EULER CHARACTERISTIC OF SPRINGER FIBERS

Transformation Groups

EULER CHARACTERISTIC OF SPRINGER FIBERS

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Abstract. For Weyl groups of classical types, we present formulas to calculate the restriction of Springer representations to a maximal parabolic subgroup of the same type. As a result, we give recursive formulas for Euler characteristics of Springer fibers for classical types. We also give tables of those for exceptional types.

1. Introduction

Suppose that we have a reductive group G over an algebraically closed field $\mathbf{k}$ and its Lie algebra $\mathfrak{g}$. Let $\mathcal{B}$ be the set of Borel subalgebras of $\mathfrak{g}$ and for any $N \in \mathfrak{g}$ define

$$\mathcal{B}_N := \{\mathfrak{b} \in \mathcal{B} \mid N \in \mathfrak{b}\}$$

which we call the Springer fiber corresponding to N. It is currently one of the main objects in (geometric) representation theory. In this paper we deal with the case when N is nilpotent.

Let W be the Weyl group of G. In [20] Springer defined an action of W on the cohomology of Springer fibers when $\mathbf{k}$ is an algebraic closure of some finite field, which is now called the Springer representation. It was reconstructed in a more general setting by Lusztig [11] using the theory of perverse sheaves, which differs from [20] by tensoring with the sign character of W. On the other hand, the stabilizer of N in G acts on $\mathcal{B}_N$ by $g \cdot \mathfrak{b} := \mathrm{Ad}(g)(\mathfrak{b})$, thus also acts on the cohomology of $\mathcal{B}_N$. As the action of a connected group on the cohomology of a variety is trivial, this action factors through the component group of the stabilizer of N in G, denoted A_N. It is known that the action of W and A_N on the cohomology of $\mathcal{B}_N$ commute, thus we may regard it as a representation of $W \times A_N$.

In this paper we present formulas to calculate the restriction of such representations to $W' \times A_N \subset W \times A_N$, where $W' \subset W$ is a maximal parabolic subgroup of the same type when G is of classical type. In particular, we have recursive formulas for Euler characteristics of Springer fibers. For G of exceptional type, we give a table for such multiplicities in each Springer representation in Appendix B.

In some special cases we also have such recursive formulas for the multiplicities of irreducible representations of A_N in each degree of the cohomology of Springer

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fibers. As a result we present closed formulas of such multiplicities and in particular the Betti numbers of Springer fibers for "two-row" cases.

Note that the Green functions of G can be calculated using the Lusztig-Shoji algorithm (e.g., [18], [12]), and it is also possible to obtain information about Springer representations from Green functions. But our method is more elementary and does not use orthogonality of Green functions, which is crucial for the Lusztig-Shoji algorithm.

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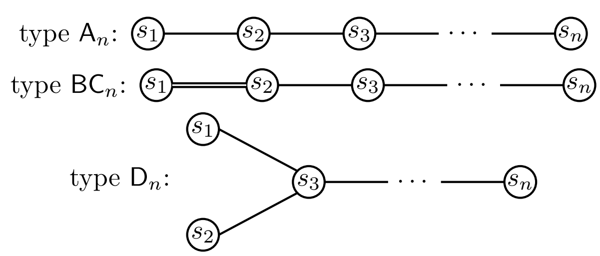
2. Notations and preliminaries

2.1.

Let $\mathbf{k}$ be an algebraically closed field. Throughout this paper we assume that G is GL_n, SO_{2n+1}, Sp_{2n}, or SO_{2n} over $\mathbf{k}$, except in Appendix B. Here $n \geqslant 1$, unless $G = \mathrm{SO}_{2n}$ in which case $n \geqslant 2$. We identify G with the set of its $\mathbf{k}$-points $G(\mathbf{k})$. We also assume that $\operatorname{char} \mathbf{k}$ is good; there is no assumption on $\operatorname{char} \mathbf{k}$ if $G = \mathrm{GL}_n$ and $\operatorname{char} \mathbf{k} \neq 2$ otherwise. If $G = \mathrm{GL}_n$ (resp. $G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$) we define V to be a $\mathbf{k}$-vector space of dimension n (resp. $2n+1, 2n, 2n$) on which G naturally acts. Also when $G = \mathrm{SO}_{2n+1}$, (resp. $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$,) V is equipped with a symmetric (resp. symplectic, symmetric) bilinear form $\langle\ ,\ \rangle$ which is invariant under the action of G, i.e., for any $v, w \in V$ and $g \in G$ we have $\langle gv, gw \rangle = \langle v, w \rangle$.

2.2.

Let W be the Weyl group of G and $S = \{s_1, \ldots, s_n\} \subset W$ be the set of simple reflections of W such that (W, S) is a Coxeter group. We choose s_i such that the labeling corresponds to one of the following Dynkin diagrams:



2.3.

Let $\mathfrak{g} = \mathfrak{gl}_n$ (resp. $\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$) be the Lie algebra of $G = \mathrm{GL}_n$ (resp. SO_{2n+1}, Sp_{2n}, SO_{2n}). Then if $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$, or $\mathfrak{so}_{2n}$ there is a natural Lie algebra action of $\mathfrak{g}$ on V which respects $\langle\ ,\ \rangle$, i.e., for any $v, w \in V$ and $N \in \mathfrak{g}$ we have $\langle Nv, w \rangle + \langle v, Nw \rangle = 0$.

2.4.

Let $\mathcal{B}$ be the flag variety of G, i.e., the set of Borel subgroups of G, or equivalently, the set of Borel subalgebras of $\mathfrak{g}$. If $G = \mathrm{GL}_n$, then $\mathcal{B}$ is isomorphic to the variety of full flags in V. If $G = \mathrm{SO}_{2n+1}$ or Sp_{2n}, then $\mathcal{B}$ is isomorphic to the variety of full isotropic flags in V. If $G = \mathrm{SO}_{2n}$, then $\mathcal{B}$ is isomorphic to the variety of isotropic flags $0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1}$ in V where $\dim V_i = i$. For $N \in \mathfrak{g}$, we write $\mathcal{B}_N$ to be the Springer fiber of N, i.e., the set of Borel subalgebras of $\mathfrak{g}$ that contains N.

2.5.

For $G = \mathrm{GL}_n$ with $n \geqslant 1$, (resp. SO_{2n+1} with $n \geqslant 1$, Sp_{2n} with $n \geqslant 1$, SO_{2n} with $n \geqslant 2$,) we define $G' = \mathrm{GL}_{n-1}$ (resp. SO_{2n-1}, $\mathrm{Sp}_{2n-2}, \mathrm{SO}_{2n-2}$) and $\mathfrak{g}' = \mathfrak{gl}_{n-1}$ (resp. $\mathfrak{g}' = \mathfrak{so}_{2n-1}$, $\mathfrak{sp}_{2n-2}, \mathfrak{so}_{2n-2}$). Also we define $\mathcal{B}'$ to be the flag variety of G' and W' to be the Weyl group of G'. We regard W' as a subgroup of W generated by $S' = \{s_1, \ldots, s_{n-1}\} \subset S$, except when $G = \mathrm{SO}_4$ in which case we regard $W' = \{\mathrm{id}\} \subset W$. If $N' \in \mathfrak{g}'$, let $\mathcal{B}'_{N'}$ be the Springer fiber corresponding to N' with respect to G'.

2.6.

For a variety X, let $H^*(X) := \sum_{i \in \mathbb{N}} (-1)^i H^i(X, \overline{\mathbb{Q}_\ell})$ be the alternating sum of ℓ-adic cohomology of X in the Grothendieck group of vector spaces. If $\mathbf{k} = \mathbb{C}$, then by comparison theorem (e.g., see [14, Thm. III.3.12] for the case when X is smooth) it is equivalent to the alternating sum of complex cohomology of X^{an}. Similarly, let $H^*_c(X) := \sum_{i \in \mathbb{N}} (-1)^i H^i_c(X, \overline{\mathbb{Q}_\ell})$ be the alternating sum of ℓ-adic cohomology with compact support of X.

2.7.

Let λ be a partition. We write $\lambda \vdash n$ or $|\lambda| = n$ if λ is a partition of n. We describe each part of λ by writing $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ and $|\lambda| = n$. Or we also write $\lambda = (1^{r_1} 2^{r_2} 3^{r_3} \cdots)$ which means that λ consists of r_1 parts of 1, r_2 parts of 2, and so on.

2.8.

We recall the correspondence between nilpotent adjoint orbits in $\mathfrak{g}$ under the adjoint action of G and partitions (see [1, Chap. 5.1]) If $G = \mathrm{GL}_n$ then such orbits are parametrized by partitions of n. This correspondence is given by taking the sizes of Jordan blocks of any element in a nilpotent orbit regarded as an endomorphism on V. Likewise, if $G = \mathrm{SO}_{2n+1}$, then nilpotent adjoint orbits in $\mathfrak{g}$ are parametrized by $\lambda = (1^{r_1} 2^{r_2} 3^{r_3} \cdots) \vdash 2n+1$ such that $r_{2i} \equiv 0 \pmod 2$ for $i \geqslant 1$. If $G = \mathrm{Sp}_{2n}$, then nilpotent adjoint orbits in $\mathfrak{g}$ are parametrized by $\lambda = (1^{r_1} 2^{r_2} 3^{r_3} \cdots) \vdash 2n$ such that $r_{2i-1} \equiv 0 \pmod 2$ for $i \geqslant 1$. In these cases we write $N_\lambda \in \mathfrak{g}$ to be such a nilpotent element corresponding to $\lambda \vdash n$. It is well defined up to adjoint action by G.

If $G = \mathrm{SO}_{2n}$, then it is almost the same as the case $G = \mathrm{SO}_{2n+1}$, so if $\lambda = (1^{r_1} 2^{r_2} 3^{r_3} \cdots) \vdash 2n$ is a partition of the sizes of Jordan blocks of some nilpotent element in $\mathfrak{g}$, then $r_{2i} \equiv 0 \pmod 2$ for $i \geqslant 1$. However this correspondence is no longer one-to-one since a partition consisting of even parts with even multiplicities, which we call *very even*, corresponds to two adjoint nilpotent orbits in $\mathfrak{g}$. Thus if $\lambda \vdash 2n$ is not very even, we write $N_\lambda \in \mathfrak{g}$ to be such a nilpotent element corresponding to λ which is again well defined up to adjoint action by G. If $\lambda \vdash 2n$ is very even, then we write $N_{\lambda+}, N_{\lambda-}$ to distinguish two such nilpotent elements corresponding to λ in different adjoint orbits. If there is no ambiguity or need to differentiate $N_{\lambda+}$ and $N_{\lambda-}$, we still write N_λ.

2.9.

Let $\widetilde{G} = \mathrm{GL}_n$ (resp. $\mathrm{O}_{2n+1}, \mathrm{Sp}_{2n}, \mathrm{O}_{2n}$) if $G = \mathrm{GL}_n$ (resp. $\mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$). For a nilpotent $N \in \mathfrak{g}$, we define $\widetilde{A}_N$ to be the component group of the stabilizer of N in $\widetilde{G}$. Also for a partition λ, define $\widetilde{A}_\lambda := \widetilde{A}_{N_\lambda}$. If $G = \mathrm{SO}_{2n}$ and λ is very even, then as $\widetilde{A}_{N_{\lambda+}} = \widetilde{A}_{N_{\lambda-}} = \{\mathrm{id}\}$, we define $\widetilde{A}_\lambda := \{\mathrm{id}\}$. If $\widetilde{G} = \mathrm{GL}_n$, any $\widetilde{A}_N$ is trivial. Otherwise if $\widetilde{G} = \mathrm{Sp}_{2n}$ (resp. O_{2n+1} or O_{2n}) and if a partition $\lambda = (1^{r_1}2^{r_2}\cdots)$ corresponds to a nilpotent element in $\mathfrak{g}$, then $\widetilde{A}_\lambda$ is a product of $\mathbb{Z}/2$ generated by z_i for each even i (resp. odd i) such that $r_i > 0$. (Here we adopt the convention that $z_i = id$ if i does not satisfy the aforementioned condition.) Likewise, we define A_N to be the component group of the stabilizer of N in G and $A_\lambda := A_{N_\lambda}$, again even when λ is very even (in which case A_λ is trivial). Then A_N can be considered as a subgroup of $\widetilde{A}_N$. If $G = \mathrm{GL}_n$ or Sp_{2n}, $A_N = \widetilde{A}_N$. Otherwise, for $\lambda = (1^{r_1}2^{r_2}\cdots)$ A_λ is the subgroup of $\widetilde{A}_\lambda$ generated by $\{z_i z_j \mid i, j \text{ odd}, r_i, r_j > 0\}$. (See [1, Chap. 6.1] or [18, 1.1] for more information.)

2.10.

For a partition λ, define $H^i(\lambda) := H^i(\mathcal{B}_{N_\lambda})$ and $H^i(\lambda+) := H^i(\mathcal{B}_{N_{\lambda+}}), H^i(\lambda-) := H^i(\mathcal{B}_{N_{\lambda-}})$ if $G = \mathrm{SO}_{2n}$ and λ is very even. We regard $H^i(\mathcal{B}_N)$ as W-modules using Springer theory, adopting the definition of [11]. We also consider the action of $\widetilde{A}_N$ on $H^*(\mathcal{B}_N)$ which is induced from the action of $\widetilde{A}_N$ on $\mathcal{B}_N$. Note that the action of W and $A_N \subset \widetilde{A}_N$ commute [20, 6.1]. We denote by $\mathbf{TSp}(\lambda)$ the character of $H^*(\lambda)$ as a $W \times A_\lambda$-module. If $G = \mathrm{SO}_{2n}$ and λ is very even, we similarly define $\mathbf{TSp}(\lambda+)$ and $\mathbf{TSp}(\lambda-)$. Note that this definition does not depend on the base field $\mathbf{k}$ insofar as $\mathrm{char}\,\mathbf{k}$ is good. Define $\mathbf{EC}(\lambda) := \dim H^*(\mathcal{B}_{N_\lambda})$ to be the ℓ-adic Euler characteristic of $\mathcal{B}_{N_\lambda}$. This is well defined even when λ is very even as $\mathcal{B}_{N_{\lambda+}}$ and $\mathcal{B}_{N_{\lambda-}}$ are isomorphic. We also define $h^k(\lambda) := \dim H^k(\lambda)$ to be the k-th Betti number of $\mathcal{B}_{N_\lambda}$.

2.11.

For a partition $\lambda = (1^{r_1}2^{r_2}3^{r_3}\cdots)$ and for $i \geqslant 1$ we define $\lambda^i, \lambda^{h,i}, \lambda^{v,i}$ as follows. (This notation will be used when we state the main theorem of this paper.)

(1) If $r_i \geqslant 1$, define $\lambda^i := (1^{r_1}2^{r_2}\cdots(i-1)^{r_{i-1}+1}i^{r_i-1}\cdots) \vdash |\lambda| - 1$. This corresponds to removing a box from the Young diagram of λ.

(2) If $r_i \geqslant 1$ and $i \geqslant 2$, we let $\lambda^{h,i} := (1^{r_1}2^{r_2}\cdots(i-2)^{r_{i-2}+1}(i-1)^{r_{i-1}}i^{r_i-1}\cdots) \vdash |\lambda| - 2$. Here the superscript "$h$" stands for "horizontal"; $\lambda^{h,i}$ is obtained by removing a horizontal domino from the Young diagram of λ and rearranging rows if necessary.

(3) If $r_i \geqslant 2$, we let $\lambda^{v,i} := (1^{r_1}2^{r_2}\cdots(i-1)^{r_{i-1}+2}i^{r_i-2}\cdots) \vdash |\lambda| - 2$. Here the superscript "$v$" stands for "vertical"; $\lambda^{v,i}$ is obtained by removing a vertical domino from the Young diagram of λ.

3. Main theorem (weak form)

The (weak version of the) main theorem in this paper is as follows.

Theorem 3.1. *Recall the notations in* 2.11.

i) *Let* $G = \mathrm{GL}_n$ *and* $\lambda = (1^{r_1}2^{r_2}3^{r_3}\cdots)$ *be a partition corresponding to a nilpotent adjoint orbit of* $\mathfrak{g}$*. Then we have*
$$\mathbf{EC}(\lambda) = \sum_{r_i \geqslant 1} r_i \mathbf{EC}(\lambda^i).$$

ii) *Let* $G = \mathrm{SO}_{2n+1}$, Sp_{2n}, *or* SO_{2n} *and* $\lambda = (1^{r_1}2^{r_2}3^{r_3}\cdots)$ *be a partition corresponding to a nilpotent adjoint orbit of* $\mathfrak{g}$*. Then we have*
$$\mathbf{EC}(\lambda) = \sum_{i\geqslant 2,\ r_i \text{ odd}} \mathbf{EC}(\lambda^{h,i}) + \sum_{i \geqslant 1,\ r_i\geqslant 2} 2\left\lfloor \frac{r_i}{2}\right\rfloor \mathbf{EC}(\lambda^{v,i}).$$
Here $\lfloor r_i/2 \rfloor$ *is the greatest integer that is not bigger than* $r_i/2$*.*

Indeed, this can be deduced from more general statements, i.e., Theorem 4.1, Theorem 4.4, Theorem 5.3, and Theorem 6.1, which provide isomorphisms of $W' \times A_\lambda$-modules. In subsequent sections we prove such generalizations.

4. Type A

For $G = \mathrm{GL}_n$, the following theorem generalizes Theorem 3.1.

Theorem 4.1. *Let* $G = \mathrm{GL}_n$ *and* $\lambda = (1^{r_1}2^{r_2}3^{r_3}\cdots) \vdash n$*. Then we have*
$$\mathrm{Res}^W_{W'} \mathbf{TSp}(\lambda) = \sum_{r_i \geqslant 1} r_i \mathbf{TSp}(\lambda^i)$$
as characters of W'*.*

Remark 1. This theorem can also be proved by a combinatorial method: see Appendix A.

Proof. The method we adopt here using certain geometric properties of Springer fibers is well known, cf. [21], [16], [19], [18], [24], etc. Let $N = N_\lambda \in \mathfrak{g}$ be a nilpotent element corresponding to $\lambda = (1^{r_1}2^{r_2}\cdots) \vdash n$ and $\mathcal{P}$ be a variety of lines in V. Then we have a natural surjective morphism $\pi : \mathcal{B} \to \mathcal{P}$ which sends $[0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V]$ to V_1. This restricts to $\pi : \mathcal{B}_N \to \mathcal{P}_N$ where $\mathcal{P}_N$ is the set of lines annihilated by N. It is easy to show that $\pi : \mathcal{B}_N \to \mathcal{P}_N$ is surjective.

Note that $\mathbb{P}(\ker N) = \mathcal{P}_N$. We have filtration of $\ker N$
$$\ker N = W_0 \supset W_1 \supset W_2 \supset \cdots$$
where $W_i = \ker N \cap \mathrm{im}\, N^i$. Now suppose W_{i-1}/W_i is nonzero. It is equivalent to N having a Jordan block of size i.

Let $\eta \colon \mathbb{P}(W_{i-1}) - \mathbb{P}(W_i) \to \mathbb{P}(W_{i-1}/W_i)$ be a canonical affine bundle with fiber W_i. We stratify $\mathbb{P}(W_{i-1}/W_i)$ into affine spaces, i.e., $\mathbb{P}(W_{i-1}/W_i) = \bigsqcup_{j=0}^{r_i-1} Y_j$ where $Y_j \simeq \mathbb{A}^j$. Also we let $\widetilde{Y}_j := \eta^{-1}(Y_j)$ and $X_j := \pi^{-1}(\widetilde{Y}_j) = (\eta\circ\pi)^{-1}(Y_j)$. Then we can choose this stratification so that $\pi : X_j \to \widetilde{Y}_j$ is a locally trivial bundle with fiber $\mathcal{B}'_{N_{\lambda^i}}$, where λ^i is defined in 2.11 (see [16]). Thus we have
$$H^k_c(X_j) \simeq \bigoplus_{k_1+k_2=k} H^{k_1}_c(\widetilde{Y}_j)\otimes H^{k_2}(\lambda^i) \simeq H^{k-2\dim W_i - 2j}(\lambda^i)$$
as a vector space. In fact, more is true: first we recall [8, Thm. 5.1].

Lemma 4.2. *The Leray sheaves $R^j\pi_!\overline{\mathbb{Q}_\ell}$ on $\mathcal{P}_N$ have structures of W'-modules such that*

(a) *for any $x \in \mathcal{P}_N$, $(R^j\pi_!\overline{\mathbb{Q}_\ell})_x \simeq H^j(\mathcal{B}'_{N'})$ as W'-modules where $H^j(\mathcal{B}'_{N'})$ is equipped with the Springer representation of W' and where N' is the image of N under the canonical quotient morphism from the maximal parabolic subalgebra corresponding to x (which contains N by definition of $\mathcal{P}_N$) to its Levi factor.*

(b) *$H^k(\mathcal{P}_N, R^j\pi_!\overline{\mathbb{Q}_\ell}) \Rightarrow H^{j+k}(\mathcal{B}_N)$ is a spectral sequence of W'-modules, where the action of W' on $H^{j+k}(\mathcal{B}_N)$ is the restriction of the Springer representation of W.*

By the first part of Lemma 4.2 we have the following result.

Lemma 4.3. *There exists a natural W'-module structure on $H^k_c(X_j)$ such that*

$$H^k_c(X_j) \simeq H^{k-2\dim W_i - 2j}(\lambda^i)$$

as W'-modules. Here the action of W' on $H^{k-2\dim W_i-2j}(\lambda^i)$ is given by the Springer representation corresponding to G'.

Now we consider the long exact sequences of the cohomology with compact support of $\mathcal{B}_N$ corresponding to the stratification by such X_j's for all i. By Lemma 4.3, these long exact sequences are defined in the category of W'-modules. Thus it follows that $H^*(\mathcal{B}_N)$ is the sum of such $H^*_c(X_j)$'s. To be more precise, let $\{A_k\}_{1\leqslant k\leqslant m}$ be the collection of all X_j over all i such that $\bigcup_{k=1}^{m'} A_k \subset \mathcal{B}_N$ is closed for any $m' \leqslant m$. Then we have a long exact sequence

$$\cdots \to H^r_c(A_{m'}) \to H^r\left(\bigcup_{k=1}^{m'} A_k\right) \to H^r\left(\bigcup_{k=1}^{m'-1} A_k\right) \to \cdots$$

of W'-modules. By taking an alternating sum, we see that

$$H^*\left(\bigcup_{k=1}^{m'} A_k\right) = H^*\left(\bigcup_{k=1}^{m'-1} A_k\right) + H^*_c(A_{m'}).$$

Using induction on m', we have

$$H^*\left(\bigcup_{k=1}^{m'} A_k\right) = \bigoplus_{k=1}^{m'} H^*_c(A_k).$$

In particular when $m' = m$ so that $\bigcup_{k=1}^{m'} A_k = \mathcal{B}_N$ we obtain the desired result. Now the second part of Lemma 4.2 and its proof implies that the W'-module structure on $H^*(\mathcal{B}_N)$, as a sum of such $H^*_c(X_j)$, coincides with the restriction to W' of the Springer representation of W on $H^*_c(\mathcal{B}_N)$.

In sum, we have

$$\mathrm{Res}^W_{W'} \mathbf{TSp}(\lambda) = \sum_{r_i \geqslant 1} r_i \mathbf{TSp}(\lambda^i)$$

as characters of W'. But this is what we want to prove. □

In fact, we may proceed further; since Springer fibers have vanishing odd cohomology by [2], it is also true for X_j. Thus each long exact sequence considered above splits into short exact sequences in even degrees. Therefore, by keeping track of degrees of each short exact sequence, we have the following theorem which generalizes Theorem 4.1.

Theorem 4.4. *For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and any $k \in \mathbb{Z}$, we have*

$$\operatorname{Res}^W_{W'} H^k(\lambda) = \bigoplus_{i=1}^{r} H^{k-2i+2}(\lambda^{\lambda_i})$$

as W'-modules. (*See* 2.11 *for the definition of* λ^{λ_i}).

Remark 2. In fact, this formula already appeared in [7, Rem. 2.4] using a similar argument. On the other hand, if we evaluate the character of each side at $id \in W'$, then it gives [4, Prop. 4.5]. This method is combinatorial, counting the number of "row-standard" tableaux of certain Young diagrams.

Example 4.5. Suppose $\mathbf{k}$ is an algebraic closure of a finite field. For $w \in W$, we define $Q_{\lambda,w}(q) = \sum_{k\in\mathbb{N}} q^k \operatorname{tr}(w, H^{2k}(\lambda))$ to be the Green function associated with w and the nilpotent element $N_\lambda \in \mathfrak{g}$ corresponding to λ. Then Theorem 4.4 implies that for $w \in W' \subset W$ we have

$$Q_{\lambda,w}(q) = \sum_{i=1}^{r} Q_{\lambda^{\lambda_i},w}(q) q^{i-1}$$

where $Q_{\lambda^{\lambda_i},w}$ is defined similarly to $Q_{\lambda,w}$. For example, if $G = \mathrm{GL}_4$, $w = (123) \in W' \subset W$, then

$$\begin{aligned}
Q_{(1,1,1,1),w}(q) &= (q^3 + q^2 + q + 1)Q_{(1,1,1),w}(q),\\
Q_{(2,1,1),w}(q) &= Q_{(1,1,1),w}(q) + (q^2 + q)Q_{(2,1),w}(q),\\
Q_{(2,2),w}(q) &= (q + 1)Q_{(2,1),w}(q),\\
Q_{(3,1),w}(q) &= Q_{(2,1),w}(q) + qQ_{(3),w},\\
Q_{(4),w}(q) &= Q_{(3),w}(q),
\end{aligned}$$

and note that we have in fact

$$\begin{aligned}
Q_{(1,1,1,1),w}(q) &= q^6 - q^4 - q^2 + 1, & Q_{(2,1,1),w}(q) &= Q_{(2,2),w}(q) = -q^2 + 1,\\
Q_{(3,1),w}(q) &= Q_{(4),w}(q) = 1, & Q_{(1,1,1),w}(q) &= q^3 - q^2 - q + 1,\\
Q_{(2,1),w}(q) &= -q + 1, & Q_{(3),w}(q) &= 1.
\end{aligned}$$

5. Type B, C and D

Now we assume that $G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}$, or SO_{2n}. We follow the argument in the previous section with some modifications. Let $N = N_\lambda \in \mathfrak{g}$ be a nilpotent element corresponding to $\lambda = (1^{r_1} 2^{r_2} \cdots)$ and $\mathcal{P}$ be a variety of isotropic lines in V. Then we have a natural surjective morphism $\pi : \mathcal{B} \to \mathcal{P}$ defined by

$$\begin{array}{lll} \text{if } G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}, & [0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n] \mapsto V_1, & \text{and} \\ \text{if } G = \mathrm{SO}_{2n}, & [0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1}] \qquad \mapsto V_1. & \end{array}$$

This restricts to $\pi \colon \mathcal{B}_N \to \mathcal{P}_N$ where $\mathcal{P}_N$ is the set of isotropic lines annihilated by N. It is easy to show that $\pi \colon \mathcal{B}_N \to \mathcal{P}_N$ is surjective.

[21] defined a stratification on $\mathcal{P}_N$ such that π is locally trivial on each stratum, which is revisited in [17] and [18]. We recall their results as follows. Start with a filtration

$$\ker N = W_0 \supset W_1 \supset W_2 \supset \cdots$$

where $W_i = \ker N \cap \operatorname{im} N^i$. Now suppose W_{i-1}/W_i is nonzero. It is equivalent to N having a Jordan block of size i.

[21] also defined a (non-degenerate) bilinear form $(\, , \,)$ on W_{i-1}/W_i which is symmetric if i is odd (resp. even) and $G = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}$ (resp. $G = \mathrm{Sp}_{2n}$). Otherwise it is symplectic. If $(\, , \,)$ is symmetric, then the set of isotropic lines in $\mathbb{P}(W_{i-1}/W_i)$ forms a quadric hypersurface, which is nonsingular if $\dim W_{i-1}/W_i \geqslant 3$, a union of two points if $\dim W_{i-1}/W_i = 2$, and empty if $\dim W_{i-1}/W_i = 1$. If $(\, , \,)$ is symplectic, then any $x \in \mathbb{P}(W_{i-1}/W_i)$ is isotropic.

There is a canonical affine bundle $\eta : \mathbb{P}(W_{i-1}) - \mathbb{P}(W_i) \to \mathbb{P}(W_{i-1}/W_i)$ with fiber isomorphic to W_i. Now we define Y or Y_j to be one of the strata of $\mathbb{P}(W_{i-1}/W_i)$ in each case below, following the argument in [18], and let $\widetilde{Y} := \eta^{-1}(Y)$, $\widetilde{Y}_j := \eta^{-1}(Y_j)$. (See 2.11 for the definition of $\lambda^{h,i}$ and $\lambda^{v,i}$.)

Case I. Suppose $(\, , \,)$ is symmetric. Let Q be the set of isotropic lines with respect to $(\, , \,)$ in $\mathbb{P}(W_{i-1}/W_i)$ and $C := \mathbb{P}(W_{i-1}/W_i) - Q$. There is a stratification

$$\begin{array}{l} Q = Q_0 \supset Q_1 \supset \cdots \supset Q_{m-1} \supset Q_{m+1} \supset \cdots \supset Q_{r_i - 1} \supset Q_{r_i} = \varnothing, \\ C = C_0 \supset C_1 \supset \cdots \supset C_{r_i - m - 1} \supset C_{r_i - m} = \varnothing \end{array}$$

defined in [21] or [18], where $m = \lfloor r_i/2 \rfloor$.

(a$_1$) $Y_j = Q_{j-1} - Q_j$ for $j \neq m, m+1$, or $Y_m = Q_{m-1} - Q_{m+1}$ when r_i is odd. Then the fiber of π at any point in $\widetilde{Y}_j$ is isomorphic to $\mathcal{B}'_{N_{\lambda^{v,i}}}$. Also we have

$$Y_j \simeq \mathbb{A}^{r_i - j - 1} \ \text{ if } 1 \leqslant j \leqslant m, \quad Y_j \simeq \mathbb{A}^{r_i - j} \ \text{ if } m+2 \leqslant j \leqslant r_i.$$

(a$_2$) $Y = Q_{m-1} - Q_{m+1}$ when r_i is even. Then the fiber of π at any point in $\widetilde{Y}$ is isomorphic to $\mathcal{B}'_{N_{\lambda^{v,i}}}$. Also $Y \simeq \mathbb{A}^{r_i - m - 1} \sqcup \mathbb{A}^{r_i - m - 1} = \mathbb{A}^{m-1} \sqcup \mathbb{A}^{m-1}$.

(b$_1$) $Y_j = C_{j-1} - C_j$ for $j \neq m+1$. Then the fiber of π at any point in $\widetilde{Y}_j$ is isomorphic to $\mathcal{B}'_{N_{\lambda^{h,i}}}$. Also we have $Y_j \simeq \mathbb{A}^{r_i - j} - \mathbb{A}^{r_i - j - 1}$.

(b$_2$) $Y = C_{r_i - m - 1} = C_m$ when r_i is odd. Then the fiber of π at any point in $\widetilde{Y}$ is isomorphic to $\mathcal{B}'_{N_{\lambda^{h,i}}}$. Also $Y \simeq \mathbb{A}^m$.

Case II. Suppose (,) is symplectic and stratify $\mathbb{P}(W_{i-1}/W_i)$ with respect to some symplectic basis, say

$$\mathbb{P}(W_{i-1}/W_i) = Z_0 \supset Z_1 \supset \cdots \supset Z_{r_i-1} \supset Z_{r_i} = \varnothing.$$

Let $Y_j = Z_{j-1} - Z_j$ be one of the strata. Then the fiber of π at any point in $\widetilde{Y}_j$ is isomorphic to $\mathcal{B}'_{N_{\lambda^{v,i}}}$. Also $Y_j \simeq \mathbb{A}^{r_i-j}$.

Let $X := \pi^{-1}(\widetilde{Y}) = (\eta \circ \pi)^{-1}(Y)$ and $X_j := \pi^{-1}(\widetilde{Y}_j) = (\eta \circ \pi)^{-1}(Y_j)$. Then we have the following lemma. (Here we do not differentiate $N_{\lambda+}$ and $N_{\lambda-}$ even when $G = \mathrm{SO}_{2n}$ and λ is very even, as they make no difference in the following statement. It is similar when $\lambda^{v,i}$ is very even. $\lambda^{h,i}$ cannot be very even in any case.) Note that $\lambda^{h,i}$ and $\lambda^{v,i}$ correspond to some nilpotent elements in $\mathfrak{g}'$.

Lemma 5.1. *Let z_a (resp. z'_a) be the generators of $\widetilde{A}_\lambda$ (resp. $\widetilde{A}_{\lambda^{h,i}}$ or $\widetilde{A}_{\lambda^{v,i}}$) following the notations of* 2.9. *Then we have the following isomorphisms of $\widetilde{A}_\lambda$-modules.*

Case I:

(a$_1$) $H^k_c(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j}(\lambda^{v,i})$.
(a$_2$) $H^k_c(X) \simeq H^{k-2\dim W_i - 2\dim Y}(\lambda^{v,i}) \oplus H^{k-2\dim W_i - 2\dim Y}(\lambda^{v,i})$.
(b$_1$) *If k is even,* $H^k_c(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j}(\lambda^{h,i})^\tau$.
If k is odd, $H^k_c(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j+1}(\lambda^{h,i})^\tau$.
(b$_2$) $H^k_c(X) \simeq H^{k-2\dim W_i - 2\dim Y}(\lambda^{h,i})$.

Case II: $H^k_c(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j}(\lambda^{v,i})$.

Here $\tau = z'_i z'_{i-2} \in \widetilde{A}_{\lambda^{h,i}}$. In each case the action of $z_a \in \widetilde{A}_\lambda$ on the right-hand side is defined as follows.

Case I:

(a$_1$) *z_a acts by z'_a.*
(a$_2$) *z_a acts by z'_a for $a \neq i$, and z_i permutes two summands.*
(b$_1$) *z_a acts by $z'_a \mod \tau$ unless r_i is even, $j = r_i/2$, $a = i$, and k is odd, in which case z_i acts by $-z'_i \mod \tau$, i.e., $v \mapsto -z'_i(v)$.* (*Note that the action of z'_i and z'_{i-2} on the right-hand side are the same.*)
(b$_2$) *z_a acts by z'_a for $a \neq i$, and z_i acts by z'_{i-2}.*

Case II: *z_a acts by z'_a.*

Proof. This is exactly [18, Prop. 2.4] with some correction. (See also [24, Lem. 3.3.1].) This formula has an error on the description of the action of $\widetilde{A}_\lambda$ in Case I.(b_1); it differs by a sign from our description in some special case. This sign comes from the fact that the action of z_i on Y_j induces -1 on $H_c^{2\dim Y_j - 1}(Y_j)$, which is equivalent to the reciprocal map $z \mapsto 1/z$ on $\mathbb{A}^1 - \{0\}$. □

In fact, we have a similar result of Lemma 5.1 for W'-modules, using the argument in the previous section (mainly based on [8, Thm. 5.1] and its proof). Thus for any $Z \subset \mathcal{P}_N$, $H^i(\pi^{-1}(Z))$ has a natural W'-module structure which comes from the Springer representations corresponding to G'. Now together with Lemma 5.1 we have the following. (As before we do not differentiate $N_{\lambda+}$ and $N_{\lambda-}$ even when $G = \mathrm{SO}_{2n}$ and λ is very even. However, we need to be careful when $\lambda^{v,i}$ is very even.)

Proposition 5.2. *There are the natural W'-module structures on $H_c^k(X)$ and $H_c^k(X_j)$, such that we have the following isomorphisms of $W' \times A_\lambda$-modules.*

Case I:

(a$_1$) $H_c^k(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j}(\lambda^{v,i})$.

(a$_2$) *If $G \neq \mathrm{SO}_{2n}$ or $\lambda^{v,i}$ is not very even, then*

$$H_c^k(X) \simeq H^{k-2\dim W_i - 2\dim Y}(\lambda^{v,i}) \oplus H^{k-2\dim W_i - 2\dim Y}(\lambda^{v,i}).$$

If $G = \mathrm{SO}_{2n}$ and $\lambda^{v,i}$ is very even, then

$$H_c^k(X) \simeq H^{k-2\dim W_i - 2\dim Y}(\lambda^{v,i}+) \oplus H^{k-2\dim W_i - 2\dim Y}(\lambda^{v,i}-).$$

(b$_1$) *If k is even,* $H_c^k(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j}(\lambda^{h,i})^\tau$.
If k is odd, $H_c^k(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j+1}(\lambda^{h,i})^\tau$.

(b$_2$) $H_c^k(X) \simeq H^{k-2\dim W_i - 2\dim Y}(\lambda^{h,i})$.

Case II: $H_c^k(X_j) \simeq H^{k-2\dim W_i - 2\dim Y_j}(\lambda^{v,i})$.

Here $\tau = z_i' z_{i-2}' \in \widetilde{A}_{\lambda^{h,i}}$. The action of A_λ is the restriction of that of $\widetilde{A}_\lambda$ described in Lemma 5.1.

Proof. It can be proved similarly to Lemma 5.1, using Lemma 4.2. Also this is similar to [17, Prop. 2.8] or [18, Lem. 3.4], which only deal with cohomology on the top degree; it can be easily generalized to any degree. Note that the actions of W' and A_λ commute since the actions of W' and $A_{\lambda^{h,i}}$ or $A_{\lambda^{v,i}}$ on the cohomology of Springer fibers corresponding to G' commute by [20, 6.1]. (Note that $\tau \in A_{\lambda^{h,i}}$, thus $H^k(\lambda^{h,i})^\tau$ are still W'-modules. Also in Case I.(a_2), if $\lambda^{v,i}$ is very even, then $A_\lambda = \{*\}$. Thus the statement is still clear in this case.) □

Now we consider the long exact sequences of the cohomology with compact support of $\mathcal{B}_N$ corresponding to the stratification by X and X_j. By Proposition 5.2, these long exact sequences are defined in the category of $W' \times A_N$-modules. Thus it follows that $H^*(\mathcal{B}_N)$ is isomorphic to the sum of such $H_c^*(X)$ and $H_c^*(X_j)$ as a $W' \times A_N$-module. Furthermore, the second part of Lemma 4.2 implies that the W'-module structure on $H^*(\mathcal{B}_N)$, as a sum of such $H_c^*(X)$ and $H_c^*(X_j)$, coincides with the restriction to W' of the W-module structure defined by Springer's theory on $H^*(\mathcal{B}_N)$. (This part is completely analogous to the argument for type A in Section 4.) In sum, we have the following theorem.

Theorem 5.3. *Let $N = N_\lambda \in \mathfrak{g}$ where $\lambda = (1^{r_1}2^{r_2}\cdots)$. Define sgn_i to be the character of $W' \times A_\lambda$ such that on A_λ it is the restriction of the character of $\widetilde{A}_\lambda$ defined by*

$$\mathrm{sgn}_i(z_i) = -1, \qquad \mathrm{sgn}_i(z_j) = 1 \text{ for } i \neq j,$$

and on W' it is trivial. Also let $\tau_i := z_i' z_{i-2}' \in A_{\lambda^{h,i}}$ and $\mathbf{TSp}(\lambda^{h,i})^{\tau_i}$ be the character of $H^*(\lambda^{h,i})^{\tau_i}$. Then we have the following equalities of characters of $W' \times A_\lambda$. Here we define

$$\mathbf{TSp}(\lambda^{h,i})(-, z_\alpha) := \mathbf{TSp}(\lambda^{h,i})(-, z_\alpha') \quad \text{for } \alpha \neq i,$$
$$\mathbf{TSp}(\lambda^{v,i})(-, z_\alpha) := \mathbf{TSp}(\lambda^{v,i})(-, z_\alpha') \quad \text{for } \alpha \neq i,$$
$$\mathbf{TSp}(\lambda^{h,i})(-, z_i) := \mathbf{TSp}(\lambda^{h,i})(-, z_{i-2}'),$$
$$\mathbf{TSp}(\lambda^{v,i})(-, z_i) := \mathbf{TSp}(\lambda^{v,i})(-, z_i').$$

(a) *Let* $G = \mathrm{SO}_{2n+1}$. *Then,*

$$\begin{aligned}
&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda}\mathbf{TSp}(\lambda)\\
&\quad= \sum_{\substack{i\geqslant 2,\ i \text{ odd},\\ r_i \text{ odd}}} \big(\mathbf{TSp}(\lambda^{h,i}) + (r_i-1)\mathbf{TSp}(\lambda^{v,i})\big)\\
&\qquad+ \sum_{\substack{i\geqslant 2,\ i \text{ odd},\\ r_i \text{ even}}} \big((1-\mathrm{sgn}_i)\mathbf{TSp}(\lambda^{h,i})^{\tau_i} + (r_i-1+\mathrm{sgn}_i)\mathbf{TSp}(\lambda^{v,i})\big)\\
&\qquad+ \sum_{i \text{ even}} r_i\mathbf{TSp}(\lambda^{v,i}).
\end{aligned}$$

(b) *Let* $G = \mathrm{Sp}_{2n}$. *Then,*

$$\begin{aligned}
&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda}\mathbf{TSp}(\lambda)\\
&\quad= \sum_{\substack{i \text{ even},\\ r_i \text{ odd}}} \big(\mathbf{TSp}(\lambda^{h,i}) + (r_i-1)\mathbf{TSp}(\lambda^{v,i})\big)\\
&\qquad+ \sum_{\substack{i \text{ even},\\ r_i \text{ even}}} \big((1-\mathrm{sgn}_i)\mathbf{TSp}(\lambda^{h,i})^{\tau_i} + (r_i-1+\mathrm{sgn}_i)\mathbf{TSp}(\lambda^{v,i})\big)\\
&\qquad+ \sum_{i \text{ odd}} r_i\mathbf{TSp}(\lambda^{v,i}).
\end{aligned}$$

(c) *Let* $G = \mathrm{SO}_{2n}$ *and* λ *is not very even. Then there is at most one* $e \in \mathbb{Z}_{>0}$ *such that* $\lambda^{v,e}$ *is very even. If such* e *exists, then* e *is odd,* $r_e = 2$, *and* $A_\lambda = \{*\}$. *In this case we have*

$$\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda}\mathbf{TSp}(\lambda) = \sum_{i \text{ even}} r_i\mathbf{TSp}(\lambda^{v,i}) + \mathbf{TSp}(\lambda^{v,e}+) + \mathbf{TSp}(\lambda^{v,e}-).$$

If such e *does not exist, then we have*

$$\begin{aligned}
&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda}\mathbf{TSp}(\lambda)\\
&\quad= \sum_{\substack{i\geqslant 2,\ i \text{ odd},\\ r_i \text{ odd}}} \big(\mathbf{TSp}(\lambda^{h,i}) + (r_i-1)\mathbf{TSp}(\lambda^{v,i})\big)\\
&\qquad+ \sum_{\substack{i\geqslant 2,\ i \text{ odd},\\ r_i \text{ even}}} \big((1-\mathrm{sgn}_i)\mathbf{TSp}(\lambda^{h,i})^{\tau_i} + (r_i-1+\mathrm{sgn}_i)\mathbf{TSp}(\lambda^{v,i})\big)\\
&\qquad+ \sum_{i \text{ even}} r_i\mathbf{TSp}(\lambda^{v,i}).
\end{aligned}$$

If λ *is very even, then* $A_\lambda = \{*\}$ *and we have*

$$\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda}\mathbf{TSp}(\lambda+) = \mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda}\mathbf{TSp}(\lambda-) = \sum_{i \text{ even}} r_i\mathbf{TSp}(\lambda^{v,i}).$$

Now Theorem 3.1 is a corollary of Theorem 5.3 if we evaluate equations above at $(id, id) \in W' \times A_\lambda$.

Example 5.4. Let $G = \mathrm{Sp}_{12}$ and $\lambda = (6,4,2) \vdash 12$. Then we have

$$\begin{aligned}\mathbf{EC}(\lambda) &= \mathbf{EC}(6,4) + \mathbf{EC}(6,2,2) + \mathbf{EC}(4,4,2)\\ &= \mathbf{EC}(6,2) + 2\mathbf{EC}(6,1,1) + 2\mathbf{EC}(4,4) + \mathbf{EC}(4,2,2) + 2\mathbf{EC}(3,3,2)\\ &= 5\mathbf{EC}(6) + \mathbf{EC}(4,2) + 4\mathbf{EC}(4,1,1) + 6\mathbf{EC}(3,3) + 5\mathbf{EC}(2,2,2)\\ &= 14\mathbf{EC}(4) + 18\mathbf{EC}(2,2) + 14\mathbf{EC}(2,1,1)\\ &= 42\mathbf{EC}(2) + 50\mathbf{EC}(1,1) = 142.\end{aligned}$$

6. Betti numbers in some special cases

In some special situation, we have not only the restriction of total Springer representations, i.e., the alternating sum $H^*(\mathcal{B}_N)$ of cohomology, but also that of each degree of the cohomology. If $G = \mathrm{GL}_n$, it is already given in Theorem 4.4. Thus from now on we assume $G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}$, or SO_{2n} and find analogous formulas.

We assume each of the following cases. Let $\lambda = (1^{r_1}2^{r_2}\cdots)$ and $\xi \geqslant 1$ be the smallest integer such that $r_\xi \neq 0$.

(a) $G = \mathrm{SO}_{2n+1}$ or SO_{2n}.
- (a$_1$) ξ is even and $r_i \in \{0,1\}$ for any odd i.
- (a$_2$) $\xi > 1$ is odd, $r_i \in \{0,1\}$ for odd i different from ξ, and $r_\xi \in \{1,2,3\}$.
- (a$_3$) $\xi = 1$, $r_1 = 1$, and λ^1 (see 2.11) satisfies either (a$_1$) or (a$_2$).
- (a$_4$) $\xi = 1$, $r_1 > 1$, and $r_i \in \{0,1\}$ for odd i different from 1.

(b) $G = \mathrm{Sp}_{2n}$.
- (b$_1$) ξ is odd and $r_i \in \{0,1\}$ for any even i.
- (b$_2$) ξ is even, $r_i \in \{0,1\}$ for even i different from ξ, and $r_\xi \in \{1,2,3\}$.

Recall that in the proof of Theorem 5.3 we used the long exact sequences of $W' \times A_N$-modules to show that $H^*(\mathcal{B}_N)$ is the sum of alternating sums of the cohomology of each stratum in $\mathcal{B}_N$. In each case above, one can easily check that such a long exact sequence splits into short exact sequences using the fact that Springer fibers have vanishing odd cohomology [2]. (In general it is no longer true since X_j in Section 5 does not necessarily have such vanishing properties.) Thus in this case we have the following theorem.

Theorem 6.1. *Suppose $G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}$, or SO_{2n}. Let $\lambda = (1^{r_1}2^{r_2}\cdots)$ be a partition and $N = N_\lambda \in \mathfrak{g}$. We define $d_i := \sum_{j>i} r_j$. Also recall the definition of sgn_i and $\tau_i \in A_{\lambda^{h,i}}$ in Theorem 5.3. Here we abuse notations to denote by $H^k(\lambda), H^k(\lambda^{h,i}), H^k(\lambda^{v,i})$ the character corresponding to each representation and define the character values of elements in A_λ on $H^k(\lambda^{h,i}), H^k(\lambda^{v,i})$ as follows:*

$$\begin{aligned}&H^k(\lambda^{h,i})(-,z_\alpha) := H^k(\lambda^{h,i})(-,z'_\alpha) \quad \text{for } \alpha \neq i,\\ &H^k(\lambda^{v,i})(-,z_\alpha) := H^k(\lambda^{v,i})(-,z'_\alpha) \quad \text{for } \alpha \neq i,\\ &H^k(\lambda^{h,i})(-,z_i) := H^k(\lambda^{h,i})(-,z'_{i-2}),\\ &H^k(\lambda^{v,i})(-,z_i) := H^k(\lambda^{v,i})(-,z'_i).\end{aligned}$$

Then we have equalities of characters of $W' \times A_\lambda$ as follows.

(a) *Assume* $G = \mathrm{SO}_{2n+1}$ *or* SO_{2n}.

(a$_1$) *If* ξ *is even and* $r_i \in \{0,1\}$ *for any odd* i, *then*

$$\begin{aligned}&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\ &\quad= \sum_{\substack{i>1 \text{ odd},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ even}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i}).\end{aligned}$$

(a$_2$) *Suppose* ξ *is odd,* $\xi>1$, $r_i\in\{0,1\}$ *for odd* i *different from* ξ, *and* $r_\xi \in \{1,2,3\}$. *If* $r_\xi=1$, *then the formula in* (a$_1$) *is still valid. If* $r_\xi=2$, *then*

$$\begin{aligned}&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\ &\quad= \sum_{\substack{i>\xi \text{ odd},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ even}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i})\\ &\qquad + H^{k-2d_\xi-2}(\lambda^{h,\xi})^{\tau_\xi} - \mathrm{sgn}_\xi\, H^{k-2d_\xi}(\lambda^{h,\xi})^{\tau_\xi}\\ &\qquad + (1+\mathrm{sgn}_\xi) H^{k-2d_\xi}(\lambda^{v,\xi}).\end{aligned}$$

If $r_\xi = 3$, *then*

$$\begin{aligned}&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\ &\quad= \sum_{\substack{i>\xi \text{ odd},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ even}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i})\\ &\qquad + H^{k-2d_\xi-4}(\lambda^{h,\xi})^{\tau_\xi} - H^{k-2d_\xi-2}(\lambda^{h,\xi})^{\tau_\xi} + H^{k-2d_\xi-2}(\lambda^{h,\xi})\\ &\qquad + H^{k-2d_\xi}(\lambda^{v,\xi}) + H^{k-2d_\xi-2}(\lambda^{v,\xi}).\end{aligned}$$

(a$_3$) *Suppose* $\xi = 1$, $r_1 = 1$, *and* λ^1 *satisfies* (a$_1$) (*resp.* (a$_2$)). *Then the formula in* (a$_1$) (*resp.* (a$_2$)) *is still valid if we replace* ξ *by the smallest integer* ξ' *such that* $r_{\xi'} \neq 0$ *and* $\xi' > 1$.

(a$_4$) *Suppose* $\xi=1$, $r_1>1$, *and* $r_i\in\{0,1\}$ *for odd* i *different from* 1. *If* r_1 *is odd, then*

$$\begin{aligned}&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\ &\quad= \sum_{\substack{i>1 \text{ odd},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ even}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i})\\ &\qquad + \sum_{j=0}^{r_1-2} H^{k-2d_1-2j}(\lambda^{v,1}).\end{aligned}$$

If r_1 is even, then

$$\begin{aligned}
&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\
&\quad= \sum_{\substack{i>1 \text{ odd},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ even}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i})\\
&\qquad+ \sum_{j=0}^{r_1-2} H^{k-2d_1-2j}(\lambda^{v,1}) + \mathrm{sgn}_1\, H^{k-2d_1-r_1+2}(\lambda^{v,1}).
\end{aligned}$$

(b) *Assume $G = \mathrm{Sp}_{2n}$.*

(b$_1$) *If ξ is odd and $r_i \in \{0,1\}$ for any even i, then*

$$\begin{aligned}
&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\
&\quad= \sum_{\substack{i \text{ even},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ odd}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i}).
\end{aligned}$$

(b$_2$) *Suppose ξ is even, $r_i \in \{0,1\}$ for even i different from ξ, and $r_\xi \in \{1,2,3\}$. If $r_\xi = 1$, then the formula in* (b$_1$) *is still valid. If $r_\xi = 2$, then*

$$\begin{aligned}
&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\
&\quad= \sum_{\substack{i>\xi \text{ even},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ odd}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i})\\
&\qquad+ H^{k-2d_\xi-2}(\lambda^{h,\xi})^{\tau_\xi} - \mathrm{sgn}_\xi\, H^{k-2d_\xi}(\lambda^{h,\xi})^{\tau_\xi}\\
&\qquad+ (1+\mathrm{sgn}_\xi) H^{k-2d_\xi}(\lambda^{v,\xi}).
\end{aligned}$$

If $r_\xi = 3$, then

$$\begin{aligned}
&\mathrm{Res}^{W\times A_\lambda}_{W'\times A_\lambda} H^k(\lambda)\\
&\quad= \sum_{\substack{i>\xi \text{ even},\\ r_i=1}} H^{k-2d_i}(\lambda^{h,i}) + \sum_{i \text{ odd}} \sum_{j=0}^{r_i-1} H^{k-2d_i-2j}(\lambda^{v,i})\\
&\qquad+ H^{k-2d_\xi-4}(\lambda^{h,\xi})^{\tau_\xi} - H^{k-2d_\xi-2}(\lambda^{h,\xi})^{\tau_\xi}\\
&\qquad+ H^{k-2d_\xi-2}(\lambda^{h,\xi}) + H^{k-2d_\xi}(\lambda^{v,\xi}) + H^{k-2d_\xi-2}(\lambda^{v,\xi}).
\end{aligned}$$

In Section 8 we use these formulas to calculate the Betti numbers of Springer fibers corresponding to two-row partitions.

7. Short proof of the main theorem

In fact, the proof is simpler if we only want to prove Theorem 3.1. First we recall the following induction statement of Springer representations from [13, Thm. 1.3].

Proposition 7.1. *Let L be a Levi subgroup of a parabolic subgroup of G with its Lie algebra $\mathfrak{l}$. Let W_L be the Weyl group of L with a natural embedding $W_L \hookrightarrow W$. Let $\mathcal{B}_L$ be the variety of Borel subgroups of L. Then for $N \in \mathfrak{l} \subset \mathfrak{g}$ we have*

$$H^*(\mathcal{B}_N) \simeq \mathrm{Ind}_{W_L}^{W} H^*((\mathcal{B}_L)_N)$$

as W_L-modules.

Suppose $G = \mathrm{GL}_n$. As every nilpotent element in $\mathfrak{g}$ is a regular element in a Levi subalgebra of some parabolic subalgebra of $\mathfrak{g}$, for $\lambda = (1^{r_1}2^{r_2}\cdots)$ it is easy to show that

$$\mathbf{EC}(\lambda) = \frac{n!}{(1!)^{r_1}(2!)^{r_2}\cdots}.$$

Thus Theorem 3.1 follows from easy induction on n.

Now if $G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}$, or SO_{2n}, using Proposition 7.1 it suffices to show the statement when the given nilpotent element is distinguished. In this case we follow the argument in Section 5, which is a bit simpler than considering all the cases.

Remark 3. Note that the vanishing of odd cohomology of Springer fibers [2] implies the positivity of Euler characteristics of Springer fibers. Meanwhile, we do not need this fact in the proof of Theorem 4.1 and 5.3. Thus it gives another proof that the Euler characteristic of any Springer fiber is positive without using [2] (at least when G is of classical type).

8. Closed formula for the Betti numbers in two-row cases

Let $N = N_\lambda \in \mathfrak{g}$ be a nilpotent element corresponding to λ, which consists of two rows (or if $G = \mathrm{SO}_{2n+1}$, we assume that it consists of two rows and an additional row of length 1). Here we use Theorem 4.4 and Theorem 6.1 to give closed formulas for the multiplicities of irreducible representations of A_λ in each $H^k(\lambda)$. As a result, we also have formulas for Betti numbers of Springer fibers of such type. There are many results about the geometry of such Springer fibers, e.g., [6], [9], [5], [23], [15], [3], [25], [26], etc.

Proposition 8.1. *Let $G = \mathrm{GL}_n$ and $\lambda = (i,j) \vdash n$ such that $i \geqslant j \geqslant 0$. Then*

$$h^{2k}(\lambda) = \binom{i+j}{k} - \binom{i+j}{k-1}$$

for $0 \leqslant k \leqslant j$ and 0 otherwise.

Remark 4. This is also calculated in [4, Example 4.5].

Proof. We use induction on the rank $n = i+j$. It is trivial when $n = 1$. Now suppose $n \geqslant 2$ and that the statement is true up to rank $n-1$. If $\lambda = (i,j) \vdash n$ with $i > j$, then by Theorem 4.4 we have

$$h^{2k}(\lambda) = h^{2k}(i-1,j) + h^{2k-2}(i,j-1),$$

thus $h^{2k}(\lambda) = 0$ unless $0 \leqslant k \leqslant j$. If k is in this range then

$$h^{2k}(\lambda) = \binom{i+j-1}{k} - \binom{i+j-1}{k-2} = \binom{i+j}{k} - \binom{i+j}{k-1}.$$

Likewise, if $\lambda = (i,i)$ then

$$h^{2k}(\lambda) = h^{2k}(i,i-1) + h^{2k-2}(i,i-1) = \binom{2i}{k} - \binom{2i}{k-1}$$

if $0 \leqslant k \leqslant i$. (Note that this is true even when $k = i$ as $\binom{2i-1}{i-1} - \binom{2i-1}{i-2} = \binom{2i}{i} - \binom{2i}{i-1}$.) □

Let $G = \mathrm{SO}_{2n+1}$ and $\lambda = (i,j,1) \vdash 2n+1$ where $i \geqslant j \geqslant 1$. Then A_λ is isomorphic to

$\mathbb{Z}/2 \times \mathbb{Z}/2$	if $i > j > 1$,
$\mathbb{Z}/2$	if i odd and either $i = j > 1$ or $i > j = 1$, and
trivial	otherwise.

If $i > j > 1$, then we let $z_i z_j$ (resp. $z_j z_1$) be the generator of the first (resp. second) factor of $\mathbb{Z}/2$. Then we set

$$h^k(\lambda)_{+,+} := \langle H^k(\lambda), \mathrm{Id} \times \mathrm{Id}\rangle_{A_\lambda}, \quad h^k(\lambda)_{+,-} := \langle H^k(\lambda), \mathrm{Id} \times \mathrm{sgn}\rangle_{A_\lambda},$$
$$h^k(\lambda)_{-,+} := \langle H^k(\lambda), \mathrm{sgn} \times \mathrm{Id}\rangle_{A_\lambda}, \quad h^k(\lambda)_{-,-} := \langle H^k(\lambda), \mathrm{sgn} \times \mathrm{sgn}\rangle_{A_\lambda},$$

to be the multiplicity of $\mathrm{Id}\times\mathrm{Id}, \mathrm{Id}\times\mathrm{sgn}, \mathrm{sgn}\times\mathrm{Id}, \mathrm{sgn}\times\mathrm{sgn}$, respectively, in $H^k(\lambda)$ as an A_λ-representation. If $i = j > 1$ is odd, then we write

$$h^k(\lambda)_{+,+} := \langle H^k(\lambda), \mathrm{Id}\rangle_{A_\lambda}, \quad h^k(\lambda)_{+,-} := \langle H^k(\lambda), \mathrm{sgn}\rangle_{A_\lambda}$$

to be the multiplicity of $\mathrm{Id}, \mathrm{sgn}$, respectively, in $H^k(\lambda)$, and set $h^k(\lambda)_{-,+} = h^k(\lambda)_{-,-} = 0$. If i is odd and $j = 1$, then we let

$$H^k(\lambda)_{+,+} := \langle H^k(\lambda), \mathrm{Id}\rangle_{A_\lambda}, \quad h^k(\lambda)_{-,+} := \langle H^k(\lambda), \mathrm{sgn}\rangle_{A_\lambda}$$

be the multiplicity of $\mathrm{Id}, \mathrm{sgn}$, respectively, in $H^k(\lambda)$, and set $h^k(\lambda)_{+,-} = h^k(\lambda)_{-,-} = 0$. If A_λ is trivial, we set $h^k(\lambda)_{+,+} = h^k(\lambda)$ and $h^k(\lambda)_{+,-} = h^k(\lambda)_{-,+} = h^k(\lambda)_{-,-} = 0$. (The reason for using these weird notations will be apparent immediately.) Then we have the following.

Proposition 8.2. *Let $G = \mathrm{SO}_{2n+1}$ and $\lambda = (i,j,1) \vdash 2n+1$ such that $i \geqslant j \geqslant 1$, and i,j are odd unless $i = j$. Then $h^\alpha(\lambda) = 0$ unless $0 \leqslant \alpha \leqslant j+2$ and α even. From now on we assume $\alpha = 2k$ satisfies this condition.*

(a) *If $i > j > 1$, then we have*

$$h^{2k}(\lambda)_{+,+} = \binom{(i+j)/2}{k}, \quad h^{2k}(\lambda)_{+,-} = \binom{(i+j)/2}{k-2}, \quad h^{2k}(\lambda)_{-,-} = 0.$$

Also, $h^{2k}(\lambda)_{-,+} = 0$ unless $2k = j+1$, in which case

$$h^{j+1}(\lambda)_{-,+} = \frac{i-j}{i+j+2}\binom{(i+j+2)/2}{(j+1)/2}.$$

(b) *If $i = j$ is odd and $i > 1$, then*

$$h^{2k}(\lambda)_{+,+} = \binom{i}{k}, \quad h^{2k}(\lambda)_{+,-} = \binom{i}{k-2}.$$

(c) *If $i > 1$ is odd and $j = 1$, then*

$$h^{2k}(\lambda)_{+,+} = \binom{(i+1)/2}{k}.$$

Also $h^{2k}(\lambda)_{-,+} = 0$ unless $2k = 2$, in which case $h^2(\lambda)_{-,+} = (i-1)/2$.
(d) *If $i = j > 1$ is even, then $h^{2k}(\lambda) = \binom{i}{k}$.*
(e) $h^0((1,1,1)) = h^2((1,1,1)) = 1$.

Thus, we always have $h^{2k}(\lambda)_{-,-} = 0$ and

$$h^{2k}(\lambda)_{+,+} = \binom{(i+j)/2}{k}, \quad h^{2k}(\lambda)_{+,-} = \binom{(i+j)/2}{k-2},$$
$$h^{2k}(\lambda)_{-,+} = \delta_{2k,j+1}\frac{i-j}{i+j+2}\binom{(i+j+2)/2}{(j+1)/2}$$

if they are not a priori zero.

Proof. From Theorem 6.1 we have the following relations.

(a) If $i > j > 1$, then

$$h^{2k}(\lambda)_* = h^{2k}(i-2, j, 1)_* + h^{2k-2}(i, j-2, 1)_*$$

where $*$ can be any of $(+,+), (+,-), (-,+), (-,-)$.

(b) If $i = j$ is odd and $i > 1$, then

$$h^{2k}(\lambda)_{+,+} = h^{2k}(i-1, i-1, 1)_{+,+} + h^{2k-2}(i, i-2, 1)_{+,+} - h^{2k}(i, i-2, 1)_{+,-},$$
$$h^{2k}(\lambda)_{+,-} = h^{2k}(i-1, i-1, 1)_{+,+} + h^{2k-2}(i, i-2, 1)_{+,-} - h^{2k}(i, i-2, 1)_{+,+}.$$

(c) If $i > 1$ is odd and $j = 1$, then

$$h^{2k}(\lambda)_* = h^{2k}(i-2, 1, 1)_* + \delta_{k,1},$$

where $*$ is either $(+,+)$ or $(-,+)$.

(d) If $i = j$ is even, then

$$h^{2k}(\lambda) = h^{2k-2}(i-1, i-1, 1)_{+,+} + h^{2k}(i-1, i-1, 1)_{+,+}.$$

(e) If $i = j = 1$, then $h^0((1,1,1)) = h^2((1,1,1)) = 1$.

Now the result follows from induction on the rank n. $\square$

If $G = \mathrm{Sp}_{2n}$, it is known that the A_N-action on $H^k(\mathcal{B}_N)$ factors through the quotient by the image of $\pm I \in \mathrm{Sp}_{2n}$, where the image of $-I$ is $\prod_i z_i^{r_i} \in A_N$. We denote such a quotient by $\overline{A}_N$. If $N = N_\lambda$ for $\lambda = (i, j) \vdash n$, then $\overline{A}_\lambda \simeq \mathbb{Z}/2$ if $i, j > 0$ are both even, and otherwise $\overline{A}_\lambda$ is trivial. If $\overline{A}_\lambda \simeq \mathbb{Z}/2$, then we let $h^k(\lambda)_{\mathrm{Id}}, h^k(\lambda)_{\mathrm{sgn}}$ be the multiplicities of $\mathrm{Id}, \mathrm{sgn}$, respectively, in $H^k(\lambda)$ as a $\overline{A}_\lambda$-module. Thus in particular $h^k(\lambda) = h^k(\lambda)_{\mathrm{Id}} + h^k(\lambda)_{\mathrm{sgn}}$. If $\overline{A}_\lambda$ is trivial, we set $h^k(\lambda)_{\mathrm{Id}} := h^k(\lambda)$ and $h^k(\lambda)_{\mathrm{sgn}} := 0$.

Proposition 8.3. *Let $G = \mathrm{Sp}_{2n}$ and $\lambda = (i, j) \vdash 2n$ such that $i \geqslant j \geqslant 0$, and i, j are even unless $i = j$. Then we have $h^{2k}(\lambda)_{id} = 0$ unless $0 \leqslant 2k \leqslant j + 1$, in which case*

$$\text{if } 0 \leqslant 2k \leqslant j, \quad h^{2k}(\lambda)_{\mathrm{Id}} = \binom{\lfloor (i+1)/2 \rfloor + \lfloor (j+1)/2 \rfloor}{k},$$
$$\text{if } 2k = j + 1, \quad h^{2k}(\lambda)_{\mathrm{Id}} = \frac{1}{2}\binom{\lfloor (i+1)/2 \rfloor + \lfloor (j+1)/2 \rfloor}{k}.$$

If i, j are both even, then $h^{2k}_{\mathrm{sgn}}(\lambda) = 0$ unless $0 \leqslant 2k \leqslant j$, in which case

$$h^{2k}(\lambda)_{\mathrm{sgn}} = \binom{(i+j)/2}{k-1}.$$

Proof. From Theorem 6.1 we have the following relations.

(a) If $i > j$, then

$$h^{2k}(\lambda)_{\mathrm{Id}} = h^{2k}(i-2, j)_{\mathrm{Id}} + h^{2k-2}(i, j-2)_{\mathrm{Id}},$$
$$h^{2k}(\lambda)_{\mathrm{sgn}} = h^{2k}(i-2, j)_{\mathrm{sgn}} + h^{2k-2}(i, j-2)_{\mathrm{sgn}}.$$

(b) If $i = j$ is even, then

$$h^{2k}(\lambda)_{\mathrm{Id}} = h^{2k-2}(i, i-2)_{\mathrm{Id}} - h^{2k}(i, i-2)_{\mathrm{sgn}} + h^{2k}(i-1, j-1),$$
$$h^{2k}(\lambda)_{\mathrm{sgn}} = h^{2k-2}(i, i-2)_{\mathrm{sgn}} - h^{2k}(i, i-2)_{\mathrm{Id}} + h^{2k}(i-1, j-1),$$

(c) If $i = j$ is odd, then

$$h^{2k}(\lambda) = h^{2k-2}(i-1, i-1)_{\mathrm{Id}} + h^{2k-2}(i-1, i-1)_{\mathrm{sgn}} + h^{2k}(i-1, i-1)_{\mathrm{Id}} + h^{2k}(i-1, i-1)_{\mathrm{sgn}}.$$

Now the result follows from easy induction on n. □

If $G = \mathrm{SO}_{2n}$, it is known that the A_N-action on $H^k(\mathcal{B}_N)$ factors through the quotient by the image of $\pm I \in \mathrm{SO}_{2n}$, where the image of $-I$ is $\prod_i z_i^{r_i} \in A_N$. We denote such a quotient by $\overline{A}_N$. If $N = N_\lambda$ for $\lambda = (i, j) \vdash n$, then $\overline{A}_\lambda$ is trivial.

Proposition 8.4. *Let $G = \mathrm{SO}_{2n}$ and $\lambda = (i, j) \vdash 2n$ such that $i \geqslant j \geqslant 0$, and i, j are odd unless $i = j$. Then we have $h^{2k}(\lambda) = 0$ unless $0 \leqslant 2k \leqslant j$, in which case*

$$\text{if } 0 \leqslant 2k \leqslant j - 1, \quad h^{2k}(\lambda) = \binom{(i+j)/2}{k},$$
$$\text{if } 2k = j, \quad h^{2k}(\lambda) = \frac{1}{2}\binom{(i+j)/2}{k}.$$

Proof. From Theorem 6.1 we have the following relations.

(a) If $i > j$, then

$$h^{2k}(\lambda) = h^{2k}(i-2, j) + h^{2k-2}(i, j-2).$$

(b) If $i = j$ is odd, then

$$h^{2k}(\lambda) = h^{2k-2}(i, j-2) - h^{2k}(i, j-2) + 2h^{2k}(i-1, j-1).$$

(c) If $i = j$ is even, then

$$h^{2k}(\lambda) = h^{2k-2}(i-1, j-1) + h^{2k}(i-1, j-1).$$

Now the result follows from easy induction on n. □

Remark 5. When $G = \mathrm{Sp}_{2n}$ or SO_{2n}, the cohomology rings of Springer fibers corresponding to two-row partitions are described in [3, Thm. B] and [26, Thm. A]. Thus their Betti numbers can also be deduced from their descriptions.

A. Combinatorial proof of Theorem 4.1

Proof of Theorem 4.1. We refer to [22, Chap. 7.18]. We let $\mathcal{S}_i$ be the symmetric group of i elements. Define CF_i to be the $\mathbb{Q}$-vector space spanned by irreducible characters of $\mathcal{S}_i$ and let $CF := \bigoplus_{i\in\mathbb{N}} CF_i$. We introduce a ring structure on CF so that for $f \in CF_i$ and $g \in CF_j$, we define $f \circ g := \mathrm{Ind}_{\mathcal{S}_i\times\mathcal{S}_j}^{\mathcal{S}_{i+j}} f \times g$ and extend it to CF by linearity. Also on each CF_i we have a well-defined inner product $\langle\ ,\ \rangle$, which we extend to CF be decreeing that $\langle f, g\rangle = 0$ for $f \in CF_i, g \in CF_j$ with $i \neq j$.

Then we have a well-defined ring isomorphism

$$\mathrm{ch} : CF \to \Lambda$$

where Λ is the ring of symmetric polynomials of $x_1, x_2, \cdots$ with coefficients in $\mathbb{Q}$. It is uniquely defined by the condition that it sends $\mathrm{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_n} \mathrm{Id}_{\mathcal{S}_\lambda}$ to h_λ, the homogeneous symmetric polynomial corresponding to λ. Here $\mathcal{S}_\lambda := \mathcal{S}_1^{r_1} \times \mathcal{S}_2^{r_2} \times \cdots \subset \mathcal{S}_n$ if $\lambda = (1^{r_1}2^{r_2}\cdots) \vdash n$. Also ch respects inner products on both rings, where the inner product on Λ is defined by $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$ and extended by linearity. Here m_μ is the monomial symmetric polynomial corresponding to μ.

In order to prove the statement, it suffices to show that for any class function f of $\mathcal{S}_{n-1}$

$$\left\langle \mathbf{TSp}(\lambda), \mathrm{Ind}_{\mathcal{S}_{n-1}}^{\mathcal{S}_n} f\right\rangle = \left\langle \bigoplus_{r_i\geqslant 1} r_i \mathbf{TSp}(\lambda^i), f\right\rangle.$$

Note that $\mathbf{TSp}(\lambda) = \mathrm{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_n} \mathrm{Id}_{\mathcal{S}_\lambda}$ by Proposition 7.1. If we apply ch, then it is equivalent to

$$\langle h_\lambda, \mathrm{ch}(f)h_1\rangle = \left\langle \sum_{r_i\geqslant 1} r_i h_{\lambda^i}, \mathrm{ch}(f)\right\rangle$$

As monomial symmetric functions m_μ for $\mu \vdash n-1$ are a basis of CF_{n-1}, it suffices to check that the above formula is true when $\mathrm{ch}(f) = m_\mu$ for any $\mu \vdash n-1$. By the definition of inner product on Λ we have $\left\langle \sum_{r_i \geqslant 1} r_i h_{\lambda^i}, m_\mu \right\rangle = r_i$ if $\mu = \lambda^i$ for some $i \in \mathbb{N}$ and 0 otherwise. On the other hand, for $\mu = (1^{r'_1} 2^{r'_2} \cdots)$ it is easy to show that

$$m_\mu h_1 = \sum_{r'_i \geqslant 1} (r'_{i+1}+1) m_{(1^{r'_1} 2^{r'_2} \cdots i^{r'_i - 1} (i+1)^{r'_{i+1}+1} \cdots)} + (r'_1+1) m_{(1^{r'_1+1} 2^{r'_2} \cdots)},$$

thus $\langle h_\lambda, m_\mu h_1 \rangle = r_i$ if $\mu = \lambda^i$ for some $i \in \mathbb{N}$ and 0 otherwise. It suffices for the proof. $\square$

B. Exceptional types

In this section we assume that G is a reductive group of exceptional type, i.e., of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4$, or G_2, over $\mathbf{k}$ such that char $\mathbf{k}$ is good. Then the corresponding Green functions are completely known. The following tables give the multiplicities of each irreducible character of A_N in $H^*(\mathcal{B}_N)$ for any nilpotent $N \in \mathfrak{g}$ if G is of exceptional type, using the data of Green functions. Here we use the tables of Green functions given in [10]. Each column in the tables represents the following.

(a) N: the type of a nilpotent element $N \in \mathfrak{g}$. We use the Bala-Carter notation here.
(b) A_N: the component group of the stabilizer of N in G. A dot (.) means that A_N is trivial. Otherwise A_N is either $\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, or $\mathcal{S}_5$, where $\mathcal{S}_n$ is the symmetric group permuting n elements.
(c) ϕ: an irreducible character of A_N, i.e., $\phi \in \widehat{A_N}$. If A_N is trivial, then ϕ is the identity, and we put a dot (.) in this case. Otherwise if $A_N = \mathcal{S}_n$, then we put the partition $\lambda \vdash n$ which parametrizes ϕ. For example, n means the identity character and $11 \cdots 1$ means the sign character.
(d) mult: the multiplicity of ϕ in $H^*(\mathcal{B}_N)$, i.e., $\langle H^*(\mathcal{B}_N), \phi \rangle$.

Note that in each case the Euler characteristic of $\mathcal{B}_N$ is given by

$$\chi(\mathcal{B}_N) = \sum_{\phi \in \widehat{A_N}} (\dim \phi) \, \langle H^*(\mathcal{B}_N), \phi \rangle_{A_N}$$

where the sum is over all the irreducible representations of A_N.

TABLE 1. Type E_6

N	A_N	ϕ	mult
E_6	.	.	1
$\mathsf{E}_6(a_1)$	.	.	7
D_5	.	.	27
$\mathsf{A}_5+\mathsf{A}_1$	$\mathcal{S}_2$	2 11	57 21
A_5	.	.	72
$\mathsf{D}_5(a_1)$	.	.	162
$\mathsf{A}_4+\mathsf{A}_1$	.	.	216
D_4	.	.	270
A_4	.	.	432
$\mathsf{D}_4(a_1)$	$\mathcal{S}_3$	3 21 111	575 370 35
$\mathsf{A}_3+\mathsf{A}_1$	.	.	1080
$2\mathsf{A}_2+\mathsf{A}_1$	.	.	720
A_3	.	.	2160
$\mathsf{A}_2+2\mathsf{A}_1$	.	.	2160
$2\mathsf{A}_2$	.	.	1440
$\mathsf{A}_2+\mathsf{A}_1$	.	.	4320
A_2	$\mathcal{S}_2$	2 11	5940 2700
$3\mathsf{A}_1$	.	.	6480
$2\mathsf{A}_1$	.	.	12960
A_1	.	.	25920
A_0	.	.	51840

Table 3. Type E_7

N	A_N	ϕ	mult
E_7	.	.	1
$\mathsf{E}_7(a_1)$	.	.	8
$\mathsf{E}_7(a_2)$	.	.	35
$\mathsf{D}_6+\mathsf{A}_1$	$\mathcal{S}_2$	2 11	91 28
E_6	.	.	56
$\mathsf{E}_6(a_1)$	$\mathcal{S}_2$	2 11	232 160
D_6	.	.	126
$\mathsf{D}_6(a_1)+\mathsf{A}_1$	$\mathcal{S}_2$	2 11	456 15
A_6	.	.	576
$\mathsf{D}_6(a_1)$	.	.	882
$\mathsf{D}_5+\mathsf{A}_1$	.	.	756
$\mathsf{D}_6(a_2)+\mathsf{A}_1$	$\mathcal{S}_3$	3 21 111	1442 826 56
D_5	.	.	1512
$(\mathsf{A}_5+\mathsf{A}_1)'$	$\mathcal{S}_2$	2 11	3192 1176
$\mathsf{D}_6(a_2)$	.	.	2772
$(\mathsf{A}_5+\mathsf{A}_1)''$	.	.	2016
A_5'	.	.	4032
$\mathsf{D}_5(a_1)+\mathsf{A}_1$	.	.	4536
$\mathsf{D}_5(a_1)$	$\mathcal{S}_2$	2 11	6426 2646
$\mathsf{A}_4+\mathsf{A}_2$	.	.	4032
$\mathsf{A}_4+\mathsf{A}_1$	$\mathcal{S}_2$	2 11	7308 4788
A_5''	.	.	4032
$\mathsf{D}_4+\mathsf{A}_1$	.	.	7560
A_4	$\mathcal{S}_2$	2 11	14616 9576
$\mathsf{A}_3+\mathsf{A}_2+\mathsf{A}_1$	.	.	10080
$\mathsf{A}_3+\mathsf{A}_2$	$\mathcal{S}_2$	2 11	19530 630
D_4	.	.	15120
$\mathsf{D}_4(a_1)+\mathsf{A}_1$	$\mathcal{S}_2$	2 11	26460 11340
$\mathsf{A}_3+2\mathsf{A}_1$	.	.	30240
$\mathsf{D}_4(a_1)$	$\mathcal{S}_3$	3 21 111	32200 20720 1960
$(\mathsf{A}_3+\mathsf{A}_1)'$	.	.	60480
$2\mathsf{A}_2+\mathsf{A}_1$	.	.	40320
$(\mathsf{A}_3+\mathsf{A}_1)''$	.	.	60480
A_3	.	.	120960
$2\mathsf{A}_2$	.	.	80640
$\mathsf{A}_2+3\mathsf{A}_1$	.	.	60480
$\mathsf{A}_2+2\mathsf{A}_1$	.	.	120960
$\mathsf{A}_2+\mathsf{A}_1$	$\mathcal{S}_2$	2 11	166320 75600
$4\mathsf{A}_1$	.	.	181440
A_2	$\mathcal{S}_2$	2 11	332640 151200
$3\mathsf{A}_1'$	.	.	362880
$3\mathsf{A}_1''$	.	.	362880
$2\mathsf{A}_1$	.	.	725760
A_1	.	.	1451520
A_0	.	.	2903040

Table 5. Type E_8

N	A_N	ϕ	mult
E_8	.	.	1
$\mathsf{E}_8(a_1)$	.	.	9
$\mathsf{E}_8(a_2)$	.	.	44
$\mathsf{E}_7+\mathsf{A}_1$	$\mathcal{S}_2$	2	156
		11	36
E_7	.	.	240
D_8	$\mathcal{S}_2$	2	366
		11	231
$\mathsf{E}_7(a_1)+\mathsf{A}_1$	$\mathcal{S}_2$	2	1010
		11	50
$\mathsf{E}_7(a_1)$	.	.	1920
$\mathsf{D}_8(a_1)$	$\mathcal{S}_2$	2	1710
		11	495
D_7	.	.	2160
$\mathsf{E}_7(a_2)+\mathsf{A}_1$	$\mathcal{S}_3$	3	4284
		21	2128
		111	84
A_8	$\mathcal{S}_3$	3	5589
		21	4263
		111	594
$\mathsf{E}_7(a_2)$	.	.	8400
$\mathsf{E}_6+\mathsf{A}_1$	.	.	6720
$\mathsf{D}_7(a_1)$	$\mathcal{S}_2$	2	15780
		11	1500
$\mathsf{D}_8(a_3)$	$\mathcal{S}_3$	3	14205
		21	175
		111	1650
$\mathsf{D}_6+\mathsf{A}_1$	$\mathcal{S}_2$	2	21840
		11	6720
$\mathsf{E}_6(a_1)+\mathsf{A}_1$	$\mathcal{S}_2$	2	27840
		11	19200
A_7	.	.	17280
E_6	.	.	13440
$\mathsf{D}_7(a_2)$	$\mathcal{S}_2$	2	38880
		11	23760
D_6	.	.	30240
$\mathsf{E}_6(a_1)$	$\mathcal{S}_2$	2	55680
		11	38400
$\mathsf{D}_5+\mathsf{A}_2$	$\mathcal{S}_2$	2	58500
		11	1980
$\mathsf{D}_6(a_1)+\mathsf{A}_1$	$\mathcal{S}_2$	2	109440
		11	3600
$\mathsf{A}_6+\mathsf{A}_1$	.	.	69120
A_6	.	.	138240
$\mathsf{D}_6(a_1)$	$\mathcal{S}_2$	2	151200
		11	60480
$2\mathsf{A}_4$	$\mathcal{S}_5$	5	135240
		41	99246
		32	52206
		311	12516
		221	7686
		2111	126
		11111	0
$\mathsf{D}_5+\mathsf{A}_1$	.	.	181440
$\mathsf{A}_5+\mathsf{A}_2$	$\mathcal{S}_3$	3	346080
		21	198240
		111	13440
$\mathsf{D}_6(a_2)$	$\mathcal{S}_2$	2	408240
		11	257040
$\mathsf{A}_5+2\mathsf{A}_1$	$\mathcal{S}_2$	2	383040
		11	141120
$(\mathsf{A}_5+\mathsf{A}_1)'$	.	.	483840
$\mathsf{D}_5(a_1)+\mathsf{A}_2$	.	.	362880
D_5	.	.	362880
$\mathsf{A}_4+\mathsf{A}_3$	.	.	241920
$(\mathsf{A}_5+\mathsf{A}_1)''$	$\mathcal{S}_2$	2	766080
		11	282240
$\mathsf{D}_4+\mathsf{A}_2$	$\mathcal{S}_2$	2	574560
		11	30240
A_5	.	.	967680
$\mathsf{D}_5(a_1)+\mathsf{A}_1$	.	.	1088640
$\mathsf{A}_4+\mathsf{A}_2+\mathsf{A}_1$	.	.	483840
$\mathsf{A}_4+\mathsf{A}_2$	.	.	967680
$\mathsf{A}_4+2\mathsf{A}_1$	$\mathcal{S}_2$	2	1103760
		11	347760
$\mathsf{D}_5(a_1)$	$\mathcal{S}_2$	2	1542240
		11	635040
$\mathsf{A}_4+\mathsf{A}_1$	$\mathcal{S}_2$	2	1753920
		11	1149120
$2\mathsf{A}_3$	.	.	1209600
$\mathsf{D}_4+\mathsf{A}_1$	.	.	1814400
$\mathsf{D}_4(a_1)+\mathsf{A}_2$	$\mathcal{S}_2$	2	2116800
		11	907200
$\mathsf{A}_3+\mathsf{A}_2+\mathsf{A}_1$	.	.	2419200
A_4	$\mathcal{S}_2$	2	3507840
		11	2298240

$A_3 + A_2$	$\mathcal{S}_2$	2	4687200
		11	151200
$D_4(a_1) + A_1$	$\mathcal{S}_3$	3	3864000
		21	2486400
		111	235200
$A_3 + 2A_1$	.	.	7257600
D_4	.	.	3628800
$2A_2 + 2A_1$	.	.	4838400
$D_4(a_1)$	$\mathcal{S}_3$	3	7728000
		21	4972800
		111	470400
$A_3 + A_1$	.	.	14515200
$2A_2 + A_1$	.	.	9676800
$2A_2$	$\mathcal{S}_2$	2	13305600
		11	6048000
$A_2 + 3A_1$	.	.	14515200
A_3	.	.	29030400
$A_2 + 2A_1$	.	.	29030400
$A_2 + A_1$	$\mathcal{S}_2$	2	39916800
		11	18144000
$4A_1$	.	.	43545600
A_2	$\mathcal{S}_2$	2	79833600
		11	36288000
$3A_1$	.	.	87091200
$2A_1$	.	.	174182400
A_1	.	.	348364800
A_0	.	.	696729600

TABLE 7. Type F_4

N	A_N	ϕ	mult
F_4	.	.	1
$F_4(a_1)$	$\mathcal{S}_2$	2	5
		11	2
$F_4(a_2)$	$\mathcal{S}_2$	2	14
		11	2
B_3	.	.	24
C_3	.	.	24
$F_4(a_3)$	$\mathcal{S}_4$	4	42
		31	19
		22	10
		211	1
		1111	0
$C_3(a_1)$	$\mathcal{S}_2$	2	96
		11	24
B_2	$\mathcal{S}_2$	2	96
		11	48
$\widetilde{A}_2 + A_1$	.	.	96
$A_2 + \widetilde{A}_1$	.	.	96
$\widetilde{A}_2$	.	.	192
A_2	$\mathcal{S}_2$	2	168
		11	24
$A_1 + \widetilde{A}_1$	.	.	288
$\widetilde{A}_1$	$\mathcal{S}_2$	2	432
		11	144
A_1	.	.	576
A_0	.	.	1152

Table 9. Type G_2

N	A_N	ϕ	mult
G_2	.	.	1
$\mathsf{G}_2(a_1)$	$\mathcal{S}_3$	3	3
		21	1
		111	0
$\widetilde{\mathsf{A}}_1$	.	.	6
A_1	.	.	6
A_0	.	.	12

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