Operations in Complex-Oriented Cohomology Theories Related to Subgroups of Formal Groups

by

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A.B. Mathematics, Princeton University
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Submitted to the Department of Mathematics
in Partial Fulfillment of the Requirements
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ABSTRACT

Using the character theory of Hopkins–Kuhn–Ravenel and the total power operation in complex cobordism of tom Dieck, we develop a theory of power operations in Landweber-exact cohomology theories. We give a description of the total power operation in terms of the theory of subgoups of formal group laws developed by Lubin.

We apply this machinery in two cases. For the cohomology theory E_h, we obtain a formal-group theoretic condition on the orientation which is an obstruction to the compatibility of H_∞ structures in MU and E_h. We show that there is a unique choice of orientation for which this obstruction vanishes, allowing us to build a large family of unstable cohomology operations based on E_h. We show that the the multiplicative formal group law of K-theory satisfies our condition.

In elliptic cohomology, our machinery is naturally related to quotients of elliptic curves by finite subgroups. We adapt our machinery to elliptic cohomology, and produce both the Adams operation and versions of the Hecke operators of Baker as power operations.

Thesis Supervisor: Haynes Miller
Title: Professor of Mathematics

Acknowledgments

At a time like this, it is natural to experience gratitude in two forms, mathematical and personal. In my case the overlap is substantial, which may or may not be a good thing, but at least makes for a shorter Acknowledgments section.

I am very grateful to my advisor, Haynes Miller, for his generosity and his patience. The idea for this project came out of a course he taught my first year here. Most of the ideas he's suggested to me have yet to be studied seriously, and if I can remember just a few, I'll be in good shape for several more years.

By now, Mike Hopkins has been my teacher for five years, poor guy. My debt to him runs throughout this paper, both in citations and in the mathematics itself. Theorem 4.0.2, especially, came into being in consultation with Mike, and I never would have started on this project without [HKR91].

If the credit for the inception of this thesis goes to Haynes's course and [HKR91], then at least some of the credit for its completion goes to Dan Kan, who told me to start writing in August, which was barely in time but at least three months before I might otherwise have started. I benefitted from his counsel on several crucial occasions, and I never had to take time out of my day otherwise to do so.

One of my favorite experiences in graduate school was reading Quillen's Homotopical Algebra with Jim Turner. Thanks, Jim. Hal Sadofsky was a kind mentor and friend in my first year at MIT, and set a standard of assistance to junior graduate students which I have found impossible to match.

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Looking at my work over the last few years, it's clear that I made a lot of progress this year. Everyone who knows me well knows how much Amy helped me. Only Amy and I know how difficult I was about it.

This thesis is dedicated to my "office staff", Chester, Spork, Spike, Volleyduck, Lobster, and Opus. They never experienced any doubt. Of course, they never experienced much of anything: they have acrylic for brains.

Contents

1 Introduction

Recent work in topology [DHS88, Hop87] and in mathematical physics [Wit88, BT89] has focused attention on a collection of complex-oriented cohomology theories that includes K-theory, elliptic cohomology [Lan88, LRS88, Seg88], and various E_h. With the exception of K–theory, these theories are most easily defined using an algebraic technique due to Landweber [Lan76], and one of the fundamental questions in the subject is how to render geometric or analytic descriptions of these theories.

The algebraic description begins with a "genus", that is, a ring homomorphism

$$MU^* \xrightarrow{t_E} E^*, \tag{1.0.1}$$

where MU^* is the cobordism ring of stably almost-complex manifolds. Landweber's theorem [Lan76] is a sufficient criterion on a genus for the resulting functor

$$X \longmapsto E^* \underset{MU^*}{\otimes} MU^*X \tag{1.0.2}$$

to be a cohomology theory on finite complexes. Because of the Mayer-Vietoris axiom, this amounts to showing that tensoring with E^* via the genus t_E is exact for MU^* modules of the form MU^*X, where X is a finite complex. For this reason, a cohomology theory determined by (1.0.2) is called "exact."

Knowing only the description (1.0.2) is a little like having a Greek-Greek dictionary when one speaks only English: given such a theory E and a space X, one can attempt to compute its cohomology groups, but then would have very little idea what has been accomplished when the task is finished.

The paradigm for this problem is K-theory. The genus associated to K-theory is the Todd genus

$$MU^* \xrightarrow{\mathrm{Td}} K^*.$$

A theorem of Conner and Floyd tells us that

Theorem 1.0.3 ([CF66, Lan76]) *The map*

$$K^* \underset{MU^*}{\otimes} MU^*X \underset{\cong}{\xrightarrow{t_K}} K^*X \tag{1.0.4}$$

is an isomorphism when X is a finite complex.

The left-hand-side of (1.0.4) is the description of K-theory provided by exactness, and the proof of (1.0.4) using Landweber's theorem makes no use of vector bundles. For the theories Ell and E_h, it's as if we knew of the existence of K-theory, but not about its relationship to vector bundles.

One thorny problem in understanding the geometry of a theory like E_h is the choice of the the genus itself. In stable homotopy, this problem is usually studied in terms of the associated group law: by Quillen's theorem, specifying a genus

$$MU^* \longrightarrow E_h^*$$

is equivalent to specifying a formal group law F over E^*. More precisely, there is a formal group law F^{MU} over MU^*, and MU^* together with the formal group law F^{MU} represents functor

$$R \overset{FGL}{\longmapsto} \{\text{Formal group laws over } R\}$$

by the correspondence

$$\begin{aligned} \mathrm{Hom}_{rings}[MU^*, R] &\longrightarrow FGL(R) \\ f &\mapsto f_* F^{MU}. \end{aligned}$$

The ring E_h^* is a complete, local ring. The usual genus

$$MU^* \xrightarrow{t_h} E_h^*$$

over E_h results in a formal group law F_h whose reduction modulo the maximal ideal we call φ. It turns out that any formal group law F over E_h^* which reduces modulo the maximal ideal to φ detects the same information in stable homotopy. The formal group law F_h has the feature that it is "p-typical", and is an obvious choice to make when E_h is constructed from BP. However, geometric considerations, when they are available, lead to other genera.

For example, E_1 is p-adic K-theory. With respect to the genus t_1 of [Rav86] (see section 5.1), the p-series of the resulting formal group law F_1 satisfies

$$[p]_{F_1}(x) = px + u^{p-1}x^p \pmod{\text{degree } p+1}.$$

From the point of view of geometry, however, the most natural formal group law is the multiplicative formal group law $\mathbb{G}_m$, which is the group law classified by the Todd genus. It satisfies

$$[p]_{\mathbb{G}_m}(x) = px + u \binom{p}{2} x^2 + \ldots + u^{p-1}x^p.$$

If we were to have any hope, starting with just the description (1.0.2), of discovering the relationship between E_1 and vector bundles, we would surely have needed to start our search with the Todd genus and the multiplicative formal group law!

The main result of this paper is the description of a new canonical genus for the cohomology theories E_h. In the case that $h = 1$, we recover the multiplicative group law over p-adic K-theory.

Let F be any formal group law over E_h which reduces modulo the maximal ideal to φ. Over any complete, local E_h^*-algebra R, the elements of the maximal ideal of R, together with the addition law specified by the formal group law, form a group, which we denote $F(R^*)$.

Let D^* be the smallest extension of E_h^* which contains the roots of the p^k-series for all $k > 0$. Then ([Hop91]; see 2.3) D^* is a Galois extension of E^* with Galois group $\mathrm{Aut}_{\mathrm{gp}}[F(D^*)_{\mathrm{tors}}]$. Let $f_p(x) \in D^*[\![x]\!]$ be the power series

$$f_p(x) = \prod_{\substack{v \in F(D^*) \\ [p](v)=0}} (v \underset{F}{+} x).$$

By Galois invariance, the power series f_p in fact has coefficients in E_h^*. It is the power series considered by Lubin in his study of finite subgroups of formal groups, in the case that the subgroup is the p-torsion points of $F(D^*)$. Associated to it is a formal group law F/p which is related to F by the homomorphism of formal group laws

$$F \xrightarrow{f_p} F/p.$$

f_p and F/p are constructed so that

$${}_pF(D^*) = \mathrm{Ker}[F(D^*) \xrightarrow{f_p} F/p(D^*)].$$

On the other hand, the p-series $[p]_F(x)$ an endomorphism of F, and certainly

$${}_pF(D^*) = \mathrm{Ker}[F(D^*) \xrightarrow{[p]_F} F(D^*)].$$

Let u denote the periodicity element of E_h; we use a version of E_h in which this element has degree -2 (5.1). Our genus is given by

Theorem A (see section 5.4) *There is a unique genus*

$$MU^* \xrightarrow{t_{PO}} E_h^*$$

such that the resulting formal group law is $\star$-isomorphic to F_h and satisfies

$$[p](x) = u^{p^h-1} f_p(x). \tag{1.0.5}$$

The reason why we hope that the genus described in Theorem A is related to geometry is its relationship to "power operations", which we now describe.

Once one has a good geometric understanding of a cohomology theory, one expects to find rich additional structure in these cohomology theories coming from the geometry. In particular, it is natural to examine symmetries of the geometry in search of cohomology operations. This thesis represents an attempt to turn this idea on its head, and to study cohomology operations in exact cohomology theories as a means of learning something about their conjectural geometry.

"Power operations" is the name of a program for constructing cohomology operations in a ring theory E. The reason for singling this method out for investigation is that on the one hand, one can describe in purely algebraic terms what it is one would like to do, while on the other, in every cohomology theory where power operations have been successfully constructed, the construction relies heavily on a good description of the cohomology theory. Some examples are mod–p cohomology, in which one obtains the Steenrod operations [SE62], K-theory, where one obtains the exterior powers and Adams operations [Ati66], and complex cobordism [tDi68], where Quillen used them to prove his theorem about MU^* [Qui71].

The basic problem in constructing power operations is as follows. Let E be a cohomology theory with products, and let S_n denote the symmetric group on n letters. For any space X, S_n acts on X^n. One wants to construct a "total power operation" P_n^E which factors the n^{th} power map

$$E^*X \xrightarrow{x \mapsto x^{\times n}} E^{*n}X^n$$

through a suitable equivariant cohomology theory E_{S_n}. If no better candidate is available, one simply takes $E^*_{S_n}(X^n)$ to be $E^*(D_nX)$, where D_nX is the Borel construction

$$D_nX = ES_n \underset{S_n}{\times} X^n.$$

In other words, the idea is to find a natural transformation P_n^E filling in the diagram

$$\begin{array}{ccc} E^*X & \xrightarrow{x^{\times n}} & E^{*n}X^n \\ & \searrow^{P_n^E} & \uparrow i^* \\ & & E^{*n}_{S_n}X^n, \end{array} \qquad (1.0.6)$$

where i^* is the forgetful map

$$E^*_{S_n}(X^n) \xrightarrow{i^*} E^*(X^n).$$

In fact it is often enough to construct, for π an abelian p-group of order n, a "total power operation based on π"

$$E^*X \xrightarrow{P^E_\pi} E^{n*}D_\pi X,$$

where $D_\pi X$ is the Borel construction

$$D_\pi X = E\pi \underset{\pi}{\times} X^\pi,$$

and

$$X^\pi = \mathrm{Maps}[\pi, X]$$

denotes the left π-space obtained by using the right action of π on itself.

In any case, with P_n^E in hand, one proceeds to produce natural transformations

$$E_{S_n}^{*n}(X^n) \xrightarrow{\theta} E^*X;$$

then the composite

$$E^*X \xrightarrow{P_n^E} E_{S_n}^{*n}(X^n) \xrightarrow{\theta} E^*X$$

is an operation on E.

For example, Atiyah [Ati66] constructed a total power operation

$$KX \xrightarrow{P_n^K} K_{S_n}(X^n).$$

If

$$X \xrightarrow{\Delta} X^n$$

is the diagonal map, then $\Delta^* P_n^K$ lands in $K_{S_n}X$, which is isomorphic to $RS_n \otimes KX$:

$$KX \xrightarrow{P_n^K} K_{S_n}(X^n) \xrightarrow{\Delta^*} K_{S_n}(X) \cong RS_n \otimes KX.$$

Evaluation of characters on the class of an n-cycle in S_n yields a ring homomorphism

$$RS_n \xrightarrow{eval} \mathbb{Z},$$

and the composite

$$KX \xrightarrow{P_n^K} K_{S_n}(X^n) \xrightarrow{\Delta^*} RS_n \otimes KX \xrightarrow{eval \otimes 1} KX \tag{1.0.7}$$

turns out to be the (integral!) Adams operation Ψ^n on KX.

It is a hard problem in general to construct the total power operation P_n in an arbitrary cohomology theory (see e.g. [BMMS86]), primarily because it has never been clear what P_n should mean in any generality. One approach is to try to use tom Dieck's [tDi68, Qui71] construction of a total power operation

$$MU^{2*}X \xrightarrow{P_n^{MU}} MU^{2n*}(D_nX)$$

for complex cobordism. Then for a cohomology theory E with a complex orientation

$$MU \xrightarrow{t_E} E,$$

(e.g. any exact theory), one might ask whether a putative total power operation P_n^E be compatible in the sense that the diagram

$$\begin{array}{ccc} MU^{2*}X & \xrightarrow{P_n^{MU}} & MU^{2n*}(D_nX) \\ {\scriptstyle t_E}\downarrow & & \downarrow{\scriptstyle t_E} \\ E^{2*} & \xrightarrow{P_n^E} & E^{2n*}(D_nX) \end{array} \tag{1.0.8}$$

commutes. If moreover E is exact, then it is determined by MU via t_E (1.0.2), so P_n^E is determined by the diagram (1.0.8) if it exists.

This optimism is supported by the example of K-theory: tom Dieck's operation P_n^{MU} and Atiyah's operation P_n^K are compatible under the Atiyah-Bott-Shapiro orientation: if

$$MU^{2*}X \xrightarrow{t_{\mathrm{ABS}}} KX$$

is the Atiyah-Bott-Shapiro orientation, then

$$\begin{array}{ccc} MU^{2*}X & \xrightarrow{P_n^{MU}} & MU^{2n*}(D_nX) \\ {\scriptstyle t_{\mathrm{ABS}}}\downarrow & & \downarrow{\scriptstyle t_{\mathrm{ABS}}} \\ KX & \xrightarrow{P_n^K} & K(D_nX) \end{array} \tag{1.0.9}$$

commutes [tDi68].

In practice, though, it is a tricky matter to carry this program out, for example because P_n^{MU} isn't additive (2.1.5), and so one doesn't expect it to behave well under tensor products like (1.0.2).

1.1 Splitting the total power operation via character theory

Our idea is to appeal at this point to the character theory of Hopkins–Kuhn–Ravenel [HKR91]. To focus the discussion, suppose that E^* is a Noetherian local domain, complete with respect to its maximal ideal $\mathfrak{m}$. Suppose that the residue characteristic of E^* is p, and that the formal group law of E^* has height h. Once again, let D^* be the ring (see section 2.3) obtained from E^* by adjoining the roots of the p^k-series for all k. For every subgroup

$$H \subset F(D^*)$$

of order n, we use [HKR91] in section 3.1 to construct a character map

$$E^*(D_nX) \xrightarrow{\chi_H} D^* \underset{E^*}{\otimes} E^*X. \tag{1.1.1}$$

The χ_H are the building blocks of a "total character map" which detects $E^*(D_nX)$ rationally (for more on the total character map, see section 3.3). It turns out that D^* is faithfully flat over E^* ([Hop91]; see section 2.3), so

$$D_*X = D_* \underset{E_*}{\otimes} E_*X$$

is a homology theory.

We shall analyze the operation

$$MU^{2*}X \xrightarrow{P_n^{MU}} MU^{2n*}(D_nX) \xrightarrow{t_E} E^{2n*}(D_nX)$$

in terms of the composites

$$Q^H : MU^{2*}X \xrightarrow{t_E \circ P_n^{MU}} E^{2n*}(D_nX) \xrightarrow{\chi_H} D^{n*}X$$

as H runs over the subgroups of $F(D^*)_{tors}$. The first indication that these operations are dramatically simpler to study than P_n^{MU} is

Theorem B *The operation*

$$MU^{2*}X \xrightarrow{Q^H} D^{2n*}X$$

is additive. *In fact, it is a homomorphism of graded rings*

$$MU^{2*}X \xrightarrow{Q^H} \Phi_n D^{2*}X,$$

where for a graded object M^*, $\Phi_n M^*$ is the graded object which is M^{nk} in degree k.

1.2 The operation Q^H and Lubin's quotient formal group law.

The real power of Theorem B as a tool for understanding power operations lies in the fact that the operations Q^H have an elegant description in terms of the formal group law. Lubin ([Lub67]; see 2.2) associates to the subgroup H of F a quotient formal group law F/H and a formal homomorphism

$$F \xrightarrow{f_H} F/H,$$

both defined over D^*. They are constructed so that

$$H = \mathrm{Ker}[F(D^*) \xrightarrow{f_H} F/H(D^*)].$$

Our description of Q^H is

Theorem C *On coefficients, Q^H is determined via Quillen's theorem by the equation*

$$Q^H_* F^{MU} = F/H.$$

Moreover, if $\begin{array}{c} L \\ \downarrow \\ X \end{array}$ *is a complex line bundle, then*

$$Q^H(e_{MU}L) = f_H(e_E L) \in D^{2n}X$$

where $e_{MU}L$ and $e_E L$ are respectively the MU and E Euler classes of L.

1.3 Factoring Q^H through the orientation t_E.

In relation to our original goal of producing operations in E-cohomology, the operations Q^H have two overriding unsatisfactory features: they have the wrong source, MU, and the wrong target, D. Making the target E instead of D can be arranged: as has already been mentioned, the ring D^* is a Galois extension of E^*; the Galois group is $\mathrm{Aut}[F(D^*)_{\mathrm{tors}}]$. If $H \subset F(D^*)_{\mathrm{tors}}$ is a finite subgroup and $g \in \mathrm{Aut}[F(D^*)_{\mathrm{tors}}]$, then it is not hard to show that

$$gQ^H = Q^{gH}.$$

If ρ is a polynomial over E^* on the set of subgroups $H \subset F(D^*_{tors})$ of order n, denote by Q^ρ the resulting operation

$$MU^{2*}X \xrightarrow{Q^\rho} E^{2*}X.$$

Theorem D *If a polynomial ρ is invariant under* $\mathrm{Aut}[F(D^*)_{tors}]$, *then the operation Q^ρ factors through the natural map*

$$E^*X \longrightarrow D^*X$$

to produce an operation

$$MU^{2*}X \xrightarrow{Q^\rho} E^{2*}X,$$

where the degrees indicated by the asterisk on either side may not coincide.

For example, if $H = {}_{p^k}F(D^*)$ is the full subgroup of points of order p^k, then the operation $Q^H = Q^{p^k}$ lands in E :

$$MU^{2*}X \xrightarrow{Q^{p^k}} E^{2p^{kh}*}X.$$

To make an operation with E as source, we can take advantage of the fact that Q^H, unlike the total operation $t_E \circ P^{MU}_\pi$, is MU-linear. When E is exact (1.0.2), it suffices by Theorem B to find a ring homomorphism

$$E^* \xrightarrow{\beta^H} D^{n*}$$

such that

$$\beta^H_* F = F/H. \tag{1.3.1}$$

Note that the existence of a homomorphism β^H satisfying (1.3.1) follows from the existence of a total power operation P^E_π for E which is compatible with P^{MU}_π. For suppose one can find P^E_n and t_E such that the diagram

$$\begin{array}{ccccc} MU^{2*}X & \xrightarrow{P^{MU}_n} & MU^{2n*}D_nX & & \\ {\scriptstyle t_E}\downarrow & & \downarrow{\scriptstyle t_E} & & \\ E^{2*}X & \xrightarrow{P^E_n} & E^{2n*}D_nX & \xrightarrow{\chi_H} & D^{2n*}X \end{array} \tag{1.3.2}$$

commutes. In case X is a point space, the left vertical arrow is the genus (1.0.1), which satisfies

$$t_{E*}F^{MU} = F. \tag{1.3.3}$$

The clockwise composite is Q^H, which on coefficients classifies F/H, according to Theorem B. If E^* is concentrated in even dimensions (as it is in for E_h and Ell), then the bottom row is a ring homomorphism

$$E^* \xrightarrow{\beta^H} D^{n*}$$

which by (1.3.2), (1.3.3), and Theorem B must satisfy (1.3.1).

1.4 A new orientation on E_h.

We can now explain Theorem A; for details, see section 5. One studies homomorphisms out of E^*_h in terms of the functor it represents: recall that φ is the reduction of the group law F_h modulo the maximal ideal of E_h. Then according to [LT66], the ring E^*_h represents the functor of complete local $\mathbb{Z}_p$-algebras

$$R \longmapsto \left\{ \begin{array}{c} \star\text{-isomorphism classes of} \\ \text{lifts of } \varphi \end{array} \right\}.$$

The quotient formal group laws F/H are lifts of φ, so there is a unique homomorphism

$$E^*_h \xrightarrow{\beta^H} D^{n*}$$

such that $\beta^H_* F$ is *$\star$-isomorphic* to F/H. According to (1.3.1), we need strict equality in order for the operation Q^H to factor through E_h. When $H = {}_pF(D^*)$ is the full subgroup of points of order p, the fact that the p-series is an *endomorphism* of the formal group law implies that $\beta^p \stackrel{\text{def}}{=} \beta^H$ is (up to a multiple of the periodicity element) the identity. The grading determines that this multiple is u^{p^h-1}. Since the p-series is the unique endomorphism of a universal formal group law over E_h^* with kernel ${}_pF(D^*)$, we conclude that

$$\beta_*^p F = F/p$$

if and only if

$$[p]_F(x) = u^{p^h-1} f_p(x),$$

which is the condition of Theorem A. This orientation is the only one that has the possibility of making the diagram (1.3.2) commute, if P_n^E exists.

By combining Theorems A and B, we obtain

Theorem E *Let t_{PO} denote the orientation given by Theorem A. Then there is a unique operation*

$$E_h^{2*}X \xrightarrow{\Psi^{p^k}} E_h^{2p^{hk}*}X$$

such that

$$\begin{array}{ccc} MU^{2*}X & \xrightarrow{P^{MU}_{p^{kh}}} & MU^{2p^{hk}}(D_{p^{kh}}X) \\ {\scriptstyle t_{PO}}\downarrow & & \downarrow{\scriptstyle \chi_{p^k} \circ t_{PO}} \\ E_h^{2*}X & \xrightarrow{\Psi^{p^k}} & E_h^{2p^{hk}*}X \end{array}$$

commutes. On coefficients, Ψ^{p^k} is given by

$$\Psi^{p^k}(m) = u^{-\frac{|m|}{2}(p^{kh}-1)}m.$$

If $\begin{array}{c} L \\ \downarrow \\ X \end{array}$ *is a complex line bundle, then*

$$\Psi^{p^k}(eL) = f_{p^k}(eL) = u^{1-p^{kh}}[p^k](eL).$$

Because of its relationship to the p-series, the operation Ψ^p should be thought of as an unstable Adams operation in E_h. Thus the orientation provided by Theorem A is the unique orientation in which the Adams operation is obtained as a power operation. In theories such as E_h there is a well-known construction of a *stable* Adams operation

$$E_h \xrightarrow{\psi^p} E_h[\frac{1}{p}]$$

which is however not integral: one has to invert p in the range. It turns out that ψ^p and our Ψ^p are related by

$$\psi^p(x) = (\frac{u^{p^{kh}-1}}{p})^{\frac{|x|}{2}}\Psi^p(x),$$

so we recover the well-known fact (5.4.10) that the unstable operation

$$E_h^{2r} X \xrightarrow{p^r \psi^p} E_h^{2r} X \tag{1.4.1}$$

is integral on $2r$-dimensional classes.

When the height h is 1, the only subgroups of the formal group law are the subgroups of the form

$$_{p^k}F.$$

This is the case that corresponds to K-theory: the character map

$$E_1^* D_p X \xrightarrow{\chi_p} E_1^* X$$

corresponds to evaluation on the class of a p-cycle, and our construction is exactly analogous to Atiyah's description of the unstable integral Adams operation in K-theory; see section 5.6.

When the height is greater than 1, there are other subgroups. In that case, the diagram (1.3.2) provides for each subgroup H a condition on t_{PO} for the orientation to be compatible with P^{MU} and any (conjectural) total power operation P^E. However, already the condition (1.0.5) was enough to determine the orientation. In fact, in section 5.5 we prove

Theorem F *The formal group law over E_h obtained in Theorem A satisfies (1.3.1) for every finite subgroup H.*

With Theorem F, we can use the operations Q^H to produce an operation

$$E^* X \xrightarrow{\Psi^H} D^{n*} X$$

for every subgroup H of $F(D^*)_{tors}$. We assemble these operations into a description of a "total power operation" (5.3.5) in E_h, at least in terms of the character map of Hopkins–Kuhn–Ravenel (5.3.5). Let Λ_∞ denote an abelian group isomorphic to $\mathbb{Z}_p^h$. For π an abelian p-group, the description of $E_h^* D_\pi X$ by Hopkins–Kuhn–Ravenel is a character map

$$E_h^* D_\pi X \xrightarrow{\chi} \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^*(X^{\pi/(\alpha)}).$$

The operations Ψ^H enable us to construct an operation

$$E_h^{2*} X \xrightarrow{\overline{P}_\pi^h} \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^{2n*}(X^{\pi/(\alpha)}),$$

where n is the order of π, such that

$$\begin{array}{ccc}
MU^{2*}X & \xrightarrow{P_\pi^{MU}} & MU^{2n*}(D_\pi X) \\
{\scriptstyle t_{PO}}\downarrow & & \downarrow{\scriptstyle t_{PO}} \\
E_h^{2*}X & & E_h^{2n*}(D_\pi X) \\
 & \searrow^{\overline{P}_\pi^h} & \downarrow{\scriptstyle \chi} \\
 & & \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^{2n*}(X^{\pi/(\alpha)})
\end{array}$$

commutes. This is as close as we have been able to come to constructing a total power operation in E_h; it is recorded as Theorem 5.3.5.

Another indication of interesting applications of the operations Ψ^H is provided by

1.5 Elliptic cohomology and Hecke operators.

In the last chapter, we begin a study of the application of our theory to elliptic cohomology. The study of subgroups of formal groups is a formal analogue of the study of subgroups of elliptic curves, which are the source of "Hecke operators" [Ser70]. For details, see section 6. Let $Z_0(2)$ denote the space of pairs

$$(\Xi, \eta)$$

consisting of a lattice $\Xi \subset \mathbb{C}$ and a point of order two $\eta \in {}_2\mathbb{C}/\Xi$. An element f of Ell^{2r} is a function

$$Z_0(2) \xrightarrow{f} \mathbb{C}$$

which satisfies (among other properties: see section 6)

$$f(t(\Xi, \eta)) = t^r f(\Xi, \eta).$$

Fix an odd prime p. A subgroup H of order p of F^{Ell} is a rule with assigns to each pair (Ξ, η) a subgroup of $\mathbb{C}/\Xi$ of order p, for which we also write H by abuse of notation. Equivalently, it assigns to each lattice Ξ a super-lattice $\Xi_H \supset \Xi$ such that

$$[\Xi_H : \Xi] = p.$$

The *p-th Hecke operator* T_p applied to f is given by

$$T_p f(\Xi, \eta) = \frac{1}{p} \sum_{\substack{H \subset F^{Ell} \\ \#H = p}} f(\Xi_H, \eta).$$

It turns out that if $f \in Ell^{-2r}$ then also $T_p f \in Ell^{-2r}$, and Andrew Baker has shown [Bak90] that T_p can be extended to a stable operation on elliptic cohomology

$$Ell^* X \xrightarrow{\overline{T_p}} Ell[\frac{1}{p}]^* X;$$

trying to understand his operations was one of the starting points of this investigation. Actually, Baker uses a slightly different version of elliptic cohomology, but his construction and ours work in either context.

It turns out that Ell is more "geometric" than the E_h in that the exponential of the Euler formal group law is an elliptic function, and so is determined (up to constant multiple) by its divisor. Because of this, the Euler formal group is related to its quotient formal group law F^{Ell}/H in a simple manner, and one can write down a homomorphism

$$Ell^* \xrightarrow{\beta^H} D^{p*}$$

such that

$$\beta^H_* F^{Ell} = F^{Ell}/H.$$

Therefore one obtains an operation

$$Ell^* X \xrightarrow{\Psi^H} D^{p*} X \tag{1.5.1}$$

for each subgroup H of order p, where D^* is the range of the character map for elliptic cohomology (see [Hop89] and section 6.1). By Theorem D, the sum

$$R_p = \sum_{\substack{H \subset F^{Ell} \\ \#H = p}} \Psi^H$$

is an operation

$$Ell^* X \xrightarrow{R_p} \frac{1}{p} Ell^{p*} X$$

which is we use to recover Baker's Hecke operation in Ell (6.5.2). Actually, it seems likely that one can do better than this, and obtain integrality results for the Hecke operators analogous to the integrality of the unstable Adams operations (1.4.1). The main issue is to find a better version of the ring D^* for elliptic cohomology.

As Mike Hopkins has explained ([Hop89],see 6.2), for groups of exponent p^k this ring is roughly ring of meromorphic modular forms for the congruence subgroup $\Gamma(2p^k)$. The trick is to show, for example, that this ring is *flat* over Ell^*. Brylinski [Bry90] has studied such a ring for elliptic cohomology which does not invert p, but which only contains modular forms of even weight. Our constructions rely heavily on certain modular forms of weight -1 (6.2.6), and at the present time we can only see our way to showing that D^* is flat over Ell^* if we admit $1/p$. However, we believe that it will be possible to show that a ring without $1/p$ is flat. Our constructions lead us to conjecture that

Conjecture G *The operation $p^{r+1} T_p$ is integral on $2r$-dimensional classes.*

1.6 Organization

The rest of this paper is organized as follows. In section 2 we collect facts about tom Dieck's total power operation in MU (2.1) and about Lubin's theory of subgroups of formal groups (2.2) which we have used in our work. The most important facts are the computation (Proposition 2.1.9) of

$$P_{\pi}^{MU}(eL)$$

where eL is the Euler class of a line bundle $\begin{array}{c} L \\ \downarrow \\ X \end{array}$, and Lubin's theorem (2.2.2) on the existence of the quotient formal group law. In section 2.3 we describe the ring D^* which is the smallest ring over which all of the subgroups of the formal group law occur. The essential facts about D^* were provided by Mike Hopkins [Hop91].

In section 3 we use the character theory of Hopkins–Kuhn–Ravenel to study the operations Q^H. In 3.1 we construct the operation Q^H and in 3.2 we give proofs of Theorems B and C. Beyond the character theory, the essential ingredient is the similarity between the expression for $P_H^{MU}(eL)$ (2.1.9) and Lubin's homomorphism $f_H(x)$ (2.2.1). The Q^H can be used to give a description of the image of

$$MU^*X \xrightarrow{P_{\pi}^{MU}} MU^{n*}D_{\pi}X \xrightarrow{t_E} E^{n*}D_{\pi}X$$

under the character map of Hopkins–Kuhn–Ravenel . We work this out in 3.3 as Theorem 3.3.3.

In section 3.4, we study the action of $\mathrm{Gal}(D^*/E^*)$ on Q^H to obtain Theorem D.

In section 4 we explain how to use the operations Q^H to produce operations out of exact cohomology theories. The key point is to find the homomorphism β^H satisfying (1.3.1). The result (Theorem 4.0.1) is almost trivial, but it does not appear to have been used before to produce ***unstable*** operations. The operation Q^H is naturally associated to a stable operation νQ^H (4.0.6). The comparison of these two operations (4.0.7) is the basis of integrality statements like (1.4.1) and Conjecture G.

In section 5 we apply these results to E_h. After describing briefly the "completed" theory E_h, we use the statement of Theorem F to obtain the operations Ψ^H. With these we can write down (5.3.5) a total operation out of E_h^*X which lands in the character-theoretic description of $E_h^*(D_{\pi}X)$. I believe that the construction of a total operation

$$E_h^*X \xrightarrow{P_{\pi}^E} E_h^*(D_{\pi}X)$$

is within reach of the methods in this paper. Theorem 5.3.5 represents the best statement we can make at this time.

In section 5.4 we study the particular operation Ψ^p, completing the proof of Theorem E. Finally, section 5.5 is devoted to the proof of Theorems A and F. As an illustration of Theorem A, we show in 5.6 that the multiplicative formal group law satisfies (1.0.5).

In section 6 we turn to elliptic cohomology. In section 6.1, we set up the character map and the operation Q^H for elliptic cohomology, following [Hop89] but with some adjustments for improvements in the character theory and to make room for conjectural integrality statements. The main result is the comparison of Lubin's quotient formal group law F^{Ell}/H with the Euler formal group law for the quotient elliptic curve (6.3.8), and the consequent existence of the homomorphisms β^H for elliptic cohomology (6.3.16). These combine to prove Theorem 6.3.15, which is the existence of the operation Ψ^H of equation (1.5.1). In section 6.4, we use apply this theorem to the full subgroup

$$_pF^{Ell}$$

to obtain an Adams operation in Ell (6.4.2). Finally, in 6.5, we turn to the operation R_p and Hecke operations. We provide sufficient information to prove Conjecture G as soon as a sufficiently good ring D^* becomes available.

2 Prerequisites

2.1 Power operations in MU

We shall frequently write the Borel construction

$$D_\pi X,$$

when π is an abelian group, *without* reference to an ambient symmetric group. In this case, we shall mean by that notation

$$D_\pi X \stackrel{\text{def}}{=} E\pi \underset{\pi}{\times} X^\pi,$$

where

$$X^\pi = \text{Maps}[\pi, X],$$

and the action of π on X^π is the left action coming from right multiplication on π.

tom Dieck and Quillen constructed a total power operation

$$MU^{2*}X \xrightarrow{P_n^{MU}} MU^{2n*}(D_n X)$$

for complex cobordism [tDi68, Qui71], where S_n is the symmetric group on n letters.

Briefly, the construction is as follows. If a map

$$M^k \xrightarrow{f} X^l$$

of manifolds is complex-oriented, then there is a Gysin homomorphism

$$MU^*M \xrightarrow{f_*} MU^{*+l-k}X.$$

In particular,

$$f_*1 \in MU^{l-k}X.$$

Thom's theorem [Tho54] shows that every class in MU^*X can be obtained in this way.

Now suppose that

$$M \xrightarrow{f} X$$

is a complex-oriented map of even dimension $2d$. Then

$$D_\pi M \xrightarrow{D_\pi f} D_\pi X$$

inherits a complex orientation, and by definition

$$P_\pi^{MU}(f_*1) = (D_\pi f)_*1. \tag{2.1.1}$$

From this construction, one can quickly check the following properties of P_n^{MU} and P_π^{MU}. When there is no possibility of confusion, we abbreviate P_π^{MU} as P_π.

Lemma 2.1.2 *[tDi68] P_π is a "total power operation" in the sense that if*

$$X^n \xrightarrow{i} D_\pi X$$

denotes the inclusion of the fiber, then

$$i^* P_\pi z = z^{\times n}.$$

Moreover, the operation P_π is natural with respect to pull-backs: for a map $X \xrightarrow{f} Y$, the diagram

$$\begin{array}{ccc} MU^{2*}Y & \xrightarrow{P_\pi^{MU}} & MU^{2*n}(D_\pi Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (D_\pi f)^*} \\ MU^{2*}X & \xrightarrow{P_\pi^{MU}} & MU^{2*n}(D_\pi X) \end{array} \tag{2.1.3}$$

commutes.

Lemma 2.1.4 (compare [tDi68]) *P_π is multiplicative: for $x \in MU^{2k}X$ and $y \in MU^{2l}X$,*

$$P_\pi(xy) = P_\pi(x)P_\pi(y) \in MU^{2n(k+l)}D_\pi X.$$

One of the features of power operations that makes subject difficult is that they are not additive. However, the failure of P_n to be additive can be expressed as a sum of terms which are transfers. A critical feature of our operations Q^H is that they are additive. The demonstration (3.2.1) depends on

Lemma 2.1.5

$$P_n(x+y) = \sum_{j=0}^{n} Tr_{j,n}^{MU} d^* \left(P_j x \times P_{n-j} y\right),$$

where

$$Tr_{j,n}^{MU} : MU^*(ES_n \underset{S_j \times S_{n-j}}{\times} X^n) \to MU^*(D_n X) \tag{2.1.6}$$

is the MU-transfer associated to the fibration

$$S_n/(S_j \times S_{n-j}) \to ES_n \underset{S_j \times S_{n-j}}{\times} X^n \to D_n X, \tag{2.1.7}$$

and d is the map

$$ES_n \underset{S_j \times S_{n-j}}{\times} X^n \xrightarrow{d} D_j X \times D_{n-j} X.$$

P_π^{MU} on Euler classes

Now let $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ be a complex vector bundle of rank r, and let

$$eV \in MU^{2r}X$$

be its MU Euler class. Then π acts on the product $\begin{matrix} V^\pi \\ \downarrow \\ X^\pi \end{matrix}$, and the resulting Borel construction $\begin{matrix} D_\pi V \\ \downarrow \\ D_\pi X \end{matrix}$ is a complex vector bundle over $D_\pi X$ with rank nr. We have

Proposition 2.1.8 ([tDi68])

$$P_\pi(eV) = e\begin{pmatrix} D_\pi V \\ \downarrow \\ D_\pi X \end{pmatrix}.$$

Proof: Let $\zeta : X \longrightarrow V$ be the zero-section. Then the Thom class of $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ is the image of 1 under the push-forward

$$MU^*X \xrightarrow{\zeta_*} MU^{*+2r}X^V.$$

By definition, the Euler class eV is the pull-back of the Thom class by ζ; in other words,

$$eV = \zeta^*\zeta_*1.$$

We have

$$\begin{aligned} P_\pi(eV) &= (D_\pi\zeta)^*P_\pi(\zeta_*1) \\ &= (D_\pi\zeta)^*(D_\pi\zeta)_*1 \\ &= e\begin{pmatrix} D_\pi V \\ \downarrow \\ D_\pi X \end{pmatrix}, \end{aligned}$$

where the first equality is the naturality of P_π with respect to pull-backs (2.1.3), the second is the definition of P_π (2.1.1), and the last is the definition of the Euler class. □

$P_\pi^{MU}(eL)$ **when** L **is a line bundle**

Let $\begin{array}{c} L \\ \downarrow \\ X \end{array}$ be a complex line bundle and $eL \in MU^2X$ its Euler class. Let Δ denote the "diagonal map"

$$B\pi \times X \xrightarrow{\Delta} D_\pi X.$$

One of the keystones of this paper is the formal similarity between the espression for $\Delta^* P_\pi(eL)$, which we now give, and Lubin's homomorphism $f_H(x)$ from a formal group law to its quotient by a finite subgroup (2.2.1).

When A is an abelian topological group, we denote by A^* its continuous complex dual

$$A^* = \mathrm{Hom}[A, \mathbb{C}^\times].$$

Proposition 2.1.9 (compare [Qui71],p. 42)

$$\Delta^* P_\pi(eL) = \prod_{u \in \pi^*} \left[e \begin{pmatrix} E\pi \underset{u}{\times} \mathbb{C} \\ \downarrow \\ B\pi \end{pmatrix} \underset{MU}{+} eL \right] \in MU^{2n}(B\pi \times X),$$

where $x \underset{MU}{+} y$ *denotes the formal sum with respect to the formal group law of MU, and we omit symbols for the pull-backs under the projections* $B\pi \times X \to X$ *and* $B\pi \times X \to B\pi$.

Proof: By (2.1.8),

$$\begin{aligned} \Delta^* P_\pi x &= e \begin{pmatrix} \Delta^*(E\pi \underset{\pi}{\times} L^\pi) \\ \downarrow \\ B\pi \times X \end{pmatrix} \\ &= e \begin{pmatrix} Reg_\pi \otimes L \\ \downarrow \\ B\pi \times X \end{pmatrix} \\ &= e \left[\bigoplus_{u \in A^*} \begin{pmatrix} E\pi \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BA \end{pmatrix} \otimes L \right] \\ &= \prod_{u \in A^*} \left[e \begin{pmatrix} E\pi \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BA \end{pmatrix} \underset{MU}{+} eL \right]. \square \end{aligned}$$

2.2 Subgroups of formal groups

To understand the notion of a subgroup of a formal group law, it is helpful to consider the situation in which a formal group law yields an actual group (see also section 6 for the example of subgroups of the group law of an elliptic curve): let R be a Noetherian ring and $\mathfrak{m}$ a maximal ideal of R; suppose moreover that R is complete in the $\mathfrak{m}$–adic topology. Let $F(x,y) \in R[\![x,y]\!]$ be a formal group law. If $a, b \in \mathfrak{m}$ are elements of this maximal ideal then the series $F(a,b)$ converges to an element of $\mathfrak{m}$. The elements of $\mathfrak{m}$ with the addition specified by F form a group, denoted $F(\mathfrak{m})$.

Now suppose $H \subset F(\mathfrak{m})$ is a finite subgroup. Lubin's main result is to show that the quotient $F(\mathfrak{m}) \longrightarrow F(\mathfrak{m})/H$ is realizeable as a homomorphism of formal groups: that is, there is a formal group law F/H over R, a homomorphism $F \xrightarrow{f_H} F/H$, and a commutative diagram

$$\begin{array}{ccccc} H & \longrightarrow & F(\mathfrak{m}) & \longrightarrow & F(\mathfrak{m})/H \\ \| & & \| & & \downarrow \cong \\ H & \longrightarrow & F(\mathfrak{m}) & \xrightarrow{f_H} & F/H(\mathfrak{m}). \end{array}$$

In other words, every subgroup H of $F(\mathfrak{m})$ is "formal" in the sense that it occurs as the the kernel some formal homomorphism $(F \xrightarrow{f_H} F/H)(\mathfrak{m})$.

The homomorphism f_H is constructed in terms of its kernel H and by the formal group law F : one takes $f_H(x)$ to be

$$f_H(x) = \prod_{h \in H} (h \underset{F}{+} x). \tag{2.2.1}$$

Since $0 \in R$ is the identity of $F(\mathfrak{m})$, equation (2.2.1) guarantees that

$$H = \{h \in F(\mathfrak{m}) \mid f_H(h) = 0\}.$$

Also, the power series $f_H(x)$ has $f_H(0) = 0$ and leading term

$$f'_H(0) = a_H = \prod_{0 \neq h \in H} h$$

given by the product of the non-zero elements of H.

Lubin's theorem is that $f_H(x)$ is in fact a homomorphism of formal group laws.

Theorem 2.2.2 (Lubin [Lub67]) *Let R be a Noetherian domain which is complete in the topology induced by an ideal I. Let F be a formal group law over R, and suppose that H is a finite subgroup of $F(I)$. Let $f_H(x)$ be defined by*

$$f_H(x) = \prod_{h \in H} (h \underset{F}{+} x).$$

Then there is a unique formal group law F/H defined over R such that

$$f_H F(x,y) = F/H(f_H(x), f_H(y)),$$

so $H = \mathrm{Ker}(F(I) \xrightarrow{f_H} F/H(I))$.

Lubin also shows that when F is not the additive group, F/H is universal, because homomorphisms of formal groups are formally surjective:

Theorem 2.2.3 ([Lub67]) *Let R be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$. Suppose that F is a formal group law over R whose reduction to the residue field has finite height. Suppose that there are homomorphisms of formal group laws*

$$F \xrightarrow{f_i} F_i,\ i = 1, 2,$$

such that

$$\mathrm{Ker}[f_1(\mathfrak{m})] \subset \mathrm{Ker}[f_2(\mathfrak{m})].$$

Then there is a unique homomorphism of formal group laws

$$F_1 \xrightarrow{g} F_2$$

such that

$$f_2 = g \circ f_1. \ \square$$

If $H \subset G \subset F(\mathfrak{m})$, then the three finite subgroups

$$H \subset F(\mathfrak{m}),$$
$$G \subset F(\mathfrak{m}), \text{ and}$$
$$G/H \subset F/H(\mathfrak{m})$$

produce three formal homomorphisms

$$F \xrightarrow{f_H} F/H,$$
$$F \xrightarrow{f_G} F/G, \text{ and}$$
$$F/H \xrightarrow{f_{G/H}} (F/H)/(G/H).$$

An important property of Lubin's isogeny is

Proposition 2.2.4

$$f_G = f_{G/H} \circ f_H,$$

and so

$$(F/H)/(G/H) = F/G.$$

Proof: Let $S \subset G$ be a set of coset representatives of G/H. Then

$$\begin{aligned} f_G(x) &= \prod_{g\in G}(g \underset{F}{+} x) \\ &= \prod_{g\in S}\prod_{h\in H}(h \underset{F}{+} g \underset{F}{+} x) \\ &= \prod_{g\in S} f_H(g \underset{F}{+} x) \\ &= \prod_{g\in S}(f_H(g) \underset{F/H}{+} f_H(x)) \\ &= \prod_{\sigma\in G/H}(\sigma \underset{F/H}{+} f_H(x)) \\ &= f_{G/H}(f_H(x)). \ \square \end{aligned}$$

Note the similarity between the expression (2.1.9) for the total power operation $P_\pi x$ with respect to an abelian group π on the Euler class of a line bundle and the expression (2.2.1) for the homomorphism $f_H(x)$ defined by a subgoup of a formal group law. This simple observation is the key to Theorem C; see section 3.2.

Formal group laws over graded rings

Our method directly produces unstable integral operations. The formal groups that occur in complex-oriented cohomology theories are graded, and keeping careful track of the gradings will enable us to prove integrality results about associated stable operations (4.0.7) including stable Adams operations (Theorem 5.4.10) and Hecke operations (Conjecture G)). In this section we cast the theory of subgroups in a graded setting.

Suppose then that R^* is a *graded* Noetherian ring which is complete with respect to a maximal ideal $\mathfrak{m}$. Let $F(x,y) \in R^*[\![x,y]\!]$ be a formal group law which is of degree $2n$ not as a power series but as an element of the graded ring $R^*[\![x,y]\!]$, where x and y are both taken to have degree $2n$. In this case we say F is "homogeneous of degree $2n$". Notice that the coefficients of F will be concentrated in R^{2n*}.

One makes this definition so that if a and b are elements of $\mathfrak{m}$ of degree $2n$, then $F(a,b)$ will also be an element of $\mathfrak{m}$ of degree $2n$. The elements of $\mathfrak{m}$ of degree $2n$ with addition defined by F is a subgroup of $F(\mathfrak{m})$ which will be denoted $F(\mathfrak{m},2n)$. If F is a homogeneous formal group law of degree $2n$ and $f(x) = x + o(x^2) \in R^*[\![x]\!]$ is a power series which is of degree $2n$ when the degree of x is $2n$, then the power series

$$G(x,y) = fF(f^{-1}(x), f^{-1}(y))$$

is also a homogeneous formal group law of degree $2n$, and $f(x)$ defines a homomorphism

$$F(\mathfrak{m},2n) \xrightarrow{f(x)} G(\mathfrak{m},2n).$$

More generally, we have

Lemma 2.2.5 *Let $F \xrightarrow{f(x)} G$ be a homomorphism of formal group laws over R^*. Suppose that F is homogeneous of degree $2n$, and that $f(x)$ satisfies*

i. $f(x)$ has degree $2k$ when x is taken to be of degree $2n$, and

ii. the element $a = f'(0)$ of $R^{2(k-n)}$ is not a zero-divisor.

Then G is also homogeneous and has degree $2k$. Morever, if R is complete with respect to the topology defined by an ideal $\mathfrak{m}$, then $f(x)$ restricts to a homomorphism

$$F(\mathfrak{m}, 2n) \xrightarrow{f(x)} G(\mathfrak{m}, 2k).$$

Proof: Since

$$f(x) = ax + o(x^2),\ |a| = 2(k-n),$$

over $\frac{1}{a}R^*$ one can study the power series $f^{-1}(x)$ which has degree $2n$ when x is taken to have degree $2k$. Over $\frac{1}{a}R^*$, G is given by

$$G(x,y) = fF(f^{-1}(x), f^{-1}(y))$$

which is homogeneous of degree $2k$. Since a is not a zero-divisor, the localization $R^* \longrightarrow \frac{1}{a}R^*$ is injective, and G has degree $2k$ as a formal group law over R^*. This sitation is summarized, and the proof of the second part of the lemma given, by the diagram

$$\begin{array}{ccc} \mathfrak{m}^{2n} \times \mathfrak{m}^{2n} & \xrightarrow{F} & \mathfrak{m}^{2n} \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{m}^{2k} \times \mathfrak{m}^{2k} & \xrightarrow{G} & \mathfrak{m}^{2k}. \ \square \end{array}$$

Corollary 2.2.6 *In the situation of Lemma 2.2.5, the coefficients of $G(x,y)$ are concentrated in R^{2k*}.*

Now we apply (2.2.5) to Lubin's quotient formal group law F/H. Let F be a homogeneous formal group law of degree 2 over a graded domain R^* which is complete with respect to the ideal $\mathfrak{m}$. Let H be a subgroup of $F(\mathfrak{m}, 2)$ of order n. Then by (2.2.1), $f_H(x)$ has degree $2n$ when x is taken to be of degree 2, and $a_H = f'_H(0)$ has degree $2(n-2)$.

Corollary 2.2.7 *F/H is homogeneous of degree $2n$. In particular, the coefficients of F/H are concentrated in R^{2n*}.*

The formal group laws of complex-oriented cohomology theories are homogeneous of degree 2. If H is a subgroup of order n, then the quotient formal group law F/H is related to a formal group law of degree 2 which we call $\nu F/H$, the *normalized* quotient. It is defined in terms of the *normalized* homomorphism $\nu f_H(x)$,

$$\nu f_H(x) = \frac{f_H(x)}{a_H} \in R[\frac{1}{a_H}]^*[\![x]\!]. \tag{2.2.8}$$

The normalized formal group law $\nu F/H$ is defined so that

$$F \xrightarrow{\nu f_H} \nu F/H; \tag{2.2.9}$$

it is defined over $R[\frac{1}{a_H}]$ since $\nu f_H(x) = x + o(x^2)$ is an invertible power series over $R[\frac{1}{a_H}]$. By Corollary 2.2.7, $\nu F/H$ is homogeneous of degree 2.

The integrality results (4.0.7) and (5.4.10) are based on the comparison of $\nu F/H$ and F/H. By construction, they are related by

$$F \xrightarrow{\nu f_H} \nu F/H \xrightarrow{a_H x} F/H.$$

Let $\mu^H : R[\frac{1}{a_H}]^* \longrightarrow R[\frac{1}{a_H}]^{n*}$ and $\delta^H : R^{2n*} \longrightarrow R[\frac{1}{a_H}]^{2*}$ be defined by

$$\mu^H(m) = a_H^{\frac{|m|}{2}} m \tag{2.2.10}$$

$$\delta^H(m) = a_H^{-\frac{|m|}{2n}} m. \tag{2.2.11}$$

Recall (2.2.7) that the coefficients of $F/H(x,y)$ are concentrated in R^{2n*}.

Proposition 2.2.12 *F/H and $\nu F/H$ are related by*

$$F/H = \mu^H_* \nu F/H$$
$$\nu F/H = \delta^H_* F/H$$

Proof: This is a consequence of the more general

Lemma 2.2.13 *Let R^* be a graded ring in which a is a unit such that*

$$|a| = 2(n-k).$$

Suppose that F is a formal group law which is homogeneous of degree $2k$. Then F and F^{ax} are related by

$$F^{ax} = \mu^a_* F$$
$$F = \delta^a_* F^{ax},$$

where $R^{2k*} \xrightarrow{\mu^a} R^{2n*}$ and $R^{2n*} \xrightarrow{\delta^a} R^{2k*}$ are the homomorphisms

$$\mu^a(m) = a^{\frac{|m|}{2k}} m$$
$$\delta^a(m) = a^{-\frac{|m|}{2n}} m$$

Proof: We prove one case, that $F = \delta^a_* F^{ax}$. Let F^{ax} be given by

$$F^{ax}(x,y) = x + y + \sum_{i,j\geq 1} \gamma_{i,j} x^i y^j \in R^*[\![x,y]\!].$$

Note that by Lemma 2.2.5, we have

$$\gamma_{i,j} \in R^{2n(1-i-j)}.$$

It follows that

$$\delta^a_* F^{ax}(x,y) = x + y + \sum_{i,j\geq 1} \gamma_{i,j} a^{i+j-1} x^i y^j.$$

On the other hand,

$$\begin{aligned} F(x,y) &= a^{-1} F^{ax}(ax, ay) \\ &= x + y + a^{-1} \sum_{i,j\geq 1} \gamma_{i,j} (ax)^i (ay)^j \\ &= x + y + \sum_{i,j\geq 1} \gamma_{i,j} a^{i+j-1} x^i y^j. \ \square \end{aligned}$$

2.3 D^* : a universal ring for formal subgroups

In this section we assume that E^* is a complete local Noetherian domain with maximal ideal $\mathfrak{m}$ and residue characterstic p, and we suppose that F is a homogeneous formal group law over E^* of degree 2, whose mod–$\mathfrak{m}$ reduction has height h. Thus this section covers the case of E_h which we study in detail in section 5. The analogous constructions for elliptic cohomology are the subject of section 6.1.

If F is a formal group law over E^* and A is a finite abelian group, a monomorphism

$$A \hookrightarrow F(\mathfrak{m})$$

is called a "level-A structure" on F. According to (2.2.2), a level-A structure on F over a complete Noetherian domain determines quotient formal group laws

$$F/B \in E^*[\![x,y]\!]$$

for every subgroup B of A.

Since an element of $F(E^*)$ of order p^k is exactly a root of the p^k-series, which has Weierstrass degree p^{kh}, the subgroups A which occur can be studied inside the algebraic closure of (the fraction field of) E^*. Over a complete local ring E^* with maximal ideal $\mathfrak{m}$ we write $F(E^*)$ for $F(\mathfrak{m})$.

Theorem 2.3.1 ([LT65]; see also [HKR91]) *Let $\mathcal{O}^*$ be the ring of integers in the algebraic closure of the fraction field of E^*. Then*

$$\begin{aligned} {}_{p^k}F(\mathcal{O}^*, 2) &\cong (\mathbb{Z}/p^k\mathbb{Z})^h \\ F(\mathcal{O}^*, 2)_{tors} &\cong (\mathbb{Q}_p/\mathbb{Z}_p)^h. \end{aligned}$$

Let Λ_∞ be an abelian group isomorphic to $\mathbb{Z}_p^h$, so

$$\Lambda_\infty^* \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h,$$

although no explicit isomorphism has been chosen. Let $\Lambda_k^* = {}_{p^k}\Lambda_\infty^*$ be the subgroup of elements of order p^k; Λ_k^* is precisely the dual of

$$\Lambda_k = \Lambda_\infty / p^k \Lambda_\infty.$$

We can rephrase Theorem 2.3.1 as saying that there exist compatible isomorphisms

$$\Lambda_k^* \xrightarrow{\cong} {}_{p^k}F(\mathcal{O}^*, 2). \tag{2.3.2}$$

We describe next an observation of Hopkins–Kuhn–Ravenel to the effect that in the case of a ring $E^* = E^*(pt)$ which comes from a cohomology theory with a complex orientation, the orientation and the group Λ_k conspire to provide an extension D_k^* of E^* which comes equipped with a *canonical* level-Λ_k^* structure

$$\Lambda_k^* \xrightarrow[\cong]{\phi_{univ}} {}_{p^k}F(D_k^*, 2).$$

Moreoever, the ring D_k is as small as it can be.

A complex orientation on E determines a map

$$A^* \ni u \stackrel{\phi}{\longmapsto} e \begin{pmatrix} EA \times_u \mathbb{C} \\ \downarrow \\ BA \end{pmatrix} \in E^2 BA \tag{2.3.3}$$

which is an element of

$$\mathrm{Hom}_{gps}[A^*, F(E^*BA, 2)].$$

The role of E^*BA in describing subgroups of the formal group law is moderated by the following observation.

Lemma 2.3.4 ([HKR91]) *The natural transformation*

$$\mathrm{Hom}_{E^*-alg}[E^*BA,\ R^*] \to \mathrm{Hom}_{gps}[A^*, F(R^*, 2)]$$

given by

$$f \longmapsto f \circ \phi$$

is an equivalence of functors of pairs (A, R), where A is a finite abelian group, and R^* is a complete, local, graded E^*-algebra.

Lemma 2.3.4 says that E^*BA reresents the functor

$$R^* \longmapsto \mathrm{Hom}_{\mathrm{gps}}[A^*, F(R^*, 2)].$$

However, E^*BA is not a domain, so Lubin's theorem doesn't apply. Moreover, not every map

$$E^*BA \to R^*$$

represents a monomorphism. Note, however, that when R^* is a domain, then a homomorphism

$$A^* \xrightarrow{\phi} F(R^*)$$

is a monomorphism precisely when $\phi(u)$ is not a zero-divisor, for $u \neq 0$.

Now take the case of Λ_k, and let $S \subset E^*B\Lambda_k$ be the multiplicative set generated by the set

$$\{\phi(u) \in E^2B\Lambda_k \mid 0 \neq u \in \Lambda_k^*\}$$

of Euler classes of non-trivial line bundles over $B\Lambda_k$. The ring D_k^* is defined by

$$D_k^* = \mathrm{Im}[E^*B\Lambda_k \to S^{-1}E^*B\Lambda_k].$$

D_k^* comes with a homomorphism

$$\phi_{univ} : \Lambda_k^* \xrightarrow{\phi_{univ}} F(E^*B\Lambda_k, 2) \to F(D_k^*, 2) \qquad (2.3.5)$$

which is represented by the localization map

$$E^*B\Lambda_k \xrightarrow{\hat{\phi}} D_k^*.$$

The basic results about the ring D_k^* are

Theorem 2.3.6 ([Dri73, KM85, Hop91]) *The ring D_k^* is a domain and is faithfully flat over E^*. It represents the functor (of complete local E^*-algebras)*

$$D_k(R^*) = \{\textit{level-}\Lambda_k^* \textit{ structures on } F(R^*, 2)\}. \ \square$$

Proposition 2.3.7 *The homomorphism*

$$\Lambda_k^* \xrightarrow{\phi_{univ}} {}_{p^k}F(D_k^*)$$

is an isomorphism.

Proof: ϕ_{univ} is the universal monomorphism which gives Proposition 2.3.6. According to Theorem 2.3.1, the group ${}_{p^k}F(D_k^*)$ can be no larger than Λ_k^*. $\square$

We proceed to construct a ring $D^* = D^*_\infty$ with a full level Λ^*_∞-structure as a limit of the D^*_k. The inclusions

$$\Lambda^*_{k-1} \longrightarrow \Lambda^*_k \tag{2.3.8}$$

yield forgetful tranformations

$$D_k \longrightarrow D_{k-1}.$$

These are represented (2.3.6) by maps

$$D^*_{k-1} \longrightarrow D^*_k.$$

The ring D^* is the colimit

$$D^* = \mathrm{colim}(\ldots \longrightarrow D^*_{k-1} \longrightarrow D^*_k \longrightarrow \ldots). \tag{2.3.9}$$

It comes with an isomorphism

$$\Lambda^*_\infty \xrightarrow{\phi_{univ}} F(D^*, 2)_{tors}.$$

Since the $D^*_{p^k}$ are free over E^*, the ring D^* is flat over E^*, and so

Proposition 2.3.10 *The functor $D^*(-)$*

$$X \longmapsto D^* \otimes_{E^*} E^*X$$

is a cohomology theory for finite spaces.

Proposition 2.3.11 *If H is a finite subgroup of $F(D^*, 2)_{tors}$ whose exponent divides p^k, then Lubin's isogeny f_H and the quotient formal group law F/H are defined over D^*_k :*

$$f_H(x) \in D^*_k[\![x]\!] \subset D^*[\![x]\!], \text{ and}$$
$$F/H(x, y) \in D^*_k[\![x, y]\!] \subset D^*[\![x, y]\!].$$

Proof: This follows from Lubin's theorem (2.2.2) and the fact (2.3.6) that D^*_k is a Noetherian domain. □

3 Character theory

The main results of this section are Theorems B and C of the introduction. They are proved in section 3.2, after we define the operation Q^H using character theory in section 3.1.

3.1 The cohomology operation Q^H defined by a subgroup H

Generalized characters

The rings D^* and D_k^* are the natural range of the character map of Hopkins–Kuhn–Ravenel . Suppose that G is a finite group and that k is so large that p^k kills the p-torsion in G. Let

$$\Lambda_\infty \longrightarrow \Lambda_k \xrightarrow{\alpha} G$$

be a homomorphism: if one chooses an isomorphism $\mathbb{Z}_p^h \xrightarrow{\cong} \Lambda_\infty$ then α determines an h–tuple of commuting elements of G of p–power order. The character map corresponding to α is the ring homomorphism

$$E^*BG \xrightarrow{\chi_\alpha} D^*$$

given by the composite

$$E^*BG \xrightarrow{\alpha^*} E^*B\Lambda_k \xrightarrow{\widehat{\phi}} D_k^* \longrightarrow D^*.$$

Remark: Hopkins–Kuhn–Ravenel use the ring $L^* = \frac{1}{p}D^*$ as the range of their character map, because only after inverting p does the "total character map"

$$L^* \otimes_{E^*} E^*BG \xrightarrow{\chi} \prod_{\substack{\text{conj. classes} \\ \Lambda_\infty \xrightarrow{\alpha} G}} L^*$$

become an *isomorphism.* However, for our purposes it will be essential not to invert p, because we must retain the maximal ideal of D^* in order to realize the quotient formal group laws F/H as *deformations* (See (5.2.1)).

The character map defined by a formal subgroup H

Now let $H \subset F(D^*, 2)_{tors}$ be a finite subgroup of order n and exponent p^k. Specifying H is equivalent to specifying the subgroup

$$\phi_{univ}^{-1} H \subset \Lambda_k^* \subset \Lambda_\infty^*.$$

Let $\mathrm{Ann}[H] \subset \Lambda_k \subset \Lambda_\infty$ be the kernel of the evaluation map

$$\Lambda_\infty \xrightarrow{eval} \mathrm{Hom}_{gps}[\phi^{-1}H, \mathbb{C}^\times],$$

and let G be the cokernel

$$\mathrm{Ann}[H] \hookrightarrow \Lambda_\infty \xrightarrow{\alpha_H} G.$$

Lemma 3.1.1 *There is a natural isomorphism*

$$G^* \cong H. \square$$

Let χ_H be the composite

$$\begin{array}{ccccc} E^*D_GX & \xrightarrow{\Delta^*} & E^*(BG \times X) & \xrightarrow{\cong} & E^*BG \otimes_{E^*} E^*X \\ & & & & \downarrow \chi_{\alpha_H} \otimes 1 \\ & & & & D^* \otimes_{E^*} E^*X. \end{array} \tag{3.1.2}$$

The operation Q^H is the natural transformation

$$MU^{2*}X \xrightarrow{Q^H} D^{2*n}X$$

given by

$$MU^{2*}X \xrightarrow{P_G} MU^{2n*}D_GX \xrightarrow{t_E} E^{2n*}D_GX \xrightarrow{\chi_H} D^{2n*}X. \tag{3.1.3}$$

We are ready to prove

3.2 Theorems B and C.

Theorem 3.2.1 ("Theorem B") *Q^H is additive. Moreover, it is a graded ring homomorphism*

$$MU^{2*}X \xrightarrow{Q^H} \Phi_n D^{2*}X, \tag{3.2.2}$$

where $\Phi_n R^*$ is the graded ring which is R^{nk} in degree k. In particular, Q^H is a ring homomorphism

$$MU^* \xrightarrow{Q^H} \Phi_n D^*. \tag{3.2.3}$$

Theorem 3.2.4 (**"Theorem C"**) *Let* $\begin{array}{c} L \\ \downarrow \\ X \end{array}$ *be a complex line bundle, and let* $e_{MU}L$ *and* $e_E L$ *denote its Euler classes in* MU *and* E *cohomology. Then* Q^H *on Euler classes is given by*

$$Q^H(e_{MU}L) = f_H(e_E L) \in D^{2n}X. \tag{3.2.5}$$

Its effect on coefficients is determined by the equation

$$Q^H_* F^{MU} = F/H. \tag{3.2.6}$$

The proof of the additivity of Q^H in Theorem 3.2.1 imitates Atiyah's proof [Ati66] of the additivity of the Adams operations in K-theory. The basic ingredient is the formula of Hopkins–Kuhn–Ravenel for induced characters:

Theorem 3.2.7 (**[HKR91]**) *Let* G *be a finite group and let* $S \subset G$ *be a subgroup. For* $u \in E^*BS$ *and* $\Lambda_\infty \xrightarrow{f} G$ *one has the formula*

$$\chi_f(Tr^E u) = \sum_{gS \in (G/S)^{Imf}} \chi_{g^{-1}fg}(u),$$

where $Tr^E : E^*BS \longrightarrow E^*BG$ *is the transfer map in* E*-cohomology associated to the fibration* $G/S \longrightarrow BS \longrightarrow BG$.

Proof of Theorem 3.2.1: In (3.1.3), all the maps are ring homomorphisms except P_G. By Lemma 2.1.4, P_G is multiplicative. Suppose we order the elements of G, which determines an isomorphism

$$\mathrm{Aut}_{sets}\, G \xrightarrow{\cong} S_n$$

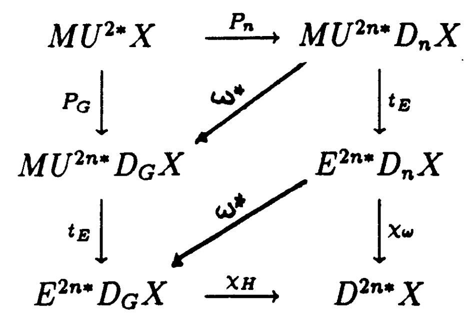
and so a monomorphism

$$G \xrightarrow{\omega} S_n.$$

Then one obtains a character map

$$E^* D_n X \xrightarrow{\chi_\omega} D^* X$$

such that the diagram



commutes. By (2.1.5), P_n is given on sums by

$$P_n(x+y) = \sum_{j=0}^{n} Tr_{j,n}^{MU}\left(P_j x \times P_{n-j} y\right),$$

so

$$Q^H(x+y) = \sum_{j=0}^{n} \chi_\omega Tr_{j,n}^{E} t_E P_j x P_{n-j} y,$$

since t_E commutes with the transfer. By the formula for induced characters (3.2.7),

$$\chi_\omega Tr_{j,n}^{E} u = \sum_{gS_j \times S_{n-j} \in (S_n/S_j \times S_{n-j})^{Im\omega}} \chi_{g^{-1}\omega g} u,$$

Since the image of G in S_n is transitive, the sum is empty unless $j = 0$ or $j = n$. So

$$\begin{aligned} Q^H(x+y) &= \chi_H t_E P_G x + \chi_H t_E P_G y \\ &= Q^H(x) + Q^H(y). \end{aligned}$$

Thus Q^H is additive. □

Proof of Theorem 3.2.4: First we prove (3.2.5); then (3.2.6) will follow from Lemma 3.2.8. In the diagram

$$\begin{array}{ccccc} MU^2X & \xrightarrow{P_G} & MU^{2n}D_GX & \xrightarrow{t_E} & E^{2n}D_GX \\ & & \downarrow{\scriptstyle \Delta^*} & & \downarrow{\scriptstyle \Delta^*} \\ & & MU^{2n}(BG \times X) & \xrightarrow{t_E} & E^{2n}(BG \times X) \\ & & & & \downarrow{\scriptstyle \cong} \\ & & & & (E^*BG \otimes_{E^*} E^*X)^{2n} \\ & & & & \downarrow{\scriptstyle \chi_{\alpha_H}} \\ & & & & D^{2n}X, \end{array}$$

the right outer arrows give Q^H. Going around the left outer arrows, we get

$$\begin{aligned} Q^H(e_{MU}L) &= \hat{\phi} t_E \Delta^* P_G e_{MU} L \\ &= \hat{\phi} t_E \prod_{u \in G^* = H} \left[e_{MU} \begin{pmatrix} EG \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BG \end{pmatrix} \underset{MU}{+} e_{MU} L \right] \\ &= \hat{\phi} \prod_{h \in H} \left[e_E \begin{pmatrix} EG \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BG \end{pmatrix} \underset{E}{+} e_E L \right] \\ &= \prod_{h \in H} \left[h \underset{E}{+} e_E L \right] \\ &= f_H(e_E L). \end{aligned}$$

where the second equality is Lemma 2.1.9.

Now recall (2.3.6) that D^* is a domain. The effect of Q^H on coefficients (3.2.6) follows from (3.2.5), Theorem 3.2.1, and

Lemma 3.2.8 *Let C be a complex-oriented cohomology theory. Suppose that*

$$MU^{2*}(-) \xrightarrow{Q} \Phi_n C^{2*}(-)$$

is a natural transformation of ring-valued functors. Suppose that the effect of Q on Euler classes is given by

$$Q(e_{MU}L) = f(e_C L), \tag{3.2.9}$$

where $f(x)$ is a homomorphism of formal group laws

$$F^C \xrightarrow{f} F'$$

such that $f'(0)$ is not a zero-divisor in $\Phi_n C^*$. Then Q is determined on MU^* as the homomorphism classifying F' :

$$Q_* F^{MU} = F'.$$

Proof: Let L_l and L_r be the two tautological bundles over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. By (3.2.9),

$$\begin{aligned} Q(e_{MU}L_l \underset{MU}{+} e_{MU}L_r) &= Q(e_{MU}\begin{bmatrix} L_l \otimes L_r \\ \downarrow \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty \end{bmatrix}) \\ &= \theta(e_C \begin{bmatrix} L_l \otimes L_r \\ \downarrow \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty \end{bmatrix}) \\ &= \theta(e_C L_l \underset{F^C}{+} e_C L_r). \end{aligned} \tag{3.2.10}$$

On the other hand, we know that Q is a ring homomorphism

$$MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \xrightarrow{Q} C^{n*}(\mathbb{CP}^\infty \times \mathbb{CP}^\infty),$$

so

$$\begin{aligned} Q(e_{MU}L_l \underset{MU}{+} e_{MU}L_r) &= Q(e_{MU}L_l) \underset{Q_*F^{MU}}{+} Q(e_{MU}L_r) \\ &= \theta(e_C L_l) \underset{Q_*F^{MU}}{+} \theta(e_C L_r). \end{aligned} \tag{3.2.11}$$

Comparing (3.2.10) and (3.2.11), we see that

$$\theta F^C(x,y) = Q_* F^{MU}(\theta(x), \theta(y)),$$

so $Q_* F^{MU} = F'$. □

3.3 The character-theoretic description of the total power operation

Hopkins–Kuhn–Ravenel give a description of the cohomology of the full Borel construction $D_\pi X$ in terms of a total character map

$$E^*(E\pi \underset{\pi}{\times} X^\pi) \xrightarrow{\chi} \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^*(X^{\pi/(\alpha)})$$

which we now describe. For a map $\Lambda_\infty \xrightarrow{\alpha} \pi$, let $G = (\mathrm{Im}\,\alpha)$ be the subgroup generated by the image of α. The isomorphism of G-sets

$$\pi \underset{G}{\cong} G \times (\pi/G)$$

gives an isomorphism

$$\mathrm{Fixed}_G[\mathrm{Maps}[\pi, X]] \cong \mathrm{Maps}[\pi/G, X]$$

between the fixed point set of X^π by the action of G and $X^{\pi/G}$. The component of χ corresponding to a map α is

$$E^*(E\pi \underset{\pi}{\times} X^\pi) \to E^*(BG \times X^{\pi/G}) \xrightarrow{\chi_\alpha} D^*(X^{\pi/G}). \tag{3.3.1}$$

Let T_π^E be the natural transformation

$$MU^*X \xrightarrow{P_\pi} MU^{n*}D_\pi X \xrightarrow{t_E} E^{n*}D_\pi X \xrightarrow{\chi} \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^{n*}X^{\pi/(\alpha)} \tag{3.3.2}$$

which is the image of the total power operation of MU under the character map of Hopkins–Kuhn–Ravenel .

Theorem 3.3.3 *For a map $\Lambda_\infty \xrightarrow{\alpha} \pi$, let $G = (\mathrm{Im}\,\alpha)$ be the subgroup generated by the image of α and let $H = (\mathrm{Im}\,\alpha^*)$ be the subgroup of Λ_∞^* generated by the image of the map*

$$\pi^* \xrightarrow{\alpha^*} \Lambda_\infty^*.$$

Suppose that $|\pi| = n$, $|G| = r$, and $[\pi : G] = n/r = k$. Then the component of T_π^E corresponding to α is given by

$$pr_\alpha T_\pi^E(z) = Q^H(z^{\times k}) \in D^{2n*}(X^k).$$

Proof: Comparison of (3.1.2) and (3.3.1) shows that if α is surjective, then

$$pr_\alpha T_\pi^E = Q^H.$$

For general α, the result follows from the case of surjective α, the definition of Q^H (3.1.3), and

Lemma 3.3.4 *In the situation of the theorem, the diagram*

$$\begin{array}{ccc} MU^{2*}X & \xrightarrow{P_\pi} & MU^{2n*}D_\pi X \\ {\scriptstyle x\mapsto x^{\times k}}\downarrow & & \downarrow \\ MU^{2k*}X^{\pi/G} & \xrightarrow{P_G} & MU^{2n*}D_G(X^{\pi/G}) \end{array}$$

commutes.

Proof: Represent a class $z \in MU^{2*}X$ as f_*1 where f is a complex-oriented map

$$Z \xrightarrow{f} X.$$

Then $P_\pi z$ is represented by the map

$$D_\pi Z \xrightarrow{D_\pi f} D_\pi X.$$

The isomorphism of G-sets

$$\pi \underset{G}{\cong} G \times \pi/G$$

provides the vertical isomorphisms in the commutative diagram

$$\begin{array}{ccc} EG \underset{G}{\times} Z^\pi & \xrightarrow{EG \underset{G}{\times} f^\pi} & EG \underset{G}{\times} X^\pi \\ \cong\downarrow & & \downarrow\cong \\ D_G(Z^{\pi/G}) & \xrightarrow{D_G(f^{\pi/G})} & D_G(X^{\pi/G}). \end{array}$$

The bottom row of the diagram represents $P_G(z^{\times k})$ by definition. The top row represents the composite

$$MU^*X \xrightarrow{P_\pi} MU^{n*}D_\pi X \to MU^{n*}D_G X$$

because it fits into the pull-back diagram

$$\begin{array}{ccc} EG \underset{G}{\times} Z^\pi & \longrightarrow & D_\pi Z \\ {\scriptstyle EG \underset{G}{\times} f^\pi}\downarrow & & \downarrow{\scriptstyle D_\pi f} \\ EG \underset{G}{\times} X^\pi & \longrightarrow & D_\pi X. \ \square \end{array}$$

Remarks: 1) For example, the component of T_π^E corresponding to the trivial map $\Lambda_\infty \xrightarrow{0} \pi$ is the n^{th} external power, which we knew already since P_π is a total power operation and evaluation at 0 corresponds to the pull-back by the inclusion of the fiber

$$X^{\pi} \longrightarrow D_{\pi}X.$$

2) Also, a computation similar to (2.1.9) shows that for

$$\Lambda_{\infty} \xrightarrow{\alpha} H \xhookrightarrow{i} G,$$

we have

$$\chi_{\alpha}(Bi)^{*}P_{G}eL = (f_{H}(eL))^{|G/H|},$$

which is a special case of Theorem 3.3.3 in view of the multiplicativity of Q^H.

3.4 Galois theory and the proof of Theorem D

The group $\mathrm{Aut}[\Lambda_k^*]$ acts on D_k^*, because it acts on the functor D_k. The fact we shall need is

Proposition 3.4.1 ([Hop91]) *The ring of* $\mathrm{Aut}[\Lambda_k^*]$*-invariants in* D_k^* *is exactly* E^*.

The group $\mathrm{Aut}[\Lambda_k^*]$ also acts on the set

$$\mathcal{SG}_k = \left\{ \begin{array}{c} \text{subgroups of} \\ \Lambda_k^* \end{array} \right\}.$$

We can use Proposition 3.4.1 and the action on $\mathcal{SG}_k$ to produce operations whose target is E rather than D. In what follows, we shall not distinguish notationally between a subgroup

$$H \subset \Lambda_k^*$$

and its image

$$\phi(H) \subset {}_{p^k}F(D_k^*, 2)$$

under the isomorphism (2.3.7).

Proposition 3.4.2 *For* $H \in \mathcal{SG}_k$ *and* $a \in \mathrm{Aut}[\Lambda_k^*]$, *we have*

$$Q^{aH} = aQ^{H},$$

where on the right the action of a refers to the action of $\mathrm{Aut}[\Lambda_k^*]$ *on the ring* D_k^* *induced by the action on the functor it represents (2.3.6, 3.4.1), together with the definition (2.3.10) of* D_k^*X *as*

$$D_k^*X = D_k^* \otimes_{E^*} E^*X.$$

Proof: It suffices to check this when X is a point space, since χ_H factors through

$$E^*(BG \times X) \cong E^*BG \otimes_{E^*} E^*X$$

and the action takes place entirely on the left of the tensor product.

$\mathrm{Aut}[\Lambda_k^*]$ also acts on Λ_k, by adjointness. The commutative square

$$\begin{array}{ccc} B\Lambda_k & \xrightarrow{Ba} & B\Lambda_k \\ {\scriptstyle B\alpha_H}\downarrow & & \downarrow{\scriptstyle B\alpha_{aH}} \\ BG & \underset{\cong}{\xrightarrow{Ba}} & BaG \end{array}$$

shows that

$$\begin{array}{ccc} E^*BaG & \xrightarrow{a^*} & E^*BG \\ {\scriptstyle \chi_{aH}}\downarrow & & \downarrow{\scriptstyle \chi_H} \\ D_k^* & \xrightarrow{a} & D_k^* \end{array}$$

commutes. The proof is complete upon observing that the diagram

$$\begin{array}{ccc} MU^{2*}X & = & MU^{2*}X \\ {\scriptstyle P_{aG}}\downarrow & & \downarrow{\scriptstyle P_G} \\ MU^{2n*}D_{aG}X & \xrightarrow{a^*} & MU^{2n*}D_GX \end{array}$$

commutes. □

Let $\mathcal{SG}_k(n)$ denote the subset of $\mathcal{SG}_k$ consisting of subgroups of order n, and let $\mathbb{Z}[\mathcal{SG}_k(n)]$ be the polynomial ring on the set $\mathcal{SG}_k(n)$. The action of $\mathrm{Aut}[\Lambda_k^*]$ on $\mathcal{SG}_k$ extends to an action of $\mathrm{Aut}[\Lambda_k^*]$ on $\mathbb{Z}[\mathcal{SG}_k(n)]$. We define the graded set Op^* by

$$Op^n = \mathbb{Z}[\mathcal{SG}_k(n)]^{\mathrm{Aut}[\Lambda_k^*]}.$$

Note that Op^n doesn't depend on k as long as $n|p^k$. An element ρ of $\mathbb{Z}[\mathcal{SG}_k(n)]$ can be represented by

$$\rho = \sum_{i\in I} a_i \prod_{H\in\alpha_i} H$$

where I is a finite set, $a_i \in \mathbb{Z}$, and the α_i are lists of elements of $\mathcal{SG}_k(n)$, with possible repetitions. For $\rho \in Op^n$, let Q^ρ be the operation

$$MU^{2*}X \xrightarrow{Q^\rho} D_k^{2*}X$$

given by

$$Q^\rho = \sum_{i\in I} a_i \prod_{H\in\alpha_i} Q^H. \tag{3.4.3}$$

In this situation, Proposition 3.4.2 imples

Theorem 3.4.4 (Theorem D) *For an element $\rho \in Op^n$, that is, a polynomial in $\mathbb{Z}[S\mathcal{G}_k(n)]$ which is invariant under* $\mathrm{Aut}[\Lambda_k^*]$, *the operation Q^ρ factors through E; that is, Q_ρ defines an operation*

$$MU^{2*}X \xrightarrow{Q_\rho} E^{2*}X.$$

Corollary 3.4.5 ("Adams operations") *The full subgroup ${}_{p^k}F(D^*,2)$ of points of order p^k defines an operation*

$$MU^{2*}X \xrightarrow{Q^{p^k}} E^{2p^{kh}*}X$$

which has all the properties described in Theorems 3.2.1 *and* 3.2.4.

Corollary 3.4.6 ("Hecke operations") *The operation*

$$R_n = \sum_{\substack{H \subset \Lambda_k^* \\ |H|=n}} Q^H$$

is an additive operation

$$MU^{2*}X \xrightarrow{R_n} E^{2n*}X$$

4 Cohomology operations out of exact cohomology theories

In this section we describe a recipe (Theorem 4.0.1) which we shall use repeatedly to produce cohomology operations in exact complex–oriented cohomology theories.

Suppose that E is an exact theory; recall (1.0.2) that this means that

$$E^*X = E^* \underset{MU^*}{\otimes} MU^*X,$$

where the tensor product is with respect to a genus

$$MU^* \longrightarrow E^*.$$

The resulting natural transformation

$$MU^*X \xrightarrow{t_E} E^*X$$

will be denoted t_E.

Let C be a cohomology theory with a ring structure, and let Q be a natural transformation of ring-valued functors

$$MU^*X \xrightarrow{Q} \Phi_n C^*X.$$

Let F^Q be the formal group law over $\Phi_n C^*$ classified by $Q(pt)$.

Such C and Q are given, for example, by D and Q^H as in Theorem 3.2.1. Other important examples are produced by Lemma 4.0.2.

Our main result is an immediate consequence Quillen's theorem and the properties of the tensor product.

Theorem 4.0.1 *The operation Q factors through t_E to an operation*

$$E^*X \xrightarrow{\Psi} \Phi_n C^*X$$

if and only if there is a ring homomorphism

$$\beta : E^* \longrightarrow \Phi_n C^*$$

such that

$$\beta_* F^E = F^Q.$$

The same result obtains if one restricts to even-dimensional classes, provided that the ring E^* is concentrated in even degrees. □

The operations

$$MU^{2*}X \xrightarrow{Q^H} \Phi_n D^{2*}X$$

of section 3.1 can be used as input to Theorem 4.0.1, and section 5 we shall produce the homomorphisms β^H required by the theorem. Another method of producing operations from Theorem 4.0.1 is provided by strict isomorphisms of the formal group law. In this case, it produces a stable operation.

Lemma 4.0.2 *Let C be a complex-oriented cohomology theory, and let*

$$e_C L \in C^2\mathbb{CP}^\infty$$

be the Euler class of the tautological bundle. For each power series

$$\theta(e_C L) = e_C L + o(e_C L)^2 \in C^2\mathbb{CP}^\infty$$

there is a unique transformation of ring theories

$$MU^*X \xrightarrow{t_\theta} C^*X$$

such that

$$t_\theta(e_{MU}L) = \theta(e_C L)$$

and

$$t_{\theta*}F^{MU} = (F^C)^\theta.$$

Proof: See, for example, Adams [Ada74]. □

Corollary 4.0.3 (e.g. Miller [Mil89]) *Let E and C be complex-oriented cohomology theories, and suppose that E is exact. Suppose that $\beta : E^* \longrightarrow C^*$ is a ring homomorphism, and that there is a strict isomorphism of formal group laws*

$$F^Q \xrightarrow{\theta} \beta_* F^E. \tag{4.0.4}$$

Then β extends to a stable natural transformation of functors of finite complexes (to rings)

$$E^*X \xrightarrow{\lambda} C^*X$$

whose effect on Euler classes is given by

$$\lambda(e_E L) = \theta(e_C L). \tag{4.0.5}$$

Proof: Use Lemma 4.0.2 to produce a natural transformation of ring theories

$$MU^*X \xrightarrow{t_\theta} C^*X$$

whose effect on Euler classes is

$$t_\theta(e_{MU}L) = \theta(e_C L).$$

By Lemma 3.2.8, the effect of t_θ on coefficients is given by

$$F^C \xrightarrow[\cong]{\theta} t_{\theta *}F^{MU}.$$

Now apply Theorem 4.0.1 to t_θ and β. □

Our main application of (4.0.2) is to the normalized quotient (2.2.8)

$$\nu f_H(x) = \frac{f_H(x)}{a_H} \in \frac{1}{a_H} D^*[\![x]\!].$$

By applying Lemma 4.0.2 to $\nu f_H(x)$ we obtain

Proposition 4.0.6 *There is a* stable *natural transformation*

$$MU^*X \xrightarrow{\nu Q^H} \frac{1}{a_H} D^*X$$

of functors of finite complexes to rings. On coefficients, it classifies the normalized *quotient*

$$\nu Q^H_* F^{MU} = \nu F/H.$$

The effect of νQ^H on the Euler class of a line bundle $\begin{array}{c} L \\ \downarrow \\ X \end{array}$ *is*

$$\nu Q^H(eL) = \nu f_H(eL).$$

Proof: By construction (2.2.8), $f_H(x) = x + o(x^2)$ is homogeneous of degree 2 when the degree of x is 2. Also by definition

$$\nu F/H = F^{\nu f_H}.$$

Lemma 4.0.2 provides us with the operation νQ^H. □

The relationship between the operations νQ^H and Q^H will allow us to prove integrality statements about stable operations.

Proposition 4.0.7 *Let $H \subset \Lambda^*_\infty$ be a subgroup of order n. On even-dimensional classes, the diagram*

$$\begin{array}{ccc} MU^{2*}X & = & MU^{2*}X \\ {\scriptstyle Q^H}\downarrow & & \downarrow{\scriptstyle \nu Q^H} \\ D^{2n*}X & \xrightarrow{\delta^H} & \frac{1}{a_H}D^{2*}X \end{array}$$

commutes, where δ^H *is the homomorphism of (2.2.10)*

$$\delta^H(m) = a_H^{-\frac{|m|}{2n}} m.$$

Proof: By the definition of f_H and νf_H, and by Theorems 3.2.4 and 4.0.6, the diagram commutes on Euler classes of line bundles. By the definition of F/H and $\nu F/H$ and Theorems 3.2.1 and 4.0.6, it commutes on coefficients. Moreover both the clockwise and couterclockwise composites are maps of rings. Since νQ^H is stable, the result follows since the facts listed so far imply that the two operations determine the same element of

$$\mathrm{Hom}_{\pi_* MU}[MU_* MU, \frac{1}{a_H} D^*]. \square$$

For odd-dimensional classes, it will be useful to know that

Proposition 4.0.8 *The operation* $a_H^r \nu Q^H$ *is integral on* $(2r-1)$*-dimensional classes:*

$$MU^{2r-1}X \xrightarrow{a_H^r \nu Q^H} D^{2nr-1}X \subset \frac{1}{a_H} D^* X.$$

Proof: Let σ denote the suspension isomorphism, and suppose $m \in MU^{2r-1}X$. By Proposition 4.0.7, we have

$$a_H^r \nu Q^H(\sigma m) = Q^H(m) \in D^{2nr}X.$$

Since νQ^H is a stable operation,

$$a_H^r \nu Q^H(\sigma m) = a_H^r \sigma \nu Q^H(m).$$

Finally, we have

$$\begin{array}{ccc} D^{2nr-1}X & \longrightarrow & \frac{1}{a_H} D^{2nr-1}X \\ \sigma \downarrow & & \downarrow \sigma \\ D^{2nr}X & \longrightarrow & \frac{1}{a_H} D^{2nr}X. \square \end{array}$$

5 Power operations in E_h

In this section we focus on the cohomology theory E_h, and study the problem of constructing a ring homomorphism

$$E_h^* \xrightarrow{\beta^H} \Phi_n D^*$$

such that

$$\beta_*^H F = F/H, \tag{5.0.1}$$

as required by Theorem 4.0.1. Our main result (Theorem 5.3.1) is that there is an (essentially unique) genus

$$MU^* \xrightarrow{t_{PO*}} E_h^*$$

such that the resulting formal group law satisfies this condition for all finite subgroups H. This is the genus announced in Theorem A.

As a result, one obtains an operation

$$E^{2*}X \xrightarrow{\Psi^H} D^{2n*}X$$

for every H. In section 5.4 we study the case of the group $H = {}_{p^k}F(D^*, 2)$, which yields the unstable Adams operation Ψ^{p^k} (Theorem E). We begin by describing

5.1 The cohomology theory E_h

Let E_h^* be the complete, Noetherian, local domain

$$E_h^* = \mathbb{Z}_p[\![u_1, \ldots, u_{h-1}]\!][u, u^{-1}],\ |u_i| = 0,\ |u| = -2.$$

Let t_{h*} be the genus

$$MU^* \to BP^* \xrightarrow{t_{h*}} E_h^*$$

which is given on the Araki generators [Rav86] by

$$t_{h*}(v_i) = \begin{cases} u_i u^{p^i-1} & i \le h-1, \\ u^{p^h-1} & i = h, \text{ and} \\ 0 & i > h. \end{cases} \tag{5.1.1}$$

It is an immediate consequence of Landweber's exact functor theorem that

Proposition 5.1.2 *The functor*

$$E_h^* X = E_h^* \underset{MU^*}{\otimes} MU^* X$$

is a cohomology theory.

The important feature of E_h is that the coefficient ring E_h^* and the formal group law $F_h = t_{h*}F^{MU}$ represent a functor. Denote by

$$K_h^* = \mathbb{F}_p[u, u^{-1}]$$

the graded residue field of E_h^*, and by

$$\varphi(x, y) \in K_h^*[\![x, y]\!]$$

the formal group law over K_h^* obtained by reducing the coefficients of F_h modulo the maximal ideal E_h^*. Note that since the formal group law F_h satisfies [Rav86]

$$[p]_{F_h}(x) = px \underset{F_h}{+} u_1 u^{p-1} x^p \underset{F_h}{+} \dots \underset{F_h}{+} u_{h-1} u^{p^{(h-1)}-1} x^{p^{(h-1)}} \underset{F_h}{+} u^{p^h-1} x^{p^h}, \qquad (5.1.3)$$

the formal group law φ satisfies

$$[p]_\varphi(x) = u^{p^h-1} x^{p^h},$$

and so has height h.

Let R^* be a complete, local, graded $\mathbb{Z}_p[u, u^{-1}]$-algebra, with residue field $\mathcal{K}^*$. When f is a power series over a local ring, we denote by $\overline{f}$ the power series over the residue field obtained by reducing the coefficients of f modulo the maximal ideal.

Definition 5.1.4 *A formal group law G over R^* which is homogeneous of degree 2 is a* deformation *of φ to R if*

$$\overline{G}(x, y) = \varphi(x, y) \in \mathcal{K}^*[\![x, y]\!].$$

Two deformations G_1 and G_2 are $\star$-isomorphic *if there is an isomorphism of formal groups*

$$G_1 \xrightarrow{f} G_2$$

such that

$$\overline{f}(x) = x.$$

The fundamental result about E_h^* is

Theorem 5.1.5 ([LT66]) *Let F be any formal group law $\star$-isomorphic to F_h. The ring E_h^* and the formal group law F represent the functor*

$$R^* \longmapsto \left\{ \begin{array}{c} \star - \textit{isomorphism classes of} \\ \textit{deformations of } \varphi \textit{ to } R^* \end{array} \right\}$$

In fact, if G is a deformation of φ to R, then there is a unique homomorphism

$$E_h^* \xrightarrow{\beta} R$$

such that for any choice of F, G is $\star$-isomorphic to $\beta_* F$ Moreover, the $\star$-isomorphism

$$G \xrightarrow{f} \beta_* F$$

is uniquely determined by G and F.

Let D_k^* be the ring of section 2.3 for the pair (E_h^*, F_h).

Corollary 5.1.6 *The ring D_k^* represents the functor*

$$R \longmapsto \left\{ \begin{array}{c} \star - \textit{isomorphism classes of} \\ \textit{deformations } G \textit{ of } \varphi \textit{ to } R^*, \\ \textit{with a level-}\Lambda_k^* \textit{ structure} \\ \Lambda_k^* \xrightarrow{\phi} G(R,2)_{tors} \end{array} \right\}.$$

Proof: This follows from Theorem 5.1.5 and Theorem 2.3.6.

5.2 Lubin's quotients F_h/H are deformations

Because the theories E_h^* are periodic, constructing a homomorphism (for a subgroup $H \subset \Lambda_\infty^*$ of order n)

$$E_h^* \xrightarrow{\beta^H} \Phi_n D^*$$

such that

$$\beta_*^H F = F/H$$

is equivalent to constructing a homomorphism

$$E_h^* \xrightarrow{\gamma^H} D^*$$

such that

$$\gamma_*^H F = (F/H)^{u^{n-1}x} :$$

by Lemma 2.2.13, β^H and γ^H are related by

$$\beta^H = \mu^n \circ \gamma^H,$$

where

$$E_h^* \xrightarrow{\mu^n} E_h^{n*}$$

is the homomorphism

$$\mu^n(m) = u^{\frac{(1-n)|m|}{2}} m.$$

For this chapter only, we take Lubin's quotient $f_H(x)$ to be homogeneous of degree 2. In other words, the homomorphism which is elsewhere written

$$u^{n-1} f_H(x).$$

Similarly, we take the quotient formal group law

$$F/H$$

to be homogeneous of degree 2, so one gets a homomorphism

$$H \longrightarrow F(D^*,2) \xrightarrow{f_H} F/H(D^*,2).$$

A $\star$-isomorphism

$$F \xrightarrow{\delta} F_h$$

yields an isomorphism

$$F(R^*,2) \underset{\cong}{\xrightarrow{\delta}} F_h(R^*,2),$$

so a level structure

$$A^* \xrightarrow{\phi} F(R^*,2)$$

is equivalent to a level structure

$$A^* \xrightarrow{\delta\phi} F_h(R^*,2).$$

The utility of Theorem 5.1.5 in our work stems from

Lemma 5.2.1 *The formal group law F/H is a deformation of φ to D^*. Moreover, the corresondence*

$$F \mapsto F/H$$

preserves $\star$-isomorphism classes.

Proof: Recall that F/H is defined by the homomorphism

$$F \xrightarrow{f_H} F/H$$

where

$$\overline{f_H}(x) = u^{n-1}x^n.$$

F/H is a deformation since $u^{n-1}x^n$ is an endomorphism of φ, so the diagram

$$\begin{array}{ccc} F & \xrightarrow{f_H(x)} & F/H \\ \downarrow & & \downarrow \\ \varphi & \xrightarrow{u^{n-1}x^n} & \varphi \end{array}$$

commutes, where the vertical arrows represent reduction modulo the maximal idea. To prove the part about preservation of $\star$-isomorphism classes, suppose that

$$F \xrightarrow{g} F'$$

is a $\star$-isomorphism. Suppose that

$$F' \xrightarrow{f'_H} F'/H$$

is the homomorphism based on the group law F'. Since

$$F(D^*,2) \xrightarrow{g} F'(D^*,2) \xrightarrow{f'_H} F'/H(D^*,2)$$

and

$$F(D^*,2) \xrightarrow{f_H} F/H(D^*,2)$$

have the same kernel, Theorem 2.2.3 implies that there is a unique isomorphism

$$F/H \xrightarrow{g'} F'/H$$

such that

$$\begin{array}{ccc} F & \xrightarrow{g} & F' \\ f_H \downarrow & & \downarrow f'_H \\ F/H & \xrightarrow{g'} & F'/H \end{array}$$

commutes. g' is a $\star$-isomorphism since

$$\overline{f}_H(x) = \overline{f}'_H(x) = u^{n-1}x^n. \ \square$$

Let

$$E_h^* \xrightarrow{\gamma^H} D^*$$

be the homomorphism determined by the $\star$-isomorphism class corresponding to H, and let

$$F/H \xrightarrow{g^H} \gamma_*^H F_h \tag{5.2.2}$$

be the $\star$-isomorphism determined by H and a group law F which is $\star$-isomorphic to F_h. Then

Lemma 5.2.3 *γ^H is the unique homomorphism*

$$E_h^* \to D^*$$

such that $\gamma_*^H F$ is $\star$-isomorphic to F/H as deformations of φ. $\square$

We really want strict equality in equation (5.2.2).

5.3 Theorem F and power operations in E_h

The main result of this chapter is

Theorem 5.3.1 ("Theorem F") *There is a unique formal group law F_{PO} over E_H which is $\star$-isomorphic to F_h and which satisfies*

$$\gamma_*^H F_{PO} = F_{PO}/H \tag{5.3.2}$$

for all finite subgroups H.

We give a proof in section 5.5. For now, we examine the consequences of Theorem 5.3.1 for cohomology operations. Let

$$MU^*X \xrightarrow{t_{PO}} E_h^*X$$

denote the orientation specified by Theorem 5.3.1. As remarked in the introduction, equation (5.3.2) is guaranteed to be satisfied by a formal group law F if there is a total power operation

$$E_h^{2*}X \xrightarrow{P_\pi^E} E_h^{2n*}D_\pi X$$

which is compatible with the orientation

$$MU^*X \xrightarrow{t} E_h^*X$$

in the sense that

$$\begin{array}{ccc} MU^{2*}X & \xrightarrow{P_\pi^{MU}} & MU^{2n*}D_\pi X \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ E_h^{2*}X & \xrightarrow{P_\pi^E} & E_h^{2n*}D_\pi X \end{array} \tag{5.3.3}$$

commutes. On the other hand, using the orientation provided by Theorem 5.3.1, the operation Q^H, and Theorem 4.0.1, we have

Corollary 5.3.4 *If $H \subset \Lambda_\infty^*$ is a finite subgroup of order n, then there is a unique operation*

$$E^{2*}X \xrightarrow{\Psi^H} D^{2n*}X$$

such that

$$\begin{array}{ccc} MU^{2*}X & & \\ {\scriptstyle t_{PO}}\downarrow & \searrow{\scriptstyle Q^H} & \\ E_h^{2*}X & \xrightarrow{\Psi^H} & D^{2n*}X \end{array}$$

commutes.

Theorem 5.3.5 *The operation*

$$MU^{2*}X \xrightarrow{T_\pi^{E_h}} \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^{2n*}X^{\pi/(\alpha)},$$

which is the image of the total power operation of MU under the character map of Hopkins–Kuhn–Ravenel (3.3.2), factors through the orientation

$$MU^*X \xrightarrow{t_{PO}} E_h^*X$$

to an operation

$$E_h^{2*}X \xrightarrow{\overline{P}_\pi^h} \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^{2n*}X^{\pi/(\alpha)}.$$

The projection to the component corresponding to a homomorphism

$$\Lambda_\infty \xrightarrow{\alpha} \pi$$

is

$$pr_\alpha \overline{P}_\pi^h(z) = \Psi^H(z^{\times k}) \in D^{n*}(X^{\pi/(\alpha)}),$$

where $k = [\pi : (\alpha)]$.

Proof: This is just Corollary 5.3.4 combined with Theorem 3.3.3. □

Corollary 5.3.4 is of particular interest in the case $H = \Lambda_k^*$, for then one obtains

5.4 The unstable Adams operation Ψ^{p^k} as a power operation.

Denote by f_p, F/p, etc., the quotient constructions corresponding to the subgroup $\Lambda_1^* \cong {}_pF(D^*, 2)$. By (3.4.5), we have

$$f_p(x) \in E_h^*[\![x]\!],$$
$$F_h/p(x,y) \in E_h^*[\![x,y]\!], \text{ and}$$
$$MU^{2*}X \xrightarrow{Q^p} E_h^{2p^h*}X.$$

Proposition 5.4.1 *For any formal group law F ⋆-isomorphic to* F_h*, there is a unique ⋆-isomorphism*

$$F/p \xrightarrow{g^p} F.$$

Proof: First of all, note that $[p]_F(x)$ is an endomorphism of F. Because of the co-equalizer diagram

$${}_pF(D^*,2) \to F(D^*,2) \underset{f_p(x)}{\overset{[p]_F(x)}{\rightrightarrows}} F(D^*,2),$$

it follows from Theorem 2.2.3 that there is a unique isomorphism of formal group laws

$$F/p \xrightarrow{g^p} F$$

such that

$$[p]_F(x) = g^p(f_p(x)). \tag{5.4.2}$$

The fact that

$$\overline{f_p}(x) = u^{p^{h-1}} x^{p^h} = \overline{[p]_F}(x),$$

shows that

$$\overline{g}^p(x) = x,$$

so g^p is a $\star$-isomorphism. □

Corollary 5.4.3 *The homomorphism γ^p determined by the deformation F/p is the identity, and β^p is the homomorphism*

$$\beta^p = \mu^{p^h}.$$

Corollary 5.4.4 *The condition*

$$\gamma^p_* F = F/p,$$

which is a special case of equation (5.3.2), is equivalent to the condition

$$f_p(x) = [p]_F(x). \tag{5.4.5}$$

Proof: The proof of (5.4.3) is just Proposition 5.4.1. (5.4.4) follows from equation (5.4.2), since with the formal group law of Theorem 5.3.1, we have

$$g^H(x) = x$$

for all H. □

Remark: Equation (5.4.5) turns out to determine the formal group law F_{PO} and figures in the proof of Theorem 5.3.1. See section 5.5.

Theorem 5.3.4 applied to Q^p, together with Corollaries 5.4.3 and 5.4.4, are the case $k = 1$ of

Theorem 5.4.6 ("Theorem E") *With the orientation*

$$MU^*X \xrightarrow{t_{PO}} E^*_h X$$

desribed in Theorem 5.3.1, the operation

$$E_h^{2*}X \xrightarrow{\Psi^{p^k}} E_h^{2p^{kh}*}X$$

obtained in Corollary 5.3.4 satisfies

$$\Psi^{p^k}(eL) = u^{1-p^{kh}}[p^k]_F(eL) \tag{5.4.7}$$

when $eL \in E_h^2X$ *is the Euler class of a complex line bundle* $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$. *On coefficients,* Ψ^{p^k} *is given by*

$$\Psi^{p^k} = \mu^{p^{kh}}.$$

The cases $k > 1$ are verified in the same manner. □

Remarks: Because of (5.4.7), we call Ψ^{p^k} the "p^kth unstable Adams operation" in E_h. Although an Adams operation could have been defined with any orientation on E_h, our orientation is the unique one in which the Adams operation is obtained as a power operation, in the spirit of [Ati66]. More familiar in the context of theories like E_h is the *stable* Adams operation

$$E_h^*X \xrightarrow{\psi^p} \frac{1}{p}E_h^*X.$$

This operation is obtained, using the method of Corollary 4.0.3, from the power series

$$\frac{[p]_F(eL)}{p} \in \frac{1}{p}E_h^2\mathbb{CP}^\infty \tag{5.4.8}$$

and the homomorphism

$$\begin{aligned} E_h^* &\to \frac{1}{p}E_h^* \\ m &\mapsto p^{-\frac{|m|}{2}}m. \end{aligned} \tag{5.4.9}$$

With the orientation t_{PO}, it coincides with the operation produced from the stable operation νQ^p (4.0.6) using the homomorphism (5.4.9). Then our comparison theorems for νQ^p and Q^p (4.0.7,4.0.8) prove that on even-dimensional classes, we have

$$\psi^p(x) = (\frac{u^{p^{kh}-1}}{p})^{\frac{|x|}{2}}\Psi^p(x),$$

and moreover that

Proposition 5.4.10 *The unstable operation $p^i\psi^p$ is integral on $2i$ and $(2i-1)$-dimensional classes:*

$$E_h^{2i}X \xrightarrow{p^i\psi^p} E_h^{2i}X \text{ and}$$
$$E_h^{2i-1}X \xrightarrow{p^i\psi^p} E_h^{2i-1}X. \ \square$$

Remark: This result is well-known to the experts; see e.g. [Wil82]. The attractive feature of our method is that it the proofs imitate the "geometric" proofs available in K-theory.

The rest of this section is devoted to a

5.5 Proof of Theorem F.

Our proof is divided into two steps, of which the most important is Theorem 5.5.1. Since the formal group laws F/H are deformations of φ (5.2.1), it shows there is a *unique* formal group law F_H which is $\star$-isomorphic to F/H and which satisfies equation (5.4.5). The case $H = 0$ yields F_{PO}. Since γ^H is a ring homomorphism, it preserves this formula for the p-series, and so must send F_{PO} to F_H. The second step (5.5.16) is to show that the p-series of F_{PO}/H also satisfies equation (5.4.5). By uniqueness, it will follow that

$$F_{PO}/H = F_H = \gamma_*^H F_{PO}.$$

Theorem 5.5.1 ("Theorem A") *Let E^* be a complete, graded, local domain which is a $\mathbb{Z}_p[u, u^{-1}]$-algebra, and let F be a deformation of φ to E^*. Let $E_1^* \supset E^*$ be an extension over which there is a level-Λ_1^* structure*

$$\Lambda_1^* \xrightarrow{\phi} {}_pF(E_1^*, 2).$$

There is exactly one representative of the $\star$-isomorphism class of F over E^* which satisfies (5.4.5).

Proof: *Existence:* It suffices to prove the statement in the universal case: suppose we can construct a formal group law F_{PO} over E_h^* satisfying (5.4.5). If

$$E_h^* \xrightarrow{\beta} E^*$$

is the homomorphism such that there is a $\star$-isomorphism

$$F \xrightarrow{g} \beta_* F_{PO},$$

then the level-Λ_1^* structure

$$\Lambda_1^* \xrightarrow{\phi} {}_pF(E_1^*, 2)$$

determines an extension of β to a homomorphism

$$D_1^* \xrightarrow{\beta} E_1^*$$

such that

$$\begin{array}{ccc}
\Lambda_1^* & = & \Lambda_1^* \\
\phi_{univ}\downarrow & & \downarrow g\phi \\
{}_pF_{PO}(D_1^*, 2) & \xrightarrow{\beta} & {}_p\beta_* F_{PO}(E_1^*, 2).
\end{array}$$

commutes. Then

$$\begin{aligned}
[p]_{\beta_* F_{PO}}(x) &= \beta_* \left[\prod_{c \in \Lambda_1^*} (x \underset{F_{PO}}{+} \phi_{univ}(c)) \right] \\
&= \prod_{c \in \Lambda_1^*} (x \underset{\beta_* F_{PO}}{+} g(\phi(c))),
\end{aligned} \tag{5.5.2}$$

which is (5.4.5) for $\beta_* F_{PO}$.

We turn to the universal case. For the purposes of the exposition, it is extremely convenient to take advantage of the distinction between formal groups and formal group *laws*: let $\mathcal{L}$ denote the Lubin–Tate formal group on E_h for lifts of the formal group law φ. For any choice of coordinate y on $\mathcal{L}$, denote by F^y the resulting formal group law. Then the homomorphism

$$\Lambda_1^* \xrightarrow{\phi} F^y(D_1^*)$$

factors as

$$\Lambda_1^* \xrightarrow{\phi} \mathcal{L}(D_1^*) \xrightarrow{y} F^y(D_1^*).$$

Under a change of coordinate

$$F^y \xrightarrow[\star-\text{iso}]{\delta} F^x,$$

the level-Λ_1^* structure changes by

$$\begin{array}{ccccc}
\Lambda_1^* & \xrightarrow{\phi} & \mathcal{L}(D_1^*) & \xrightarrow{y} & F^y(D_1^*) \\
\downarrow = & & \downarrow = & & \downarrow \delta \\
\Lambda_1^* & \xrightarrow{\phi} & \mathcal{L}(D_1^*) & \xrightarrow{x} & F^x(D_1^*)
\end{array}$$

In this language, Lubin's homomorphism f_p becomes

$$f_p^y(x) = \prod_{c \in \Lambda_1^*} (x \underset{F^y}{+} y(\phi(c))).$$

The proof is inductive, on powers of the maximal ideal I of D_1^*. Let y be any coordinate on $\mathcal{L}$ which gives a group law F^y $\star$-isomorphic to F_h. Let $g^y(t) \in E_h^*[[t]]$ denote the unique $\star$–isomorphism

$$F^y/p \xrightarrow{g^y} F^y$$

such that

$$[p]_{F^y}(t) = g^y(f_p^y(t)), \tag{5.5.3}$$

whose existence and uniqueness are the matter of Proposition 5.4.1. Write

$$g^y(t) = t + a(t).$$

Since g^y is a $\star$-isomorphism, we get automatically the case $n = 2$ of the hypothesis that

$$a(t) = \sum_{j \geq 1} a_j t^j, \text{ with } a_j \in I^{n-1}.$$

Let $\delta(t)$ be the power series

$$\delta(t) = t - a(t).$$

Since g^y is defined over E_h^*, the coordinate

$$x = \delta(y)$$

on $\mathcal{L}$ yields a formal group law F^x over E_h^*. We shall show that the $\star$-isomorphism $g^x(t)$ such that

$$[p]_{F^x}(t) = g^x(f_p^x(t)) \tag{5.5.4}$$

satisfies

$$g^x(t) \equiv t \pmod{I^n}. \tag{5.5.5}$$

By construction, δ is a homomorphism of formal group laws

$$F^y \xrightarrow{\delta} F^x.$$

Thus the diagram

$$\begin{array}{ccc} F^y & \xrightarrow{\delta} & F^x \\ {\scriptstyle [p]^y}\downarrow & & \downarrow{\scriptstyle [p]^x} \\ F^y & \xrightarrow{\delta} & F^x \end{array}$$

commutes; in other words, we have

$$\delta([p]^y(y)) = [p]^x \delta(y). \tag{5.5.6}$$

Substituting (5.5.3) and (5.5.4) into this equation yields

$$\delta[g^y(f_p^y(y))] = g^x[f_p^x(\delta(y))]. \tag{5.5.7}$$

Notice that

$$\begin{aligned} f_p^x(\delta(y)) &= u^{p^h-1} \prod_{c\in\Lambda_1^*} (\delta(y) \underset{F^x}{+} x(\phi(c))) \\ &= u^{p^h-1} \prod_{c\in\Lambda_1^*} \delta(\delta^{-1}\delta(y) \underset{F^y}{+} \delta^{-1}(x(\phi(c))) \\ &= u^{p^h-1} \prod_{c\in\Lambda_1^*} \delta(y \underset{F^y}{+} y(\phi(c))), \end{aligned}$$

so (5.5.7) becomes

$$\delta[g^y(f_p^y(y))] = g^x[u^{p^h-1} \prod_{c\in\Lambda_1^*} \delta(y \underset{F^y}{+} y(\phi(c)))]. \tag{5.5.8}$$

We can evaluate the left side modulo I^n very easily:

$$\begin{aligned} \delta(g^y(f_p^y(y))) &= \delta(f_p^y(y) + a(f_p^y(y))) \\ &\equiv \delta(f_p^y(y) + a(u^{p^h-1}y^{p^h})) \\ &\equiv f_p^y(y) + a(u^{p^h-1}y^{p^h}) - a(f_p^y(y) + a(u^{p^h-1}y^{p^h})) \\ &\equiv f_p^y(y) + a(u^{p^h-1}y^{p^h}) - a(f_p^y(y)) \\ &\equiv f_p^y(y), \end{aligned} \tag{5.5.9}$$

where the equivalences are all modulo I^n. The first equivalence uses the fact that $f_p^y(y) \equiv u^{p^h-1}y^{p^h} \pmod{I}$ plus the fact that $a(t)$ is a power series with constant term zero and coefficients in I^{n-1} (with $n \geq 2$!). The third equivalence follows from this description of $a(t)$, and the last equivalence is like the first.

All we know *a priori* about g^x is that it is a power series of the form

$$g^x(t) = t + b(t)$$

where

$$b(t) = \sum_{j\geq 1} b_j t^j,\ b_j \in I, \tag{5.5.10}$$

so we start by evaluating right-hand-side of (5.5.8) modulo I^2. We can compute $\prod_{c\in\Lambda_1^*}\delta(y \underset{F^y}{+} y(\phi(c)))$ modulo I^n :

$$\begin{aligned}
\prod_{c\in\Lambda_1^*}\delta(y \underset{F^y}{+} y(\phi(c))) &= \prod_{c\in\Lambda_1^*}[(y \underset{F^y}{+} y(\phi(c))) - a(y \underset{F^y}{+} y(\phi(c)))] \\
&\equiv \prod_c (y \underset{F^y}{+} y(\phi(c))) - \sum_c \left[\prod_{d\neq c}(y \underset{F^y}{+} y(\phi(d)))\right] a(y \underset{F^y}{+} y(\phi(c))) \\
&\equiv \prod_c (y \underset{F^y}{+} y(\phi(c))) - \sum_c y^{p^h-1}a(y) \qquad (5.5.11)\\
&\equiv \prod_c (y \underset{F^y}{+} y(\phi(c))) - p^h y^{p^h-1}a(y) \\
&\equiv \prod_c (y \underset{F^y}{+} y(\phi(c)))
\end{aligned}$$

(Recall that $p \in I$ is contained in the maximal ideal.) Continuing (5.5.11) modulo I shows that

$$\prod_{c\in\Lambda_1^*}\delta(y \underset{F^y}{+} y(\phi(c))) \equiv y^{p^h} \pmod{I}.$$

Equipped with these observations, we compute that if

$$b(t) \equiv 0 \pmod{I^{j-1}},\ j \geq 2,$$

then for $j \leq n$,

$$\begin{aligned}
g^x(u^{p^h-1}\prod_{c\in\Lambda_1^*}\delta(y \underset{F^y}{+} y(\phi(c)))) &= u^{p^h-1}\prod_{c\in\Lambda_1^*}\delta(y \underset{F^y}{+} y(\phi(c))) \\
&\quad + b(u^{p^h-1}\prod_{c\in\Lambda_1^*}\delta(y \underset{F^y}{+} y(\phi(c))) \\
&\equiv f_p^y(y) + b(u^{p^h-1}y^{p^h}) \pmod{I^j} \qquad (5.5.12)
\end{aligned}$$

Comparing (5.5.9) and (5.5.12), it follows in view of (5.5.10) that

$$b(t) \equiv 0 \pmod{I^j},$$

and inductively that

$$b(t) \equiv 0 \pmod{I^n}.$$

Uniqueness: Suppose that F^y satisfies (5.4.5) and that

$$F^y \xrightarrow{\delta} F^x$$

is a $\star$-isomorphism

$$\delta(t) = t + \sum_{j\geq 1} a_j t^j.$$

Let $\mathfrak{m}$ be the maximal ideal of E^*, and let n be as large as possible so that

$$a_j \in \mathfrak{m}^n$$

for all j. Note that $n \geq 1$, and that if $n = \infty$ then $F = F'$, so we may suppose that n is finite. Then, working modulo $\mathfrak{m}^{n+1}$, we have

$$\begin{aligned}
[p]^x(x) &= \delta([p]^y(\delta^{-1}(x))) \\
&= \delta(u^{p^h-1} \prod_{c\in\Lambda_1^*} (\delta^{-1}(x) \underset{F^y}{+} y(\phi(c))) \\
&= \delta(u^{p^h-1} \prod_{c\in\Lambda_1^*} \delta^{-1}(x \underset{F^x}{+} \delta(y(\phi(c))))) \\
&\equiv u^{p^h-1} \prod_{c\in\Lambda_1^*} \delta^{-1}(x \underset{F^x}{+} x(\phi(c))) + a(u^{p^h-1}x^{p^h}) \\
&\equiv f_p^x(x) - u^{p^h-1} \sum_{c\in\Lambda_1^*} a(x \underset{F^x}{+} x(\phi(c))) \prod_{d\neq c} (x \underset{F^x}{+} x(\phi(c))) + a(u^{p^h-1}x^{p^h}) \\
&\equiv f_p^x(x) - u^{p^h-1}p^h a(x) x^{p^h-1} + a(u^{p^h-1}x^{p^h}) \\
&\equiv f_p^x(x) + a(u^{p^h-1}x^{p^h}),
\end{aligned}$$

so F^x fails to satisfy (5.4.5). □

Denote by F_{PO} the formal group law over E_h^* $\star$-isomorphic to F_h which is constructed in Theorem 5.5.1. For a finite subgroup $H \subset \Lambda_\infty^*$ of exponent p^k and order n and for $l \geq k$ we get level-Λ_l/H structures on the formal group law F_{PO}/H via

$$\begin{array}{ccc}
H & = & H \\
\downarrow & & \downarrow \phi \\
\Lambda_l^* & \xrightarrow{\phi} & F_{PO}(D_l^*, 2) \\
\downarrow & & \downarrow f_H \\
\Lambda_l^*/H & \xrightarrow{\phi/H} & F_{PO}/H(D_l^*, 2).
\end{array} \tag{5.5.13}$$

Corollary 5.5.14 *The formal group law $\gamma_*^H F_{PO}$ is the* unique *formal group law in the $\star$-isomorphism class determined by H over D_k^* such that over D_{k+1}^*,*

$$[p]_{\gamma_*^H F_{PO}}(x) = \prod_{\substack{c\in\Lambda_{k+1}^*/H \\ pc=0}} (x \underset{\gamma_*^H F_{PO}}{+} \delta\phi/H(c)), \tag{5.5.15}$$

where δ is the $\star$-isomorphism

$$F_{PO}/H \xrightarrow{\delta} \gamma_*^H F_{PO}$$

Proof: Let Λ_1' denote the group

$$\Lambda_1' = {}_p(\Lambda_{k+1}/H).$$

Any isomorphism

$$\Lambda_1^* \xrightarrow{\cong} \Lambda_1'$$

determines a level-Λ_1 structure

$$\Lambda_1^* \xrightarrow{\cong} \Lambda_1' \xrightarrow{\delta\phi/H} F_{PO}/H(D_{k+1}^*, 2).$$

and so an extension of γ^H to a homomorphism

$$D_1^* \xrightarrow{\gamma^H} D_{k+1}^*.$$

We are now in the situation in the beginning of the proof of Theorem 5.3.1; compare (5.5.2). □

The proof of Theorem 5.3.1 is completed by

Proposition 5.5.16 *For any finite subgroup $H \subset \Lambda_\infty^*$, the quotient formal group law F_{PO}/H satisfies (5.5.15).*

Proof: The quotient F_{PO}/H is determined by the homomorphism f_H. The p-series $[p]_{F_{PO}/H}(x)$ is determined by the functional equation

$$\begin{array}{ccc} F_{PO} & \xrightarrow{f_H} & F_{PO}/H \\ {\scriptstyle [p]_{F_{PO}}}\downarrow & & \downarrow{\scriptstyle [p]_{F_{PO}}} \\ F_{PO} & \xrightarrow{f_H} & F_{PO}/H. \end{array} \tag{5.5.17}$$

Let $Z(x)$ be the product

$$Z(x) = u^{n-1} \prod_{\substack{c \in \Lambda_{k+1}^*/H \\ pc=0}} (x \underset{F_{PO}/H}{+} \phi/H(c)).$$

We are going to show that $Z(x)$ satisfies (5.5.17). Let $p^{-1}H$ denote the subgroup

$$p^{-1}H = \{c \mid pc \in H\} \subset \Lambda_{k+1}^*.$$

Then $Z(x)$ is exactly the Lubin isogeny

$$Z(x) = f_{p^{-1}H/H}(x)$$

for the group law F_{PO}/H. The composition rule (2.2.4) applied to this case is

$$f_{p^{-1}H/H}(f_H(x)) = f_{p^{-1}H}.$$

On the other hand, by construction

$$[p]_{F_{PO}}(x) = f_p(x),$$

so

$$\begin{aligned} f_H([p]_{F_{PO}}(x)) &= f_H(f_p(x)) \\ &= f_{p^{-1}H}(x). \ \Box \end{aligned}$$

5.6 Example: K-theory

The group law of K-theory is the multiplicative group law

$$x \underset{K}{+} y = x + y - vxy,$$

where $v \in K^{-2}$ is the Bott element. This group law arises from the fact that the Euler class of a line bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ is

$$eL = v^{-1}(1 - L).$$

The roots of the p-series of F^K are

$$v^{-1}(1 - \zeta^j),\ 0 \leq j < p,$$

where $\zeta = e^{2\pi i/p}$.

Suppose $p = 2$. Then

$$[2](x) = 2x - vx^2.$$

On the other hand,

$$\begin{aligned} f_2(x) &= v^{-1}(1 - L)[v^{-1}(1 - L) \underset{K}{+} 2v^{-1}] \\ &= v^{-2}(1 - L)[1 - L + 2 - 2 + 2L] \\ &= v^{-2}(1 - L^2) \\ &= v^{-1}[2](x). \end{aligned}$$

Similarly, for p odd, we have

$$[p](x) = v^{-1}(1 - L^p)$$

while

$$
\begin{aligned}
f_p(x) &= \prod_{j=0}^{p-1}[v^{-1}(1-L) \underset{K}{+} v^{-1}(1-\zeta^j)] \\
&= v^{-p}\prod_{j=0}^{p-1}(1-\zeta^j L) \\
&= v^{-p}\zeta^{p(p-1)/2}\prod_{j=0}^{p-1}(\zeta^{-j}-L) \\
&= v^{-p}(1-L^p) \\
&= v^{1-p}[p](x).
\end{aligned}
$$

Thus the multiplicative group law is the unique formal group law over (p-adic) K-theory which is $\star$-isomorphic to F_1 and which satisfies (5.4.5). In the case of E_1, Theorem A picks out the multiplicative formal group law.

The construction of the Adams operation Ψ^p given as Theorem E in p-adic K-theory is in the same spirit as Atiyah's construction [Ati66]. In the case of the multiplicative group law, the only finite subgroups of the formal group law are the groups ${}_nF^K$, since the height of the group law is one. For greater heights, however, there are other interesting subgroups of order n. As an example, we turn now to the study of elliptic cohomology.

6 Elliptic cohomology

Mike Hopkins [Hop89] has described the application of the character theory of Hopkins–Kuhn–Ravenel to the elliptic cohomology theory Ell of Landweber–Ravenel–Stong [LRS88]. In this section we add this information to the ideas developed in sections 3 and 4 to produce two sorts of operations in elliptic cohomology. In section 6.4 we produce an "Adams operation"

$$Ell^{2*}X \xrightarrow{\Psi^n} \frac{1}{p}Ell^{2p^2*}X$$

whose effect on the Euler class of a line bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ is given by

$$\Psi^p(eL) = \varepsilon^{\frac{1-p^2}{4}}[p]_{Ell}(eL),$$

where ε is a unit in Ell^*; see below. The technique is the same as that of Theorem 5.4.6.

In the case of K–theory, this is essentially the only kind of power operation there is, because the only subgroup of $\mathbb{G}_m$ of order n is the group ${}_n\mathbb{G}_m$. In the case of the formal group law F^{Ell} of elliptic cohomology, however, we have

$${}_nF^{Ell} \cong (\mathbb{Z}/n\mathbb{Z})^2,$$

so there are interesting subgroups of order n. These non–trivial subgroups are the source of *Hecke operators*. As operations on Elliptic cohomology, they have been constructed by Andrew Baker [Bak90]. In section 6.5 we show how to realize these operations as power operations. To prepare for the discussion in sections 6.4 and 6.5, we adapt in sections 6.1 and 6.2 the machinery developed in section 3 to the case of elliptic cohomology.

In our work on E_h, the operations Q^H were only a small part of the battle, and the really interesting problem was to produce the homomorphism

$$E_h^* \xrightarrow{\beta^H} D^{n*}$$

such that

$$\beta_*^H F_{PO} = F_{PO}/H.$$

Part of the reason why we think that the orientation F_{PO} which we discovered in the process is "analytic" is that in the case of K-theory, we recover the multiplicative group. Another reason is that in the case of Ell, the exponential of the formal group law is an elliptic function, so it is determined up to a constant multiple by its divisor. This simple fact identifies the quotient formal group law F^{Ell}/H, and enables us to produce the homomorphisms β^H (section 6.3).

6.1 A review of elliptic cohomology

We use the version of elliptic cohomology whose coefficient ring is the ring of modular forms over $\mathbb{Z}[\frac{1}{2}]$ for curves with $\Gamma_0(2)$ structure, with possible poles at the cusps; thus

$$Ell^* \cong \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon, \Delta^{\pm 1}]/(\Delta = \varepsilon(\delta^2 - \varepsilon)),$$

where $\delta \in Ell^{-4}$ and $\varepsilon \in Ell^{-8}$ are modular forms of weight 2 and 4 repspectively. Let Z_0 denote the space of pairs

$$(\Xi, \eta)$$

where Ξ is a lattice in $\mathbb{C}$, and η is a non-trivial point of order two of $\mathbb{C}/\Xi$. Z_0 comes with an action of $\mathbb{C}^\times$ by homotheties:

$$t(\Xi, \eta) = (t\Xi, t\eta),\ t \in \mathbb{C}^\times.$$

A modular function f of weight r is a function

$$Z_0(2) \xrightarrow{f} \mathbb{C}$$

such that for $t \in \mathbb{C}^\times$,

$$f(t(\Xi, \eta)) = t^{-r} f(\Xi, \eta). \tag{6.1.1}$$

Let $\mathfrak{h}$ denote the complex upper half-plane; then the quotient $Z_0/\mathbb{C}^\times$ is the modular curve

$$Y_0 = \Gamma_0(2)\backslash\mathfrak{h}.$$

For $\tau \in \mathfrak{h}$ let $[\tau]$ denote its orbit modulo $\Gamma_0(2)$. The projection $Z_0 \longrightarrow Y_0$ comes with a section

$$[\tau] \overset{\sigma}{\mapsto} (4\pi i\tau\mathbb{Z} + 4\pi i\mathbb{Z}, 2\pi i). \tag{6.1.2}$$

Via the section σ and the projection $\mathfrak{h} \xrightarrow{\pi} Y_0$, a modular function f pulls back to a function $\tilde{f}$ on $\mathfrak{h}$ such that

$$\tilde{f}(\alpha\tau) = (c\tau + d)^r f(\tau),$$

where

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2).$$

A *meromorphic modular form* $f \in Ell^{-2r}$ is a modular function of weight r on Z_0 such that $\tilde{f}$ is an analytic function on $\mathfrak{h}$ whose q-expansions at the two inequivalent cusps of $\Gamma_0(2)$ have coefficients in $\mathbb{Z}[\frac{1}{2}]$.

There is a formal group law F^{Ell} defined over Ell^* called the Euler formal group law [Igu59], given by

$$F^{Ell}(x_1, x_2) = \frac{x_1\sqrt{R(x_2)} + x_2\sqrt{R(x_1)}}{1 - \varepsilon x_1^2 x_2^2}, \tag{6.1.3}$$

where

$$R(x) = 1 - 2\delta x^2 + \varepsilon x^4.$$

This determines a genus

$$MU^* \xrightarrow{t_{Ell*}} Ell^*.$$

Once again the exact functor theorem applies and shows

Theorem 6.1.4 ([LRS88, Lan88]) *The functor*

$$X \mapsto Ell^*X \stackrel{def}{=} Ell^* \underset{MU^*}{\otimes} MU^*X$$

is a cohomology theory on finite complexes. □

Remark: Franke [Fra92] has recently shown that the extension of $Ell^*(-)$ to infinite complexes is unique.

The exponential of F^{Ell} is a power series

$$s(z) \in \mathbb{Q} \otimes Ell^*[\![z]\!]$$

which has a characterization as an elliptic function.

Lemma 6.1.5 ([CC88, Zag88]) *For pair* $(\Xi, \eta) \in Z_0$, *there is a unique meromorphic function*

$$s(-, \Xi, \eta) : \mathbb{C} \rightarrow \mathbb{C}$$

which satisfies

i. $s(z + \xi, \Xi, \eta) = s(z, \Xi, \eta)$ *for* $\xi \in \Xi$,

ii. $s(z + \eta, \Xi, \eta) = -s(z, \Xi, \eta)$, *and*

iii. $s(z, \Xi, \eta) = z + o(z^2)$.

The function $s(-, \Xi, \eta)$ and its derivative satisfy the functional equation

$$s'^2 = 1 - 2\delta(\Xi, \eta)s^2 + \varepsilon(\Xi, \eta)s^4, \tag{6.1.6}$$

and so uniformize the *Jacobi quartic*

$$C : y^2 = 1 - 2\delta(\Xi, \eta)x^2 + \varepsilon(\Xi, \eta)x^4. \tag{6.1.7}$$

via the map

$$\mathbb{C}/\Xi \xrightarrow{(s(-,\Xi,\eta),s'(-,\Xi,\eta),1)} C \subset \mathbb{P}^2. \tag{6.1.8}$$

As written, the Jacobi quartic has a singularity at ∞, but this singularity is minor. The compactification of C in $\mathbb{P}^3$ via the map

$$C \xrightarrow{(1,x,x^2,y)} \mathbb{P}^3$$

is a smooth curve C_0 such that

$$C_0 \cap \{X_0 \neq 0\} \cong C.$$

So by choosing the origin to be $(0,1,1)$, the Jacobi quartic becomes an elliptic curve; as such, it is a group. Because of (6.1.5, iii), the function s determines a parameter near the origin, with respect to which the group law of the curve (6.1.7) becomes the Euler formal group law specialized at (Ξ, η) :

$$F^{Ell}(x_1, x_2) = \frac{x_1 y_1 + x_2 y_2}{1 - \varepsilon(\Xi, \eta) x_1^2 x_2^2}. \tag{6.1.9}$$

As promised, [Zag88] s is the *exponential* for the Euler formal group law:

$$s(z + w, \Xi, \eta) = s(z, \Xi, \eta) \underset{F^{Ell}}{+} s(w, \Xi, \eta), \tag{6.1.10}$$

so the uniformization (6.1.8) is a homomorphism of groups.

We shall use two additional facts about F^{Ell}. First, equation (6.1.6) and the fact that s is characterized by the three properties listed above imply, in view of the weights of δ and ε, that s satisfies the homogeneity property

$$s(tz, t\Xi, t\eta) = ts(z, \Xi, \eta). \tag{6.1.11}$$

The second fact about the group law F^{Ell} is that for N odd, the series $[N]_{F^{Ell}}(x)$ has a "Weierstrass factorization" already over Ell^* :

Theorem 6.1.12 ([Igu59]) *There are polynomials*

$$p_N(x), g_N(x) \in Ell^*[x]$$

with

$$p_N(x) = Nx + \ldots \pm \varepsilon^{\frac{N^2-1}{4}} x^{N^2} \text{ and}$$
$$g_N(0) = 1$$

such that

$$[N]_{F^{Ell}}(x) = \frac{p_N(x)}{g_N(x)}.$$

6.2 The operations Q^H in the case of elliptic cohomology

The ring "D^*" for elliptic cohomology

Now let p be an odd prime. As in section 2.3, we let Λ_∞ be an abelian group isomorphic to $\mathbb{Z}_p^2$, so that its dual is

$$\Lambda_\infty^* \cong (\mathbb{Q}_p/\mathbb{Z}_p)^2.$$

We denote by

$$\Lambda_k^* = {}_{p^k}\Lambda_\infty^*$$

the p^k–torsion subgroup of Λ_∞^*. For the purposes of this chapter, it will be convenient to choose explicit compatible isomorphisms

$$\Lambda_k^* \cong (\mathbb{Z}/p^k)^2;$$

then we refer to the chosen generators of Λ_k^* as e_k and f_k.

In this version of elliptic cohomology, also we have to keep track of the point η of order 2, so we let Λ_k' be the family of groups

$$\Lambda_k' = (\mathbb{Z}/2p^k)^2$$

with generators e_{2k} and f_{2k}. The point of order 2, η, will be the point $p^k e_{2k}$. Thus there are compatible inclusions

$$\begin{array}{ccc} \Lambda_k^* & \longrightarrow & \Lambda_k' \\ \downarrow & & \downarrow \\ \Lambda_{k+1}^* & \longrightarrow & \Lambda_{k+1}' \end{array}$$

under which

$$\begin{aligned} e_k &\mapsto 2e_{2k} \text{ and} \\ f_k &\mapsto 2f_{2k}. \end{aligned}$$

Let $Z(2p^k)$ be the space of pairs

$$(\Xi, \alpha)$$

consisting of a lattice $\Xi \subset \mathbb{C}$ and an isomorphism

$$\Lambda_k' \xrightarrow{\alpha} {}_{2p^k}(\mathbb{C}/\Xi)$$

such that

$$w_{2p^k}(\alpha(e_{2k}), \alpha(f_{pk})) = \zeta_{2p^k} \stackrel{def}{=} e^{\pi i/p^k}, \qquad (6.2.1)$$

where w_N is the *Weil pairing* for points of order N of $\mathbb{C}/\Xi$ given by ([KM85],p. 90)

$$w_N(v_1/N, v_2/N) = \exp(\frac{2\pi i}{N} \cdot \frac{\mathrm{Im}(\overline{v_1}v_2)}{A(\Xi)}),$$

where $v_1, v_2 \in \Xi$ and $A(\Xi)$ is the area of a fundamental parallelogram for Ξ.

$\mathbb{C}^\times$ acts on $Z(2p^k)$ by homotheties; that is,

$$t(\Xi, \alpha) = (t\Xi, t\alpha),$$

and the quotient $Z(2p^k)/\mathbb{C}^\times$ is the modular curve

$$Y(2p^k) = \Gamma(2p^k)\backslash\mathfrak{h}.$$

The projection $Z(2p^k) \to Y(2p^k)$ comes with a section

$$\begin{array}{rcl} Y(2p^k) & \xrightarrow{\sigma} & Z(2p^k) \\ [\tau] & \mapsto & (4p^k\pi i\tau\mathbb{Z} + 4\pi i\mathbb{Z}, \alpha(bf+ce) = 2\pi i\tau b + \frac{2\pi i c}{p^k}). \end{array}$$

The group $SL_2\mathbb{Z}/2p^k \cong SL_2\mathbb{Z}/p^k \times SL_2\mathbb{Z}/2$ acts on $Z(2p^k)$. Let $G_k \subset SL_2\mathbb{Z}/2 \subset SL_2\mathbb{Z}/2p^k$ be the subgroup (isomorphic to $\mathbb{Z}/2$) such that

$$g(\eta) = \eta.$$

Let $Z_0(p^k)$ be the orbit space of the action of G_k on $Z(2p^k)$. It is the space of triples

$$(\Xi, \alpha, \eta)$$

where Ξ is a lattice in $\mathbb{C}$, η is a non-trivial point of order two in $\mathbb{C}/\Xi$, and α is an isomorphism

$$\Lambda_k^* \xrightarrow{\alpha} {}_{p^k}\mathbb{C}/\Xi$$

such that

$$w_{p^k}(\alpha(e_k), \alpha(f_k)) = \zeta_{p^k}.$$

Once again, there is a map

$$\begin{array}{rcl} Y(2p^k) & \xrightarrow{\sigma} & Z_0(p^k) \\ [\tau] & \mapsto & (4p^k\pi i\tau\mathbb{Z} + 4\pi i\mathbb{Z}, \alpha(bf_k+ce_k) = 4\pi i\tau b + \frac{4\pi i c}{p^k}, 2\pi i) \end{array} \tag{6.2.2}$$

A "meromorphic modular form of level $2p^k$ and weight r" is a function f on $Z(2p^k)$ which behaves as (6.1.1) with respect to homotheties and whose pull-back via the section σ is analytic on $\mathfrak{h}$ and has q-expansions at every cusp which are finite-tailed laurent series with coefficients in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]$. The ring of meromorphic modular forms of level $2p^k$ will be denoted $Ell^*_{\Gamma(2p^k)}$.

The ring which is the range of the character map for elliptic cohomology is the ring of modular functions on $Z_0(p^k)$ which pull back via σ to analytic functions on $\mathfrak{h}$ whose q-epansions at the cusps of $Y(2p^k)$ have coefficients in the ring $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]$. It is exactly the subring $(Ell^*_{\Gamma(2p^k)})^{G_k}$ of $Ell^*_{\Gamma(2p^k)}$ invariant under the action of G_k. We denote this ring by $Ell^*_{p^k}$.

The map "forget α" gives a projection $Z_0(p^k) \rightarrow Z_0$ and exibits $Ell^*_{p^k}$ as an Ell^*-algebra. Brylinski has shown

Proposition 6.2.3 ([Bry90]; see also [KM85]) *$Ell^*_{p^k}$ is a faithfully flat Ell^*-algebra.*

Proof: The proof of [Bry90] is based on the study of the moduli schemes for level structures on elliptic curves [KM85]. His proof can be imitated in our situation, with one important modification. Brylinski studies modular forms of *even* weight, and obtains a ring $Ell^*_{\Gamma(N)}$ of modular forms whose q-expansions have coefficients in $\mathbb{Z}[\frac{1}{2}, \zeta_N]$. We shall need modular forms of *odd* weight, namely the functions $\phi(a)$ (6.2.5). For a moduli space M with a universal curve $E \xrightarrow{\pi} M$, one defines the invertible sheaf

$$\underline{\omega} = \pi_* \Omega^1_{E/M}.$$

Let T be the circle bundle of ω. It is a principal $\mathbb{G}_m$-bundle over M, and one can view modular functions as functions on the space T, graded by the $\mathbb{G}_m$-action. In order study q-expansions of modular forms in this setting, one must be able to extend the sheaf ω to a sheaf on the *compactified* moduli space $\overline{M}$, as the cusps are precisely the points one adds in compactifying. Actually, only the square $\underline{\omega}^{\otimes 2}$ extends in general ([KM85], section 10.13), which enables Brylinski to study modular forms of even weight. Fortunately, there is a natural extension of $\underline{\omega}$ in case the moduli problem is "elliptic curves with level $2p^k$-structure over $\mathbf{Z}[\frac{1}{2p}, \zeta_{2p^k}]$," ([KM85],(10.13.9.1), [Kat73]) and this moduli space can be used in Brylinksi's argument, now for the ring of modular forms with possibly odd weight. Unfortunately, we have had to invert p. See the remarks at the end of this chapter. □

We shall need the additional facts that

Proposition 6.2.4 (compare 3.4.1) *$Ell^*_{p^k}$ is a domain (since it is a subring of a ring of functions on $Y(2p^k)$). The group*

$$\mathrm{Aut}[Z_0(p^k)/Z_0] \stackrel{def}{=} SL_2 Z/p^k \times \mathrm{Gal}(\mathbf{Z}[\zeta_{2p^k}]/\mathbf{Z})$$

acts on $Ell^*_{p^k}$, and the ring of invariants is $\frac{1}{p}Ell^*$.

Proof: By definition, an element of $Ell^*_{p^k}$ which is invariant under $SL_2\mathbf{Z}/p^k$ is a modular form for the group $\Gamma_0(2)$ whose q-expansions have coefficients in $\mathbf{Z}[\frac{1}{2p}, \zeta_{2p^k}]$.

Denote this ring by

$$M(\Gamma_0(2), \mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]).$$

Then [Kat73]

$$M(\Gamma_0(2), \mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]) = \mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}] \underset{\mathbb{Z}[\frac{1}{2}]}{\otimes} Ell^*.$$

(Actually, Katz shows this for the ring of *cusp* forms for the congruence subgroup $\Gamma(2)$. His argument applies also to the group $\Gamma_0(2)$. Then [Bak90] we can use multiples of Δ, which is a cusp form, to convert a modular form f to a cusp form $\Delta^d f$. Since $\Delta^{-1} \in Ell^{16}$ is already a defined over $\mathbb{Z}[\frac{1}{2}]$, the result follows for the full ring of modular forms). □

The character map for elliptic cohomology

We need an assignment

$$\Lambda_k^* \xrightarrow{\phi} Ell_{p^k}^*$$

that will play the role of the map ϕ_{univ} constructed in (2.3.5). The key idea is to use the exponential map for elliptic cohomology to find roots of the p^k–series, in the same way that roots of the p^k–series in K-theory are

$$1 - e^{\frac{2\pi i j}{p^k}}, \; 0 \leq j < p^k.$$

We define the map ϕ to be

$$\phi(a) = \{(\Xi, \alpha, \eta) \mapsto s(\alpha(a), \Xi, \eta).\} \tag{6.2.5}$$

This definition only makes $\phi(a)$ a complex-valued function on $Z_0(p^k)$. Notice, however, that by the homogeneity property of s (6.1.11), ϕ satisfies

$$\phi(a)(t\Xi, t\alpha, t\eta) = t\phi(a)(\Xi, \alpha, \eta),$$

so $\phi(a)$ has the right behavior with respect to homotheties to be an element of $Ell_{p^k}^2$.

Proposition 6.2.6 $\phi(a)$ *is an element of* $Ell_{p^k}^2$.

Proof: It remains to show that the q-expansions of $\phi(a)$ have coefficients in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]$ at every cusp. To simplify the notation, let $N = p^k$. The cusps of $Y(2N)$ correspond to choices of level structure on the "Tate curve" Tate(q^{2N}) [Kat73]. Via the map σ (6.2.2), we are led to consider the lattice

$$\Xi = 4\pi i N\tau\mathbb{Z} + 4\pi i\mathbb{Z}$$

and the function

$$\tilde{s}(z,q) = s(z,\Xi,2\pi i),$$

where $q = e^{2\pi i\tau}$. According to [CC88, Zag88], $\tilde{s}(z,q)$ is given by

$$\tilde{s}(z,q) = (e^{\frac{z}{2}} - e^{-\frac{z}{2}}) \prod_{m\geq 1} \left\{ \frac{(1-q^{Nm}e^{z})(1-q^{Nm}e^{-z})}{(1-q^{Nm})^2} \right\}^{(-1)^m}.$$

Then

$$\begin{aligned} \phi(bf_k + ce_k a)(\Xi, 2\pi i, \alpha) &= s(4\pi i b\tau + \frac{4\pi i c}{N}, \Xi, 2\pi i) \\ &= (\zeta^c q^b - \zeta^{-c}q^{-b}) \qquad (6.2.7) \\ &\prod_{m\geq 1} \left\{ \frac{(1-q^{mN+2b}\zeta^{2c})(1-q^{mN-2b}\zeta^{-2c})}{(1-q^{mN})^2} \right\}^{(-1)^m}, \end{aligned}$$

where $\zeta = e^{\frac{2\pi i}{p^k}}$ is a p^k-th root of unity. For each point $a \in (\mathbb{Z}/p^k)^2$, then, the expression (6.2.7) has coefficients of q in $\mathbb{Z}[\frac{1}{2}, \zeta_N]$. Fixing this level p^k structure and letting a vary over $(\mathbb{Z}/p^k)^2$ is equivalent to fixing an a and varying over all level p^k structures. There are other cusps which correspond to varying the point of order 2. However, it in fact suffices to check each $\phi(a)$ at one cusp, because of the q-expansion principle ([Kat73]): by choosing a value of the Weil pairing (6.2.1), we have restricted ourselves to one component of the moduli space Katz calls $\overline{M}_{2p^k}$. □

The significance of the modular forms $\phi(a)$ is that they are the roots of the p^k series of F^{Ell}:

$$\begin{aligned} [p^k]_{F^{Ell}}(\phi(a))(\Xi,\alpha,\eta) &= [p^k]_{F^{Ell}} s(\alpha(a),\Xi,\eta) \\ &= s(p^k\alpha(a),\Xi,\eta) = 0. \end{aligned}$$

Proposition 6.2.8 *Let $N = p^k$. The p^{2k} modular forms $\phi(a)$ are the roots of f_N:*

$$f_N(x) = \pm\varepsilon^{\frac{N^2-1}{4}} \prod_{a\in\Lambda_k^*} (x - \phi(a)). \ \square$$

Let $S = \mathrm{Im}(\phi)$ be the image of the map (of sets) ϕ. Notice that for any two elements $\phi(a)$ and $\phi(b)$, their formal sum is

$$\begin{aligned} \phi(a+b)(\Xi,\alpha,\eta) &= s(\alpha(a+b),\Xi,\eta) \\ &= s(\alpha(a)+\alpha(b),\Xi,\eta) \\ &= s(\alpha(a),\Xi,\eta) \underset{F^{Ell}}{+} s(\alpha(b),\Xi,\eta) \\ &= (\phi(a) \underset{F^{Ell}}{+} \phi(b))(\Xi,\alpha,\eta). \end{aligned}$$

In particular, $\phi(a) \underset{F^{Ell}}{+} \phi(b) = \phi(a+b)$ is an element of S. Let $F^{Ell}(S)$ be the group made of the set S together with the formal sum F^{Ell}. We have shown

Proposition 6.2.9 *ϕ is an isomorphism of groups*

$$\Lambda_k^* \xrightarrow{\phi} F^{Ell}(S).$$

In order to construct the character map, we have to complete $Ell^*_{p^k}$ with respect to the ideal I generated by the set S. Let $\widehat{Ell}^*_{p^k}$ denote the completion of $Ell^*_{p^k}$ with respect to I, and $\widehat{I}$ the extension of I to $\widehat{Ell}^*_{p^k}$.

Proposition 6.2.10 *ϕ induces an isomorphism*

$$\Lambda_k^* \xrightarrow{\phi} {}_{p^k}F^{Ell}(\widehat{I}, 2).$$

Moreover, ϕ is represented (in the sense of (2.3.4)) by a homomorphism

$$Ell^* B\Lambda_k \xrightarrow{\widehat{\phi}} \widehat{Ell}^*_{p^k}.$$

Proof: Ell_{p^k} is a domain, so by Krull's theorem

$$Ell^*_{p^k} \longrightarrow \widehat{Ell}^*_{p^k}$$

is injective. The first part then follows from (6.2.9) since by Igusa's theorem (6.1.12), $[p^k]_{F^{Ell}}(x)$ has p^{2k} roots. The definition of the map $\widehat{\phi}$ is (compare 2.3.3)

$$e\begin{pmatrix} E\Lambda_k \times_a \mathbb{C} \\ \downarrow \\ B\Lambda_k \end{pmatrix} \longmapsto \phi(a) \tag{6.2.11}$$

for $a \in \Lambda_k^*$. It defines a map since power series in the $\phi(a)$ converge in $\widehat{Ell}^*_{p^k}$. □

Theorems B and C for *Ell*.

We can use the maps ϕ and $\widehat{\phi}$ to imitate the constructions in section 3.1, and so to obtain versions of Theorems B and C. Let $H \subset \Lambda_k^*$ be a subgroup of order n.

Theorem 6.2.12 *There is a natural transformation*

$$MU^{2*}X \xrightarrow{Q^H} \Phi_n \widehat{Ell}^{2*}_{p^k} X$$

of functors of finite spaces to graded rings, whose effect on the Euler classes of a line bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ *is*

$$Q^H(eL) = f_H(eL) = \prod_{h \in H} (\phi(h) \underset{F^{Ell}}{+} eL).$$

Its effect on coefficients is determined by the equation

$$Q^H_* F^{MU} = F^{Ell}/H.$$

6.3 The Euler form of the quotient formal group law.

In the case of elliptic cohomology, the theory of elliptic functions enables us to show that the quotient formal group law F^{Ell}/H takes a particularly nice form. A finite subgroup of odd order

$$H \subset \mathbb{C}/\Xi$$

determines a quotient map

$$H \to \mathbb{C}/\Xi \to \mathbb{C}/\Xi_H,$$

where Ξ_H denotes the lattice in $\mathbb{C}$ obtained from the union of Ξ and the preimage of H under $\mathbb{C} \to \mathbb{C}/\Xi$, and we have denoted again by η the image of the point $\eta \in \mathbb{C}/\Xi$ under the projection. This quotient uniformizes the Jacobi quartic

$$C_H : y^2 = 1 - 2\delta(\Xi_H, \eta)x^2 + \varepsilon(\Xi_H, \eta)x^4, \tag{6.3.1}$$

which is the quotient of C by the subgroup H, via the map

$$\mathbb{C}/\Xi_H \xrightarrow{(s(-,\Xi_H,\eta),s'(-,\Xi_H,\eta),1)} C_H \subset \mathbf{P}^2. \tag{6.3.2}$$

With respect to the parametrization (6.3.2), the group law of the curve becomes the Euler formal group law for the quotient curve:

$$F_H^{Ell}(x_1, x_2) = \frac{x_1\sqrt{R_H(x_2)} + x_2\sqrt{R_H(x_1)}}{1 - \varepsilon(\Xi_H, \eta)x_1^2x_2^2}, \tag{6.3.3}$$

where

$$R_H(x) = 1 - 2\delta(\Xi_H, \eta)x^2 + \varepsilon(\Xi_H, \eta)x^4.$$

Now suppose the $H \subset \Lambda_k^*$. Then by using the level structure component of $Z_0(p^k)$, we can associate a quotient curve to each triple (Ξ, α, η).

Proposition 6.3.4 *For $H \subset \Lambda_k^*$ and a modular form $f \in Ell^{-2r}$, the modular function $\gamma^H f$ given by*

$$\gamma^H f(\Xi, \alpha, \eta) = f(\Xi_H, \eta) \tag{6.3.5}$$

is an element of $Ell_{p^k}^{-2r}$.

Proof: The only thing to show is that $\gamma^H f$ has q-expansions in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]((q))$. By decomposing the subgoup H via short exact sequences, it suffices to prove this when the order of H is p. Suppose that

$$f(\Xi, 2\pi i) = \sum_{n \gg -\infty} a_n q^n \in \mathbb{Z}[\frac{1}{2}]((q)),$$

where $q = e^{2\pi i\tau}$ and once again Ξ is the lattice

$$\Xi = 4\pi i\tau\mathbb{Z} + 4\pi i\mathbb{Z}.$$

The q-expansions we need to check are obtained by evaluating f on the quotients by subgroups of order p of $\mathbb{C}/\Xi(p)$, where $\Xi(p)$ is the lattice

$$\Xi(p) = 4\pi ip\tau\mathbb{Z} + 4\pi i\mathbb{Z},$$

so

$$f(\Xi(p), 2\pi i) = \sum_{n \succ -\infty} a_n q^{pn}.$$

For any level–p structure on $\mathbb{C}/\Xi(p)$, the quotients by subgroups of order p correspond to the lattices

$$\Xi_j = 4\pi i(\tau + \frac{j}{p})\mathbb{Z} + 4\pi i\mathbb{Z},\ 0 \le j < p, \text{ and}$$
$$\Xi_p = 4\pi ip\tau\mathbb{Z} + \frac{4\pi i}{p}\mathbb{Z}.$$

We have

$$f(\Xi_j, 2\pi i) = \sum_{n \succ -\infty} a_n q^n \zeta^{2nj},\ 0 \le j < p, \text{ and} \tag{6.3.6}$$
$$f(\Xi_p, 2\pi i) = f(\frac{1}{p}(4\pi ip^2\tau\mathbb{Z} + 4\pi i\mathbb{Z}, 2\pi i)) = p^r \sum_{n \succ -\infty} a_n q^{p^2 n},$$

where $\zeta = e^{2\pi i/p}$. □

Corollary 6.3.7 *The formal group law F_H^{Ell} is defined over $Ell^*_{p^k}$. It is classified by the homomorphism*

$$Ell^* \xrightarrow{\gamma^H} Ell^*_{p^k}.$$

By the miracle of complex analysis, the group law F_H^{Ell} is very simply related to Lubin's quotient F^{Ell}/H. Let

$$a_H = f'_H(0) = \prod_{0 \neq h \in H} \phi(h) \in Ell^{2n-2}_{p^k}.$$

Theorem 6.3.8 *F^{Ell}/H is related to the formal group law F_H^{Ell} by*

$$F_H^{Ell} \xrightarrow{a_H x} F^{Ell}/H.$$

In other words, F_H^{Ell} is exactly the "normalized" quotient formal group law $\nu F^{Ell}/H$ of (2.2.9).

Proof: Since the ring $\widehat{Ell}^*_{p^k}$ is torsion-free (6.2.4), formal group laws over $\widehat{Ell}^*_{p^k}$ are determined by their exponentials over $\mathbb{Q} \otimes \widehat{Ell}^*_{p^k}$. Let $t(z)$ denote the power series

$$t(z) = f_H(s(z)) \in \mathbb{Q} \otimes \widehat{Ell}^*_{p^k}[\![z]\!],$$

and let $\mathbb{G}_a$ denote the additive group. Then t, s, and f_H fit into a diagram of formal group laws

$$\begin{array}{ccc} \mathbb{G}_a & \xrightarrow{s} & F^{Ell} \\ = \big\downarrow & & \big\downarrow f_H \\ \mathbb{G}_a & \xrightarrow{t} & F^{Ell}/H \end{array}$$

The function $t(z)$ is an element of $\mathbb{Q} \otimes \widehat{Ell}^*_{p^k}[\![z]\!]$, so we evaluate it on the triple $(\Xi, \alpha, \eta) \in Z_0(p^k)$.

$$\begin{aligned} t(z)(\Xi,\alpha,\eta) &= \prod_{h\in H} (\phi(h) +_{F^{Ell}} s(z))(\Xi,\alpha,\eta) \\ &= \prod_{h\in H} (s(\alpha(h),\Xi,\eta) +_{F^{Ell}} s(z,\Xi,\eta)) \\ &= \prod_{h\in H} s(z+\alpha(h),\Xi,\eta). \end{aligned} \tag{6.3.9}$$

Now we use the key fact that as a function on $\mathbb{C}/\Xi$, $s(z,\Xi,\eta)$ is characterized by Lemma 6.1.5. Think of z as a complex parameter. Then by (6.3.9), $t(z)(\Xi,\alpha,\eta)$ is a complex-valued function

$$\mathbb{C} \xrightarrow{t(-)(\Xi,\alpha,\eta)} \mathbb{C}$$

which satisfies

$$t(z+\xi) = t(z),\ \xi \in \Xi, \tag{6.3.10}$$

$$\begin{aligned} t(z+\eta) &= \prod_{h\in H} s(z+\eta,\Xi,\eta) \\ &= (-1)^n \prod_{h\in H} s(z,\Xi,\eta) \\ &= -t(z), \text{ and} \end{aligned} \tag{6.3.11}$$

$$t(z+\alpha(h)) = t(z),\ h \in H. \tag{6.3.12}$$

Together, equations (6.3.10-6.3.12) imply that for each triple (Ξ,α,η), $t(-,\Xi,\alpha,\eta)$ is a constant multiple of the function $s(-,\Xi_H,\eta) = \gamma^H_* s(-,\Xi,\eta)$. We can compute this multiple using the last part of Lemma 6.1.5:

$$t(z,\Xi,\alpha,\eta) = c(\Xi,\alpha,\eta)s(z,\Xi_H,\eta)$$

where

$$\begin{aligned} c(\Xi,\alpha,\eta) &= \lim_{z\to 0}\frac{t(z,\Xi,\alpha,\eta)}{s(z,\Xi_H,\eta)} \\ &= \lim_{z\to 0}\frac{t'(z,\Xi,\alpha,\eta)}{s'(z,\Xi_H,\eta)} \\ &= \prod_{0\neq h\in H} s(\alpha(h),\Xi,\eta) \\ &= a_H(\Xi,\alpha,\eta). \end{aligned}$$

It follows that

$$F^{Ell}/H = \mathbb{G}_a^t = (F_H^{Ell})^{a_H x},$$

since t fits into the diagram of formal group laws over $\mathbb{Q}\otimes\widehat{Ell}^*_{p^k}$

$$\begin{array}{ccc} \mathbb{G}_a & \xrightarrow{\gamma^H_* s} & F_H^{Ell} \\ {\scriptstyle s}\downarrow & \searrow^{t} & \downarrow{\scriptstyle a_H} \\ F^{Ell} & \xrightarrow{f_H} & F^{Ell}/H. \ \square \end{array} \tag{6.3.13}$$

Proposition 6.3.14 *The stable operation*

$$MU^*X \xrightarrow{\nu Q^H} \frac{1}{a_H}\widehat{Ell}^*_{p^k}X$$

factors through the map

$$\frac{1}{a_H}Ell^*_{p^k}X \to \frac{1}{a_H}\widehat{Ell}^*_{p^k}kX.$$

Moreover, νQ^H factors through the orientation

$$MU^*X \xrightarrow{t_{Ell}} Ell^*X$$

to produce a stable ring operation

$$Ell^* \xrightarrow{\nu\Psi^H} \frac{1}{a_H}Ell^*_{p^k}X$$

for X a finite complex.

Proof: By (6.3.8), the homomorphism of formal group laws

$$F \to F_H^{Ell}$$

is given by the normalized quotient

$$\nu f_H(x) = \frac{1}{a_H}f_H(x) \in \frac{1}{a_H}Ell^*_{p^k}[\![x]\!],$$

and so one gets the operation νQ^H by (4.0.2). By (6.3.7), the ring homomorphism which classifies F_H^{Ell} is

$$\gamma^H \circ t_{Ell*}.$$

Since it factors through t_{Ell*}, we get $\nu\Psi^H$ by (4.0.3). □

Factoring Q^H through the orientation t_{Ell}

Combining (6.3.8) and (6.3.7) with (6.2.12), we obtain

Theorem 6.3.15 *Let $H \subset \Lambda_k^*$ be a subgroup of order n. The operation*

$$MU^{2*}X \xrightarrow{Q^H} \widehat{Ell}_{p^k}^{2n*}X$$

factors through the orientation

$$MU^*X \xrightarrow{t_{Ell}} Ell^*X$$

to produce an natural transformation of functors of finite complexes to rings

$$Ell^{2*}X \xrightarrow{\Psi^H} \widehat{Ell}_{p^k}^{2n*}X.$$

In particular, if $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ *is a complex line bundle, then*

$$\Psi^H(eL) = f_H(eL) \in Ell_{p^k}^{2n}X.$$

On coefficients, Ψ^H is given by

$$\Psi^H(m) = \beta^H(m) = (a_H)^{\frac{|m|}{2}}\gamma^H(m); \tag{6.3.16}$$

more succinctly, we have the equation

$$\beta_*^H F^{Ell} = F^{Ell}/H. \ \square$$

6.4 An example: the Adams operation.

When the machinery of 6.2 is applied to the full group Λ_k^*, one obtains an Adams operation for Ell. Let $N = p^k$, and write Ψ^N for $\Psi^{\Lambda_k^*}$. The main task is to use (6.3.8) to identify the formal group law F^{Ell}/N.

Lemma 6.4.1 *For the full subgroup Λ_k^* of N torsion points, the constant a_N of (6.3.8) is given by*

$$a_N = \prod_{\substack{0 \neq h \\ [N](h)=0}} \phi(h) = \pm N \varepsilon^{\frac{1-N^2}{4}}$$

Proof: The $\phi(h)$ are exactly the non-zero roots of the N–series for F^{Ell} by (6.2.9). The result follows from (6.2.8). □

Theorem 6.4.2 *The operation*

$$Ell^{2*}X \xrightarrow{\Psi^N} \widehat{Ell}_{p^k}^{2N^2*}X$$

factors through an operation

$$Ell^{2*}X \xrightarrow{\Psi^N} \frac{1}{p} Ell^{2N^2*}X.$$

The effect of Ψ^N on the Euler class of a line bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ is given by

$$\Psi^N(eL) = f_N(eL) = \pm\varepsilon^{\frac{1-N^2}{4}}[N]_{F^{Ell}}(eL). \tag{6.4.3}$$

On coefficients, Ψ^N acts by

$$\Psi^N(f) = (\pm\varepsilon^{\frac{1-N^2}{4}})^{\frac{|f|}{2}} f.$$

The associated stable operation

$$Ell^*(-) \xrightarrow{\nu\Psi^N} \frac{1}{p} Ell^*(-)$$

is the usual stable Adams operation whose effect on Euler classes is given by

$$\Psi^N(eL) = \frac{1}{N}[N]_{F^{Ell}}(eL). \tag{6.4.4}$$

Proof: According to (6.3.14), the effect of $\nu\Psi^N$ on coefficients is given by

$$\begin{aligned} &Ell^* \xrightarrow{\gamma^N} \frac{1}{p} Ell^* \\ &f \mapsto \{(\Xi, \eta) \mapsto f(\frac{1}{N}(\Xi, \eta)) = N^{-\frac{|f|}{2}} f(\Xi, \eta)\}. \end{aligned} \tag{6.4.5}$$

Moreover, the formal group law F_N^{Ell} has exponential

$$\begin{aligned}\exp_{F_N}(z) &= s(z, \frac{1}{N}\Xi, \frac{1}{N}\eta) \\ &= \frac{1}{N}s(Nz, \Xi, \eta) \\ &= \frac{1}{N}[N]_{Ell}s(z, \Xi, \eta).\end{aligned}$$

which proves (6.4.4). Since $\nu\Psi^N$ on coefficients and on Euler classes lands in $\frac{1}{p}Ell$, it is in fact an operation

$$Ell^*X \xrightarrow{\nu\Psi^H} \frac{1}{p}Ell^*X.$$

According to (4.0.7), νQ^N and Q^N are related by

$$\nu Q^N(x) = a_N^{-\frac{|x|N^2}{2}} Q^N(x).$$

Since a_N is a unit in $Ell^*_{p^k}$, it follows that

$$\begin{aligned}Q^N(x) &= a_N^{\frac{|x|}{2}} \nu Q^N(x) \\ &= (\pm N\varepsilon^{\frac{1-N^2}{4}})^{\frac{|x|}{2}} \nu Q^N(x) \in \frac{1}{p}Ell^{2N^2|x|}X.\end{aligned}$$

The proof is complete once we observe that by (6.3.8), the homomorphism

$$Ell^* \xrightarrow{\beta^N} \frac{1}{p}Ell^{N^2*}$$

such that

$$\beta^N_* F^{Ell} = F^{Ell}/N$$

is

$$\begin{aligned}\beta^N(f) &= a_N^{\frac{|x|}{2}} \gamma^N f \\ &= (\pm\varepsilon^{\frac{1-N^2}{4}})^{\frac{|f|}{2}} f \; \square\end{aligned}$$

6.5 Hecke operators as power operations

Associated to a finite subgroup $H \subset \Lambda_k^*$ and the Jacobi quartic C we have seen that there is a quotient curve C_H. The Hecke operator T_n on a modular form f is obtained by evaluating f on the quotient curve C_H as H runs over all subgroups of a given order. Specifically, the p-th Hecke operator T_p is given by

$$T_p f = \frac{1}{p} \sum_{\substack{H \subset \Lambda_1 \\ \#H = p}} \gamma^H f, \tag{6.5.1}$$

where γ^H is the map defined in (6.3.5). Indeed, the computations (6.3.6) in the proof of Lemma 6.3.4 are exactly the famous computation of the effect of T_p on the q-expansion of a modular form [Ser70].

Let T_p be the operation

$$T_p = \frac{1}{p} \sum_{\substack{H \subset \Lambda_1 \\ \#H = p}} \nu \Psi^H.$$

Theorem 6.5.2 (compare [Bak90]) *The operation*

$$T_p = \frac{1}{p} \sum_{\substack{H \subset \Lambda_1 \\ \#H = p}} \nu \Psi^H$$

is an additive operation

$$Ell^* X \longrightarrow \frac{1}{p} Ell^* X$$

Its effect on coefficients is given by (6.5.1).

Proof: The only thing to check is the claim about the range: *a priori*, the operation lands in

$$\frac{1}{A} Ell_p^* X,$$

where

$$A = \prod_{\substack{H \subset \Lambda_1 \\ \#H = p}} a_H.$$

At the end of this section we shall prove

Proposition 6.5.3 *Each of the constants a_H is a unit in Ell_p^*.*

Granting this, the operation T_p lands in $Ell_p^* X$. Visibly, it is invariant under $SL_2 Z/p$, so it lands in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p}] \otimes Ell^*(-)$ by (3.4.6). It is invariant under $\mathrm{Aut}[Z_0(p^k)/Z_0]$ because it is on coefficients and on Euler classes: on coefficients by the formulas (6.3.6), and on Euler classes because by equation (6.3.13), we have

$$f_H(s(z)) = a_H \gamma_*^H s(z) \in \mathbb{Q} \otimes Ell_p^* [\![z]\!]. \ \square$$

This operation is entirely analogous in its effect to Baker's [Bak90] Hecke operation $\overline{T}_p$ for his slightly different version of elliptic cohomology. What is new here is the realization of the operation as a sum of power operations $\nu \Psi^H$. It shows that the Hecke operator is a very natural operation in elliptic cohomology in much the way that the Adams operation is a natural operation in K-theory.

Our description of the situation in elliptic cohomology is far from ideal. A better approach would have used as the range of the character map a ring in which p is not a unit: for example, the ring of meromorphic modular forms for $Z_0(p^k)$ whose q-expansions have coefficients lying in the ring $\mathbb{Z}[\frac{1}{2}][\zeta_{2p^k}]$. Then, as in the case of K-theory and the E_h, one could use the unstable operation Ψ^H to prove integrality theorems about their stable counterparts. For example, we conjecture

Conjecture ("Conjecture G") *The operation $p^{r+1}T_p$ is integral on $2r$-dimensional classes.*

This conjecture would followsimmediately from the computation of the constants a_H in the proof of Proposition 6.5.3, below, once a better version of $Ell^*_{p^k}$ is available. Additionally, we would get the integrality theorem about the Adams operation Ψ^N analogous to (5.4.10).

All the constructions in this chapter would apply with this more stringent integrality condition: most crucially, the computations in (6.2.6) show that the roots of the p-series $\phi(a)$ have q-expansions with coefficients in $\mathbb{Z}[\frac{1}{2}][\zeta_{2p^k}]$. What is missing is a proof that the ring of such modular forms is flat over Ell^* and so defines a cohomology theory. In fact, Brylinski shows that such a ring is flat, but only for modular forms with even weight. I believe that the right ring for this discussion exists, is within reach, and is even possibly interesting in its own right; however, I don't know enough to say more about it, let alone use it, at this time.

Finally, I owe the reader a

6.6 Proof of Proposition 6.5.3.

This is simply a matter of computation. Let $\Xi(p)$ be the lattice

$$\Xi(p) = 4\pi i p\tau\mathbf{Z} + 4\pi i\mathbf{Z}.$$

Then the sugroups of $\mathbf{C}/\Xi(p)$ of order p correspond the superlattices

$$\Xi_j = 4\pi i(\tau + \frac{j}{p})\mathbb{Z} + 4\pi i\mathbf{Z},\ 0 \le j < p, \text{ and}$$
$$\Xi_p = 4\pi i p\tau\mathbb{Z} + \frac{4\pi i}{p}\mathbb{Z}.$$

Varying the level structure interchanges these lattices and so the constants a_H, so it is enough to fix the level structure, and check that $a_H(\Xi(p), \alpha, 2\pi i)$ is a unit in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]((q))$ for various H. Let the level structure be given by

$$\alpha(bf + ce) = 4\pi i\tau b + \frac{4\pi i c}{p}.$$

Let H be the subgroup of Λ_1^* generated by $f + je$, where $0 \le j < p-1$. Then we claim that

$$a_H(\Xi(p), \alpha, 2\pi i) = (-1)^{(1-p)/2}(\zeta q^{-1})^{(p^2-1)/4} \prod_{n \ge 1} \left[\frac{1 - q^n \zeta^{nj}}{(1 - q^{pn})^p} \right]^{2(-1)^n}, \qquad (6.6.1)$$

where $\zeta = e^{2\pi i/p}$. This is even a unit in $\mathbb{Z}[\frac{1}{2}][\zeta]((q))$. We do this in a series of steps. First of all, by (6.2.7),

$$\begin{aligned} a_H(\Xi(p), \alpha, 2\pi i) &= \prod_{r=1}^{p-1} s(4\pi i r\tau + \frac{4\pi i j r}{p}, \Xi(p), 2\pi i) \\ &= \prod_{r=1}^{p-1} \frac{(\zeta^{2jr} q^{2r} - 1)}{\zeta^{jr} q^r} \prod_{m \ge 1} \left\{ \frac{(1 - q^{mp+2r}\zeta^{2jr})(1 - q^{mp-2r}\zeta^{-2jr})}{(1 - q^{mp})^2} \right\}^{(-1)^m}. \qquad (6.6.2) \end{aligned}$$

We move the product over r inside, and collect terms. The terms of the form $(1-q^{mp})^2$ contribute a factor

$$\prod_{s \ge 1} \left[\frac{(1 - q^{2s-1})}{(1 - q^{2s})} \right]^{2(p-1)} \qquad (6.6.3)$$

to the infinite product. We treat the other part of the infinite product in two steps, the numerator and denominator. The numerator corresponds to m even in (6.6.2). It produces a term of the form

$$\prod_{s \ge 1} (1 - q^{2sp+2}\zeta^{2j}) \cdot \ldots \cdot (1 - q^{2sp+2(p-1)}\zeta^{2j(p-1)})$$
$$(1 - q^{2sp-2}\zeta^{-2j}) \cdot \ldots \cdot (1 - q^{2sp-2(p-1)}\zeta^{-2j(p-1)}).$$

Collecting terms in q^{2l}, we find for each l prime to p and also greater than p a term of the form

$$(1 - q^{2l}\zeta^{2jl})^2,$$

where one factor comes from the $(1 - q^{2sp+2r}\zeta^{2r})$ side, and the other from the $(1 - q^{2sp-2r}\zeta^{-2r})$ side.

When l is less than p, we get since $s \ge 1$ only one term of the form

$$1 - q^{2l}\zeta^{2jl}.$$

Here, however, the numerator of the leading factor

$$\prod_{r=1}^{p-1} \frac{(\zeta^{2jr} q^{2r} - 1)}{\zeta^{jr} q^r}$$

comes to the rescue, and contributes the missing terms

$$(\zeta^{2j}q^2 - 1)(\zeta^{4j}q^4 - 1)\ldots$$

up to a factor of $(-1)^{p-1} = 1$!

The denominator (m odd in (6.6.2)) is a product of the form

$$\prod_{s\geq 1}(1 - q^{(2s-1)p+2}\zeta^{2j}) \cdot \ldots \cdot (1 - q^{(2s-1)p+2(p-1)}\zeta^{2j(p-1)})$$
$$(1 - q^{(2s-1)p-2}\zeta^{-2j}) \cdot \ldots \cdot (1 - q^{(2s-1)p-2(p-1)}\zeta^{-2j(p-1)}). \quad (6.6.4)$$

Proceeding analogously, we find, upon collecting terms of the form $(1 - q^{2l-1}\zeta^{j(2l-1)})$, that for $2l-1$ prime to and greater than p, we get a term of the form

$$(1 - q^{2l-1}\zeta^{j(2l-1)})^2,$$

where one factor comes from the $(1 - q^{(2s-1)p+2r}\zeta^{2r})$ and the other from the $(1 - q^{(2s-1)p-2r}\zeta^{-2r})$ side. Now however there are no terms of the form $(1 - q^{2l-1}\zeta^{j(2l-1)})$ at all for $2l - 1 < p$. However, there are messy terms arising on the $(1 - q^{(2s-1)p-2r}\zeta^{-2r})$ side when $s = 1$: the factor is

$$(1 - q^{p-2}\zeta^{-2j})(1 - q^{p-4}\zeta^{-4})\ldots(1 - q^{-p+2}\zeta^{-2(p-1)j}). \quad (6.6.5)$$

It includes negative exponents of q, which would be very bad in the denominator. Once again, the denominator of the leading factor saves the day, this time by contributing a factor of

$$q^{1+2+\ldots+(p-1)}.$$

Multiplying (6.6.5) by this factor yields

$$\begin{aligned} &q^{2+4+\ldots+(p-1)}(q^{p-2} - \zeta^{-2(p-1)j})(q^{p-4} - \zeta^{-2(p-2)j}) \\ &\quad\ldots(q - \zeta^{(p+1)j})(1 - q\zeta^{(p-1)j})\ldots(1 - q^{(p-2)}\zeta^{-2j}) \\ &\quad = q^{(p^2-1)/4}(-1)^{(p-1)/2}\zeta^{-(p^2-1)j/4}(1 - q\zeta^j)^2\ldots(1 - q^{(p-2)}\zeta^{-2j})^2 \end{aligned}$$

which are exactly the missing terms.

Assembling the numerator, denominator, and (6.6.3) gives

$$\begin{aligned} a_H(\Xi(p), \alpha, 2\pi i) &= (-1)^{(1-p)/2}(\zeta q^{-1})^{(p^2-1)/4}\prod_{s\geq 1}\left[\frac{(1 - q^{2s-1})}{(1 - q^{2s})}\right]\prod_{\substack{s\geq 1\\(s,p)=1}}(1 - q^s\zeta^{sj})^{2(-1)^s} \\ &= (-1)^{(1-p)/2}(\zeta q^{-1})^{(p^2-1)/4}\prod_{n\geq 1}\left[\frac{1 - q^n\zeta^{nj}}{(1 - q^{pn})^p}\right]^{2(-1)^n} \end{aligned}$$

which is (6.6.1).

The remaining subgroup of Λ_1^* is the one generated by e. In that case, we have

$$a_H(\Xi p, \alpha, 2\pi i) = \prod_{r=1}^{p-1} \frac{(\zeta^{2r}-1)}{\zeta^r} \prod_{m\geq 1} \left\{ \frac{(1-q^{mp}\zeta^{2r})(1-q^{mp}\zeta^{-2r})}{(1-q^{mp})^2} \right\}^{(-1)^m}$$

Since p is odd, we get

$$\zeta^{1+2+\ldots+(p-1)} = \zeta^{p(p-1)/2} = 1$$

and

$$\prod_{r=1}^{p-1} (\zeta^{2r}-1) = p.$$

Also,

$$\begin{aligned} \prod_{r=1}^{p-1} (1-q^{mp}\zeta^{\pm 2r}) &= 1 + q^{mp} + \ldots + q^{mp(p-1)} \\ &= \frac{1-q^{mp^2}}{1-q^{mp}}. \end{aligned}$$

Putting these together, we find

$$a_H(\Xi p, \alpha, 2\pi i) = p \prod_{m\geq 1} \left[\frac{1-q^{mp^2}}{(1-q^{mp})^p} \right]^{2(-1)^m},$$

which is a unit in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]((q))$. □

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