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A Note on the Modularization of Lattices

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A Note on the Modularization of Lattices

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Abstract Valuations on finite lattices have been known for a long time. In this paper, we present a combinatorial procedure called modularization that associates a modular lattice to any given finite lattice such that they have the same valuation polytopes.

Keywords lattice, modular lattice, valuation polytope

1 Introduction and theorem statement

Poset polytopes are a class of important polytopes in algebraic and enumerative combinatorics since their geometric properties are helpful towards understanding the combinatorial properties of the posets. The most well-known polytopes among them are order polytopes and chain polytopes, introduced by R. Stanley [8], where lots of combinatorial statistics including descriptions of vertices, mixed volumes, and Ehrhart polynomials are studied. People are also interested in other poset polytopes including valuation polytopes (which are defined for lattices), flow polytopes (for which we consider a poset as a directed graph) and double order polytopes and double chain polytopes [2]. Some results regarding their Ehrhart polynomials are summarized by F. Liu [7].

In this paper, we will be looking at valuation polytopes. As a motivating fact, it is known that vertices of the valuation polytope of $J(P)$, the lattice of order ideals of a poset P , are in one-to-one correspondence with chains in P [3]. Beyond that, little is known. Let L denote a finite lattice for notation throughout the paper.

Definition 1 (valuation) A *valuation* on a finite lattice L is a function $v : L \rightarrow \mathbb{R}$ such that $v(s) + v(t) = v(s \vee t) + v(s \wedge t)$ for all $s, t \in L$.

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Definition 2 (valuation polytope) The *valuation polytope* of a finite lattice L is the space of all valuations v on L such that $v(s) \in [0, 1]$ for all $s \in L$.

If L is a modular lattice, a complete description of all valuations on L is provided in [1] (Theorem 7 on p. 236). If L is further a distributive lattice, then by the fundamental theorem for finite distributive lattices, L is isomorphic to $J(P)$, the lattice of order ideals of some poset P . Then each valuation v on L is determined uniquely by its values on the join-irreducibles of L , and can be written out nicely [9] (Ex. 3.94 on p. 374). A description of vertices of valuation polytopes of finite distributive lattices (with a slightly different definition) was conjectured by L. D. Geissinger [4] and proved by H. Dobbartin [3]. However, even in the case of finite distributive lattices, many desirable geometric properties of the valuation polytopes, including descriptions of facets, volumes and Ehrhart polynomials remain unknown except for some trivial cases [9] (Ex. 4.58 on p. 541-542).

In this paper, we are more interested in the case where L is not necessarily modular. We make a step forward by showing that there is a well-defined combinatorial procedure called “modularization” to change L into a modular lattice L_M such that their valuation polytopes are the same up to some linear equivalence. Recall that two polytopes $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ are called *linearly equivalent* if there exists a linear map $\varphi : \mathbb{R}^p \rightarrow \mathbb{R}^q$ on their ambient spaces that maps bijectively from P to Q . Notice that linear equivalence is a stronger condition than combinatorial equivalence. Thus, we reduce the general study of valuation polytopes to those of modular lattices. More formally, our main result is stated as follows.

Theorem 1 *Let L be an arbitrary finite lattice. Then there exists a modular lattice L_M such that the valuation polytope of L_M is linearly equivalent to the valuation polytope of L .*

2 Proof of main theorem

Let L be a finite lattice. We say that x is *covered by* y in L , or y *covers* x , denoted $x \lessdot y$, if $x < y$ and there doesn't exist $z \in L$ such that $x < z < y$. Let \mathcal{H} be the graph of the Hasse diagram of L which indicates all covering relations in L .

Definition 3 We define a subgraph \mathcal{H}_M , whose vertex set is the whole lattice L , of \mathcal{H} generated inductively in the following way:

1. Connect $x \lessdot y$ in \mathcal{H}_M if there exists $z \in L$ such that $x \vee z = y \vee z$ and $x \wedge z = y \wedge z$.
2. Connect $a \lessdot b$ in \mathcal{H}_M if there exists $x \lessdot y$ already connected in \mathcal{H}_M such that $a \vee y = b$ and $a \wedge y = x$ or $b \vee x = y$ and $b \wedge x = a$.
3. Repeat Step 2 until no more edges can be added to \mathcal{H}_M .

We see that \mathcal{H}_M as above is well-defined (Definition 3). In both step 1 and step 2, the order of adding edges does not make a difference since if an edge

can be added at some point, it can always be added at any later point no matter how many more edges have joined \mathcal{H}_M .

Lemma 1 *If v is a valuation on L , then v is constant on each connected component of \mathcal{H}_M .*

Proof It suffices to show that if $x < y$ is in \mathcal{H}_M , then $v(x) = v(y)$. Let's fix an order in which edges are added to \mathcal{H}_M , as in Definition 3, and use induction with respect to this order. If $x < y$ is added to \mathcal{H}_M in Step 1, then by definition, $v(x) + v(z) = v(x \vee z) + v(x \wedge z) = v(y \vee z) + v(y \wedge z) = v(y) + v(z)$, so $v(x) = v(y)$. If $a < b$ is added to \mathcal{H}_M in Step 2, then there exist $x < y$ in \mathcal{H}_M already such that $a \vee y = b$ and $a \wedge y = x$ or $b \vee x = y$ and $b \wedge x = a$. We either have $v(a) + v(y) = v(a \vee y) + v(a \wedge y) = v(b) + v(x)$ or $v(b) + v(x) = v(b \vee x) + v(b \wedge x) = v(y) + v(a)$. In both cases, by induction hypothesis that $v(x) = v(y)$, we obtain $v(a) = v(b)$.

Definition 4 (Modularization) Let L_M be the set of connected components of \mathcal{H}_M defined in Definition 3. We can define a partial order \leq_M on L_M : $A \leq_M B$ in L_M if there exists $a \in A$ and $b \in B$ such that $a \leq b$ in L . Such L_M is called the *modularization* of L .

Intuitively, \mathcal{H}_M partitions L into connected components, thus providing an equivalence relation whereas L_M is a set of equivalence classes. Our main task in this section is to show that we can indeed “quotient out” L by \mathcal{H}_M , where the resulting “quotient” L_M behaves well.

So far, our definition for \leq_M is just a binary relation that is reflexive. We have yet to justify that it is indeed a partial order. We will develop some lemmas and show that (L_M, \leq_M) is a well-defined poset and further a well-defined lattice that is modular.

Example 1 Consider the following lattice L shown in Figure 1 (left). To obtain \mathcal{H}_M , in step 1 of Definition 3, we need to connect x and y since $x \vee d = y \vee d$ and $x \wedge d = y \wedge d$. Similarly connect $y < z$. Then in step 2, we connect $a < b$ since $x < y$ is already in \mathcal{H}_M and $a \vee y = b$ and $a \wedge y = x$. Similarly add $c < d$ to \mathcal{H}_M . As we can see in this case, L_M is a chain of length 2, which is indeed a modular lattice.

Example 2 Consider another example where the lattice L is shown in Figure 2 (left), obtained from a Boolean algebra B_3 by deleting one covering relation $3 < 23$ between the middle ranks. In step 1 of Definition 3, we connect $1 < 13$ since $1 \vee 23 = 13 \vee 23 = 123$, $1 \wedge 23 = 13 \wedge 23 = \emptyset$ and similarly connect $2 < 12$ because of 3 , $3 < 13$ because of 2 , $2 < 23$ because of 13 . In step 2, we connect $\emptyset < 1$ since $3 < 13$ is already connected and that $1 \vee 3 = 13$, $1 \wedge 3 = \emptyset$, and similarly connect $0 < 3$, $12 < 123$, $23 < 123$.

We will be using notations from Definition 3 and Definition 4 throughout the section.

The following Lemma 2 plays an important role. For elements $x < y$, a *maximal chain* connecting x and y is a subgraph of \mathcal{H} of the form $x = z_0 < \dots < z_n = y$.

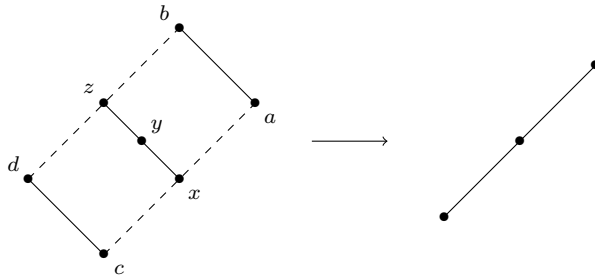


Fig. 1 Left: Hasse diagram for L with \mathcal{H}_M drawn in solid lines. Right: L_M .

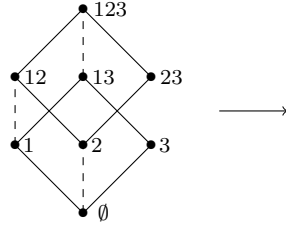


Fig. 2 Left: Hasse diagram for another example L with \mathcal{H}_M in solid lines. Right: L_M .

Lemma 2 *If $x < y$ is connected in \mathcal{H}_M and $x \leq z$, then every maximal chain connecting z and $y \vee z$ is contained in \mathcal{H}_M . Dually, if $x < y$ in \mathcal{H}_M and $z \leq y$, then every maximal chain connecting $x \wedge z$ and z is in \mathcal{H}_M .*

Proof By symmetry, it suffices to prove the first statement. Take any such maximal chain $z = z_0 < \dots < z_n = y \vee z$ and use induction on n . The base case $n = 0$ is vacuously true so assume $n \geq 1$. This means that z is not greater than y . Note that $y \wedge z$ is lower-bounded by x . Since L is a lattice and y covers x , we deduce that $x = y \wedge z$. Consider z_1 that covers z . Clearly $z_1 \vee y = y \vee z$. If $z_1 \geq y$, then $y \vee z = z_1$, meaning that $n = 1$. By Definition 3, as $x < y$ is connected in \mathcal{H}_M , $y \wedge z = x$, $y \vee z = z_1$, then $z < z_1$ must be in \mathcal{H}_M as well. If $z_1 \not\geq y$, then $z_1 \wedge y = x$. As $z \wedge y = z_1 \wedge y$ and $z \vee y = z_1 \vee y$, $z < z_1$ needs to be in \mathcal{H}_M . In both cases, $z < z_1$ is in \mathcal{H}_M and then apply induction hypothesis on the chain $z_1 < \dots < y \vee z$ yields the desired result.

Lemma 3 *Let A and B be two distinct connected components of \mathcal{H}_M . If $a \in A$ and $b \in B$ satisfies $a < b$, then there do not exist $a' \in A$ and $b' \in B$ such that $a' > b'$.*

Proof For an element $x \in L$, let $\rho(x)$ denote its *rank*, which is defined to be the maximum length of paths from $\hat{0}$ to x , where $\hat{0}$ is the minimum of L . Therefore, if $x > y$, then $\rho(x) > \rho(y)$.

Assume the opposite that for some distinct connected components A, B of \mathcal{H}_M , there exist $a, a' \in A$, $b, b' \in B$ such that $a < b$ and $a' > b'$. Let n be the smallest nonnegative integer such that such objects exist and that a and a'

can be connected to each other in \mathcal{H}_M via a path of length n . Let such path be $a = a_0, a_1, \dots, a_n = a'$. If $n \geq 1$, consider a_1 . If $a_1 \leq a_0$, then we have a clear contradiction with the minimality of n . If $a_1 \geq a_0$, by Lemma 2, $b \vee a_1$ is in the connected component B . Replacing b by $b \vee a_1$ and a by a_1 , we find a shorter path, contradicting the minimality of n as well. As a result, $n = 0$, meaning that $a' = a$.

Find such a described in the last paragraph with the smallest possible rank $\rho(a)$. As $a > b'$, we know $\rho(a) > \rho(b')$. Consider the path in \mathcal{H}_M from b to b' and let it be $b = b_0, b_1, \dots, b_m = b'$ where $b_i \leq b_{i+1}$ or $b_i \geq b_{i+1}$. Let's define a_0, \dots, a_m inductively as follows. Start with $a_0 = a$. For each $i \geq 1$,

$$a_i = \begin{cases} a_{i-1}, & \text{if } b_i \geq a_{i-1} \\ b_i \wedge a_{i-1}, & \text{if } b_i \not\geq a_{i-1} \end{cases}.$$

We then show that $a_i \in A$ and $a_i \leq b_i$ for $i = 0, \dots, m$ by induction. The base case $i = 0$ is true by construction so assume $i \geq 1$. If $b_i \geq a_{i-1}$, then $a_i = a_{i-1} \leq b_i$ and is in A . If $b_i \not\geq a_{i-1}$, as $b_{i-1} \geq a_{i-1}$ by induction hypothesis, we must have $b_i \leq b_{i-1}$. By Lemma 2, $a_i = b_i \wedge a_{i-1}$ is in the same connected component as a_{i-1} , and clearly $a_i \leq b_i$. Therefore, the induction step carries through. In the end, $a_m \leq b_m = b'$ and that $a_m \in A$. However, $a > b' > a_m$ and $\rho(b') < \rho(a)$, contradicting the minimality of $\rho(a)$.

Recall some basic concepts about lattice congruences. We say that an equivalence relation Θ is a *congruence relation* if $a_1 \equiv b_1(\Theta)$, $a_2 \equiv b_2(\Theta)$ implies $a_1 \vee a_2 \equiv b_1 \vee b_2(\Theta)$ and $a_1 \wedge a_2 \equiv b_1 \wedge b_2(\Theta)$. Once we have a congruence relation Θ , the quotient L/Θ , whose elements are equivalence classes of Θ denoted as $[a]\Theta$ for $a \in L$, is naturally equipped with a meet and a join operation by $[a]\Theta \vee [b]\Theta = [a \vee b]\Theta$, $[a]\Theta \wedge [b]\Theta = [a \wedge b]\Theta$ that makes L/Θ into a lattice (Chapter 1 of [5]). Such lattice L/Θ is called a *quotient lattice* or a *factor lattice* of L .

Lemma 4 \mathcal{H}_M is a congruence relation on L . Consequently, L_M is a lattice.

Proof Choose two arbitrary connected components A, B of \mathcal{H}_M . We just need to show that if $a, a' \in A$ and $b, b' \in B$, then $a \vee b$ and $a' \vee b'$ are in the same connected component of \mathcal{H}_M , and so are $a \wedge b$ and $a' \wedge b'$. It suffices to prove this claim when $a = a'$. Since we can connect b and b' by a path of covering relations in \mathcal{H}_M , by induction, it suffices to show the case when $b \leq b'$. Let $c = a \vee b$. Then $c' := a \vee b' = a \vee b \vee b' = c \vee b'$. By Lemma 2, as $c \geq b$ and $b \leq b'$ are in \mathcal{H}_M , we know that c and $c' = c \vee b'$ are in the same connected component of \mathcal{H}_M as desired. Dually, $a \wedge b$ and $a \wedge b'$ are in the same connected component as well.

Lemma 5 Every connected component of \mathcal{H}_M is a convex sublattice of L .

Proof See Chapter 1 of [5].

By Lemma 5, each connected component A of \mathcal{H}_M has a maximum and a minimum and let us denote them by A_{\max} and A_{\min} respectively.

Lemma 6 In L_M , $A \leq_M B$ if and only if $A_{\max} \leq B_{\max}$ if and only if $A_{\min} \leq B_{\min}$.

Proof The “if” direction follows from Definition 4. Now assume that $A \leq_M B$, which means that there exists $a \in A$ and $b \in B$ such that $a \leq b$. By Lemma 4, $A_{\min} \wedge B_{\min}$ is in the same connected component as $a \wedge b = a$. Thus, $A_{\min} \wedge B_{\min} \in A$ and by the existence of A_{\min} , $A_{\min} \wedge B_{\min} \geq A_{\min}$, meaning that $A_{\min} \leq B_{\min}$. Similarly, $A_{\max} \leq B_{\max}$ as desired.

Remark 1 Lemma 6 implies that (L_M, \leq_M) is isomorphic to the induced subposet of L by choosing the minimum elements of each connected component of \mathcal{H}_M . However, the subposet of L formed by the minimum elements of each connected component of \mathcal{H}_M is not necessarily a sublattice of L . Consider the following example shown in Figure 3, which also demonstrates that the subposet of L formed by the minimum elements of each connected component of \mathcal{H}_M may not be a sublattice of L .

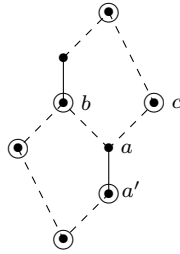


Fig. 3 A lattice where \mathcal{H}_M is drawn in solid lines, with minimum elements of each connected component circled.

In the induced subposet of L consisting of the minimum elements of each connected component of \mathcal{H}_M , there are a', b, c . But $b \wedge c = a$, not a' . So this induced subposet is not a sublattice of L .

Theorem 2 (L_M, \leq_M) is a well-defined modular lattice.

Proof We have seen that L_M is a lattice so the main task is to show that L_M is also modular. A classical theorem (Theorem I.12 of [1]) states that a lattice is modular if and only if it is free of sublattices isomorphic to N_5 (Figure 4).

Assume the opposite that L_M contains a sublattice N_5 , consisting of distinct connected components A, B, C, D, E of \mathcal{H}_M such that $A < B$, $A \vee C = B \vee C = D$, $A \wedge C = B \wedge C = E$, shown in Figure 4. Use the notation as in Lemma 6. Let $b = A_{\max} \vee B_{\min} \in B$ and $d = b \vee C_{\max} \in D$ since meet and join operations in L_M are compatible with those in L . By the same reasoning, $A_{\max} \vee C_{\max} \in D$ so $A_{\max} \vee C_{\max} \geq D_{\min} \geq B_{\min}$. Therefore,

$$A_{\max} \vee C_{\max} = B_{\min} \vee (A_{\max} \vee C_{\max}) = (B_{\min} \vee A_{\max}) \vee C_{\max} = b \vee C_{\max} = d.$$

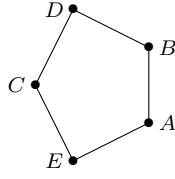


Fig. 4 N_5 : obstruction to modularity

Next we claim that $A_{\max} \wedge C_{\max} = E_{\max}$. As $A_{\max} \wedge C_{\max} \in E$, $A_{\max} \wedge C_{\max} \leq E_{\max}$. And as $A_{\max}, C_{\max} \geq E_{\max}$ by Lemma 6, $A_{\max} \wedge C_{\max} \geq E_{\max}$. So they are equal. Then as $b \geq A_{\max}$, $b \wedge C_{\max} \geq A_{\max} \wedge C_{\max} = E_{\max}$ and since $b \wedge C_{\max} \in E$, we have $b \wedge C_{\max} = E_{\max}$ as well. Now, we have found $A_{\max}, b, C_{\max} \in L$ such that $A_{\max} < b$, $A_{\max} \vee C_{\max} = b \vee C_{\max} = d$ and $A_{\max} \wedge C_{\max} = b \wedge C_{\max} = E_{\max}$. Take a maximal chain $A_{\max} = x_0 < \dots < x_m = b$ in L . It is clear that for any x_i in this chain, $x_i \vee C_{\max} = d$ and $x_i \wedge C_{\max} = E_{\max}$ so every edge $x_i < x_{i+1}$ is contained in \mathcal{H}_M . This contradicts with the assumption that A and B are two distinct connected components. As a result, L_M is modular.

Now we are ready to prove the main theorem.

Proof (Proof of Theorem 1) For a finite lattice L , let \mathcal{V}_L denote the valuation polytope of L (Definition 2). It suffices to show that \mathcal{V}_L and \mathcal{V}_{L_M} are linearly equivalent. In fact, they are the same polytope embedded in possibly different Euclidean spaces.

Lemma 1 states that every valuation v on L needs to be constant on each connected component of \mathcal{H}_M . Then define a linear map $f : \mathcal{V}_L \rightarrow \mathcal{V}_{L_M}$ by $v \mapsto (A \mapsto v(A))$ where A is any element in the connected component A . Let $v \in \mathcal{V}_L$ and $v_M = f(v)$. Let's check that $v_M \in \mathcal{V}_{L_M}$. Choose arbitrary $A, B \in L_M$ and pick $a \in A$ and $b \in B$. By the proof of Theorem 2, $A \wedge B$ and $A \vee B$ in L_M are the connected components of \mathcal{H}_M containing $a \wedge b$ and $a \vee b$ respectively. Since $v(a) + v(b) = v(a \wedge b) + v(a \vee b)$, it follows that $v_M(A) + v_M(B) = v_M(A \wedge B) + v_M(A \vee B)$. Also as $v(a) \in [0, 1]$, we have $v(A) \in [0, 1]$ as well. Therefore, f is well-defined. An inverse of f is given by $v_M \mapsto (a \mapsto v_M(A))$, where A is the connected component containing a . Checking that it is well-defined is analogous and checking that it is indeed the inverse of f is straightforward.

Remark 2 The modularization construction (Definition 4) of L_M provides the maximal modular quotient (Notation 8.7 of [6]) of the lattice L . Indeed, let Θ be any congruence relation on L such that the quotient L/Θ is modular. If for some $x < y$, there exists z such that $x \vee z = y \vee z$ and $x \wedge z = y \wedge z$, then $[x]\Theta \leq [y]\Theta$ and the modular law give

$$[x]\Theta \vee ([z]\Theta \wedge [y]\Theta) = ([x]\Theta \vee [z]\Theta) \wedge [y]\Theta.$$

LHS equals $[x \vee (z \wedge y)]\Theta = [x \vee (z \wedge x)]\Theta = [x]\Theta$ while RHS equals $[(x \vee z) \wedge y]\Theta = [(y \vee z) \wedge y]\Theta = [y]\Theta$. Therefore, $x \equiv y(\Theta)$. Next, if for some $a \leq b$ such that there exists $x \leq y$ with $x \equiv y(\Theta)$ and $a \vee y = b$, $a \wedge y = x$ (or $b \vee x = y$, $b \wedge x = a$), then $[a]\Theta \wedge [y]\Theta = [a \wedge y]\Theta = [x]\Theta = [y]\Theta$ so $[a]\Theta \geq [y]\Theta$. This means $[b]\Theta = [a]\Theta \vee [y]\Theta = [a]\Theta$, so $a \equiv b(\Theta)$. By comparing to Definition 3, we see that \mathcal{H}_M must be a refinement of Θ .

Remark 3 If L is distributive, each valuation v on L is uniquely determined by its values on the join-irreducibles and is nicely written out (Ex. 3.94 of [9]). We would love to extend our construction towards distributive lattices but this seems much harder. It is well-known that a modular lattice is distributive if and only if it is free of the diamond lattice M_3 (Figure 5). In case of M_3 ,

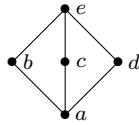


Fig. 5 M_3 : obstruction to distributivity (given modularity)

a valuation needs to satisfy $v(b) = v(c) = v(d)$ so we would like b, c, d to be in the same equivalence class. However, there is one more linear relation $v(a) + v(e) = 2v(b)$ so it is not very clear what can be done. In particular, the valuation polytope of the modular lattice L shown in Figure 6 has a different combinatorial type (a different face poset) than the valuation polytope of any distributive lattice. This is checked by explicitly computing the face poset of the valuation polytope of $J(P)$ (see Section 1) for all posets P with $\#P$ smaller than the dimension of the valuation polytope of L , which is 4.

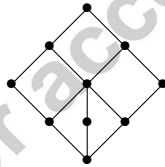


Fig. 6 A modular lattice that is not distributive

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