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## COMPLEXITY OF PARABOLIC SYSTEMS

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**ABSTRACT.** We first bound the codimension of an ancient mean curvature flow by the entropy. As a consequence, all blowups lie in a Euclidean subspace whose dimension is bounded by the entropy and dimension of the evolving submanifolds. This drastically reduces the complexity of the system. We use this in a major application of our new methods to give the first general bounds on generic singularities of surfaces in arbitrary codimension.

We also show sharp bounds for codimension in arguably some of the most important situations of general ancient flows. Namely, we prove that in any dimension and codimension any ancient flow that is cylindrical at  $-\infty$  must be a flow of hypersurfaces in a Euclidean subspace. This extends well-known classification results to higher codimension.

The bound on the codimension in terms of the entropy is a special case of sharp bounds for spectral counting functions for shrinkers and, more generally, ancient flows. Shrinkers are solutions that evolve by scaling and are the singularity models for the flow.

We show rigidity of cylinders as shrinkers in all dimension and all codimension in a very strong sense: Any shrinker, even in a large dimensional space, that is sufficiently close to a cylinder on a large enough, but compact, set is itself a cylinder. This is an important tool in the theory and is key for regularity; cf. [CM11].

## 0. INTRODUCTION

We introduce a new circle of ideas that gives a new way of attacking mean curvature flow (MCF) in higher codimension.

Higher codimension MCF is a complicated nonlinear parabolic system where much less is known than for hypersurfaces. The complexity of the system increases as the codimension increases. We show that blowups of higher codimension MCF have much smaller codimension than the original flow. In many important instances, we show that blowups are evolving hypersurfaces in a Euclidean subspace even when the original flow is far from being hypersurfaces. Another major point of this article is to give the first general bounds on generic singularities of surfaces in arbitrary codimension.

One way of thinking about MCF is as a one-parameter family of submanifolds  $M_t \subset \mathbf{R}^N$  evolving so that the position vector  $x \in M_t^n$  satisfies the nonlinear heat equation

$$(0.1) \quad (\partial_t - \Delta_{M_t})x = 0.$$

This equation is nonlinear since the Laplacian depends on the evolving submanifold  $M_t$ . Many fundamental results and tools about elliptic PDEs have originated in the study of the minimal surface equation. In much the same way, MCF is one of the most fundamental parabolic systems. New results and tools are expected to apply to a variety of other systems.

The key to understanding MCF is to understand blowups. Blowups are limits of parabolic rescalings of the flow that magnify around a sequence of points converging to a singular point. Such limits are defined for all negative times and said to be *ancient*. When the point

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is fixed, the limit is a *shrinker* that evolves by rescaling. This paper deals with understanding both general ancient flows, and shrinkers, in all codimension.

There is a Lyapunov function for the flow that is particularly useful. To define it recall that the Gaussian surface area  $F$  of an  $n$ -dimensional submanifold  $\Sigma^n \subset \mathbf{R}^N$  is

$$(0.2) \quad F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|x|^2}{4}}.$$

Following [CM8], the entropy  $\lambda$  is the supremum of  $F$  over all translations and dilations

$$(0.3) \quad \lambda(\Sigma) = \sup_{c, x_0} F(c\Sigma + x_0).$$

By Huisken's monotonicity, [Hu], it follows that  $\lambda$  is monotone nonincreasing under the flow. From this, and lower semi continuity of  $\lambda$ , we have that all blowups have entropy bounded by that of the initial submanifold in a MCF.

A great deal of fundamental results with wide-ranging applications have been obtained for evolving hypersurfaces. However, the theory in higher codimension is notoriously difficult and very little has been known. We will introduce a number of new tools to deal with flows in higher codimension. As a result, we are able to attack a number of outstanding issues and problems in higher codimension. We begin by bounding the codimension of any blowup by the entropy. This already gives a drastic simplification. After that, we prove sharp spectral bounds. We give the first general bounds for generic singularities in all codimension. We show that, in some of the most important cases of general ancient flows, the evolving submanifolds are hypersurfaces inside some affine subset. We also show a very strong rigidity theorem for cylinders in all codimension.

The first part of the paper deals with establishing codimension and various analytic bounds in terms of the entropy and other low regularity quantities. The second part, deals with establishing a priori entropy and related bounds in all codimension, allowing us to apply the first part.

**0.1. Liouville properties.** Let  $M_t^n \subset \mathbf{R}^N$  be an ancient MCF of  $n$ -dimensional submanifolds with entropies  $\lambda(M_t) \leq \lambda_0 < \infty$ . Ancient flows are solutions that exist for all negative times. The space  $\mathcal{P}_d$  of polynomial growth caloric functions consists of  $u(x, t)$  on  $\cup_t M_t \times \{t\}$  so that  $(\partial_t - \Delta_{M_t})u = 0$  and there exists  $C$  depending on  $u$  with

$$(0.4) \quad |u(x, t)| \leq C(1 + |x|^d + |t|^{\frac{d}{2}}) \text{ for all } (x, t) \text{ with } x \in M_t, t < 0.$$

Motivated by [CM1]–[CM5], similar spaces were considered in Calle's thesis [Ca1], [Ca2].

Our first theorem is a sharp bound for a parabolic “counting function” on ancient MCF (in all of these results, the time slices  $M_t$  are allowed to be non-compact):

**Theorem 0.5.** There exists  $C_n$  so that if  $M_t^n \subset \mathbf{R}^N$  is an ancient MCF with  $\lambda(M_t) \leq \lambda_0$  and  $d \geq 1$ , then  $\dim \mathcal{P}_d \leq C_n \lambda_0 d^n$ .

The dependence on  $d$  is sharp on Euclidean space, where  $\mathcal{P}_d(\mathbf{R}^n)$  consists of the classical caloric polynomials. For a fixed manifold with  $\text{Ric} \geq 0$  that is time-independent, the related bound  $C d^{n+1}$  was proven by Lin and Zhang, [LZ], adapting the arguments of [CM1]–[CM5] for harmonic functions. The sharp bound  $C d^n$  in that case was proven in [CM12]. These time-independent bounds use the commutativity of  $\Delta$  and  $\partial_t$  and do not apply here. Instead a key here is a new localization inequality for the Gaussian  $L^2$  norm. This new approach

allows us to obtain the optimal dependence; see [CM13] for more. Similar localization ideas also play a role later in this paper.

Theorem 0.5 has a number of applications, including bounds for the associated heat kernel. One remarkable consequence with  $d = 1$  is a bound for the codimension. This is because the flow sits inside a linear subspace of dimension at most  $\dim \mathcal{P}_1$  since a linear relation for coordinate functions specifies a hyperplane containing the flow.

**Corollary 0.6.** There exists  $C_n$  so that if  $M_t^n \subset \mathbf{R}^N$  is an ancient MCF, then it is contained in a Euclidean subspace of dimension  $\leq C_n \sup_t \lambda(M_t)$ .

Singularities are modeled by shrinkers  $\Sigma$  that evolve by scaling. The most fundamental shrinkers are cylinders  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$ , but there are many others including all  $n$ -dimensional minimal submanifolds of the sphere  $\partial B_{\sqrt{2n}} \subset \mathbf{R}^N$ . Let  $\Sigma^n \subset \mathbf{R}^N$  be a shrinker with finite entropy  $\lambda(\Sigma)$ . As in [CM8], the drift Laplacian (Ornstein-Uhlenbeck operator)  $\mathcal{L} = \Delta - \frac{1}{2} \nabla_{x^T}$  is self-adjoint with respect to the Gaussian inner product  $\int_{\Sigma} u v e^{-\frac{|x|^2}{4}}$ . Let  $\|u\|_{L^2}$  denote the Gaussian  $L^2$  norm. We will say that  $u$  is a  $\mu$ -eigenfunction if  $\mathcal{L} u = -\mu u$  and  $0 < \|u\|_{L^2} < \infty$ . The *spectral counting function*  $\mathcal{N}(\mu)$  is the number of eigenvalues  $\mu_i \leq \mu$  counted with multiplicity. The next result bounds  $\mathcal{N}$ :

**Theorem 0.7.** There exists  $C_n$  so that the counting function for  $\mathcal{L}$  on an  $n$ -dimensional shrinker  $\Sigma^n \subset \mathbf{R}^N$  satisfies  $\mathcal{N}(\mu) \leq C_n \lambda(\Sigma) \mu^n$  for  $\mu \geq \frac{1}{2}$ .

The dependence on  $\mu$  is sharp even on Euclidean space. A key component in the proof is a sharp polynomial growth bound for eigenfunctions of  $\mathcal{L}$  on any shrinker. This result is of independent interest. It too is sharp on  $\mathbf{R}^n$  and shows that any eigenfunction on any shrinker grows polynomially of degree at most twice the eigenvalue (see Theorem 2.1).

Specializing Theorem 0.7 to  $\mu = \frac{1}{2}$  gives:

**Corollary 0.8.** There exists  $C_n$  so that if  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker, then it is contained in a Euclidean subspace of dimension  $\leq C_n \lambda(\Sigma)$ .

In the second part, we will use Corollary 0.8 to show that all closed 2-dimensional singularities for higher codimension mean curvature flow that cannot be perturbed away have uniform entropy bounds and lie in a linear subspace of small dimension. This gives the first general bounds for generic singularities in higher codimension; see subsection 0.2 for more.

Our estimates in Corollaries 0.6 and 0.8 are linear in the entropy. The corresponding linear estimate for algebraic varieties in complex projective space follows from Bézout's theorem; see corollary 18.12 in [Ha]. When  $\Sigma \subset \partial B_{\sqrt{2n}} \subset \mathbf{R}^N$  is a closed  $n$ -dimensional minimal submanifold of the sphere and the entropy reduces to the volume, this estimate follows Cheng-Li-Yau, [CgLYa].

**0.2. Generic singularities in higher codimension.** Even for hypersurfaces, examples show that singularities of MCF are too numerous to classify. The hope is that the generic ones that cannot be perturbed away are much simpler. Indeed for hypersurfaces in all dimensions generic singularities have been classified in [CM8]. These are round generalized cylinders  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$ .

We show in part 2 of this paper that the only closed 2-dimensional generic singularities, i.e.,  $F$ -stable shrinkers, have a uniform entropy bound and lie in a small linear subspace.

The entropy and dimension of the subspace are both  $\leq C(1 + \gamma)$  for a universal constant  $C$  and genus  $\gamma$ .

**Theorem 0.9.** There exists a universal constant  $C$  so that if  $\Sigma^2 \subset \mathbf{R}^N$  is a closed  $F$ -stable shrinker of genus  $g$  and  $N \geq C \lambda(\Sigma)$ , then  $\Sigma \subset \mathcal{V}$  where  $\mathcal{V}$  is a linear subspace and

$$(0.10) \quad \lambda(\Sigma) \leq C(1 + \gamma),$$

$$(0.11) \quad \dim \mathcal{V} \leq C(1 + \gamma).$$

This gives the first general bounds on generic singularities of surfaces in arbitrary codimension. When  $\Sigma$  is diffeomorphic to a sphere, (0.10) becomes

$$(0.12) \quad \lambda(\Sigma) < 4 = e \lambda(\mathbf{S}_2^2).$$

The sharp constant is unknown, but (0.12) is at most off by a factor of  $e$ . Theorem 0.9 holds even when the  $F$ -index is not zero, with  $C$  depending on the index.

There is no analog of (0.10) for minimal surfaces in  $\mathbf{R}^4$ . Namely, viewing  $\mathbf{R}^4$  as  $\mathbf{C}^2$  one sees that for each integer  $m$  the parametrized complex submanifold  $z \rightarrow (z, z^m)$  is a stable minimal variety that is topologically a plane. It has  $\text{Area}(B_r \cap \Sigma) \geq C m r^2$  for  $r \geq 1$ . In contrast, Theorem 0.9 implies that  $\text{Area}(B_r \cap \Sigma) \leq C(1 + \gamma) r^2$  for a closed stable 2-dimensional shrinker  $\Sigma$  of genus  $\gamma$ . Similarly, there is no analog of (0.11) for minimal surfaces. Indeed, for each  $m$ , the parametrized surface  $z \rightarrow (z, z^2, z^3, \dots, z^{m+1})$  is a stable minimal variety that is topologically a plane. Its real codimension is  $2m$  and it is not contained in a proper subspace. In contrast to (0.10) and (0.11), very little is known about stable minimal surfaces in higher codimension. A notable exception is a result of Micallef, [Mi], that a stable oriented parabolic minimal surface in  $\mathbf{R}^4$  is complex for some orthogonal complex structure.

Entropy is bounded from below by the Gaussian Willmore functional, see Lemma 12.1 below. We will also prove a sharp upper bound for the Gaussian Willmore functional  $W$

$$(0.13) \quad W(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} |\mathbf{H}|^2 e^{-\frac{|x|^2}{4}}.$$

The next theorem gives a sharp bound for  $W$ , in arbitrary codimension, for stable shrinkers that are topological spheres.

**Theorem 0.14.** If  $\Sigma^2 \subset \mathbf{R}^N$  is a  $F$ -stable shrinker diffeomorphic to a sphere, then  $W(\Sigma) \leq W(\mathbf{S}_2^2)$ . With equality if and only if  $\Sigma = \mathbf{S}_2^2 \subset \mathbf{R}^3$  up to rotation.

We will also prove  $W$  bounds for surfaces of any genus, see Theorem 12.4, in addition to several other entropy and eigenvalue bounds.

**0.3. Sharp bound for codimension.** The next result gives sharp bounds for codimension in arguably some of the most important situations for general ancient flows. The bound in Theorem 0.5 is sharp in the exponent of  $d$  and, thus, asymptotically sharp as  $d \rightarrow \infty$ . The next result is more delicate and obtains sharp constants for  $d$  fixed.

Suppose that  $M_t^n \subset \mathbf{R}^N$  is an ancient MCF with  $\sup_t \lambda(M_t) < \infty$ . For each constant  $c > 0$  define the flow  $M_{c,t}$  by  $M_{c,t} = \frac{1}{c} M_{c^2 t}$ . It follows that  $M_{c,t}$  is an ancient MCF as well. Since  $\sup_t \lambda(M_t) < \infty$ , it follows from Huisken's monotonicity, [Hu], and work of Ilmanen, [I], White, [W3], that every sequence  $c_i \rightarrow \infty$  has a subsequence (also denoted by  $c_i$ ) so that

$M_{c_i,t}$  converges to a shrinker  $M_{\infty,t}$  (so  $M_{\infty,t} = \sqrt{-t} M_{\infty,-1}$ ) with  $\sup_t \lambda(M_{\infty,t}) \leq \sup_t \lambda(M_t)$ . We will say that such a  $M_{\infty,t}$  is a tangent flow at  $-\infty$  of the original flow. We next give a sharp bound for the codimension:

**Theorem 0.15.** If  $M_t^n \subset \mathbf{R}^N$  is an ancient MCF and one tangent flow at  $-\infty$  is a cylinder  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$ , then  $M_t$  is a flow of hypersurfaces in a Euclidean subspace.

We believe that Theorem 0.15 will have wide ranging consequences for MCF in higher codimension. We will try here to briefly explain some of these (see Section 8 for more).

Using Angenent-Daskalopoulos-Sesum, [ADS], Brendle-Choi, [BCh], and Choi-Haslhofer-Hershkovits, [ChHH], we get uniqueness for ancient flows of surfaces in higher codimension:

**Corollary 0.16.** If  $M_t^2 \subset \mathbf{R}^N$  is an ancient MCF of surfaces and one tangent flow at  $-\infty$  is a cylinder  $\mathbf{S}_{\sqrt{2}}^1 \times \mathbf{R}$ , then  $M_t \subset \mathbf{R}^3 \subset \mathbf{R}^N$  for some 3-plane  $\mathbf{R}^3$ . Therefore, by [ChHH]  $M_t$  is either shrinking round cylinders, or the ancient ovals, or the bowl soliton.

White, [W2], and Haslhofer-Hershkovits, [HH], constructed ancient MCF of closed hypersurfaces that for time zero disappear in a round point and at time  $-\infty$  are shrinking cylinders. These are the ancient ovals. Hershkovits, [H], showed (see also Haslhofer, [Has]) that the bowl soliton in  $\mathbf{R}^3$  is the unique translating solution of MCF which has the family of shrinking cylinders as an asymptotic shrinker at  $-\infty$ .

**0.4. Rigidity of cylinders.** Our next result plays a key role in the proof of Theorem 0.15 and in the regularity of MCF in higher codimension, cf. [CM11]. This result shows that cylinders are rigid in a very strong sense: Any shrinker, even in a large dimensional space, that is sufficiently close to a cylinder on a large enough, but compact, set is itself a cylinder. To state the theorem, let  $\mathcal{C}_{n,N}$  be the collection of all  $\mathbf{R}^N$  rotations of  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$  for  $k = 1, \dots, n$ .

**Theorem 0.17.** There exists  $R_N$  so that if  $\Sigma^n \subset \mathbf{R}^N$  is a complete shrinker with finite entropy and there exists  $\mathcal{C} \in \mathcal{C}_{n,N}$  so that  $B_{R_N} \cap \Sigma$  is a graph over  $\mathcal{C}$  of a normal vector field  $V$  with  $\|V\|_{C^{2,\alpha}} \leq R_N^{-1}$ , then  $\Sigma \in \mathcal{C}_{n,N}$ .

The rigidity of cylinders in codimension one was proven in [CIM]. To prove Theorem 0.17, we show that a shrinker, even in high codimension, that is close to a cylinder on a large bounded set must be a hypersurface in some Euclidean subspace.

One of several reasons that cylinders are significant is that they are the most prevalent singularities. By uniqueness of solutions to ODEs, any shrinking curve in  $\mathbf{R}^N$  is planar. From this and dimension reduction, it is expected that for MCF in all codimension the most prevalent singularities are  $\gamma \times \mathbf{R}^{n-1}$ . Here  $\gamma$  is a closed planar curve (Abresch-Langer) that is a round circle if embedded or stable, [CM8], or with  $\lambda(\gamma) < 2$ .

**0.5. Outline.** Part 1, consisting of sections 1 through 8, focuses on caloric functions of polynomial growth on ancient flows as well as on eigenfunctions on shrinkers. Section 1 recalls fundamental properties of shrinkers including the concentration of  $L^2$  eigenfunctions. The next section proves a sharp bound for the growth of  $L^2$  eigenfunctions on a shrinker. Section 3 introduces Gaussian inner products and establishes the framework for separating caloric functions of polynomial growth, including a localization inequality. Section 4 proves finite dimensionality for the space of caloric functions of polynomial growth, with the dimension



depending on the entropy and the rate of growth. The dependence on the rate of growth is sharp. The following section proves sharp bounds for the spectral counting function on a shrinker. Section 6 proves that a shrinker, even in high codimension, that is close to a cylinder on a sufficiently large bounded set must be a hypersurface in some Euclidean subspace and, thus by [CIM], must be a cylinder. In Section 7, we prove that an ancient flow that is cylindrical at time  $-\infty$  must be a flow of hypersurfaces in some linear subspace; this is done by proving an optimal estimate on the space of linear functions of caloric growth. Section 8 contains conjectures and some brief remarks.

Part 2, consisting of sections 9 through 13, bounds the entropy for stable shrinking surfaces in arbitrary codimension and, in combination with the results of the first part, bounds the codimension. This gives the first restrictions on stable singularities in higher codimension.

Some of the results of this paper are surveyed in the article [CM10].

## Part 1. Complexity of parabolic systems

We begin by introducing a new circle of ideas that gives a new way of attacking MCF in higher codimension.

### 1. THE OPERATOR $\mathcal{L}$ ON THE GAUSSIAN SPACE ON SHRINKERS

Singularities are modeled by shrinkers that are special solutions to the flow that evolve by rescaling. The shrinker equation is  $\mathbf{H} = \frac{x^\perp}{2}$ , where  $\mathbf{H} = -\text{Tr } A$  is the mean curvature vector,  $A$  the second fundamental form, and  $x^\perp$  is the perpendicular part of  $x$ .<sup>1</sup> Set  $f = \frac{|x|^2}{4}$ , so the Gaussian weight is  $e^{-f}$ . As in lemma 3.20 in [CM8], the coordinate functions  $x_i$  are  $\frac{1}{2}$ -eigenfunctions and  $|x|^2 - 2n$  is a 1-eigenfunction for  $\mathcal{L}$  on any shrinker with finite entropy. We will need some standard facts about  $L^2$  eigenfunctions (cf. section 3 in [CM9]):

**Lemma 1.1.** If  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker and  $u$  is a  $L^2$   $\mu$ -eigenfunction, then  $u \in W^{1,2}$  and  $\int |\nabla u|^2 e^{-f} = \mu \int u^2 e^{-f}$ . If  $v$  is a  $\nu$ -eigenfunction with  $\nu \neq \mu$ , then

$$(1.2) \quad 0 = \int u v e^{-f} = \int \langle \nabla u, \nabla v \rangle e^{-f}.$$

*Proof.* Let  $\eta$  be a compactly supported function with  $\eta^2 \leq 1$  and  $|\nabla \eta| \leq 1$ . Taking the divergence of  $u \nabla u \eta^2 e^{-f}$  and applying Stokes' theorem gives that

$$(1.3) \quad \int |\nabla u|^2 \eta^2 e^{-f} - \mu \int u^2 \eta^2 e^{-f} = -2 \int u \eta \langle \nabla u, \nabla \eta \rangle e^{-f}.$$

Applying the absorbing inequality  $2ab \leq \frac{a^2}{2} + 2b^2$  and then using  $\eta^2 \leq 1$  and  $|\nabla \eta| \leq 1$  gives

$$(1.4) \quad \int |\nabla u|^2 \eta^2 e^{-f} \leq (\mu + 2) \int u^2 e^{-f} + \frac{1}{2} \int |\nabla u|^2 \eta^2 e^{-f}.$$

Absorbing the last term on the right and taking  $\eta$ 's converging to one everywhere, we conclude that  $\int |\nabla u|^2 e^{-f} < \infty$ . Once we have this,  $|u| |\nabla u|$  is also integrable, so the right-hand side of (1.3) goes to zero as  $\eta \rightarrow 1$ . We conclude that  $\int |\nabla u|^2 e^{-f} = \mu \int u^2 e^{-f}$ . Finally, (1.2) follows from the symmetry of  $\mathcal{L}$ .  $\square$

<sup>1</sup>See [AHW], [AS], [LL] and [Wa] for results on higher codimension MCF.



We show next that the  $W^{1,2}$  norm of an eigenfunction concentrates in a bounded set; this concentration has been used on manifolds a number of times, going back at least to [E3]. The next lemma and corollary apply to  $W^{1,2}$  functions that are either entire or defined on a compact subdomain and vanish on the boundary.

**Lemma 1.5.** If  $u$  is a  $W^{1,2}$  function on a shrinker  $\Sigma^n \subset \mathbf{R}^N$ , then

$$(1.6) \quad \int |x|^2 u^2 e^{-f} \leq 4n \int u^2 e^{-f} + 16 \int |\nabla u|^2 e^{-f}.$$

Moreover, for any  $r > 2$ , we have

$$(1.7) \quad \int_{\Sigma \setminus B_r} |\nabla u|^2 e^{-f} \leq \int_{\Sigma \setminus B_{r-1}} (5u^2 + (\mathcal{L}u)^2) e^{-f}.$$

*Proof.* Let  $\eta$  be a compactly supported function with  $\eta^2 \leq 1$ ,  $\eta \equiv 1$  on a large ball  $B_R$ , and  $|\nabla \eta| \leq 1$ . Since  $\mathcal{L}|x|^2 = 2n - |x|^2$ , taking the divergence of  $\eta^2 u^2 x^T e^{-f}$  and applying Stokes' theorem and the absorbing inequalities  $2ab \leq \frac{a^2}{4} + 4b^2$  and  $2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$  gives

$$(1.8) \quad \begin{aligned} \frac{1}{2} \int \eta^2 |x|^2 u^2 e^{-f} &= n \int \eta^2 u^2 e^{-f} + 2 \int \eta^2 u \langle \nabla u, x^T \rangle e^{-f} + 2 \int \eta u^2 \langle x^T, \nabla \psi \rangle e^{-f} \\ &\leq n \int \eta^2 u^2 e^{-f} + \left( \frac{1}{4} + \epsilon \right) \int \eta^2 u^2 |x|^2 e^{-f} + 4 \int \eta^2 |\nabla u|^2 e^{-f} + \frac{1}{\epsilon} \int |\nabla \eta|^2 u^2 e^{-f}. \end{aligned}$$

Taking  $0 < \epsilon < \frac{1}{8}$  fixed and letting  $R \rightarrow \infty$ , the last term goes to zero by the dominated convergence theorem since  $u \in L^2$  and we get that

$$(1.9) \quad \left( \frac{1}{4} - \epsilon \right) \int |x|^2 u^2 e^{-f} \leq n \int u^2 e^{-f} + 4 \int |\nabla u|^2 e^{-f}.$$

The first claim follows since this holds for all  $\epsilon > 0$ . For the second claim, let  $\psi$  be zero on  $B_{r-1}$  and identically one outside of  $B_r$ . Taking the divergence of  $\psi^2 u \nabla u e^{-f}$  and applying Stokes' theorem and the absorbing inequalities  $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$  and  $2ab \leq \frac{a^2}{2} + 2b^2$  gives

$$(1.10) \quad \begin{aligned} \int \psi^2 |\nabla u|^2 e^{-f} &= - \int \psi^2 u \mathcal{L}u e^{-f} - 2 \int \psi u \langle \nabla \psi, \nabla u \rangle e^{-f} \\ &\leq \frac{1}{2} \int \psi^2 (u^2 + (\mathcal{L}u)^2) e^{-f} + \frac{1}{2} \int \psi^2 |\nabla u|^2 e^{-f} + 2 \int u^2 |\nabla \psi|^2 e^{-f}. \end{aligned}$$

Simplifying this and taking  $\psi$  to cut off linearly gives the second claim.  $\square$

One immediate consequence of Lemma 1.5 (with  $u \equiv 1$ ) is that if  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker with entropy  $\lambda < \infty$ , then  $\lambda$  is bounded in terms of the volume of  $B_r \cap \Sigma$  for  $r > \sqrt{4n}$ .

**Corollary 1.11.** If  $\mathcal{L}u = -\mu u$  on a shrinker  $\Sigma^n \subset \mathbf{R}^N$  and  $\|u\|_{L^2} = 1$ , then for any  $r > 2$

$$(1.12) \quad \int_{\Sigma \setminus B_r} \{u^2 + |\nabla u|^2\} e^{-f} \leq (6 + \mu^2) \frac{4(n + 4\mu)}{(r - 1)^2}.$$

*Proof.* Lemma 1.1 gives that  $\|\nabla u\|_{L^2}^2 = \mu$ . Thus, Lemma 1.5 gives that

$$(1.13) \quad (r-1)^2 \int_{\Sigma \setminus B_{r-1}} u^2 e^{-f} \leq \int |x|^2 u^2 e^{-f} \leq 4n + 16\mu,$$

$$(1.14) \quad \int_{\Sigma \setminus B_r} |\nabla u|^2 e^{-f} \leq \int_{\Sigma \setminus B_{r-1}} (5 + \mu^2) u^2 e^{-f}.$$

Combining these gives the corollary.  $\square$

## 2. SHARP POLYNOMIAL GROWTH OF EIGENFUNCTIONS

On  $\mathbf{R}^n$ , the  $L^2$  space is spanned by eigenfunctions for  $\mathcal{L}$  and these are polynomials of degree twice the eigenvalue. Moreover, on any shrinker, the coordinate functions are eigenfunctions with eigenvalue  $\frac{1}{2}$  and  $|x|^2 - 2n$  is an eigenfunction with eigenvalue 1. In both cases, the degree is twice the eigenvalue. The next theorem shows that  $L^2$  eigenfunctions on a shrinker always grow at most polynomially with degree twice the eigenvalue.

**Theorem 2.1.** If  $\mathcal{L}u = -\mu u$  on a shrinker  $\Sigma^n \subset \mathbf{R}^N$  and  $\|u\|_{L^2} < \infty$ , then

$$(2.2) \quad u^2(x) \leq C_n \lambda(\Sigma) \|u\|_{L^2(\Sigma)}^2 (4 + |x|^2)^{2\mu}.$$

The key to Theorem 2.1 will be to use parabolic estimates on an associated solution of the heat equation on the self-shrinking MCF.

**2.1. Separation of variables solutions.** If  $u$  is an eigenfunction on  $\mathbf{R}^n$  with  $\mathcal{L}u = -\mu u$ , then we get a separation of variables solution  $v(x, t)$  of the heat equation

$$(2.3) \quad v(x, t) = (-t)^\mu u\left(\frac{x}{\sqrt{-t}}\right).$$

On  $\mathbf{R}$ ,  $\mathcal{L}x = -\frac{1}{2}x$  gives  $v(x, t) = x$ , while  $\mathcal{L}(x^2 - 2) = -(x^2 - 2)$  gives  $v(x, t) = x^2 + 2t$ .

Suppose that  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker and define a MCF of sets  $\Sigma_t = \sqrt{-t}\Sigma$ . Let  $M_t$  be a MCF associated to  $\Sigma_t$ , so  $M_t$  is parametrized by motion in the normal direction. As sets  $M_t$  and  $\Sigma_t$  are the same, but they are parameterized differently.

**Lemma 2.4.** If  $u$  is a function on  $\Sigma$  with  $\mathcal{L}u = -\mu u$ , then  $v$  given on the  $\Sigma_t$ 's by

$$(2.5) \quad v(y, t) = (-t)^\mu u\left(\frac{y}{\sqrt{-t}}\right)$$

satisfies  $(\partial_t - \Delta_{M_t})v = 0$  on the MCF  $M_t$ .

*Proof.* Given  $t < 0$  and a point  $y \in \Sigma_t$ , we get that

$$(2.6) \quad \mathbf{H}_{\Sigma_t}(y) = \frac{1}{\sqrt{-t}} \mathbf{H}_\Sigma\left(\frac{y}{\sqrt{-t}}\right) = \frac{1}{\sqrt{-t}} \frac{\left[\frac{y}{\sqrt{-t}}\right]^\perp}{2} = \frac{y^\perp}{-2t}.$$

Here, we have freely used that the normal projection  $(\cdot)^\perp$  operator is invariant under dilation and, thus, is the same at corresponding points in  $\Sigma$  and  $\Sigma_t$ . Since  $\mathcal{L}u = -\mu u$  on  $\Sigma$ , we have

$$(2.7) \quad \Delta_\Sigma u(x) = \frac{1}{2} \langle \nabla_\Sigma u(x), x^T \rangle - \mu u(x).$$

At  $y \in \Sigma_t$ , we use the chain rule and (2.7) to compute the  $\Sigma_t$  Laplacian of  $v$

$$\begin{aligned} \Delta_{\Sigma_t} v(y, t) &= (-t)^\mu \Delta_{\Sigma_t} \left[ u \left( \frac{y}{\sqrt{-t}} \right) \right] = (-t)^{\mu-1} [\Delta_{\Sigma} u] \left( \frac{y}{\sqrt{-t}} \right) \\ (2.8) \quad &= (-t)^{\mu-1} \left[ \frac{1}{2} \left\langle \nabla_{\Sigma} u \left( \frac{y}{\sqrt{-t}} \right), \frac{y^T}{\sqrt{-t}} \right\rangle - \mu u \left( \frac{y}{\sqrt{-t}} \right) \right]. \end{aligned}$$

If  $y(t) \in \Sigma_t$  evolves by MCF  $y_t = -\mathbf{H}_{\Sigma_t}(y)$ , then (2.6) gives

$$(2.9) \quad \partial_t \left( \frac{y}{\sqrt{-t}} \right) = \frac{1}{2} (-t)^{-\frac{3}{2}} y + \frac{y_t}{\sqrt{-t}} = \frac{1}{2} (-t)^{-\frac{3}{2}} y + \frac{y^\perp}{2t \sqrt{-t}} = \frac{1}{\sqrt{-t}} \left( \frac{y^T}{-2t} \right).$$

Therefore, using the chain rule and then (2.9) gives

$$\begin{aligned} \partial_t [v(y, t)] &= \partial_t \left[ (-t)^\mu u \left( \frac{y}{\sqrt{-t}} \right) \right] = (-t)^\mu \left\langle \nabla_{\Sigma} u \left( \frac{y}{\sqrt{-t}} \right), \partial_t \left( \frac{y}{\sqrt{-t}} \right) \right\rangle - \mu (-t)^{\mu-1} u \left( \frac{y}{\sqrt{-t}} \right) \\ (2.10) \quad &= (-t)^{\mu-1} \left\langle \nabla_{\Sigma} u \left( \frac{y}{\sqrt{-t}} \right), \frac{y^T}{2 \sqrt{-t}} \right\rangle - \mu (-t)^{\mu-1} u \left( \frac{y}{\sqrt{-t}} \right). \end{aligned}$$

Combining (2.8) and (2.10), we see that  $(\partial_t - \Delta_{\Sigma_t}) v = 0$  on the MCF.  $\square$

## 2.2. Sharp polynomial growth of drift eigenfunctions.

**Lemma 2.11.** If  $(\partial_t - \Delta_{M_t}) w = 0$  on a MCF  $M_t$  and  $q \geq 1$ , then  $(\partial_t - \Delta_{M_t}) |w|^q \leq 0$ .

*Proof.* Given any function  $v : \mathbf{R} \rightarrow \mathbf{R}$ , set  $h = v(w^2)$ . Differentiating gives

$$(2.12) \quad h_t = v'(w^2) 2 w w_t,$$

$$(2.13) \quad \nabla_{M_t} h = v'(w^2) 2 w \nabla_{M_t} w,$$

$$(2.14) \quad \Delta_{M_t} h = v'(w^2) (2 |\nabla_{M_t} w|^2 + 2 w \Delta_{M_t} w) + v''(w^2) 4 w^2 |\nabla_{M_t} w|^2.$$

Therefore, using that  $(\partial_t - \Delta_{M_t}) w = 0$ , we have

$$(2.15) \quad h_t - \Delta_{M_t} h = -2 [v'(w^2) + 2 v''(w^2) w^2] |\nabla_{M_t} w|^2.$$

In particular, we have  $h_t - \Delta_{M_t} h \leq 0$  as long as

$$(2.16) \quad v'(s) + 2 s v''(s) \geq 0.$$

Now, we set  $v(s) = s^{\frac{q}{2}}$  with  $q \geq 1$ , so that  $v'(s) = \frac{q}{2} s^{\frac{q-2}{2}}$  and  $v''(s) = \frac{q(q-2)}{4} s^{\frac{q-4}{2}}$ . Using this in (2.16) gives

$$(2.17) \quad v'(s) + 2 s v''(s) = \frac{q}{2} \left[ s^{\frac{q-2}{2}} + 2 s \frac{(q-2)}{2} s^{\frac{q-4}{2}} \right] = \frac{q}{2} s^{\frac{q-2}{2}} [q-1].$$

This is nonnegative for  $q \geq 1$  as long as it is defined (i.e.,  $s > 0$  when  $q$  is small). The general case follows by approximation.  $\square$

*Proof of Theorem 2.1.* Set  $\Sigma_t = \sqrt{-t} \Sigma$  and let  $M_t$  be the associated MCF. By Lemma 2.4, the function  $v(y, t) = (-t)^\mu u \left( \frac{y}{\sqrt{-t}} \right)$  satisfies  $(\partial_t - \Delta_{M_t}) v = 0$  on  $M_t$ . Thus, Lemma 2.11 gives that  $(\partial_t - \Delta_{M_t}) |v| \leq 0$  on  $M_t$ , so the weighted monotonicity formula (theorem 4.13 in

[E1], cf. [Hu]; see (3.1) below) applies to  $|v|$ . Therefore, given any  $x_0 \in \Sigma = \Sigma_{-1}$ , we get for all  $t < -1$  that

$$(2.18) \quad \begin{aligned} |u|(x_0) &= |v(x_0, -1)| \leq (4\pi(-1-t))^{-\frac{n}{2}} \int_{\Sigma_t} |v(y, t)| e^{\frac{|y-x_0|^2}{4(t+1)}} \\ &= (-t)^\mu (4\pi(-1-t))^{-\frac{n}{2}} \int_{\Sigma_t} \left| u\left(\frac{y}{\sqrt{-t}}\right) \right| e^{\frac{|y-x_0|^2}{4(t+1)}}. \end{aligned}$$

Making the change of variables  $x = \frac{y}{\sqrt{-t}}$ , we get

$$(2.19) \quad |u|(x_0) = \frac{(-t)^{\mu+\frac{n}{2}}}{(4\pi(-1-t))^{\frac{n}{2}}} \int_{\Sigma} |u(x)| e^{\frac{|\sqrt{-t}x-x_0|^2}{4(t+1)}} = \frac{(-t)^\mu}{(4\pi(1+t^{-1}))^{\frac{n}{2}}} \int_{\Sigma} |u(x)| e^{-\frac{|x-\frac{x_0}{\sqrt{-t}}|^2}{4(1+t^{-1})}}.$$

We will take  $t < -4$ . Using this and expanding the square gives

$$(2.20) \quad |u|(x_0) \leq (-t)^\mu \int_{\Sigma} |u| e^{\frac{\langle x, \frac{x_0}{\sqrt{-t}} \rangle}{2(1+t^{-1})}} e^{-\frac{|x|^2}{4(1+t^{-1})}}.$$

We will apply the Cauchy-Schwarz inequality to the last term, writing the integrand as the product of  $|u| e^{-\frac{|x|^2}{8}}$  and  $e^{\frac{\langle x, \frac{x_0}{\sqrt{-t}} \rangle}{2(1+t^{-1})}} e^{-\frac{(1-t^{-1})|x|^2}{8(1+t^{-1})}}$ . The first term just gives  $\|u\|_{L^2}$ , as desired. To bound the second term, we use the absorbing inequality

$$(2.21) \quad \left| \left\langle x, \frac{x_0}{\sqrt{-t}} \right\rangle \right| \leq \frac{|x|^2}{8} + 2 \frac{|x_0|^2}{-t}.$$

By section 1 in [CM8] we can bound  $\text{Vol}(B_s \cap M_t)$  in terms of a dimensional constant times the entropy times  $s^n$ . Combining this all gives

$$(2.22) \quad \int_{\Sigma} e^{\frac{\langle x, \frac{x_0}{\sqrt{-t}} \rangle}{2(1+t^{-1})}} e^{-\frac{(1-t^{-1})|x|^2}{8(1+t^{-1})}} \leq e^{2\frac{|x_0|^2}{-t-1}} \int_{\Sigma} e^{-\frac{(1-2t^{-1})|x|^2}{8(1+t^{-1})}} \leq e^{\frac{2|x_0|^2}{-t-1}} \int_{\Sigma} e^{-\frac{|x|^2}{8}} \leq C_n \lambda(\Sigma) e^{\frac{2|x_0|^2}{-t-1}}.$$

Finally, using this back in (2.20) and taking  $t = -4 - |x_0|^2$

$$(2.23) \quad u^2(x_0) \leq (-t)^{2\mu} \|u\|_{L^2}^2 C_n \lambda(\Sigma) e^{\frac{2|x_0|^2}{-t-1}} \leq (|x_0|^2 + 4)^{2\mu} \|u\|_{L^2}^2 C_n \lambda(\Sigma) e^2.$$

□

**Corollary 2.24.** If  $\mathcal{L}u = -\mu u$  on a shrinker  $\Sigma^n \subset \mathbf{R}^N$  and  $\|u\|_{L^2} < \infty$ , then  $v$  given on  $\Sigma_t = \sqrt{-t}\Sigma$ 's by  $v(y, t) = (-t)^\mu u\left(\frac{y}{\sqrt{-t}}\right)$  is in  $\mathcal{P}_{2\mu}$  and satisfies

$$(2.25) \quad v^2(y, t) \leq C_n \lambda(\Sigma) \|u\|_{L^2(\Sigma)}^2 (-4t + |y|^2)^{2\mu}.$$

*Proof.* This follows by combining Theorem 2.1 and Lemma 2.4. □

### 3. GROWTH AND GAUSSIAN INNER PRODUCTS

In this section,  $M_t^n \subset \mathbf{R}^N$  is an ancient MCF with finite entropy and  $\phi = \mathbf{H} + \frac{x^\perp}{2t}$ . We will study the growth of caloric functions on  $M_t$  in the Gaussian  $L^2$  norm. The key result, inspired by [CM2], uses linear independence and polynomial growth to produce orthonormal caloric functions with a fixed doubling property. This will be used in the next section to bound  $\dim \mathcal{P}_d$  and then used later for sharp bounds on  $\mathcal{P}_1$ .

We will need the weighted Huisken monotonicity formula (theorem 4.13 in [E1], cf. [Hu])

$$(3.1) \quad \frac{d}{dt} \left\{ (-4\pi t)^{-\frac{n}{2}} \int_{M_t} v e^{\frac{|x|^2}{4t}} \right\} = (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \{ (v_t - \Delta v) - v |\phi|^2 \} e^{\frac{|x|^2}{4t}}.$$

**3.1. Polynomial growth.** Given  $u, v \in L^2(M_t)$ , define a bilinear form  $J$  and associated quadratic form  $I$  by

$$(3.2) \quad J_t(u, v) = (-4\pi t)^{-\frac{n}{2}} \int_{M_t} u v e^{\frac{|x|^2}{4t}},$$

$$(3.3) \quad I_u(t) = (-4\pi t)^{-\frac{n}{2}} \int_{M_t} u^2 e^{\frac{|x|^2}{4t}}.$$

The next lemma shows that  $I_u$  is monotone and grows polynomially when  $u \in \mathcal{P}_d$ .

**Lemma 3.4.** If  $u \in \mathcal{P}_d$ , then there exists  $C_{u,n,d}$  so that

$$(3.5) \quad I_u(t) \leq C_{u,n,d} \lambda(M_t) (1-t)^d,$$

$$(3.6) \quad I'_u(t) = (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \{ -2|\nabla u|^2 - u^2 |\phi|^2 \} e^{\frac{|x|^2}{4t}} \leq 0.$$

*Proof.* Since  $u \in \mathcal{P}_d$ , there is some  $C_u$  so that  $|u(x, t)| \leq C_u (1 + |x|^d + |t|^{\frac{d}{2}})$  and, thus,

$$(3.7) \quad \begin{aligned} I_u(t) &\leq C_u (-t)^d (-4\pi t)^{-\frac{n}{2}} \int_{M_t} e^{\frac{|x|^2}{4t}} + C_u (-4\pi t)^{-\frac{n}{2}} \int_{M_t} (1 + |x|^d)^2 e^{\frac{|x|^2}{4t}} \\ &\leq C_u \lambda(M_t) (-t)^d + C_u (4\pi)^{-\frac{n}{2}} \int_{\frac{M_t}{\sqrt{-t}}} (1 + (-t)^{\frac{d}{2}} |y|^d)^2 e^{-\frac{|y|^2}{4}} \leq C_{u,n,d} \lambda(M_t) (1-t)^d, \end{aligned}$$

where  $C_{u,n,d}$  depends on  $u, n$  and  $d$  but not on  $t$ . Applying (3.1) with  $v = u^2$  gives (3.6).  $\square$

**3.2. General constructions.** Let  $u_0 \equiv 1, u_1, \dots, u_\ell \in \mathcal{P}_d(M_t)$  be linearly independent. These are independent, but not necessarily orthogonal. To separate them, we will use ideas introduced in section 4 in [CM2] for studying harmonic functions.

Following definition 4.2 in [CM2], for each  $t_0$  we set  $w_{0,t_0} = u_0 = 1$  and then inductively define  $w_{i,t_0}$  by choosing coefficients  $\lambda_{j,i}(t_0) \in \mathbf{R}$  so that

$$(3.8) \quad w_{i,t_0}(x, t) \equiv u_i(x, t) - \sum_{j=0}^{i-1} \lambda_{j,i}(t_0) u_j(x, t)$$

is  $J_{t_0}$ -orthogonal to  $u_0, \dots, u_{i-1}$ . Finally, set  $f_i(t_0) = I_{w_{i,t_0}}(t_0)$ .

Following proposition 4.7 in [CM2], we get the following properties:

- (1) If  $t_0 \leq t_1$ , then  $f_i(t_1) = I_{w_{i,t_1}}(t_1) \leq I_{w_{i,t_0}}(t_1) \leq f_i(t_0)$ .
- (2) For each  $i$ , there exist  $T_i$  and  $C_i$  so that for  $t \leq T_i$  we have  $0 < f_i(t) \leq C_i (1-t)^d$ .

The next lemma is a variation on proposition 4.16 in [CM2] adapted to our situation:

**Lemma 3.9.** Given  $\delta > 0$  and  $\Omega > 1$ , there exist  $m_q \rightarrow \infty$  so that  $v_1, \dots, v_\ell$  defined by  $v_i = \frac{w_{i, -\Omega^{m_q+1}}}{\sqrt{f_i(-\Omega^{m_q+1})}}$  satisfy

$$(3.10) \quad J_{-\Omega^{m_q+1}}(v_i, v_j) = \delta_{ij} \text{ and } \sum_{i=1}^{\ell} I_{v_i}(-\Omega^{m_q}) \geq \ell \Omega^{-d-\delta}.$$

*Proof.* By (1) and (2), the sequence  $a_m = \Pi_{i=1}^{\ell} f_i(-\Omega^m)$  is non-decreasing. By (2) positive for  $m$  large, and  $a_m \leq C(1 + \Omega^m)^{d\ell}$ . Therefore, there must exist  $m_q \rightarrow \infty$  where

$$(3.11) \quad a_{m_q+1} \leq \Omega^{d\ell+\delta} a_{m_q}.$$

If this was not the case, then we would get some  $\bar{m}$  so that  $a_{m+1} \geq \Omega^{d\ell+\delta} a_m$  for every  $m \geq \bar{m}$ . Iterating this forces  $a_m$  to grow and, eventually, contradict  $a_m \leq C(1 + \Omega^m)^{d\ell}$ .

We will show (3.10) holds for  $m_q$  satisfying (3.11). Namely, (1) and (3.11) give

$$(3.12) \quad \prod_{i=1}^{\ell} I_{v_i}(-\Omega^{m_q}) \geq \prod_{i=1}^{\ell} \frac{f_i(-\Omega^{m_q})}{f_i(-\Omega^{m_q+1})} = \frac{a_{m_q}}{a_{m_q+1}} \geq \Omega^{-d\ell-\delta}.$$

Finally, combining this with the arithmetic-geometric mean inequality gives

$$(3.13) \quad \frac{1}{\ell} \sum_{i=1}^{\ell} I_{v_i}(-\Omega^{m_q}) \geq \left( \prod_{i=1}^{\ell} I_{v_i}(-\Omega^{m_q}) \right)^{\frac{1}{\ell}} \geq \Omega^{-d-\frac{\delta}{\ell}}.$$

□

**3.3. Localization.** We will need the following localization inequality:

**Lemma 3.14.** Given any function  $u$  on  $M_t$ , we have

$$(3.15) \quad \frac{1}{-t} \int_{M_t} |x|^2 u^2 e^{\frac{|x|^2}{4t}} \leq 4n \int_{M_t} u^2 e^{\frac{|x|^2}{4t}} - 4t \int_{M_t} (4|\nabla u|^2 + u^2 |\phi|^2) e^{\frac{|x|^2}{4t}}.$$

*Proof.* Using that  $(\partial_t - \Delta_{M_t})|x|^2 = -2n$  and  $x_t = -\mathbf{H}$  on  $M_t$ , we get

$$(3.16) \quad \begin{aligned} 2e^{-\frac{|x|^2}{4t}} \operatorname{div}_{M_t} \left( u^2 x^T e^{\frac{|x|^2}{4t}} \right) &= 4u \langle \nabla u, x^T \rangle + u^2 \left( \Delta_{M_t} |x|^2 + \frac{|x^T|^2}{t} \right) \\ &= 4u \langle \nabla u, x^T \rangle + u^2 \left( 2n - 2\langle x^\perp, \mathbf{H} \rangle + \frac{|x^T|^2}{t} \right) \\ &= 4u \langle \nabla u, x^T \rangle + u^2 \left( 2n - 2\langle x^\perp, \phi \rangle + \frac{|x|^2}{t} \right). \end{aligned}$$

Using the absorbing inequality twice gives

$$(3.17) \quad \begin{aligned} |4u \langle \nabla u, x^T \rangle - 2u^2 \langle x^\perp, \phi \rangle| &\leq \frac{u^2 |x^T|^2}{2|t|} - 8t |\nabla u|^2 + \frac{u^2 |x^\perp|^2}{2|t|} - 2tu^2 |\phi|^2 \\ &= \frac{u^2 |x|^2}{2|t|} - 8t |\nabla u|^2 - 2tu^2 |\phi|^2. \end{aligned}$$

Inserting this in (3.16) and applying the divergence theorem gives the lemma. □



#### 4. SHARP BOUNDS FOR $\dim \mathcal{P}_d$ ON AN ANCIENT MCF

In this section,  $M_t^n \subset \mathbf{R}^N$  is an ancient MCF with  $\lambda(M_t) \leq \lambda_0$  for all  $t$ . We will need the following local meanvalue inequality (proposition 2.1 in [E2]; cf. [Hu]):

**Lemma 4.1.** There exists  $c$  depending on  $n$  so that if  $(\partial_t - \Delta)u = 0$ , then for any  $\rho > 0$

$$(4.2) \quad u^2(x_0, t_0) \leq \frac{c}{\rho^{n+2}} \int_{t_0-\rho^2}^{t_0} \int_{B_\rho(x_0) \cap M_t} u^2.$$

*Proof of Theorem 0.5.* Suppose that  $d \geq 1$  and  $u_0 \equiv 1, u_1, \dots, u_\ell$  are linearly independent functions in  $\mathcal{P}_d(M_t)$ . We will prove that there is a constant  $C_n$  so that  $\ell \leq C_n \lambda_0 d^n$ .

The first step is to apply Lemma 3.9 with  $\Omega = 1 + \frac{3}{d}$  and  $\delta = d$  to get  $m_q \rightarrow \infty$  so that  $v_1, \dots, v_\ell$  defined by  $v_i = \frac{w_{i, -\Omega^{m_q+1}}}{\sqrt{f_i(-\Omega^{m_q+1})}}$  satisfy

$$(4.3) \quad J_{-\Omega^{m_q+1}}(v_i, v_j) = \delta_{ij} \text{ and } \sum_{i=1}^{\ell} I_{v_i}(-\Omega^{m_q}) \geq \ell \Omega^{-2d} \geq e^{-6} \ell.$$

Integrating  $I'_{v_i}$  from  $-(1 + 1/d)\Omega^{m_q}$  to  $-\Omega^{m_q}$ , there exists  $t_0 \in [-(1 + 1/d)\Omega^{m_q}, -\Omega^{m_q}]$  with

$$(4.4) \quad \begin{aligned} \frac{\Omega^{m_q}}{d} \sum_{i=1}^{\ell} (-4\pi t_0)^{-\frac{n}{2}} \int_{M_{t_0}} \{2|\nabla v_i|^2 + v_i^2 |\phi|^2\} e^{\frac{|x|^2}{4t_0}} &= \frac{\Omega^{m_q}}{d} \sum_{i=1}^{\ell} |I'_{v_i}|(t_0) \\ &\leq \int_{-(1+1/d)\Omega^{m_q}}^{-\Omega^{m_q}} \sum_{i=1}^{\ell} |I'_{v_i}|(t) dt \leq \ell. \end{aligned}$$

Since  $I_{v_i}$  is monotone and  $\frac{|t_0|}{\Omega^{m_q}} \in [1, 1 + 1/d]$ , (4.3) and (4.4) give

$$(4.5) \quad e^{-6} \ell \leq \sum_{i=1}^{\ell} I_{v_i}(t_0),$$

$$(4.6) \quad -t_0 (-4\pi t_0)^{-\frac{n}{2}} \sum_{i=1}^{\ell} \int_{M_{t_0}} \{2|\nabla v_i|^2 + v_i^2 |\phi|^2\} e^{\frac{|x|^2}{4t_0}} \leq d \ell \left(1 + \frac{1}{d}\right) \leq 2d \ell.$$

Applying the localization inequality Lemma 3.14 to each  $v_i$  gives

$$(-4\pi t_0)^{-\frac{n}{2}} \int_{M_{t_0}} \frac{|x|^2}{-t_0} v_i^2 e^{\frac{|x|^2}{4t_0}} \leq 4n I_{v_i}(t_0) - t_0 (-4\pi t_0)^{-\frac{n}{2}} \int_{M_{t_0}} (16|\nabla v_i|^2 + 4v_i^2 |\phi|^2) e^{\frac{|x|^2}{4t_0}}.$$

Summing this over  $i$  and then using (4.6), we conclude that

$$(4.7) \quad (-4\pi t_0)^{-\frac{n}{2}} \int_{M_{t_0}} \frac{|x|^2}{-t_0} \sum_{i=1}^{\ell} v_i^2 e^{\frac{|x|^2}{4t_0}} \leq (4n + 16d) \ell \leq (4n + 16) d \ell.$$

Now, define the function  $K(x, t) = \sum_{i=1}^{\ell} v_i^2(x, t)$  to be the “trace of the Bergman kernel”. Equation (4.5) gives

$$(4.8) \quad e^{-6} \ell \leq (-4\pi t_0)^{-\frac{n}{2}} \int_{M_{t_0}} K e^{\frac{|x|^2}{4t_0}}.$$

To bound  $\ell$ , we will combine (4.8) with an upper bound on the integral of  $K$ . We will divide the integral into an inner ball of radius proportional to  $\sqrt{-d t_0}$  and an integral outside.

Set  $\Lambda = e^6(8n + 32)$ . It follows from (4.7) that

$$(4.9) \quad (-4\pi t_0)^{-\frac{n}{2}} \int_{M_{t_0} \setminus B_{\sqrt{-\Lambda d t_0}}} K e^{\frac{|x|^2}{4t_0}} \leq \frac{(4n + 16)d}{\Lambda d} \ell \leq \frac{e^{-6}}{2} \ell.$$

Suppose, on the other hand, that  $x_0 \in B_{\sqrt{-\Lambda d t_0}}$ . Since  $K(x_0, t_0)$  is the trace of a quadratic form, there exist coefficients  $a_1, \dots, a_\ell$  so that  $\sum a_i^2 = 1$  and  $u(x, t) = \sum_{i=1}^\ell a_i v_i(x, t)$  satisfies  $K(x_0, t_0) = u^2(x_0, t_0)$ . Moreover, monotonicity of  $I_u$  and (4.3) give

$$(4.10) \quad \sup_{t \geq (1+1/d)t_0} I_u(t) \leq I_u((1 + 1/d)t_0) \leq I_u(-\Omega^{m_q+1}) = 1.$$

Set  $\rho = \frac{\sqrt{-t_0}}{\sqrt{d}}$  and observe that there is a constant  $c_n$ , depending just on  $n$ , so that

$$(4.11) \quad (-4\pi t_0)^{-\frac{n}{2}} e^{\frac{|x_0|^2}{4t_0}} \leq c_n \sup_{B_\rho(x_0) \times [t_0 - \rho^2, t_0]} \left\{ (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}} \right\}.$$

Lemma 4.1 gives  $c$  depending on  $n$  so that

$$(4.12) \quad u^2(x_0, t_0) \leq \frac{c}{\rho^{n+2}} \int_{t_0 - \rho^2}^{t_0} \int_{B_\rho(x_0) \cap M_t} u^2.$$

Combining this with (4.11) and the bound (4.10) on  $I_u$  gives

$$(4.13) \quad \begin{aligned} (-4\pi t_0)^{-\frac{n}{2}} e^{\frac{|x_0|^2}{4t_0}} u^2(x_0, t_0) &\leq \frac{c c_n}{\rho^{n+2}} \int_{t_0 - \rho^2}^{t_0} (-4\pi t)^{-\frac{n}{2}} \int_{B_\rho(x_0) \cap M_t} u^2 e^{\frac{|x|^2}{4t}} \leq \frac{c c_n}{\rho^{n+2}} \int_{t_0 - \rho^2}^{t_0} I_u(t) \\ &\leq \frac{c c_n}{\rho^n} I_u(t_0 - \rho^2) \leq \frac{c c_n}{\rho^n} = c c_n \left( \frac{d}{-t_0} \right)^{\frac{n}{2}}. \end{aligned}$$

Integrating this bound over  $x_0 \in B_{\sqrt{-\Lambda d t_0}} \cap M_{t_0}$  gives

$$(4.14) \quad (-4\pi t_0)^{-\frac{n}{2}} \int_{B_{\sqrt{-\Lambda d t_0}} \cap M_{t_0}} K e^{\frac{|x|^2}{4t_0}} \leq c c_n \left( \frac{d}{-t_0} \right)^{\frac{n}{2}} \text{Vol} \left( B_{\sqrt{-\Lambda d t_0}} \cap M_{t_0} \right) \leq C_n \lambda_0 d^n.$$

Using the lower bound from (4.8) and combining (4.9) with (4.14), we see that

$$(4.15) \quad e^{-6} \ell \leq (-4\pi t_0)^{-\frac{n}{2}} \int_{M_{t_0}} K e^{\frac{|x|^2}{4t_0}} \leq \frac{e^{-6}}{2} \ell + C_n \lambda_0 d^n.$$

We can absorb the first term on the right and the theorem follows.  $\square$

## 5. ENTROPY CONTROLS SPECTRAL MULTIPLICITY AND HEAT KERNEL

We will next bound the counting function on a shrinker and then estimate the heat kernel.

*Proof of Theorem 0.7.* The shrinker  $\Sigma$  gives rise to a MCF  $M_t$  where each  $M_t$  is given as a set by  $\sqrt{-t}\Sigma$ . Fix some  $\mu \geq \frac{1}{2}$ . For each  $L^2$ -eigenvalue  $\mu_i \leq \mu$  of  $\mathcal{L}$  on  $\Sigma$ , let  $u_i$  be

an eigenfunction with  $\|u_i\|_{L^2(\Sigma)} = 1$ . Corollary 2.24 then gives  $w_i \in \mathcal{P}_{2\mu_i}(M_t)$  defined by  $w_i(y, t) = (-t)^{\mu_i} u_i\left(\frac{y}{\sqrt{-t}}\right)$ . Combining this with Theorem 0.5 gives  $C = C(n)$  so that

$$(5.1) \quad \mathcal{N}(\mu) \leq \dim \mathcal{P}_{2\mu}(M_t) \leq C \lambda(\Sigma) \mu^n.$$

□

From Theorem 0.7, and the proof of the Courant nodal domain theorem, [CtHi], we get:

**Corollary 5.2.** If  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker, then any hyperplane through the origin cannot divide  $\Sigma$  into more than  $C_n \lambda(\Sigma)$  many components.

*Proof.* After a rotation, we may assume that the hyperplane is  $\{x_1 = 0\}$ . Since  $\mathcal{L} x_1 = -\frac{1}{2} x_1$  and  $\mathcal{N}\left(\frac{1}{2}\right) \leq C_n \lambda(\Sigma)$  by Theorem 0.7, the claim follows from the argument in the Courant nodal domain theorem for the operator  $\mathcal{L}$ ; see page 45 of [Cg] for a proof for  $\Delta$ . □

In the case where  $\Sigma^n \subset B_{\sqrt{2n}} \subset \mathbf{R}^N$  is a closed minimal submanifold and the entropy reduces to the volume, this result was established by Cheng-Li-Yau in corollary 6 of [CgLYa].

**5.1. Drift heat kernel on shrinkers.** In [CgLYa], Cheng, Li and Yau proved heat kernel estimates on closed spherical minimal submanifolds. Although there are many of these submanifolds, they form a relatively small subset of all closed shrinkers. In addition, there are many non-compact shrinkers. On a closed manifold, the heat kernel is given by  $H(x, y, t) = \sum_i e^{-\mu_i t} u_i(x) u_i(y)$  where the  $u_i$ 's are eigenfunctions with eigenvalues  $\mu_i$ . In general, the heat kernel on a non-compact manifold cannot be constructed this way unless the eigenvalues go to infinity at a rate. The heat kernel has four properties:  $H_t = \mathcal{L}H$ ,  $H(x, y, t) = H(y, x, t)$ , the reproducing property as  $t \rightarrow 0$ , and the semi-group property.

We next estimate the drift heat kernel  $H$  on a shrinker in arbitrary codimension and show that  $H$  is given explicitly in terms of the eigenfunctions. To construct  $H$ , we need that the spectrum is discrete, which was proven by Cheng-Zhou, [CxZh].

**Theorem 5.3.** Let  $\Sigma^n \subset \mathbf{R}^N$  be a shrinker with finite entropy. There is a complete basis of  $W^{1,2}$  eigenfunctions  $u_i$  for  $\mathcal{L}$  with eigenvalues  $\mu_i$  and  $\|u_i\|_{L^2} = 1$ . The heat kernel  $H(x, y, t)$  for  $\partial_t - \mathcal{L}$  exists and is given by

$$(5.4) \quad H(x, y, t) = \sum_i e^{-\mu_i t} u_i(x) u_i(y).$$

*Proof.* Let  $\mu_i^j$  be the Dirichlet eigenfunctions for  $\mathcal{L}$  on  $B_j \cap \Sigma$  and let  $u_i^j$  be the corresponding eigenfunctions with  $\|u_i^j\|_{L^2} = 1$ . By domain monotonicity of eigenvalues,  $\mu_i^j$  is non-increasing in  $j$  and we get limits  $\mu_i = \lim_{j \rightarrow \infty} \mu_i^j$ . For each  $i$ , elliptic theory gives uniform estimates for the  $u_i^j$  on compact subsets and, thus, Arzela-Ascoli gives limiting functions  $u_i$  with  $\mathcal{L}u_i = -\mu_i u_i$  with  $\|u_i\|_{L^2} \leq 1$ . Corollary 1.11 gives that  $\|u_i\|_{L^2} = 1$  and  $\|\nabla u_i\|_{L^2} \neq 0$  for  $i > 0$ , as desired. The  $\mu_i$  must go to infinity by [CxZh]; this also follows from Theorem 0.7.

We will show that the  $u_i$ 's are complete. If this was not the case, then there would be some with  $\|w\|_{L^2} = 1$ ,  $\|\nabla w\|_{L^2} < \infty$ , and

$$(5.5) \quad \int_{\Sigma} u_i w e^{-f} = 0 \text{ for every } i.$$

Since  $\mu_i \rightarrow \infty$ , we can fix  $k$  so that  $\mu_k > 2 \|\nabla w\|_{L^2}^2$ . The first claim in Lemma 1.5 gives

$$(5.6) \quad \int |x|^2 w^2 e^{-f} \leq 4n + 16 \int |\nabla w|^2 e^{-f}.$$

Let  $\phi_j$  be a cutoff function that is one on  $B_{j-1}$  and zero on  $\partial B_j$  and set  $w_j = \phi_j w$ . It follows from (5.5), (5.6) and the uniform convergence on compact sets of  $u_i^j$ 's to  $u_i$  that

$$(5.7) \quad \lim_{j \rightarrow \infty} \|w_j\|_{L^2} = 1,$$

$$(5.8) \quad \limsup_{j \rightarrow \infty} \|\nabla w_j\|_{L^2}^2 \leq \|\nabla w\|_{L^2}^2,$$

$$(5.9) \quad \lim_{j \rightarrow \infty} \int_{\Sigma} u_i^j w_j e^{-f} = 0 \text{ for } i \leq k.$$

In particular, we can choose some  $j$  large so that the orthogonal projection  $\bar{w}_j$  of  $w_j$  onto the eigenspaces with  $\mu_i^j$  with  $i > k$  has

$$(5.10) \quad \frac{3}{4} < \|\bar{w}_j\|_{L^2}^2 \text{ and } \|\nabla \bar{w}_j\|_{L^2}^2 \leq \frac{5}{4} \|\nabla w\|_{L^2}^2.$$

However, since  $\mu_k^j > \mu_k > 2 \|\nabla w\|_{L^2}^2$ , the variational characterization of eigenvalues gives

$$(5.11) \quad 2 \|\nabla w\|_{L^2}^2 \|\bar{w}_j\|_{L^2}^2 \leq \|\nabla \bar{w}_j\|_{L^2}^2.$$

This contradicts (5.10), so we conclude that the  $u_i$ 's are complete.

To see that the sum (5.4) converges for each  $t > 0$ , observe that elliptic theory and the bounds  $\int_{B_R} u_i^2 \leq e^{\frac{R^2}{4}}$  and  $\int_{B_R} |\nabla u_i|^2 \leq \mu_i e^{\frac{R^2}{4}}$  give  $c = c(R)$  so that

$$(5.12) \quad \mu_i \sup_{B_R} |u_i|^2 + \sup_{B_R} |\nabla u_i|^2 \leq c \mu_i^{\frac{n}{2}+1}.$$

Theorem 0.7 gives  $C = C(n)$  so that  $\mathcal{N}(m) \leq C \lambda m^n$ , so we have

$$(5.13) \quad \begin{aligned} \sup_{B_R \times B_R} |H|(x, y, t) &\leq \sum_m \left\{ \sum_{\mu_i \in [m-1, m]} c \mu_i^{\frac{n}{2}} e^{-\mu_i t} \right\} \leq c \sum_m \mathcal{N}(m) m^{\frac{n}{2}} e^{-(m-1)t} \\ &\leq c C \lambda e^t \sum_m m^{\frac{3n}{2}} e^{-mt}. \end{aligned}$$

This is finite for each  $t > 0$ . Arguing similarly gives estimates also for higher derivatives, so Arzela-Ascoli gives convergence of (5.4) on compact subsets. It then follows that  $H$  has the semi-group property and satisfies the drift heat equation. Finally, the reproducing property at  $t = 0$  follows from the completeness of the eigenvalues.  $\square$

## 6. RIGIDITY

In this section, we show that a shrinker, even in high codimension, that is close to a cylinder on a sufficiently large bounded set must be a hypersurface in some Euclidean subspace. It then follows from [CIM] that it must be a cylinder; cf. [CM9], [GKS].

**6.1. Convergence of the spectrum.** By Theorem 5.3, the operator  $\mathcal{L}$  on a shrinker has eigenvalues  $0 = \mu_0 < \mu_1 \leq \dots$  going to infinity and a complete basis of  $L^2$  eigenfunctions. Given  $r > \sqrt{2n}$ , let  $\beta_i^r$  be the Dirichlet eigenvalues of  $\mathcal{L}$  on  $B_r \cap \Sigma$ . We show next that the Dirichlet spectrum converges uniformly.

**Lemma 6.1.** Given  $k, \delta > 0$  and  $n$ , there exists  $\bar{r} = \bar{r}(\mu_k, n, \delta)$  so that for  $\bar{r} \leq r$

$$(6.2) \quad \mu_i \leq \beta_i^r \leq \mu_i + \delta \text{ for every } i \leq k.$$

*Proof.* Domain monotonicity of eigenvalues gives for  $r_1 < r_2$  and every  $i$

$$(6.3) \quad \mu_i \leq \beta_i^{r_2} \leq \beta_i^{r_1}.$$

This gives the first inequality in (6.2). Let  $u_1, \dots, u_k$  satisfy  $\mathcal{L} u_i = -\mu_i u_i$  and  $\|u_i\|_{L^2} = 1$ . Corollary 1.11 gives  $c = c(n, \mu_k) > 0$  so that

$$(6.4) \quad \int_{\Sigma \setminus B_r} (u_i^2 + |\nabla u_i|^2) e^{-f} \leq \frac{c}{(r-1)^2} \text{ for } i = 1, \dots, k.$$

Let  $\psi$  be a linear cutoff function that is one on  $B_r$  and zero outside of  $B_{r+1}$  and set  $v_i = \psi u_i$ . The  $v_i$ 's are supported in  $B_{r+1}$  and we get for each  $i$  that

$$(6.5) \quad \|u_i - v_i\|_{W^{1,2}}^2 \leq \int ((1-\psi)^2 u_i^2 + 2(1-\psi)^2 |\nabla u_i|^2 + 2u_i^2 |\nabla \psi|^2) e^{-f} \leq \frac{5c}{(r-1)^2}.$$

Since (6.5) implies that the  $u_i$  and  $v_i$  are close both in  $L^2$  and for the energy, we get the last inequality in (6.2) for each  $i$  for  $r$  sufficiently large depending on  $\mu_k, n$  and  $\delta$ .  $\square$

**6.2. Stability of eigenvalues.** In this subsection,  $\Gamma^n \subset \mathbf{R}^N$  is a smooth complete shrinker with finite entropy; let  $\mu_i^\Gamma$  denote its eigenvalues.

**Definition 6.6.** We will say that  $\Sigma$  and  $\Gamma$  are  $(\epsilon, R, C^1)$ -close if  $B_R \cap \Sigma$  can be written as a normal graph of a vector field  $U$  over (a subset of)  $\Gamma$  and  $\|U\|_{C^1} \leq \epsilon$  and, likewise,  $B_R \cap \Gamma$  is a graph over  $\Sigma$ .

The definition of  $(\epsilon, R, C^1)$ -close gives  $C^1$  control in a compact set, allowing wild differences outside of this set. The next proposition shows that this is enough to get spectral stability.

**Proposition 6.7.** Given  $k$  and  $\delta > 0$ , there exist  $\epsilon$  and  $R$  depending on  $\delta, k, \Gamma$  so that if a shrinker  $\Sigma^n$  is  $(\epsilon, R, C^1)$ -close to  $\Gamma$  and  $\lambda(\Sigma) < \infty$ , then for  $i \leq k$

$$(6.8) \quad |\mu_i^\Gamma - \mu_i^\Sigma| \leq \delta.$$

*Proof.* Lemma 6.1 gives  $\bar{r} = \bar{r}(\mu_k^\Gamma, n, \delta)$  so that for  $\bar{r} \leq r$

$$(6.9) \quad \mu_i^\Gamma \leq \beta_i^{\Gamma, r} \leq \mu_i^\Gamma + \frac{\delta}{3} \text{ for every } i \leq k.$$

Moreover, given a fixed  $r \leq R$ , then for  $\epsilon > 0$  small enough we can identify  $B_r \cap \Gamma$  and  $B_r \cap \Sigma$  and, moreover, this identification is almost an isometry on  $L^2$  and almost preserves the energy. It follows that we can arrange that

$$(6.10) \quad |\beta_i^{\Gamma, r} - \beta_i^{\Sigma, r}| \leq \frac{\delta}{3}.$$

Combining (6.9), (6.10) and the fact that  $\mu_i^\Sigma \leq \beta_i^{\Sigma,r}$ , we get an upper bound on  $\mu_k^\Sigma$ . Hence, we can apply Lemma 6.1 to get  $r$  large enough that

$$(6.11) \quad \mu_i^\Sigma \leq \beta_i^{\Sigma,r} \leq \mu_i^\Sigma + \frac{\delta}{3} \text{ for every } i \leq k.$$

Finally, combining (6.9), (6.10) and (6.11) gives the proposition.  $\square$

An immediate corollary is the lower semi-continuity of the spectral multiplicity (here  $d(\mu)$  will denote the multiplicity of an eigenvalue  $\mu$ ):

**Corollary 6.12.** Given  $\mu$ , there exist  $\epsilon, R > 0$  depending on  $\mu$  and  $\Gamma$  so that if a shrinker  $\Sigma^n$  with  $\lambda(\Sigma) < \infty$  is  $(\epsilon, R, C^1)$ -close to  $\Gamma$ , then  $d_\Sigma(\mu) \leq d_\Gamma(\mu)$ .

*Proof.* Theorem 0.7 gives that the  $\mu_i^\Gamma$ 's go to infinity, so we can  $\delta > 0$  so that  $\Gamma$  has no eigenvalues in  $[\mu - 2\delta, \mu] \cup (\mu, \mu + 2\delta]$ . It follows that

$$(6.13) \quad d_\Gamma(\mu) = \mathcal{N}_\Gamma(\mu + 2\delta) - \mathcal{N}_\Gamma(\mu - 2\delta).$$

Using Proposition 6.7 with  $k = \mathcal{N}_\Gamma(\mu + 2\delta) + 1$  gives  $\epsilon > 0$  and  $R$  so that

$$(6.14) \quad |\mu_i^\Gamma - \mu_i^\Sigma| < \delta \text{ for } i \leq k.$$

This implies that  $\mathcal{N}_\Sigma(\mu + \delta) \leq \mathcal{N}_\Gamma(\mu + 2\delta)$  and  $\mathcal{N}_\Gamma(\mu - 2\delta) \leq \mathcal{N}_\Sigma(\mu - \delta)$  and, thus,

$$(6.15) \quad d_\Sigma(\mu) \leq \mathcal{N}_\Sigma(\mu + \delta) - \mathcal{N}_\Sigma(\mu - \delta) \leq \mathcal{N}_\Gamma(\mu + 2\delta) - \mathcal{N}_\Gamma(\mu - 2\delta).$$

The corollary follows by combining this with (6.13).  $\square$

In [dCW], do Carmo-Wallach construct families of minimal submanifolds of the sphere, each isometric to the same round sphere, generalizing results of Calabi, [Ca]. The boundary immersions of the families in [dCW] lie in a lower-dimensional affine space. Obviously, they have the same volume and, since they are contained in spheres, also the same entropy. Thus, the number of linearly independent coordinate functions can vary along a family.

**6.3. Shrinking curves.** We next recall the classification of shrinking curves and some of their elementary properties. These results will not be used in the main theorems, but will be used in the next subsection to give a type of rigidity that illustrates ideas in a simple case and that could be useful in later applications.

In [AL], Abresch-Langer classified closed shrinking curves in  $\mathbf{R}^2$ . The only embedded one is the circle  $\mathbf{S}^1_{\sqrt{2}}$ . There are immersed solutions  $\gamma_{m,\ell}$  with  $\frac{1}{2} < \frac{m}{\ell} < \frac{\sqrt{2}}{2}$  where  $m$  is the rotation index and  $\ell$  is the number of periods of its curvature function. Moreover, each  $\gamma_{m,\ell}$  is convex and has self-intersections, so  $\lambda(\gamma_{m,\ell}) > 2$ .

**Lemma 6.16.** If  $\gamma^1 \subset \mathbf{R}^N$  is a complete immersed shrinker and  $\lambda(\gamma) < \infty$ , then  $\gamma$  is a rotation of either  $\mathbf{R}$ ,  $\mathbf{S}^1_{\sqrt{2}}$ , or one of the  $\gamma_{m,\ell}$ 's.

*Proof.* Suppose first that  $\gamma \subset \mathbf{R}^2$  with unit normal  $\mathbf{n}$  and let  $k = \frac{1}{2}\langle x, \mathbf{n} \rangle$  be its geodesic curvature. By [AL], the quantity  $k e^{-\frac{|x|^2}{4}}$  is constant (this follows from differentiating the equation  $k = \frac{1}{2}\langle x, \mathbf{n} \rangle$ ). If  $k$  ever vanishes, then it is identically zero and  $\gamma = \mathbf{R}$ . Otherwise, we can assume that  $k > 0$  and there is a constant  $c > 0$  with

$$(6.17) \quad c e^{\frac{|x|^2}{4}} = k \leq \frac{|x|}{2}.$$



It follows that  $k \geq c > 0$  and  $|x|$ , and thus also  $k$ , are bounded from above. Thus,  $\gamma$  is a convex curve in a bounded region. Since  $\lambda(\gamma) < \infty$  and  $\mathcal{L}|x|^2 = 2 - |x|^2$ , we have

$$(6.18) \quad 4c^2 \int_{\gamma} e^{\frac{|x|^2}{4}} = 4 \int_{\gamma} k^2 e^{-\frac{|x|^2}{4}} \leq \int_{\gamma} |x|^2 e^{-\frac{|x|^2}{4}} = 2 \int_{\gamma} e^{-\frac{|x|^2}{4}} = 2\lambda(\gamma).$$

Therefore,  $\gamma$  has finite length and must be one of the Abresch-Langer curves. Finally, by uniqueness for ODE's, these are also the only shrinking curves in  $\mathbf{R}^N$  (up to rotation).  $\square$

The next lemma shows that coordinate functions generate the entire  $\frac{1}{2}$ -eigenspace on the product of an Abresch-Langer curve with  $\mathbf{R}^{n-1}$ :

**Lemma 6.19.** For any  $n, m$  and  $\ell$ , we have  $d_{\gamma_{m,\ell} \times \mathbf{R}^{n-1}}(\frac{1}{2}) = n + 1$ .

*Proof.* Set  $\gamma = \gamma_{m,\ell}$  and  $\Gamma = \gamma_{m,\ell} \times \mathbf{R}^{n-1}$ . Following lemma 3.26 in [CM9], let  $y_i$  be coordinates on  $\mathbf{R}^{n-1}$  so that  $\mathcal{L}$  splits as

$$(6.20) \quad \mathcal{L} = \mathcal{L}_{\gamma} + \mathcal{L}_y,$$

where  $\mathcal{L}_{\gamma}$  is the drift operator on  $\gamma \subset \mathbf{R}^2$  and  $\mathcal{L}_y$  is the drift operator on  $\mathbf{R}^{n-1}$ . Suppose that  $u$  is an  $L^2$  and, thus also  $W^{1,2}$ , function on  $\Gamma$  satisfying  $\mathcal{L}u = -\frac{1}{2}u$ . Set  $u_i = \frac{\partial u}{\partial y_i}$  and observe that  $\mathcal{L}u_i = 0$ . It follows that  $u_i$  is constant. Since this holds for each  $i$ ,  $u = \sum_i a_i y_i + g$  where the  $a_i$ 's are constants and  $g$  is a function on  $\gamma$ . Consequently, to prove the lemma, we must show that the two coordinate functions generate the entire  $\frac{1}{2}$ -eigenspace on  $\gamma \subset \mathbf{R}^2$ . However, this follows immediately from uniqueness for the second order ODE  $\mathcal{L}_{\gamma}g = -\frac{1}{2}g$ .  $\square$

#### 6.4. Rigidity of spheres and cylinders.

**Corollary 6.21.** Given  $k < n$ , there exists  $R > 2n$  so if  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker with  $\lambda(\Sigma) < \infty$  and  $B_R \cap \Sigma$  is the graph of a vector field  $U$  over  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$  with  $\|U\|_{C^1} < 1/R$ , then there is a  $(n+1)$ -dimensional Euclidean space  $\mathcal{W}$  so that  $\Sigma \subset \mathcal{W}$ .

*Proof.* On the cylinder  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$  (see, e.g., section 3 in [CM9]), the low eigenvalues of  $\mathcal{L}$  are  $\mu_0 = 0$ , given by the constants,  $\frac{1}{2}$  with multiplicity  $n+1$ , and then there is a gap to 1. More precisely, it follows from lemma 3.26 in [CM9] (and its proof<sup>2</sup>) that:

- If  $u \in W^{1,2}$  has  $\mathcal{L}u = -\frac{1}{2}u$ , then  $u = f(\theta) + \sum a_j y_j$  where  $f$  is a  $\frac{1}{2}$ -eigenfunction on  $\mathbf{S}^k$ ,  $a_j$  are constants, and  $y_j$  are coordinate functions on the axis  $\mathbf{R}^{n-k}$ .

It follows that  $d_{\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}}(\frac{1}{2}) = n+1$  and Corollary 6.12 with  $\mu = \frac{1}{2}$  completes the proof.  $\square$

*Proof of Theorem 0.17.* As long as  $R$  is large enough, Corollary 6.21 gives that  $\Sigma$  is a hyperplane in some affine  $(n+1)$ -space. By assumption, it is also close to a cylinder. We will apply the main rigidity theorem for hypersurfaces from [CIM], but first need to establish a uniform bound for  $\int_{\Sigma} e^{-f}$  (which gives an entropy bound). However, this follows from the closeness to a cylinder in  $B_R$ , for  $R$  large, since  $\int_{\Sigma} (|x|^2 - 2n) e^{-f} = 0$ .

In the case of a sphere (i.e.,  $k = n$ ), we can argue directly without [CIM]. Namely, since  $\Sigma$  is a shrinker, we have  $\mathcal{L}(|x|^2 - 2n) = -(|x|^2 - 2n)$ . However, 1 is not in the spectrum for  $\mathbf{S}_{\sqrt{2n}}^n$ , so Corollary 6.12 gives 1 is also not in the spectrum of  $\Sigma$  for  $\Sigma$  sufficiently close. It follows that  $|x|^2 - 2n \equiv 0$  and  $\Sigma \subset \partial B_{\sqrt{2n}}$ .  $\square$

<sup>2</sup>Lemma 3.26 in [CM9] deals with eigenvalue 1; obvious modifications give eigenvalue  $\frac{1}{2}$  as well.

We get the corresponding statement for products with the Abresch-Langer curves  $\gamma_{m,\ell}$ :

**Corollary 6.22.** Given  $m, \ell, n$ , there exists  $R > 2n$  so if  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker with  $\lambda(\Sigma) < \infty$  and  $B_R \cap \Sigma$  is a parameterized graph of a vector field  $U$  over  $\gamma_{m,\ell} \times \mathbf{R}^{n-1}$  with  $\|U\|_{C^1} < 1/R$ , then there is a  $(n+1)$ -dimensional Euclidean space  $\mathcal{W}$  so that  $\Sigma \subset \mathcal{W}$ .

*Proof.* This follows by combining Lemma 6.19 and Corollary 6.12 with  $\mu = \frac{1}{2}$ .  $\square$

The rigidity should also hold for  $\gamma_{m,\ell} \times \mathbf{R}^{n-1}$  by combining Corollary 6.22 with a modification of [CIM]. See [CM14] for rigidity of cylinders in Ricci flow and [CM15] for uniqueness.

## 7. SHARP BOUNDS FOR CODIMENSION

In this section, we show that an ancient MCF  $M_t$  that is cylindrical at  $-\infty$  must be a flow of hypersurfaces in a Euclidean subspace. Since the constant function and the Euclidean coordinate functions are in  $\mathcal{P}_1(M_t)$ , the point will be to use the asymptotic cylindrical structure to prove  $\dim \mathcal{P}_1(M_t) \leq n+2$ . This estimate is absolutely sharp and quite delicate.

The argument is by contradiction, showing that  $\dim \mathcal{P}_1(M_t) \geq n+3$  forces one of the functions to grow faster than linear. Roughly speaking, we divide the  $(n+3)$ -dimensional space into the constant function,  $n+1$  functions that are roughly linear, and a last function orthogonal to both constants and linear functions. Orthogonality is not preserved in time, but we show that it is roughly preserved in the next subsection. After this, we show that the orthogonality leads to growth. Namely, using a Gaussian Poincaré inequality, we show that each of the functions orthogonal to constants must grow at least linearly in Lemma 7.30. The quadratic lower bound for growth, Lemma 7.54, will come from a Gaussian Poincaré inequality that holds on cylinders for functions that are orthogonal to both constants and to the coordinate functions. Finally, in the last subsection, we combine these growth estimates with the results of Section 3 to get the desired contradiction.

There are two subtle points that are worth highlighting. First, the lower bounds for growth hold only as long as the orthogonality does; thus, we can only work on a fixed scale. This is why we need Section 3 to find the right scale. Second, the results of Section 3 give bounds on the trace of the projection kernel for  $\mathcal{P}_1$ , so to get very sharp upper bounds on any individual function requires very sharp lower bounds on the remaining functions. This is why we must show that all but one of the functions grows at least linearly.

In this section,  $M_t^n \subset \mathbf{R}^N$  is an ancient MCF with  $\lambda(M_t) \leq \lambda_0$  and  $\phi = \mathbf{H} + \frac{x^\perp}{2t}$ .

**7.1. Preserving orthogonality.** The next lemma shows that a caloric function that integrates to zero on one time slice must remain nearly orthogonal to constants, with the error bounded by the change in the Gaussian area  $I_1(t)$ .

**Lemma 7.1.** If  $(\partial_t - \Delta)u = 0$  and  $J_{t_1}(u, 1) = 0$  for some  $t_1 < 0$ , then for any  $t_2 \in [t_1, 0)$

$$(7.2) \quad |J_{t_2}(u, 1)|^2 \leq I_u(t_1) |I_1(t_1) - I_1(t_2)|.$$

*Proof.* Given  $t \in [t_1, t_2]$ , (3.1) gives the derivative

$$(7.3) \quad \frac{d}{dt} J_t(u, 1) = -(-4\pi t)^{-\frac{n}{2}} \int_{M_t} u |\phi|^2 e^{\frac{|x|^2}{4t}}.$$

Since  $J_{t_1}(u, 1) = 0$ , integrating from  $t_1$  to  $t_2$  and using the Cauchy-Schwarz inequality gives

$$(7.4) \quad |J_{t_2}(u, 1)|^2 \leq \left\{ \int_{t_1}^{t_2} (-4\pi t)^{-\frac{n}{2}} \int_{M_t} u^2 |\phi|^2 e^{\frac{|x|^2}{4t}} \right\} \left\{ \int_{t_1}^{t_2} (-4\pi t)^{-\frac{n}{2}} \int_{M_t} |\phi|^2 e^{\frac{|x|^2}{4t}} \right\}.$$

By (3.1), the last integral on the right is bounded by  $|I_1(t_1) - I_1(t_2)|$ . Similarly, by Lemma 3.4, the first integral is bounded by  $\left| \int_{t_1}^{t_2} I'_u(t) \right| \leq I_u(t_1) - I_u(t_2)$ .  $\square$

In the next two lemmas,  $V \in \mathbf{S}^{N-1}$  is a unit vector and  $v$  the linear function  $v(x) = \langle x, V \rangle$ . Let  $\mathcal{L}_t = \Delta + \frac{1}{2t} \nabla_{x^T}$  be the drift operator that is symmetric for  $e^{\frac{|x|^2}{4t}}$ . We will use that

$$(7.5) \quad \mathcal{L}_t v = \operatorname{div}_{M_t} V^T + \frac{1}{2t} \langle x^T, V \rangle = -\langle V, \mathbf{H} \rangle + \langle \frac{x^T}{2t}, V \rangle = \frac{v}{2t} - \langle \phi, V \rangle.$$

**Lemma 7.6.** We get for  $t_1 < t_2 < 0$  that

$$\int_{t_1}^{t_2} (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{v^2}{-t} |\phi|^2 e^{\frac{|x|^2}{4t}} dt \leq \left| \frac{I_v(t_1)}{t_1} - \frac{I_v(t_2)}{t_2} \right| + C_n \sqrt{\lambda_0} \left( \frac{t_1}{t_2} \right)^{\frac{1}{2}} |I_1(t_1) - I_1(t_2)|^{\frac{1}{2}}.$$

*Proof.* Using (3.1) for  $\frac{v^2}{-t}$ , then the divergence theorem and (7.5) gives

$$(7.7) \quad \begin{aligned} \frac{d}{dt} \left( \frac{I_v(t)}{-t} \right) &= (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \left\{ \frac{v^2}{t^2} - \frac{2|\nabla v|^2}{-t} - \frac{v^2}{-t} |\phi|^2 \right\} e^{\frac{|x|^2}{4t}} \\ &= (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \left\{ -\frac{2v \langle \phi, V \rangle}{-t} - \frac{v^2}{-t} |\phi|^2 \right\} e^{\frac{|x|^2}{4t}}. \end{aligned}$$

Using that  $|V| = 1$ , we get the absorbing inequality for any  $\epsilon > 0$

$$(7.8) \quad \left| \frac{2v \langle \phi, V \rangle}{-t} \right| \leq \epsilon \frac{v^2}{t^2} + \frac{1}{\epsilon} |\phi|^2.$$

Using (7.8) in (7.7), we get that

$$(7.9) \quad (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{v^2}{-t} |\phi|^2 e^{\frac{|x|^2}{4t}} \leq -\frac{d}{dt} \left( \frac{I_v(t)}{-t} \right) + \epsilon \frac{I_v(t)}{t^2} + \frac{1}{\epsilon} (-4\pi t)^{-\frac{n}{2}} \int_{M_t} |\phi|^2 e^{\frac{|x|^2}{4t}}.$$

Integrating this from  $t_1$  to  $t_2$  and using the monotonicity of  $I_v$  to bound the second term on the right and Huisken's monotonicity on the last term gives

$$\int_{t_1}^{t_2} (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{v^2}{-t} |\phi|^2 e^{\frac{|x|^2}{4t}} dt \leq \left| \frac{I_v(t_1)}{t_1} - \frac{I_v(t_2)}{t_2} \right| + \epsilon \frac{I_v(t_1)}{-t_2} + \frac{|I_1(t_1) - I_1(t_2)|}{\epsilon}.$$

The lemma follows by using that  $I_v(t) \leq -C_n \lambda_0 t$  (cf. (3.7)) and optimizing  $\epsilon$ .  $\square$

The next lemma shows that the inner product of a caloric function with a fixed linear function grows approximately linearly in  $t$ .

**Lemma 7.10.** If  $u_t = \Delta u$  and  $t_1 < t_2 < 0$ , then for any  $\epsilon_1 \in (0, 1/2)$  and all  $t \in [t_1, t_2]$

$$\begin{aligned} \sqrt{-t_2} \left| \frac{J_{t_1}(u, v)}{t_1} - \frac{J_t(u, v)}{t} \right| &\leq \frac{5}{2} \epsilon_1 I_u(t_1) + \frac{1}{\epsilon_1} |I_1(t_1) - I_1(t_2)| \\ &+ \frac{1}{2\epsilon_1} \left\{ \left| \frac{I_v(t_1)}{t_1} - \frac{I_v(t_2)}{t_2} \right| + C_n \sqrt{\lambda_0} \left( \frac{t_1}{t_2} \right)^{\frac{1}{2}} |I_1(t_1) - I_1(t_2)|^{\frac{1}{2}} \right\}. \end{aligned}$$

*Proof.* Since  $u$  and  $v$  satisfy the heat equation, applying (3.1) to  $v u$  gives

$$(7.11) \quad \frac{d}{dt} J_t(u, v) = -(-4\pi t)^{-\frac{n}{2}} \int_{M_t} \{2 \langle \nabla u, \nabla v \rangle + u v |\phi|^2\} e^{\frac{|x|^2}{4t}}.$$

Using (7.5) and the divergence theorem on the first term in (7.11) gives

$$(7.12) \quad \begin{aligned} \frac{d}{dt} J_t(u, v) &= (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \{2 u \mathcal{L}_t v - u v |\phi|^2\} e^{\frac{|x|^2}{4t}} \\ &= \frac{1}{t} J_t(u, v) - (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \{2 u \langle \phi, V \rangle + u v |\phi|^2\} e^{\frac{|x|^2}{4t}}. \end{aligned}$$

Using absorbing inequalities on the last integral, we get for any  $\epsilon_1 > 0$  that

$$\left| \frac{d}{dt} \frac{J_t(u, v)}{t} \right| \leq \epsilon_1 \frac{I_u(t)}{(-t)^{3/2}} + (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \left\{ \frac{|\phi|^2}{\epsilon_1 \sqrt{-t}} + \left( \frac{\epsilon_1 u^2}{2\sqrt{-t}} + \frac{v^2}{2\epsilon_1 (-t)^{3/2}} \right) |\phi|^2 \right\} e^{\frac{|x|^2}{4t}}.$$

Integrating in  $t$ , using the monotonicity of  $I_u$  on the first term, (3.1) on the next two terms, and Lemma 7.6 on the last term gives for any  $t \in [t_1, t_2]$  that

$$\begin{aligned} \sqrt{-t_2} \left| \frac{J_{t_1}(u, v)}{t_1} - \frac{J_t(u, v)}{t} \right| &\leq 2\epsilon_1 I_u(t_1) + \frac{1}{\epsilon_1} |I_1(t_1) - I_1(t_2)| + \frac{\epsilon_1}{2} |I_u(t_1) - I_u(t_2)| \\ &+ \frac{1}{2\epsilon_1} \left\{ \left| \frac{I_v(t_1)}{t_1} - \frac{I_v(t_2)}{t_2} \right| + C_n \sqrt{\lambda_0} \left( \frac{t_1}{t_2} \right)^{\frac{1}{2}} |I_1(t_1) - I_1(t_2)|^{\frac{1}{2}} \right\}. \end{aligned}$$

□

**7.2. Poincaré inequalities.** The first eigenvalue on a cylinder is  $\frac{1}{2}$ , with the eigenspace spanned by the  $(n+1)$  coordinate functions, and the next eigenvalue is 1. The next lemma gives corresponding Poincaré inequalities for submanifolds close to a cylinder on a fixed large set. This requires that the function satisfies a “localization inequality” (cf. Lemma 3.14):

$$(7.13) \quad (-4\pi t)^{-\frac{n}{2}} \int_{\Gamma} \left( \frac{|x|^2}{-t} u^2 - t |\nabla u|^2 \right) e^{\frac{|x|^2}{4t}} \leq C_0 (-4\pi t)^{-\frac{n}{2}} \int_{\Gamma} u^2 e^{\frac{|x|^2}{4t}} < \infty.$$

In the next lemma,  $\Gamma^n \subset \mathbf{R}^N$  is a submanifold with  $\lambda(\Gamma) \leq \lambda_0 < \infty$  and  $B_{R_\mu} \cap \frac{\Gamma}{\sqrt{-t}}$  is a  $C^1$  graph over a cylinder with norm at most  $\epsilon_\mu$  and  $t < 0$  is a constant.

**Lemma 7.14.** Given  $C_0$  and  $\mu > 0$ , there exists  $\epsilon_\mu > 0$  and  $R_\mu > 0$  so that if  $u$  is a  $W^{1,2}$  function satisfying  $\int_{\Gamma} u e^{\frac{|x|^2}{4t}} = 0$  and (7.13), then

$$(7.15) \quad (1 - \mu) (-4\pi t)^{-\frac{n}{2}} \int_{\Gamma} u^2 e^{\frac{|x|^2}{4t}} \leq -2t (-4\pi t)^{-\frac{n}{2}} \int_{\Gamma} |\nabla u|^2 e^{\frac{|x|^2}{4t}}.$$

If in addition  $\int_{\Gamma} u x_i e^{\frac{|x|^2}{4t}} = 0$  for the coordinates  $x_1, \dots, x_{n+1}$  on the cylinder, then

$$(7.16) \quad (1 - \mu) (-4\pi t)^{-\frac{n}{2}} \int_{\Gamma} u^2 e^{\frac{|x|^2}{4t}} \leq -t (-4\pi t)^{-\frac{n}{2}} \int_{\Gamma} |\nabla u|^2 e^{\frac{|x|^2}{4t}}.$$

*Proof.* Since the statement is scale-invariant, we can assume that  $t = -1$ . To shorten notation, let  $\oint$  denote the Gaussian integral

$$(7.17) \quad \oint w \equiv (4\pi)^{-\frac{n}{2}} \int w e^{-f}.$$

Let  $L$  be a large integer to be chosen and choose  $R \in \{L, L+1, \dots, 2L-1\}$  with

$$(7.18) \quad \oint_{B_{R+1} \cap \Gamma \setminus B_R} |\nabla u|^2 \leq \frac{1}{L} \oint_{B_{2L} \cap \Gamma \setminus B_L} |\nabla u|^2 \leq \frac{1}{L} \oint_{\Gamma} |\nabla u|^2.$$

Combining this with the localization inequality (7.13) gives

$$(7.19) \quad \oint_{B_{R+1} \cap \Gamma \setminus B_R} |\nabla u|^2 \leq \frac{C_0}{L} \oint_{\Gamma} u^2.$$

Let  $\psi$  be a linear cutoff function from  $R$  to  $R+1$  and define  $w = \psi u$ . The localization inequality (7.13) gives

$$(7.20) \quad \oint_{\Gamma} |u - w|^2 \leq \oint_{\Gamma \setminus B_R} u^2 \leq \frac{C_0}{R^2} \oint_{\Gamma} u^2.$$

As long as  $B_{2R} \cap \Gamma$  is a small  $C^1$  graph over the cylinder  $\Sigma$ , we can transplant the function  $w$  to a function  $\bar{w}$  on  $\Sigma$  which is supported inside  $B_{2R}$ . Moreover, the distortion of the measure and the gradient are as small as we want if we make  $\epsilon_\mu$  small enough. In particular, there is a continuous function  $\eta(R, \epsilon_\mu)$  with  $\eta(R, 0) = 0$  so that

$$(7.21) \quad \left| \oint_{\Gamma} w^2 - \oint_{\Sigma} \bar{w}^2 \right| \leq \eta \oint_{\Gamma} w^2,$$

$$(7.22) \quad \left| \oint_{\Gamma} |\nabla w|^2 - \oint_{\Sigma} |\nabla \bar{w}|^2 \right| \leq \eta \oint_{\Gamma} |\nabla w|^2,$$

$$(7.23) \quad \left| \oint_{\Gamma} w - \oint_{\Sigma} \bar{w} \right|^2 \leq \eta \oint_{\Gamma} w^2.$$

The first non-zero eigenvalue of the cylinder  $\Sigma$  is  $\frac{1}{2}$ , so  $\bar{v}$  satisfies

$$(7.24) \quad \oint_{\Sigma} \bar{w}^2 \leq \frac{(\oint_{\Sigma} \bar{w})^2}{\lambda(\Sigma)} + 2 \oint_{\Sigma} |\nabla \bar{w}|^2 \leq \left( \oint_{\Sigma} \bar{w} \right)^2 + 2 \oint_{\Sigma} |\nabla \bar{w}|^2.$$

Combining this with (7.21), (7.22), (7.23) and the squared triangle inequality gives

$$(7.25) \quad \begin{aligned} (1 - \eta) \oint_{\Gamma} w^2 &\leq \oint_{\Sigma} \bar{w}^2 \leq 2 \oint_{\Sigma} |\nabla \bar{w}|^2 + \left( \oint_{\Sigma} \bar{w} \right)^2 \\ &\leq 2(1 + \eta) \oint_{\Gamma} |\nabla w|^2 + 2 \left( \eta \oint_{\Gamma} w^2 + \left[ \oint_{\Gamma} w \right]^2 \right). \end{aligned}$$

Absorbing the middle term, using  $\oint_{\Gamma} u = 0$ , (7.20) and the Cauchy-Schwarz inequality gives

$$(7.26) \quad \begin{aligned} (1 - 3\eta) \oint_{\Gamma} w^2 &\leq 2(1 + \eta) \oint_{\Gamma} |\nabla w|^2 + 2 \left[ \oint_{\Gamma} (u - w) \right]^2 \\ &\leq 2(1 + \eta) \oint_{\Gamma} |\nabla w|^2 + \frac{C}{R^2} \oint_{\Gamma} u^2. \end{aligned}$$

Using (7.20) to bound  $\|u\|_{L^2}$  by  $\|w\|_{L^2}$  and splitting  $\oint_{\Gamma} |\nabla w|^2$  into inner and outer parts

$$(7.27) \quad \begin{aligned} \left(1 - \frac{C_0}{R^2}\right) (1 - 3\eta) \oint_{\Gamma} u^2 &\leq (1 - 3\eta) \oint_{\Gamma} w^2 \\ &\leq 2(1 + \eta) \oint_{B_R \cap \Gamma} |\nabla w|^2 + \frac{C}{R^2} \oint_{\Gamma} u^2 + 2(1 + \eta) \oint_{\Gamma \setminus B_R} |\nabla w|^2. \end{aligned}$$

The first term is bounded by  $2(1 + \eta) \oint_{\Gamma} |\nabla u|^2$  since  $u = w$  on  $B_R \cap \Gamma$ . The second can be absorbed on the left. For the last, we use (7.19) and (7.20) to get

$$(7.28) \quad \oint_{\Gamma \setminus B_R} |\nabla w|^2 \leq 2 \oint_{\Gamma \setminus B_R} (u^2 + |\nabla u|^2) \leq 2 \left( \frac{C_0}{R^2} + \frac{C_0}{L} \right) \oint_{\Gamma} u^2.$$

Choosing  $L$  large and then taking  $\eta$  small enough, this gives (7.15).

Suppose in addition that  $\oint_{\Gamma} u x_i = 0$  for the coordinates  $x_1, \dots, x_{n+1}$  on  $\Sigma$ . These  $x_i$ 's are a basis for the first non-zero eigenspace of  $\Sigma$  and the next eigenvalue is 1. Thus, if  $\zeta$  is a function on  $\Sigma$  that integrates to zero against 1 and  $x_1, \dots, x_{n+1}$ , then  $\oint_{\Sigma} \zeta^2 \leq \oint_{\Sigma} |\nabla \zeta|^2$ . For each of these  $x_i$ 's, using that the support of  $w$  is a graph over  $\Sigma$  gives

$$(7.29) \quad \left( \oint_{\Gamma} w x_i - \oint_{\Sigma} \bar{w} x_i \right)^2 \leq \eta \oint_{\Gamma} w^2.$$

We can now argue as above to get (7.16).  $\square$

**7.3. Lower bounds for growth.** The next lemma shows that any caloric function that has integral zero on a slice must grow at least linearly if the flow is close to cylindrical.

**Lemma 7.30.** Given  $\mu \in (0, 1/2)$ , there exist  $\epsilon_{\mu} > 0$  and  $R_{\mu} > 0$  so that if

- $B_{R_{\mu}} \cap \frac{M_t}{\sqrt{-t}}$  is an  $\epsilon_{\mu}$   $C^1$ -graph over a cylinder for  $t_1 \leq t \leq t_2 < 0$ ,
- $u_t = \Delta u$ ,  $I_u(t_1) = 1$ , and  $J_{t_1}(u, 1) = 0$ ,

then  $I_u(t_2) \leq \left(\frac{t_1}{t_2}\right)^{\mu-1} + 2 |I_1(t_1) - I_1(t_2)|$ .

*Proof.* Set  $\kappa = |I_1(t_1) - I_1(t_2)|$ . We are done if  $I_u(t_2) \leq 2\kappa$ , so we can assume

$$(7.31) \quad 2\kappa < I_u(t_2) \leq I_u(t) \text{ for all } t \in [t_1, t_2].$$

When (7.31) holds, we will prove that  $I_u$  satisfies the differential inequality

$$(7.32) \quad ((-t)^{\mu-1} I_u)' = (-t)^{\mu-1} \left( I_u' + (\mu - 1) \frac{I_u}{t} \right) \leq \kappa (-t)^{\mu-2}.$$

Once we have (7.32), then integrating from  $t_1$  to  $t_2$  gives

$$(7.33) \quad (-t_2)^{\mu-1} I_u(t_2) - (-t_1)^{\mu-1} I_u(t_1) \leq \kappa \int_{t_1}^{t_2} (-t)^{\mu-2} dt \leq \frac{\kappa (-t_2)^{\mu-1}}{1 - \mu}.$$



The lemma follows from this since  $I_u(t_1) = 1$  and  $\mu \in (0, 1/2)$ .

It remains to prove (7.32). If  $I'_u(t) < \frac{I_u}{t}$ , then (7.32) holds. Hence, suppose that

$$(7.34) \quad I'_u(t) \geq \frac{I_u}{t}.$$

In this case  $t I'_u(t) \leq I_u(t)$  and (3.6) gives

$$(7.35) \quad -t (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \{2|\nabla u|^2 + u^2 |\phi|^2\} e^{\frac{|x|^2}{4t}} = t I'_u(t) \leq I_u(t).$$

Combining (7.35) with Lemma 3.14, gives

$$(7.36) \quad \begin{aligned} \frac{(-4\pi t)^{-\frac{n}{2}}}{-t} \int_{M_t} |x|^2 u^2 e^{\frac{|x|^2}{4t}} &\leq 4n I_u(t) - 4t (-4\pi t)^{-\frac{n}{2}} \int_{M_t} (4|\nabla u|^2 + u^2 |\phi|^2) e^{\frac{|x|^2}{4t}} \\ &\leq (4n + 8) I_u(t). \end{aligned}$$

For each  $t \in [t_1, t_2]$ , Lemma 7.1 gives that

$$(7.37) \quad |J_t(u, 1)|^2 \leq \kappa.$$

The function  $v = u - \frac{J_t(u, 1)}{I_1(t)}$  integrates to zero on  $M_t$  and, using (7.37) and (7.31),

$$(7.38) \quad I_u(t) = I_v(t) + \frac{J_t^2(u, 1)}{I_1(t)} \leq I_v(t) + \kappa < I_v(t) + \frac{1}{2} I_u(t).$$

From this, we conclude that

$$(7.39) \quad I_u(t) \leq 2 I_v(t).$$

We will show that  $v$  satisfies the localization inequality (7.13). The energy bound in (7.13) follows from (7.35) and (7.39) since  $|\nabla v|^2 = |\nabla u|^2$ . The squared triangle inequality for  $v$ , (7.36), and using the entropy bound on  $M_t$  to bound  $(-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} e^{\frac{|x|^2}{4t}}$  give

$$(7.40) \quad \begin{aligned} \frac{(-4\pi t)^{-\frac{n}{2}}}{-t} \int_{M_t} |x|^2 v^2 e^{\frac{|x|^2}{4t}} &\leq 2 \frac{(-4\pi t)^{-\frac{n}{2}}}{-t} \int_{M_t} |x|^2 (u^2 + \kappa) e^{\frac{|x|^2}{4t}} \\ &\leq 2(4n + 8) I_u(t) + C \kappa. \end{aligned}$$

Using (7.31) and (7.39), we get the remaining estimate for (7.13). Thus, if  $M_t/\sqrt{-t}$  is sufficiently close to cylindrical, Lemma 7.14, (7.37) and the equality in (7.35) give

$$(7.41) \quad (1 - \mu) I_u(t) \leq \frac{J_t^2(u, 1)}{I_1(t)} - 2t (-4\pi t)^{-\frac{n}{2}} \int_{M_t} |\nabla u|^2 e^{\frac{|x|^2}{4t}} \leq \kappa + t I'_u(t).$$

This gives (7.32) in the remaining case (7.34), completing the proof.  $\square$

**7.4. Projecting orthogonally to linear functions.** Let  $x_1, \dots, x_{n+1}$  be the coordinates on the cylinder  $\Sigma = \mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k} \subset \mathbf{R}^{n+1} \subset \mathbf{R}^N$ . Given  $u$  on  $M_t$ , define  $\zeta = \zeta(t) \in \mathbf{R}^{n+2}$  by

$$(7.42) \quad \zeta_0 = J_t(u, 1) \text{ and } \zeta_i = \frac{J_t(u, x_i)}{\sqrt{-t}} \text{ for } i = 1, \dots, n+1.$$

The function  $\frac{x_i}{\sqrt{-t}}$  is normalized to be roughly unit size on  $M_t$  and constant size on a self-shrinking flow. Let  $a = a(t) \in \mathbf{R}^{n+2}$  be coefficients so that  $v = u - a_0 - \sum_{i=1}^{n+1} a_i \frac{x_i}{\sqrt{-t}}$  is the  $J_t$ -projection of  $u$  orthogonally to  $\{1, \frac{x_1}{\sqrt{-t}}, \dots, \frac{x_{n+1}}{\sqrt{-t}}\}$ .

If  $\zeta = 0$ , then  $a = 0$  and  $v = u$ . The next lemma shows that  $u$  and  $v$  are close if  $\zeta$  is small.

**Lemma 7.43.** There exist  $R_0, \epsilon_0 > 0$  so that if  $B_{R_0} \cap \frac{M_t}{\sqrt{-t}}$  is an  $\epsilon_0$   $C^1$ -graph over  $\Sigma$ , then

$$(7.44) \quad I_u(t) \leq I_v(t) + |\zeta|^2,$$

$$(7.45) \quad (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} v^2 e^{\frac{|x|^2}{4t}} \leq C_n \lambda_0 |\zeta|^2 + (n+3) (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} u^2 e^{\frac{|x|^2}{4t}},$$

$$(7.46) \quad -t |I_{|\nabla u|}(t) - I_{|\nabla v|}(t)| \leq (n+2) \lambda_0 |\zeta|^2 + 2|\zeta| (-\lambda_0 t I_{u|\phi|}(t))^{\frac{1}{2}}.$$

*Proof.* We will need some calculations on  $\Sigma$ . Let  $g_{ij}$  denote the matrix of  $(4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$  Gaussian inner products of  $\{1, x_1, \dots, x_{n+1}\}$  on  $\Sigma$ :

$$(7.47) \quad \frac{g_{ij}}{\lambda(\mathbf{S}^k)} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j = 0 \\ \frac{2k}{k+1} & \text{if } i = j \leq k+1 \\ 2 & \text{if } k+1 < i = j \leq n+1 \end{cases}$$

By (7.47),  $g_{ij}$  is invertible and the largest eigenvalue of  $g_{ij}^{-1}$  is  $\frac{1}{\lambda(\mathbf{S}^k)} < \frac{1}{\sqrt{2}}$ . Thus, for  $R_0$  large and  $\epsilon_0$  small, the matrix  $\bar{g}_{ij} = \bar{g}_{ij}(t)$  of  $J_t$ -inner products of  $\{1, \frac{x_1}{\sqrt{-t}}, \dots, \frac{x_{n+1}}{\sqrt{-t}}\}$  on  $M_t$  is invertible and the largest eigenvalue of  $\bar{g}_{ij}^{-1}$  is  $< 1$  in norm. Thus, since  $\zeta = \bar{g}(a)$ , we have

$$(7.48) \quad a = \bar{g}^{-1}(\zeta) \text{ and } |a|^2 = \sum_i a_i^2 \leq |\zeta|^2.$$

Since  $I_u(t) = I_v(t) + I_{u-v}(t)$  and  $I_{u-v}(t) = \sum_{i,j} a_i a_j \bar{g}_{ij} = \langle \zeta, a \rangle \leq |\zeta|^2$ , so (7.48) gives (7.44).

Using the Cauchy-Schwarz inequality, (7.48) and the entropy bound for  $M_t$  gives

$$(7.49) \quad \begin{aligned} (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} v^2 e^{\frac{|x|^2}{4t}} &\leq (n+3) (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} \left( u^2 + a_0^2 + \sum \frac{a_i^2 x_i^2}{-t} \right) e^{\frac{|x|^2}{4t}} \\ &\leq C_n \lambda_0 |\zeta|^2 + (n+3) (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} u^2 e^{\frac{|x|^2}{4t}}. \end{aligned}$$

To compare the energy of  $u$  and  $v$ , we first write

$$(7.50) \quad I_{|\nabla v|} - I_{|\nabla u|} = I_{|\nabla(u-v)|} - 2 (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \langle \nabla(u-v), \nabla u \rangle e^{\frac{|x|^2}{4t}}.$$

We bound the first term on the right using the Cauchy-Schwarz inequality, (7.48) and the entropy bound for  $M_t$

$$(7.51) \quad I_{|\nabla(u-v)|} \leq (n+1) \sum_i a_i^2 I_{\frac{|\nabla x_i|}{\sqrt{-t}}} \leq (n+1) \frac{\lambda_0}{-t} \sum_i a_i^2 \leq (n+1) \frac{\lambda_0}{-t} |\zeta|^2.$$

Since  $\mathcal{L}_t x_i = \frac{x_i}{2t} - \langle \phi, \partial_i \rangle$  by (7.5) with  $V = \partial_i$ , Stokes's theorem and the definition of  $\zeta$  give

$$(7.52) \quad \begin{aligned} 2 (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \langle \nabla(u-v), \nabla u \rangle e^{\frac{|x|^2}{4t}} &= -2 J_t(u, \mathcal{L}_t(u-v)) \\ &= \sum_{i=1}^{n+1} \frac{a_i \zeta_i}{-t} + 2 \sum_{i=1}^{n+1} a_i J_t(u, \frac{\langle \phi, \partial_i \rangle}{\sqrt{-t}}). \end{aligned}$$

Using this and (7.51) in (7.50), the bound (7.48) and the Cauchy-Schwarz inequality gives

$$(7.53) \quad -t |I_{|\nabla u|} - I_{|\nabla v|}| \leq (n+1) \lambda_0 |\zeta|^2 + 2 |\zeta|^2 - \lambda_0 t I_u |\phi|.$$

□

**7.5. Quadratic growth.** Using a variation of Lemma 7.30, we will show: If  $M_t$  is close to a cylinder  $\Sigma \subset \mathbf{R}^{n+1} \subset \mathbf{R}^N$  and  $u$  is a caloric function on  $M_t$  that is orthogonal to  $\{1, \frac{x_1}{\sqrt{-t}}, \dots, \frac{x_{n+1}}{\sqrt{-t}}\}$ , then  $u$  grows essentially quadratically. The growth comes from a Poincaré inequality on  $\Sigma$  for functions orthogonal to  $\{1, \frac{x_1}{\sqrt{-t}}, \dots, \frac{x_{n+1}}{\sqrt{-t}}\}$ .

Let  $\kappa$  be the vector in  $\mathbf{R}^{n+2}$  given by  $\kappa_0 = |I_1(t_1) - I_1(t_2)|$  and  $\kappa_i = \left| \frac{I_{x_i}(t_1)}{t_1} - \frac{I_{x_i}(t_2)}{t_2} \right|$  for  $i = 1, \dots, n+1$ . The vector  $\kappa$  vanishes when  $M_t$  is self-shrinking.

**Lemma 7.54.** Given  $\mu \in (0, 1/4)$ , there exist  $\epsilon_\mu > 0$ ,  $R_\mu > 0$  and  $C'_n$  so that if

- $B_{R_\mu} \cap \frac{M_t}{\sqrt{-t}}$  is an  $\epsilon_\mu$   $C^1$ -graph over  $\Sigma$  for  $t_1 \leq t \leq t_2 < 0$ ,
- $u_t = \Delta u$ ,  $I_u(t_1) = 1$ , and  $J_{t_1}(u, 1) = J_{t_1}(u, x_i) = 0$  for  $i = 1, \dots, (n+1)$ ,

then  $I_u(t_2) \leq \left(\frac{t_1}{t_2}\right)^{2\mu-2} + C'_n (2 + \mu^{-1}) \lambda_0^2 \sqrt{|\kappa|} \left(\frac{t_1}{t_2}\right)^2$ .

*Proof.* Let  $C'_n$  be a large constant to be chosen, depending just on  $n$ . Set  $\omega \equiv \lambda_0 \sqrt{|\kappa|} \left(\frac{t_1}{t_2}\right)^2$ . We are done if  $I_u(t_2) \leq C'_n \lambda_0 \omega$ , so we can assume that

$$(7.55) \quad C'_n \lambda_0 \omega < I_u(t_2) \leq I_u(t) \text{ for all } t \in [t_1, t_2].$$

Following Lemma 7.30, we will show that  $I_u$  satisfies a differential inequality that integrates to give the lemma. Namely, we will show that there exist  $\bar{C}_n$  so that

$$(7.56) \quad (2 - 2\mu) I_u(t) \leq 2 (2 + \mu^{-1}) \bar{C}_n \lambda_0 \omega + t I'_u(t)$$

This gives that

$$(7.57) \quad ((-t)^{2\mu-2} I_u)' = (-t)^{2\mu-3} \{(2 - 2\mu) I_u(t) - t I'_u(t)\} \leq 2 (2 + \mu^{-1}) \bar{C}_n \lambda_0 \omega (-t)^{2\mu-3}.$$

Integrating (7.57) from  $t_1$  to  $t_2$  gives

$$(7.58) \quad (-t_2)^{2\mu-2} I_u(t_2) - (-t_1)^{2\mu-2} I_u(t_1) \leq 2 (2 + \mu^{-1}) \bar{C}_n \lambda_0 \omega \int_{t_1}^{t_2} (-t)^{2\mu-3} dt.$$

The lemma follows from this since  $I_u(t_1) = 1$  and  $\mu \in (0, 1/4)$ .

It remains to prove (7.56). If  $t I'_u(t) > 2 I_u$ , then (7.56) holds. Hence, suppose that  $t I'_u(t) \leq 2 I_u(t)$  and, thus, (3.6) gives

$$(7.59) \quad -2t I_{|\nabla u|}(t) - t I_{u|\phi|}(t) = t I'_u(t) \leq 2 I_u(t).$$

Combining (7.59) with the localization inequality for  $u$  in Lemma 3.14, gives

$$(7.60) \quad \frac{(-4\pi t)^{-\frac{n}{2}}}{-t} \int_{M_t} |x|^2 u^2 e^{\frac{|x|^2}{4t}} \leq 4n I_u(t) - 4t (4 I_{|\nabla u|} + I_{u|\phi|}) \leq (4n + 16) I_u(t).$$

The Poincaré inequality (7.16) for  $u$  would imply (7.56) if  $J_t(u, 1) = J_t(u, x_i) = 0$  ( $u$  has localization by (7.60)). Since this may not be the case, let  $v$  be the  $J_t$ -projection of  $u$  orthogonal to  $\{1, \frac{x_1}{\sqrt{-t}}, \dots, \frac{x_{n+1}}{\sqrt{-t}}\}$ . We will prove localization for  $v$  to get (7.16) for  $v$  and then deduce (7.56).

Since  $J_{t_1}(u, x_i) = 0$ , Lemma 7.10 gives for any  $\epsilon_1 \in (0, 1/2)$  and  $t \in [t_1, t_2]$

$$(7.61) \quad \sqrt{-t_2} \left| \frac{J_t(u, x_i)}{t} \right| \leq \frac{5}{2} \epsilon_1 + \frac{\kappa_0}{\epsilon_1} + \frac{1}{2\epsilon_1} \left\{ \kappa_i + C_n \sqrt{\lambda_0} \left( \frac{t_1}{t_2} \right)^{\frac{1}{2}} \kappa_0^{\frac{1}{2}} \right\}.$$

We can assume that  $|\kappa| < 1/16$  since there is nothing to prove if  $|\kappa|$  is bounded away from 0. Taking  $\epsilon_1 = |\kappa|^{\frac{1}{4}}$  in (7.61) gives

$$(7.62) \quad \sqrt{-t_2} \left| \frac{J_t(u, x_i)}{t} \right| \leq C_n \sqrt{\lambda_0} \left( \frac{t_1}{t_2} \right)^{\frac{1}{2}} |\kappa|^{\frac{1}{4}}.$$

Since Lemma 7.1 gives  $J_t^2(u, 1) \leq \kappa_0$ , (7.62) gives a constant  $C_n$  so  $\zeta$  from (7.42) satisfies

$$(7.63) \quad |\zeta|^2 \leq C_n \lambda_0 \left( \frac{t_1}{t_2} \right)^2 \sqrt{|\kappa|} \equiv C_n \omega.$$

Using this in Lemma 7.43 gives  $\bar{C}_n$  so that

$$(7.64) \quad I_u(t) \leq I_v(t) + \bar{C}_n \omega,$$

$$(7.65) \quad (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} v^2 e^{\frac{|x|^2}{4t}} \leq \bar{C}_n \lambda_0 \omega + (n+3) (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} u^2 e^{\frac{|x|^2}{4t}},$$

$$(7.66) \quad -t |I_{|\nabla u|}(t) - I_{|\nabla v|}(t)| \leq \bar{C}_n \lambda_0 \omega + \sqrt{\bar{C}_n \omega} (-\lambda_0 t I_{u|\phi|}(t))^{\frac{1}{2}}.$$

As long as we choose  $C'_n > 2\bar{C}_n$ , then (7.55) and (7.64) guarantee that

$$(7.67) \quad I_v(t) \leq I_u(t) \leq 2 I_v(t).$$

Similarly, (7.55), (7.65) and (7.60) give

$$(7.68) \quad (-4\pi t)^{-\frac{n}{2}} \int_{M_t} \frac{|x|^2}{-t} v^2 e^{\frac{|x|^2}{4t}} \leq I_u(t) + (n+3) (4n+16) I_u(t).$$

Using (7.59) in (7.66) gives

$$(7.69) \quad -t |I_{|\nabla u|}(t) - I_{|\nabla v|}(t)| \leq \bar{C}_n \lambda_0 \omega + \sqrt{\bar{C}_n \omega} (2\lambda_0 I_u(t))^{\frac{1}{2}}.$$

By (7.59), (7.67), (7.68), (7.69),  $v$  satisfies the localization inequality (7.13). Therefore, we can apply (7.16) in Lemma 7.14 to get

$$(7.70) \quad (1 - \mu/2) I_v(t) \leq -t I_{|\nabla v|}(t).$$

Using (7.64) and (7.69), (7.70) implies that

$$(7.71) \quad \begin{aligned} (1 - \mu/2) I_u(t) &\leq (1 - \mu/2) I_v(t) + \bar{C}_n \omega \leq \bar{C}_n \omega - t I_{|\nabla v|}(t) \\ &\leq 2 \bar{C}_n \lambda_0 \omega + \sqrt{\bar{C}_n \omega} (2 \lambda_0 I_u(t))^{\frac{1}{2}} - t I_{|\nabla u|}(t). \end{aligned}$$

Using an absorbing inequality on the middle term gives

$$(7.72) \quad (1 - \mu) I_u(t) \leq (2 + \mu^{-1}) \bar{C}_n \lambda_0 \omega - t I_{|\nabla u|}(t) \leq (2 + \mu^{-1}) \bar{C}_n \lambda_0 \omega + \frac{t}{2} I'_u(t).$$

This gives the desired differential inequality, completing the proof.  $\square$

**7.6. Sharp bounds for codimension.** The following implies Theorem 0.15:

**Theorem 7.73.** If a tangent flow of  $M_t$  at  $-\infty$  is a cylinder, then  $\dim \mathcal{P}_1(M_t) = n + 2$ .

By a cylinder, we mean a multiplicity one cylinder. The space  $\mathcal{P}_1(M_t)$  always includes the constant function and the linearly independent coordinate functions, so  $\dim \mathcal{P}_1(M_t)$  is  $(n + 1)$  for an  $n$ -plane and at least  $(n + 2)$  otherwise. The point is to use the asymptotic cylindrical structure to prove  $\dim \mathcal{P}_1(M_t) \leq n + 2$ .

We will need the following uniqueness of blowup type for  $M_t$ :

**Lemma 7.74.** Suppose that  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$  is a tangent flow at  $-\infty$ . Given  $\epsilon > 0$  and  $\Lambda > 1$ , there exists  $T < 0$  so that if  $t_0 < T$ , then there is a rotation  $\mathcal{R}$  of  $\mathbf{R}^N$  so that  $B_\Lambda \cap \frac{M_t}{\sqrt{-t}}$  is a graph over  $\mathcal{R}(\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k})$  with  $C^2$  norm at most  $\epsilon$  for every  $t \in [\Lambda^2 t_0, t_0]$ .

*Proof.* This follows from the rigidity of the cylinder of Theorem 0.17 and White's curvature estimate [W1] (cf. corollary 0.3 in [CIM]).  $\square$

*Proof of Theorem 7.73.* We will get a contradiction if  $u_0 \equiv 1$ ,  $u_1, \dots, u_{n+2}$  are linearly independent functions in  $\mathcal{P}_1(M_t)$ .

Given  $\mu > 0$  and  $\Omega > 1$ , Lemma 3.9 with  $d = 1$  gives  $m_q \rightarrow \infty$  so that  $v_1, \dots, v_{n+2}$  defined by  $v_i = \frac{w_{i, -\Omega^{m_q+1}}}{\sqrt{f_i(-\Omega^{m_q+1})}}$  satisfy

$$(7.75) \quad J_{-\Omega^{m_q+1}}(v_i, v_j) = \delta_{ij} \text{ and } \sum_{i=1}^{n+2} I_{v_i}(-\Omega^{m_q}) \geq (n + 2) \Omega^{-1-\mu}.$$

For  $m_q$  sufficiently large, Lemma 7.74 gives that  $\frac{M_t}{\sqrt{-t}}$  is as close as we want to a cylinder  $\Sigma_q$  (a priori, the cylinder can vary with  $q$ ). Let  $x_1, \dots, x_{n+1}$  be the coordinate functions for the cylinder  $\Sigma_q$ . Make an orthogonal change of basis of  $v_1, \dots, v_{n+2}$  so that

$$(7.76) \quad J_{-\Omega^{m_q+1}}(v_{n+2}, x_i) = 0 \text{ for } i = 1, \dots, n + 1.$$

Since trace is invariant under orthogonal changes, (7.75) still holds.

Every  $v_i$  is  $J_{-\Omega^{m_q+1}}$ -orthogonal to the constants for  $i \geq 1$ . Thus, given  $\mu \in (0, 1/2)$ , then for every  $m_q$  sufficiently large we can apply Lemma 7.30 to get that

$$(7.77) \quad I_{v_i}(-\Omega^{m_q}) \leq \Omega^{\mu-1} + 2 \kappa_0.$$

However,  $v_{n+2}$  is also orthogonal to the  $x_i$ 's and, thus, the stronger Lemma 7.54 gives

$$(7.78) \quad I_{v_{n+2}}(-\Omega^q) \leq \Omega^{2\mu-2} + C'_n (2 + \mu^{-1}) \lambda_0^2 \sqrt{|\kappa|} \Omega^2.$$

Using (7.77) for  $i \leq n+1$  and (7.78) for  $i = n+2$  gives

$$(7.79) \quad \sum_{i=1}^{n+2} I_{v_i}(-\Omega^{m_q}) \leq (n+1) \Omega^{\mu-1} + 2(n+1) \kappa_0 + \Omega^{2\mu-2} + C'_n (2 + \mu^{-1}) \lambda_0^2 \sqrt{|\kappa|} \Omega^2.$$

Combining this with the lower bound (7.75) gives

$$(7.80) \quad (n+2) \Omega^{-1-\mu} \leq (n+1) \Omega^{\mu-1} + 2(n+1) |\kappa| + \Omega^{2\mu-2} + C'_n \left(2 + \frac{1}{\mu}\right) \lambda_0^2 \sqrt{|\kappa|} \Omega^2.$$

This gives a contradiction. To see this, fix any  $\Omega > 1$  and then choose  $\mu > 0$  small so that

$$(7.81) \quad (n+2) \Omega^{-1-\mu} > (n+1) \Omega^{\mu-1} + \Omega^{2\mu-2}.$$

Then take  $q$  large enough so that  $|\kappa|$  is small enough to contradict (7.80).  $\square$

## 8. CONJECTURES

Using [CIMW] and Brendle, [B], Bernstein-Wang, [BW3], showed that any shrinker in  $\mathbf{R}^3$  with entropy  $\leq \lambda(\mathbf{S}^1) + \epsilon$ , is a flat plane, round sphere, or round cylinder. We believe that there should be a similar classification in low dimension and any codimension of low entropy shrinkers (cf. conjecture 0.10 in [CIMW]):

**Conjecture 8.1.** There exists  $\epsilon > 0$  so that for  $n \leq 4$  and any codimension, the only shrinkers with entropy  $< \lambda(\mathbf{S}^1) + \epsilon$  are round generalized cylinders,  $\mathbf{S}_{\sqrt{2k}}^k \times \mathbf{R}^{n-k}$ .

We conjecture that for any  $n$  the round  $\mathbf{S}^n$  has the least entropy of any closed shrinker<sup>3</sup>  $\Sigma^n \subset \mathbf{R}^N$ . This was proven for hypersurfaces in [CIMW]; see also [HW]. The “Simons cone” over  $\mathbf{S}^2 \times \mathbf{S}^2$  has entropy  $< \lambda(\mathbf{S}^1)$ , see [CIMW]. So already for  $n = 5$ , round cylinders do give not a complete list of the lowest entropy shrinkers. Conjecture 8.1 is known for  $n = 1$  since shrinking curves are planar and have entropy  $\geq \lambda(\mathbf{S}^1)$ .

It would be interesting to estimate the optimal constant  $C_n$  in Corollary 0.6. A sufficiently strong bound could have strong implications for blowups near cylindrical singularities.

### Part 2. Entropy and codimension bounds for generic singularities

Even for hypersurfaces, singularities of MCF are too numerous to classify. The hope is that the generic ones that cannot be perturbed away are much simpler. Using the first part, we give the first general bounds on generic singularities of surfaces in arbitrary codimension.

Throughout this part,  $\Sigma^n \subset \mathbf{R}^N$  will be an immersed shrinker with finite entropy. Because of the lack of the maximum principle in higher codimension, embeddedness is not preserved and, thus, is not natural to assume. Shrinking curves are automatically planar and the only  $F$ -stable ones are lines and circles by Corollary 11.12 below.

<sup>3</sup>In [CIMW], it was conjectured that the round  $\mathbf{S}^n$  minimizes entropy among closed hypersurfaces for  $n \leq 6$ . This was proven by Bernstein-Wang, [BW1]. Zhu later proved this for all  $n$  in [Z]; cf. [BW2], [KZ]. We conjecture that this holds in all codimension.



## 9. SHRINKERS

Shrinkers are characterized variationally as critical points of the Gaussian area  $F$ . The shrinker equation is  $\mathbf{H} \equiv \sum_{i=1}^n A_{ii} = \frac{1}{2} x^\perp$ , where  $e_i$  is an orthonormal frame for  $\Sigma$  and the second fundamental form is given by  $A_{ij} = A(e_i, e_j) = \nabla_{e_i}^\perp e_j$ . Following [CM8] and [CM11], define the second-variation operator  $L$  by

$$(9.1) \quad L = \mathcal{L} + \frac{1}{2} + \sum_{k,\ell} \langle \cdot, A_{k\ell} \rangle A_{k\ell}.$$

Note that  $L$  is symmetric with respect to the Gaussian inner product on normal vector fields. The second variation  $\delta^2 F(u)$  of  $F$  in the normal direction  $u$  is ([CM8], [CM11], [AHW], [AS], [LL])

$$(9.2) \quad \delta^2 F(u) = -(4\pi)^{-\frac{n}{2}} \int \langle u, L u \rangle e^{-f}.$$

The second variation is negative along translations and dilations, so there are no stable shrinkers in the usual sense. As in [CM8], a shrinker is said to be  $F$ -stable if the second variation is nonnegative perpendicular to these unstable directions. A shrinker is *entropy-stable* if it is a local minimum for the entropy  $\lambda$ . Entropy-unstable shrinkers are singularities that can be perturbed away, whereas entropy-stable ones cannot; see [CM8], [CM10]. By section 7 in [CM8], entropy-stable and  $F$ -stable are equivalent for closed shrinkers. By [CM8], spheres and planes are the only  $F$ -stable hypersurfaces. (This was generalized to higher codimension when  $\mathbf{H}$  does not vanish and the principal normal is parallel in [AHW]; see also [AS], [LL].) It is easy to see that spheres and planes are  $F$ -stable in any codimension.

There are several ways to show that a shrinker  $\Sigma$  is  $F$ -unstable. The first, essentially the definition, is to find a normal vector field  $u \in L^2$  that is  $L^2$ -orthogonal to  $\mathbf{H}$  and translations and so that the second variation of  $F$  in the direction of  $u$  is negative. For instance,  $u \in L^2$  with  $L u = \mu u$  with  $\mu > 1$  implies  $F$ -instability. The second is to find  $u \in L^2$  orthogonal to  $\mathbf{H}$  and “below the translations”, i.e., with  $\int \langle u, L u \rangle e^{-f} > \frac{1}{2} \int |u|^2 e^{-f}$ . In codimension one,  $L$  becomes an operator on functions and the lowest eigenfunction does not vanish. In [CM8], we used this to conclude that  $F$ -stability implied mean convexity for hypersurfaces. Because of the vector-valued nature of things, there is no analog of this in higher codimension unless one assumes that the principal normal is parallel (see [AHW]).

**9.1. Simons type equations for  $A$  and translations.** One of the important tools in [CM8], [CM9] and [CM11] was a series of elliptic equations for various geometric objects on a shrinker, including the second fundamental form, mean curvature and translation vector fields. Namely, if  $V^\perp$  is the normal part of  $V \in \mathbf{R}^N$ , then

$$(9.3) \quad (L A)_{ij} = A_{ij} + 2 \sum_{k,\ell} \langle A_{j\ell}, A_{ik} \rangle A_{\ell k} - \sum_{m,\ell} \{ \langle A_{m\ell}, A_{il} \rangle A_{jm} + \langle A_{j\ell}, A_{m\ell} \rangle A_{mi} \},$$

$$(9.4) \quad L \mathbf{H} = \mathbf{H} \text{ and } L V^\perp = \frac{1}{2} V^\perp.$$

One consequence of (9.4) and symmetry of  $L$  is that  $\mathbf{H}$  and  $V^\perp$  are orthogonal with respect to the Gaussian inner product. These are proved for hypersurfaces in theorem 5.2 and lemma 10.8 in [CM8]. For higher codimension, see proposition 3.6 in [CM11], [AHW], [AS], or [LL].

**Lemma 9.5.** (cf. (5.6), (5.11) in [CM8]) The derivatives of  $\mathbf{H}$  and  $V^\perp$  are

$$(9.6) \quad \nabla \mathbf{H} = -\langle \mathbf{H}, A(\cdot, \cdot) \rangle - \frac{1}{2} A(x^T, \cdot) \text{ and } \nabla^\perp V^\perp = -A(\cdot, V^T).$$

*Proof.* Let  $e_j$  be an orthonormal frame for  $\Sigma$  and differentiate the shrinker equation

$$(9.7) \quad \begin{aligned} 2 \nabla_{e_i} \mathbf{H} &= \nabla_{e_i} x^\perp = \nabla_{e_i} (x - \langle x, e_j \rangle e_j) = e_i - \langle e_i, e_j \rangle e_j - \langle x, \nabla_{e_i} e_j \rangle e_j - \langle x, e_j \rangle \nabla_{e_i} e_j \\ &= -\langle x, \nabla_{e_i} e_j \rangle e_j - \langle x, e_j \rangle \nabla_{e_i} e_j. \end{aligned}$$

Now fix a point  $p$  and choose the frame  $e_i$  so that  $\nabla_{e_i}^T e_j = 0$  at  $p$ . It follows that (at  $p$ )  $\nabla_{e_i} e_j = \nabla_{e_i}^\perp e_j = A(e_i, e_j)$ , so we get that

$$(9.8) \quad 2 \nabla_{e_i} \mathbf{H} = -\langle x, A(e_i, e_j) \rangle e_j - \langle x, e_j \rangle A(e_i, e_j) = -2 \langle \mathbf{H}, A(e_i, \cdot) \rangle - A(e_i, x^T).$$

The first claim follows. Next, we have  $\nabla_{e_i}^\perp V^\perp = -\nabla_{e_i}^\perp V^T = -A(e_i, V^T)$ .  $\square$

**Lemma 9.9.** Given a function  $\phi$  and  $V \in \mathbf{R}^N$ , the second variation of  $F$  in the direction of  $\phi V^\perp$  and  $\phi \mathbf{H}$  are

$$(9.10) \quad \delta^2 F(\phi V^\perp) = (4\pi)^{-\frac{n}{2}} \int \left[ |\nabla \phi|^2 - \frac{1}{2} \phi^2 \right] |V^\perp|^2 e^{-f}.$$

$$(9.11) \quad \delta^2 F(\phi \mathbf{H}) = (4\pi)^{-\frac{n}{2}} \int [|\nabla \phi|^2 - \phi^2] |\mathbf{H}|^2 e^{-f}.$$

*Proof.* Given any normal section  $u$ , the Leibniz rule  $\mathcal{L}(\phi u) = \phi \mathcal{L} u + (\mathcal{L} \phi) u + 2 \nabla_{\nabla^\perp \phi} u$  gives

$$(9.12) \quad \begin{aligned} (4\pi)^{\frac{n}{2}} \delta^2 F(\phi u) &= - \int \langle \phi u, L(\phi u) \rangle e^{-f} = - \int (\phi^2 \langle u, L u \rangle + (\phi \mathcal{L} \phi) |u|^2 + 2 \langle \phi u, \nabla_{\nabla^\perp \phi} u \rangle) e^{-f} \\ &= - \int (\phi^2 \langle u, L u \rangle + (\phi \mathcal{L} \phi) |u|^2 + \langle \phi \nabla \phi, \nabla |u|^2 \rangle) e^{-f} \\ &= \int (|\nabla \phi|^2 |u|^2 - \phi^2 \langle u, L u \rangle) e^{-f}, \end{aligned}$$

where the last equality used integration by parts. The claims follow from applying (9.4) and (9.12) with  $u = V^\perp$  and  $u = \mathbf{H}$ .  $\square$

## 10. THE HESSIAN EQUATION

We will see that shrinkers satisfy a Hessian equation. Define the symmetric 2-tensor  $A^\mathbf{H} = \langle A, \mathbf{H} \rangle$  and define a symmetric operator  $A^2$  on tangent vector fields by

$$(10.1) \quad \langle e_i, A^2(e_j) \rangle \equiv \langle A_{ik}, A_{kj} \rangle.$$

**Proposition 10.2.** If  $\Gamma^n \subset \mathbf{R}^N$  is a shrinker or an  $n$ -plane, then

$$(10.3) \quad \text{Hess}_f^\Gamma - A^\mathbf{H} = \frac{1}{2} \langle \cdot, \cdot \rangle.$$

For hypersurfaces, the converse also holds.

It is interesting to compare this with shrinking solitons for the Ricci flow, [Ham]. A gradient shrinking soliton is a manifold  $M$ , metric  $g$  and function  $f$  satisfying  $\text{Hess}_f + \text{Ric} = \frac{1}{2} g$ .

**Lemma 10.4.** If  $\Gamma^n \subset \mathbf{R}^N$ , then  $\text{Hess}_{|x|^2}^\Gamma = 2\langle \cdot, \cdot \rangle + 2\langle x^\perp, A \rangle$ .

*Proof.* Given an orthonormal frame  $e_i$  for  $\Gamma$ , we compute

$$(10.5) \quad \frac{1}{2} \text{Hess}_{|x|^2}^\Gamma(e_i, e_j) = \langle \nabla_{e_i} x^T, e_j \rangle = \langle e_i - \nabla_{e_i} x^\perp, e_j \rangle = \delta_{ij} + \langle \nabla_{e_i} e_j, x^\perp \rangle.$$

□

*Proof of Proposition 10.2.* Equation (10.3) holds on a shrinker since Lemma 10.4 gives

$$(10.6) \quad \frac{1}{2} \langle \cdot, \cdot \rangle + A^{\mathbf{H}} - \text{Hess}_f^\Gamma = \langle \mathbf{H} - \frac{1}{2} x^\perp, A \rangle.$$

For the converse, suppose that (10.3) holds and  $\Gamma$  is a hypersurface with unit normal  $\mathbf{n}$ . It follows that at every point either  $A = 0$  or  $\Gamma$  satisfies the shrinker equation. If  $A \equiv 0$ , then  $\Sigma$  is a hyperplane. When  $A$  is not identically zero, then let  $\mathcal{S} = \{\mathbf{H} - \frac{1}{2} x^\perp = 0\}$  be where  $\Gamma$  satisfies the shrinker equation. This must be nonempty and closed. We will argue by contradiction to show that  $\mathcal{S} = \Gamma$ . Let  $U$  be a component of the (necessarily open) complement of  $\mathcal{S}$ . Note that  $U$  is path connected since it is connected and locally path-connected by theorem 25.5 in [Mu]. It follows that  $A = 0$  on  $U$  and, thus, that  $\langle x, \mathbf{n} \rangle$  is constant on  $U$ . Since  $U$  cannot be all of  $\Gamma$  (since  $\mathcal{S}$  is nonempty), there must be a boundary point  $p \in \mathcal{S} \cap \partial U$ . Since the set where  $A = 0$  is closed, we see that  $\mathbf{H}(p) = 0$  and, thus, that  $\langle x, \mathbf{n} \rangle(p) = 0$ . It follows that  $\langle x, \mathbf{n} \rangle \equiv 0$  on all of  $U$  and, thus, that  $\Gamma$  satisfies the shrinker equation in  $U$ , giving the desired contradiction. □

The next lemma recalls the standard Gauss equation for the Ricci curvature  $\text{Ric}$  and scalar curvature  $S$ . By convention, the Riemann tensor is given in an orthonormal frame  $e_j$  by

$$(10.7) \quad R_{ijkl} = \langle \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{e_i} \nabla_{e_j} e_k - \nabla_{[e_j, e_i]} e_k, e_l \rangle,$$

and the Ricci tensor is  $\text{Ric}_{ij} = \sum_k R_{kikj}$ .

**Lemma 10.8.** If  $\Gamma^n \subset \mathbf{R}^N$ , then  $\text{Ric} = -A^2 - A^{\mathbf{H}}$  and  $S = H^2 - |A|^2$ .

*Proof.* The Gauss equation gives  $R_{ijkn} = \langle A_{ik}, A_{jn} \rangle - \langle A_{jk}, A_{in} \rangle$ . Summing this over  $j = n$  and using that  $A_{jj} = -\mathbf{H}$ , we get

$$(10.9) \quad \text{Ric}_{ik} = R_{ijkj} = \langle A_{ik}, A_{jj} \rangle - \langle A_{jk}, A_{ij} \rangle = -A_{ik}^{\mathbf{H}} - (A^2)_{ik}.$$

This gives the first claim. Taking the trace gives the second claim. □

**Corollary 10.10.** If  $\Sigma^n \subset \mathbf{R}^N$  is a shrinker, then

$$(10.11) \quad \text{Hess}_f + \text{Ric} = \frac{1}{2} \langle \cdot, \cdot \rangle - A^2 \leq \frac{1}{2} \langle \cdot, \cdot \rangle.$$

*Proof.* Lemma 10.8 and Proposition 10.2 give that

$$(10.12) \quad \text{Ric} + \text{Hess}_f = -A^2 - A^{\mathbf{H}} + \text{Hess}_f = \frac{1}{2} \langle \cdot, \cdot \rangle - A^2.$$

For a tangent vector  $V$ , we have  $\langle A^2(V), V \rangle = |A(V)|^2$ , giving the inequality. □

### 10.1. Minimal submanifolds in spheres.

**Lemma 10.13.** For a submanifold  $\Gamma^n \subset \partial B_{\sqrt{2n}} \subset \mathbf{R}^N$  the following are equivalent:

- (A)  $\Gamma$  is a shrinker in  $\mathbf{R}^N$ .
- (B)  $\Gamma$  is a minimal submanifold of the sphere  $\partial B_{\sqrt{2n}} \subset \mathbf{R}^N$ .
- (C)  $A^{\mathbf{H}} = -\frac{1}{2} \langle \cdot, \cdot \rangle$ .

*Proof.* The equivalence of (A) and (B) is well known. (A) implies (B) since the  $F$  functional is equivalent to area for spherical submanifolds. Fix a point  $p$  in  $\Gamma$  and let  $e_i$  be an orthonormal frame for  $\Gamma$  with  $\nabla_{e_i}^T e_j = 0$  at  $p$ . Since the  $e_i$ 's are tangent also to the sphere, we have  $\langle e_i, x \rangle = 0$ . Differentiating this gives

$$(10.14) \quad -n = \langle \nabla_{e_i} e_i, x \rangle = -\langle \mathbf{H}, x \rangle.$$

In (B),  $\mathbf{H} = u x$  for a function  $u$  on  $\Sigma$ . By (10.14),  $u \equiv \frac{1}{2}$ , giving (A). Furthermore, (A) and Proposition 10.2 imply (C). Finally, we will show that (C) implies (A) and (B). Taking the trace of (C) gives that  $|\mathbf{H}|^2 = \frac{n}{2}$ . Since  $|x|^2 \equiv 2n$  on  $\Gamma$ ,  $x$  is normal to  $\Gamma$  so

$$(10.15) \quad \langle \mathbf{H}, x \rangle = -\langle \nabla_{e_i} e_i, x \rangle = \langle e_i, \nabla_{e_i} x \rangle = n.$$

It follows that

$$(10.16) \quad \left| \mathbf{H} - \frac{x}{2} \right|^2 = |\mathbf{H}|^2 + \frac{|x|^2}{4} - \langle \mathbf{H}, x \rangle = \frac{n}{2} + \frac{2n}{4} - n = 0.$$

We conclude that  $\mathbf{H} = \frac{x}{2}$ , giving (A) and (B) and, thus, completing the proof.  $\square$

## 11. PROOF OF THEOREM 0.9

Recall that  $u \in L^2$  is an eigenfunction of  $\mathcal{L}$  with eigenvalue  $\mu$  if  $\mathcal{L}u + \mu u = 0$ . Let  $\mu_0 = 0 < \mu_1 \leq \mu_2 \leq \dots$  be the eigenvalues of  $\mathcal{L}$  on  $\Sigma$  and  $u_0, u_1, \dots$  the corresponding  $L^2$ -orthonormal eigenfunctions (note that  $u_0$  is constant).

**Proposition 11.1.** If  $\Sigma^n \subset \mathbf{R}^N$  is contained in a proper linear subspace  $\mathcal{V} \subset \mathbf{R}^N$  and is  $F$ -stable, then  $\mu_1 \geq \frac{1}{2}$ .

*Proof.* Let  $\phi$  be an eigenfunction with  $\mathcal{L}\phi = -\mu\phi$  and  $\mu > 0$ . Let  $E \in \mathcal{V}^\perp \subset \mathbf{R}^N$  be a unit vector. Observe that

$$(11.2) \quad L(\phi E) = \left[ \left( \mathcal{L} + \frac{1}{2} \right) \phi \right] E = \left( \frac{1}{2} - \mu \right) \phi E.$$

We will show that  $\phi E$  is an allowable variation, i.e., is orthogonal to  $\mathbf{H}$  and all translations. Since  $\Sigma \subset \mathcal{V}$ , we have at each point that  $\mathbf{H}$  is parallel to  $\mathcal{V}$  and, thus, that  $\langle \mathbf{H}, E \rangle = 0$  point-wise. Let  $V$  be any vector parallel to  $\mathcal{V}$  and note that  $V^T$  must also be parallel to  $\mathcal{V}$ . Thus,  $\langle E, V^\perp \rangle = 0$  point-wise. Finally, the last translation vector field is  $E$  itself and  $\int \langle \phi E, E \rangle e^{-f} = \int \phi e^{-f} = 0$ . Since  $\phi E$  is allowable,  $F$ -stability and (11.2) give

$$(11.3) \quad 0 \geq \int \langle \phi E, L(\phi E) \rangle = \int \left( \frac{1}{2} - \mu \right) \phi^2.$$

$\square$

When  $\Sigma$  has  $F$ -index  $I > 0$ , then a variation of Proposition 11.1 gives  $\mu_{I+1} \geq \frac{1}{2}$ .

Next, we adapt a result of Korevaar, [K] (see [GNY]; cf. [He], [YY]) to this setting:

**Proposition 11.4.** There is a universal constant  $C$  such that if  $\Sigma^2 \subset \mathbf{R}^N$  is closed, then

$$(11.5) \quad \mu_k(\Sigma) \lambda(\Sigma) \leq C(1 + \gamma)k.$$

*Proof.* Let  $g$  be the metric on  $\Sigma$  and define the conformal metric  $g_1 = e^{-f}g$ . Let  $dv_g$  and  $dv_{g_1}$  be the corresponding area elements. Note that  $\lambda(\Sigma) = \frac{1}{4\pi} \text{Area}_{g_1}(\Sigma)$ . Since  $e^{-f} \leq 1$ , we have for any function  $u$  that

$$(11.6) \quad \int |\nabla_g u|^2 e^{-f} dv_g \leq \int |\nabla_g u|^2 dv_g = \int |\nabla_{g_1} u|^2 dv_{g_1},$$

$$(11.7) \quad \int u^2 e^{-f} dv_g = \int u^2 dv_{g_1}.$$

Thus, for each  $k$ , it follows that  $\mu_k = \mu_k(\mathcal{L}) \leq \mu_k(\Delta_{g_1})$ . Finally, [K] gives that

$$(11.8) \quad \mu_k(\Delta_{g_1}) \leq \frac{C(1 + \gamma)k}{\text{Area}_{g_1}(\Sigma)} = \frac{C(1 + \gamma)k}{4\pi \lambda(\Sigma)}.$$

□

*Proof of Theorem 0.9.* Corollary 0.8 gives  $C_1$  so that if  $N \geq C_1 \lambda_\Sigma$ , then there is a proper linear subspace  $\mathcal{V} \subset \mathbf{R}^N$  so that  $\Sigma \subset \mathcal{V}$ . Combining Propositions 11.1 and 11.4 gives

$$(11.9) \quad \frac{1}{2} \leq \mu_1(\mathcal{L}) \leq \frac{C(1 + \gamma)}{\lambda(\Sigma)}.$$

The second claim follows from the first and Corollary 0.8. □

When  $\Sigma$  is diffeomorphic to a sphere, we can argue as above and use [He] to obtain (0.12).

**Conjecture 11.10.** Theorem 0.9 holds for complete  $n$ -dimensional  $\lambda$ -stable shrinkers.

**11.1. Spectrum of  $\mathcal{L}$  and  $L$  for curves.** In [AL], Abresch-Langer classified shrinking curves. The embedded ones are the circle and lines. By Lemma 6.16, every other shrinking curve with  $\lambda < \infty$  is closed, planar, and strictly convex with Gauss map of degree at least two.

**Lemma 11.11.** If  $\gamma \subset \mathbf{R}^2$  is a closed shrinker and  $\gamma \neq \mathbf{S}_{\sqrt{2}}^1$ , then:

- (1) The lowest eigenvalue of  $L$  is  $-1$  and the next is less than  $-\frac{1}{2}$ .
- (2) The lowest eigenvalue of  $\mathcal{L}$  is  $0$  and the next is less than  $\frac{1}{2}$ .

*Proof.* Let  $\mathbf{n}$  be the outward pointing unit normal. Since  $\gamma$  is strictly convex,  $H = \langle \mathbf{H}, \mathbf{n} \rangle$  is positive. By [CM8],  $LH = H$  and, thus,  $H$  is the lowest eigenfunction for  $L$ .

Let  $E_1, E_2$  be the standard basis for  $\mathbf{R}^2$ . By [CM8], the translation  $u_i = \langle \mathbf{n}, E_i \rangle$  is a  $-\frac{1}{2}$ -eigenfunction  $Lu_i = \frac{1}{2}u_i$ . Since  $\mathbf{n}$  is monotone as a map from  $\mathbf{S}^1$  to  $\mathbf{S}^1$  with degree at least two,  $u_i$  has at least four nodal domains. The Courant nodal domain theorem then gives that there must be another eigenvalue below  $-\frac{1}{2}$ . This gives (1).

For part (2), observe that  $x_i = \langle E_i, x \rangle$  is a  $\frac{1}{2}$ -eigenfunction  $\mathcal{L}x_i = -\frac{1}{2}x_i$ . Since  $\gamma$  is strictly convex and  $\mathbf{n}$  has degree at least two,  $x_i$  has at least two positive local maxima on  $\gamma$  and a negative local minimum between each maxima. From this, we see that  $x_i$  has at least four nodal domains and (2) now follows from the Courant nodal domain theorem. □

**Corollary 11.12.** If  $\gamma \subset \mathbf{R}^2$  is a  $F$ -stable shrinker with  $\lambda(\gamma) < \infty$ , then  $\gamma = \mathbf{R}$  or  $\gamma = \mathbf{S}_{\sqrt{2}}^1$ .

*Proof.* We can assume  $\gamma$  is closed since otherwise  $\lambda(\gamma) < \infty$  implies that  $\gamma = \mathbf{R}$ . If  $\gamma \neq \mathbf{S}_{\sqrt{2}}^1$ , then Lemma 11.11 gives an eigenvalue for  $L$  strictly between  $-1$  and  $-\frac{1}{2}$ . The corresponding eigenfunction gives a negative variation that is orthogonal to  $\mathbf{H}$  and to translations.  $\square$

## 12. SHARP BOUNDS FOR THE GAUSSIAN WILLMORE FUNCTIONAL

In general,  $W$  is always bounded by entropy (cf. corollary 3.34 in [CM8]):

**Lemma 12.1.** If  $\Sigma^n \subset \mathbf{R}^N$ , then  $2W(\Sigma) \leq n\lambda(\Sigma)$ . Equality holds if and only if  $\Sigma \subset \partial B_{\sqrt{2n}}$ .

*Proof.* Using that  $4|\mathbf{H}|^2 = |x|^2 - |x^T|^2$  and  $\mathcal{L}|x|^2 = 2n - |x|^2$ , we get

$$(12.2) \quad 16 \int_{\Sigma} |\mathbf{H}|^2 e^{-f} = 8n \int_{\Sigma} e^{-f} - 4 \int_{\Sigma} |x^T|^2 e^{-f} = 8n \int_{\Sigma} e^{-f} - \int_{\Sigma} (|x|^2 - 2n)^2 e^{-f}.$$

$\square$

In the rest of this section  $\Sigma^n \subset \mathbf{R}^N$  is closed. Given  $\Sigma^2$  with genus  $\gamma$ , define  $C_{YY}$  by

$$(12.3) \quad C_{YY} = \begin{cases} 2 & \text{if } \gamma = 0 \\ \gamma + 3 & \text{if } \gamma > 0 \end{cases}.$$

**Theorem 12.4.** If  $\Sigma$  is a  $F$ -stable closed surface with genus  $\gamma$ , then

$$(12.5) \quad W(\Sigma) \leq \begin{cases} \frac{2C_{YY}}{e} & \text{if } \Sigma \text{ is oriented.} \\ \frac{4C_{YY}}{e} & \text{if } \Sigma \text{ is unoriented.} \end{cases}$$

Let  $\Lambda$  be set of smooth functions  $u$  on  $\Sigma$  with  $\int_{\Sigma} u |\mathbf{H}|^2 e^{-f} = 0$  and let  $\Lambda^* \subset \Lambda$  be the  $u$ 's with  $\int_{\Sigma} u^2 |\mathbf{H}|^2 e^{-f} > 0$ . Define  $\mu_{|\mathbf{H}|^2} \geq 0$  by

$$(12.6) \quad \mu_{|\mathbf{H}|^2} = \inf_{u \in \Lambda^*} \frac{\int_{\Sigma} |\nabla u|^2 |\mathbf{H}|^2 e^{-f}}{\int_{\Sigma} u^2 |\mathbf{H}|^2 e^{-f}}.$$

When  $|\mathbf{H}| > 0$ , then this infimum is achieved and  $\mu_{|\mathbf{H}|^2}$  is the first positive eigenvalue for the drift operator  $\mathcal{L}_{|\mathbf{H}|^2} = \mathcal{L} + \nabla_{\nabla \log |\mathbf{H}|^2}$  for the weight  $|\mathbf{H}|^2 e^{-f}$ .

**Lemma 12.7.** If  $\mu_{|\mathbf{H}|^2} < \frac{1}{2}$ , then  $\Sigma$  is  $F$ -unstable.

*Proof.* Since  $\mu_{|\mathbf{H}|^2} < \frac{1}{2}$ , there exists a function  $u$  with

$$(12.8) \quad \int |\nabla u|^2 |\mathbf{H}|^2 e^{-f} < \frac{1}{2} \int u^2 |\mathbf{H}|^2 e^{-f},$$

$$(12.9) \quad \int u |\mathbf{H}|^2 e^{-f} = 0.$$

The equality gives that  $u\mathbf{H}$  is  $L^2$ -orthogonal to  $\mathbf{H}$ . Using (9.11) and (12.8) gives

$$(12.10) \quad (4\pi)^{\frac{n}{2}} \delta^2 F(u\mathbf{H}) = \int [|\nabla u|^2 - u^2] |\mathbf{H}|^2 e^{-f} < -\frac{1}{2} \int u^2 |\mathbf{H}|^2 e^{-f}.$$

Let  $V^\perp$  with  $V \in \mathbf{R}^N$  be the  $L^2$ -projection of  $u\mathbf{H}$  to the space of translations. Since  $\mathbf{H}$  is orthogonal to  $V^\perp$ , it follows that  $u\mathbf{H} - V^\perp$  is orthogonal to both  $\mathbf{H}$  and translations. We will show that  $\delta^2 F(u\mathbf{H} - V^\perp) < 0$ . Using (9.2), (9.4), symmetry of  $L$ , and (12.10), we have

$$(12.11) \quad \begin{aligned} (4\pi)^{\frac{n}{2}} \delta^2 F(u\mathbf{H} - V^\perp) &= - \int \langle (u\mathbf{H} - V^\perp), L(u\mathbf{H}) \rangle e^{-f} = (4\pi)^{\frac{n}{2}} \delta^2 F(u\mathbf{H}) + \int \langle V^\perp, L(u\mathbf{H}) \rangle e^{-f} \\ &= (4\pi)^{\frac{n}{2}} \delta^2 F(u\mathbf{H}) + \frac{1}{2} \int \langle V^\perp, u\mathbf{H} \rangle e^{-f} < -\frac{1}{2} \int u^2 |\mathbf{H}|^2 e^{-f} + \frac{1}{2} \int |V^\perp|^2 e^{-f}. \end{aligned}$$

Since  $\|V^\perp\|_{L^2} \leq \|u\mathbf{H}\|_{L^2}$ , it follows that  $\Sigma$  is  $F$ -unstable.  $\square$

Similarly, we define higher  $\mu_{k,|\mathbf{H}|^2}$ 's to be the infimum over  $k$ -dimensional families in (12.6).

**Corollary 12.12.** If  $\Sigma^n \subset \mathbf{R}^N$  is  $F$ -stable, then  $\mu_{|\mathbf{H}|^2, N+1} \geq 1$ .

*Proof.* Suppose not. Since the space of translations is  $N$ -dimensional, we can find a function  $\phi$  so that  $\phi\mathbf{H}$  is orthogonal to translations (and  $\mathbf{H}$ ) and, moreover,

$$(12.13) \quad \int |\nabla \phi|^2 |\mathbf{H}|^2 e^{-f} < \int \phi^2 |\mathbf{H}|^2 e^{-f}.$$

Stability implies that  $\delta^2(\phi\mathbf{H}) \geq 0$  which contradicts this and (9.11).  $\square$

**Lemma 12.14.** For all  $r > 0$ , we have  $r^2 e^{-\frac{r^2}{4}} \leq \frac{4}{e}$ , with equality if and only if  $r = 2$ .

*Proof.* Set  $h(r) = r^2 e^{-\frac{r^2}{4}}$ , then  $h'(r) = 2r \left(1 - \frac{r^2}{4}\right) e^{-\frac{r^2}{4}}$ . It follows that  $h(r) \leq h(2)$ .  $\square$

*Proof of Theorem 12.4.* We will assume first that  $\Sigma$  is a topological sphere and roughly follow the argument of Hersch, [He] (page 240 in [CM7]; cf. [ChY], [CM6]). Let  $g$  be the metric on  $\Sigma$ . Since  $\Sigma$  is a sphere, there is a conformal diffeomorphism  $\Phi : \Sigma \rightarrow \mathbf{S}^2 \subset \mathbf{R}^3$ . The group of conformal transformations of  $\mathbf{S}^2$  contains a subgroup parametrized on the ball  $B_1 \subset \mathbf{R}^3$  with  $z \in B_1$  corresponding to a “dilation”  $\Psi_z$  in the direction of  $\frac{z}{|z|}$  with  $|z|$  determining the amount of the dilation (these are the  $\psi(x, t)$ 's on page 240 in [CM6]). As  $|z| \rightarrow 1$ ,  $\Psi_z$  takes  $\mathbf{S}^2 \setminus \{\frac{-z}{|z|}\}$  to  $\frac{z}{|z|}$ . Define a map  $\mathcal{A} : B_1 \rightarrow \mathbf{R}^3$  by

$$(12.15) \quad \mathcal{A}(z) = \frac{1}{\int_\Sigma |\mathbf{H}|^2 e^{-f}} \int_\Sigma (x_i \circ \Psi_z \circ \Phi) |\mathbf{H}|^2 e^{-f}.$$

It follows that  $\mathcal{A}$  extends continuously to  $\partial B_1 = \mathbf{S}^2$  to be the identity on  $\partial B_1$ . Elementary topology gives some  $\bar{z} \in B_1$  so that  $\mathcal{A}(\bar{z}) = 0$ . Define  $u_i$  on  $\Sigma$  by  $u_i = x_i \circ \Psi_{\bar{z}} \circ \Phi$  so that

$$(12.16) \quad \int_\Sigma u_i |\mathbf{H}|^2 e^{-f} = 0.$$

Therefore,  $F$ -stability, (12.16) and Lemma 12.7 imply that for each  $i$

$$(12.17) \quad \int_\Sigma u_i^2 |\mathbf{H}|^2 e^{-f} \leq 2 \int_\Sigma |\nabla_g u_i|^2 |\mathbf{H}|^2 e^{-f}.$$

Summing over  $i$  and using that  $\sum_i u_i^2 \equiv 1$  gives

$$(12.18) \quad 4\pi W(\Sigma) = \int_\Sigma |\mathbf{H}|^2 e^{-f} \leq 2 \sum_i \int_\Sigma |\nabla_g u_i|^2 |\mathbf{H}|^2 e^{-f}.$$



Since  $4|\mathbf{H}|^2 \leq |x|^2$ , Lemma 12.14 implies that  $|\mathbf{H}|^2 e^{-f} \leq e^{-1}$ . Using this in (12.18) and then conformal invariance of the energy gives

$$(12.19) \quad 4\pi W(\Sigma) \leq \frac{2}{e} \sum_i \int_{\Sigma} |\nabla_g u_i|^2 = \frac{2}{e} \sum_i \int_{\mathbf{S}^2} |\nabla x_i|^2 = \frac{16\pi}{e}.$$

When  $\Sigma$  is not a sphere, then we follow Yang-Yau, [YY] (see also [EI], remark 1.2 in [Ka]), and replace  $\Phi$  by a branched conformal map whose degree is bounded in the terms of  $\gamma$ . The degree comes in as a factor in the equalities in (12.19), increasing the estimate for  $W$ .  $\square$

*Proof of Theorem 0.14.* The case  $\gamma = 0$  of (12.5) gives the inequality. Suppose now that  $\Sigma$  realizes equality. First, we must have equality in  $4|\mathbf{H}|^2 \leq |x|^2$  and, thus,  $|x^T|^2 \equiv 0$  and  $\Sigma$  is contained in a sphere. We also get equality in Lemma 12.14 so this sphere has radius 2. By Lemma 10.13,  $\Sigma$  is minimal in  $\partial B_2 \subset \mathbf{R}^N$ . Moreover, equality also implies that  $\Sigma$  has the same area as  $\mathbf{S}_2^2$  and, thus,  $\Sigma = \mathbf{S}_2^2$  by Cheng-Li-Yau, [CGLYa].  $\square$

**12.1. When  $\mu_{|\mathbf{H}|^2} = \frac{1}{2}$ .** When we analyzed the case of equality in the bound for the Gaussian Willmore functional, one of the things that came out along the proof was that  $\mu_{|\mathbf{H}|^2} = \frac{1}{2}$  with multiplicity three and the eigenfunctions spanned the tangent space at each point. We next analyze the borderline case where  $\mathcal{L}_{|\mathbf{H}|^2}$  has eigenvalue  $\mu_{|\mathbf{H}|^2} = \frac{1}{2}$  more generally. Recall that the principal normal  $\mathbf{N} = \frac{\mathbf{H}}{|\mathbf{H}|}$  is defined wherever  $\mathbf{H} \neq 0$ .

**Lemma 12.20.** If  $\Sigma$  is  $F$ -stable and  $\mu_{|\mathbf{H}|^2} = \frac{1}{2}$ , then for any eigenfunction  $\phi$  of  $\mathcal{L}_{|\mathbf{H}|^2}$  with eigenvalue  $\frac{1}{2}$  there exists a vector  $V \in \mathbf{R}^N$  such that  $\phi \mathbf{H} = V^\perp$  and  $\nabla_{\nabla^T \phi}^\perp \mathbf{N} = 0$ .

*Proof.* Using (9.11), integration by parts, and  $\mathcal{L}_{|\mathbf{H}|^2} \phi = -\frac{1}{2} \phi$  gives

$$(12.21) \quad (4\pi)^{\frac{n}{2}} \delta^2 F(\phi \mathbf{H}) = \int [|\nabla \phi|^2 - \phi^2] |\mathbf{H}|^2 e^{-f} = -\frac{1}{2} \int \phi^2 |\mathbf{H}|^2 e^{-f}.$$

Choose  $V \in \mathbf{R}^N$  so that  $V^\perp$  is the  $L^2$ -projection of  $\phi \mathbf{H}$  to the space of translations. Since  $V^\perp$  and  $\phi \mathbf{H}$  are orthogonal to  $\mathbf{H}$ , it follows that  $\phi \mathbf{H} - V^\perp$  is orthogonal to both  $\mathbf{H}$  and translations. Thus, stability, symmetry of  $L$ ,  $L V^\perp = \frac{1}{2} V^\perp$  and (12.21) give

$$(12.22) \quad \begin{aligned} 0 &\leq (4\pi)^{\frac{n}{2}} \delta^2 F(\phi \mathbf{H} - V^\perp) = - \int_{\Sigma} \langle \phi \mathbf{H} - V^\perp, L(\phi \mathbf{H}) \rangle e^{-f} \\ &= -\frac{1}{2} \|\phi \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \int_{\Sigma} \langle V^\perp, \phi \mathbf{H} \rangle e^{-f} = -\frac{1}{2} \|\phi \mathbf{H}\|_{L^2}^2 + \frac{1}{2} \|V^\perp\|_{L^2}^2. \end{aligned}$$

It follows that  $\|V^\perp\|_{L^2} = \|\phi \mathbf{H}\|_{L^2}$  and, thus, that  $\phi \mathbf{H} = V^\perp$  and  $L(\phi \mathbf{H}) = \frac{1}{2} \phi \mathbf{H}$ . The second claim follows from Leibniz' rule and  $L(\phi \mathbf{H}) = \frac{1}{2} \phi \mathbf{H}$

$$\begin{aligned} \frac{1}{2} \phi \mathbf{H} &= L(\phi \mathbf{H}) = \phi L \mathbf{H} + (\mathcal{L} \phi) \mathbf{H} + 2 \nabla_{\nabla^T \phi}^\perp \mathbf{H} \\ &= (\phi + \mathcal{L}_{|\mathbf{H}|^2} \phi) \mathbf{H} - |\mathbf{H}|^{-2} \langle \nabla |\mathbf{H}|^2, \nabla^T \phi \rangle \mathbf{H} + 2 \nabla_{\nabla^T \phi}^\perp \mathbf{H} = \frac{1}{2} \phi \mathbf{H} + 2 |\mathbf{H}| \nabla_{\nabla^T \phi}^\perp \mathbf{N}. \end{aligned}$$

$\square$

**12.2. Frenet-Serret equations for shrinkers.** In  $\mathbf{R}^3$ , the Frenet-Serret frame for a curve  $\gamma$  parametrized by arclength is the orthonormal frame for  $\mathbf{R}^3$  along  $\gamma$  consisting of the unit tangent  $\gamma'$ , the unit normal  $\mathbf{n} \equiv \frac{\gamma''}{|\gamma''|}$ , and the binormal  $\mathbf{b} \equiv \gamma' \times \mathbf{n}$ . The Frenet-Serret formulas are ( $k = |\gamma''|$ ):

$$(12.23) \quad \gamma'' = k \mathbf{n},$$

$$(12.24) \quad \mathbf{n}' = -k \gamma' + \tau \mathbf{b},$$

$$(12.25) \quad \mathbf{b}' = -\tau \mathbf{n},$$

where  $\tau$  is the torsion of  $\gamma$ . We give an analog of these for an oriented shrinker  $\Sigma^n \subset \mathbf{R}^{n+2}$ . Let  $J$  be the almost-complex structure of the (oriented) normal bundle. Using  $J$ , we get a well-defined binormal  $\mathbf{B} = J\mathbf{N}$ . Observe that  $\langle \mathbf{B}, x \rangle = \langle \mathbf{B}, x^\perp \rangle = 2 \langle \mathbf{B}, \mathbf{H} \rangle = 0$ , so that  $\mathbf{B}$  is always tangent to a sphere centered at 0. We get the following Frenet-Serret type formulas:

$$(12.26) \quad \nabla \mathbf{N} = \tau \mathbf{B} - \langle \mathbf{N}, A(\cdot, \cdot) \rangle,$$

$$(12.27) \quad \nabla \mathbf{B} = -\tau \mathbf{N} - \langle \mathbf{B}, A(\cdot, \cdot) \rangle.$$

It remains to compute the torsion. Given a tangent vector  $V$ , Lemma 9.5 gives

$$(12.28) \quad \tau(V) = \langle \nabla_V^\perp \mathbf{N}, \mathbf{B} \rangle = \langle \frac{\nabla_V^\perp \mathbf{H}}{|\mathbf{H}|}, \mathbf{B} \rangle = -\frac{1}{2} \langle \frac{A(x^T, V)}{|\mathbf{H}|}, \mathbf{B} \rangle.$$

**Corollary 12.29.**  $\langle A, \mathbf{B} \rangle = 0$  if and only if  $\Sigma$  is a hypersurface in a hyperplane.

*Proof.* By (12.27) and (12.28),  $\langle A, \mathbf{B} \rangle = 0$  if and only if  $\mathbf{B}$  is a constant vector.  $\square$

**Theorem 12.30.** If  $\Sigma^2 \subset \mathbf{R}^4$  is  $F$ -stable, closed, oriented, and  $\mu_{|\mathbf{H}|^2} = \frac{1}{2}$ , then  $\Sigma^2 = \mathbf{S}_2^2$ .

*Proof.* We will show that  $\langle A, \mathbf{B} \rangle = 0$  on an open set. Once we have this, Corollary 12.29 and unique continuation imply that  $\Sigma$  is contained in a hyperplane and then [CM6] and  $F$ -stability give that it is spherical or planar. Let  $\phi$  be an eigenfunction as in Lemma 12.20, so  $\phi \mathbf{H} = V^\perp$  for  $V \in \mathbf{R}^4$ . Differentiating gives

$$(12.31) \quad -\frac{\phi}{2} \langle A, \mathbf{B} \rangle (x^T, \cdot) = \langle \nabla(\phi \mathbf{H}), \mathbf{B} \rangle = \langle \nabla V^\perp, \mathbf{B} \rangle = -\langle A, \mathbf{B} \rangle (V^T, \cdot).$$

It follows that  $\frac{\phi}{2} x^T - V^T$  is in the kernel of  $\langle A, \mathbf{B} \rangle$  at each point. If  $\frac{\phi}{2} x^T - V^T$  vanishes everywhere, then so does  $\frac{\phi}{2} x - V$  and, thus,  $\phi x = 2V$  is constant. This is impossible, so there must be an open set  $\Omega$  where  $\frac{\phi}{2} x^T - V^T \neq 0$ . However, the two by two matrix  $\langle A, \mathbf{B} \rangle$  is symmetric and trace-free, so it is either invertible or zero. Since it has nontrivial kernel in  $\Omega$ , we see that  $\langle A, \mathbf{B} \rangle \equiv 0$  in  $\Omega$ . This completes the proof.  $\square$

### 13. ENTROPY BOUNDS

In this section we prove eigenvalue and entropy bounds without assuming that the dimension  $N$  of the ambient Euclidean space is large compared with the entropy of the shrinker.

**Theorem 13.1.** If  $\Sigma^n \subset \mathbf{R}^N$  is  $F$ -stable with finite entropy and  $N \geq 2n$ , then  $\mu_{2nN} \geq \frac{1}{4}$ .

**Corollary 13.2.** There is a universal constant  $C$  so that if  $\Sigma^2 \subset \mathbf{R}^N$  is closed and  $F$ -stable of genus  $\gamma$ , then  $\lambda(\Sigma) \leq C(1 + \gamma)N$ .

In the next lemma,  $E_i$  is an orthonormal basis for  $\mathbf{R}^N$ .

**Lemma 13.3.** If  $\mathcal{V} \subset \mathbf{R}^N$  is an  $n$ -dimensional linear subspace,  $\Pi$  and  $\Pi^\perp$  are orthogonal projections to  $\mathcal{V}$  and  $\mathcal{V}^\perp$ , then for any  $k \in \mathbf{Z}$  with  $1 \leq k \leq N - n$

$$(13.4) \quad k \leq \sum_{i=1}^{n+k} |\Pi^\perp(E_i)|^2 \leq n + k.$$

*Proof.* Since  $\sum_{i=1}^N |\Pi(E_i)|^2$  is the trace of a quadratic form, it is independent of the choice of basis. Choosing the basis  $\bar{E}_i$  so that  $\bar{E}_1, \dots, \bar{E}_n \in \mathcal{V}$  and the rest are in  $\mathcal{V}^\perp$ , we see that

$$(13.5) \quad \sum_{i=1}^N |\Pi(E_i)|^2 = \sum_{i=1}^N |\Pi(\bar{E}_i)|^2 = n.$$

Using this, we see that

$$\sum_{i=1}^{n+k} |\Pi^\perp(E_i)|^2 = \sum_{i=1}^{n+k} (1 - |\Pi(E_i)|^2) = (n+k) - \sum_{i=1}^{n+k} |\Pi(E_i)|^2 \geq (n+k) - \sum_{i=1}^N |\Pi(E_i)|^2 = k.$$

This gives the first inequality in (13.4). The second inequality is immediate.  $\square$

**Lemma 13.6.** Suppose that  $\Sigma^n \subset \mathbf{R}^N$  is  $F$ -stable,  $k \in \mathbf{Z}$  and  $1 \leq k \leq N - n$ . If  $\phi \in L^2$  is a function so that

$$(13.7) \quad \int_{\Sigma} \phi \langle E_j^\perp, \mathbf{H} \rangle e^{-f} = 0 \text{ for } j = 1, \dots, n+k,$$

$$(13.8) \quad \int_{\Sigma} \phi \langle E_j^\perp, E_\ell^\perp \rangle e^{-f} = 0 \text{ for } (j, \ell) \in \{1, \dots, n+k\} \times \{1, \dots, N\},$$

$$\text{then } \int_{\Sigma} \phi^2 e^{-f} \leq 2 \left( \frac{n}{k} + 1 \right) \int_{\Sigma} |\nabla \phi|^2 e^{-f}.$$

*Proof.* By (13.7) and (13.8), the vector field  $\phi E_j^\perp$  is orthogonal to  $\mathbf{H}$  and to translations for each  $j = 1, \dots, n+k$ . By the definition of  $F$ -stability and Lemma 9.9

$$(13.9) \quad 0 \leq (4\pi)^{\frac{n}{2}} \delta^2 F(\phi E_j^\perp) = \int \left[ |\nabla \phi|^2 - \frac{1}{2} \phi^2 \right] |E_j^\perp|^2 e^{-f}.$$

Finally, we sum (13.9) over  $j \leq n+k$  and use  $k \leq \sum_{j=1}^{n+k} |E_j^\perp|^2 \leq n+k$  by Lemma 13.3.  $\square$

**Corollary 13.10.** If  $\Sigma^n \subset \mathbf{R}^N$  has  $F$ -index  $I$ , then for any  $k \in \mathbf{Z}$  with  $1 \leq k \leq N - n$

$$(13.11) \quad \mu_{(n+k)(N+I-\frac{1}{2}(n+k-3))} \geq \frac{k}{2(n+k)}.$$

*Proof.* We will first show the corollary when  $I = 0$ . For a fixed  $\phi$ , (13.7) and (13.8) give  $n+k$  and  $\frac{1}{2}(n+k-1)(n+k) + (n+k) + (n+k)(N - (n+k)) = (n+k)(N - \frac{1}{2}(n+k-1))$  homogeneous linear equations. So  $(n+k)(N - \frac{1}{2}(n+k-3))$  linear equations. Thus, we can choose a linear combination  $\phi = \sum a_i u_i$  of the functions

$$(13.12) \quad u_0, u_1, \dots, u_{(n+k)(N-\frac{1}{2}(n+k-3))}$$

with  $\sum a_i^2 = 1$  and so  $\phi$  satisfies (13.7) and (13.8). Lemma 13.6 with this  $\phi$  gives

$$\begin{aligned} 1 &= \int \phi^2 e^{-f} \leq 2 \left( \frac{n}{k} + 1 \right) \int |\nabla \phi|^2 e^{-f} = 2 \left( \frac{n}{k} + 1 \right) \sum (a_i^2 \mu_i) \\ (13.13) \quad &\leq 2 \left( \frac{n}{k} + 1 \right) \mu_{(n+k)(N-\frac{1}{2}(n+k-3))} \sum a_i^2 = 2 \left( \frac{n}{k} + 1 \right) \mu_{(n+k)(N-\frac{1}{2}(n+k-3))}. \end{aligned}$$

The case where  $I > 0$  follows with obvious modifications.  $\square$

Specializing to  $k = n$  and  $I = 0$  gives Theorem 13.1.

*Proof of Corollary 13.2.* Corollary 13.10 gives for  $n = 2$  that  $\mu_{2(2N-1)} \geq \frac{1}{4}$ . Combining this with Proposition 11.4 gives the corollary.  $\square$

Corollary 13.2 extends easily to give general entropy bounds in terms of the index  $I > 0$ .

**Conjecture 13.14.** There exist  $\alpha < 1$  and  $C_\alpha = C_\alpha(\alpha, \gamma)$  so that if  $\Sigma^2 \subset \mathbf{R}^N$ , then the multiplicity of the  $\frac{1}{2}$  eigenvalue for  $\mathcal{L}$  is at most  $C_\alpha \lambda^\alpha(\Sigma)$ . If so, then Corollary 0.8 would give Theorem 0.9 without the assumption  $N \geq C \lambda(\Sigma)$ .

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