^K(\mathcal{S}^N)$ is compact for the supremum norm.

Finally we will prove part (I). Let $f \in \mathcal{L}^K(\mathcal{S}^N)$, and for each $k \in \mathbb{N}$ let f^k be as in Lemma 2.6. By Lemma 2.8 and the Borel-Cantelli lemma we can find an increasing sequence $\{k_l\}_{l \in \mathbb{N}}$, $k_l \in \mathbb{N}$, such that $\alpha(f^{k_l}) \rightarrow \alpha(f)$ a.s. as $l \rightarrow \infty$. By part (II) we have $\alpha(f^k) \rightarrow \tilde{\alpha}(f)$ a.s. as $k \rightarrow \infty$. Part (I) now follows by the triangle inequality. \square

2.3 Existence and uniqueness of solutions of the continuum Schelling model

In this section we will prove Theorem 2.1, i.e., we will prove existence and uniqueness of solutions of the initial value problem (6), (8). Our strategy is to prove that the operator ϕ^ω in the equivalent problem (10) is a.s. Lipschitz continuous (Proposition 2.12), which allows us to conclude the proof of Theorem 2.1 by applying a variant of the Picard-Lindelöf theorem (Theorem 2.13). We first prove the result for the torus, and then we use some notion of locality for the one-dimensional problem to transfer the result to the case $N = 1$ and domain \mathbb{R} .

First we will see that Theorem 2.1 does not hold for all choices of initial data, i.e., there exist initial data for which (6) does not have a unique solution. Furthermore, solutions of (6) do not in general vary continuously with the initial data. We will illustrate these properties of (6) by considering the model for $N = 1$, $M = 2$, $\mathcal{N} = \mathcal{N}_\infty$, and $V = \mathcal{S}$. Let \hat{Y} be as in (11). The initial data $\hat{Y}(\cdot, 0) \equiv \epsilon$ and $\hat{Y}(\cdot, 0) \equiv -\epsilon$ for $\epsilon > 0$, give solutions $\hat{Y}(x, t) = \epsilon + 2t$ and $\hat{Y}(x, t) = -\epsilon - 2t$, respectively, so the solution does not vary continuously with the initial data. As an example of initial data for which Theorem 2.1 does not hold, assume $n := 3R/4 \in \mathbb{N}$, and define periodic initial data as follows (see Figure 2.3)

$$\hat{Y}(x, 0) = \begin{cases} -1 + 3(x - 4k/3) & \text{for } x \in \frac{2}{3}[2k, 2k + 1], k \in \{0, \dots, n - 1\}, \\ 1 - 3(x - 4k/3 - 2/3) & \text{for } x \in \frac{2}{3}[2k + 1, 2k + 2], k \in \{0, \dots, n - 1\}. \end{cases} \quad (23)$$

Then we have for all $t \in [0, 3/2)$,

$$\frac{\partial \hat{Y}}{\partial t}(x, t) = \begin{cases} 2/3 - 2(x - 4k/3) & \text{for } x \in \frac{2}{3}[2k, 2k + 1], k \in \{0, \dots, n - 1\}, \\ -2/3 + 2(x - 4k/3 - 2/3) & \text{for } x \in \frac{2}{3}[2k + 1, 2k + 2], k \in \{0, \dots, n - 1\}. \end{cases}$$

For $t = 3/2$ we have $\hat{Y}(\cdot, t) \equiv 0$. If we only allow for solutions satisfying (6) for all $t \geq 0$, we have no solutions since (6) is not satisfied at $t = 3/2$. If we allow the time derivative not to exist for the single time $t = 3/2$, solutions are not unique, e.g. $\hat{Y}(t, x) = 2(t - 3/2)$ and $\hat{Y}(t, x) = -2(t - 3/2)$ are both solutions for $t \geq 3/2$. We do not encounter these problems for the Gaussian initial data (8). The problems in the examples above

arise since $\widehat{Y}(x, t)$ is close to 0 for many $x \in \mathcal{S}^N$ simultaneously, and Theorem 2.4 implies that this is not the case for the Gaussian initial data.

In order to establish that ϕ^ω is a.s. Lipschitz continuous, we first prove a weaker property, namely a.s. continuity.

Lemma 2.11. *For any $\omega \in \Omega$ let ϕ^ω be defined by (9). For any $K > 0$ the map $\phi^\omega|_{\mathcal{L}_M^K(\mathcal{S}^N)} : \mathcal{L}_M^K(\mathcal{S}^N) \rightarrow \mathcal{L}_M^1(\mathcal{S}^N)$ is a.s. continuous for the supremum norm.*

Proof. By the definition of ϕ^ω , for any $\epsilon > 0$ and $y \in \mathcal{L}^K(\mathcal{S}^N)$ it holds a.s. that

$$\begin{aligned} & \sup_{\tilde{y} \in \mathcal{L}(\mathcal{S}^N), \|\tilde{y}-y\|_{L^\infty} < \epsilon} \|\phi^\omega(y) - \phi^\omega(\tilde{y})\|_{L^\infty} \\ & \leq \sup_{\tilde{y} \in \mathcal{L}(\mathcal{S}^N), \|\tilde{y}-y\|_{L^\infty} < \epsilon} \sum_{\substack{1 \leq m, m' \leq M, \\ m \neq m'}} \int_{\mathcal{S}^N} \mathbf{1}_{B_m(x)+y_m(x) \geq B_{m'}(x)+y_{m'}(x); B_m(x)+\tilde{y}_m(x) \leq B_{m'}(x)+\tilde{y}_{m'}(x)} dx \\ & \leq \sum_{\substack{1 \leq m, m' \leq M, \\ m \neq m'}} \int_{\mathcal{S}^N} \mathbf{1}_{|(B_{m'}(x)-B_m(x))-(y_m(x)-y_{m'}(x))| \leq 2\epsilon} dx. \end{aligned} \tag{24}$$

We want to show that a.s., for all $y \in \mathcal{L}^K(\mathcal{S}^N)$ the right side of (24) converges to 0 as $\epsilon \rightarrow 0$. Fix $m, m' \in [M]$, $m \neq m'$. The random field $\tilde{B}(x) := B_{m'}(x) - B_m(x)$ has the law of a constant multiple of B_m . Also note that $y_m - y_{m'} \in \mathcal{L}^{2K}(\mathcal{S}^N)$. Assume $\epsilon = 2^{-k}$ for $k \in \mathbb{N}$. On the event G_k^1 of Lemma 2.10 (with $2K$ instead of K , \tilde{B} instead of B , and $\epsilon = 1/1000$),

$$\int_{\mathcal{S}^N} \mathbf{1}_{|(B_{m'}(x)-B_m(x))-(y_m(x)-y_{m'}(x))| \leq 2\epsilon} dx \leq 2^{k(N-1/2+1/1000)} \times 2^{-kN} = 2^{-k/2+k/1000}$$

for all $y \in \mathcal{L}^K(\mathcal{S}^N)$. The lemma now follows by Lemma 2.10. □

We prove Lipschitz continuity of ϕ^ω by using the continuity result of Lemma 2.11, and by using Theorem 2.4 (III) to argue Lipschitz continuity of ϕ^ω on a dense set of functions.

Proposition 2.12. *For any $\omega \in \Omega$ let ϕ^ω be defined by (9). For any $K > 0$ the map $\phi^\omega|_{\mathcal{L}_M^K(\mathcal{S}^N)} : \mathcal{L}_M^K(\mathcal{S}^N) \rightarrow \mathcal{L}_M^1(\mathcal{S}^N)$ is a.s. Lipschitz continuous for the supremum norm.*

Proof. Recall the set $\tilde{\mathcal{L}}^K(\mathcal{S}^N)$ defined by (17). By Lemmas 2.6 and 2.11 it is sufficient to prove a.s. Lipschitz continuity on $\tilde{\mathcal{L}}^K(\mathcal{S}^N)$. By (24) it is sufficient to prove that for any $m, m' \in [M]$, $m \neq m'$, and $\tilde{B} = B_m - B_{m'}$,

$$\sup_{f \in \tilde{\mathcal{L}}^{2K}(\mathcal{S}^N)} \int_{\mathcal{S}^N} \mathbf{1}_{|\tilde{B}(x)-f(x)| \leq \epsilon} dx \leq \epsilon, \tag{25}$$

where the implicit constant is independent of ϵ , but depends on \tilde{B} . The occupation times formula (13) implies that a.s. for fixed f and with $\alpha(f, \cdot, \cdot)$ denoting the occupation kernel of $\tilde{B} - f$,

$$\int_{\mathcal{S}^N} \mathbf{1}_{|\tilde{B}(x)-f(x)| \leq \epsilon} dx = \int_{-\epsilon}^{\epsilon} \alpha(f, a, \mathcal{S}^N) da.$$

For fixed $a \in \mathbb{R}$, by the definition of α it holds a.s. that $\alpha(f, a, \mathcal{S}^N) = \alpha(f + a, 0, \mathcal{S}^N)$. By Theorem 2.4 (I) and with $\tilde{\alpha}$ as in this theorem, we have $\alpha(f + a, 0, \mathcal{S}^N) = \tilde{\alpha}(f + a)$ a.s. Therefore,

$$\int_{\mathcal{S}^N} \mathbf{1}_{|\tilde{B}(x)-f(x)| \leq \epsilon} dx = \int_{-\epsilon}^{\epsilon} \tilde{\alpha}(f + a) da \leq 2\epsilon \sup_{f \in \mathcal{L}^{K+\epsilon}(\mathcal{S}^N)} \tilde{\alpha}(f),$$

which completes the proof of the lemma upon an application Theorem 2.4 (III). □

We will deduce Theorem 2.1 from the following Banach space version of the theorem known as the Picard-Lindelöf theorem in the theory of differential equations. The theorem is proved in the same manner as the Picard-Lindelöf theorem, i.e., by defining a contraction mapping from the integral version of (10), showing that the Picard iterates converge to a solution, and deducing uniqueness from the contraction property, see e.g. [AMR88, Lemma 4.1.6]. The integral in (iii) is the Bochner integral.

Theorem 2.13. *Let $(\widehat{\mathcal{C}}, \|\cdot\|)$ be a Banach space, $\widehat{\mathcal{L}} \subset \widehat{\mathcal{C}}$, $y_0 \in \widehat{\mathcal{L}}$, $t_0 \in \mathbb{R}$, $\Delta t > 0$, and $I = [t_0 - \Delta t, t_0 + \Delta t]$. Let $\phi : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ be a map satisfying the following properties:*

- (i) ϕ is uniformly Lipschitz continuous on the closure of $\widehat{\mathcal{L}}$,
- (ii) $\sup_{y \in \widehat{\mathcal{L}}} \|\phi \circ y\| < \infty$, and
- (iii) $y_0 + \int_{t_0}^t \phi(y) ds \in \widehat{\mathcal{L}}$ for any $t \in I$ and any continuous curve $(y(s))_{t_0 \leq s \leq t}$ with values in $\widehat{\mathcal{L}}$.

Then there is a unique curve $(y(t))_{t \in I}$, such that $y(t_0) = y_0$, $\frac{\partial y}{\partial t}(t) = \phi(y(t))$, and $y(t) \in \widehat{\mathcal{L}}$ for all $t \in I$.

Proof. The conditions $\frac{\partial y}{\partial t} = \phi(y(t))$ and $y(t_0) = y_0$ are equivalent to the following

$$y(t) = y_0 + \int_{t_0}^t \phi(y(s)) ds. \tag{26}$$

Define $y_0(t) = y_0$ for all $t \in I$, and for $n \in \mathbb{N}$ and $t \in I$ define $y_n(t)$ by induction:

$$y_{n+1}(t) = y_0 + \int_{t_0}^t \phi(y_n(s)) ds. \tag{27}$$

By assumption (iii) we have $y_n(t) \in \widehat{\mathcal{L}}$ for all $n \in \mathbb{N}$ and $t \in I$. Let $C_1 \geq 0$ be the Lipschitz constant of ϕ on $\widehat{\mathcal{L}}$, and let $C_2 \in [0, \infty)$ be defined by $C_2 := \sup_{y \in \widehat{\mathcal{L}}} \|\phi(y)\|$. Then $\|y_1(t) - y_0\| \leq C_2|t - t_0|$ for all $t \in I$ by (27). By assumption (i) and induction on n , we get further

$$\|y_{n+1}(t) - y_n(t)\| \leq \int_{t_0 \wedge t}^{t_0 \vee t} \|\phi(y_n(s)) - \phi(y_{n-1}(s))\| ds \leq C_2 C_1^n |t - t_0|^{n+1} / (n+1)!.$$

Since $\widehat{\mathcal{C}}$ is complete it follows that $(y_n(t))_{t \in I}$ converges uniformly to a curve $(y(t))_{t \in I}$, such that $y(t) \in \widehat{\mathcal{C}}$ for any $t \in I$. Further, by sending $n \rightarrow \infty$ in (27) it follows by continuity of ϕ on the closure of $\widehat{\mathcal{L}}$ and by the dominated convergence theorem for the Bochner integral that y satisfies the integral equation (26). We have $y(t) \in \widehat{\mathcal{L}}$ for all $t \in I$ by assumption (iii). This concludes the proof of existence of solutions.

To obtain uniqueness of solutions let $(\tilde{y}(s))_{s \in I}$ for $\tilde{y} \in \widehat{\mathcal{L}}$ be another solution. By (26) we have $\|y_0(t) - \tilde{y}(t)\| \leq C_2|t - t_0|$ for all $t \in I$. By assumption (i) and induction we get further that for any $n \in \mathbb{N}$,

$$\|y_n(t) - \tilde{y}(t)\| \leq \int_{t_0 \wedge t}^{t_0 \vee t} \|\phi(y_n(s)) - \phi(\tilde{y}(s))\| ds \leq C_2 C_1^n |t - t_0|^{n+1} / (n+1)!.$$

Letting $n \rightarrow \infty$ it follows that $y = \tilde{y}$. □

Theorem 2.1 follows from Theorem 2.13 applied with $\widehat{\mathcal{C}} = \mathcal{C}_M(\mathcal{S}^N)$, $\widehat{\mathcal{L}} = \mathcal{L}_M^K(\mathcal{S}^N)$ for some $K > 0$, and $\phi = \phi^\omega$ as defined by (9).

Proof of Theorem 2.1. First consider the case when $V = \mathcal{S}$. Assume the assertion of the theorem is not true, and let $\tilde{t} \in [0, \infty)$ be the supremum of times $t \geq 0$ for which (10) has a unique solution y in $[0, t]$. Choose an arbitrary $\Delta t > 0$, let $K > \tilde{t} + \Delta t$, and define $t_0 = \tilde{t} - (\Delta t)/2$. We will prove that the assumptions of Theorem 2.13 are a.s. satisfied with $\phi = \phi^\omega$, $y_0 = y(t_0)$, $\widehat{\mathcal{L}} = \mathcal{L}_M^K(\mathcal{S}^N)$, $\widehat{\mathcal{C}} = \mathcal{C}_M(\mathcal{S}^N)$ and $I = [t_0 - \Delta t, t_0 + \Delta t]$. We equip $\widehat{\mathcal{C}}$ with the norm $\|f\| = \max_{1 \leq m \leq M} \|f_m\|_{L^\infty(\mathcal{S}^N)}$ for $f = (f_1, \dots, f_M) \in \widehat{\mathcal{C}}$. Condition (i) holds by

Proposition 2.12. Condition (ii) is satisfied since $\|\phi^\omega(y)\| \leq 2^N$ for any $y \in \mathcal{L}_M^K(\mathcal{S}^N)$ (in fact, this bound holds even for $y \in \mathcal{C}_M(\mathcal{S}^N)$). Finally observe that (iii) holds since for any $t \in I$,

$$\left\| y(t_0) + \int_{t_0}^t \phi^\omega(y(\cdot, s)) ds \right\| \leq 2^N t_0 + \Delta t 2^N,$$

and since the function $y(t_0) + \int_{t_0}^t \phi^\omega(y(\cdot, s)) ds$ is Lipschitz continuous with Lipschitz constant at most $2^{N-1}\tilde{t} + 2^{N-1}\Delta t$ in each coordinate. Theorem 2.13 now implies that (10) has a unique solution on I , which is a contradiction. This completes the proof of the theorem for the case $V = \mathcal{S}$.

Now consider the case $N = 1$ and $V = \mathbb{R}$. First we prove uniqueness. It is sufficient to show that given any $\epsilon > 0$ solutions are unique on $[-\epsilon^{-1}, \epsilon^{-1}] \times \mathbb{R}_+$ with probability at least $1 - \epsilon$. Let $R \in \{3, 4, \dots\}$ be sufficiently large such that with probability at least $1 - \epsilon$ there are real numbers x_i (which are random and measurable with respect to $\sigma(B)$) for $i = 1, 2, 3, 4$ such that $-R/2 + 1 < x_1 < x_2 - 1 < -\epsilon^{-1} - 1 < \epsilon^{-1} + 1 < x_3 + 1 < x_4 < R/2 - 1$, and such that $x \mapsto p(B(x))$ is constant on the intervals $[x_1, x_2]$ and $[x_3, x_4]$. Consider the Schelling model on the torus \mathcal{S} of width R , and let $(\tilde{B}(x))_{x \in [-R/2, R/2]}$ be the initial values. Couple \tilde{B} and B such that $\tilde{B}|_{[-R/2+1, R/2-1]} = B|_{[-R/2+1, R/2-1]}$ a.s., and observe that if $Y : \mathbb{R} \times \mathbb{R}_+$ solves the Schelling model (6), (8) on \mathbb{R} , then $Y|_{[x_2, x_3]}$ is a solution to the Schelling model on \mathcal{S} restricted to $[x_2, x_3]$. Here we use that if $x \mapsto p(B(x))$ is constant on an interval of length > 1 then Y (resp. \tilde{Y}) evolves independently to the left and to the right of this interval. By uniqueness of solutions to the Schelling model on \mathcal{S} , we obtain uniqueness of solutions to the Schelling model on \mathbb{R} .

Existence follows by a similar argument. Let $\epsilon > 0$, $R \in \{3, 4, \dots\}$, \tilde{B} , and $x_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$ be as in the previous paragraph. It is sufficient to prove existence of Y restricted to $[x_2, x_3]$, since the real line a.s. can be divided into countably many disjoint intervals, such that each interval either (i) has length > 1 and is such that $x \mapsto p(B(x))$ is constant on the interval, or (ii) is between two intervals of type (i). If we find a solution on each interval of type (ii) we can get a global solution by concatenating the solution from the different intervals, since $p(Y(x, t))$ is constant for all $t \geq 0$ and all x in an interval of type (i). By existence of solutions to the Schelling model on the torus, we define $Y|_{[x_2, x_3]}$ to be equal to the solution of the Schelling model on the torus restricted to $[x_2, x_3]$, which concludes the proof. \square

2.4 Long-time behavior of the one-dimensional continuum Schelling model

The main result in this section is the following proposition, which says that in the one-dimensional continuum Schelling model with $M = 2$ opinions, the spatial domain of Y can be written as the union of intervals of length strictly greater than one on which one opinion dominates as $t \rightarrow \infty$.

Proposition 2.14. *Let $V = \mathbb{R}$ or let $V = \mathcal{S}$. Let Y be the solution of the initial value problem (6), (8) on V with $M = 2$, $N = 1$ and $\mathcal{N} = \mathcal{N}_\infty$, and for $m = 1, 2$ define*

$$A_m := \left\{ x \in \mathbb{R} : \lim_{t \rightarrow \infty} p(Y(x, t)) = m \right\}.$$

Then $\mathbb{R} = \overline{A_1} \cup A_2$ a.s., the boundary ∂A_1 is a.s. equal to the boundary ∂A_2 , and this boundary a.s. consists of a countable collection of points such that the distance between any two of these points is strictly greater than one.

The proposition implies that the complement of $\partial A_1 = \partial A_2$ is a sequence of open intervals, each of length greater than 1, which alternately belong to A_1 and A_2 . We make no statement about whether the limit does or does not exist at the boundary points themselves.

Remark 2.15. The reason the proposition is only stated for $M = 2$ is that a particular form of monotonicity of (6) holds only for $M = 2$. More precisely, if $M = 2$, Y solves (6), (8), and \tilde{Y} solves (6) with initial data $\tilde{Y}^1(\cdot, 0) = B + f$ and $\tilde{Y}^2(\cdot, 0) = B - f$ for a strictly positive function f (with f chosen such that \tilde{Y} is well-defined), then $(\tilde{Y}^1 - \tilde{Y}^2) - (Y^1 - Y^2)$ is a strictly positive function which is increasing in t . However, we do believe that the proposition also holds for $M > 2$, and if we had established the proposition for all M , then Theorem 1.3 would hold for all $M \in \{2, 3, \dots\}$.

We briefly outline the proof of Proposition 2.14 before we proceed, and we begin with some notation. We say that an opinion m *dominates* in an interval J in the limit as $t \rightarrow \infty$ if the fraction of $(x, t) \in J \times [0, T]$ for which $p(Y(x, t)) = m$ converges to 1 as $T \rightarrow \infty$ (equivalently, $r(m, J, T) \rightarrow 1$ with the notation introduced below). Intuitively, this means that individuals in the interval J are (regardless of how they started out) increasingly tending to switch their opinions to m in the large T limit.

The first part of the argument is to show that if a certain opinion m dominates in an interval J of length ≥ 1 in the limit as $t \rightarrow \infty$, and J is not contained in a larger interval satisfying this property, then the interval of length 1 immediately to the right (or left) of J is dominated by some other opinion m' as $t \rightarrow \infty$.

This result is stated in Lemma 2.20 (which is in turn immediate from Lemmas 2.18 and 2.19 below). As explained right after Lemma 2.20, from this lemma we can deduce a weak variant of Proposition 2.14 which holds for all M .

We conclude the proof of Proposition 2.14 by a perturbative approach. We show that if the result of Proposition 2.14 does not hold, then it will hold for a slight perturbation of the initial data which favors one opinion more. We deduce from this that the set of initial data on which the proposition does not hold is exceptional. If we increase the initial bias towards opinion (say) 1 uniformly by ϵ , then we can find some positive (random) number $\tilde{\ell}(t)$ such that for any x the measure of the set $\{x' \in \mathcal{N}(x) : p(Y(x', t)) = 1\}$ increases by at least $\tilde{\ell}(t)$. By a detailed analysis of the differential equation we can show that $\inf_{t \geq 0} \tilde{\ell}(t) > 0$, which we use to show that there exists at least one interval of length > 1 on which the bias converges, and further (using Lemma 2.20) that this property must hold everywhere.

The following lemma is immediate from (6), and will be used throughout the proof of the proposition. It says that if we have an interval J of length at least one on which one opinion dominates (in the sense that $p(Y(\cdot, t))$ is constant on the interval), then this opinion will dominate J for all times $t' \geq t$. The lemma will be useful at several occasions, e.g. since it allows us to isolate the evolution of the differential equation on certain compact intervals (which is important when relating the model on \mathcal{S} and on \mathbb{R}), and since it simplifies certain parts of our perturbative argument by proving the occurrence of intervals J as in the lemma for certain perturbations of Y .

Lemma 2.16. *Let $V = \mathbb{R}$ or let $V = \mathcal{S}$. Let Y be a solution of the initial value problem (6), (8) on V with $N = 1$ and $\mathcal{N} = \mathcal{N}_\infty$. Let $t \geq 0$, let $m \in [M]$, and let $J \subset V$ be an interval of length ≥ 1 such that $p(Y(x, t)) = m$ for all $x \in J$. Then $p(Y(x, t')) = m$ for all $t' \geq t$ and $x \in J$.*

Remark 2.17. For the discrete Schelling model on \mathbb{Z} the final configuration of opinions will always consist of intervals of length at least $w + 1$ in which all nodes have the same limiting opinion. This can be seen by the following argument. If there is an interval of length $w + 1$ where all nodes have the same opinion m , then no nodes in this interval will ever change their opinion. Furthermore, if i is the first node to the right of this interval for which m is not the limiting opinion, then nodes $i, i + 1, \dots, i + w$ must all have the same limiting opinion; otherwise node i would not be satisfied in the final configuration. Since we consider the model on \mathbb{Z} , there will always be some interval of length $w + 1$ where all nodes have the same opinion in the initial configuration. By induction on the nodes to the left and right, respectively, of this interval, it follows that all nodes are contained in an interval of length at least $w + 1$ in which all nodes have the same limiting opinion. The two lemmas we will prove next (which imply Lemma 2.20 when combined) are continuum analogs of this result.

Lemma 2.18 says that if some opinion m dominates in an interval $I = (a - 1, a)$ of length exactly 1, but $p \circ Y$ does not converge pointwise to m in I , then there is some opinion $m' \neq m$ which dominates in the interval $\tilde{I} = (a, a + 1)$. We give a brief outline of the proof in the simplified setting where $M = 2$ and $m = 1$. For $m \in [M]$, an interval $J \subset \mathbb{R}$, and $t \geq 0$, define

$$r(m, J, t) := \frac{1}{|J|t} \int_0^t \int_J \mathbb{1}_{p(Y(x, t'))=m} dx dt'. \tag{28}$$

Observe that for $x \in I$ and $t \gg 1$, we see from (11) that $\hat{Y}(x, t)$ is approximately equal to $t(r(1, I, t) - r(2, \mathcal{N}(x) \setminus I, t))$. If $r(1, I, t)$ is very close to 1, but $\hat{Y}(x, t) < 0$, then we must have $r(2, \mathcal{N}(x) \setminus I, t)$ very

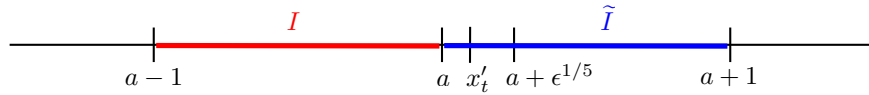


Figure 7: Illustration of objects in the statement and proof of Lemma 2.18. We assume that opinion m dominates in the interval I , in the sense that $r(m, I, t) \rightarrow 1$ as $t \rightarrow \infty$. We also assume that there exists an $x \in I$ for which $p(Y(x, t))$ does not converge to m . Using the latter assumption, we prove that if we pick some sufficiently small $\epsilon > 0$ then for every sufficiently large t (where in particular t is large enough so that $r(m, I, t) > 1 - \epsilon$) there exists an x_t in the slightly translated interval $(a - 1 + \sqrt{\epsilon}, a + \sqrt{\epsilon})$ such that $p(Y(x_t, t)) \neq m$. Using this, we will then prove existence of $m' \in [M] \setminus \{m\}$ and $x'_t \in (a, a + \epsilon^{1/5})$ for all sufficiently large $t > 0$, such that $r(m', x'_t, t) > 1 - \epsilon^{1/5}$. Then we use the existence of x'_t to prove that $\lim_{t \rightarrow \infty} r(m', \tilde{I}, t) = 1$. Note that by symmetry, we could have made the analogous argument with \tilde{I} on the left side of I , instead of the right side.

close to 1. We can use this to argue existence of x' satisfying $0 < x' - a \ll 1$ such that $r(2, x', t)$ is close to 1. By (11) and since $\mathcal{N}(x')$ is approximately equal to $I \cup \tilde{I}$, we see that $\hat{Y}(x', t)$ is approximately equal to $t(r(1, I, t) - r(2, \tilde{I}, t))$. We can deduce from this that $r(2, \tilde{I}, t)$ is close to 1, so opinion 2 dominates in the interval \tilde{I} .

Lemma 2.18. *Let Y be a solution of (6) with continuous initial data (chosen such that we have existence of solutions of (6)), $N = 1$, $\mathcal{N} = \mathcal{N}_\infty$, and either $V = \mathcal{S}$ or $V = \mathbb{R}$. For $m \in [M]$, an interval $J \subset \mathbb{R}$, and $t \geq 0$, define $r(m, J, t)$ by (28). For $a \in \mathbb{R}$ define $I := (a - 1, a)$ and $\tilde{I} := (a, a + 1)$. Assume there exists an $m \in [M]$ such that $\lim_{t \rightarrow \infty} r(m, I, t) = 1$, and that there exists $x \in I$ for which the limit $\lim_{t \rightarrow \infty} p(Y(x, t))$ either does not exist or takes a value different from m . Then there exists an $m' \in [M]$, $m' \neq m$, such that $\lim_{t \rightarrow \infty} r(m', \tilde{I}, t) = 1$.*

Proof. Define $K := \sup_{x \in I \cup \tilde{I}} \|Y(x, 0)\|_1$. Let $\epsilon \in (0, 1/10)$, and define $d_\epsilon := \epsilon^{1/2}$ and $d'_\epsilon := \epsilon^{1/5}$. For each $t \geq 0$ we define $x_t \in \mathbb{R}$ by

$$x_t := \inf\{x \geq a - 1 + d_\epsilon : p(Y(x, t)) \neq m\},$$

and let $m'_t \in [M] \setminus \{m\}$ be such that $Y_m(x_t, t) \leq Y_{m'_t}(x_t, t)$. Note that we can find such an m'_t since Y is continuous.

For $m \in [M]$, $x \in \mathbb{R}$ and $t \geq 0$ define

$$r(m, x, t) := \frac{1}{t} \int_0^t \mathbb{1}_{p(Y(x, t'))=m} dt'. \tag{29}$$

We abuse notation slightly by letting r denote both this function and the function in the statement of the lemma.

We will prove that for all sufficiently large $t \geq 0$ there exists $x'_t \in (a, a + d'_\epsilon)$ satisfying $r(m'_t, x'_t, t) > 1 - d'_\epsilon$. The following relation, which follows directly from (6) and holds for any $m_1, m_2 \in [M]$, will be used multiple times throughout the proof of this result

$$\frac{1}{t}(Y_{m_1}(x_t, t) - Y_{m_2}(x_t, t)) - \frac{1}{t}(Y_{m_1}(x_t, 0) - Y_{m_2}(x_t, 0)) = r(m_1, \mathcal{N}(x_t), t) - r(m_2, \mathcal{N}(x_t), t). \tag{30}$$

We consider the following three cases separately: (I) $a - 1 + d_\epsilon \leq x_t < a - 1 + d'_\epsilon$, (II) $a - 1 + d'_\epsilon \leq x_t < a$, (III) $a \leq x_t \leq a + d_\epsilon$. One of the cases (I)-(III) must occur by the following argument, i.e., we cannot have $x_t > a + d_\epsilon$. If $x_t > a + d_\epsilon$, we would have an interval of length > 1 , such that $p(Y(x, t)) = m$ for all x in the interval. Therefore, by Lemma 2.16 we would have $p(Y(x, t')) = m$ for all $t' \geq t$ and all x in the interval. Since $\lim_{t \rightarrow \infty} r(m, I, t) = 1$, and by the differential equation (6) and Lemma 2.16, it would follow that $\lim_{t \rightarrow \infty} p(Y(x, t)) = m$ for all $x \in I$, which is a contradiction to the assumptions of the lemma.

First consider cases (I) and (II) defined above. Define $I_L := (x_t - 1, a - 1)$, $I_R := (a, x_t + 1)$ and $d_t := x_t - (a - 1) \geq d_\epsilon$. Note that $\mathcal{N}(x_t) = \overline{I_L \cup I \cup I_R}$, $|I_R| = d_t$ and $|I_L| = 1 - d_t$. By (30),

$$\begin{aligned} & \frac{1}{t} \left(Y_{m'_t}(x_t, t) - Y_m(x_t, t) \right) - \frac{1}{t} \left(Y_{m'_t}(x_t, 0) - Y_m(x_t, 0) \right) \\ &= r(m'_t, I, t) - r(m, I, t) + |I_L| r(m'_t, I_L, t) - |I_L| r(m, I_L, t) + |I_R| r(m'_t, I_R, t) - |I_R| r(m, I_R, t). \end{aligned}$$

Therefore, for all sufficiently large t (chosen such that $r(m, I, t) > 1 - \epsilon$, which implies $r(m'_t, I, t) < \epsilon$),

$$\begin{aligned} |I_R| r(m'_t, I_R, t) &= \frac{1}{t} \left(Y_{m'_t}(x_t, t) - Y_m(x_t, t) \right) - \frac{1}{t} \left(Y_{m'_t}(x_t, 0) - Y_m(x_t, 0) \right) - r(m'_t, I, t) + r(m, I, t) \\ &\quad - |I_L| r(m'_t, I_L, t) + |I_L| r(m, I_L, t) + |I_R| r(m, I_R, t) \\ &> 0 - 2K/t - \epsilon + (1 - \epsilon) - (1 - d_t) + 0 + 0 \\ &= -2K/t - 2\epsilon + d_t. \end{aligned}$$

If there is no appropriate x'_t and case (I) occurs,

$$d_t(1 - d'_\epsilon) \geq d_t \sup_{x \in [a, a+d_t]} r(m'_t, x, t) \geq |I_R| r(m'_t, I_R, t) > -2K/t - 2\epsilon + d_t,$$

which is a contradiction for sufficiently large t , since $d_t d'_\epsilon \geq 2K/t + 2\epsilon$ for all large t . If there is no appropriate x'_t and case (II) occurs,

$$d_t - (d'_\epsilon)^2 \geq d'_\epsilon \sup_{x \in [a, a+d'_\epsilon]} r(m'_t, x, t) + (d_t - d'_\epsilon) \sup_{x \in [a+d'_\epsilon, a+d_t]} r(m'_t, x, t) \geq |I_R| r(m'_t, I_R, t) \geq 2K/t - 2\epsilon + d_t,$$

which is a contradiction for sufficiently large t , since $d'_\epsilon d'_\epsilon \geq K/t + \epsilon$ for all large t .

In case (III) define $d_t := x_t - a < d_\epsilon$, $I_L := I \cap \mathcal{N}(x_t) = (x_t - 1, a)$ and $I_R := \mathcal{N}(x_t) \setminus I_L = (a, x_t + 1)$. Note that $|I_L| = 1 - d_t$ and $|I_R| = 1 + d_t$. By (30), for any sufficiently large t (such that $|I_L| r(m, I_L, t) > 1 - d_t - \epsilon$)

$$\begin{aligned} |I_R| r(m'_t, I_R, t) &= \frac{1}{t} \left(Y_{m'_t}(x_t, t) - Y_m(x_t, t) \right) - \frac{1}{t} \left(Y_{m'_t}(x_t, 0) - Y_m(x_t, 0) \right) \\ &\quad - |I_L| r(m'_t, I_L, t) + |I_L| r(m, I_L, t) + |I_R| r(m, I_R, t) \\ &\geq -2K/t + 0 - \epsilon + (1 - d_t - \epsilon) + 0, \end{aligned}$$

so if there is no appropriate x'_t ,

$$\begin{aligned} 1 + d_t - (d'_\epsilon)^2 &= d'_\epsilon(1 - d'_\epsilon) + (1 + d_t - d'_\epsilon) \\ &\geq d'_\epsilon \sup_{x \in [a, a+d'_\epsilon]} r(m'_t, x, t) + (1 + d_t - d'_\epsilon) \inf_{x \in I_R \setminus [a, a+d'_\epsilon]} r(m'_t, x, t) \\ &\geq |I_R| r(m'_t, I_R, t) \\ &\geq -2K/t - 2\epsilon - d_t + 1, \end{aligned}$$

which implies $(d'_\epsilon)^2 \leq 2K/t + 2\epsilon + 2d_t$. This is a contradiction for sufficiently large t . We conclude that an appropriate x'_t exists for all large t in all cases (I)-(III).

For any $t \geq 0$ define

$$S_t^\epsilon := \{s \in (K/\epsilon, \infty) : r(m'_t, \tilde{I}, s) < 1 - 3\epsilon - 3d'_\epsilon, r(m, I, s) > 1 - \epsilon\}.$$

We will prove that $|S_t^\epsilon \cap [0, t]|/t < d'_\epsilon$ for all sufficiently large $t > 0$. For any $t > 0$ sufficiently large such that x'_t exists and $r(m, I, s) > 1 - \epsilon$, and for any $s \in S_t^\epsilon$ it follows from (30) that

$$\begin{aligned} \frac{1}{s} \left(Y_m(x'_t, s) - Y_{m'_t}(x'_t, s) \right) &= \frac{1}{s} \left(Y_m(x'_t, 0) - Y_{m'_t}(x'_t, 0) \right) + (x'_t - a) r(m, [a + 1, x'_t + 1], s) \\ &\quad - (x'_t - a) r(m'_t, [a + 1, x'_t + 1], s) + r(m, \tilde{I}, s) - r(m'_t, \tilde{I}, s) \\ &\quad + (1 + a - x'_t) r(m, [x'_t - 1, a], s) - (1 + a - x'_t) r(m'_t, [x'_t - 1, a], s) \\ &\geq -2K/s + 0 - d'_\epsilon + 0 - (1 - 3\epsilon - 3d'_\epsilon) + (1 - \epsilon - d'_\epsilon) - \epsilon \\ &> 0, \end{aligned}$$

where we used the following estimates to obtain the first inequality

$$\begin{aligned} (1 + a - x'_t)r(m, [x'_t - 1, a], s) &= r(m, I, s) - (x'_t - a)r(m, [a - 1, x'_t - 1], s) \geq (1 - \epsilon) - d'_\epsilon, \\ (1 + a - x'_t)r(m'_t, [x'_t - 1, a], s) &= r(m_t, I, s) - (x'_t - a)r(m_t, [a - 1, x'_t - 1], s) \leq (1 - r(m, I, s)) - 0 \leq \epsilon. \end{aligned}$$

Therefore $Y_m(x'_t, s) > Y_{m'_t}(x'_t, s)$ for all sufficiently large t and $s \in S_t^\epsilon$, and it follows from the definition of r that $r(m'_t, x'_t, t) \leq 1 - |S_t^\epsilon \cap [0, t]|/t$ for all sufficiently large t . Since $1 - d'_\epsilon < r(m'_t, x'_t, t)$ by definition of x'_t it follows that $|S_t^\epsilon \cap [0, t]|/t < d'_\epsilon$ for all sufficiently large t .

Note that if ϵ is sufficiently small, and $t, t' \geq 0$ are such that $m'_t \neq m'_{t'}$, then it follows from the definition of S_t^ϵ and $S_{t'}^\epsilon$ that $(S_t^\epsilon)^c \cap (S_{t'}^\epsilon)^c \subset \{s \geq 0 : r(m, I, s) \leq 1 - \epsilon\}$. Therefore the estimate $|S_t^\epsilon \cap [0, t]|/t < d'_\epsilon$ for all sufficiently large t and the assumption $\lim_{s \rightarrow \infty} r(m, I, s) = 1$ imply that there is an $m' \in [M]$ such that $m'_t = m'$ for all sufficiently large t . Define

$$S'_\epsilon := \{s \in (K/\epsilon, \infty) : r(m', \tilde{I}, s) < 1 - 3\epsilon - 3d'_\epsilon\},$$

and note that $|S'_\epsilon \cap [0, t]|/t < d'_\epsilon$ for all sufficiently large t .

Let $\tilde{\epsilon} > 0$. In order to complete the proof of the lemma it is sufficient to show that the set of $t' > 0$ such that $r(m', \tilde{I}, t') < 1 - \tilde{\epsilon}$ is bounded from above. Let $t' > 0$ be such that $r(m', \tilde{I}, t') < 1 - \tilde{\epsilon}$. Choose $\epsilon > 0$ such that $\tilde{\epsilon} > 10d'_\epsilon$. By definition of r , for any $t \in [(1 - \frac{\epsilon}{10})t', t']$ we have $r(m', \tilde{I}, t) < 1 - \frac{1}{2}\tilde{\epsilon}$. By definition of ϵ we have $3\epsilon + 3d'_\epsilon < \frac{1}{2}\tilde{\epsilon}$, so $r(m', \tilde{I}, t) < 1 - 3\epsilon - 3d'_\epsilon$ for all $t \in [(1 - \frac{\epsilon}{10})t', t']$. By definition of S'_ϵ this implies that $|S'_\epsilon \cap [0, t']|/t' > \tilde{\epsilon}/10$. On the other hand we know from the preceding paragraph that $|S'_\epsilon \cap [0, t]|/t < d'_\epsilon < \tilde{\epsilon}/10$ for all sufficiently large t , which completes the proof of the lemma. \square

Lemma 2.19. *Let Y be the solution of the initial value problem (6), (8) for $N = 1$, $\mathcal{N} = \mathcal{N}_\infty$, and either $V = \mathcal{S}$ or $V = \mathbb{R}$. Assume $a \in \mathbb{R}$ and $m \in [M]$ are such that $\lim_{t \rightarrow \infty} p(Y(x, t)) = m$ for all $x \in I := (a - 1, a)$, and that for $x > a$ arbitrarily close to a , either the limit $\lim_{t \rightarrow \infty} p(Y(x, t))$ does not exist or $\lim_{t \rightarrow \infty} p(Y(x, t)) \neq m$. Then there is an $m' \in [M]$, $m' \neq m$, such that, in the notation of Lemma 2.18, we have $\lim_{t \rightarrow \infty} r(m', \tilde{I}, t) = 1$ for $\tilde{I} := (a, a + 1)$.*

Proof. For each $t \geq 0$ we can find an $x_t \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} x_t = a$ and $m_t := p(Y(x_t, t)) \neq m$. By the identity (30) and letting $o_t(1)$ denote a term which converges to 0 as $t \rightarrow \infty$,

$$\begin{aligned} 0 &\leq \frac{1}{t}(Y_{m_t}(x_t, t) - Y_m(x_t, t)) \\ &= \frac{1}{t}(Y_{m_t}(x_t, 0) - Y_m(x_t, 0)) + |\mathcal{N}(x_t) \cap I| \cdot r(m_t, \mathcal{N}(x_t) \cap I, t) + |\mathcal{N}(x_t) \cap \tilde{I}| \cdot r(m_t, \mathcal{N}(x_t) \cap \tilde{I}, t) \\ &\quad - |\mathcal{N}(x_t) \cap I| \cdot r(m, \mathcal{N}(x_t) \cap I, t) - |\mathcal{N}(x_t) \cap \tilde{I}| \cdot r(m, \mathcal{N}(x_t) \cap \tilde{I}, t) + o_t(1) \\ &= r(m_t, I, t) + r(m_t, \tilde{I}, t) - r(m, I, t) - r(m, \tilde{I}, t) + o_t(1). \end{aligned}$$

Since $\lim_{t \rightarrow \infty} r(m, I, t) = 1$, which implies $\lim_{t \rightarrow \infty} r(m_t, I, t) = 0$, it follows that $\lim_{t \rightarrow \infty} r(m_t, \tilde{I}, t) = 1$ and $\lim_{t \rightarrow \infty} r(m, \tilde{I}, t) = 0$. Since $\sum_{k=1}^M r(k, \tilde{I}, t) = 1$ this implies further that there is an $m' \in [M]$, $m' \neq m$, such that $m_t = m'$ for all sufficiently large $t > 0$. It follows that $\lim_{t \rightarrow \infty} r(m', \tilde{I}, t) = 1$. \square

The following lemma is immediate from Lemmas 2.18 and 2.19. It says that if some opinion m dominates an interval I (in the sense that $\lim_{t \rightarrow \infty} r(m, I, t) = 1$), and I is not contained in a larger interval satisfying this property, then some other opinion m' will dominate the interval of length 1 immediately to the right of I . (By symmetry, the same property holds for the interval of length 1 immediately to the left of I .)

Lemma 2.20. *Let $V = \mathcal{S}$ or $V = \mathbb{R}$, and consider the initial value problem (6), (8) for $N = 1$ and $\mathcal{N} = \mathcal{N}_\infty$. Assume $m \in [M]$ and $I = [a_1, a_2] \subset \mathbb{R}$ is an interval of length ≥ 1 such that $\lim_{t \rightarrow \infty} r(m, I, t) = 1$, and such that I is not contained in any larger interval satisfying this property. Then there is an $m' \in [M] \setminus \{m\}$ such that $\lim_{t \rightarrow \infty} r(m', [a_2, a_2 + 1], t) = 1$.*

We can deduce a weak version of Proposition 2.14 for the case $V = \mathbb{R}$ from Lemma 2.20. This weak version of Proposition 2.14 is the last result of the section which holds also for $M > 2$. By Lemma 2.16 we know that there will be some interval I satisfying the conditions of Lemma 2.20. Lemma 2.20 therefore says that all $x \in \mathbb{R}$ will be contained in an interval $J \subset \mathbb{R}$ of length ≥ 1 , such that for some $m' \in [M]$, we have $\lim_{t \rightarrow \infty} r(m', J, t) = 1$. In particular, both this lemma and Proposition 2.14 say that the limiting states of the continuum Schelling model on \mathbb{R} can be divided into intervals of length ≥ 1 such that each interval is associated with a particular limiting opinion. The lemma is weaker than Proposition 2.14 in two ways: First, we do not prove that each interval has length strictly larger than 1, and second, instead of proving that $p \circ Y$ converges pointwise on the intervals we prove a weaker result expressed in terms of the function r . Both these stronger properties are needed when we apply Proposition 2.14 in our proof of Theorem 1.3.

In the following lemma we consider two solutions Y^{ϵ_1} and Y^{ϵ_2} for perturbed initial data such that $Y^{\epsilon_1} > Y^{\epsilon_2}$. Roughly speaking, we let $h(t)$ denote the minimal distance between the solutions at time t and let $\ell(t)$ denote the smallest distance between a pair of points where Y^{ϵ_1} is negative and Y^{ϵ_2} is positive, respectively. We lower bound $\ell(t)$ in terms of $h(t)$ by using (among other properties) Lipschitz continuity of $x \mapsto Y^{\epsilon_k}(x, t) - Y^{\epsilon_k}(x', t)$ for $k = 1, 2$. For $x, x' \in \mathcal{S}$ we define $|x - x'|$ to be equal to $\inf_{k \in \mathbb{N}} |x - x' - Rk|$ when we identify \mathcal{S} with the interval $[0, R)$.

Lemma 2.21. *Let $M = 2$, and consider the initial value problem (11) with $N = 1$, $\mathcal{N} = \mathcal{N}_\infty$, and either $V = \mathcal{S}$ or $V = \mathbb{R}$, but with the following perturbed initial data for some $K > 0$*

$$\begin{aligned} \widehat{Y}(x, 0) &= \widehat{B}(x) + \epsilon\lambda([-K + 1, K + 1] \cap [x - 1, x + 1]) & \text{for } V = \mathbb{R}, \\ \widehat{Y}(x, 0) &= \widehat{B}(x) + \epsilon & \text{for } V = \mathcal{S}. \end{aligned} \tag{31}$$

For $\epsilon_1 > \epsilon_2 > 0$ let \widehat{Y}^{ϵ_1} and \widehat{Y}^{ϵ_2} denote the solution of (11), (31) with $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$, respectively. For $V = \mathbb{R}$ let $J \subset [-K, K]$ be an interval which may depend on \widehat{B} , ϵ_1 , and ϵ_2 , and for $V = \mathcal{S}$ let $J = \mathcal{S}$. For $t \geq 0$ define

$$h(t) := \inf\{\widehat{Y}^{\epsilon_1}(x, t) - \widehat{Y}^{\epsilon_2}(x, t) : x \in J\}, \quad \ell(t) := \inf\{|x - x'| : x, x' \in J, \widehat{Y}^{\epsilon_1}(x, t) < 0, \widehat{Y}^{\epsilon_2}(x', t) > 0\},$$

where the infimum over the empty set is defined to be ∞ . There are $c_0, c_1 > 0$ depending on \widehat{B} , ϵ_1 , and ϵ_2 , such that for all $t > 0$,

$$\ell(t) \geq \max\left\{\frac{h(t) - c_1}{2t}; \min\left\{c_0; \frac{h(t)}{2.01t}\right\}\right\}.$$

Proof. For all $x \in J$ we have $\widehat{Y}^{\epsilon_1}(x, 0) \geq \widehat{Y}^{\epsilon_2}(x, 0) + (\epsilon_1 - \epsilon_2)$. Since the right side of the differential equation (11) is monotone in Y , this implies that the function $t \mapsto \widehat{Y}^{\epsilon_1}(x, t) - \widehat{Y}^{\epsilon_2}(x, t)$ is increasing for each $x \in J$.

Define $c_1 := \sup\{|\widehat{B}(x) - \widehat{B}(x')| : x, x' \in J\}$. Let $\widehat{y}, \widetilde{y} : V \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be such that $\widehat{Y}^{\epsilon_1}(x, t) = \widehat{Y}^{\epsilon_1}(x, 0) + \widehat{y}(x, t)$ and $\widehat{Y}^{\epsilon_2}(x, t) = \widehat{Y}^{\epsilon_2}(x, 0) + \widetilde{y}(x, t)$, and recall that $\widehat{y}(\cdot, t)$ and $\widetilde{y}(\cdot, t)$ are Lipschitz continuous with constant $2t$ by Theorem 2.1. For $x, x' \in J$ and $t > 0$ satisfying $\widehat{Y}^{\epsilon_2}(x, t) \geq 0$ and $|x - x'| \leq \frac{h(t) - c_1}{2t}$, we have

$$\begin{aligned} \widehat{Y}^{\epsilon_1}(x', t) &= (\widehat{B}(x') - \widehat{B}(x)) + (\widehat{y}(x', t) - \widehat{y}(x, t)) + (\widehat{Y}^{\epsilon_1}(x, t) - \widehat{Y}^{\epsilon_2}(x, t)) + \widehat{Y}^{\epsilon_2}(x, t) \\ &\geq -c_1 - 2t|x - x'| + h(t) + 0 \geq 0, \end{aligned}$$

which implies $\ell(t) \geq \frac{h(t) - c_1}{2t}$.

Define $c_0 := \left(\frac{\epsilon_1 - \epsilon_2}{1000c_2}\right)^3$, where c_2 is the 1/3-Hölder constant for \widehat{B} on J , and observe that $c_0 \leq \left(\frac{h(t)}{1000c_2}\right)^3$ for all $t \geq 0$, so $c_2c_0^{1/3} \leq \frac{h(t)}{1000}$. For $x, x' \in J$ satisfying $\widehat{Y}^{\epsilon_2}(x, t) \geq 0$ and $|x - x'| \leq \min\left\{c_0; \frac{h(t)}{2.01t}\right\}$, we have

$$\begin{aligned} \widehat{Y}^{\epsilon_1}(x', t) &= (\widehat{B}(x') - \widehat{B}(x)) + (\widehat{y}(x', t) - \widehat{y}(x, t)) + (\widehat{Y}^{\epsilon_1}(x, t) - \widehat{Y}^{\epsilon_2}(x, t)) + \widehat{Y}^{\epsilon_2}(x, t) \\ &\geq -c_2|x - x'|^{1/3} - 2t|x - x'| + h(t) + 0 \geq -\frac{h(t)}{1000} - 2t\frac{h(t)}{2.01t} + h(t) \geq 0, \end{aligned}$$

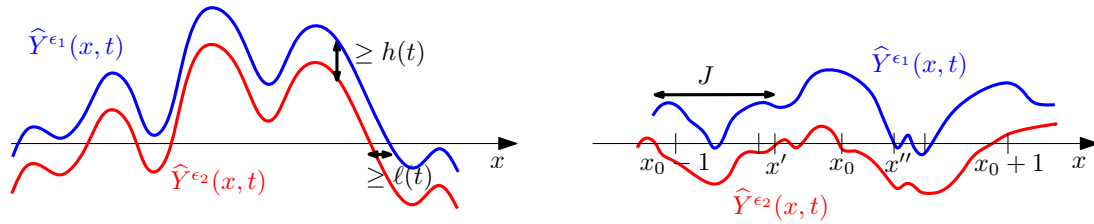


Figure 8: For $\epsilon_1 > \epsilon_2 > 0$ we consider two solutions \hat{Y}^{ϵ_1} and \hat{Y}^{ϵ_2} of (11) with initial data perturbed by ϵ_1 and ϵ_2 , respectively. We prove Proposition 2.14 by showing that the event considered in the proposition must occur for at least one of \hat{Y}^{ϵ_1} and \hat{Y}^{ϵ_2} . Left: Illustration of h and ℓ defined in Lemma 2.21. Right: Illustration of the proof of Proposition 2.14, case (b).

which implies $\ell(t) \geq \min \{c_0; \frac{h(t)}{2.01t}\}$. Combining the above two bounds for $\ell(t)$ we obtain the lemma. \square

The next lemma says that if Y is a solution of (11) and there is some interval of length greater than 1 on which one opinion dominates as $t \rightarrow \infty$ (in the sense that the limit $\lim_{t \rightarrow \infty} p(Y(x, t))$ exists and is constant for all x in the interval), then the event considered in Proposition 2.14 occurs a.s.

Lemma 2.22. *Let $M = 2$, and consider the initial value problem (11) with $N = 1$, $\mathcal{N} = \mathcal{N}_\infty$, and either $V = \mathcal{S}$ or $V = \mathbb{R}$. Let E be the event that (in the notation of Proposition 2.14), $\mathbb{R} \setminus (A_1 \cup A_2)$ has measure zero, and each set A_1 and A_2 can be written as the union of intervals of length > 1 . Let E' be the event that at least one of the sets A_1 and A_2 contains an interval of length > 1 . Then $\mathbb{P}[E' \cap E^c] = 0$.*

Proof. For $\epsilon > 0$ and fixed $K > 1$ consider the initial value problem (11) with perturbed initial data (31). Let $A_1(\epsilon)$ and $A_2(\epsilon)$ be defined just as the sets A_1 and A_2 , respectively, in the statement of Proposition 2.14. Let $\tilde{E}(\epsilon)$ be the event that the following two properties hold: (i) if $V = \mathcal{S}$ then at least one of the sets $A_1(\epsilon)$ and $A_2(\epsilon)$ contains an interval of length > 1 , and if $V = \mathbb{R}$ then at least one of the sets $A_1(\epsilon) \cap [-K, K]$ and $A_2(\epsilon) \cap [-K, K]$ contains an interval of length > 1 , and (ii) the origin is not contained in an interval I of length > 1 such that $\lim_{t \rightarrow \infty} \text{sign } Y^\epsilon(x, t)$ exists and is equal for all $x \in I$. Since K was arbitrary and by translation invariance in law in the space variable, in order to complete the proof of the lemma it is sufficient to show that $\mathbb{P}[\tilde{E}(0)] = 0$.

First we will reduce the lemma to proving

$$\mathbb{P}[\tilde{E}(\epsilon_1) \cap \tilde{E}(\epsilon_2)] = 0 \quad \forall \epsilon_1 > \epsilon_2 > 0. \tag{32}$$

Assume $\mathbb{P}[\tilde{E}(0)] > 0$. Observe that the initial data (31) are absolutely continuous with respect to the initial data with $\epsilon = 0$. By absolute continuity, we can find a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ and $p > 0$ such that $\epsilon_{n_1} \neq \epsilon_{n_2}$ for $n_1 \neq n_2$ and $\mathbb{P}[\tilde{E}(\epsilon_n)] > p$ for all $n \in \mathbb{N}$. If we assume (32) holds this leads to a contradiction, since for any $n \in \mathbb{N}$,

$$\mathbb{P} \left[\bigcup_{k=1}^n \tilde{E}(\epsilon_k) \right] = \sum_{k=1}^n \mathbb{P}[\tilde{E}(\epsilon_n)] \geq np,$$

which converges to ∞ as $n \rightarrow \infty$. We conclude that the lemma will follow once we have established (32).

Let \hat{Y}^{ϵ_1} (resp. \hat{Y}^{ϵ_2}) denote the solution of (11) with perturbed initial data for $\epsilon = \epsilon_1$ (resp. $\epsilon = \epsilon_2$). We will assume that both events $\tilde{E}(\epsilon_1)$ and $\tilde{E}(\epsilon_2)$ occur, and want to derive a contradiction. For an interval $I \subset V$ and $t \geq 0$ define

$$r(I, t) := \frac{1}{\lambda(I)t} \int_0^t \int_I \text{sign}(\hat{Y}^{\epsilon_2}(x, t')) dx dt'.$$

By Lemma 2.20 and by absolute continuity of the initial data, we know that on the event $\tilde{E}(\epsilon_2)$ there is a.s. an interval I of length ≥ 1 containing the origin, such that either $\lim_{t \rightarrow \infty} r(I, t) = 1$ or $\lim_{t \rightarrow \infty} r(I, t) = -1$.

Without loss of generality we assume $\lim_{t \rightarrow \infty} r(I, t) = -1$; the case $\lim_{t \rightarrow \infty} r(I, t) = 1$ can be treated similarly. We also assume that I is the maximal open interval satisfying this property, and let a be the left end-point of the interval. By Lemma 2.20, $\lim_{t \rightarrow \infty} r([a - 1, a], t) = 1$.

If $\lambda(I) > 1$, it is immediate from the definition of r and (11) that $\lim_{t \rightarrow \infty} \text{sign } \widehat{Y}^{\epsilon_2}(x, t) = -1$ for all $x \in I$, which contradicts (ii) in the definition of $E(\epsilon_2)$. Therefore we will assume $\lambda(I) = 1$. To conclude the proof of the lemma it is sufficient to derive a contradiction to the occurrence of $E(\epsilon_1)$.

Let $J = [a - 1/2, a + 1/2]$, and let h and ℓ be as in Lemma 2.21. Let $\mathcal{T} \subset \mathbb{R}_+$ be the set of times $t \geq 0$ for which we can find a $x' \in [a - 1/10, a + 1/10]$ such that \widehat{Y}^{ϵ_2} takes both positive and negative values arbitrarily close to x' . Since $\lim_{t \rightarrow \infty} r([a, a + 1/2], t) = -1$ and $\lim_{t \rightarrow \infty} r([a - 1/2, a], t) = 1$, we have $\lim_{t \rightarrow \infty} \lambda(\mathcal{T} \cap [0, t])/t \rightarrow 1$.

Let $t \in \mathcal{T}$, and consider two cases: (i) there is some interval $J' \subset J$ of length $\geq \ell(t)$ on which $Y^{\epsilon_2}(\cdot, t)$ is negative, and (ii) otherwise. In case (ii), by definition of ℓ there will be some interval J'' containing J of length strictly larger than 1 on which $Y^{\epsilon_1}(\cdot, t)$ is positive. By Lemma 2.16, $Y^{\epsilon_1}(x, t')$ will be positive for all $t' \geq t$ and $x \in J''$. By using this, $\lim_{t' \rightarrow \infty} r([a + 1, a + 2], t') = 1$ (which follows from Lemma 2.20), $Y^{\epsilon_1} \geq Y^{\epsilon_2}$, and (11), we get that $\lim_{t' \rightarrow \infty} \text{sign } Y^{\epsilon_1}(\cdot, t') = 1$ on $[a - 1, a + 2]$, which contradicts $E(\epsilon_1)$, so case (ii) cannot occur.

In case (i), assume the interval J' is chosen as large as possible, i.e., it is the largest interval in J on which $Y^{\epsilon_2}(\cdot, t)$ is negative. Then each end-point of J' is either contained in $\{a - 1/2, a + 1/2\}$ or is a point x' such that $Y^{\epsilon_2}(x', t) = 0$. By the definition of \mathcal{T} , at least one end-point of J' must be a point x' such that $Y^{\epsilon_2}(x', t) = 0$. If x' is the left (resp. right) end-point of J' then it follows from the definition of ℓ and J' that on $(x', x' + \ell(t))$ (resp. $(x' - \ell(t), x')$) we have $\widehat{Y}^{\epsilon_1}(x, t) > 0$ and $\widehat{Y}^{\epsilon_2}(x, t) < 0$. In particular,

$$\lambda(\{x \in J : \widehat{Y}^{\epsilon_1}(x, t) > 0, \widehat{Y}^{\epsilon_2}(x, t) < 0\}) \geq \ell(t).$$

By (11) we get further that for all $t \in \mathcal{T}$,

$$\frac{dh}{dt}(t) \geq 2\lambda(\{x \in J : \widehat{Y}^{\epsilon_1}(x, t) > 0, \widehat{Y}^{\epsilon_2}(x, t) < 0\}) \geq 2\ell(t).$$

Since $\ell(t) \geq \min\left\{c_0; \frac{h(t)}{2.01t}\right\}$, this implies that $\lim_{t \rightarrow \infty} h(t) = \infty$.

By the lower bound for ℓ in Lemma 2.21 we also have $\frac{dh}{dt}(t) \geq 2\ell(t) \geq \frac{h(t) - c_1}{t}$ for all $t \in \mathcal{T}$, and since $\lim_{t \rightarrow \infty} h(t) = \infty$ this implies that $h(t) \geq ct$ for some random constant $c > 0$ and all $t \geq 0$. Lemma 2.20 implies that $\lim_{t \rightarrow \infty} r([a - 1, a] \cup [a + 1, a + 2], t) = 1$. By this result, (11), and $\lim_{t \rightarrow \infty} r(I, t) = -1$,

$$\inf_{x \in I} \frac{1}{t} \widehat{Y}^{\epsilon_2}(x, t) \geq \inf_{x \in I} \frac{1}{t} \widehat{Y}^{\epsilon_2}(x, 0) + r(I, t) + \inf_{I' \subset [a-1, a] \cup [a+1, a+2] : \lambda(I')=1} r(I', t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

It follows that $\inf_{x \in J} \widehat{Y}^{\epsilon_1}(x, t) \geq \inf_{x \in J} \widehat{Y}^{\epsilon_2}(x, t) + h(t) > 0$ for all sufficiently large t . This implies (using (11) and Lemma 2.16) that $\inf_{x \in [a-1, a+2]} \widehat{Y}^{\epsilon_1}(x, t) > 0$ for all sufficiently large t , which is a contradiction to the condition (ii) in the definition of $E'(\epsilon_1)$. \square

By Lemma 2.22, in order to complete the proof of Proposition 2.14 we need only show that the event E' in that lemma occurs a.s. For $V = \mathbb{R}$ this is immediate, while for the torus case $V = \mathcal{S}$ we will apply another perturbative argument along with Lemmas 2.21 and 2.22.

Proof of Proposition 2.14. For the case when $V = \mathbb{R}$ the proposition follows immediately from Lemma 2.22, since with probability 1 there will be some interval in the initial data of length greater than 1 on which the sign of Y is constant, and by Lemma 2.16 this implies the occurrence of E' , so we have $\mathbb{P}[E'] = 1$.

Let $V = \mathcal{S}$. For $\epsilon > 0$ let $E'(\epsilon)$ be the event of Lemma 2.22, but for the perturbed initial data. It is sufficient to prove that for any $\epsilon_1 > \epsilon_2 > 0$, we have $\mathbb{P}[E'(\epsilon_1)^c \cap E'(\epsilon_2)^c] = 0$, since this implies that $\mathbb{P}[(E')^c] = 0$ (see the argument in the second paragraph in the proof of Lemma 2.22 for a similar argument). We assume that both $E'(\epsilon_2)^c$ and $E'(\epsilon_1)^c$ occur, and will derive a contradiction.

First we will argue that the following inequality holds

$$\frac{dh}{dt} \geq 4\ell(t). \tag{33}$$

By (11) we have

$$\frac{dh}{dt} \geq 2 \inf_{x \in \mathcal{S}} \lambda(\{x' \in \mathcal{N}(x) : \widehat{Y}^{\epsilon_1}(x', t) > 0, \widehat{Y}^{\epsilon_2}(x', t) < 0\}). \tag{34}$$

Since $E'(\epsilon_1)^c$ occurs, each closed interval $I' \subset \mathcal{S}$ of length ≥ 1 must intersect some interval of length $\geq 2\ell(t)$ on which $\widehat{Y}^{\epsilon_2}(x, t) < 0$; otherwise we would have $\widehat{Y}^{\epsilon_1}(x, t) > 0$ on I' . For some fixed $x_0 \in \mathcal{S}$ the interval $[x_0 - 1/2, x_0 + 1/2]$ therefore intersect some interval of length $\geq 2\ell(t)$ on which $\widehat{Y}^{\epsilon_2}(x, t) < 0$. Let J be an interval satisfying this property and which is maximal in the sense that J is not contained inside any other interval satisfying the property. We have either (a) $J \subset (x_0 - 1, x_0 + 1)$, (b) $\{x_0 - 1, x_0 + 1\} \cap J \neq \emptyset$. If case (a) occurs, then the following holds by the definition of ℓ

$$\lambda(\{x' \in \mathcal{N}(x_0) : \widehat{Y}^{\epsilon_1}(x', t) > 0, \widehat{Y}^{\epsilon_2}(x', t) < 0\}) \geq 2\ell(t). \tag{35}$$

Next assume case (b) occurs, and without loss of generality assume $(x_0 - 1) \in J$. See Figure 8 for an illustration. We have $\ell(t) \leq 1/2$; otherwise $E'(\epsilon_2)^c$ and $E'(\epsilon_1)^c$ cannot both occur, since \widehat{Y}^{ϵ_1} would be positive on any maximal interval of length < 1 on which \widehat{Y}^{ϵ_2} is negative. Let $x' \in [x_0 - 1/2, x_0)$ be the right end-point of J , and observe that \widehat{Y}^{ϵ_1} is positive and \widehat{Y}^{ϵ_2} negative on the interval $(x' - \ell(t), x')$. Also observe that $(x' - \ell(t), x') \subset \mathcal{N}(x_0)$ since $x' \in [x_0 - 1/2, x_0)$ and $\ell(t) \leq 1/2$.

Define $x'' := \inf\{x > x' : \widehat{Y}^{\epsilon_1}(x, t) < 0\}$. We have $x'' < x' + 1 - \ell(t) < x_0 + 1 - \ell(t)$, where the first inequality holds since $\widehat{Y}^{\epsilon_1}(x', t) > 0$ and $E'(\epsilon_1)^c$ occurs. We also have $x'' > x' + 2\ell(t)$; otherwise we would have $\widehat{Y}^{\epsilon_2} < 0$ on $(x' - \ell(t), x' + \ell(t))$ by the definition of $\ell(t)$, which contradicts the definition of x' . It follows that \widehat{Y}^{ϵ_1} is positive and \widehat{Y}^{ϵ_2} negative on the interval $(x'' - \ell(t), x'') \subset \mathcal{N}(x_0)$. Since the intervals $(x' - \ell(t), x')$ and $(x'' - \ell(t), x'')$ are disjoint, we see that (35) holds also in case (b). We obtain (33) by combining (34) and (35).

Next we argue that $h(t) \geq 2c_0t$ for all $t \geq 0$. Let $\tau = \inf\{t \geq 0 : h(t) \leq 2c_0t\}$. We see that $\tau = \infty$, since $h(0) > 0$, and since (33) and Lemma 2.21 imply that for $t \in [0, \tau]$,

$$\frac{dh}{dt} \geq 4 \min \left\{ c_0; \frac{h(t)}{2.01t} \right\} \geq \frac{8}{2.01}c_0.$$

We conclude that $h(t) \geq 2c_0t$ for all $t \geq 0$. In particular, we have $\lim_{t \rightarrow \infty} h(t) = \infty$, so by Lemma 2.21 we have $\ell(t) \geq \frac{h(t) - c_1}{2t} \geq \frac{h(t)}{2.01t}$ for all sufficiently large t , which implies by (33) that $\frac{dh}{dt} \geq \frac{4h(t)}{2.01t}$ for all sufficiently large t . Further we get $h(t) \geq ct^{4/2.01}$ for some random constant $c > 0$. The function h cannot grow superlinearly, since \widehat{Y}^{ϵ_2} and \widehat{Y}^{ϵ_1} grow at most linearly in time. Therefore we obtain a contradiction, which concludes the proof. \square

3 The discrete Schelling model

3.1 The early phase of the discrete Schelling model

The main purpose of this section is to prove the following proposition, which says that the solution of the initial value problem (6), (8) describes the early phase of the discrete Schelling model well for large w . Recall that we rescaled time by $w^{-N/2}$ when we defined Y^w . Therefore the proposition only provides information about times up to order $w^{-N/2}$ for the discrete Schelling model. However, as we will see in Section 3.2, in the one-dimensional model the limiting opinion of the nodes is essentially determined at times of order $w^{-N/2}$ in the sense that the bias of most nodes is unchanged after time $Cw^{-N/2}$ for $C \gg 1$.

Proposition 3.1. *Let $N \in \mathbb{N}$ and $V = \mathcal{S}$, or let $N = 1$ and $V = \mathbb{R}$. Let $M \in \{2, 3, \dots\}$, let \mathcal{N} be as defined in Section 1.3, let Y be the solution of the initial value problem (6), (8), and let Y^w be given by (4). There is a coupling of Y and Y^w for $w \in \mathbb{N}$ such that a.s. Y^w converges uniformly to Y on compact subsets of $V^N \times \mathbb{R}_+$ as $w \rightarrow \infty$.*

We will see in the proof of the proposition that it is sufficient to construct a coupling of the *initial* data for the discrete and continuum Schelling model; given the initial data, the evolution of Y is deterministic and the evolution of Y^w is approximately deterministic. The following lemma says that the initial data can be coupled.

Lemma 3.2. *Let $V = \mathcal{S}$ or $V = \mathbb{R}$. There is a uniquely defined continuous centered Gaussian (N, M) -random field B on V^N with covariances given by (7). The initial data $Y^w(\cdot, 0)$ of the normalized discrete bias function defined by (4) converges in distribution to B .*

Proof. We first define a smoothed version $\check{Y}^w \in \mathcal{C}_M(V^N)$ of $Y^w(\cdot, 0)$, and prove convergence of \check{Y}^w to B . For any $i \in \mathbb{Z}^N$ let $A(i)$ denote the square of side length 1 centered at i , and for $i \in \mathbb{Z}^N$ let $w\mathcal{N} + i = \{x \in \mathbb{R}^N : w^{-1}(x - i) \in \mathcal{N}\}$. If $V = \mathcal{S}$ (resp. $V = \mathbb{R}$) define $\mathcal{V} = \mathfrak{S}$ (resp. $\mathcal{V} = \mathbb{Z}$). For $m \in [M]$ define the smoothed unscaled bias function $\check{\mathcal{Y}}_m : \mathcal{V} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\check{\mathcal{Y}}_m(i, t) = \sum_{j \in \mathcal{V}} \lambda(A(j) \cap (w\mathcal{N} + i)) \left(\mathbb{1}_{X(j,t)=m} - \frac{1}{M} \right), \quad i \in \mathcal{V}, t \geq 0.$$

Then define \check{Y}_m^w by (4) and (5), but using $\check{\mathcal{Y}}$ instead of \mathcal{Y} . It follows by e.g. [AP86] that for each $m \in [M]$, \check{Y}_m^w converges in law to B_m in $\mathcal{C}(V^N)$. Note in particular that the entropy integral considered in [AP86] is finite as required, since $\partial\mathcal{N}$ has an upper Minkowski dimension strictly smaller than 2. By this convergence result, we see that the law of $\check{Y}^w(\cdot, 0)$ is tight in $\mathcal{C}_M(V^N)$. By convergence of the finite dimensional distributions and hence uniqueness of the limit, we get that $(B(x))_{x \in V^N}$ exists and that $\check{Y}^w(\cdot, 0)$ converges in law to B in $\mathcal{C}_M(V^N)$. Continuity of B follows e.g. by applying the Kolmogorov-Chentsov theorem as in the construction of Brownian motion, see the proof of Lemma 2.9.

To conclude the proof it is sufficient to show that $Y_m^w(\cdot, 0) - \check{Y}_m^w$ converges in law to 0 as $w \rightarrow \infty$ for any $m \in [M]$. Let $q_w \in \mathbb{N}$ denote the following measure for the number of lattice points which are near the boundary of $w\mathcal{N}$

$$q_w = |\{j \in \mathbb{Z}^N : 0 < \lambda(A(j) \cap (w\mathcal{N})) < 1\}|.$$

Observe that for each i , $Y_m^w(i, 0) - \check{Y}_m^w(i)$ is the weighted sum of q_w i.i.d. centered random variables with values in $\{-\frac{1}{M}, \frac{M-1}{M}\}$, divided by $w^{N/2}$, where the weights are in $[0, 1]$. Since the upper Minkowski dimension of $\partial\mathcal{N}$ is smaller than N , it holds that $q_w = O(w^{N-\epsilon})$ for some $\epsilon > 0$. Therefore, for any $i \in \mathcal{V}$, we have $\mathbb{P}[|Y_m^w(i, 0) - \check{Y}_m^w(i)| > w^{-\epsilon/3}] \preceq \exp(-w^{-\epsilon/10})$. A union bound now gives that $Y_m^w(\cdot, 0) - \check{Y}_m^w$ converges in law to 0 as $w \rightarrow \infty$. \square

For times of order $O(w^{-N/2})$ (in particular, for times of order $o(1)$), in any fixed set the fraction of nodes of each opinion $m \in [M]$ is approximately M^{-1} , since few nodes have changed their opinion since time $t = 0$. This is quantified in the following lemma. The lemma is useful since it will imply that each time the clock of some node i rings and $t = O(w^{-N/2})$, the probability that its opinion is different from its bias (i.e., $X(i, t) \neq p(\mathcal{Y}(i, t))$) is approximately $1 - M^{-1}$.

Lemma 3.3. *Let M, N, R, \mathcal{R}, w and \mathcal{N} be as in Section 1.3, and consider the Schelling model on $V = \mathcal{S}$. Divide \mathcal{S}^N into disjoint cubes of side length L^{-1} for $L = \lceil w^{1/2} \rceil$. For $T > 0$ let $\widehat{E}(T)$ be the event that for each cube U , each $m \in [M]$, and each $t \in [0, T]$,*

$$\left| \frac{|\{i \in \mathfrak{S}^N : w^{-1}i \in U, X(i, w^{-N/2}t) = m\}|}{|\{i \in \mathfrak{S}^N : w^{-1}i \in U\}|} - M^{-1} \right| < w^{-0.1}.$$

Then $\mathbb{P}[\widehat{E}(T)] \rightarrow 1$ as $w \rightarrow \infty$, and the rate of convergence depends only on T, R, M, N .

Proof. Throughout the proof, when we say that an event occurs with probability converging to 1 as $w \rightarrow \infty$, we require that the rate at which the convergence holds depends only on T, R, M, N . By concentration for the sum of independent Bernoulli random variables, with probability converging to 1 as $w \rightarrow \infty$ the following holds at time $t = 0$, simultaneously for all cubes U and each $m \in [M]$

$$\left| \frac{|\{i \in \mathfrak{S}^N : w^{-1}i \in U, X(i, 0) = m\}|}{|\{i \in \mathfrak{S}^N : w^{-1}i \in U\}|} - M^{-1} \right| < \frac{1}{2}w^{-0.1}. \quad (36)$$

Note that each cube contains $w^N L^{-N} = \Theta(w^{N/2})$ nodes, so the expected number of nodes in each cube whose clock rings before time T is $\Theta(w^{N/2} \cdot w^{-N/2}) = \Theta(1)$. Let $\xi(U)$ denote the number of times that the clock of some node in U rings before time T . By using concentration for the sum of independent Bernoulli random variables again, with probability converging to 1 as $w \rightarrow \infty$, we have $\xi(U) \leq 0.1w^{0.01}$ for all U . On this event and the event in (36), by the triangle inequality,

$$\begin{aligned} & \left| \frac{|\{i \in \mathfrak{S}^N : w^{-1}i \in U, X(i, w^{-N/2}t) = m\}|}{|\{i \in \mathfrak{S}^N : w^{-1}i \in U\}|} - M^{-1} \right| \\ & < \left| \frac{|\{i \in \mathfrak{S}^N : w^{-1}i \in U, X(i, 0) = m\}|}{|\{i \in \mathfrak{S}^N : w^{-1}i \in U\}|} - M^{-1} \right| + \frac{|\{i \in \mathfrak{S}^N : w^{-1}i \in U, X(i, 0) \neq X(i, w^{-N/2}t)\}|}{|\{i \in \mathfrak{S}^N : w^{-1}i \in U\}|} \\ & < \frac{1}{2}w^{-0.1} + \frac{\xi(U)}{w^N L^{-N}} < w^{-0.1}. \end{aligned}$$

□

The evolution of Y^w is random, while the evolution of Y (given its random initial condition) is deterministic. The next lemma states that if Y^w and Y are coupled in such a way that they are likely to be close at time t_0 , then it is likely that they remain close at time $t_0 + \Delta t$. Informally, this means that the evolution of Y^w is *approximately* deterministic and *approximately* follows the same evolution rule as Y . One reason the lemma is challenging to prove is that the evolution of Y^w (resp. Y) may be very sensitive to small perturbations when the bias is approximately as strong towards two different opinions. To bound the effect of this we will use Theorem 2.4, which will imply that the measure of the set of points at which this happens is not too large, uniformly for t in a compact set.

Lemma 3.4. *Let M, N, R, \mathcal{R}, w , and \mathcal{N} be as in Section 1.3, and consider the Schelling model on V^N for $V = \mathcal{S}$. Let $T > 0, \Delta t > 0$ and $t_0 \in \{0, \Delta t, 2\Delta t, \dots, \lceil T/\Delta t \rceil \Delta t\}$. Consider an arbitrary coupling of Y^w (which is defined by (4)) and Y (which solves (6), (8)). Let $(\mathcal{F}_t)_{t \geq 0}$ denote the filtration which contains all information about the discrete and continuum model until (rescaled) time t , i.e.,*

$$\mathcal{F}_t = \sigma\left(Y|_{\mathcal{S}^N \times [0, t]}, X|_{\mathfrak{S}^N \times [0, tw^{-N/2}]}, \{(j, s) \in \mathcal{R} : s \in [0, tw^{-N/2}]\}\right).$$

There exists a random function $v : \mathbb{N} \rightarrow [0, 1]$ and a random constant $c > 1$, such that $\lim_{w \rightarrow \infty} v(w) = 1$ a.s., and such that for all $w \in \mathbb{N}$ and with $\hat{E}(t_0)$ as in Lemma 3.3 and $E(t) := \|Y^w(\cdot, t) - Y(\cdot, t)\|_{L^\infty(\mathcal{S}^N)}$,

$$\mathbb{1}_{E(t_0) < \delta; \hat{E}(t_0)} \mathbb{P}\left[E(t_0 + \Delta t) \leq E(t_0)(1 + c\Delta t) + c\Delta t^2 \mid \mathcal{F}_{t_0}\right] \geq \mathbb{1}_{E(t_0) < \delta; \hat{E}(t_0)} v(w).$$

The constant c depends on T and the σ -algebra generated by $(B(x))_{x \in \mathcal{S}^N}$, and the function v depends on $T, \delta, \Delta t$ and the σ -algebra generated by $(B(x))_{x \in \mathcal{S}^N}$.

Lemma 3.4 will follow almost immediately from the following lemma. For any $f \in \mathcal{C}_M(\mathcal{S}^N)$ and $w \in \mathbb{N}$ define $\|f\|_{L^\infty(\mathfrak{S}^N)} := \sup_{m \in [M]} \sup_{i \in \mathfrak{S}^N} |f_m(w^{-1}i)|$.

Lemma 3.5. *The result of Lemma 3.4 holds if we replace $L^\infty(\mathcal{S}^N)$ by $L^\infty(\mathfrak{S}^N)$ in the second indented equation.*

The next lemma, which is a discrete version of the estimate (25), will help us to prove Lemma 3.5.

Lemma 3.6. For any $T > 0$ and $\Delta t > 0$ there are random constants c and w_0 such that for any $\delta \in (0, 1)$, if

$$\tilde{A}_{t,\delta,w} := \{i \in \mathfrak{S}^N : \exists m, m' \in [M], m \neq m', \text{ such that } |Y_m(iw^{-1}, t) - Y_{m'}(iw^{-1}, t)| < \delta\}, \quad t \geq 0,$$

then $|\tilde{A}_{t,\delta,w}| < c\delta w^N$ for all $t \in \{0, \Delta t, 2\Delta t, \dots, \lceil T/\Delta t \rceil \Delta t\}$ and all sufficiently large $w \geq w_0$. The constant c satisfies the same properties as in Lemma 3.4, and the constant w_0 depends on T , Δt , and the σ -algebra generated by $(B(x))_{x \in \mathfrak{S}^N}$.

Proof. Let $m, m' \in [M]$ satisfy $m \neq m'$. Then the field $Y_m(\cdot, t) - Y_{m'}(\cdot, t)$ has the law of a constant multiple of B_m plus some element of $\mathcal{L}^{2t}(\mathfrak{S}^N)$. The estimate (25) implies that

$$\lambda(A_{t,\delta}) < \frac{1}{2}c\delta, \quad A_{t,\delta} := \{x \in \mathfrak{S}^N : |Y_m(x, t) - Y_{m'}(x, t)| < \delta\}. \quad (37)$$

By e.g. [Str11, Lemma 4.2.6] and since $A_{t,\delta}$ is open,

$$\lambda(A_{t,\delta}) = \lim_{w \rightarrow \infty} \sum_{i \in \mathfrak{S}^N} w^{-N} \mathbf{1}_{|Y_m(iw^{-1}, t) - Y_{m'}(iw^{-1}, t)| < \delta} = \lim_{w \rightarrow \infty} w^{-N} |\tilde{A}_{t,\delta,w}|.$$

Therefore $|\tilde{A}_{t,\delta,w}| < c\delta w^N$ for all sufficiently large values of w for fixed $t \in [0, T]$, so $|\tilde{A}_{t,\delta,w}| < c\delta w^N$ for all sufficiently large values of w and all $t \in \{0, \Delta t, 2\Delta t, \dots, \lceil T/\Delta t \rceil \Delta t\}$. \square

Proof of Lemma 3.5. Throughout the proof the implicit constant of \preceq will depend only on T and $(B(x))_{x \in \mathfrak{S}^N}$. The function v will change throughout the proof, but will always satisfy the properties of the function v in the statement of the lemma. The expression "for all sufficiently large w " means that a statement is true for all $w \succeq 1$. Fix $m \in [M]$. For any $i \in \tilde{S}$ let $U_{i,t} = \mathbf{1}_{X(i, w^{-N/2}t^-) \neq m}$, and let V_i be i.i.d. Bernoulli random variables with $\mathbb{P}[V_i = 0] = M^{-1}$ and $\mathbb{P}[V_i = 1] = 1 - M^{-1}$. Here $X(i, w^{-N/2}t^-)$ is equal to $\lim_{s \uparrow t} X(i, w^{-N/2}s)$. Recall the definition (2) of \mathcal{R} , and for $i \in \mathfrak{S}^N$ define the following sets \mathcal{R}' , \mathcal{R}'_0 , \mathcal{R}'_i , $\mathcal{R}'_{0,i} \subset \mathcal{R}$

$$\begin{aligned} \mathcal{R}' &:= \{(j, tw^{-N/2}) \in \mathcal{R} : t \in (t_0, t_0 + \Delta t)\}, & \mathcal{R}'_0 &:= \{(j, t) \in \mathcal{R}' : \exists s \in [t_0, t] \text{ such that } (j, s) \in \mathcal{R}'\}, \\ \mathcal{R}'_i &:= \{(j, t) \in \mathcal{R}' : j \in \mathfrak{N}(i)\}, & \mathcal{R}'_{0,i} &:= \{(j, t) \in \mathcal{R}'_0 : j \in \mathfrak{N}(i)\} = \mathcal{R}'_0 \cap \mathcal{R}'_i. \end{aligned}$$

Note that several of the random variables or sets we have defined above depend on w , but we have chosen not to indicate the w dependence in order to simplify notation. We have

$$\begin{aligned} &\|(Y_m^w(\cdot, t_0 + \Delta t) - Y_m(\cdot, t_0 + \Delta t)) - (Y_m^w(\cdot, t_0) - Y_m(\cdot, t_0))\|_{L^\infty(\mathfrak{S}^N)} \\ &= \sup_{j \in \mathfrak{S}^N} \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j} \mathbf{1}_{X(i, w^{-N/2}t^-) \neq X(i, w^{-N/2}t) = m} - w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j} \mathbf{1}_{m = X(i, w^{-N/2}t^-) \neq X(i, w^{-N/2}t)} \right. \\ &\quad \left. - (1 - M^{-1}) \int_{t_0}^{t_0 + \Delta t} \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y(x,t)) = m} dx dt + M^{-1} \int_{t_0}^{t_0 + \Delta t} \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y(x,t)) \neq m} dx dt \right|. \end{aligned}$$

To conclude the proof of the lemma it is sufficient to show that with probability $> v(w)$, the right side is $\preceq \Delta t(\Delta t + \delta)$. By the triangle inequality, if $X_0(j), X_2(j), \dots, X_n(j)$ is any sequence of real numbers for $j \in \mathfrak{S}^N$ and $A_k = \sup_{j \in \mathfrak{S}^N} |X_{k-1}(j) - X_k(j)|$, then

$$\sup_{j \in \mathfrak{S}^N} |X_n(j) - X_0(j)| \leq \sum_{k=1}^n A_k.$$

We apply this principle with certain expressions in place of the $X_k(i)$, where each expression in some sense approximates the amount the bias function evaluated at i has changed during $(t_0, t_0 + \Delta t]$. This gives

$$\|(Y_m^w(\cdot, t_0 + \Delta t) - Y_m(\cdot, t_0 + \Delta t)) - (Y_m^w(\cdot, t_0) - Y_m(\cdot, t_0))\|_{L^\infty(\mathfrak{S}^N)} \leq \sum_{k=1}^7 (A_k + A'_k), \quad (38)$$

where

$$\begin{aligned}
 A_1 &= \sup_{j \in \mathfrak{S}^N} \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j} \mathbf{1}_{X(i,w^{-N/2}t^-) \neq X(i,w^{-N/2}t) = m} - w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j} \mathbf{1}_{p(Y^w(iw^{-1},t)) = m} U_{i,t} \right|, \\
 A_2 &= \sup_{j \in \mathfrak{S}^N} \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j} \mathbf{1}_{p(Y^w(iw^{-1},t)) = m} U_{i,t} - w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} U_{i,t} \right|, \\
 A_3 &= \sup_{j \in \mathfrak{S}^N} \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} U_{i,t} - w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_{0,j}} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} U_{i,t_0} \right| \\
 &= \sup_{j \in \mathfrak{S}^N} \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_j \setminus \mathcal{R}'_{0,j}} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} U_{i,t} \right|, \\
 A_4 &= \sup_{j \in \mathfrak{S}^N} \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_{0,j}} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} U_{i,t_0} - w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_{0,j}} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} V_i \right| \\
 &= \sup_{j \in \mathfrak{S}^N} \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_{0,j}} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} (U_{i,t_0} - V_i) \right|, \\
 A_5 &= \sup_{j \in \mathfrak{S}^N} A_{5,j}, \quad A_{5,j} = \left| w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_{0,j}} \mathbf{1}_{p(Y^w(iw^{-1},t_0)) = m} V_i - \Delta t (1 - M^{-1}) \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y^w(x,t_0)) = m} dx \right|, \\
 A_6 &= (1 - M^{-1}) \Delta t \sup_{j \in \mathfrak{S}^N} \left| \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y^w(x,t_0)) = m} dx - \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y(x,t_0)) = m} dx \right| \\
 &= (1 - M^{-1}) \Delta t \sup_{j \in \mathfrak{S}^N} \left| \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y^w(x,t_0)) = m} - \mathbf{1}_{p(Y(x,t_0)) = m} dx \right|, \\
 A_7 &= (1 - M^{-1}) \sup_{j \in \mathfrak{S}^N} \left| \Delta t \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y(x,t_0)) = m} dx - \int_{t_0}^{t_0 + \Delta t} \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y(x,t)) = m} dx dt \right|,
 \end{aligned}$$

where we view iw^{-1} as an element of \mathcal{S}^N by identifying \mathcal{S} (resp. \mathfrak{S}) with $[0, R)$ (resp. $\{0, \dots, Rw - 1\}$). Define A'_v exactly as A_v , except that $= m$ is replaced by $\neq m$, $1 - M^{-1}$ is replaced by M^{-1} , $U_{i,t}$ is replaced by $1 - U_{i,t}$, and V_i is replaced by $1 - V_i$. We have chosen to split the terms as above since the accumulation of error during $(t_0, t_0 + \Delta t]$ (as defined on the left side of (38)) is caused by several factors, while each term A_k is of the form $A_k = \sup_{i \in \mathfrak{S}^N} |X_{k-1}(j) - X_k(j)|$ such that each difference $X_{k-1}(j) - X_k(j)$ is more straightforward to bound.

In our bound of A_5 it will become clear why the natural scaling of time in (4) is $w^{-N/2}$. Recall that we can write A_5 on the form $A_5 = \sup_{j \in \mathfrak{S}^N} |X_4(j) - X_5(j)|$. In this notation, the term $X_5(j)$ is of order $O(\Delta t)$. The sum appearing in the definition of $X_4(j)$ is of order $w^{N/2} \Delta t$, since during an unscaled time interval of duration $w^{-N/2} \Delta t$ (equivalently, a rescaled time interval of duration Δt) the fraction of nodes whose Poisson clock rings is approximately $w^{-N/2} \Delta t$, and the sum is over $\Theta(w^N)$ nodes. Therefore the order of magnitude of $X_4(j)$ is $O(w^{-N/2} \cdot w^{N/2} \Delta t) = O(\Delta t)$, i.e., of the same order as $X_5(j)$ as desired.

For each $j \in \{1, \dots, 7\}$ we will show that with probability $> v(w)$ we have $A_j \preceq \Delta t(\Delta t + \delta)$. The term A'_j can be bounded exactly as A_j for all j , and the proof of its bound will therefore be omitted.

First we show that $A_1 \preceq \Delta t(\delta + \Delta t)$ with probability $> v(w)$. Recall the update rules described in (i)-(iii) in Section 1.3. Notice that the two sums in the definition of A_1 are identical, except possibly for the contribution from terms where $p(Y^w(iw^{-1}, t)) = 0$, i.e., from situations where there is not a unique

opinion which is the most common opinion in the neighborhood of j . We will bound A_1 by using that there are very few nodes for which $p(Y^w(iw^{-1}, t)) = 0$. We may assume that $|\mathcal{R}'| < 2R^N \Delta t w^{N/2}$, since $|\mathcal{R}'|$ is a Poisson random variable with parameter $R^N \Delta t w^{N/2}$, which gives $\mathbb{P}[|\mathcal{R}'| < 2R^N \Delta t w^{N/2}] \geq v(w)$. This implies $\|Y^w(\cdot, t) - Y(\cdot, t)\|_{L^\infty(\mathcal{S}^N)} \leq \delta + c\Delta t$ for all $t \in (t_0, t_0 + \Delta t]$ and some appropriate c as in the statement of the lemma. Under these assumptions,

$$A_1 \leq w^{-N/2} \sum_{(i,t) \in \mathcal{R}'} \mathbb{1}_{p(Y^w(iw^{-1}, t))=0} \leq w^{-N/2} \sum_{(i,t) \in \mathcal{R}'} \sum_{1 \leq m, m' \leq M, m \neq m'} \mathbb{1}_{|Y_{m'}(iw^{-1}, t) - Y_m(iw^{-1}, t)| \leq 2\delta + 2c\Delta t}.$$

By independence of \mathcal{R}' and \mathcal{F}_{t_0} , conditioned on \mathcal{F}_{t_0} the right side is stochastically dominated by the sum of i.i.d. $\{0, 1\}$ -valued random variables; this can be seen by ordering the terms by the value of t and use the memoryless property of the Poisson clocks. By Lemma 3.6 the probability that each of these i.i.d. random variables equals 1 is $\leq \delta + \Delta t$. The bound for A_1 now follows by the assumed upper bound for $|\mathcal{R}'|$ and a Chernoff bound.

Next we bound A_2 . The first sum defining A_2 represents the actual update rule of the dynamics (except for the case of $p(Y^w(iw^{-1}, t)) = 0$, which is handled above), while the second sum defining A_2 represents an update rule where the nodes use their bias at time t_0 rather than their bias at time t to determine whether they update their opinion. We bound A_2 by using that (due to (25) and the assumption $\|Y^w(\cdot, t_0) - Y(\cdot, t_0)\|_{L^\infty(\mathcal{S}^N)} < \delta$) most nodes have a rather significant bias towards one opinion at time t_0 , and this is unlikely to change throughout the considered time interval since the number of rings $|\mathcal{R}'|$ in the neighborhood of j is not too large. As above we assume $|\mathcal{R}'| < 2R^N \Delta t w^{N/2}$. Define $\tilde{A} \subset \mathcal{S}^N$ by

$$\tilde{A} = \{i \in \mathcal{S}^N : \exists m' \in [M] \setminus \{m\} \text{ such that } |Y_{m'}^w(w^{-1}i, t_0) - Y_m^w(w^{-1}i, t_0)| \leq 2w^{-N/2}|\mathcal{R}'|\}.$$

We claim that $|\tilde{A}| \leq (\Delta t + \delta)w^N$ for all sufficiently large w . If $i \in \tilde{A}$ the assumption $\|Y(\cdot, t_0) - Y^w(\cdot, t_0)\|_{L^\infty(\mathcal{S}^N)} < \delta$ implies that there exists $m' \in [M] \setminus \{m\}$, such that

$$|Y_m(iw^{-1}, t_0) - Y_{m'}(iw^{-1}, t_0)| \leq 2w^{-N/2}|\mathcal{R}'| + 2\delta < 4R^N \Delta t + 2\delta. \quad (39)$$

It follows by Lemma 3.6 that the set of nodes i satisfying (39) is $\leq w^N(\Delta t + \delta)$, and our claim follows.

If $i \in \mathcal{S}^N$ is such that there is a $t \in (t_0, t_0 + \Delta t]$ for which $p(Y^w(iw^{-1}, t)) \neq p(Y^w(iw^{-1}, t_0))$, then we must have $i \in \tilde{A}$ by the definition of Y^w . Therefore

$$A_2 \leq w^{-N/2} \sum_{(i,t) \in \mathcal{R}'} \mathbf{1}_{i \in \tilde{A}}.$$

We conclude the bound for A_2 by using independence of \mathcal{R}' and \tilde{A} and proceeding exactly as in the proof of A_1 .

Next we claim that with probability $> v(w)$ we have $A_3 < (\Delta t)^2$. Note that the two sums defining A_3 are identical, except that in the second sum we only consider terms coming from the *first* time a node rings during $(t_0, t_0 + \Delta t]$. We bound A_3 by using that there are few clocks which ring two or more times. For each $i \in \mathcal{S}^N$ let P_i be the Poisson random variable with parameter $\Delta t w^{-N/2}$ which denotes the number of rings of Poisson clock i during the interval $(t_0, t_0 + \Delta t]$. Then

$$A_3 \leq w^{-N/2} \sum_{i \in \mathcal{S}^N} \max\{P_i - 1, 0\}.$$

Since $\mathbb{E}[\max\{P_i - 1, 0\}] \leq (\Delta t)^2 w^{-N}$, we have $\mathbb{E}[A_3] \leq (\Delta t)^2 w^{-N/2}$. By Chebyshev's inequality $\mathbb{P}[A_3 \geq (\Delta t)^2] \leq w^{-N/2}$, which implies our claim.

Next we will bound A_4 . In the neighborhood of each node i the fraction of nodes which has each opinion is approximately $1/M$. Furthermore, as we will argue below, the same property holds if we only consider nodes for which the bias is m . Therefore, each time the clock of some node with bias m is ringing, the probability that this node has an opinion different from m (which will cause the node to change opinion)

is approximately $1 - M^{-1}$. The term A_4 quantifies the error we make by assuming this probability is exactly $1 - M^{-1}$, independently for each clock ring. We will bound A_4 by approximating the region where $p(Y^w(iw^{-1}, t_0)) = m$ by small cubes of side length L^{-1} , and by proving that when a node i is sampled uniformly from one of these cubes and $L \ll w$, then U_{i,t_0} and V_i have approximately the same distribution. As in our proof for the bound of A_3 we can assume $|\mathcal{R}'| < 2R^N \Delta t w^{N/2}$. Let $L = \lceil w^{1/2} \rceil$ and divide \mathcal{S}^N into $(RL)^N$ disjoint cubes of side length L^{-1} . For any $i \in \mathfrak{S}^N$ let I_i^L denote the cube containing iw^{-1} . Define $A \subset \mathcal{S}$ and $\tilde{A} \subset \mathfrak{S}^N$ by

$$A = \{x \in \mathcal{S}^N : p(Y(x, t_0)) = m\}, \quad \tilde{A} = \{i \in \mathfrak{S}^N : I_i^L \subset A\}.$$

We have

$$A_4 \leq w^{-N/2} \sum_{(i,t) \in \mathcal{R}'} |\mathbf{1}_{p(Y^w(iw^{-1}, t_0))=m} - \mathbf{1}_{i \in \tilde{A}}| + \sup_{j \in \mathfrak{S}^N} w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_{0,j}, i \in \tilde{A}} (U_{i,t_0} - V_i). \quad (40)$$

We will prove that $A_4 < \delta \Delta t$ with probability $> v(w)$. Any $i \in \mathfrak{S}^N$ for which $\mathbf{1}_{p(Y^w(iw^{-1}, t_0))=m} \neq \mathbf{1}_{i \in \tilde{A}}$ must satisfy one of the following conditions: (i) $p(Y^w(iw^{-1}, t_0)) \neq p(Y(iw^{-1}, t_0))$, or (ii) $p(Y(iw^{-1}, t_0)) = m$ and $i \notin \tilde{A}$.

We will prove that the number of nodes satisfying one of the conditions (i)-(ii) is $\leq w^N \delta$ with probability $> v(w)$. If $i \in \mathfrak{S}^N$ satisfies (i) and $E(t_0) < \delta$, there is an $m' \in [M]$, $m' \neq m$, such that $|Y_m(iw^{-1}, t_0) - Y_{m'}(iw^{-1}, t_0)| < 2\delta$, and the wanted result follows by Lemma 3.6. If i satisfies (ii), there is a $m' \neq m$ such that the function $Y_m(\cdot, t_0) - Y_{m'}(\cdot, t_0)$ intersects zero in I_i^L . By our estimates for the event G_k^1 in Proposition 2.10, it holds with probability $> v(w)$ that the number of such cubes is $< L^{N-1/2+1/100}$ for all $t \in [0, T]$. Using $L = \lceil w^{1/2} \rceil$, it follows that for $w > \delta^{-4}$ the number of nodes $i \in \tilde{\mathfrak{S}}$ satisfying (ii) is $\leq L^{N-1/2+1/100} (w/L)^N \leq w^N \delta$ with probability $> v(w)$. This completes the proof that the number of nodes satisfying one of the conditions (i)-(ii) is $\leq w^N \delta$ with probability $> v(w)$.

Given any $i \in \mathfrak{S}^N$ the events $\{\exists t \in (t_0, t_0 + \Delta t] \text{ such that } (i, t) \in \mathcal{R}'\}$ and $\mathbf{1}_{p(Y^w(i, t_0))=m} \neq \mathbf{1}_{i \in \tilde{A}}$ are independent. Proceeding as when bounding A_2 and A_3 , we see that the first term on the right side of (40) is $\leq \delta \Delta t$ with probability $> v(w)$.

Next we will prove that the second term on the right side of (40) converges to 0 in probability as $w \rightarrow \infty$. On the event $\hat{E}(t_0)$, for each $m \in [M]$ the number of nodes in I_j^L with type m at time t_0 is $M^{-1} \cdot \#I_j^L \cdot (1 + O(w^{-0.1}))$. Therefore the difference in probability between the events $\{U_{i,t_0} = 1\}$ and $\{V_i = 1\}$ is $\leq w^{-0.1}$ when we sample i uniformly from one of the cubes I_j^L . Since \mathcal{R}' is independent of U_{i,t_0} and V_i for all $i \in \mathfrak{N}(j)$, the second term on the right side of (40) is stochastically dominated by $w^{-N/2}$ times the sum of $< |\mathcal{R}'| \leq \Delta t w^{N/2}$ i.i.d. random variables taking values in $\{-1, 0, 1\}$ and with expectation $\leq w^{-0.1}$. Our claim follows by a Chernoff bound and a union bound.

Next we claim that $A_5 < \delta \Delta t$ with probability $> v(w)$. The integral in the definition of A_5 is the exact continuum analogue of the sum: For each node i having bias m at time t_0 , the probability that $V_i = 1$ and that the clock of i rings during a time interval of rescaled length Δt (i.e., unscaled time $w^{-N/2} \Delta t$) is approximately equal to $(1 - M^{-1})w^{-N/2} \Delta t$. Therefore, for each node with bias m we get an expected contribution to the first term defining A_5 which is equal to approximately $(1 - M^{-1})w^{-N} \Delta t = \Delta t (1 - M^{-1}) \int_{x: \|x - w^{-1}i\|_\infty < w^{-1/2}} dx$. Summing over all nodes for which the bias is m at time t_0 , we get a heuristic justification for why A_5 is small.

We will now make the heuristic argument in the above paragraph more precise. If $|\mathcal{R}'_{0,j}| \geq \Delta t 2^N w^{N/2}$ let $\tilde{\mathcal{R}}'_{0,j}$ denote the first $\lceil \Delta t 2^N w^{N/2} \rceil$ rings of the Poisson clocks during the interval $(t_0, t_0 + \Delta t]$, and if $|\mathcal{R}'_{0,j}| < \Delta t 2^N w^{N/2}$ let $\tilde{\mathcal{R}}'_{0,j}$ denote the union of $\mathcal{R}'_{0,j}$ and $(\lceil \Delta t 2^N w^{N/2} \rceil - |\mathcal{R}'_{0,j}|)$ pairs $(i, t_0 + \Delta t)$, where the i 's are pairwise different and sampled independently and uniformly from $\mathfrak{N}(j)$. By the triangle inequality

and letting Δ denote symmetric difference,

$$\begin{aligned}
 A_{5,j} &\leq \left(w^{-N/2} \sum_{(i,t) \in \mathcal{R}'_{0,j} \Delta \widetilde{\mathcal{R}}'_{0,j}} V_i \right) \\
 &+ \left| w^{-N/2} \sum_{(i,t) \in \widetilde{\mathcal{R}}'_{0,j}} \mathbf{1}_{p(Y^w(iw^{-1},t_0))=m} V_i - \Delta t (1 - M^{-1}) \int_{x \in \mathcal{N}(jw^{-1})} \mathbf{1}_{p(Y^w(iw^{-1},t_0))=m} dx \right|.
 \end{aligned} \tag{41}$$

We will prove that $\mathbb{P}[A_{5,j} > w^{-1/100}]$ decays faster than any power of w when $w \rightarrow \infty$, which is sufficient to complete the proof of our bound for A_5 . We see immediately that the first term on the right side of (41) decays sufficiently fast. By independence of $\mathcal{R}'_{0,j}$ and \mathcal{F}_{t_0} , the second sum on the right side of (41) is, conditioned on \mathcal{F}_{t_0} , equal in law to $w^{-N/2}$ times the sum of $\Delta t 2^N w^{N/2}$ independent bounded centered random variables. We obtain the desired bound by a Chernoff bound.

Now we will prove that $A_6 \leq \delta \Delta t$ with probability $> v(w)$. The difference between the considered integrals is whether we let Y^w or Y determine the bias of the nodes. We bound this term by using that Y^w and Y are close, and that at most points the nodes have a rather clear bias towards one opinion, so the slight difference between Y^w and Y does not change the bias at most points (i.e., $p(Y^w(x, t_0)) = p(Y(x, t_0))$ for most x). By first using $\|Y_m^w(\cdot, t_0) - Y_m(\cdot, t_0)\|_{L^\infty(\mathcal{S}^N)} < \delta$ and (24), and then using $Y_m - B_m \in \mathcal{L}_M^t(\mathcal{S}^N)$ and (25) for all $t \in (t_0, t_0 + \Delta t]$, we get

$$\begin{aligned}
 A_6 &\leq \Delta t \sum_{1 \leq m, m' \leq M, m \neq m'} \sup_{x \in \mathcal{S}^N} \int_{|x-x'| \leq 1} \mathbf{1}_{|Y_m(t_0, x') - Y_{m'}(t_0, x')| \leq 2\delta} dx' \\
 &\leq \Delta t \sum_{1 \leq m, m' \leq M, m \neq m'} \sup_{f \in \mathcal{L}^{2t_0}(\mathcal{S}^N)} \int_{\mathcal{S}^N} \mathbf{1}_{|(B_m(x) - B_{m'}(x)) - f(x)| < 2\delta} dx \\
 &\leq \delta \Delta t.
 \end{aligned}$$

Finally we will bound A_7 . The first integral corresponds to a model where the updates are determined by the biases at time t_0 (rather than time $t \in (t_0, t_0 + \Delta t]$), while the second integral represents the true dynamics. The difference is small for similar reasons as A_2 is small: At most points $x \in \mathcal{N}(jw^{-1})$ one element of the vector $Y(x, t_0)$ is significantly larger than the other elements, so $t \mapsto p(Y(x, t))$ will be constant for $t \in (t_0, t_0 + \Delta t]$ since the time derivative of Y is bounded. More precisely, we can use Lemma 2.12 and $\|Y_m(\cdot, t) - Y_m(\cdot, t_0)\|_{L^\infty(\mathcal{S}^N)} \leq 2\Delta t$ for all $t \in (t_0, t_0 + \Delta t]$, to argue that $A_7 \leq (\Delta t)^2$.

Combining the above estimates for A_j , $j = 1, \dots, 7$, we obtain the lemma by a union bound. \square

Proof of Lemma 3.4. For any $x \in \mathcal{S}^N$ there are $\alpha_i(x) \in [0, 1]$ and $\mathfrak{r}_i(x) \in \mathfrak{S}^N$ for $i = 1, \dots, 2^N$ such that $\|\mathfrak{r}_i(x) - x\|_\infty \leq w^{-1}$ and $\sum_{i=1}^{2^N} \alpha_i(x) = 1$, and such that for any $t \geq 0$, $Y^w(x, t) = \sum_{i=1}^{2^N} \alpha_i(x) Y^w(\mathfrak{r}_i(x), t)$. For any $x \in \mathcal{S}^N$ define $\Delta Y^w(x) := Y^w(x, t_0 + \Delta t) - Y^w(x, t_0)$ and $\Delta Y(x) := Y(x, t_0 + \Delta t) - Y(x, t_0)$. Observe that

$$\Delta Y^w(x) - \Delta Y(x) = \sum_{i=1}^{2^N} \alpha_i(x) (\Delta Y^w(\mathfrak{r}_i(x)) - \Delta Y(\mathfrak{r}_i(x))) + \sum_{i=1}^{2^N} \alpha_i(x) (\Delta Y(\mathfrak{r}_i(x)) - \Delta Y(x)). \tag{42}$$

By uniform continuity of Y , which follows from uniform continuity of B ,

$$\sup_{x \in \mathcal{S}^N} \sum_i \alpha_i(x) (\Delta Y(\mathfrak{r}_i(x)) - \Delta Y(x)) \rightarrow 0 \quad \text{as } w \rightarrow \infty, \tag{43}$$

and the rate of convergence depends only on T and the σ -algebra generated by $(B(x))_{x \in \mathcal{S}^N}$. By Lemma 3.5,

$$\mathbb{1}_{E(t_0) < \delta; \widehat{E}(t_0)} \mathbb{P} \left[\sup_{x \in \mathfrak{S}^N} \|\Delta Y^w(x) - \Delta Y(x)\| \leq c \Delta t (\Delta t + \delta) \mid \mathcal{F}_{t_0} \right] \geq \mathbb{1}_{E(t_0) < \delta; \widehat{E}(t_0)} v(w). \tag{44}$$

We obtain the desired bound for $\Delta Y^w(x) - \Delta Y(x)$ by combining (42), (43) and (44). \square

The following lemma will be needed to transfer the result of Proposition 3.1 from \mathcal{S} to \mathbb{R} . It says that a discrete version of Lemma 2.16 holds with high probability for large w .

Lemma 3.7. *Consider the Schelling model on \mathfrak{S} (resp. \mathbb{Z}) with (in the notation of Section 1.3) $N = 1$, $M \in \{2, 3, \dots\}$ and $\mathcal{N} = \mathcal{N}_\infty$. Let $Y^w \in \mathcal{C}_M(\mathcal{S} \times \mathbb{R}_+)$ (resp. $Y^w \in \mathcal{C}_M(\mathbb{R} \times \mathbb{R}_+)$) be given by (4). Fix an interval $I \subset \mathbb{R}$ of length > 1 and some $m \in [M]$. For any $\epsilon > 0$ and $t \geq 0$ define the event E_t^ϵ by*

$$E_t^\epsilon := \left\{ Y_m^w(x, t) > \sup_{m' \in [M] \setminus \{m\}} Y_{m'}^w(x, t) + \epsilon, \forall x \in I \right\}.$$

Then for any $\epsilon > 0$, $\lim_{w \rightarrow \infty} \mathbb{P}[E_0^\epsilon \cap (\bigcup_{t \in [0, \epsilon^{-1}]} (E_t^0)^c)] = 0$.

Proof. Define $\tilde{Y}_m^w(x, t) := Y_m^w(x, t) - \sup_{m' \in [M] \setminus \{m\}} Y_{m'}^w(x, t)$. For any interval $I \subset \mathbb{R}$ and $w \in \mathbb{N}$, let $wI = \{wx : x \in I\}$. Define stopping times T and T_i for $i \in wI$ by

$$T_i = \inf\{t \geq 0 : \tilde{Y}_m^w(w^{-1}i, t) \leq 0\}, \quad T = \inf_{i \in (wI)} T_i.$$

Since $\bigcup_{t \in [0, \epsilon^{-1}]} (E_t^0)^c \subset \{T \leq \epsilon^{-1}\}$ it is sufficient by a union bound to prove that for each fixed $i \in wI$

$$\log \mathbb{P}[E_0^\epsilon; T_i = T \leq \epsilon^{-1}] \preceq -w,$$

where the implicit constant can depend on all parameters except w and i . Letting $t_n := \epsilon^{-1}w^{-0.1}n$ for $n \in \{0, 1, \dots, \lceil w^{0.1} \rceil\}$, we observe that

$$\begin{aligned} \{E_0^\epsilon; T_i = T \in [t_n, t_{n+1}]\} &\subset \left\{ E_0^\epsilon; t_n \leq T_i = T; \tilde{Y}_m^w(w^{-1}i, t_n) < \frac{1}{2} \tilde{Y}_m^w(w^{-1}i, 0) \right\} \\ &\cup \left\{ E_0^\epsilon; |\{(j, w^{-1/2}t) \in \mathcal{R} : j \in \mathfrak{N}(i), t \in [t_n, t_{n+1}]\}| w^{-1/2} > \frac{1}{100} \tilde{Y}_m^w(w^{-1}i, 0) \right\}. \end{aligned}$$

It follows by a union bound that

$$\begin{aligned} \mathbb{P}[E_0^\epsilon; T_i = T \leq \epsilon^{-1}] &\leq \sum_{n=0}^{\lceil w^{0.1} \rceil - 1} \mathbb{P} \left[E_0^\epsilon; t_n \leq T_i = T; \tilde{Y}_m^w(w^{-1}i, t_n) < \frac{1}{2} \tilde{Y}_m^w(w^{-1}i, 0) \right] \\ &+ \sum_{n=0}^{\lceil w^{0.1} \rceil - 1} \mathbb{P} \left[E_0^\epsilon; |\{(j, w^{-1/2}t) \in \mathcal{R} : j \in \mathfrak{N}(i), t \in [t_n, t_{n+1}]\}| w^{-1/2} > \frac{1}{100} \tilde{Y}_m^w(w^{-1}i, 0) \right]. \end{aligned}$$

Since $\tilde{Y}_m^w(w^{-1}i, 0) > \epsilon$ on the event E_0^ϵ , the logarithm of the last sum is $\preceq -w$, so to conclude the proof of the lemma it is sufficient to show that for each fixed $n \in \{0, 1, \dots, \lceil w^{0.1} \rceil - 1\}$,

$$\log \mathbb{P} \left[E_0^\epsilon; t_n \leq T_i = T; \tilde{Y}_m^w(w^{-1}i, t_n) < \frac{1}{2} \tilde{Y}_m^w(w^{-1}i, 0) \right] \preceq -w, \quad (45)$$

where the implicit constant can depend on all parameters except w , i , and n . Fix $n \in \{0, 1, \dots, \lceil w^{0.1} \rceil - 1\}$, and define

$$\begin{aligned} \mathfrak{N}^+ &:= \{j \in \mathfrak{N}(i) : X(j, 0) \neq m, jw \in I\}, & \mathcal{R}^+ &:= \{j \in \mathfrak{N}^+ : \exists t \in [0, t_n] \text{ such that } (j, tw^{-1/2}) \in \mathcal{R}\}, \\ \mathfrak{N}^- &:= \{j \in \mathfrak{N}(i) : X(j, 0) = m, jw \notin I\}, & \mathcal{R}^- &:= \{j \in \mathfrak{N}^- : \exists t \in [0, t_n] \text{ such that } (j, tw^{-1/2}) \in \mathcal{R}\}. \end{aligned}$$

By large deviation estimates for Bernoulli random variables,

$$\log \mathbb{P}[\widehat{E}^c] \preceq -w, \quad \widehat{E} := \left\{ \left| |\mathcal{R}^\pm| - (1 - e^{-w^{-1/2}t_n}) |\mathfrak{N}^\pm| \right| < w^{0.1} \right\}. \quad (46)$$

Furthermore, observe that

$$\begin{aligned} |\mathfrak{N}^+| - |\mathfrak{N}^-| &= |\{j \in \mathfrak{N}(i) : (jw) \in I\}| - |\{j \in \mathfrak{N}(i) : X(j, 0) = m\}| \\ &\geq (w + 1) - \left(\frac{2w + 1}{M} + w^{1/2} Y_m^w(w^{-1}i, 0) \right) \\ &> -w^{1/2} Y_m^w(w^{-1}i, 0), \end{aligned} \tag{47}$$

where the inequality on the second line follows by the definition of Y^w . By (47), the definition of \widehat{E} , $|1 - e^{-w^{-1/2}t_n}| \leq 2\epsilon^{-1}w^{-1/2}$, and

$$\mathbb{1}_{t_n \leq T} \widetilde{Y}_m^w(w^{-1}i, t_n) \geq \mathbb{1}_{t_n \leq T} \left(\widetilde{Y}_m^w(w^{-1}i, 0) + 2w^{-1/2}|\mathcal{R}^+| - 2w^{-1/2}|\mathcal{R}^-| \right),$$

it follows that on the event $\{E_0^\epsilon; t_n \leq T_i = T; \widehat{E}\}$,

$$\widetilde{Y}_m^w(w^{-1}i, t_n) \geq \widetilde{Y}_m^w(w^{-1}i, 0) - 2\epsilon^{-1}w^{-1/2}Y_m^w(w^{-1}i, 0) - 4w^{-0.4}.$$

Since $\log \mathbb{P}[2\epsilon^{-1}w^{-1/2}Y_m^w(w^{-1}i, 0) > \widetilde{Y}_m^w(w^{-1}i, 0)] \leq -w$, this result and (46) implies (45). \square

Proposition 3.1 now follows by iterating the estimate of Lemma 3.4.

Proof of Proposition 3.1. First consider the case $V = \mathcal{S}$ and $N \in \mathbb{N}$. By Skorokhod embedding we may couple the discrete and continuum Schelling model such that the convergence in law in Lemma 3.2 is almost sure. Let $T, \Delta t > 0$. Conditioned on B , let c and v be the (random) constant and function, respectively, of Lemma 3.4. Recall that c depends on B and T , while v depends on $B, T, \Delta t$, and the error $E(t_0)$ with E as in Lemma 3.4. By Lemma 3.4 with $t_0 = 0$ and with notation as in that lemma,

$$\mathbb{1}_{E(0) < \delta; \widehat{E}(T)} \mathbb{P} \left[\|Y^w(\cdot, \Delta t) - Y(\cdot, \Delta t)\|_{L^\infty(\mathcal{S}^N)} < c^2(\Delta t)^3 + 2c(\Delta t)^2 \mid \mathcal{F}_0 \right] > \mathbb{1}_{E(0) < \delta; \widehat{E}(T)} v(w).$$

Iterating the result of Lemma 3.4, we get further that for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{1}_{E(0) < \delta; \widehat{E}(T)} \mathbb{P} \left[\|Y^w(\cdot, \tilde{n}\Delta t) - Y(\cdot, \tilde{n}\Delta t)\|_{L^\infty(\mathcal{S}^N)} < \Delta t(1 + c\Delta t)^{\tilde{n}+1} - \Delta t, \quad \tilde{n} \in \{0, \dots, n\} \mid \mathcal{F}_0 \right] \\ > \mathbb{1}_{E(0) < \delta; \widehat{E}(T)} v(w)^n. \end{aligned}$$

We need $n_0 := \lceil T/\Delta t \rceil$ time steps to reach time T , so conditioned on \mathcal{F}_0 and on the event $\{E(0) < \delta\} \cap \widehat{E}(T)$, with probability at least $v(w)^{n_0}$ and for $\Delta t < 1/(100c)$,

$$\|Y^w(\cdot, \tilde{n}\Delta t) - Y(\cdot, \tilde{n}\Delta t)\|_{L^\infty(\mathcal{S}^N)} < \Delta t(1 + c\Delta t)^{n_0+1} - \Delta t < (2e^{cT} - 1)\Delta t, \quad \forall \tilde{n} \in \{0, \dots, n_0\}. \tag{48}$$

With probability converging to 1 as $w \rightarrow \infty$, for any interval $I = [\Delta t\tilde{n}, \Delta t(\tilde{n} + 1)]$ and node $i \in \mathcal{S}^N$, the total number of times during I at which the Poisson clock of a node in $\mathfrak{N}(i)$ rings, is $\leq 10^N w^{N/2} \Delta t$. Therefore, with probability converging to 1 as $w \rightarrow \infty$,

$$\sup_{i \in \mathcal{S}^N} \sup_{0 \leq n \leq n_0} \sup_{d \in [0, \Delta t]} \|Y^w(iw^{-1}, n\Delta t + d) - Y^w(iw^{-1}, n\Delta t)\|_{L^\infty(\mathcal{S}^N)} \leq 10^N \Delta t.$$

Combining this estimate with (48), for any given $\epsilon_0 > 0$ and for all w sufficiently large as compared to ϵ_0 ,

$$\mathbb{1}_{E(0) < \delta; \widehat{E}(T)} \mathbb{P} \left[\sup_{x \in \mathcal{S}^N} \sup_{t \in [0, T]} \|Y(x, t) - Y^w(x, t)\|_{L^\infty(\mathcal{S}^N)} \leq (2e^{cT} + 10^N)\Delta t \mid \mathcal{F}_0 \right] > \mathbb{1}_{E(0) < \delta; \widehat{E}(T)} (v(w)^{n_0} - \epsilon_0).$$

Since $\mathbb{P}[E(0) < \delta; \widehat{E}(T)] \rightarrow 1$ as $w \rightarrow \infty$, for all sufficiently large w ,

$$\mathbb{P} \left[\sup_{x \in \mathcal{S}^N} \sup_{t \in [0, T]} \|Y(x, t) - Y^w(x, t)\|_{L^\infty(\mathcal{S}^N)} \leq (2e^{cT} + 10^N)\Delta t \right] > \mathbb{E}[v(w)^{n_0} - 2\epsilon_0].$$

We first make $(2e^{cT} + 10^N)\Delta t$ arbitrarily small by decreasing Δt , and then we make $v(w)^{n_0} - 2\epsilon_0$ arbitrarily close to 1 by sending $\epsilon_0 \rightarrow 0$ and $w \rightarrow \infty$. It follows that $\sup_{t \in [0, T]} \|Y^w(\cdot, t) - Y(\cdot, t)\|_{L^\infty(\mathcal{S}^N)} \rightarrow 0$ in probability. By the Skorokhod representation theorem we can couple the model for different values of w , such that we obtain a.s. convergence. This concludes the proof in the case $V = \mathcal{S}$.

Now consider the case $V = \mathbb{R}$ and $N = 1$. Let $\epsilon > 0$. For $R > 2(\epsilon^{-1} + 2)$ define the event E_R by

$$E_R = \{\exists a^- \in [-R/2 + 2, -\epsilon^{-1}], a^+ \in [\epsilon^{-1}, R/2 - 2] : Y_1(x, 0) > \sup_{m' \in \{2, \dots, M\}} Y_{m'}(x, 0) + \epsilon, \\ \forall x \in [a^- - 2, a^-] \cup [a^+, a^+ + 2]\}.$$

Choose R sufficiently large such that $\mathbb{P}[E_R] > 1 - \epsilon/2$. Let Y (resp. \hat{Y}) denote the solution of (6), (8) on \mathbb{R} (resp. $\mathcal{S} = [-R/2, R/2]$), and let Y^w (resp. \hat{Y}^w) be given by (4) for the Schelling model on \mathbb{Z} (resp. \mathfrak{S}). We will argue that we can couple Y, \hat{Y}, Y^w , and \hat{Y}^w such that with probability at least $1 - \epsilon$, $Y^w \rightarrow Y$ uniformly on $[-\epsilon^{-1}, \epsilon^{-1}] \times [0, \epsilon^{-1}]$. This will be sufficient to complete the proof of the proposition since ϵ was arbitrary.

By the convergence result for the torus proved above, we can couple \hat{Y}^w and \hat{Y} such that $\hat{Y}^w|_{[-R/2, R/2] \times [0, \epsilon^{-1}]}$ converges uniformly to $\hat{Y}|_{[-R/2, R/2] \times [0, \epsilon^{-1}]}$. Furthermore, on E_R we can couple Y and \hat{Y} such that $Y|_{[a^-, a^+]} = \hat{Y}|_{[a^-, a^+]}$, since the law of the initial conditions are the same, and since Lemma 2.16 implies that $p(Y(x, t)) = p(\hat{Y}(x, t)) = 1$ for all $x \in [a^- - 1, a^-] \cup [a^+, a^+ + 1]$ and $t \geq 0$. To complete the proof of the proposition it is sufficient to prove that on E_R we can couple Y^w and \hat{Y}^w such that $Y^w|_{[a^-, a^+] \times [1, \epsilon^{-1}]} = \hat{Y}^w|_{[a^-, a^+] \times [1, \epsilon^{-1}]}$ with probability at least $1 - \epsilon/2$.

Consider a coupling of Y^w and \hat{Y}^w such that the initial opinion of the nodes corresponding to the interval $[a^- - 1, a^+ + 1]$ is identical for the models on \mathbb{Z} and \mathfrak{S} , and such that the set of rings of Poisson clocks corresponding to this interval, i.e. the set $\{(i, t) \in \mathcal{R} : (a^- - 1)w \leq i \leq (a^+ + 1)w\}$, is the same for the models on \mathbb{Z} and \mathfrak{S} . We also assume that draws as described in (iii) of Section 1.3 are resolved in the same way. By Lemma 3.7, $p(Y^w(x, t)) = p(\hat{Y}^w(x, t)) = 1$ for all $x \in [a^- - 2, a^-] \cup [a^+, a^+ + 2]$ and $t \geq 0$ with probability at least $1 - \epsilon/2$ for sufficiently large w . On the event that this happens $Y^w|_{[a^-, a^+] \times [0, \epsilon^{-1}]} = \hat{Y}^w|_{[a^-, a^+] \times [0, \epsilon^{-1}]}$ for all $t \in [0, \epsilon^{-1}]$, so we have obtained an appropriate coupling. \square

3.2 Limiting states for the one-dimensional discrete Schelling model

In this section we will first conclude the proof of Theorem 1.3. Then we will prove that the opinion of each node in the Schelling model in any dimension converges a.s., and we will present a result on stable configurations in the higher-dimensional Schelling model.

The main inputs to our proof of Theorem 1.3 are Propositions 2.14 and 3.1. We consider a coupling of the discrete and continuum Schelling model as in Proposition 3.1, and choose a sufficiently large $t \geq 0$ such that the limiting configuration of the continuum Schelling model described in Proposition 2.14 is almost obtained; more precisely, we choose t sufficiently large such that with high probability 0 is contained in an interval I' of length strictly larger than 1 on which $p \circ Y(\cdot, t)$ is constant. Let $m \in [M]$ denote the value of $p \circ Y(\cdot, t)$ in I' . Recall that by the scaling we used when defining Y^w in (4), a time t for Y corresponds to time $tw^{-1/2}$ for the discrete Schelling model.

To conclude the proof it will be sufficient to prove that $p \circ Y^w = m$ in some slightly smaller interval $I'' \subset I'$ (still of length > 1) until all nodes in I'' have changed opinion to m . We will first prove Lemma 3.8, which says, roughly speaking, that $p \circ Y = m$ in some interval I (satisfying $I'' \subset I \subset I'$) for a macroscopic time with high probability. Then we prove (Lemma 3.9) that conditioned on the event of Lemma 3.8, $p \circ Y = m$ in I'' for a time interval with length of order $w^{0.01}$; this duration is sufficient to guarantee that at the end of this time interval, with high probability all nodes in I'' have changed their opinion to m . The latter result is proved by considering $\lceil w^{0.02} \rceil$ consecutive time intervals each of length $w^{-0.01}$. In each of these length $w^{-0.01}$ intervals we allow the interval on which $p \circ Y = m$ to shrink slightly. For nodes i bounded away from the boundary of the interval, we can guarantee that $p \circ Y = m$ by using (among other properties) that the fraction of nodes in $\mathfrak{N}(i)$ which have a bias towards m is strictly larger than $1/2$ throughout the time interval we consider; therefore the bias of i towards m will have an upwards drift and never become negative. For

nodes i near the boundary of our interval, however, up to half of the nodes in $\mathfrak{N}(i)$ may have a bias towards another opinion than m , so we do not necessarily have an upward drift, and the node may eventually get a bias towards another opinion. For such nodes we can guarantee that the bias will not become negative too fast, by using that the node typically has a strong bias towards m at the beginning of the time interval we consider. We show that the interval on which $p \circ Y = m$ shrinks sufficiently slowly, such that all nodes on the subinterval I'' of length > 1 get opinion m before the interval gets a length smaller than 1.

Define

$$\begin{aligned} \bar{\mathcal{Y}}_m^w(i, t) &:= \left(\sum_{j \in \mathfrak{N}(i)} \mathbb{1}_{X(j,t)=m} \right) - \sup_{m' \in [M] \setminus \{m\}} \left(\sum_{j \in \mathfrak{N}(i)} \mathbb{1}_{X(j,t)=m'} \right) \\ &= w^{1/2} \left(Y_m^w(i/w, tw^{1/2}) - \sup_{m' \in [M] \setminus \{m\}} Y_{m'}^w(i/w, tw^{1/2}) \right). \end{aligned} \tag{49}$$

Observe that if $\bar{\mathcal{Y}}_m^w(i, t) > 0$ then m is the most common opinion in the neighborhood of node i at time t . In the statement and proof of the following lemma, $wI = \{wx : x \in I\}$ for any interval $I \subset \mathbb{R}$ and $w \in \mathbb{N}$. The lemma says, roughly speaking, that if the event of Proposition 3.1 (or its analog for $M > 2$) occurs for the continuum Schelling model, then it is very likely that after some small macroscopic time a typical point is contained in an interval I of length strictly greater than 1 on which the bias is constant.

Lemma 3.8. *Couple the discrete and continuum Schelling model on V as described in Proposition 3.1, where $V = \mathbb{R}$ or $V = \mathcal{S}$, and $N = 1$, $M \in \{2, 3, \dots\}$, and $\mathcal{N} = \mathcal{N}_\infty$. Let $\{A_1, \dots, A_M\}$ be as defined in Proposition 2.14, and let \hat{E} be the event that the set $\mathbb{R} \setminus \cup_{1 \leq m \leq M} A_m$ has measure zero, and that each set A_m can be written as the union of intervals of length > 1 . If \hat{E} occurs, choose $m \in [M]$ and an interval $I' \subset V$ in a $\sigma(B)$ -measurable way such that I' is a connected component of A_m . Let $c_1, c_2 \in (0, 1/10)$ and let $I \subset I'$ be the open interval with left (resp. right) end-point at distance c_2 from the left (resp. right) end-point of I' . Let $E = E_{c_1, c_2}^w$ be the event that \hat{E} occurs, that I has length larger than $1 + c_2$ and is contained in $[-c_2^{-1}, c_2^{-1}]$, and that $\bar{\mathcal{Y}}_m^w(i, t) > 0$ for all $i \in (wI) \cap \mathbb{Z}$ and $t \in [c_1^{-1/2} w^{-1/2}, c_1]$. Then $\lim_{c_2 \rightarrow 0} \lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P}[E \cup (\hat{E})^c] = 1$.*

Proof. First we give a brief outline of the proof. For small c_1 and large w it holds with high probability (by Proposition 3.1) that all nodes in wI have a large bias towards opinion m at time $c_1^{-1/2} w^{-1/2}$. In particular, $w^{-1/2} \bar{\mathcal{Y}}_m^w(i, c_1^{-1/2} w^{-1/2}) \gg 1$ for nodes i in wI . We consider the system until, roughly speaking, the first time $\hat{T} > c_1^{-1/2} w^{-1/2}$ at which $\bar{\mathcal{Y}}_m^w(i, \hat{T}) \leq 0$ for some node i in wI . More precisely, we will consider a slightly differently defined random variable $\hat{\mathcal{Y}}_{m, m'}^w(i, t)$ instead of $\bar{\mathcal{Y}}_m^w(i, t)$ in order to simplify the estimates (see below for the precise definition). Until this time all nodes in wI will have a bias towards m . We show that $\hat{T} > c_1$ with high probability by arguing that each individual node i in wI is unlikely to be the *first* node in wI for which $\hat{\mathcal{Y}}_{m, m'}^w(i, t) \leq 0$ for $t \in [c_1^{-1/2} w^{-1/2}, c_1]$. Let the nodes i_1^* and i_2^* represent the two end-points of $(wI) \cap \mathbb{Z}$. At time $t = c_1^{-1/2} w^{-1/2}$, with high probability at least half of the neighbors of i_1^* have a bias towards m . Using this and that the types are approximately uniformly distributed on $[M]$, one can show that $\hat{\mathcal{Y}}_{m, m'}^w(i_1^*, t)$ stochastically dominates a random walk with increments that have negative expectation with a magnitude of order $O(w^{-1/2})$. This implies that $\hat{\mathcal{Y}}_{m, m'}^w(i_1^*, t)$ will not reach zero before time $c_1 \ll 1$ with high probability. We argue similarly for $i = i_2^*$. If i is contained in wI and has distance $\Omega(w)$ from the boundary of wI , then $\hat{\mathcal{Y}}_{m, m'}^w(i, t)$ has an upward drift for $t \in [c_1^{-1/2} w^{-1/2}, \hat{T}]$, since the fraction of neighbors of i which have a bias towards m is uniformly above $1/2$; therefore $\hat{\mathcal{Y}}_{m, m'}^w(i, t)$ is very unlikely to get negative before time c_1 . If i is close to the boundaries of wI , but not equal to i_1^* or i_2^* , we conclude that $\hat{\mathcal{Y}}_{m, m'}^w(i, t)$ is unlikely to get negative by comparing with i_1^* or i_2^* .

Probabilistic estimates involving both I and the *initial data* of Y^w need to be handled with care since I depends on Y and therefore through the coupling also on the initial data of Y^w . Some places we deal with

this by taking a union bound (truncating on the event that $I \subset [-c_2^{-1}, c_2^{-1}]$), and other places we use that the considered random variables have a scaling limit as $w \rightarrow \infty$. On the other hand, the randomness of the Poisson clocks is independent of I .

Note that the lemma clearly holds if $\mathbb{P}[\widehat{E}] = 0$, so we may assume $\mathbb{P}[\widehat{E}] > 0$. We will work on the event \widehat{E} throughout the proof of the lemma. Let i_1^* (resp. i_2^*) be the smallest (resp. largest) element of $(wI) \cap \mathbb{Z}$, and let \mathcal{R} denote the set of rings as defined in (2).

Define the following random variables $A_1, A_2 \in \{0, 1, 2, \dots\}$, where $|\cdot|$ denotes the number of elements in a set

$$\begin{aligned} A_1 &:= |\{(j, t) \in \mathcal{R} : c_1^{-1/2}w^{-1/2} \leq t \leq c_1, j \in \{i_1^* - w, \dots, i_2^* + w\}\}|, \\ A_2 &:= |\{(j, t) \in \mathcal{R} : j \in \{i_1^* - w, \dots, i_2^* + w\}, 0 \leq t \leq c_1^{-1/2}w^{-1/2}, \\ &\quad \exists(j, t') \in \mathcal{R} \text{ such that } c_1^{-1/2}w^{-1/2} \leq t' \leq c_1\}|, \end{aligned} \tag{50}$$

In other words, A_1 is the number of times that some Poisson clock is ringing between the times $c_1^{-1/2}w^{-1/2}$ and c_1 , if we only consider the clock of nodes in $\{i_1^* - w, \dots, i_2^* + w\}$. Furthermore, A_2 is the number of Poisson clock rings by the same set of nodes which happen before time $c_1^{-1/2}w^{-1/2}$, such that the same clock rings again between the times $c_1^{-1/2}w^{-1/2}$ and c_1 .

In the initial data, for any node $i \in \mathbb{Z}$ it holds with high probability that the fraction of nodes in (a subinterval of) its neighborhood with opinion $m' \in [M]$ is approximately M^{-1} . We define a random variable $A_3 \in [0, 1]$ in order to quantify the extent to which this is true for the two nodes i_1^* and i_2^*

$$\begin{aligned} A_3 &= \inf \left\{ a \geq 0 : \left| \left| \{i \in \mathcal{N}(i_k^*) \cap (wI) : X(i, 0) = m'\} \right| w^{-1} - M^{-1} \right| \leq aw^{-1/2}, \quad k = 1, 2, m' \in [M] \right. \\ &\quad \left. \left| \left| \{i \in \mathcal{N}(i_k^*) \setminus (wI) : X(i, 0) = m'\} \right| w^{-1} - M^{-1} \right| \leq aw^{-1/2}, \quad k = 1, 2, m' \in [M] \right\}. \end{aligned} \tag{51}$$

Since Y is an approximation to Y^w , and since it follows from occurrence of \widehat{E} and (6) that $\lim_{t \rightarrow \infty} Y_m(x, t) = \infty$ and $\lim_{t \rightarrow \infty} Y_{m'}(x, t) = -\infty$ for $m' \neq m$ and $x \in I$, we expect that $Y_m^w(x, t) - \sum_{m' \neq m} Y_{m'}^w(x, t) \rightarrow \infty$ as $t \rightarrow \infty$ for all $x \in I$. We introduce a random variable A_4 in order to quantify this. Namely, we define $A_4 \in \mathbb{R}$ by

$$A_4 := \inf_{x \in I} \left[Y_m^w(x, c_1^{-1/2}) - \sum_{m' \neq m} Y_{m'}^w(x, c_1^{-1/2}) \right].$$

Recall that for any i and any fixed subset of $\mathcal{N}(i)$, the fraction of nodes with initial type equal to m is $1/M$ in expectation. We now define an event \overline{E} which, roughly speaking, says that for all $i \in (wI)$ and the sets $\mathcal{N}(i) \cap (wI)$ and $\mathcal{N}(i) \setminus wI$, the fraction of nodes with initial type equal to m does not deviate from its expectation by more than $w^{-1/4}$. More precisely, \overline{E} is defined as follows.

$$\begin{aligned} \overline{E} &:= \left\{ \forall i \in (wI) \cap \mathbb{Z}, \forall m' \in [M], \right. \\ &\quad \left| \frac{|\{j \in \mathcal{N}(i) \cap (wI) : X(j, 0) = m'\}|}{|\mathcal{N}(i) \cap (wI)|} - M^{-1} \right| \vee \left| \frac{|\{j \in \mathcal{N}(i) \setminus (wI) : X(j, 0) = m'\}|}{|\mathcal{N}(i) \setminus (wI)|} - M^{-1} \right| < w^{-1/4} \left. \right\}. \end{aligned}$$

We will now define an event \widetilde{E} which we will truncate on throughout part of the proof. Let $\widetilde{E} = \widetilde{E}_{c_1, c_2}^w$ be defined by

$$\begin{aligned} \widetilde{E} &= \{A_1 < 2c_1c_2^{-1}w\} \cap \{A_2 < 2c_1^{1/2}c_2^{-1}w^{1/2}\} \cap \{A_3 < c_1^{-1/10}\} \\ &\quad \cap \{A_4 > 2\} \cap \{1 + c_2 < \lambda(I); I \subset [-c_2^{-1}, c_2^{-1}]\} \cap \overline{E}. \end{aligned} \tag{52}$$

First we will argue that

$$\lim_{c_2 \rightarrow 0} \lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P}[\widetilde{E}] = 1. \tag{53}$$

The probability of the fifth event on the right side of (52) is independent of w and c_1 , and converges to 1 as $c_2 \rightarrow 0$. The probability of the first, second, and sixth event on the right side of (52) converge to 1 as $w \rightarrow \infty$ for any fixed $c_1, c_2 \in (0, 1/10)$ if we assume occurrence of the fifth event. (Here we use occurrence of the fifth event since it allows us bound the probability of \bar{E}^c by taking a union bound over $i \in w[-c_2^{-1}, c_2^{-1}]$.) The probability of the third event on the right side of (52) converges to a constant as $w \rightarrow \infty$, and it converges to 1 when first $w \rightarrow \infty$ and then $c_1 \rightarrow 0$. It is immediate by occurrence of \hat{E} and from (6) that $Y_m(\cdot, t) - \sum_{m' \neq m} Y_{m'}(\cdot, t) \rightarrow \infty$ uniformly on I as $t \rightarrow \infty$. Therefore it follows from Proposition 3.1 that the probability of the fourth event on the right side of (52) converges to a constant as $w \rightarrow \infty$, and it converges to 1 as first $w \rightarrow \infty$ and then $c_1 \rightarrow 0$. Combining the above estimates we get (53).

Fix $m' \in [M] \setminus \{m\}$. Consider a modified dynamic where, for every time $t > c_1^{-1/2}w^{-1/2}$ at which the clock of a node $j \notin wI$ (resp. $j \in wI$) rings, this node updates its type to m' (resp. m), even if it has a bias towards some other type. We refer to this dynamic as the m' -modified dynamic, as opposed to the *original* dynamic. Define $\hat{\mathcal{Y}}_{m,m'}^w(i, t)$ as follows, where we assume the opinions $X(j, t)$ have been updated according to the m' -modified dynamic

$$\hat{\mathcal{Y}}_{m,m'}^w(i, t) := \left(\sum_{j \in \mathfrak{N}(i)} \mathbb{1}_{X(j,t)=m} \right) - \left(\sum_{j \in \mathfrak{N}(i)} \mathbb{1}_{X(j,t)=m'} \right).$$

Note that for any i and for t such that $p(Y(j, t')) = m$ for all $j \in wI$ and $t' \in (c_1^{-1/2}w^{-1/2}, t]$,

$$\inf_{m'' \in [M] \setminus \{m\}} \hat{\mathcal{Y}}_{m,m''}^w(i, t) \leq \bar{\mathcal{Y}}_m^w(i, t). \quad (54)$$

(Here $\hat{\mathcal{Y}}_{m,m''}^w(i, t)$ is defined with the m'' -modified dynamic, while $\bar{\mathcal{Y}}_m^w(i, t)$ is defined with the original dynamic.) Let the event $E_{m'}$ be defined exactly as the event E in the statement of the lemma, except that we require $\hat{\mathcal{Y}}_{m,m'}^w(i, t) > 0$ instead of $\bar{\mathcal{Y}}_m^w(i, t) > 0$. Since m' is arbitrary and by (54), in order to conclude the proof of the lemma it is sufficient to show that

$$\lim_{c_2 \rightarrow 0} \lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P}[E_{m'} \cup \hat{E}^c] = 1. \quad (55)$$

For any $j \in \mathbb{N}$ let t_j^1 be the j th smallest element of $\{t \geq c_1^{-1/2}w^{-1/2} : \exists x \in \mathcal{N}(i_1^*) \text{ such that } (x, t) \in \mathcal{R}\}$. In other words, t_j^1 is the j th time after time $c_1^{-1/2}w^{-1/2}$ at which the clock of some node in $\mathcal{N}(i_1^*)$ rings. Then define $i_j^1 \in \mathcal{N}(i_1^*)$ such that $(i_j^1, t_j^1) \in \mathcal{R}$, and define $R_j^1 := \hat{\mathcal{Y}}_{m,m'}^w(i_1^*, t_j^1) - \hat{\mathcal{Y}}_{m,m'}^w(i_1^*, (t_j^1)^-)$, where $\hat{\mathcal{Y}}_{m,m'}^w(i_1^*, (t_j^1)^-) := \lim_{t \uparrow t_j^1} \hat{\mathcal{Y}}_{m,m'}^w(i_1^*, t)$. Define (i_j^2, t_j^2) (resp. R_j^2) exactly as (i_j^1, t_j^1) (resp. R_j^1), but with i_2^* in place of i_1^* . Now define the following stopping times T_j for $j = 1, 2, 3$

$$\begin{aligned} T_1 &:= \inf \left\{ t \geq c_1^{-1/2}w^{-1/2} : \sum_{j: t_j^1 \leq t} R_j^1 \leq -w^{1/2} \right\}, \\ T_2 &:= \inf \left\{ t \geq c_1^{-1/2}w^{-1/2} : \sum_{j: t_j^2 \leq t} R_j^2 \leq -w^{1/2} \right\}, \\ T_3 &:= \inf \left\{ t \geq c_1^{-1/2}w^{-1/2} : \exists i \in wI \text{ such that } i \in \{i_1^* + c_2w + 1, i_1^* + c_2w + 2, \dots, i_2^* - c_2w - 1\} \right. \\ &\quad \left. \text{and } \hat{\mathcal{Y}}_{m,m'}^w(i, t) \leq 0 \right\}. \end{aligned} \quad (56)$$

Our strategy for establishing (55) is to show that on the high probability event \tilde{E} , for all nodes $j \in (wI) \cap \mathbb{Z}$ we have $\hat{\mathcal{Y}}_{m,m'}^w(j, t) > 0$ until time $T_1 \wedge T_2 \wedge T_3$, and at the same time each of the times T_1, T_2, T_3 is very unlikely to be the *smallest* time in $\{T_1, T_2, T_3, c_1\}$. Combining these results give that with high probability, for all nodes $j \in (wI) \cap \mathbb{Z}$ we have $\hat{\mathcal{Y}}_{m,m'}^w(j, t) > 0$ until time c_1 .

First we will argue that

$$\widehat{\mathcal{Y}}_{m,m'}^w(i, t) > 0 \quad \forall i \in wI \quad \text{if} \quad t < T_1 \wedge T_2 \wedge T_3 \text{ and } \widetilde{E} \text{ occurs.} \quad (57)$$

To prove this it is sufficient to show that if \widetilde{E} occurs and $t < T_1 \wedge T_2 \wedge T_3$, then $\widehat{\mathcal{Y}}_{m,m'}^w(i, t) > 0$ for any $i \in \{i_1^*, \dots, i_1^* + c_2 w\} \cup \{i_2^* - c_2 w, \dots, i_2^*\}$, since the inequality $\widehat{\mathcal{Y}}_{m,m'}^w(i, t) > 0$ clearly holds for other i by the definition of T_3 . We will prove this result for $i \in \{i_1^*, \dots, i_1^* + c_2 w\}$, which is sufficient by symmetry. It is sufficient to show that

$$\widehat{\mathcal{Y}}_{m,m'}^w(i, t) - \widehat{\mathcal{Y}}_{m,m'}^w(i, c_1^{-1/2} w^{-1/2}) \geq \sum_{j: t_j^1 \leq t} R_j^1, \quad (58)$$

since by the definition of T_1 and $\widehat{\mathcal{Y}}_{m,m'}^w(i, c_1^{-1/2} w^{-1/2}) \geq A_4 w^{1/2} > 2w^{1/2}$, it gives $\widehat{\mathcal{Y}}_{m,m'}^w(i, t) > 0$. Consider a time $t' \in (c_1^{-1/2} w^{-1/2}, t]$ at which the clock of some node $j \in \mathcal{N}(i)$ rings. Then exactly one of the following holds: (i) $j \in \{i_1^*, i_1^* + 1, \dots, i_1^* + w\}$, (ii) $j > i_1^* + w$, (iii) $j \in \{i - w, i - w + 1, \dots, i_1^* - 1\}$. In case (ii) the increment made by $\widehat{\mathcal{Y}}_{m,m'}^w(i, \cdot)$ is non-negative. The sum of the increments made by $\widehat{\mathcal{Y}}_{m,m'}^w(i, \cdot)$ in cases (i) and (iii) stochastically dominates the right side of (58). In fact, the two sums are identical if we do not consider the contribution to the latter sum from nodes in $\{i_1^* - w, i_1^* - w + 1, \dots, i - w - 1\}$; note that these terms are always ≤ 0 due to the definition of the m' -modified dynamic. This implies the inequality in (58), which concludes the proof of (57).

Since (57) implies that $\widetilde{E} \cap \{c_1 < T_1 \wedge T_2 \wedge T_3\} \subset E_{m'}$, a union bound gives

$$\begin{aligned} \mathbb{P}[E_{m'}^c \cap \widehat{E}] &\leq \mathbb{P}[\widetilde{E}^c \cap \widehat{E}] + \mathbb{P}[\widetilde{E}; T_1 \leq T_2 \wedge T_3; T_1 \leq c_1; \widehat{E}] + \mathbb{P}[\widetilde{E}; T_2 \leq T_1 \wedge T_3; T_2 \leq c_1; \widehat{E}] \\ &\quad + \mathbb{P}[\widetilde{E}; T_3 \leq T_1 \wedge T_2; T_3 \leq c_1; \widehat{E}]. \end{aligned} \quad (59)$$

In order to prove (55) it is sufficient to show that each of the four terms on the right side converge to 0 as we send $w \rightarrow \infty$, $c_1 \rightarrow 0$, and $c_2 \rightarrow 0$ (in that order). We will bound the probability of each term on the right side separately. By (53), and since the second and third terms on the right side have the same probability, it is sufficient to bound the second term and the fourth term on the right side of (59).

First we bound the second term on the right side of (59). Recall the definition of $(i_j^1, t_j^1)_{j \in \mathbb{N}}$ above, and observe that the following hold, where $X(i_j^1, (t_j^1)^-) = \lim_{t \uparrow t_j^1} X(i_j^1, t)$:

- (i) If $X(i_j^1, (t_j^1)^-) = m$ and $i_j^1 \notin (wI) \cap \mathbb{Z}$ then $R_j = -2$;
- (ii) if $X(i_j^1, (t_j^1)^-) = m'$ and $i_j^1 \notin (wI) \cap \mathbb{Z}$ then $R_j = 0$;
- (iii) if $X(i_j^1, (t_j^1)^-) \notin \{m, m'\}$ and $i_j^1 \notin (wI) \cap \mathbb{Z}$ then $R_j = -1$;
- (iv) if $X(i_j^1, (t_j^1)^-) = m$ and $i_j^1 \in (wI) \cap \mathbb{Z}$ then $R_j = 0$;
- (v) if $X(i_j^1, (t_j^1)^-) = m'$, and $i_j^1 \in (wI) \cap \mathbb{Z}$ then $R_j = 2$;
- (vi) if $X(i_j^1, (t_j^1)^-) \notin \{m, m'\}$ and $i_j^1 \in (wI) \cap \mathbb{Z}$ then $R_j = 1$.

Letting \mathcal{F}_j denote the σ -algebra containing all information until times which are strictly smaller than t_j^1 , consider the martingale $M_k = \sum_{j=1}^k (R_j^1 - \mathbb{E}[R_j^1 | \mathcal{F}_j])$. Since this is a martingale with increments bounded by 2, Freedman's inequality [Fre75] gives

$$\lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P} \left[\sup_{1 \leq k \leq 2c_1 c_2^{-1} w} |M_k| > c_1^{1/10} w^{1/2}; \widehat{E} \right] = 0.$$

On the event \widetilde{E} (in particular, the requirement on A_2 and A_3), the fraction of nodes j in wI (resp. $\mathcal{N}(i_1^*) \setminus (wI)$) for which $X(j, c_1^{-1/2} w^{-1/2}) = m$ (resp. $X(j, c_1^{-1/2} w^{-1/2}) = m'$) differs from M^{-1} by at most $(2c_1^{1/2} c_2^{-1} + c_1^{-1/10}) w^{-1/2}$. Using this and that the number of times each event (i)-(vi) above occurs in any interval $(c_1^{-1/10} w^{1/2}, s)$ concentrates around its mean,

$$\lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P} \left[\inf_{1 \leq k \leq 2c_1 c_2^{-1} w} \sum_{j=1}^k \mathbb{E}[R_j^1 | \mathcal{F}_j] < -20c_1^{-1/10} w^{-1/2} \cdot 2c_1 c_2^{-1} w; \widehat{E}; \widetilde{E} \right] = 0.$$

Combining the above,

$$\begin{aligned}
 & \lim_{c_2 \rightarrow 0} \lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P}[\widehat{E}; \widetilde{E}; T_1 \leq T_2 \wedge T_3; T_1 \leq c_1] \\
 & \leq \lim_{c_2 \rightarrow 0} \lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P} \left[\widehat{E}; \widetilde{E}; T_1 \leq T_2 \wedge T_3; \inf_{c_1^{-1/2} w^{-1/2} \leq t \leq c_1 \wedge T_1} \left(\sum_{j: t_j^1 \leq t} R_j^1 \right) < -\frac{1}{2} w^{1/2} \right] \\
 & \leq \lim_{c_2 \rightarrow 0} \lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P} \left[\widehat{E}; \widetilde{E}; T_1 \leq T_2 \wedge T_3; \inf_{c_1^{-1/2} w^{-1/2} \leq k \leq 2c_1 c_2^{-1} w} \left(M_k + \sum_{j=1}^k \mathbb{E}[R_j^1 | \mathcal{F}_j] \right) < -\frac{1}{2} w^{1/2} \right] \\
 & = 0,
 \end{aligned}$$

which bounds the probability of the second term on the right side of (59).

In order to bound the fourth term on the right side of (59), a union bound and $\widetilde{E} \subset \{I \subset [-c_2, c_2]\}$ imply that it is sufficient to prove that for fixed $c_1, c_2 \in (0, 1/10)$ and fixed $i \in [-c_2 w, c_2 w] \cap \mathbb{Z}$, defining

$$\begin{aligned}
 T & := \inf\{t \geq c_1^{-1/2} w^{-1/2} : \widehat{\mathcal{Y}}_{m,m'}^w(i, t) \leq 0\}, \\
 E' & := \widetilde{E} \cap \{T = T_3 \leq T_1 \wedge T_2\} \cap \{i \in \{i_1^* + c_2 w + 1, i_1^* + c_2 w + 2, \dots, i_2^* - c_2 w - 1\}\},
 \end{aligned} \tag{60}$$

we have

$$\log \mathbb{P}[E'; \widehat{E}; T \leq c_1] \preceq -w, \tag{61}$$

where the implicit constant is independent of w , but may depend on c_1, c_2 . We prove this by a similar approach as for the second term on the right side of (59), and will therefore only give a brief justification. For any $j \in \mathbb{N}$ let t_j' be the j th smallest element of $\{t \geq c_1^{-1/2} w^{-1/2} : \exists x \in \mathfrak{N}(i) \text{ such that } (x, t) \in \mathcal{R}\}$. Then define $i_j' \in \mathfrak{N}(i)$ such that $(i_j', t_j') \in \mathcal{R}$, and define $R_j^1 := \widehat{\mathcal{Y}}_{m,m'}^w(i, t_j') - \widehat{\mathcal{Y}}_{m,m'}^w(i, (t_j')^-)$. The random variables R_j^1 are supported on $\{-2, -1, 0, 1, 2\}$ and they have uniformly positive expectation on the event E' since $\lambda(I) > 1 + c_2$ and all nodes in I have bias towards m . Using $A_1 < 2c_1 c_2^{-1} w$ and $A_2 < 2c_1^{1/2} c_2^{-1} w^{1/2}$ on E' ,

$$\mathbb{1}_{E'} \inf_{c_1^{-1/2} w^{-1/2} \leq t \leq c_1 \wedge T} \left(\widehat{\mathcal{Y}}_{m,m'}^w(i, t) - \widehat{\mathcal{Y}}_{m,m'}^w(i, c_1^{-1/2} w^{-1/2}) \right) \geq \mathbb{1}_{E'} \min_{1 \leq k \leq 2c_1 c_2^{-1} w} \sum_{j=1}^k R_j^1 - 2c_1^{1/2} c_2^{-1} w^{1/2}.$$

Another application of Freedman's inequality now gives

$$\begin{aligned}
 \mathbb{P}[E'; \widehat{E}; T \leq c_1] & \leq \mathbb{P} \left[E'; \widehat{E}; \inf_{c_1^{-1/2} w^{-1/2} \leq t \leq c_1 \wedge T} \left(\widehat{\mathcal{Y}}_{m,m'}^w(i, t) - \widehat{\mathcal{Y}}_{m,m'}^w(i, c_1^{-1/2} w^{-1/2}) \right) < -w^{1/2} \right] \\
 & = \exp(-\Theta(w)),
 \end{aligned}$$

which implies (61). □

Lemma 3.8 says that after some small macroscopic time, with high probability all nodes in the interval I has a bias towards m . The following lemma will imply that this is still true for slightly smaller intervals I_n at time $t_n := nw^{-0.01}$

Lemma 3.9. *Consider the setting described in Lemma 3.8, and let I, c_1, c_2 , and E be as in that lemma. Define open intervals I_n for $n \in \{0, \dots, \lceil w^{0.02} \rceil\}$ inductively by $I_0 := I$, and by letting $I_n \subset I_{n-1}$ be the open interval such that the left (resp. right) end-point of I_n has distance $w^{-0.1}$ from the left (resp. right) end-point of I_{n-1} . For $n \in \{1, \dots, \lceil w^{0.02} \rceil\}$ define $t_n := nw^{-0.01}$, and let E_n be the following event*

$$E_n = \{\overline{\mathcal{Y}}_m^w(i, t) > 0, \forall i \in (wI_n), \forall t \text{ such that } c_1^{-1/2} w^{-1/2} \leq t \leq t_n\}.$$

Then $\log \mathbb{P}[E \cap E_{n-1} \cap E_n^c] \preceq -w$ for all $n \in \{1, \dots, \lceil w^{0.02} \rceil\}$, where the implicit constant is independent of w and n , but may depend on all other constants, including c_1 and c_2 .

Proof. We first give a brief outline of the proof. As in the proof of Lemma 3.8, we consider some time T which represents the first time at which $\bar{\mathcal{Y}}_m^w(i, t) \leq 0$ for some i in wI_n , and we show that for each i in wI_n it is very unlikely that i is the *first* node for which this happens for $T \leq t_n$. We assume the event $E \cap E_{n-1}$ occurs. We divide $\mathfrak{N}(i)$ into three parts (see $\mathfrak{N}_1(i)$, $\mathfrak{N}_2(i)$, $\mathfrak{N}_3(i)$ below). For each $k \in \{1, 2, 3\}$ we can obtain a good (lower) bound on the contribution to $\bar{\mathcal{Y}}_m^w(i, t_{n-1})$ coming from nodes in $\mathfrak{N}_k(i)$, since we know approximately how many Poisson clocks which have been ringing before time t_{n-1} , and since nodes in I_{n-1} (corresponding to $\mathfrak{N}_1(i) \cup \mathfrak{N}_3(i)$, plus maybe part of $\mathfrak{N}_2(i)$) have a bias towards m during $[c_1^{-1/2}w^{-1/2}, t_{n-1}]$ on the event $E \cap E_{n-1}$. Conditioning on the fraction of nodes in $\mathfrak{N}_k(i)$, $k \in \{1, 2, 3\}$, of opinion m at time t_{n-1} , we compare the evolution of $\bar{\mathcal{Y}}_m^w(i, t)$ on $[t_{n-1}, T]$ to a random walk of a certain step size distribution, and we argue that this random walk is unlikely to hit 0 before time t_n , which completes the proof.

As in the proof of Lemma 3.8 we may assume $\mathbb{P}[\hat{E}] > 0$. We work on the event \hat{E} throughout the proof; in particular, some variables we define may exist only conditional on \hat{E} . Define the following stopping times for $i \in [-c_2^{-1}w, c_2^{-1}w] \cap \mathbb{Z}$

$$T_i := \inf\{t \geq t_{n-1} : \bar{\mathcal{Y}}_m^w(i, t) \leq 0\}, \quad T := \inf\{T_i : i \in (wI_n)\},$$

Fix $i \in [c_2^{-1}w, c_2^{-1}w] \cap \mathbb{Z}$. By a union bound it is sufficient to prove the following estimate, where the implicit constant is independent of w , n , and i

$$\log \mathbb{P}[E_n^i; T_i < t_n] \preceq -w, \quad E_n^i := E \cup E_{n-1} \cup \{i \in (wI_n)\} \cup \{T_i = T < t_n\}. \quad (62)$$

This estimate says that each node i is very unlikely to be the *first* node for which $\bar{\mathcal{Y}}_m^w(i, t) \leq 0$. Divide the neighborhood $\mathfrak{N}(i)$ into three disjoint parts $\mathfrak{N}_k(i)$ for $k = 1, 2, 3$ satisfying the following requirements; existence of appropriate neighborhoods is immediate by using the definition of I_n and $\lambda(I_n) > 1$

$$\begin{aligned} |\mathfrak{N}_1(i)| &= w, & |\mathfrak{N}_2(i)| &= w - w^{0.9} + 1, & |\mathfrak{N}_3(i)| &= w^{0.9}, \\ \mathfrak{N}_1(i) &\subset \{j \in \mathfrak{N}(i) : j \in (wI_n)\}, & \mathfrak{N}_3(i) &\subset \{j \in \mathfrak{N}(i) : j \in (wI_{n-1})\}. \end{aligned}$$

Note that on $E \cap E_{n-1}$, nodes in $\mathfrak{N}_1(i)$ have bias towards m until time t_n , nodes in $\mathfrak{N}_3(i)$ have bias towards m until time t_{n-1} , while we have no information about the bias of nodes in $\mathfrak{N}_2(i)$. For any $t \geq 0$ define $\bar{\mathcal{Y}}_m^{w,k}(i, t)$ for $k = 1, 2, 3$ by

$$\bar{\mathcal{Y}}_m^{w,k}(i, t) := \sum_{j \in \mathfrak{N}_k(i)} \mathbb{1}_{X(j,t)=m} - \sum_{j \in \mathfrak{N}_k(i)} \mathbb{1}_{X(j,t)=m'}, \quad (63)$$

where $m' \in [M] \setminus \{m\}$ is chosen such that the second sum is maximized if we sum over $j \in \mathfrak{N}(i)$ (rather than $j \in \mathfrak{N}_k(i)$). In other words, $\bar{\mathcal{Y}}_m^{w,k}(i, t)$ is defined just as $\bar{\mathcal{Y}}_m^w(i, t)$, except that we only consider nodes in $\mathfrak{N}_k(i)$. Note that $\bar{\mathcal{Y}}_m^w(i, t) = \sum_{k=1}^3 \bar{\mathcal{Y}}_m^{w,k}(i, t)$. On $E \cap E_{n-1}$ the nodes in $\mathfrak{N}_1(i)$ have bias m throughout $[c_1^{-1/2}w^{-1/2}, t_{n-1}]$. For each node the probability that its clock rings during $[0, t_{n-1}]$ is $(1 - e^{-t_{n-1}})$, and when this happens for the first time for some $t \in [c_1^{-1/2}w^{-1/2}, t_{n-1}]$ the node changes its opinion to m iff its current opinion is different. When this happens $\bar{\mathcal{Y}}_m^{w,1}(i, \cdot)$ increases by either 1 or 2, depending on whether the opinion of the node right before the clock ring was equal to the maximizing m' in (63) or not. Except on an event of exponentially small probability, for large w the number of nodes in $\mathfrak{N}_1(i)$ with an opinion equal to each $m'' \in [M]$ at time $c_1^{-1/2}w^{-1/2}$ differ from $M^{-1}w$ by at most $\frac{1}{100}w^{1/2+1/100}$, and the number of nodes in $\mathfrak{N}_1(i)$ whose Poisson clock rings during $t \in [c_1^{-1/2}w^{-1/2}, t_{n-1}]$ differ from $(1 - e^{-t_{n-1}})w$ by at most $\frac{1}{100}w^{1/2+1/100}$. Therefore

$$\log \mathbb{P}\left[E; E_{n-1}; \left|\bar{\mathcal{Y}}_m^{w,1}(i, t_{n-1}) - (1 - e^{-t_{n-1}})w\right| > w^{1/2+1/100}\right] \preceq -w. \quad (64)$$

By a similar argument the following inequalities hold. Note that we only get a lower bound for $\bar{\mathcal{Y}}_m^{w,2}(i, t_{n-1})$

since we do not know the bias of the nodes in $\mathfrak{N}_2(i)$ throughout $[0, t_{n-1}]$

$$\begin{aligned} \log \mathbb{P} \left[E; E_{n-1}; \bar{\mathcal{Y}}_m^{w,2}(i, t_{n-1}) < -(1 - e^{-t_{n-1}})(w - w^{0.9}) - w^{1/2+1/100} \right] &\preceq -w, \\ \log \mathbb{P} \left[E; E_{n-1}; \left| \bar{\mathcal{Y}}_m^{w,3}(i, t_{n-1}) - (1 - e^{-t_{n-1}})w^{0.9} \right| > w^{1/2+1/100} \right] &\preceq -w. \end{aligned} \quad (65)$$

We first consider the case $M = 2$. Defining $\mathcal{T}^w := \{t_{n-1} + w^{-0.3}, t_{n-1} + 2w^{-0.3}, \dots, t_{n-1} + \lceil w^{0.3-0.01} \rceil w^{-0.3}\}$, observe by a union bound that for all sufficiently large w

$$\begin{aligned} \mathbb{P} \left(E_n^i; \inf_{t_{n-1} \leq t \leq T_i \wedge t_n} \bar{\mathcal{Y}}_m^w(i, t) \leq 0 \right) &\leq \sum_{s \in \mathcal{T}^w} \mathbb{P} \left(E_n^i; s < T_i; \bar{\mathcal{Y}}_m^w(i, s) < \frac{1}{2} \bar{\mathcal{Y}}_m^w(i, t_{n-1}); \bar{\mathcal{Y}}_m^w(i, t_{n-1}) > w^{0.85} \right) \\ &+ \sum_{s \in \mathcal{T}^w} \mathbb{P} \left(E_n^i; |\{(j, t) \in \mathcal{R} : j \in \mathfrak{N}(i), t \in [s - w^{-0.3}, s]\}| > w^{0.8} \right) \\ &+ \mathbb{P} \left(E_n^i; \bar{\mathcal{Y}}_m^w(i, t_{n-1}) \leq w^{0.85} \right). \end{aligned}$$

The logarithm of the second sum on the right side is $\preceq -w$. The logarithm of the third term on the right side is also $\preceq -w$ by (64)-(65), so to prove (62), and thereby complete the proof of the lemma, it is sufficient to prove the following estimate for any $s \in \mathcal{T}^w$

$$\log \mathbb{P} \left(E_n^i; s < T_i; \bar{\mathcal{Y}}_m^w(i, s) < \frac{1}{2} \bar{\mathcal{Y}}_m^w(i, t_{n-1}); \bar{\mathcal{Y}}_m^w(i, t_{n-1}) > w^{0.85} \right) \preceq -w. \quad (66)$$

Fix $s \in \mathcal{T}^w$, and define the following random variables

$$\begin{aligned} R^+ &:= |\{j \in \mathfrak{N}^+(i) : \exists t \in (t_{n-1}, s] \text{ such that } (j, t) \in \mathcal{R}\}|, \quad \mathfrak{N}^+(i) := \{j \in \mathfrak{N}_1(i) : X(j, t_{m-1}) \neq m\}, \\ R^- &:= |\{j \in \mathfrak{N}^-(i) : \exists t \in (t_{n-1}, s] \text{ such that } (j, t) \in \mathcal{R}\}|, \quad \mathfrak{N}^-(i) := \{j \in \mathfrak{N}_2(i) \cup \mathfrak{N}_3(i) : X(j, t_{m-1}) = m\}. \end{aligned}$$

In other words, R^+ (resp. R^-) is the number of times during $(t_{n-1}, s]$ at which the clock of some node in $\mathfrak{N}_1(i)$ (resp. $\mathfrak{N}_2(i) \cup \mathfrak{N}_3(i)$) is ringing and this node has opinion different from m (resp. equal to m). Observe that

$$E_n^i \cap \{s < T_i\} \cap \left\{ \bar{\mathcal{Y}}_m^w(i, s) < \frac{1}{2} \bar{\mathcal{Y}}_m^w(i, t_{n-1}) \right\} \subset E_n^i \cap \{s < T_i\} \cap \left\{ R^+ - R^- < -\frac{1}{4} \bar{\mathcal{Y}}_m^w(i, s) \right\}. \quad (67)$$

By the definition of $\bar{\mathcal{Y}}_m^{w,k}(i, t_{n-1})$,

$$|\mathfrak{N}^+(i)| = \frac{1}{2}(w - \bar{\mathcal{Y}}_m^{w,1}(i, t_{n-1})), \quad |\mathfrak{N}^-(i)| = \frac{1}{2}(w + \bar{\mathcal{Y}}_m^{w,2}(i, t_{n-1}) + \bar{\mathcal{Y}}_m^{w,3}(i, t_{n-1})). \quad (68)$$

By large deviation estimates for Bernoulli random variables

$$\begin{aligned} \log \mathbb{P}[R^+ < (1 - e^{-(s-t_{n-1})})|\mathfrak{N}^+(i)| - w^{1/2+1/100}] &\preceq -w, \\ \log \mathbb{P}[R^- > (1 - e^{-(s-t_{n-1})})|\mathfrak{N}^-(i)| + w^{1/2+1/100}] &\preceq -w, \end{aligned}$$

so using (68) and $1 - e^{-(s-t_{n-1})} < 2w^{-0.01}$, for all sufficiently large w ,

$$\log \mathbb{P} \left[E_n^i; s < T_i; R^+ - R^- < -\frac{1}{4} \bar{\mathcal{Y}}_m^w(i, s) \right] \preceq -w.$$

We obtain (66) by using this estimate and (67).

We now explain how to modify the argument above to the case $M > 2$. Again it is sufficient to prove (66). Let $m' \in [M] \setminus \{m\}$ be the optimal value of m' in (49) at time $t = s$ and define $\mathfrak{M} := [M] \setminus \{m, m'\}$. Instead of considering R^+ and R^- as in the case $M = 2$, we define $R_{m'}^+, R_{\mathfrak{M}}^+, R_m^-, R_{\mathfrak{M}}^-$ as above using

$$\begin{aligned} \mathfrak{N}_{m'}^+(i) &:= \{j \in \mathfrak{N}_1(i) : X(j, t_{m-1}) = m'\}, & \mathfrak{N}_{\mathfrak{M}}^+(i) &:= \{j \in \mathfrak{N}_1(i) : X(j, t_{m-1}) \in \mathfrak{M}\}, \\ \mathfrak{N}_m^-(i) &:= \{j \in \mathfrak{N}_2(i) \cup \mathfrak{N}_3(i) : X(j, t_{m-1}) = m\}, & \mathfrak{N}_{\mathfrak{M}}^-(i) &:= \{j \in \mathfrak{N}_2(i) \cup \mathfrak{N}_3(i) : X(j, t_{m-1}) \in \mathfrak{M}\}. \end{aligned} \quad (69)$$

As in the case $M = 2$ and with $\widehat{\mathcal{Y}}_{m,m'}^w(i,t)$ as in the proof of Lemma 3.8, we can express $\widehat{\mathcal{Y}}_{m,m'}^w(i,t_{n-1})$ in terms of the size of the sets in (69). By using that the random variables $R_{m'}^+, R_{\mathfrak{M}}^+, R_m^-, R_{\mathfrak{M}}^-$ concentrate around their mean given the configuration at time t_{n-1} , we can conclude as before. \square

Proof of Theorem 1.3. Couple the discrete and continuum Schelling model as in Proposition 3.1. Almost surely there is an $m^* \in \{1, 2\}$ such that $0 \in A_{m^*}$. Let $I' \subset A_{m^*}$ (resp. $I^w \subset A_{m^*}^w$) be the connected component of A_{m^*} (resp. $A_{m^*}^w$) containing the origin, where I^w is empty if $0 \notin A_{m^*}^w$. Let $\epsilon > 0$. Define $I \subset I'$ by $I := \{x \in I' : \text{dist}(x, (I')^c) > \epsilon\}$. By translation invariance in law of both the discrete and continuum Schelling model it is sufficient to prove that $\mathbb{P}[I \subset I^w] > 1 - \epsilon$ for all sufficiently large $w \in \mathbb{N}$.

Consider the objects defined in the statement of Lemma 3.8 for I' the connected component of A_{m^*} containing the origin. Since $M = 2$ we know by Proposition 2.14 that $\mathbb{P}[\widehat{E}] = 1$, so $\lim_{c_2 \rightarrow 0} \lim_{c_1 \rightarrow 0} \lim_{w \rightarrow \infty} \mathbb{P}[E] = 1$. Let $c_1, c_2 \in (0, 1/2)$ be such that $\lim_{w \rightarrow \infty} \mathbb{P}[E] > \epsilon/100$ and such that $c_2 \ll \epsilon$. Observe that if

$$E' := \left\{ \forall i \in (wI) \cap \mathbb{Z}, \exists t \in (c_1^{-1/2} w^{-1/2}, w^{0.01}) \text{ such that } (i, t) \in \mathcal{R} \right\}$$

and the events E_n for $n \in \{0, \dots, \lceil w^{0.02} \rceil\}$ are defined as in Lemma 3.9, then $E' \cap E_{\lceil w^{0.02} \rceil} \subset \{I \subset I^w\}$. It follows by a union bound that

$$\mathbb{P}[I \not\subset I^w] \leq \mathbb{P}[(E')^c] + \mathbb{P}[E^c] + \mathbb{P}[E; E_0^c] + \sum_{n=1}^{\lceil w^{0.02} \rceil} \mathbb{P}[E; E_{n-1}; E_n^c]. \quad (70)$$

The first term on the right side of (70) converges to 0 as $w \rightarrow \infty$, and the second term on the right side of (70) is smaller than $\epsilon/3$ for all sufficiently large w by our choice of c_1 and c_2 . The third term on the right side of (70) is identically equal to zero. The last term on the right side of (70) is smaller than $\epsilon/3$ for all sufficiently large w by Lemma 3.9. Therefore $\mathbb{P}[I \subset I^w] > 1 - \epsilon$ for all sufficiently large w , which concludes the proof. \square

The following proposition says that the opinion of the nodes in the discrete Schelling model converge a.s. For the case $M = 2$ it was proved in [DS93, Theorem 1]. See e.g. [Mor95, GH00, TT15] for other related results. Our proof is inspired by the earlier proofs, in particular the proof found in [TT15]. Observe that the proposition below is not used in our proof of our main results, and is included only as a statement of independent interest.

Proposition 3.10. *Consider the Schelling model on either \mathbb{Z}^N or on the torus \mathfrak{S}^N where (in the notation on Section 1.3) \mathcal{N} is invariant upon reflection through the origin, $N \in \mathbb{N}$, and $M \in \{2, 3, \dots\}$. For each node i the opinion $X(i, t)$ converges a.s. as $t \rightarrow \infty$, i.e., for each node i there is a random time $T > 0$ such that $X(i, t) = X(i, T)$ for all $t \geq T$.*

Proof. The proof is identical for the two cases \mathbb{Z}^N and \mathfrak{S}^N . We will only present it for the case of \mathbb{Z}^N , but the result for \mathfrak{S}^N follows by replacing \mathbb{Z}^N by \mathfrak{S}^N throughout the proof. Let E be the set of undirected edges of the graph on which the Schelling model takes place, i.e. $(i, j) = (j, i) \in E$ for $i, j \in \mathbb{Z}^N$ iff $j \in \mathfrak{N}(i)$ (equivalently, since \mathcal{N} is invariant upon reflection through the origin, $i \in \mathfrak{N}(j)$). For each $i, j \in \mathbb{Z}^N$ such that $(i, j) \in E$, associate a positive real number w_{ij} , such that for any i, j, k for which $(i, j), (i, k) \in E$

$$\frac{w_{ij}}{w_{ik}} < \frac{(2w+1)^N + 1}{(2w+1)^N - 1}. \quad (71)$$

Choose the w_{ij} 's such that $\sum_{(i,j) \in E} w_{ij} < \infty$. The existence of appropriate w_{ij} satisfying these properties follows by [TT15, Proposition 3.4], since the degree of each node is bounded by $(2w+1)^N$, and since our graph satisfies the growth criterion considered in [TT15]. For each $i \in \mathbb{Z}^N$ and $t \geq 0$ define J_t^i by

$$J_t^i := \sum_{j \in \mathfrak{N}(i)} w_{ij} \mathbf{1}_{X(j,t) \neq X(i,t)} - \sum_{j \in \mathfrak{N}(i)} w_{ij} \mathbf{1}_{X(j,t^-) \neq X(i,t^-)}.$$

If t is a time at which the Poisson clock of node i rings, the first (resp. second) term on the right side expresses how many neighbors of i that disagree with i after (resp. before) i updates its opinion at time t . Assume the clock of node i rings at time t , and consider the three rules (i)-(iii) from Section 1.3 that i follows when updating its opinion. By our constraint (71), $J_t^i < 0$ if one of the rules (i) or (iii) apply, while $J_t^i = 0$ if i does not change its opinion, which is the case when (ii) applies. Given a node i and the value of w_{ij} for all $j \in \mathfrak{N}(i)$, the number of possible values for J_t^i is finite. It follows that we can find a real number $\epsilon_i > 0$, such that we have either $J_t^i < -\epsilon_i$ or $J_t^i = 0$ for all times $t \geq 0$ for which the clock of node i rings.

Next define the Lyapunov function $L : [0, \infty) \rightarrow \mathbb{R}_+$ by

$$L_t = \sum_{(i,j) \in E} w_{ij} \mathbf{1}_{X(i,t) \neq X(j,t)}.$$

Note that $L_t < \infty$ for all $t \geq 0$ by our assumption of summability of w_{ij} . Since $L_t - L_{t^-} = J_t^i$ if the clock of node i rings at time t , L_t is decreasing in t . It follows that there exists some $L \geq 0$ such that $\lim_{t \rightarrow \infty} L_t = L$. Fix $i \in \mathbb{Z}^N$, and let $T > 0$ be such that $L_t - L < \epsilon_i$ for all $t \geq T$. Since $L_t - L_{t^-} = J_t^i$ for any time t at which the clock of node i rings, we see that $J_t^i = 0$ for all times $t > T$. It follows that i never updates its opinion after time T , which completes the proof of the lemma. \square

We end the section with a result which may be related to the limiting opinions of the Schelling model on \mathbb{Z}^N for $N \geq 2$ and $M = 2$. We say that $A \subset \mathbb{Z}^N$ is *connected* if, for any two $i, j \in A$, there is an $n \in \mathbb{N}$ and a sequence $\{i^k\}_{0 \leq k \leq n}$ such that $i^0 = i$, $i^n = j$, $i^k \in A$ and $\|i^k - i^{k-1}\|_1 \leq 1$ for all $k \in \{1, \dots, n\}$. We say that $A \subset \mathbb{Z}^N$ is *stable* if all nodes of \mathbb{Z}^N agree with the most common opinion in their neighborhood when all nodes in A have opinion 1 and all nodes in $\mathbb{Z}^N \setminus A$ have opinion 2. Note that the definition of stability depends on \mathcal{N} and w . The *diameter* of a set $A \subset \mathbb{Z}^N$ is defined by $\sup_{i,j \in A} \|i - j\|_\infty$. We say that A is a *smallest stable shape* for the Schelling model if it is a connected stable subset of \mathbb{Z}^N of minimal diameter.

We observed before the statement of Lemma 2.18 that in the one-dimensional Schelling model on \mathbb{Z} the final configuration of opinions consists of monochromatic intervals of length at least $w + 1$. We also observe that the smallest stable shapes for the Schelling model on \mathbb{Z} are sets $A \subset \mathbb{Z}$ consisting of $w + 1$ consecutive integers. In particular, this means that all nodes are part of a monochromatic stable shape in the final configuration. By Theorem 1.3 the blocks with constant opinion in the final configuration of the Schelling model on \mathbb{Z} or \mathfrak{S} have length of order w .

One might guess that the smallest stable shape is related to the diameter of a typical cluster in the limiting configuration of opinions also in higher dimensions. There exist stable configurations where the cluster sizes are smaller than the diameter of a stable shape (e.g. a checkerboard configuration when \mathcal{N} has the shape of a cube), but these seem unlikely to occur since they are typically unstable, in the sense that changing the opinion of a small number of nodes may cause a large cluster of nodes to obtain the same opinion.

We thank Omer Tamuz for suggesting the approach used in the upper bound of the following proposition.

Proposition 3.11. *Let $N > 1$, $p \in [1, \infty]$, and assume $\mathcal{N} = \mathcal{N}_p := \{x \in \mathbb{R}^N : \|x\|_p < 1\}$. The diameter d of a smallest stable shape for the Schelling model satisfies $w^2 \preceq d \preceq w^{N+1}$, where the implicit constants can be chosen independently of p and w , but the constant in the upper bound may depend on N .*

Proof. We will prove the lower bound $w^2 \preceq d$ by induction on the dimension N . We will only do the case $p < \infty$, but the case $p = \infty$ can be done in exactly the same way. We start with the case $N = 2$. See Figure 9 for an illustration.

Assume A is a stable shape, and define i_1^k and i_2 as follows for $k = \{1, \dots, \lfloor w/2 \rfloor\}$

$$i_2 := \min\{i_2 \in \mathbb{Z} : \exists i_1 \in \mathbb{Z} \text{ such that } (i_1, i_2) \in A\}, \quad i_1^k := \min\{i_1 \in \mathbb{Z} : (i_1, i_2 + k - 1) \in A\}.$$

Then define $i^k := (i_1^k, i_2 + k - 1) \in A$. We will prove by induction on k that $i_1^k - i_1^{k+1} \geq w - k$ for all $k \in \{1, \dots, \lfloor w/2 \rfloor\}$, which is sufficient to obtain the lower bound $w^2 \preceq d$ since it implies that $i_1^1 - i_1^{\lfloor w/2 \rfloor} \succeq w^2$. For any $i = (i_1, i_2) \in \mathbb{Z}^2$ define

$$\mathfrak{N}^-(i) = \{(i'_1, i'_2) \in \mathfrak{N}(i) : i'_2 < i_2 \vee ((i'_2 = i_2) \wedge (i'_1 < i_1))\}, \quad \mathfrak{N}^+(i) = \mathfrak{N}(i) \setminus \mathfrak{N}^-(i).$$

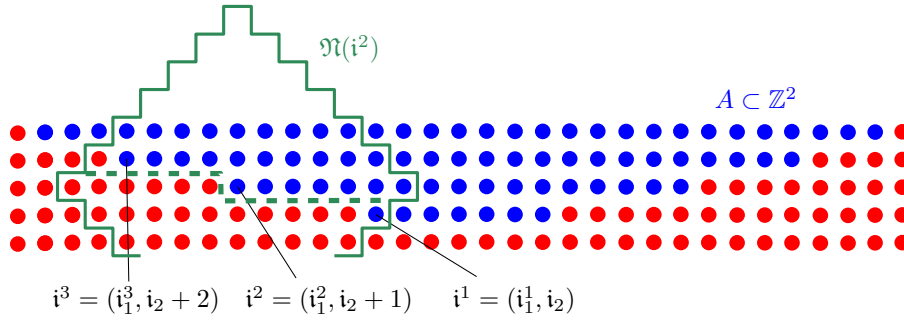


Figure 9: Illustration of the lower bound in Proposition 3.11 for $N = 2$, $w = 6$, and $p = 1$. The points of \mathbb{Z}^2 marked in blue is a subset of a stable shape A , while elements of $\mathbb{Z} \setminus A$ are shown in red. The green dotted line separates $\mathfrak{N}^-(i^2)$ (lower part) and $\mathfrak{N}^+(i^2)$ (upper part).

First let $k = 1$. By definition of i_2 and i_1^1 , $A \cap \mathfrak{N}^-(i^1) = \emptyset$. Since i^1 agrees with the most common opinion in its neighborhood this implies $A \cap \mathfrak{N}^+(i^1) = \mathfrak{N}^+(i^1)$. In particular, this implies by the definition of i_1^2 and since $(i_1^1 - w + 1, i_2 + 1) \in \mathfrak{N}^+(i^1)$ that $i_1^1 - i_1^2 \geq w - 1$.

Now assume $k > 1$ and that $i_1^\ell - i_1^{\ell+1} \geq w - \ell$ for $\ell \in \{1, \dots, k - 1\}$. This assumption implies by the definition of i_1^ℓ that

$$A \cap \mathfrak{N}^-(i^k) \subset \{(i_1^k + w - k + 1, i_2 + k - 2), (i_1^k + w - k + 2, i_2 + k - 2), \dots, (i_1^k + w - 1, i_2 + k - 2)\},$$

in particular, $|A \cap \mathfrak{N}^-(i^k)| \leq k$. Since $i^k \in A$ agrees with the most common opinion in its neighborhood by stability of A , we must have $|\mathfrak{N}^+(i^k) \setminus A| \leq k - 1$. By $\mathcal{N} = \mathcal{N}_p$ for $p \in [1, \infty)$,

$$\{(i_1^k - w + 1, i_2 + k), (i_1^k - w + 2, i_2 + k), \dots, (i_1^k - w + k, i_2 + k)\} \subset \mathfrak{N}^+(i^k),$$

where we note that the set on the left side has k elements. Since $|\mathfrak{N}^+(i^k) \setminus A| \leq k - 1$ this implies $i_1^k - i_1^{k+1} \geq w - k$. This completes our proof by induction, and hence completes the proof of the lower bound for d in the case when $N = 2$.

Now assume the lower bound $w^2 \preceq d$ has been proved for dimension $2, \dots, N - 1$ for some $N > 2$. We want to show that it also holds in dimension N . Define

$$i_N := \min\{i_N \in \mathbb{Z} : \exists i_1, \dots, i_{N-1} \in \mathbb{Z} \text{ such that } (i_1, i_2, \dots, i_N) \in A\}.$$

Then

$$\{i = (i_1, \dots, i_N) \in A : i_N < i_N\} = \emptyset,$$

so since all elements of $A \cap \{i = (i_1, \dots, i_N) \in \mathbb{Z}^N : i_N = i_N\}$ agree with the most common opinion in its neighborhood and by, for any $i' = (i'_1, \dots, i'_N) \in \mathbb{Z}^N$, symmetry of $\mathfrak{N}(i')$ upon reflection through the plane $i_N = i'_N$, the following $(N - 1)$ -dimensional set must be stable if the neighborhood of any $j \in \mathbb{Z}^{N-1}$ is given by $\{j' \in \mathbb{Z}^{N-1} : \|j - j'\|_p < w\}$

$$\{(i_1, \dots, i_{N-1}) \in \mathbb{Z}^{N-1} : (i_1, \dots, i_{N-1}, i_N) \in A\}.$$

By the induction hypothesis this set has diameter $\succeq w^2$, so A also has diameter $\succeq w^2$. This concludes the proof of the lower bound by induction.

Now we will prove the upper bound $d \preceq w^{N+1}$. Let $r = 2^{2N}w^{N+1}$, and define $A_0 = \{i \in \mathbb{Z}^N : \|i\|_\infty \leq r\}$. Let all nodes in A_0 have opinion 1, and let all nodes in $\mathbb{Z}^N \setminus A_0$ have opinion 2. We define decreasing sets $A_n \subset \mathbb{Z}^N$, $n \in \mathbb{N}$, by induction as follows. For each $n \in \mathbb{N}$ we choose one element of $i \in A_{n-1}$ which does not agree with the most common opinion in its neighborhood, we change the opinion of this node to 2, and we define $A_n = A_{n-1} \setminus \{i\}$. We continue this procedure until $A_n = \emptyset$ or until all nodes in A_n agree with the

most common opinion in its neighborhood. Note that throughout the process all nodes not in A_n agree with the most common opinion in their neighborhood. Let \tilde{n} denote the time at which the process terminates. We will prove by contradiction that $A_{\tilde{n}} \neq \emptyset$. This will imply the existence of a stable shape of diameter $\preceq r \asymp w^{N+1}$.

Define

$$E_n = \{(i, j) : i \in A_n, j \in \mathbb{Z}^N \setminus A_n\}.$$

By our choice of the node i in each step, the sequence $(|E_n|)_{1 \leq n \leq \tilde{n}}$ is strictly decreasing. Assuming $A_{\tilde{n}} = \emptyset$ this implies $|A_0| \leq |E_0|$. We have $|A_0| = (2r + 1)^N$. We can find a constant $C_N > 0$ depending on N , such that there are $< C_N(2r + 1)^{N-1}w$ nodes in A_0 which have a neighbor in $\mathbb{Z}^N \setminus A_0$. By using this and that $|\mathfrak{N}(i)| \leq (2w + 1)^N$ for any $i \in \mathfrak{S}^N$, we get $|E_0| \leq C_N(2r + 1)^{N-1}w(2w + 1)^N$. Using $|A_0| \leq |E_0|$ these estimates imply $(2r + 1) \leq C_N w(2w + 1)^N$. This is a contradiction to our definition of r , and we conclude that $A_{\tilde{n}} \neq \emptyset$. \square

4 Open problems

This paper explains the limiting configuration of opinions in the Schelling model when $N = 1$ and $M = 2$ (see Theorem 1.3). One open problem is to understand the limiting configuration of opinions in cases where $N \geq 2$ and/or $M > 2$. In particular, it remains an open question to understand the following situations: (i) \mathfrak{S}^N for $N \geq 2$, (ii) \mathbb{Z}^N for $N \geq 2$, and (iii) \mathfrak{S} or \mathbb{Z} for $M > 2$.

Case (i) could possibly be understood by studying the long-time behavior of solutions Y of the initial value problem (6), (8) for $N \geq 2$. If we knew that the limit $Y(x) := \lim_{t \rightarrow \infty} p(Y(x, t))$ exists for almost every $x \in \mathcal{S}^N$ a.s., the field $(Y(x))_{x \in \mathcal{S}^N}$ would likely describe law of the limiting opinions in the discrete model. Observe that non-trivial limiting configurations (i.e., limiting configurations with more than one limiting opinion) happen with positive probability, e.g. if the torus width R is at least 3, and the initial data are such that $p(Y((x_1, \dots, x_N), 0))$ equals 1 (resp. 2) for $x_1 \in [0, 1]$ (resp. $x_1 \in [R - 5/4, R - 1/4]$). For such initial data we will have $p(Y((x_1, \dots, x_N), t))$ equal to 1 (resp. 2) for $x_1 \in [0, 1]$ (resp. $x_1 \in [R - 5/4, R - 1/4]$) and all $t \geq 0$.

The continuum Schelling model (6), (8) may be less helpful for understanding case (ii). Even if we had established existence and uniqueness of solutions of (6), (8) on \mathbb{R}^N for $N \geq 2$ (see Theorem 2.1), the solution Y may be of only limited help for understanding the final configurations of opinions in the discrete model. Observe that there are no bounded continuum stable shapes, where a continuum stable shape is a set $D \subset \mathbb{R}^N$ which is such that if $m \in [M]$, $t_0 \geq 0$ and $p(Y(x, t_0)) = m$ for all $x \in D$ then $p(Y(x, t)) = m$ for all $t \geq t_0$ (see above Proposition 3.11 for the discrete definition). Since there are no bounded continuum stable shapes, we expect that the limit $\lim_{t \rightarrow \infty} p(Y(x, t))$ a.s. does not exist for any fixed $x \in \mathbb{R}^N$, at least when $M = 2$. The existence of this limit is necessary in order for the continuum Schelling model to describe the limiting configurations of opinions in the discrete model.

The continuum Schelling model (6), (8) is related to the discrete Schelling model upon rescaling the lattice by w^{-1} . If we proved a scaling limit result for the discrete model by proving convergence of the opinions in the continuum model, the diameter of a typical cluster in the discrete model would therefore be of order w . Figure 5 suggests that the diameter of the limiting clusters on \mathbb{Z}^2 grow superlinearly in w . The typical cluster size may be related to the size of the smallest (discrete) stable shape for the model; see Proposition 3.11 for upper and lower bounds on the diameter of the smallest stable shape. An independently interesting problem (which involves no probability) is to resolve the sizable discrepancy between the upper and lower bounds in Proposition 3.11. One could try to explicitly construct the minimal stable shape for each given w , and compute its size.

Case (iii) could be understood by studying the long-time behavior of the solutions of (6), (8) for $N = 1$. We believe Theorem 1.3 also holds for $M > 2$, i.e., the limiting opinions in the Schelling model have a scaling limit upon rescaling the lattice by w^{-1} , and the limiting law can be described by an M -tuple (A_1, \dots, A_M) , where the sets A_m have a.s. disjoint interior and can be written as the union of intervals each of length larger than 1 a.s. This version of Theorem 1.3 with $M > 2$ would be immediate from the approach in Section 3.2 if we had established the corresponding version of Proposition 2.14 (see Remark 2.15).

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