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STOCHASTIC HEAT EQUATION LIMIT OF A $(2 + 1)$ D GROWTH MODEL

ALEXEI BORODIN, IVAN CORWIN, AND FABIO LUCIO TONINELLI

ABSTRACT. We determine a $q \rightarrow 1$ limit of the two-dimensional q -Whittaker driven particle system on the torus studied previously in [4]. This has an interpretation as a $(2 + 1)$ -dimensional stochastic interface growth model, that is believed to belong to the so-called anisotropic Kardar-Parisi-Zhang (KPZ) class. This limit falls into a general class of two-dimensional systems of driven linear SDEs which have stationary measures on gradients. Taking the number of particles to infinity we demonstrate Gaussian free field type fluctuations for the stationary measure. Considering the temporal evolution of the stationary measure, we determine that along characteristics, correlations are asymptotically given by those of the $(2 + 1)$ -dimensional additive stochastic heat equation. This confirms (for this model) the prediction that the non-linearity for the anisotropic KPZ equation in $(2 + 1)$ -dimension is irrelevant.

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1. INTRODUCTION

The two-dimensional q -Whittaker particle system on the torus was introduced by Corwin and Toninelli [4]. The state of this system is interlacing collections of particles on the two-dimensional torus. Particles jump right by one on their row according to exponential clocks whose rates are determined by certain nearest neighbor inter-particle distances – see (2.8) – as well as a parameter $q \in [0, 1)$. As discussed in [4], the dynamics can be seen also as a stochastic growth process for a random discrete $(2 + 1)$ -dimensional interface. The mapping is based on the fact that interlacing particle configurations correspond to perfect matchings (dimer coverings) of the hexagonal lattice, and the associated integer-valued height function provides the discrete interface. In the present work, we adopt the point of view of the particle system rather than that of the growth model.

These dynamics originated in Borodin and Corwin’s study of Macdonald processes [1] wherein they were defined on certain triangular arrays of interlacing particles. Based on that inspiration as well as a recent treatment by Toninelli in [14] of the $q = 0$ case of this system, [4] proposed and then verified that certain local Gibbs measures are stationary for the dynamics – see (2.9) and Proposition 1 for a summary of those results.

When $q = 0$, [14] determined that stationary measures are simply uniform measures on particle configurations, which are known to enjoy a relation to determinantal point processes and to have Gaussian Free Field type fluctuations in the infinite volume limit for the torus. In [14] it was also shown that the dynamics remain well-defined in the infinite volume limit and bounds on the scale of fluctuations of the associated height function for the system were determined. In particular, it was shown that height function fluctuations grow at a smaller rate than any polynomial in time t (and that the bound can be improved to $O(\sqrt{\log t})$ in a certain range of particle densities).

The $q = 0$ model is predicted (cf. [3]) to be in the anisotropic $(2 + 1)$ -dimensional Kardar-Parisi-Zhang universality class which is represented by the stochastic PDE

$$\frac{\partial h}{\partial t}(t; x, y) = \frac{1}{2} \Delta h(t; x, y) + (\nabla h, Q \nabla h)(t; x, y) + \xi(t; x, y) \quad (1.1)$$

where $h(t; x, y)$ represents a height function at time t and position (x, y) , Δ is the Laplacian in x and y , Q is a 2×2 matrix with signature $(+, -)$ and $\xi(t; x, y)$ is a space-time white noise. This is called anisotropic because of the mixed signature on the non-linearity, whereas when the signature of Q is $(+, +)$ or $(-, -)$ the model is called “isotropic”. In 1991, Wolf [15] predicted that the fluctuations of the anisotropic equation should grow like $\sqrt{\log t}$ and moreover that the non-linearity should be irrelevant and the long-time behavior of the system should be exactly as that of the two-dimensional additive stochastic heat equation (i.e., the equation with Q set to zero).

Wolf’s prediction of $\sqrt{\log t}$ fluctuations was demonstrated numerically soon after by Halpin-Healy and Assdah [10]. Prähofer and Spohn [12] considered a microscopic model related to the Gates-Westcott model, and demonstrated through exact calculation this fluctuation scaling. Borodin and Ferrari [3] studied a (non-periodic) triangular array variant of the $q = 0$ case of the q -Whittaker particle system and, using the technology of Schur processes they proved the $\sqrt{\log t}$ scaling and further demonstrated convergence to a Gaussian free field as time goes to infinity. The occurrence of a Gaussian free field is consistent with the prediction of convergence to the additive stochastic heat equation since the Gaussian free field is stationary for the time evolution of that stochastic PDE. It should be noted that since [3] dealt with dynamics on a triangular array of particles, the Gaussian Free Field fluctuations only appear after a suitable coordinate change. This coordinate change is not visible from the renormalization group arguments of Wolff. In this paper we focus on translation invariant models for which no coordinate change is necessary.

We should note that (1.1) is not, a priori, well-defined because of the non-linearity and the fact that solutions are not function valued, but rather generalized functions (like the Gaussian free field). We are not aware of any rigorous treatment of this equation, though it may eventually fall into the class of systems which can be defined through Hairer’s regularity structures [9]. As such, none of Wolf’s predictions have been proved for (1.1) itself. Let us also note that the story is quite different when the model is isotropic – see recent numerical studies of Halpin-Healy [11].

To our knowledge, the present paper, together with the forthcoming work of Borodin, Corwin and Ferrari [2] in the context of these dynamics on triangular arrays, is the first instance in which the full space-time picture has been rigorously established for a model in the $(2 + 1)$ -dimensional anisotropic KPZ class converging to the additive stochastic heat equation. To be upfront about things, we do not prove this convergence as a process (which would require some additional tightness estimates), but rather in terms of the correlation structure for Gaussian processes.

The initial motivation for this paper was the desire to extend the study of [3, 14] to the $q \neq 0$ case. In that case the Schur / determinantal point process structure is lost. To overcome this impediment, we decided to consider a Gaussian limit of the model, hoping that calculations there would become sufficiently doable without said structure. In particular, we consider the $q = e^{-\epsilon} \rightarrow 1$ limit of the particle system, as we simultaneously scale the torus width and height like ϵ^{-1} . We start particles spaced on the ϵ^{-1} scale according to a certain crystalline configuration (see the beginning of Section 3) with smaller $\epsilon^{-1/2}$ scale fluctuations. Speeding time up by ϵ^{-1} , we prove (Theorem 1) that particle positions (multiplied by ϵ) have asymptotically a constant speed v and that fluctuations (multiplied by $\epsilon^{1/2}$) converge (as a space-time process) to a limiting system of SDEs. Likewise, under this scaling the stationary measure on the q -Whittaker system converges to a Gaussian measure (Lemma 3). (Note: we do not prove that the stationary measure concentrates on the crystalline configuration, though it is certainly compelling to conjecture this).

Once in the setting of SDEs with Gaussian stationary measures (in fact, the stationary measures are on gradients) we are able to use Fourier transforms to explicitly compute the space-time correlations as the number of particles goes to infinity (Theorem 2) as well as the correlations and Gaussian free field limit of the stationary measure (Theorem 3). Theorem 2 has a number of corollaries. Corollary 1 shows that for fixed, large time t , fluctuations grow like $\sqrt{\log t}$ and correlations decay in a spatial range of order $t^{1/2}$. Corollary 2 considers the correlations along space-time lines. There exists a distinguished direction U along which correlations exist in a temporal scale of order

t and a spatial scale of order $t^{1/2}$. In fact, Corollary 3 shows that in this scale, the correlations converge to those of the (2 + 1)-dimensional additive stochastic heat equation – thus validating Wolf’s prediction for this model. On the other hand, Corollary 2 also shows that for space-time direction not equal to U , the correlations decay to zero on a $t^{1/2}$ time-scale, thus much faster than along U .

We call the direction U the “characteristic” direction. The reason is that the lines $y = Ut$ are the characteristic curves of a PDE that we conjecture to describe the hydrodynamic limit of our model when the initial condition is not crystalline, see Section 3.2. There is a close analogy with what happens in the context of (1 + 1)-dimensional particle systems. In that setting, the hydrodynamics are described by Hamilton-Jacobi conservation laws which can be solved by computing the characteristics and propagating initial data along them. Characteristics are computed as the derivative of the flux with respect to the local slope. At a more microscopic level, initial data fluctuations are propagated along characteristics. In particular, one has “slow decorrelation” [7, 5] along characteristics whereby fluctuations along these space-time directions decorrelate far slower than along other space-time directions. The phenomenon of slow decorrelation along characteristics was conjectured (with some supporting evidence) in [3] to hold for the two-dimensional q -Whittaker particle system at $q = 0$.

The aforementioned results concerning correlations of two-dimensional systems of SDEs are actually proved below in much broader generality. In particular, all results are proved provided that the SDEs take the form of (3.6) with the matrix A satisfying the conditions of Proposition 2. This could reflect the expected universality of the (2 + 1)-dimensional anisotropic KPZ class. It would be compelling to see if any of these universality results can be proved directly for the general q system, without first taking the SDE limit. Without the Gaussian structure, though, it is unclear how to proceed in this goal.

As mentioned above, in the $q = 0$ case, there exists a triangular array variant of the q -Whittaker particle system which was studied at length in [3] using Schur processes. The triangular variant of the general q case relates to q -Whittaker processes [1] and though the system is no longer determinantal, there are many useful formulas provided through the technology of Macdonald processes. In a parallel paper to this, Borodin, Corwin and Ferrari [2] develop the analogous $q \rightarrow 1$ limit of this triangular variant of the particle system and explore the limits of the exact formulas and their applications (in particular, concerning the asymptotic behavior of correlations along certain space-time directions).

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2. MODEL AND NOTATION

We start by recalling the definition of the q -Whittaker particle system on the torus, and its stationary measure as defined in [4]. We consider an interacting particle system in which particles live on the $L \times N$ discrete torus $\mathbb{T}_{L,N} = \mathbb{Z}/(L\mathbb{Z}) \times \mathbb{Z}/(N\mathbb{Z})$. The horizontal size is L and the vertical size is N .

The particle configuration space will be denoted $\Omega_{L,N;m_1,m_2}$, and it depends on two integers $1 < m_1 < L$ and $1 \leq m_2 < N$ such that

$$m_1/L + m_2/N < 1. \quad (2.1)$$

At each site $x = (x_1, x_2) \in \mathbb{T}_{L,N}$ there is at most one particle. On each row there are exactly m_1 particles. We exclude $m_1 = 1$ and $m_1 = L$ to avoid trivialities. The parameter m_2 has a more topological nature and its meaning will be explained below.

The horizontal position of particle p is denoted $x_p \in \mathbb{Z}/(L\mathbb{Z})$. Particle positions are interlaced, in the following sense. Given particle p (say on row i), we let p_1, p_4 denote its right/left neighbor on the same row (note that if $m_1 = 2$ then $p_1 = p_4$). Then, we require that in row $i - 1$ there is exactly one particle, labeled p_2 , whose position satisfies

$$x_{p_2} \in \{x_p + 1, x_p + 2, \dots, x_{p_1}\} \quad (2.2)$$

and exactly one particle, labeled p_3 , satisfying

$$x_{p_3} \in \{x_{p_4} + 1, x_{p_4} + 2, \dots, x_p\}. \quad (2.3)$$

See Figure 1. Note that, automatically, in row $i + 1$ there are exactly one particle p_5 and one particle p_6 satisfying respectively

$$x_{p_5} \in \{x_{p_4}, \dots, x_p - 1\}, \quad x_{p_6} \in \{x_p, \dots, x_{p_1} - 1\}. \quad (2.4)$$

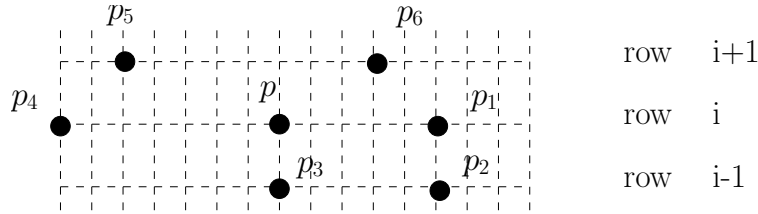


FIGURE 1. The neighbors p_1, \dots, p_6 of particle labeled p . Note that conditions (2.2), (2.3) allow $C_p := x_p - x_{p_3} = 0$ but they impose $B_p + 1 := x_{p_2} - x_p \geq 1$.

We define non-negative integers A_p, \dots, F_p as

$$\begin{aligned} A_p &= x_{p_1} - x_p - 1; & B_p &= x_{p_2} - x_p - 1; & C_p &= x_p - x_{p_3} \\ D_p &= x_p - x_{p_4} - 1; & E_p &= x_p - x_{p_5} - 1; & F_p &= x_{p_6} - x_p. \end{aligned} \quad (2.5)$$

The particles p_1, \dots, p_6 are the six neighbors of p , labeled clockwise starting from the one on the right. The definition of the dynamics will be such that the labels of the neighbors of a particle p do not change with time (particles will not jump over each other or change interlacements).

Let $\Omega_{L,N;m_1}$ be the set of particle occupation functions, i.e. of functions $\eta : \mathbb{T}_{L,N} \mapsto \{0, 1\}$, with m_1 particles (i.e. occupation variables equal to 1) per row, whose positions satisfy the constraints (2.2)-(2.4). The set $\Omega_{L,N;m_1}$ decomposes into disjoint ‘sectors’:

$$\Omega_{L,N;m_1} = \bigcup_{m_2} \Omega_{L,N;m_1,m_2} \quad (2.6)$$

as follows. Given any particle p , connect p to its up-right neighbor p_6 , then p_6 with its own up-right neighbor and repeat the operation until the path Γ thus obtained gets back to the starting particle p . Note that Γ forms a simple loop: otherwise, there would be a particle r which is reached along Γ from two different particles r', r'' . This is impossible, since both r' and r'' would be the lower-left neighbor r_3 of r . Call $N_v \in \mathbb{N} \cup \{0\}$, $N_h \in \mathbb{N} \cup \{0\}$ the vertical and horizontal winding numbers of Γ

around the torus $\mathbb{T}_{L,N}$. It is easy to see that N_h, N_v are independent of the chosen initial particle p . As discussed in [4, Remark 2],

$$m_2 := m_1 \frac{N_h}{N_v} \quad (2.7)$$

is an integer and it satisfies (2.1). The set $\Omega_{L,N;m_1,k}$ is defined as the subset of $\Omega_{L,N;m_1}$ with $m_2 = k$. Each sector $\Omega_{L,N;m_1,m_2}$ will remain invariant under our dynamics.

Let us briefly remark that the particle configurations we are considering can also be mapped onto dimers on the periodized $L \times N$ hexagonal lattice. This perspective is explained in [4] (see in particular Fig. 2 there) wherein $n_1 := m_1 N$ corresponds to the number of vertical dimers, $n_2 := m_2 L$ to the number of north-west dimers and $n_3 := NL - n_1 - n_2$ the number of north-east dimers. We will not pursue this perspective any further here.

Given a configuration $\eta \in \Omega_{L,N;m_1,m_2}$, draw a directed upward edge from any particle label r to its up-right neighbor r_6 if $F_r = 0$ (in which case r and r_6 have the same horizontal position). For the particle labeled by p let V_p^+ be the set that includes p plus the particle labels that can be reached from p by following upward oriented edges. The dynamics we consider is a continuous-time Markov chain on $\Omega_{L,N;m_1,m_2}$. For each p , there is an exponential clock of rate

$$\frac{(1 - q^{B_p})(1 - q^{D_{p+1}})}{(1 - q^{C_p+1})} \quad (2.8)$$

with $q \in [0, 1)$. When said clock rings, all particles with label $r \in V_p^+$ shift by $(1, 0)$. Note that the rate is zero if $B_p = 0$. This prevents particles from overlapping after the move. Note also that after the move, the configuration is still in $\Omega_{L,N;m_1,m_2}$. This is discussed in more detail in [4]. Another way to understand the dynamics is that when particle p moves, provided its up-right neighbor $p_6 \in V_p^+$, then C_{p_6} becomes -1 and the jump rate for p_6 becomes infinite, and hence it immediately moves (and so on for all other $r \in V_p^+$). These dynamics are called the q -Whittaker particle system on the torus. As a side remark, let us add that, in terms of dimer model, shift to the right by $+1$ of a family V_p^+ corresponds to increasing the height by 1 in $|V_p^+|$ faces of the hexagonal lattice.

Given $q \in [0, 1)$, let π be the probability measure on $\Omega_{L,N;m_1,m_2}$ defined as

$$\pi(\sigma) := \frac{1}{Z_{L,N;m_1,m_2}} \prod_p \frac{(q; q)_{A_p}}{(q; q)_{B_p} (q; q)_{C_p}} \mathbf{1}_{\{\sigma \in \Omega_{L,N;m_1,m_2}\}} \quad (2.9)$$

where $(q; q)_n = (1 - q)(1 - q^2) \dots (1 - q^n)$ and $Z_{L,N;m_1,m_2}$ is the normalizing constant necessary to make this a probability measure. The main result of [4] (Theorem 1 therein) is:

Proposition 1. *The probability law π is stationary in time for q -Whittaker particle system on the torus.*

Clearly, the measure π is not reversible, since the process is totally asymmetric.

We will consider a certain $q \rightarrow 1$ limit of the particle system. To fix scalings, for $\epsilon > 0$ let $q = \exp(-\epsilon)$. For simplicity (to avoid a plethora of $[\cdot]$), we will assume ϵ^{-1} is an integer, though all results hold for arbitrary ϵ . Further, let $L = \epsilon^{-1} \ell$ for some integer ℓ . On each row there are $m := m_1 = n_1/N$ particles, with m an integer of order ℓ . Also for simplicity we take $N = m$, so altogether we have $n_1 = m^2 = O(\ell^2)$ particles. Taking N to be another multiple of m would not change our results qualitatively.

Remark 1. *On \mathbb{Z}^2 , introduce the equivalence relation \sim such that $p \sim p'$ iff $p = p' + (j_1 m - j_2 m_2, j_2 m)$ for some $j_1, j_2 \in \mathbb{Z}$. Observe that if we take $N = m$ steps along Γ starting from p , we get a particle p' that is the j^{th} right neighbor of p on the same row, for some $0 \leq j \leq m$. Actually, one has $j = m_2$, since $N_v j = N_h m$. Particles will be given a label $p \in \mathbb{Z}^2$ such that*

$p_1 = p + (1, 0), p_2 = p + (1, -1), p_3 = p + (0, -1), p_4 = p - (1, 0), p_5 = p + (-1, 1), p_6 = p + (0, 1)$, with an arbitrary choice of which particle is labeled $p = (0, 0)$. Thanks to the above observation, given integers j_1, j_2 , particle $p + (j_1 m_1, j_2 N)$ is to be identified with $p + (j_2 m_2, 0)$. In other words, particles are labeled by $p \in \mathcal{R}_m := \mathbb{Z}^2 / \sim$ (the quotient set of \mathbb{Z}^2 by \sim , which contains m^2 equivalence classes).

We will first take the limit $\epsilon \rightarrow 0$ with m and m_2 fixed, and then $m \rightarrow \infty$ with the ratio m_2/m bounded away from 0 and 1 if we want to take an infinite-volume limit ($m_2/m \leq 1$ by definition, recall (2.1) and $m = N$).

The average inter-particle distance along a row is

$$\frac{D}{\epsilon} := \frac{L}{m} = \frac{\ell}{\epsilon m} \quad (2.10)$$

so that, with reference to (2.5), the average value of D_p is $\epsilon^{-1}D$.

The average value of C_p is instead fixed by choosing m_2 suitably, as follows. Let $C \in (0, D)$ be defined by

$$\frac{C}{D} = \frac{m_2}{m} \quad (2.11)$$

and let $B = D - C$. (Note that, as long as the integers m, m_2 are kept finite, C can take only a finite set of values in $(0, D)$; such set becomes dense in $(0, D)$ if we allow m, m_2 to diverge with a fixed ratio).

We claim that the average value of C_p in any particle configuration is $\epsilon^{-1}C$. Indeed, take the path Γ defined after (2.6). It visits exactly $N_v \times N = mN_v$ particles and its total displacement to the right equals $N_h \times L$. On the other hand, $N_h \times L$ is also the sum of the C_p over all particles in Γ . From this, we see that the average value of C_p is

$$\frac{N_h L}{N_v m} = \frac{m_2 L}{m^2} = \epsilon^{-1} \frac{m_2}{m} \frac{\epsilon L}{m} = \epsilon^{-1} \frac{m_2}{m} D \quad (2.12)$$

where in the first equality we used (2.7), and in the third (2.10). Eq. (2.11) then allows to conclude.

Similarly, we see that the average of B_p is $\epsilon^{-1}B$. On the other hand, the averages of D_p, E_p, F_p equal by definition the averages of A_p, B_p, C_p respectively.

3. CONVERGENCE TO A SYSTEM OF SDES

We will start the dynamics from an initial condition where each particle $p \in \mathcal{R}_m$ is within distance $O(\epsilon^{-1/2})$ from its ‘‘ideal position’’ X_p in a perfect ‘‘crystalline configuration’’ where $D_p = \epsilon^{-1}D, B_p = \epsilon^{-1}B, C_p = \epsilon^{-1}C$ for every p . Assuming without loss of generality that $X_0 = 0$, we have for $p = (p^{(1)}, p^{(2)})$

$$X_p = p^{(1)} \epsilon^{-1}D + p^{(2)} \epsilon^{-1}C \quad (3.1)$$

where the r.h.s. has to be taken modulo $\epsilon^{-1}\ell$.

Our first result (proven in Section 5) says that particles move macroscopically with a deterministic speed $v(C, D) > 0$ and that, in the time-scale of order ϵ^{-1} , fluctuations around such hydrodynamic limit are of order $\epsilon^{-1/2}$ and converge to a system of linear SDEs.

Theorem 1. Fix $N = m$ and ℓ , so that the lattice $\Lambda_{L, N}$ depends only on ϵ . Let $\{\bar{\xi}_p\}_{p \in \mathcal{R}_m} \in \mathbb{R}^{\mathcal{R}_m}$. Consider an initial configuration σ_0 such that, defining

$$\eta_p := \eta_{p, \epsilon} = \sqrt{\epsilon}(x_p - X_p), \quad (3.2)$$

one has

$$\lim_{\epsilon \rightarrow 0} \eta_p = \bar{\xi}_p \quad \forall p \in \mathcal{R}_m. \quad (3.3)$$

Define

$$v(C, D) = \frac{(1 - \exp(-B))(1 - \exp(-D))}{1 - \exp(-C)} \quad (3.4)$$

(recall that $B = D - C$) and

$$\eta_{p,t} = \sqrt{\epsilon} (x_p(t/\epsilon) - X_p - \epsilon^{-1}vt), \quad t \geq 0. \quad (3.5)$$

Then, the random process $\{\eta_{p,\cdot}\}_{p \in \mathcal{R}_m}$ converges weakly as $\epsilon \rightarrow 0$ to the solution of the system of linear stochastic differential equations

$$\begin{cases} d\xi_{p,t} = \sqrt{v} dW_{p,t} + \sum_{p'} A_{p,p'} \xi_{p',t} dt \\ \xi_{p,0} = \bar{\xi}_p \end{cases} \quad (3.6)$$

with

$$A_{p,p'} = \delta_{p'=p} \left(\frac{e^{-D}(1 - e^{-B})}{1 - e^{-C}} - \frac{e^{-B}(1 - e^{-D})}{1 - e^{-C}} - \frac{e^{-C}(1 - e^{-B})(1 - e^{-D})}{(1 - e^{-C})^2} \right) \quad (3.7)$$

$$+ \delta_{p'=p+(1,-1)} \frac{e^{-B}(1 - e^{-D})}{1 - e^{-C}} - \delta_{p'=p-(1,0)} \frac{e^{-D}(1 - e^{-B})}{1 - e^{-C}} \quad (3.8)$$

$$+ \delta_{p'=p-(0,1)} \frac{e^{-C}(1 - e^{-B})(1 - e^{-D})}{(1 - e^{-C})^2} \quad (3.9)$$

and where $dW_{p,t}$ are independent white noises indexed by p , and one-dimensional in time t .

Remark 2. Note that the matrix A is not symmetric, so that the diffusion (3.6) is not reversible with respect to its stationary measure described by Theorem 3. In other words, the irreversibility of the microscopic dynamics survives also in the Gaussian limit.

3.1. Properties of the matrix A . It is convenient to work in Fourier space. For this purpose, let

$$f_k : \mathcal{R}_m \mapsto \mathbb{C}, \quad f_k : p \mapsto \frac{1}{m} e^{-ipk}. \quad (3.10)$$

The set $\{f_k\}_{k \in \mathcal{K}_m}$, where

$$\mathcal{K}_m = \left\{ \left(\frac{C}{D} r_1 + r_2 \right), r_1, r_2 \in \mathbb{Z}, -m/2 \leq r_1, r_2 < m/2 \right\}, \quad (3.11)$$

forms an orthonormal basis of $\mathbb{C}^{\mathcal{R}_m}$.

Remark 3. Note that \mathcal{K}_m was chosen such that, if we define $f_k(p) = (1/m)e^{-ipk}$ for every $p \in \mathbb{Z}^2$, then $f_k(p) = f_k(p')$ if $p \sim p'$ (use that $m_2/m = C/D$, see (2.11)). Also, observe that $|\mathcal{K}_m| = m^2$.

Define

$$\hat{\xi}_{k,t} = \sum_{p \in \mathcal{R}_m} \xi_{p,t} f_k(p), \quad (3.12)$$

so that

$$\xi_{p,t} = \sum_{k \in \mathcal{K}_m} \hat{\xi}_{k,t} \overline{f_k(p)}, \quad \text{and} \quad \hat{\xi}_{-k,t} = \overline{\hat{\xi}_{k,t}}. \quad (3.13)$$

Let also

$$\hat{A}(k) = \sum_p A_{p,0} e^{-ipk}, \quad k \in \mathbb{R}^2. \quad (3.14)$$

In our specific case, one sees that

$$\hat{A}(k) = A_{0,0} + A_{0,(1,-1)}e^{-i(k_1-k_2)} + A_{0,(-1,0)}e^{ik_1} + A_{0,(0,-1)}e^{ik_2} \quad (3.15)$$

$$\hat{A}(k) + \hat{A}(-k) = 2[A_{0,0} + A_{0,(1,-1)} \cos(k_1 - k_2)] \quad (3.16)$$

$$+ A_{0,(-1,0)} \cos(k_1) + A_{0,(0,-1)} \cos(k_2)]. \quad (3.17)$$

Observe that we defined $\hat{A}(k)$ for any $k \in \mathbb{R}^2$ and not just for $k \in \mathcal{K}_m$.

Proposition 2. *The matrix A satisfies the following properties:*

- (1) *Translation invariance: $A_{p,p'} = A_{p+r,p'+r}$ for every r ;*
- (2) $\hat{A}(0) = \sum_{p'} A_{p,p'} = 0$;
- (3)

$$\hat{R}(k) := \hat{A}(k) + \hat{A}(-k) \leq 0 \quad (3.18)$$

and the only zero of $\hat{R}(k)$ on $[-\pi, \pi]^2$ is at $k = 0$;

(4)

$$\hat{R}(k) = \hat{W}(k) + O(|k|^3) = (k, \hat{W}k) + O(|k|^3), \quad k \rightarrow 0, \quad (3.19)$$

with \hat{W} a strictly negative definite 2×2 matrix and $\hat{W}(k)$ the associated quadratic form.

- (5) *The function $\hat{A}(\cdot)$ is 2π -periodic and C^∞ on $[-\pi, \pi]^2$.*

All properties are trivial to check, except for (3) which is proven in Appendix B. The fact that

$$\hat{W}(k) := -A_{0,(1,-1)}(k_1 - k_2)^2 - A_{0,(-1,0)}k_1^2 - A_{0,(0,-1)}k_2^2 \quad (3.20)$$

is a strictly negative definite quadratic form follows from negativity of $\hat{R}(k)$, together with the fact that

$$\det(\hat{W}) = \frac{e^{-D}(1 - e^{-D})(1 - e^{-B})^2}{(1 - e^{-C})^2} =: w^2 > 0. \quad (3.21)$$

For later convenience, let the 2×2 matrix V be such that

$$V\hat{W}V^T = -I. \quad (3.22)$$

Let us also define

$$U := i\nabla\hat{A}(0) = \sum_p pA_{0,p} = (A_{0,(1,-1)} - A_{0,(-1,0)}, -A_{0,(1,-1)} - A_{0,(0,-1)}) \in \mathbb{R}^2, \quad (3.23)$$

and note that $U \neq 0$ for any choice of $B, C, D = B + C$.

Remark 4. *A few comments are in order:*

- *Property (1) (translation invariance of A) originates from the fact that we are considering fluctuations around a “trivial”, translation invariant hydrodynamic limit where particles have positions $X_p + \epsilon^{-1}vt$ and are therefore equi-spaced at all times. This is the property that makes it convenient to work in Fourier space (e.g., property (1) is behind the fact that Eq. (6.1) is diagonal in k);*
- *Property (2) means that the drift of a particle p is unchanged if all particles are globally shifted by the same amount. This is a consequence of the fact that transition rates (2.8) of the microscopic particle process depend only on inter-particle distances, and would hold even if we studied fluctuations w.r.t. a non-translation invariant hydrodynamic limit (cf. Section 3.2);*

- *Property (3) guarantees that there exists a stationary measure for gradients $\xi_p - \xi_{p'}$, see Theorem 3 and formula (6.9). Negative-definiteness of $\hat{A}(k) + \hat{A}(-k)$ corresponds to the fact that in the hydrodynamic scaling the particle configuration is “crystalline” at all times, and that crystalline configurations are an (at least local) maximum of the stationary measure $\pi(\cdot)$, see Lemma 3.*

3.2. A conjectural hydrodynamic equation. We will briefly consider the hydrodynamic behavior of the q -Whittaker particle system with general initial data (which may not be close to the crystalline configurations considered above). We provide a heuristic derivation of the hydrodynamic (law of large numbers) PDE satisfied by the limit. For us, the purpose of this derivation is to justify and explain our use of the word “characteristic” to describe the direction U of slow decorrelation. In this hydrodynamic limit we will first take $\epsilon \rightarrow 0$ and then $m \rightarrow \infty$. With this in mind, let us rescale space and time by m/ϵ and define

$$h(\tau, y) = (\epsilon/m)x_{\lfloor ym \rfloor}(\tau m/\epsilon)$$

with $y \in [0, 1]^2$, $\tau \geq 0$, and where the r.h.s. has to be taken modulo $(\ell/\epsilon) \times (\epsilon/m) = D$ because x_p is defined modulo $L = \ell/\epsilon$. Note that, as a consequence of Remark 1, $h(\tau, \cdot)$ satisfies

$$h(\tau, y + (j_1, j_2)) = h(\tau, y + (Cj_2/D, 0)), \quad j_1, j_2 \in \mathbb{Z}.$$

If at time zero the configuration satisfies the conditions of Theorem 1, then $h(0, y)$ is close to a linear function:

$$H(0, y) := \lim_{m \rightarrow \infty} \lim_{\epsilon \rightarrow 0} h(0, y) = Dy_1 + Cy_2.$$

In this case, it follows from Theorem 1 that, for $\tau > 0$, the limit

$$H(\tau, y) := \lim_{m \rightarrow \infty} \lim_{\epsilon \rightarrow 0} h(\tau, y) \tag{3.24}$$

exists and solves

$$\partial_\tau H = v \tag{3.25}$$

with $v(C, D)$ as in (3.4). Given that $D = \partial_{y_1} H$ and $C = \partial_{y_2} H$, and using $B = D - C$, we see that

$$v(C, D) = v(\nabla H) = \frac{(1 - \exp(\partial_{y_2} H - \partial_{y_1} H))(1 - \exp(-\partial_{y_1} H))}{(1 - \exp(-\partial_{y_2} H))}. \tag{3.26}$$

If we assume instead that the initial condition satisfies

$$\lim_{m \rightarrow \infty} \lim_{\epsilon \rightarrow 0} h(0, y) = H(0, y),$$

with $H(0, \cdot)$ some smooth enough but non-linear function, then it is natural to conjecture that the limit (3.24) exists and still satisfies (3.25), with $v = v(\nabla H)$ equal to the r.h.s. of (3.26). It is easy to see that the characteristic lines of the PDE (3.25) are the straight lines $y = Ut$ with U as in (3.23). In fact, the characteristic lines are determined by

$$y_i(t) = t \nabla_i v, \quad i = 1, 2$$

with $\nabla_i v$ the derivative of $v = v(\nabla h)$ with respect to its i -th argument. Explicitly,

$$\nabla_1 v = \frac{e^{\partial_{y_2} H - \partial_{y_1} H} (1 - e^{-\partial_{y_1} H})}{1 - e^{-\partial_{y_2} H}} + \frac{e^{-\partial_{y_1} H} (1 - e^{\partial_{y_2} H - \partial_{y_1} H})}{1 - e^{-\partial_{y_2} H}} = A_{0, (1, -1)} - A_{0, (-1, 0)} = U_1$$

and similarly $\nabla_2 v = U_2$. We emphasize that the identity $\partial_i v = U_i$ is not a coincidence. Indeed, view the hydrodynamic speed v as a function of the relative horizontal distances between particle, say, 0 and the other particles p (in our case, the dependence is only through the three neighbors p_2, p_3, p_4 of 0). On one hand, since the SDEs (3.6) describe a linearization of the stochastic dynamics around the hydrodynamic limit, $A_{0,p}$ is obtained taking the derivative of v w.r.t. the position x_p of particle

p , with the others kept fixed. On the other hand, if the slope $\nabla_i H$ is changed by ϵ , the distance between particles 0 and $p = (p^{(1)}, p^{(2)})$ changes by $\epsilon \times p^{(i)}$. Therefore,

$$\nabla_i v = \sum_{p=(p^{(1)}, p^{(2)})} p^{(i)} A_{0,p} = U_i. \quad (3.27)$$

This shows that the identity $U_i = \nabla_i v$ is not related to the particular form of the function $v(\cdot)$. If we had another interacting particle process for which we could prove convergence to a hydrodynamic equation and convergence of fluctuations to a system of linear SDEs with a matrix A satisfying the conditions in Proposition 2, we would have automatically that the direction of slow decay of correlations would coincide with the characteristic lines of the PDE.

4. CORRELATIONS OF THE GAUSSIAN SYSTEM

In this section we study the space-time correlations of the SDE system (3.6). Since the equations are linear, they can be solved explicitly.

We formulate the results in wider generality. Again, the solution to (3.6) with initial condition $\bar{\xi}$ is denoted $\{\xi_{p,t}\}_{t \geq 0, p \in \mathcal{R}_m}$ and we let $\mathbb{P}_{\bar{\xi}}$ be its law. The matrix $A = \{A_{p,p'}\}_{p,p' \in \mathcal{R}_m}$ is not necessarily given by (3.7) but is required to satisfy properties (1)–(5) of Proposition 2. Theorems 2 and 3 hold in this generality.

Let $\mathcal{C}^{\bar{\xi}}(t, s)$ be the $m^2 \times m^2$ symmetric matrix

$$\mathcal{C}_{p,p'}^{\bar{\xi}}(t, s) = \mathbb{E}_{\bar{\xi}}[\xi_{p,t} \xi_{p',s}], \quad p, p' \in \mathcal{R}_m \quad (4.1)$$

and $\mathcal{M}^{\bar{\xi}}(t) \in \mathbb{R}^m$ be the vector

$$\mathcal{M}_p^{\bar{\xi}}(t) = \mathbb{E}_{\bar{\xi}}[\xi_{p,t}], \quad p \in \mathcal{R}_m \quad (4.2)$$

with of course $\mathcal{C}_{p,p'}^{\bar{\xi}}(0, 0) = \bar{\xi}_p \bar{\xi}_{p'}$, $\mathcal{M}_p^{\bar{\xi}}(0) = \bar{\xi}_p$. By Itô's lemma,

$$\frac{d}{dt} \mathcal{M}_p^{\bar{\xi}} = \sum_{p'} A_{p,p'} \mathcal{M}_{p'}^{\bar{\xi}}(t), \quad (4.3)$$

$$\frac{d}{dt} \mathcal{C}^{\bar{\xi}}(t, t) = v \mathbb{I} + \mathcal{C}^{\bar{\xi}}(t, t) A^T + A \mathcal{C}^{\bar{\xi}}(t, t),$$

$$\frac{d}{dt} \mathcal{C}(t, s) = A \mathcal{C}(t, s), \quad t > s,$$

with A^T the transpose of A and \mathbb{I} the identity matrix.

Let also

$$\mathcal{W}_{p,p'}^{\bar{\xi}}(t, s) := \mathcal{C}_{p,p'}^{\bar{\xi}}(t, s) - \mathcal{M}_p^{\bar{\xi}}(t) \mathcal{M}_{p'}^{\bar{\xi}}(s). \quad (4.4)$$

Theorem 2. *Let A satisfy the properties of Proposition 2. Then, $\mathcal{W}_{p,p+y}^{\bar{\xi}}$ does not depend on $\bar{\xi}$ or p and the limit*

$$\mathbb{W}_y(t, s) = \lim_{m \rightarrow \infty} \mathcal{W}_{p,p+y}^{\bar{\xi}}(t, s) \quad (4.5)$$

exists for any given $y \in \mathbb{Z}^2$. Moreover,

$$\mathbb{W}_y(t, s) = \frac{v}{4\pi w} \int_{1+(t-s)/2}^{1+(t+s)/2} \frac{e^{-|H|^2/(4a)}}{a} da + j(t, s, y), \quad (4.6)$$

where $w > 0$ is defined in (3.21), $H = Vy + (t-s)VU$, V is defined in (3.22), $U = i\nabla \hat{A}(0)$, and j satisfies

$$\sup_{t,s,y} |j(t, s, y)| < \infty, \quad \lim_{\max(t-s, |y|) \rightarrow \infty} j(t, s, y) = 0. \quad (4.7)$$

If the matrix A is the one given in (3.7), using (3.21) and (3.4), one sees that

$$\frac{v}{w} = \sqrt{e^D - 1}. \quad (4.8)$$

From (4.6) one can obtain all desired asymptotics. For instance, one can obtain equal-time correlations.

Corollary 1 (Equal-time correlations). *For $y = 0$ one has*

$$\limsup_{t \rightarrow \infty} \left| \mathbb{W}_0(t, t) - \frac{v}{4\pi w} \log t \right| < \infty. \quad (4.9)$$

For $y \neq 0$ one finds, with $Y = Vy$ and after the change of variables $a = |Y|^2/(4x)$,

$$\mathbb{W}_y(t, t) = \frac{v}{4\pi w} \int_{|Y|^2/4(1+t)}^{|Y|^2/4} \frac{e^{-x}}{x} dx + j(t, t, y). \quad (4.10)$$

This implies that

$$\lim_{\substack{|y| \rightarrow \infty, t \rightarrow \infty \\ |y| = O(\sqrt{t})}} \left(\mathbb{W}_y(t, t) - \frac{v}{4\pi w} \log(4(t+1)/|Y|^2) \right) = 0 \quad (4.11)$$

where we used the fact that

$$c_- \leq \frac{|Y|}{|y|} \leq c_+ \quad (4.12)$$

for some non-zero constants c_{\pm} (the lower bound holds because the determinant of V is not zero).

If instead both $|y|, t$ diverge and $|y| \gg \sqrt{t}$, then $\mathbb{W}_y(t, t) = o(1)$.

The same results hold if $t - s = O(1)$.

As for correlations at different times, the following result shows that the behavior is special along the space-time lines $y = tU$ (that will be called ‘‘characteristics’’, in view of the discussion in Section 3.2), with $U = i\nabla \hat{A}(0) = \sum_p p A_{p,0} \in \mathbb{R}^2$. We will assume that $t - s \gg 1$, since the case $t - s = O(1)$ is effectively covered by the previous Corollary.

Corollary 2 (Correlations along the characteristics). *We deduce from (4.6)*

$$\limsup_{t-s \rightarrow \infty} \left(\mathbb{W}_{\lfloor U(t-s) \rfloor}(t, s) - \frac{v}{4\pi w} \log \frac{t+s}{t-s} \right) = 0, \quad (4.13)$$

in particular $\mathbb{W}_{\lfloor U(t-s) \rfloor}(t, s)$ is large if $t - s \ll t$.

If instead $u \neq U$, we have

$$\limsup_{t-s \rightarrow \infty} \left(\mathbb{W}_{\lfloor u(t-s) \rfloor}(t, s) - \frac{v}{4\pi w} \int_{\frac{(t-s)^2 |V(U-u)|^2}{2(t+s)}}^{\infty} \frac{e^{-x}}{x} dx \right) = 0; \quad (4.14)$$

from this we deduce that

$$\limsup_{t \rightarrow \infty, (t-s)/\sqrt{t} \rightarrow \infty} \mathbb{W}_{\lfloor u(t-s) \rfloor}(t, s) = 0, \quad (4.15)$$

while

$$\limsup_{t-s \rightarrow \infty, t-s=O(\sqrt{t})} \left| \mathbb{W}_{\lfloor u(t-s) \rfloor}(t, s) - \frac{v}{4\pi w} [\log t - 2 \log(t-s)] \right| < \infty. \quad (4.16)$$

Summarizing: along the characteristic correlations are large as soon as $(t - s) \ll t$ and grow proportionally to $\log t$ if $t - s \leq t^a$, $a < 1$. For all other space-time directions, correlations are large only if $t - s \ll \sqrt{t}$.

Equation (4.13) shows that, along the characteristic, there is a limit for the correlation as $t, s \rightarrow \infty$ with $t - s$ and $t + s$ of comparable size. This suggests that in this scaling the whole fluctuation field near the characteristic has a non-trivial limit process which, in the following statement, we identify as being related to the Stochastic Heat Equation.

Corollary 3. *Define*

$$h_{x,t} = a\xi_{[tU+V^{-1}x],t}, \quad a = \sqrt{\frac{4\pi w}{8v}}. \quad (4.17)$$

Then, for any given x, y , $0 < s < t$, from (4.6) we get

$$\lim_{\delta \rightarrow 0} \left[\mathbb{E}_{\xi}(h_{\delta^{-1/2}x, \delta^{-1}t} h_{\delta^{-1/2}y, \delta^{-1}s}) - \mathbb{E}_{\xi}(h_{\delta^{-1/2}x, \delta^{-1}t}) \mathbb{E}_{\xi}(h_{\delta^{-1/2}y, \delta^{-1}s}) \right] \quad (4.18)$$

$$= \frac{1}{8} \int_{(t-s)/2}^{(t+s)/2} \frac{e^{-|x-y|^2/(4a)}}{a} da. \quad (4.19)$$

In other words, in this limit the randomly evolving height field h_{\cdot} has the same space-time correlations as the additive stochastic heat equation in 2 spatial dimensions,

$$\partial_t h = \Delta h + \dot{W} \quad (4.20)$$

with \dot{W} the $(2+1)$ -dimensional space-time white noise (compare (4.18) with the formal expression derived in [8, Formula (2.8)] for the covariance of the stochastic heat equation).

Remark 5. If the particle label p is d -dimensional, $d \neq 2$, and the matrix A satisfies the analog the properties (1)–(5) of Proposition 2 (with $[-\pi, \pi]^2$ replaced by $[-\pi, \pi]^d$), then one can check that Theorem 2 still holds, with (4.6) replaced by

$$\mathbb{W}_y(t, s) = \frac{\pi v}{(2\pi)^d w} \int_{1+(t-s)/2}^{1+(t+s)/2} \frac{e^{-|H|^2/(4a)}}{a^{d/2}} da + j(t, s, y), \quad y \in \mathbb{Z}^d. \quad (4.21)$$

We leave it to an interested reader to derive the analogs of Corollaries 1–3 for $d \neq 2$.

4.1. Stationary measure. We cannot expect that there is a stationary measure for $\{\xi_p\}_p$: since there is invariance by global shifts on \mathbb{R} of all positions ξ_p , the inverse covariance matrix will have a zero mode (like a Gaussian Free Field not pinned to zero anywhere). However, the stationary measure on gradients of ξ will be well-defined. Not surprisingly, its scaling limit (which requires letting $m \rightarrow \infty$) is the standard massless Gaussian Free Field (see e.g. [13] for definitions), up to an affine transformation of coordinates by the matrix V (cf. (4.29)).

Theorem 3. *Let A satisfy the properties of Proposition 2. There exists a unique stationary measure μ for the gradients $(\xi_p - \xi_{p'})_{p, p' \in \mathcal{R}_m}$. This measure is Gaussian and its mean and covariances are given by*

$$\mathbb{E}_{\mu}(\xi_p - \xi_{p'}) = 0 \quad \forall p, p' \quad (4.22)$$

and

$$\text{Cov}_{\mu}[(\xi_{y_1} - \xi_{y_2}); (\xi_{y_3} - \xi_{y_4})] = -\frac{v}{m^2} \sum_{k \in \mathcal{K}_m} \frac{(e^{iky_1} - e^{iky_2})(e^{-iky_3} - e^{-iky_4})}{\hat{R}(k)} \quad (4.23)$$

$$\xrightarrow{m \rightarrow \infty} -\frac{v}{(2\pi)^2} \int_{[-\pi, \pi]^2} dk \frac{(e^{iky_1} - e^{iky_2})(e^{-iky_3} - e^{-iky_4})}{\hat{R}(k)} \quad (4.24)$$

$$\equiv \text{Cov}_{\mu_{\infty}}[(\xi_{y_1} - \xi_{y_2}); (\xi_{y_3} - \xi_{y_4})]. \quad (4.25)$$

Moreover, one has

$$\text{Cov}_{\mu_\infty}[(\xi_{y_1} - \xi_{y_2}); (\xi_{y_3} - \xi_{y_4})] = \frac{v}{2\pi w} \log \frac{1 + |V(y_1 - y_4)||V(y_3 - y_2)|}{1 + |V(y_1 - y_3)||V(y_2 - y_4)|} + R_{y_1, y_2, y_3, y_4} \quad (4.26)$$

where V is the matrix in (3.22) and¹

$$R_{y_1, y_2, y_3, y_4} = O(1/(1 + \min(|y_1 - y_3|, |y_2 - y_4|, |y_1 - y_4|, |y_2 - y_3|))). \quad (4.27)$$

From this we deduce a convergence to a massless GFF on \mathbb{R}^2 , of covariance proportional to

$$-\log |V(x - y)|,$$

in the following sense: Let $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be a C^∞ , compactly supported function such that $\int_{\mathbb{R}^2} \phi(x) dx = 0$. Then, the zero-average random function

$$\xi_\phi := \delta^2 \sum_p \phi(\delta p)(\xi_p - \xi_0) \quad (4.28)$$

converges in law, in the limit $\lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty}$, to a centered Gaussian random variable of variance

$$-\frac{v}{2\pi w} \int \phi(x)\phi(y) \log |V(x - y)| dx dy. \quad (4.29)$$

Remark that, given $\phi^{(i)}$, $i = 1, 2$ satisfying the same assumptions as ϕ above, the limit covariance $\lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \mathbb{E}_\mu(\xi_{\phi^{(1)}} \xi_{\phi^{(2)}})$ can be simply deduced via

$$2\mathbb{E}_\mu(\xi_{\phi^{(1)}} \xi_{\phi^{(2)}}) = \mathbb{E}_\mu(\xi_{\phi^{(1)}}^2) + \mathbb{E}_\mu(\xi_{\phi^{(2)}}^2) - \mathbb{E}_\mu((\xi_{\phi^{(1)} - \phi^{(2)}})^2). \quad (4.30)$$

In particular, (4.26) gives

$$\limsup_{|y| \rightarrow \infty} \left| \text{Var}_{\mu_\infty}[\xi_0 - \xi_y] - \frac{v}{\pi w} \log |Vy| \right| < \infty \quad (4.31)$$

(using (4.12), one can replace Vy with y).

5. CONVERGENCE TO THE SDES: PROOF OF THEOREM 1

5.1. **A tightness estimate.** Let

$$\mathcal{G} = \left\{ \sigma \in \Omega_{L, N; m_1, m_2} : \forall p, \left| B_p - \frac{B}{\epsilon} \right| \leq \frac{\log(1/\epsilon)}{\sqrt{\epsilon}}, \left| D_p - \frac{D}{\epsilon} \right| \leq \frac{\log(1/\epsilon)}{\sqrt{\epsilon}} \right\}. \quad (5.1)$$

Note that the initial condition σ_0 is well inside \mathcal{G} , cf. (3.3). The crucial ingredient in the proof of Theorem 1 is the following a-priori tightness estimate:

Lemma 1. *Let $\partial\mathcal{G} \subset \mathcal{G}$ denote the inner boundary of \mathcal{G} (the set of configurations from which the dynamics can exit \mathcal{G} with a single update) and define the stopping time*

$$\tau_{\mathcal{G}} = \inf\{t > 0 : \sigma(t) \in \partial\mathcal{G}\}. \quad (5.2)$$

Then,

$$\lim_{\epsilon \rightarrow 0} \mathbb{P}_{\sigma_0}(\tau_{\mathcal{G}} \leq \epsilon^{-2}) = 0. \quad (5.3)$$

The time ϵ^{-2} could be replaced by any ϵ^{-a} . What we need is that $a > 1$, so that this time is much larger than $1/\epsilon$, the time-scale of convergence to the SDEs.

Proof of Lemma 1. Let us start with the following:

¹here, $|y_i - y_j|$ denotes the Euclidean distance between y_i and y_j on \mathbb{Z}^2 and not on the ‘torus’ \mathcal{R}_m (recall that the limit $m \rightarrow \infty$ has already been taken).

Lemma 2. *Let $a = \epsilon^{-1}b + X$ with $b > 0$ and $\sqrt{\epsilon}|X| \ll \epsilon^{-1/10}$. The following asymptotic expansion holds as $\epsilon \rightarrow 0$:*

$$(q; q)_a = \exp \left[\frac{1}{\epsilon} \sum_{n \geq 1} \frac{1}{n^2} e^{-bn} - \frac{1}{2} \sum_{n \geq 1} \frac{1}{n} e^{-bn} \right] \quad (5.4)$$

$$\left. - X \sum_{n \geq 1} \frac{1}{n} e^{-bn} + \epsilon \frac{X^2}{2} \frac{e^{-b}}{1 - e^{-b}} + C(\epsilon) + o(1) \right] \quad (5.5)$$

with $C(\epsilon)$ independent of b and X .

See Appendix A for the proof.

Lemma 3. *If $\sigma \in \mathcal{G}$ and if as in (3.3) we let $\eta_p := \sqrt{\epsilon}(x_p - \epsilon^{-1}P_p)$, then*

$$\pi(\sigma) = \frac{1}{Z} \exp \left(v \sum_{k \in \mathcal{K}_m} \hat{R}(k) |\hat{\eta}_k|^2 + o(1) \right) \quad (5.6)$$

where $\pi(\cdot)$ is defined in (2.9) and, as in (3.12), we set

$$\hat{\eta}_k = \sum_{p \in \mathcal{R}_m} \eta_p f_k(p), \quad \eta_p = \sum_{k \in \mathcal{K}_m} \hat{\eta}_k \overline{f_k(p)}. \quad (5.7)$$

We recall from Proposition 2 that $\hat{R}(k) \leq 0$ and vanishes only for $k = 0$.

Proof of Lemma 3. From Lemma 2 we see that

$$\pi(\sigma) = \frac{1}{Z} \exp \left(-\frac{1}{2}(\eta, Q\eta) + o(1) \right) \quad (5.8)$$

with Z a normalization constant independent of σ and

$$-(\eta, Q\eta) = \sum_p (\eta_p - \eta_{p+(1,0)})^2 \frac{e^{-D}}{1 - e^{-D}} \quad (5.9)$$

$$- \sum_p (\eta_p - \eta_{p+(1,-1)})^2 \frac{e^{-B}}{1 - e^{-B}} - \sum_p (\eta_p - \eta_{p+(0,-1)})^2 \frac{e^{-C}}{1 - e^{-C}}. \quad (5.10)$$

This can be rewritten in Fourier space as

$$-\frac{1}{2}(\eta, Q\eta) = \sum_k |\hat{\eta}_k|^2 \hat{Q}(k) \quad (5.11)$$

$$\hat{Q}(k) = \frac{e^{-D}}{1 - e^{-D}} (1 - \cos(k_1)) - \frac{e^{-B}}{1 - e^{-B}} (1 - \cos(k_1 - k_2)) \quad (5.12)$$

$$- \frac{e^{-C}}{1 - e^{-C}} (1 - \cos(k_2)). \quad (5.13)$$

Here, we use for instance that

$$\sum_p (\eta_p - \eta_{p+(1,0)})^2 = \sum_p \sum_{k, k' \in \mathcal{K}_m} \hat{\eta}_k \hat{\eta}_{-k'} \overline{f_k(p)} f_{k'}(p) (1 - e^{ik_1})(1 - e^{-ik'_1}) \quad (5.14)$$

$$= \sum_{k \in \mathcal{K}_m} |\hat{\eta}_k|^2 (2 - 2 \cos(k_1)) \quad (5.15)$$

where we used orthonormality of $\{f_k(\cdot)\}_{k \in \mathcal{K}_m}$ and the second of (3.13). One checks that

$$\hat{Q}(k) = v(\hat{A}(k) + \hat{A}(-k)) = v\hat{R}(k) \quad (5.16)$$

and the proof is concluded. \square

Given any Markov chain with stationary measure π , generator \mathcal{L} and two states x, y , we have for any $t > 0$

$$P_t(x, y) \leq \frac{\pi(y)}{\pi(x)}. \quad (5.17)$$

To see this, write

$$P_t(x, y) = [\delta_x e^{t\mathcal{L}}](y) \leq \left[\frac{\pi}{\pi(x)} e^{t\mathcal{L}} \right](y) = \frac{\pi(y)}{\pi(x)} \quad (5.18)$$

where we used $\pi(\cdot)/\pi(x) \geq \delta_x(\cdot)$. For the initial configuration, one has from (5.8)

$$\pi(\sigma_0) \geq \frac{1}{Z} e^{-K_1} \quad (5.19)$$

for some finite constant K_1 depending on $\bar{\xi}$ (and, typically, of order m^2). If instead $\sigma \in \partial\mathcal{G}$, then one has

$$\pi(\sigma) \leq \frac{1}{Z} e^{-K_2(\log \epsilon)^2} \quad (5.20)$$

for some strictly positive K_2 (this is proven below). Finally, observe that

$$|\partial\mathcal{G}| = O(\epsilon^{-K_3}) \quad (5.21)$$

for some constant K_3 depending on m . This is trivial: indeed, the total number of configurations is bounded by $(\ell/\epsilon)^{m^2}$ (recall the definition of the model, the fact that ℓ is independent of ϵ and that there are m^2 particles). Using (5.17), (5.19), (5.20) and (5.21), we deduce

$$\mathbb{P}(\sigma(t) \in \partial\mathcal{G}) \leq \exp(-K_4(\log \epsilon)^2). \quad (5.22)$$

As a consequence,

$$\mathbb{E} \left[\int_0^{\epsilon^{-2}+1} 1_{\{\sigma(t) \in \partial\mathcal{G}\}} dt \right] \leq (\epsilon^{-2} + 1) \exp(-K_4(\log \epsilon)^2). \quad (5.23)$$

Next observe that, if $\tau_{\mathcal{G}} < \epsilon^{-2}$, then there exists a probability at least $\delta > 0$ independent of ϵ that the total time spent in $\partial\mathcal{G}$ up to time $\epsilon^{-2} + 1$ is at least $\delta \times \epsilon$. This is simply because the maximal transition rate of the Markov chain is of order ϵ^{-1} (this is the case when C_p in (2.8) is of order 1). In conclusion,

$$\mathbb{E} \left[\int_0^{\epsilon^{-2}+1} 1_{\sigma(t) \in \partial\mathcal{G}} dt \middle| \tau_{\mathcal{G}} \leq \epsilon^{-2} \right] \geq \delta^2 \epsilon. \quad (5.24)$$

Together with (5.23) we deduce

$$\mathbb{P}(\tau_{\mathcal{G}} \leq \epsilon^{-2}) \leq \frac{1}{\delta^2 \epsilon} \mathbb{E} \left[\int_0^{\epsilon^{-2}+1} 1_{\{\sigma(t) \in \partial\mathcal{G}\}} dt \right] \leq \exp(-K_5(\log \epsilon)^2) \quad (5.25)$$

and (5.3) is proven.

It remains to prove (5.20). Recall that $(\hat{A}(k) + \hat{A}(-k))$ is negative for every $k \in \mathcal{K}_m$ and vanishes only for $k = 0$, so that

$$-\frac{1}{2}(\eta, Q\eta) \leq -\delta \sum_{k \in \mathcal{K}_m, k \neq 0} |\hat{\eta}_k|^2 \quad (5.26)$$

for some positive δ that depends only on the number of particles m^2 .

If $\sigma \in \partial\mathcal{G}$ we must have either $|\eta_p - \eta_{p+(1,0)}| \geq (1/2) \log(1/\epsilon)$ or $|\eta_p - \eta_{p+(1,-1)}| \geq (1/2) \log(1/\epsilon)$ for some p . Assume to fix ideas that the former is the case. Then, from (5.7) (and writing \sum_k instead of $\sum_{k \in \mathcal{K}_m}$)

$$\frac{1}{2} \log(1/\epsilon) \leq |\eta_p - \eta_{p+(1,0)}| = \left| \sum_k \hat{\eta}_k \overline{f_k(p)} (e^{ik_1} - 1) \right| \quad (5.27)$$

$$= \left| \sum_{k \neq 0} \hat{\eta}_k \overline{f_k(p)} (e^{ik_1} - 1) \right| \leq 2 \sqrt{\sum_{k \neq 0} |\hat{\eta}_k|^2} \sqrt{\sum_k |f_k(p)|^2} = 2 \sqrt{\sum_{k \neq 0} |\hat{\eta}_k|^2}. \quad (5.28)$$

Then, (5.20) immediately follows. \square

5.2. Proof of Theorem 1. Let $U(t) = \{U_p(t)\}_{p \in \mathcal{R}_m}$ be defined as

$$U_p(t) = \frac{1}{\sqrt{\epsilon}} \int_0^t [-v + r_p(\sigma(s/\epsilon))] ds \quad (5.29)$$

where $r_p(\sigma)$ is the rate at which particle p jumps $+1$ to the right in the configuration σ (this includes the event that it jumps because it is pushed by another particle, i.e. because clock of particle p' rings and $p \in V_{p'}^+$). Note that $\{M_p(t)\}_t \equiv \{\eta_{p,t} - U_p(t)\}_t$ is a martingale, since we have

$$\frac{d}{dt} U_p(t) = \partial_s \mathbb{E}[\eta_{p,t+s} | \{\eta_{p',t}\}_{p'}] \Big|_{s=0^+}. \quad (5.30)$$

Next, define $V(t) = \{V_{p,p'}(t)\}_{p,p'}$ as solution to

$$\frac{d}{dt} V_{p,p'}(t) = \partial_s \mathbb{E}[M_{p,t+s} M_{p',t+s} | \{\eta_{r,t}\}_r] \Big|_{s=0^+} \quad V_{p,p'}(0) = 0, \quad (5.31)$$

so that $\{M_p(t)M_{p'}(t) - V_{p,p'}(t)\}_t$ is again a martingale for every (p, p') .

We will apply [6, Theorem 4.1, Chapter 7], that gives a set of sufficient conditions on the processes η, U and V that imply that $\{\eta_{p,t}\}_{t \geq 0, p \in \mathcal{R}_m}$ converges weakly to the solution of (3.6)². In particular, conditions (4.3)–(4.5) in [6] are trivial. For (4.6), we have to check that, for any $T > 0$,

$$\sup_{t \leq T} \left| U_p(t) - \int_0^t \sum_{p'} A_{p,p'} \eta_{p',s} ds \right| \xrightarrow{\mathbb{P}} 0 \quad (5.32)$$

as $\epsilon \rightarrow 0$. On the event $\{\tau_{\mathcal{G}} > \epsilon^{-2}\}$, whose probability tends to 1 as $\epsilon \rightarrow 0$ thanks to Lemma 1, no particle can push any other in configuration $\sigma(s/\epsilon)$ for any $s/\epsilon \leq \epsilon^{-2}$ (particles are far away from each other and all the families V_p^+ defined just before (2.8)) include only the particle p itself). Then, $r_p(\sigma(s/\epsilon))$ equals the r.h.s. of (2.8) for any $s \leq T$. We have

$$r_p(\sigma(s/\epsilon)) = \frac{(1 - e^{-B - \sqrt{\epsilon}(\eta_{p+(1,-1),s} - \eta_{p,s})})(1 - e^{-D - \sqrt{\epsilon}(\eta_{p,s} - \eta_{p-(1,0),s}) - \epsilon})}{1 - e^{-C - \sqrt{\epsilon}(\eta_{p,s} - \eta_{p-(0,1),s}) - \epsilon}} \quad (5.33)$$

$$= v + \sqrt{\epsilon} \left[e^{-B} \frac{1 - e^{-D}}{1 - e^{-C}} (\eta_{p+(1,-1),s} - \eta_{p,s}) + e^{-D} \frac{1 - e^{-B}}{1 - e^{-C}} (\eta_{p,s/\epsilon} - \eta_{p-(1,0),s}) \right. \quad (5.34)$$

$$\left. - e^{-C} \frac{(1 - e^{-B})(1 - e^{-D})}{(1 - e^{-C})^2} (\eta_{p,s} - \eta_{p-(0,1),s}) \right] + O(\epsilon \log(1/\epsilon)) \quad (5.35)$$

$$= v + \sqrt{\epsilon} \sum_{p'} A_{p,p'} \eta_{p',s} + O(\epsilon \log(1/\epsilon)) \quad (5.36)$$

where $O(\epsilon \log(1/\epsilon))$ is uniform on $s \leq T$. We used that, on \mathcal{G} , $|\eta_{p-b} - \eta_p| \leq \log(1/\epsilon)$, $b \in \{(-1, 1), (1, 0), (0, 1)\}$. Plugging this into (5.29) we get (5.32).

²A warning on notations: in [6], η is called X , U is called B and V is called A

Finally, condition (4.7) in [6, Theorem 4.1, Chapter 7] amounts in our case to requiring that

$$\sup_{t \leq T} |V_{p,p'}(t) - vt\delta_{p,p'}| \xrightarrow{\mathbb{P}} 0. \quad (5.37)$$

Indeed, one has (using that $\{M_{p,t}\}_{t \geq 0}$ is a martingale)

$$\frac{d}{dt} V_{p,p'}(t) = \partial_s \mathbb{E} [N_{p,t+s} N_{p',t+s} | \{\eta_{r,t}\}_r] |_{s=0^+} \quad (5.38)$$

where

$$N_{p,t+s} = \sqrt{\epsilon} x_p(\epsilon^{-1}(t+s)) - \frac{1}{\sqrt{\epsilon}} \int_0^{t+s} r_p(\sigma(\epsilon^{-1}u)) du.$$

Note that, by definition of the jump rate $r_p(\sigma)$,

$$\partial_s \mathbb{E} \left[x_p(\epsilon^{-1}(t+s)) \int_0^{t+s} r_{p'}(\sigma(\epsilon^{-1}u)) du \middle| \{\eta_{r,t}\}_r \right]_{s=0^+} \quad (5.39)$$

$$= \epsilon^{-1} r_p(\epsilon^{-1}t) \int_0^t r_{p'}(\sigma(\epsilon^{-1}u)) du + x_p(\sigma(\epsilon^{-1}t)) r_{p'}(\sigma(\epsilon^{-1}t)). \quad (5.40)$$

Therefore, one sees that

$$\frac{d}{dt} V_{p,p'}(t) = -x_p(t/\epsilon) r_{p'}(\sigma(t/\epsilon)) - x_{p'}(t/\epsilon) r_p(\sigma(t/\epsilon)) \quad (5.41)$$

$$+ \epsilon \partial_s \mathbb{E} [x_p((t+s)/\epsilon) x_{p'}((t+s)/\epsilon) | \{\eta_{r,t}\}_r] |_{s=0^+}. \quad (5.42)$$

If $p = p'$, this gives

$$\frac{d}{dt} V_{p,p'}(t) = -2x_p(t/\epsilon) r_p(\sigma(t/\epsilon)) + r_p(\sigma(t/\epsilon))(1 + 2x_p(t/\epsilon)) \quad (5.43)$$

$$= r_p(\sigma(t/\epsilon)) \quad (5.44)$$

where in the first step we used the fact that when x_p jumps +1 (which happens with rate r_p), x_p^2 increases by $(x_p + 1)^2 - x_p^2 = 1 + 2x_p$. Recall that, on the event $\{\tau_G \geq \epsilon^{-2}\}$, we have $r_p(\sigma(t/\epsilon)) = v + o(1)$, cf. (5.33). Since the probability of $\{\tau_G \geq \epsilon^{-2}\}$ tends to 1, this implies (5.37) for $p = p'$. As for $p \neq p'$, on the event $\{\tau_G \geq \epsilon^{-2}\}$ the particles p and p' cannot jump simultaneously since all particles are well spaced all the time so that no particle can push any other. Then, on this event,

$$\epsilon \partial_s \mathbb{E} [x_p((t+s)/\epsilon) x_{p'}((t+s)/\epsilon) | \{\eta_{r,t}\}_r] |_{s=0^+} \quad (5.45)$$

$$= r_{p'}(\sigma(t/\epsilon)) x_p(t/\epsilon) + r_p(\sigma(t/\epsilon)) x_{p'}(t/\epsilon) \quad (5.46)$$

so that $\frac{d}{dt} V_{p,p'}(t) = 0$ for every $t \leq T$ and (5.37) follows.

6. SPACE-TIME CORRELATIONS OF THE SDES

6.1. Proof of Theorems 2. In Fourier space, Eqs. (4.3) give

$$\frac{d}{dt} \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t}) = \hat{A}(k) \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t}), \quad (6.1)$$

$$\frac{d}{dt} \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t} \hat{\xi}_{k',t}) = v \delta_{k+k'=0} + \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t} \hat{\xi}_{k',t}) (\hat{A}(k') + \hat{A}(k)) \quad (6.2)$$

$$\frac{d}{dt} \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t} \hat{\xi}_{k',s}) = \hat{A}(k) \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t} \hat{\xi}_{k',s}), \quad t > s \quad (6.3)$$

whose solution is (just differentiate w.r.t. t to check)

$$\mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t}) = e^{\hat{A}(k)t} \hat{\xi}_{k,0} \quad (6.4)$$

$$\mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t} \hat{\xi}_{k',t}) = v \delta_{k=-k'} \frac{e^{\hat{R}(k)t} - 1}{\hat{R}(k)} + \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t}) \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k',t}), \quad (6.5)$$

$$\mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t} \hat{\xi}_{k',s}) = v \delta_{k=-k'} e^{\hat{A}(k)(t-s)} \frac{e^{\hat{R}(k)s} - 1}{\hat{R}(k)} + \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t}) \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k',s}), \quad t \geq s. \quad (6.6)$$

Note that it immediately follows that $\mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t} \hat{\xi}_{k',s}) - \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k,t}) \mathbb{E}_{\bar{\xi}}(\hat{\xi}_{k',s})$ (and therefore $\mathcal{W}_{p,p'}^{\bar{\xi}}(t,s)$) is independent of the initial condition $\bar{\xi}$.

Using the first of (3.13) together with (6.6) we find, for $t \geq s$,

$$\mathcal{W}_{p,p+y}^{\bar{\xi}}(t,s) = \frac{v}{m^2} \sum_{k \in \mathcal{K}_m} e^{-iky} e^{\hat{A}(k)(t-s)} \frac{e^{\hat{R}(k)s} - 1}{\hat{R}(k)} \quad (6.7)$$

$$\xrightarrow{m \rightarrow \infty} \frac{v}{(2\pi)^2} \int_{[-\pi, \pi]^2} dk \frac{e^{\hat{R}(k)s} - 1}{\hat{R}(k)} e^{(t-s)\hat{A}(k)} e^{-iky}. \quad (6.8)$$

Observe that, as $m \rightarrow \infty$, the set \mathcal{K}_m fills the parallelogram

$$\mathcal{K}_\infty := \{(k_1, k_2) \in \mathbb{R}^2 : |k_1| \leq \pi, |k_2 - (C/D)k_1| \leq \pi\}.$$

However, since the integrand of (6.8) is 2π -periodic, integrating over \mathcal{K}_∞ or on $[-\pi, \pi]^2$ gives the same result.

Recall property (3) in Proposition 2. The singularity at $(0,0)$ is integrable and the dominant contribution to the integral comes from $k \sim 0$. From (6.8) one deduces (4.6), see Appendix C.1 for details.

6.2. Proof of Theorem 3. The gradients $\xi_{p,t} - \xi_{p',t}$ can be written via (3.13) as linear combinations of the Fourier components $\{\hat{\xi}_{k,t}\}_{k \in \mathcal{K}_m, k \neq 0}$. Since the random variables $\hat{\xi}_{k,t}$ solve a set of linear SDEs, their invariant measure μ (if it exists) is Gaussian. Stationarity of μ implies, via (6.1) and (6.2), that for every $k, k' \in \mathcal{K}_m, k, k' \neq 0$,

$$\mathbb{E}_\mu(\hat{\xi}_k) = 0 \quad (6.9)$$

$$\mathbb{E}_\mu(\hat{\xi}_k \hat{\xi}_{k'}) = -\delta_{k=-k'} \frac{v}{\hat{A}(k) + \hat{A}(-k)}. \quad (6.10)$$

where in the first equality we used that the fact that $\hat{A}(k) \neq 0$ whenever $k \neq 0$. Since a Gaussian measure is uniquely characterized by mean and variance, uniqueness of the invariant measure also follows. Using (3.13) we see that (4.22) and (4.24) hold. Again, the dominant contribution comes from $k \sim 0$ and one obtains the asymptotics (4.26), see Appendix C.2.

Next we prove the claim about the limit behavior of the random variable ξ_ϕ . Since ξ_ϕ is Gaussian and centered for any δ, m , it suffices to prove that its variance converges to (4.29). Note that the support S_ϕ of the function $\phi(\cdot)$ has a diameter of order $\delta^{-1} \ll m$. Let p_0 be a point outside S_ϕ , at distance of order $1/\delta$ from the origin. Write

$$\mathbb{E}_\mu(\xi_\phi^2) = \delta^4 \sum_{p,p'} \phi(\delta p) \phi(\delta p') \mathbb{E}_\mu((\xi_p - \xi_0)(\xi_{p'} - \xi_{p_0})) + o(1) \quad (6.11)$$

where we used

$$\delta^2 \sum_{p'} \phi(\delta p') = O(\delta) \quad (6.12)$$

(because the integral of ϕ is zero and the function is smooth) together with $\text{Cov}_{\mu_\infty}((\xi_p - \xi_0); (\xi_0 - \xi_{p_0})) \leq C \log(\delta^{-1})$ for p in the support of $\phi(\delta \cdot)$, which follows from (4.31) and Cauchy-Schwarz. Using (4.26) we have

$$\mathbb{E}_\mu(\xi_\phi^2) = \frac{v}{2\pi w} \delta^4 \sum_{p,p'} \phi(\delta p) \phi(\delta p') \log \frac{|V(p-p_0)| |Vq|}{|V(p-p')| |Vp_0|} + o(1) \quad (6.13)$$

$$= \frac{v}{2\pi w} \delta^4 \sum_{p,p'} \phi(\delta p) \phi(\delta p') \log \frac{|V\delta(p-p_0)| |V\delta p'|}{|V\delta(p-p')| |V\delta p_0|} + o(1) \quad (6.14)$$

where in the first step we used

$$\|\phi\|_\infty^2 \delta^4 \sum_{p,p' \in S_\phi} \frac{1}{\min(1, |p-p_0|, |p'|, |p-p'|, |p_0|)} \stackrel{\delta \rightarrow 0}{=} o(1) \quad (6.15)$$

The sum in (6.14) can be written as

$$-\frac{v}{2\pi w} \delta^4 \sum_{p,p'} \phi(\delta p) \phi(\delta p') \log |V\delta(p-p')| + o(1). \quad (6.16)$$

Indeed, the terms proportional to $\log |V(p-p_0)|$, $\log |Vp'|$ and $\log |Vp_0|$ are independent of at least one of the two summation variables p, p' : then, using once more (6.12) together with e.g. $|\log |V\delta p'|| = O(\log(1/\delta))$, (6.16) follows.

The sum in (6.16) is the Riemann approximation of the convergent integral (4.29).

APPENDIX A. PROOF OF LEMMA 2

Set $\delta = 1/10$. To get the asymptotics of $(q; q)_a$, write

$$\log(q; q)_a = \sum_{i=1}^a \log(1 - q^i) = - \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^a q^{ni} = - \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^n} (1 - q^{na}). \quad (A.1)$$

Then we recall that $a = \epsilon^{-1}b + X$ and that $q = e^{-\epsilon}$ and we write

$$\log(q; q)_a = R_1 + R_2 := - \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^n} (1 - e^{-bn}) + \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^n} e^{-bn} (e^{-nX\epsilon} - 1). \quad (A.2)$$

We have, using $q^n/(1 - q^n) \sim (1 - \epsilon n/2)/(\epsilon n)$ for $\epsilon n \ll 1$,

$$R_1 = C(\epsilon) + \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^n} e^{-bn} = C(\epsilon) + \frac{1}{\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-bn} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-bn}}{n} + o(1) \quad (A.3)$$

with $C(\epsilon) = - \sum_n (1/n) q^n / (1 - q^n)$ independent of b, X . As for R_2 , we claim that

$$R_2 = -X \sum_{n=1}^{\infty} \frac{e^{-bn}}{n} + \epsilon \frac{X^2}{2} \sum_{n=1}^{\infty} e^{-bn} + o(1) \quad (A.4)$$

which, together with (A.3), concludes the proof of the Lemma.

To get (A.4), remark first of all that

$$\sum_{n=\epsilon^{-1/2+\delta}}^{\infty} \frac{1}{n} \frac{e^{-\epsilon n}}{1 - e^{-\epsilon n}} e^{-bn} (e^{-nX\epsilon} - 1) = o(1) \quad (A.5)$$

because $b > 0$ and $X\epsilon = o(1)$. Next, for $n \leq \epsilon^{-1/2+\delta}$ one has $|Xn\epsilon| = o(1)$. Then,

$$\sum_{n=1}^{\epsilon^{-1/2+\delta}} \frac{1}{n} \frac{e^{-\epsilon n}}{1 - e^{-\epsilon n}} e^{-bn} (e^{-nX\epsilon} - 1) \quad (\text{A.6})$$

$$\sim \sum_{n=1}^{\epsilon^{-1/2+\delta}} \frac{1 - \epsilon n/2}{\epsilon n^2} e^{-bn} (-Xn\epsilon + \frac{(Xn\epsilon)^2}{2} + O(|Xn\epsilon|^3)). \quad (\text{A.7})$$

The terms proportional to X and X^2 give the r.h.s. of (A.4). As for the rest, it is $o(1)$: just recall that $\delta = 1/10$ and observe that

$$|X|^3 \epsilon^2 \ll \epsilon^{2-3/2-3\delta} = o(1). \quad (\text{A.8})$$

APPENDIX B. NEGATIVITY OF $\hat{R}(k)$

Recall (cf. (5.16)) that $\hat{R}(k) = (1/v)\hat{Q}(k)$. We claim that the only stationary points of $\hat{Q}(k)$ are $k^{(1)} = (0, 0)$, $k^{(2)} = (0, \pi)$, $k^{(3)} = (\pi, 0)$, $k^{(4)} = (\pi, \pi)$ (modulo 2π). It is trivial to check that $\hat{Q}(k^{(i)}) < 0$ for $i = 2, 3, 4$ (use that $B, C < D$ and that $x \mapsto f(x) := e^{-x}/(1 - e^{-x})$ is decreasing on \mathbb{R}^+) while of course $\hat{Q}(0, 0) = 0$. Recall also (cf. (3.21)) that the Hessian of \hat{R} at $k = 0$ is not zero, which implies that $\hat{R}(k)$ is strictly negative outside $k = 0$.

Letting $X = \sin(k_1)$, $Y = \sin(k_2)$, the stationary points of \hat{Q} must satisfy

$$f(D)X = f(C)Y \quad (\text{B.1})$$

$$f(D)X = \pm f(B)(X\sqrt{1 - Y^2} \pm Y\sqrt{1 - X^2}). \quad (\text{B.2})$$

If we exclude the solution $X = Y = 0$ (which corresponds to $k = k^{(i)}$, $i = 1, \dots, 4$), (B.1) implies that

$$X = \pm \frac{\sqrt{\Delta}}{2f(D - C)f(C)f(D)^2} \quad (\text{B.3})$$

$$\Delta = -\frac{(1 + e^C)(e^C + e^D)(-3e^C + e^{2C} + e^D + e^{C+D})}{(1 - e^C)^2(e^C - e^D)^2(1 - e^D)^4} \quad (\text{B.4})$$

where we used $B = D - C$. However, one sees immediately that $\Delta < 0$, since $C < D$, so that (B.1) does not give real solutions.

APPENDIX C. INTEGRAL ASYMPTOTICS

C.1. Proof of (4.6). Let $\chi : [-\pi, \pi]^2 \rightarrow [0, 1]$ be a C^∞ cutoff function such that $\chi(k) = 1$ for $|k| < 1/2$ and $\chi(k) = 0$ for $|k| \geq 1$. We first write the r.h.s. of (6.8) as

$$\frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} dk \chi(k) \frac{e^{\hat{R}(k)s} - 1}{\hat{R}(k)} e^{(t-s)\hat{A}(k)} e^{-iky} + j_1(t, s, y) \quad (\text{C.1})$$

where j_1 , as well as the error terms j_2, j_3, \dots below, satisfies

$$j_1 = O(1) \quad \text{and} \quad |j_1| \rightarrow 0 \quad \text{if} \quad \max(|y|, t - s) \rightarrow \infty. \quad (\text{C.2})$$

To see this, remark first of all that

$$\hat{A}(k) = \frac{1}{2}\hat{R}(k) + \frac{1}{2}(\hat{A}(k) - \hat{A}(-k)) \quad (\text{C.3})$$

and

$$\hat{A}(k) - \hat{A}(-k) \in i\mathbb{R}, \quad \hat{R}(k) < 0, k \neq 0. \quad (\text{C.4})$$

Actually,

$$\frac{\hat{A}(k) - \hat{A}(-k)}{2} \stackrel{k \rightarrow 0}{\cong} -i(k, U) + O(|k|^3). \quad (\text{C.5})$$

The function $(1 - \chi(k))/\hat{R}(k)$ is a C^∞ function on $[-\pi, \pi]^2$, so that

$$\int_{[-\pi, \pi]^2} dk \frac{e^{(t-s)\hat{A}(k)}}{\hat{R}(k)} e^{-iky} (1 - \chi(k))$$

is $o(1)$ when either $t - s \rightarrow \infty$ (using $\hat{R}(k) < 0$) or when $t - s = O(1)$ and $|y| \rightarrow \infty$ (the Fourier coefficients of a C^∞ function on the torus decay faster than any inverse power). Also,

$$\int_{[-\pi, \pi]^2} dk (1 - \chi(k)) \frac{e^{\hat{R}(k)s}}{\hat{R}(k)} e^{(t-s)\hat{A}(k)} e^{-iky} \quad (\text{C.6})$$

is $o(1)$ when either s or $t - s$ diverge; when $s, t - s = O(1)$ the integral in (C.6) is again $o(1)$ when $|y| \rightarrow \infty$ by decay of Fourier coefficients. Eq. (C.1) follows.

Next, the integral in (C.1) gives

$$\frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} dk \chi(k) \frac{e^{\hat{R}(k)s} - 1}{\hat{W}(k)} e^{(t-s)\hat{A}(k)} e^{-iky} + j_2(t, s, y). \quad (\text{C.7})$$

For this, just note that, since $\hat{R}(k)$ is smooth and even in k ,

$$\frac{\hat{W}(k) - \hat{R}(k)}{\hat{W}(k)\hat{R}(k)} = O(1)$$

and

$$\nabla_k \frac{\hat{W}(k) - \hat{R}(k)}{\hat{W}(k)\hat{R}(k)} = O(1/|k|).$$

This immediately implies

$$\int_{\mathbb{R}^2} dk \chi(k) e^{\hat{R}(k)s + \hat{A}(k)(t-s)} \left(\frac{1}{\hat{W}(k)} - \frac{1}{\hat{R}(k)} \right) e^{-iky} = o(1) \quad (\text{C.8})$$

if $\max(t, t - s) \rightarrow \infty$ (use dominated convergence, together with

$$e^{\hat{R}(k)s + \hat{A}(k)(t-s)} \rightarrow 0 \quad \text{for all } k \neq 0 \quad \text{if } \max(t, t - s) \rightarrow \infty).$$

When $\max(t, t - s) = O(1)$ then the l.h.s. of (C.8) is $(1/|y|)$ (just do an integration by parts). One bounds

$$\int_{\mathbb{R}^2} dk \chi(k) e^{\hat{A}(k)(t-s)} \left(\frac{1}{\hat{W}(k)} - \frac{1}{\hat{R}(k)} \right) e^{-iky} \quad (\text{C.9})$$

similarly and (C.7) follows.

As a third step, write the integral in (C.7) as

$$\frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} dk \chi(k) \frac{e^{\hat{W}(k)s} - 1}{\hat{W}(k)} e^{(t-s)\hat{A}(k)} e^{-iky} + j_3(t, s, y). \quad (\text{C.10})$$

In fact, assume first that $s \rightarrow \infty$. Then, the contribution to the integral

$$\int_{\mathbb{R}^2} dk \chi(k) \frac{e^{\hat{R}(k)s} - e^{\hat{W}(k)s}}{\hat{W}(k)} e^{(t-s)\hat{A}(k)} e^{-iky} \quad (\text{C.11})$$

from the region $|k| \geq s^{-1/2+\epsilon}$, $\epsilon > 0$ is negligible because we have $\hat{R}(k), \hat{W}(k) \leq -C|k|^2$. The contribution from $|k| \leq s^{-1/2+\epsilon}$ is also negligible, this time because

$$\frac{e^{\hat{W}(k)s} - e^{\hat{R}(k)s}}{\hat{W}(k)} = e^{\hat{W}(k)s} \frac{1 - e^{(\hat{R}(k) - \hat{W}(k))s}}{\hat{W}(k)} = O(s|k|) = O(s^{1/2+\epsilon})$$

(use that $s(\hat{R}(k) - \hat{W}(k)) = O(s|k|^3) = o(1)$ and $\exp(\hat{W}(k)s) \leq 1$) so that the contribution to the integral is $O(s^{-1/2+3\epsilon}) = o(1)$ if ϵ is small enough. The proof of (C.10) in the case $s = O(1)$ is simpler and follows that of (C.1) or (C.7).

With similar considerations, one rewrites (C.10) as

$$\frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} dk \chi(k) \frac{e^{\hat{W}(k)s} - 1}{\hat{W}(k)} e^{(t-s)\hat{W}(k)/2} e^{-iky - i(k,U)(t-s)} + j_4(t, s, y) \quad (\text{C.12})$$

$$= \frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} dk \chi(k) \frac{e^{\hat{W}(k)s} - 1}{\hat{W}(k)} e^{\hat{W}(k) + (t-s)\hat{W}(k)/2} e^{-iky - i(k,U)(t-s)} + j_5(t, s, y) \quad (\text{C.13})$$

$$= \frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} dk \frac{e^{\hat{W}(k)s} - 1}{\hat{W}(k)} e^{\hat{W}(k) + (t-s)\hat{W}(k)/2} e^{-iky - i(k,U)(t-s)} + j_6(t, s, y). \quad (\text{C.14})$$

In the first step, we used (C.3)–(C.5) to replace $\hat{A}(k)$ with its second-order Taylor expansion $\hat{W}(k)/2 - i(k, U)$. In the second, we used that $(\exp(\hat{W}(k)) - 1)/\hat{W}(k)$ is smooth at $k \sim 0$ in order to multiply by $\exp(\hat{W}(k))$. In the third we remove the cutoff function: the integral with $\chi(\cdot)$ replaced by $1 - \chi(\cdot)$ is $o(1)$ either because $|y| \rightarrow \infty$ (integrate by parts w.r.t. k) or because $(t-s) \rightarrow \infty$, so that $\exp((t-s)\hat{W}(k)) \rightarrow 0$ (apply dominated convergence).

Remark 6. *The reason why in the second step in (C.12) we multiplied by $\exp(\hat{W}(k))$ is that otherwise we could not remove the cutoff function $\chi(\cdot)$ since for $t = s$ the integrand would decay only as $O(|k|^{-2})$ at infinity and the integral would not converge.*

The integral can now be computed explicitly. Recall that $\hat{W}(k) = (k, \hat{W}k)$ with \hat{W} a strictly negative definite symmetric matrix. From definition (3.22) we have

$$|\det(V)| = \frac{1}{w}. \quad (\text{C.15})$$

Changing variables as $k = V^T w$ and putting $H = Vy + (t-s)VU$, the integral in the r.h.s. of (C.12) becomes

$$-|\det(V)| \frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(w,H)} \frac{e^{-s|w|^2} - 1}{|w|^2} e^{-|w|^2(1+(t-s)/2)} dw \quad (\text{C.16})$$

$$= -|\det(V)| \frac{v}{(2\pi)^2} \int_1^\infty da \int_{\mathbb{R}^2} e^{-i(w,H)} (e^{-s|w|^2} - 1) e^{-(a+(t-s)/2)|w|^2} dw \quad (\text{C.17})$$

$$= -\pi |\det(V)| \frac{v}{(2\pi)^2} \int_1^\infty \left[\frac{e^{-|H|^2/(4(a+(t+s)/2))}}{a + (t+s)/2} - \frac{e^{-|H|^2/(4(a+(t-s)/2))}}{a + (t-s)/2} \right] da \quad (\text{C.18})$$

$$= |\det(V)| \frac{v}{4\pi} \int_{1+(t-s)/2}^{1+(t+s)/2} \frac{e^{-|H|^2/(4a)}}{a} da \quad (\text{C.19})$$

Together with (C.15), Eq. (4.6) then follows.

C.2. **Proof of (4.26).** Similar (actually simpler) arguments as those leading to (C.12) show that the integral in (4.24) equals

$$-\frac{v}{(2\pi)^2} \int_{\mathbb{R}^2} dk \frac{(e^{iky_1} - e^{iky_2})(e^{-iky_3} - e^{-iky_4})}{\hat{W}(k)} e^{\hat{W}(k)} + R_{y_1, y_2, y_3, y_4} \quad (\text{C.20})$$

with R satisfying (4.27). The factor $\exp(\hat{W}(k))$ appears for the same reasons as in Remark 6. With the same change of coordinates $k = V^T w$ as before, the integral in (C.20) becomes

$$\frac{v}{(2\pi)^2 w} \int_{\mathbb{R}^2} \frac{(e^{i(w, Y_1)} - e^{i(w, Y_2)})(e^{-i(w, Y_3)} - e^{-i(w, Y_4)})}{|w|^2} e^{-|w|^2} dw \quad (\text{C.21})$$

with $Y_i = Vy_i$. This equals

$$\frac{v}{4\pi w} \int_1^\infty da \frac{e^{-|Y_1 - Y_3|^2/(4a)} - e^{-|Y_1 - Y_4|^2/(4a)} + e^{-|Y_2 - Y_4|^2/(4a)} - e^{-|Y_2 - Y_3|^2/(4a)}}{a} \quad (\text{C.22})$$

$$= \frac{v}{(2\pi)^2 w} \left[-\Gamma\left(0, \frac{|Y_1 - Y_3|^2}{4}\right) + \Gamma\left(0, \frac{|Y_1 - Y_4|^2}{4}\right) \right. \quad (\text{C.23})$$

$$\left. -\Gamma\left(0, \frac{|Y_2 - Y_4|^2}{4}\right) + \Gamma\left(0, \frac{|Y_2 - Y_3|^2}{4}\right) + 2 \log \frac{|Y_1 - Y_4||Y_2 - Y_3|}{|Y_1 - Y_3||Y_2 - Y_4|} \right], \quad (\text{C.24})$$

where $\Gamma(0, x) := \int_x^\infty e^{-t}/t dt$ is the incomplete Gamma function. Using the exponential decay of $\Gamma(0, x)$ as $x \rightarrow \infty$, we get immediately (4.26) (the “+1”s in (4.26) takes care of the case where $Y_1 = Y_3$ and/or $Y_2 = Y_4$: in fact, $-\Gamma(0, x^2/4) + \log(1/x^2)$ has a finite limit as $x \rightarrow 0$).

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