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Enhanced gauge symmetry in 6D F-theory models and tuned elliptic Calabi-Yau threefolds

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ABSTRACT: We systematically analyze the local combinations of gauge groups and matter that can arise in 6D F-theory models over a fixed base. We compare the low-energy constraints of anomaly cancellation to explicit F-theory constructions using Weierstrass and Tate forms, and identify some new local structures in the “swampland” of 6D supergravity and SCFT models that appear consistent from low-energy considerations but do not have known F-theory realizations. In particular, we classify and carry out a local analysis of all enhancements of the irreducible gauge and matter contributions from “non-Higgsable clusters,” and on isolated curves and pairs of intersecting rational curves of arbitrary self-intersection. Such enhancements correspond physically to unHiggsings, and mathematically to tunings of the Weierstrass model of an elliptic CY threefold. We determine the shift in Hodge numbers of the elliptic threefold associated with each enhancement. We also consider local tunings on curves that have higher genus or intersect multiple other curves, codimension two tunings that give transitions in the F-theory matter content, tunings of abelian factors in the gauge group, and generalizations of the “ E_8 ” rule to include tunings and curves of self-intersection zero. These tools can be combined into an algorithm that in principle enables a finite and systematic classification of all elliptic CY threefolds and corresponding 6D F-theory SUGRA models over a given compact base (modulo some technical caveats in various special circumstances), and are also relevant to the classification of 6D SCFT’s. To illustrate the utility of these results, we identify some large example classes of known CY threefolds in the Kreuzer-Skarke database as Weierstrass models over complex surface bases with specific simple tunings, and we survey the range of tunings possible over one specific base.

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1 Introduction

Compactifications have long played a fundamental role within string theory, beginning as a necessary ingredient for approaching 4D phenomenology. Evolving perspectives on string theory have yielded increasingly rich classes of compactifications. With the advent of F-theory [1, 2], a spatially varying axio-dilaton profile in type IIB string theory has enabled the construction of lower-dimensional theories with a wide range of gauge symmetries and

matter content in a non-perturbative and unifying framework. In recent years, following extensive work on increasingly phenomenologically coherent F-theory GUT models [3, 4], there has been a renaissance of work on both phenomenological and more foundational aspects of F-theory. For 4D $\mathcal{N} = 1$ F-theory constructions, fluxes produce a superpotential that lifts many moduli; non-perturbative 7-brane world-volume dynamics are difficult to control; and other constructions appear to give large classes of vacua with no F-theory dual. These difficulties render the exploration and global understanding of the full set of 4D $\mathcal{N} = 1$ compactifications quite subtle. In 6D, on the other hand, the class of models resulting from F-theory compactifications is tightly controlled. In particular, there is a very close connection between the geometry of complex surfaces and the physics of 6D supergravity theories; for example, low-energy consistency conditions such as the Green-Schwarz mechanism of anomaly cancellation [5, 6], which are quite stringent, can be mapped directly to geometric constraints [7–11].

The moduli space of 6D supergravities arising from F-theory compactifications is a connected space, with different branches connected through superconformal fixed points by tensionless string transitions [12, 2, 9]. The possibility of a complete and explicit description of this space seems in principle feasible. F-theory compactifications to 6D are realized by compactifying type IIB string theory on a complex surface base B that supports an elliptic Calabi-Yau threefold. In recent work, significant progress has been made on a systematic classification of all possible such compact smooth base surfaces B [13–16]. If one can in addition understand the allowed tunings of different elliptic fibrations over each of these bases, corresponding to F-theory models with distinct spectra, one would have a very clear handle on this moduli space. Some progress in this direction was made in [17]. One of the primary goals of this paper is to expand the set of tools for the systematic understanding of such tunings, which correspond physically to enhancing the gauge and matter content of the supergravity model. Recently, similar methods have been used to explore six-dimensional superconformal field theories (SCFT’s) [18, 19]. These non-compact F-theory models are in some ways simpler, yet in other ways less constrained, than their supergravity counterparts; for these theories as well, the classification problem can be divided into two independent sub-problems: the classification of the base geometry, and the classification of tuned enhancements of gauge symmetry and matter content, given a base geometry. Parts of this paper that are relevant to SCFT’s have significant overlap with [19], which appeared while this work was in progress.

Mathematically, the classification of 6D F-theory models corresponds to the classification of Weierstrass models for elliptic Calabi-Yau threefolds. Every elliptic Calabi-Yau threefold has a Weierstrass model realization [20], so this problem is tantamount to (though slightly distinct from) the problem of classifying elliptic Calabi-Yau threefolds. The minimal model program for surfaces and the work of Grassi [21] show that all base surfaces that support elliptic CY threefolds can be constructed as blow-ups of a small finite set of minimal surfaces. It was proven by Gross [22] that the number of birational equivalence classes of elliptically fibered Calabi-Yau threefolds is finite. A more constructive argument was given in [9], which shows that there are a finite number of distinct families of Weierstrass models corresponding

to distinct 6D F-theory spectra; these Weierstrass models can be in principle classified by first constructing all bases B and then finding all allowed tunings over each B . It is the latter problem, of tuning Weierstrass models over a given base B , that we address in the present work.

There are thus three distinct classification problems that we wish to make progress towards by the systematic study of allowed tunings in 6D F-theory models: First, the classification of elliptic Calabi-Yau threefolds; second, the related classification of 6D supergravity models; and third, the classification of 6D SCFT's. It has been conjectured that all quantum consistent 6D supergravity theories and all 6D superconformal field theories have realizations in F-theory [23, 8, 18]. Thus, a complete classification of allowed tunings may help to solve all three of these classification problems. While we do not completely solve these classification problems here, we develop a systematic description of possible local tunings, which we incorporate into an algorithmic framework for approaching these problems, with unresolved issues isolated into specific technical questions that can be addressed in further work. Throughout the paper, we also identify and focus particular attention on theories in the “swampland” [24], which are apparently consistent from low-energy considerations, yet do not have an F-theory description.

The outline of this paper is the following. We first review the basics of F-theory constructions in §2, including the traditional interpretation in terms of a IIB theory with a varying axio-dilaton profile. We review the structure of non-Higgsable clusters, 6D supergravity, and 6D SCFTs. We also include a brief review of some aspects of algebraic geometry and toric geometry, setting notation for the following calculations. In §3, we present a more in-depth summary of our classification strategy, the components of which are developed in the following sections. Sections §4 through §6 contain the core results of the paper: they consist respectively of constraints for tunings of enhanced gauge groups on isolated divisors (curves), constraints for tunings on multiple-divisor clusters, and constraints determining when two or more of the previously discussed tunings may be achieved simultaneously. These constraints are all local in that they require the simultaneous consideration of local configurations of intersecting divisors in the base. For many of these cases, particularly those where the curves involved form a non-Higgsable cluster, we explicitly check that local models for all of the possible gauge groups allowed by the constraints can be realized explicitly by tuning monomials in a Weierstrass model. In §7 and §8 we describe the tunings of exotic matter and abelian gauge factors, which go beyond the simple Kodaira classification of gauge groups and generic matter. In §9, we assemble all these pieces into a systematic algorithm for determining a finite set of possible Weierstrass models over a given base. In some cases, this list may represent a superset of the true set of allowed F-theory models; certain combinations of local structures must be checked explicitly to verify the existence of a global Weierstrass model. In §10, we illustrate the utility of these rules by investigating two classes of tuned models in the Kreuzer-Skarke database, and exploring the range of tunings possible over the Hirzebruch surface base $B = \mathbb{F}_{12}$. Lastly, §11 reviews our results and their limitations, and explores some potentially interesting future directions.

As this paper was in the final stages of completion, we learned of independent concurrent

work by Morrison and Rudelius [25] that also considers the generalization of the E_8 rule to tuned gauge groups.

2 Physical and geometric background

2.1 F-theory preliminaries

We begin with a brief review of some basic aspects of F-theory; more extensive pedagogical introductions can be found in [26–28]. In its original form, F-theory [1, 2] is an attempt to capture and codify geometrically the $SL(2, \mathbb{Z})$ S-symmetry of type IIB string theory in a way that greatly broadens the class of manifolds on which the IIB theory can be compactified while preserving symmetry, by allowing variation of the axiodilaton field $\tau := C_0 + ie^{-\phi}$ to compensate for curvature in the compactification space. Under the IIB $SL(2, \mathbb{Z})$ symmetry, the axiodilaton transforms as

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \quad (2.1)$$

for $ab - cd = \pm 1$ (all four constants in \mathbb{Z}). Because the true moduli space of IIB is a quotient of the τ plane by this symmetry, and this is nothing but the (complex) moduli space of the torus, the axiodilaton at each point in space-time can be identified with an elliptic curve \mathbb{E} . Allowing this coupling to depend on position yields an elliptic fibration (with a well defined global zero-section) over the compact space B of the IIB compactification. In practice, when we form an F-theory model by starting with the IIB theory compactified on a “base” B , with varying axio-dilaton profile as above, as a shorthand we say “F-theory compactified on X ” where X is the total space of the elliptic fibration $\mathbb{E} \hookrightarrow X \xrightarrow{\pi} B$.

The study of F-theory is morally the study of elliptically fibered (with section) Calabi-Yau manifolds. (Throughout this paper, we focus on (complex) threefolds.) Essential physical data of the resulting low-energy theory on $\mathbb{R}^{1,5}$ are encoded in the singularities of the elliptic fibration, namely the loci where one or both cycles of the torus degenerate. Because singularities in the elliptic fibration are accompanied by monodromies of τ , it is natural to interpret these singularity loci as the support of 7-branes within the theory. In terms of the two generators $T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $SL(2, \mathbb{Z})$, a monodromy of $pT_1 + qT_2$ corresponds to a so-called (p, q) 7-brane. (Only the familiar $(1, 0)$ D7 brane is visible in perturbative IIB string theory.) Just as it was discovered early on that when a stack of parallel D-branes coincide, the open strings stretched between them have endpoint degrees of freedom that fill out the gauge sector of an $SU(N)$ gauge theory; so too was it realized that these generalized branes could encode more general gauge symmetries. Although many perspectives exist on this result ([2, 27]), the most useful is a long-known mathematical result that completely characterizes codimension one singularities of elliptic fibrations: the Kodaira classification (see *e.g.* [29]).

To discuss the Kodaira classification, it is necessary to recall a convenient description of an elliptic curve. In the weighted projective space $\mathbb{P}^{[2,3,1]}$, an elliptic curve can be written in

Type	ord (f)	ord (g)	ord (Δ)	singularity	nonabelian symmetry algebra
I_0	≥ 0	≥ 0	0	none	none
I_n	0	0	$n \geq 2$	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}(\lfloor n/2 \rfloor)$
II	≥ 1	1	2	none	none
III	1	≥ 2	3	A_1	$\mathfrak{su}(2)$
IV	≥ 2	2	4	A_2	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$
I_0^*	≥ 2	≥ 3	6	D_4	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or \mathfrak{g}_2
I_n^*	2	3	$n \geq 7$	D_{n-2}	$\mathfrak{so}(2n-4)$ or $\mathfrak{so}(2n-5)$
IV^*	≥ 3	4	8	\mathfrak{e}_6	\mathfrak{e}_6 or \mathfrak{f}_4
III^*	3	≥ 5	9	\mathfrak{e}_7	\mathfrak{e}_7
II^*	≥ 4	5	10	\mathfrak{e}_8	\mathfrak{e}_8
non-min	≥ 4	≥ 6	≥ 12	does not occur in F-theory	

Table 1. Table of codimension one singularity types for elliptic fibrations and associated nonabelian symmetry algebras. In cases where the algebra is not determined uniquely by the orders of vanishing of f, g , the precise gauge algebra is fixed by monodromy conditions that can be identified from the form of the Weierstrass model.

Weierstrass form by the equation

$$y^2 = x^3 + fx + g \quad (2.2)$$

where (x, y, t) are generalized homogeneous coordinates on $\mathbb{P}^{[2,3,1]}$ with the respective weights of the equivalence relation defining the projective space; we work in an affine chart where $t = 1$. Indeed, from this description it is obvious that f must have weight 4 and g weight 6. Because f and g together determine the complex structure of the torus, allowing this structure to change as a function of position (z, w) on the base B amounts to promoting f and g to functions $f(z, w)$ and $g(z, w)$ on the base. In fact, considering the defining equations as living in $\mathbb{P}^{[2,3,1]} \times B$, these coefficients must be sections of line bundles $f \in \mathcal{O}(-4K)$, $g \in \mathcal{O}(-6K)$ for the defining equation to yield a Calabi-Yau total space, where K is the canonical class of the base.

To understand the singularities of the defining equation, we rely on the discriminant Δ to locate where its zeroes coincide with those of its first derivatives. This locus is defined to be the set of points in B where the following equation holds:

$$0 = \Delta = 4f^3 + 27g^2. \quad (2.3)$$

Being generically codimension one, this locus corresponds to an effective divisor on B . On each irreducible component Σ of this divisor, a simple gauge algebra factor can reside. The Kodaira classification makes the connection explicit by associating to a given singularity a corresponding gauge algebra, according to the orders of vanishing of f, g , and Δ on the associated divisor in the base. The possibilities are listed in table 1. As discussed later, there are a few cases with ambiguities that arise from monodromies of the defining equation in the fiber over the singularity; the procedure that allows one to discriminate between these cases is known as

the Tate algorithm [30–32]. Note that the Kodaira singularity type only determines the Lie algebra of the resulting nonabelian group G . In many situations we will not be careful about this distinction; so in general, for example, we may discuss tuning an $SU(N)$ gauge group, though the actual group may have a quotient $G = SU(N)/\Gamma$ by a discrete finite subgroup. In a few cases where this distinction is relevant we comment explicitly on the issue.

In some circumstances, it is convenient to describe Weierstrass models starting from a more general form of the equation for an elliptic curve on $\mathbb{P}^{[2,3,1]}$, known as the *Tate form*

$$y^2 + a_1yx + a_3y = x^3 + a_2x^2 + a_4x + a_6. \quad (2.4)$$

Here $a_k \in \mathcal{O}(-kK)$. Given such a form, it is straightforward to transform into Weierstrass form by completing the square in y to remove the terms linear in y , and then shifting x to remove the quadratic term in x . In the resulting Weierstrass form, f, g can then be expressed in terms of the a_k [31].

The advantage of Tate form is that certain Kodaira singularity types can be tuned more readily by choosing the sections a_k to vanish to a given order on a divisor of interest than by constructing the corresponding Weierstrass model. For example, if we wish to tune a gauge algebra $\mathfrak{su}(6)$ on a divisor Σ defined in local coordinates by $\Sigma = \{z = 0\}$, in Weierstrass form f and g are described locally by functions that can be expressed as power series in z , $f = f_0 + f_1z + f_2z^2$, etc.. The condition that Δ vanish to order 5 in z while $f_0, g_0 \neq 0$ imposes a series of nontrivial algebraic conditions on the f_k, g_k coefficient functions. While these algebraic equations can be solved explicitly when Σ is smooth [33], the resulting algebraic structures are rather complex. In Tate form, on the other hand, the classical algebras $\mathfrak{sp}(n)$, $\mathfrak{su}(n)$, and $\mathfrak{so}(n)$ can all be tuned simply by choosing the leading coefficients in an expansion of the a_k to vanish to an appropriate order. Table 2 gives the orders to which the a_k must vanish to ensure the appropriate classical algebra. Note that in each case we have only given the minimal required orders of vanishing. Note also that while in most cases tuning a Tate form guarantees the desired Kodaira singularity type of the resulting Weierstrass model, there are some exceptions. In some cases the resulting Weierstrass model will have extra singularities; we encounter some examples of this in §4. In other cases, there are Weierstrass models with a given gauge group that do not follow from the Tate form [33]. Thus, the Weierstrass form is more complete, but in many cases the Tate formulation gives a simpler way of constructing certain kinds of tunings.

It is important to emphasize that the coefficients in the Weierstrass form map directly to neutral scalar fields in 6D F-theory models, so the Weierstrass form is useful in computing the spectrum of a theory and verifying anomaly cancellation; this is much more difficult in Tate form, where there is some redundancy in the parameterization for any given Weierstrass model.

As an example of Tate form, we can tune an $\mathfrak{su}(2)$ on the divisor $\{s = 0\}$ in local coordinates by choosing the Tate model

$$y^2 + 2xy + sy = x^3 + 2x^2 + sx + s^2. \quad (2.5)$$

Group	a_1	a_2	a_3	a_4	a_6	Δ
$\mathfrak{su}(2) = \mathfrak{sp}(1)$	0	0	1	1	2	2
$\mathfrak{sp}(n)$	0	0	n	n	$2n$	$2n$
$\mathfrak{su}(n)$	0	1	$\lfloor n/2 \rfloor$	$\lfloor (n+1)/2 \rfloor$	n	n
\mathfrak{g}_2	1	1	2	2	3	6
$\mathfrak{so}(7), \mathfrak{so}(8)^*$	1	1	2	2	4	6
$\mathfrak{so}(4n+1), \mathfrak{so}(4n+2)^*$	1	1	n	$n+1$	$2n$	$2n+3$
$\mathfrak{so}(4n+3), \mathfrak{so}(4n+4)^*$	1	1	$n+1$	$n+1$	$2n+1$	$2n+4$
\mathfrak{f}_4	1	2	2	3	4	8
\mathfrak{e}_6	1	2	2	3	5	8
\mathfrak{e}_7	1	2	3	3	5	9
\mathfrak{e}_8	1	2	3	4	5	10
non-min.	1	2	3	4	6	12

Table 2. Table of minimal vanishing orders needed for realizing algebras using Tate form. Algebras marked with * require additional monodromy conditions. In particular, for all the $\mathfrak{so}(N)$ algebras with N even, there are monodromy conditions that remove the ambiguity between $\mathfrak{so}(2n-1)$ and $\mathfrak{so}(2n)$. Table follows [32], with minor improvements as discussed in §4.2.

Converting to Weierstrass form we have

$$y^2 = x^3 + (-3 + 2s)x + (2 - 2s + 5s^2/4), \quad (2.6)$$

and $\Delta = 99s^2 + \mathcal{O}(s^3)$, so indeed the discriminant cancels to order s^2 and the Weierstrass model has a Kodaira type I_2 singularity encoding an $\mathfrak{su}(2)$ gauge algebra.

In most cases, tuning a singularity in Tate form is equivalent to tuning the same singularity in Weierstrass form in the most generic way. For example, $N \leq 5$, the Tate tuning of $\mathfrak{su}(N)$ and the Weierstrass tuning of $\mathfrak{su}(N)$ are equivalent on a smooth divisor; this can be seen explicitly by matching the terms in the analyses of [32, 33] using the dictionary provided on page 22 of [33]. There are more possibilities for Weierstrass tunings beginning at $\mathfrak{su}(6)$; note, however, that for tuning on a curve Σ of self-intersection $\Sigma \cdot \Sigma = -2$, where f_0, g_0 are constant on Σ since the normal bundle is equal to the canonical bundle, the Tate and Weierstrass forms are equivalent. This fact will be relevant in the later analysis. Examples of non-Tate tunings have recently been explored in [33–36] and in virtually all known cases involve non-generic types of matter.

2.2 Algebraic geometry preliminaries

To apply the Kodaira classification in various contexts, it is useful to have available some well known tools from algebraic geometry. We first briefly review relevant aspects of the geometry of the base surfaces in which we are interested. We then discuss general arguments that allow one to deduce the existence of non-Higgsable clusters (NHCs): groups of divisors over which

even a generic fibration has a singularity that corresponds to a nontrivial gauge algebra. Then we introduce a few relevant aspects of toric geometry that allow one to explicitly execute a given local tuned gauge algebra enhancement (increasing the Kodaira singularity) at the level of coordinates; generally, such computations can be used to explicitly determine that a given tuned fibration is possible either locally or globally in a geometry with a local or global toric description. We primarily focus on local constructions in this paper, though in some situations global analysis on a toric base is also relevant.

We are interested in complex surfaces B that can act as the base of an elliptically fibered Calabi-Yau threefold. We thus focus on rational surfaces that can be realized by blowing up \mathbb{P}^2 or \mathbb{F}_m , $m \leq 12$ at a finite number of points. We review a few basic facts about such surfaces (for more details see *e.g.* [16]). Divisors in a complex surface are integer linear combinations of irreducible algebraic curves on B . The set of homology classes of curves in B form a signature $(1, T)$ integer lattice $\Gamma = H_2(B, \mathbb{Z}) = \mathbb{Z}^{1+T}$ where $T = h^{1,1}(B) - 1$. The intersection form on Γ is unimodular, and for $T \neq 1$ can be written as $\text{diag}(+1, -1, -1, \dots, -1)$. (For Hirzebruch surfaces \mathbb{F}_m with m even, the intersection form is the matrix $((01)(10))$.) The canonical class K satisfies $K \cdot K = 9 - T$, and can be put into the form $(3, -1, -1, \dots, -1)$ when the intersection form is diagonal as above, and in the form $(2, 2)$ for even Hirzebruch surfaces. The set of effective curves, which can be realized algebraically in B , form a cone in the homology lattice. In F-theory, gauge groups can only be tuned on effective curves, so these are the curves on which we focus attention. As an example of a set of allowed bases and their effective cones, the Hirzebruch surfaces \mathbb{F}_m have a cone of effective curves generated by the curves S, F where $S \cdot S = -m$, $S \cdot F = 1$, $F \cdot F = 0$, and can support elliptic Calabi-Yau threefolds when $m = 0, \dots, 8, 12$.

The Zariski decomposition [37] enables one to write $-kK$ of the base in an explicit form that allows one to read off minimal (generic) degrees of vanishing of f , g and Δ on a given irreducible divisor. Given any effective divisor, in particular $-kK$, we can expand it over the rational numbers as a combination of irreducible effective divisors. We can write

$$-kK = \sum_{i=1}^N \sigma_i \Sigma_i + X \quad (2.7)$$

where $\{\Sigma_i\}$ is the set of irreducible effective divisors of negative self-intersection, each of which must be rigid, and $X \cdot \Sigma_i, \Sigma_i \cdot \Sigma_j \geq 0$. By the Riemann-Roch formula, curves of genus 0 satisfy

$$-2 = 2g - 2 = \Sigma \cdot (K + \Sigma) \quad (2.8)$$

(implying, *e.g.*, that a -2 curve Σ satisfies $K \cdot \Sigma = 0$). Taking the intersection product of (2.7) with a $-n$ curve $\Sigma = \Sigma_1$ yields (in the case $N = 1$)

$$-k(n - 2) \geq \sigma(-n) \quad (2.9)$$

This immediately implies $\sigma \geq k(n - 2)/n$, so that for $n \geq 3$, for example, we have $\sigma \geq \frac{4}{3}$, ≥ 2 for $k = 4, 6$, respectively. A section of the line bundle $\mathcal{O}(-kK)$ thus vanishes to at least order

Cluster	gauge algebra	r	V	H_{charged}
(-12)	\mathfrak{e}_8	8	248	0
(-8)	\mathfrak{e}_7	7	133	0
(-7)	\mathfrak{e}_7	7	133	28
(-6)	\mathfrak{e}_6	6	78	0
(-5)	\mathfrak{f}_4	4	52	0
(-4)	$\mathfrak{so}(8)$	4	28	0
(-3, -2, -2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	3	17	8
(-3, -2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	3	17	8
(-3)	$\mathfrak{su}(3)$	2	8	0
(-2, -3, -2)	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	5	27	16
(-2, -2, \dots, -2)	no gauge group	0	0	0

Table 3. List of “non-Higgsable clusters” of irreducible effective divisors with self-intersection -2 or below, and corresponding contributions to the gauge algebra and matter content of the 6D theory associated with F-theory compactifications on a generic elliptic fibration (with section) over a base containing each cluster. The quantities r and V denote the rank and dimension of the nonabelian gauge algebra, and H_{charged} denotes the number of charged hypermultiplet matter fields associated with intersections between the curves supporting the gauge group factors.

$[\sigma]$ on each Σ ; therefore, for $n = 3$ we are in case *IV* of the Kodaira classification, for which the algebra is $\mathfrak{su}(3)$. (In principle, one must perform an additional calculation using the Tate algorithm to distinguish this from $\mathfrak{su}(2)$. We will describe how this is done shortly.) This reasoning can be applied to deduce the existence of all the non-Higgsable clusters [13]. These are clusters of mutually intersecting divisors of self-intersections ≤ -2 that are forced, by this geometric mechanism, to support gauge algebras even for a generic fibration. By demanding that no points in B reach a singularity type with $\text{ord } f \geq 4, \text{ord } g \geq 6$, one can derive a complete set of constraints for when these NHCs can be connected by -1 curves.¹ The NHCs are listed in table 3.

For genus 0 curves that intersect their neighbors, one can elaborate on the previous formula 2.7. Taking for example $N = 3$ (*i.e.* including two neighbors), and assuming that the curve Σ intersects each of the two curves $\Sigma_{R,L}$ with multiplicity one, we have $-kK = \sigma_L \Sigma_L + \sigma \Sigma + \sigma_R \Sigma_R + X$. Intersecting with the curve Σ , which we take to have self-intersection

¹It is important to understand why -1 curves play a pivotal role here. Any two non-Higgsable Kodaira type singularities that are independently consistent can also be simultaneously realized on divisors that are separated by a ≥ 0 curve. (For instance, when the base is a Hirzebruch surface \mathbb{F}_n , which is a \mathbb{P}^1 bundle over \mathbb{P}^1 , the fiber is a 0 curve.) On the other hand, two NHCs cannot be separated by a curve of self-intersection < -1 , since then the resulting collection would itself be one larger NHC. And not all combinations of NHC’s can be separated by a -1 curve. These facts isolate -1 curves as a particularly interesting intermediate situation whose cases must be studied with care.

$-n$, yields

$$\begin{aligned} -k(n-2) &\geq -n\sigma + \sigma_L + \sigma_R \\ \sigma &\geq n^{-1}(k(n-2) + \sigma_L + \sigma_R) \end{aligned} \quad (2.10)$$

This inequality demonstrates that the orders of f and g on neighboring divisors influence the minimum (generic) order of f and g on Σ itself; the higher these orders become on neighboring divisors, the higher must be the order on Σ . We will see the utility of this in many of the following calculations.

We mention here that this kind of analysis can be rephrased in terms of more explicit sheaves. Instead of speaking only of sections of $\mathcal{O}(-kK)$ on the base, it is possible to describe the leading nonvanishing term in f, g around any given divisor in terms of sections of a line bundle over that divisor. To this end, consider a divisor Σ of interest, which can locally be defined as the set $\{z = 0\}$ for some coordinate z . Then any section $s \in \mathcal{O}(-kK)$ can be expanded as a Taylor series in z : $s = \sum_{i=0} s_i z^i$ locally. As derived in [38], the leading nonvanishing coefficient s_i in this expansion may be considered as a section of a sheaf defined over the rational curve Σ ; moreover this sheaf is explicitly given as

$$s_i \in \mathcal{O}_{\Sigma=\mathbb{P}^1}(2k + (k-i)n - \sum_j \phi_j) \quad (2.11)$$

In this formula, n is the self-intersection number of Σ and the sum adds the orders ϕ_j of s on Σ_j for all neighbors Σ_j of Σ (with appropriate multiplicity if the intersection has multiplicity greater than one). We will also have use for this formulation in what follows. Just like the above Zariski formula, it can be used to determine the minimal order of vanishing of f and g on a divisor of interest, incorporating information about the orders of f and g on neighboring divisors; this task is easily accomplished by identifying the smallest i such that $s_i \in \mathcal{O}(m)$ for nonnegative m . This is the first nonvanishing term in the expansion $s = \sum_i s_i z^i$ and therefore the order of s on Σ is i . (One can check that this reproduces the above formula 2.10.)

The preceding analysis is useful in determining the leading nonvanishing terms in f, g on each divisor and the corresponding non-Higgsable gauge groups over the given base. In order to analyze tunings of the Weierstrass model over various divisors, while this abstract approach is in principle possible to extend and implement, it is helpful to have a more explicit presentation of the sections s_i in terms of monomials. When Σ is a rational curve (equivalent to \mathbb{P}^1), and any other curves with which it intersects are other rational curves connected by single transverse intersections in a linear chain, we can give a complete and explicit description of the local coordinates on Σ and its neighbors using the framework of toric geometry. In particular, in this case, we may complete the local coordinate system around $\Sigma = \{z = 0\}$ with a coordinate w on Σ (which could be a local defining coordinate for one of Σ 's neighbors). Then the statement $f_i \in \mathcal{O}(j)$ says that f_i is an order j polynomial in w , and the expressions (2.11) are precisely reproduced by an analysis in local toric coordinates. Furthermore, this expression holds for all values of i , not just the first nonvanishing term, since the toric coordinates act as

global coordinates. In the following analysis, therefore, we focus on explicit local constructions of tunings in the toric context, and freely use the language of toric geometry, which we now review briefly. Our use of toric geometry should always be understood as a convenient way to do calculations in local coordinates that are valid for genus zero curves intersecting with multiplicity one. This kind of local analysis thus allows us to compute tunings on sets of curves that can be locally described torically, even if the full base geometry is not a toric surface. When the base is itself a compact toric variety, toric coordinates can be used to cover the full base and we can completely control the Weierstrass model in terms of monomials in the toric language.

Here we recall some notions and notations from toric geometry; interested readers may consult excellent references such as [39] for more background. Most of the relevant concepts are described in this context and in more detail in [14]. A toric variety can be described by a fan, which for a two (complex) dimensional variety is characterized by a collection of r integral vectors $\{v_i\}_{i=1}^r$ in the lattice $N = \mathbb{Z}^2$, each of which represents a rational curve in a toric surface. We restrict attention to smooth toric varieties, where v_i, v_{i+1} span a unit cell in the lattice, associated with a 2D cone describing a point in the toric variety where a pair of local coordinates vanish. A rational curve of self-intersection $-n$ satisfies $nv_j = v_{j-1} + v_{j+1}$. A compact toric variety also has a 2D cone connecting v_r, v_1 . The principal formula we will borrow from toric geometry describes a basis of sections of line bundles over a toric variety, with fixed vanishing order on D_j ,

$$\mathcal{S}(-kK)_{D_j, n_j} = \text{span}(\{m \in M \mid m \cdot v_i \geq -n \text{ \& } m \cdot v_j = -k + n_j\}) \quad (2.12)$$

The *lhs* denotes sections of $-kK$ that vanish to order exactly n_j on the particular divisor D_j associated to v_j , where $-K = D_j + \sum_{i \neq j} D_i$. (Taken together for all $n_j \geq 0$, this reproduces the full collection of sections of $\mathcal{O}(-kK)$ without poles.) The additional constraints indexed by i correspond to conditions imposed from other toric rays. The *rhs* is the span of a basis of sections of $-nK$ with the desired orders of vanishing. Finally, M denotes the dual lattice to $N = \mathbb{Z}^2$. In §4 and beyond, this formula is used frequently.

Note that unlike in [14], we are not necessarily performing a global analysis of toric monomials. For a local analysis on a single divisor we only include the rays $i = j \pm 1$ adjacent to v_j in the toric fan, while by increasing the number of rays we can include further adjacent divisors in a linear chain, or by including all rays in the toric fan we can consider a global analysis on a toric base B .

2.3 6D supergravity

In the classification of 6D supergravity (SUGRA) vacua, one can bring to bear the additional tool of anomaly cancellation, which turns out to be quite powerful. The Green-Schwarz mechanism is possible in 6D if and only if the anomaly polynomial factorizes, which can be rephrased as a set of equations on various group theory quantities derived from simple factors of the gauge group and their representations [5, 6, 40]. In fact, these equations are

restrictive enough to strongly constrain the set of possible 6D supergravity theories that can be realized from F-theory or any other approach [41, 9]. These relations can furthermore determine uniquely the matter content of the 6D theory in many cases. Vectors in the anomaly polynomial, which lie in the lattice of charged dyonic strings, map directly to certain divisors in $H_2(B, \mathbb{Z})$, which for F-theory constructions enables computation of the low-energy spectrum of the theory and associated constraints purely in terms of easily computed quantities in the base; this greatly simplifies the implementation of anomaly constraints in the F-theory context. The close connection between F-theory geometry and 6D supergravity theories is described in, among other places, [7, 9, 10]. A detailed description of the low-energy supergravity action for 6D F-theory compactifications from the M-theory perspective is given in [42].

For an F-theory compactification on a base B with canonical class K and nonabelian gauge group factors G_i associated with codimension one singularities on divisors Σ_i , the anomaly cancellation conditions are [5, 6, 40], as summarized in [9]

$$H - V = 273 - 29T \quad (2.13)$$

$$0 = B_{\text{adj}}^i - \sum_{\mathbf{R}} x_{\mathbf{R}}^i B_{\mathbf{R}}^i \quad (2.14)$$

$$K \cdot K = 9 - T \quad (2.15)$$

$$-K \cdot \Sigma_i = \frac{1}{6} \lambda_i \left(\sum_{\mathbf{R}} x_{\mathbf{R}}^i A_{\mathbf{R}}^i - A_{\text{adj}}^i \right) \quad (2.16)$$

$$\Sigma_i \cdot \Sigma_i = \frac{1}{3} \lambda_i^2 \left(\sum_{\mathbf{R}} x_{\mathbf{R}}^i C_{\mathbf{R}}^i - C_{\text{adj}}^i \right) \quad (2.17)$$

$$\Sigma_i \cdot \Sigma_j = \lambda_i \lambda_j \sum_{\mathbf{RS}} x_{\mathbf{RS}}^{ij} A_{\mathbf{R}}^i A_{\mathbf{S}}^j \quad (2.18)$$

where $A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}$ are group theory coefficients defined through

$$\text{tr}_{\mathbf{R}} F^2 = A_{\mathbf{R}} \text{tr} F^2 \quad (2.19)$$

$$\text{tr}_{\mathbf{R}} F^4 = B_{\mathbf{R}} \text{tr} F^4 + C_{\mathbf{R}} (\text{tr} F^2)^2, \quad (2.20)$$

λ_i are numerical constants associated with the different types of gauge group factors (*e.g.*, $\lambda = 1$ for $SU(N)$, 2 for $SO(N)$ and G_2, \dots), and where $x_{\mathbf{R}}^i$ and $x_{\mathbf{RS}}^{ij}$ denote the number of matter fields that transform in each irreducible representation \mathbf{R} of the gauge group factor G_i and (\mathbf{R}, \mathbf{S}) of $G_i \otimes G_j$ respectively. (The unadorned “tr” above denotes a trace in the fundamental representation.) Note that for groups such as $SU(2)$ and $SU(3)$, which lack a fourth order invariant, $B_{\mathbf{R}} = 0$ and there is no condition (2.14). The group theory coefficients for matter representations that appear in generic tunings of the various nonabelian group factors and the values of λ for different groups are compiled in Appendix B.

For groups such as $SU(N)$, $N > 2$, which have a quartic Casimir, the coefficients $B_{\mathbf{R}}$ exist and are nonzero for most representations. For such groups, the anomaly conditions give three independent constraints on the matter spectrum. Thus, these constraints can always be solved

in terms of three basic representations. For each such group, generic F-theory tunings will produce matter in a standard set of representations; for example, for $SU(N)$, a generic tuning gives a combination of matter in the fundamental, two-index antisymmetric, and adjoint representations. For generic tunings, the number of adjoint representations is given by the genus of the curve supporting the group. Any other, more exotic, representation will always be *anomaly equivalent* [9, 10] to a linear combination of the three basic representations. We primarily focus here on the generic representation content associated with minimal (Tate) tunings; exotic matter is discussed in §7. Note that many groups, particularly the exceptional groups and $SU(2)$, have no quartic Casimir, and thus (2.14) identically vanishes. For these groups, there are only two independent anomaly constraints, and the generic matter content consists of only the fundamental and adjoint representations. This is discussed further in §4.4.

It is also worthwhile to comment briefly here on the purely gravitational anomaly condition (2.13). For a global F-theory model this constrains the number of moduli in the theory. While we are primarily focused on local constraints here, it must be kept in mind that a global model must satisfy (2.13), and in principle this constraint can produce additional limits on what may be tunable in a given Weierstrass model. In fact, in most cases it seems that the limits on tuning can be determined purely from combinations of local constraints, so that the gravitational anomaly is generally automatically satisfied by an F-theory model when all local constraints are satisfied; it is not, however, proven at this time that this must always be true. We focus on deriving local constraints in this paper, but occasionally reference the connection to the global gravitational anomaly constraint.

In this paper we use the anomaly cancellation conditions to help constrain the possibilities for F-theory tunings. We are also interested in exploring the “swampland” [24] of models that appear consistent from known low-energy considerations but are not realized in F-theory. The 6D anomaly conditions as well as other constraints such as the sign of the gauge kinetic term can be used to strongly constrain 6D supergravity theories based on the consistency of the low-energy theory. All consistent F-theory models should satisfy these constraints; otherwise F-theory would be an intrinsically inconsistent theory of quantum gravity. It has been conjectured [23] that all consistent 6D $\mathcal{N} = 1$ supergravity theories have a description in string theory. Given the close correspondence between the low-energy theory and the geometry of F-theory, and the fact that essentially all known consistent 6D SUGRA spectra that come from string theory can be realized in F-theory, it seems that F-theory may have the ability to realize the full moduli space of consistent 6D supergravity theories. Thus, we highlight particularly those cases where a given tuning seems consistent from low-energy considerations but does not have a known construction through an F-theory Weierstrass model.

2.4 6D SCFTs

In [18], Heckman, Morrison, and Vafa proposed a method of generating 6D SCFTs through F-theory. Here we perform only a cursory review. One of the crucial ingredients in the classification of [18], as in the classification of 6D supergravity theories, is the set of non-Higgsable clusters, which form basic units for composing 6D SCFTs.

To decouple gravity, F-theory is taken on a non-compact manifold (cross $\mathbb{R}^{5,1}$) containing some set of seven-branes wrapped on various closed cycles in the base. This defines a field theory, which should flow to an SCFT under RG. Length scales are removed by simultaneously contracting all the relevant 2-cycles (divisors) in the base geometry to zero size. Whether this is possible in a given geometry can be determined by investigating the adjacency matrix [18] with entries defined by

$$A_{ij} := -(D_i \cap D_j) \quad (2.21)$$

If this matrix is positive definite, then all two-cycles can be contracted simultaneously; otherwise, they cannot. It is interesting to note that no closed circuit of two-cycles with nontrivial π_1 can satisfy this condition. On a compact base, such cycles of divisors always exist.

The part of the classification that we carry out in this paper that relates to tunings on local configurations of negative self-intersection curves can be applied to the construction and classification of 6D SCFT's. In a recent and quite comprehensive work [19], the authors adopted a related (“atomic”) perspective on classifying 6D SCFTs via the F theory construction. This work was posted during the completion of this paper, and overlaps with the relevant parts of this work. Where there is overlap, our results are in agreement with those of [19]. Our investigation differs in some aspects, mainly related to the fact that we do not restrict to the study of SCFTs but are instead interested in using these tunings in SUGRA as well, so that we are studying a much broader range of possible tunings, including on curves of nonnegative self-intersection, and computing Hodge number shifts, which are irrelevant for 6D SCFTs. Our results also extend those of [19] in that while most of the computations in that paper were based on field theory considerations, particularly anomaly cancellation, we have also explicitly analyzed the local geometry in all the cases relevant to 6D SCFTs. This more detailed analysis confirms the close correspondence between field theory and geometry in those situations relevant to SCFTs, but also highlights some specific new cases where field theory and geometry seem to disagree.

The superconformal field theory perspective also suggests an additional class of low-energy constraints that do not follow directly from anomalies. In particular, it was argued in [18] that the detailed constraints identified in [13] on the combinations of non-Higgsable clusters that can be connected by a -1 curve can be understood from an “ E_8 ” rule stating that the global symmetry of the SCFT on any contracted -1 curve not carrying a gauge group should be E_8 , so that the combination of gauge algebras of other curves intersected by the -1 curve should be a subalgebra of E_8 . This logic suggests that even tuned gauge groups on curves intersecting a -1 curve without its own gauge algebra should obey the same constraint. In this paper (§6.3) we explore the extent to which this extension of the E_8 rule holds for tuned F-theory models, and speculate on an extension to curves of self-intersection 0. While we find that the E_8 rule is satisfied for tuned models as well as for NHC's, we also identify some cases of tunings that satisfy this rule but do not admit realization in F-theory using Tate-based tunings, posing a puzzle for low-energy consistency conditions.

2.5 Calabi-Yau threefolds

One of the primary goals of this work is to use tunings as a means of exploring and classifying the space of elliptic Calabi-Yau threefolds. For any given elliptically fibered CY threefold X with a Weierstrass description over a given base B , the Hodge numbers of X can be read off from the form of the singularities and the corresponding data of the low-energy theory. A succinct description of the Hodge numbers of X can be given using the geometry-F-theory correspondence [2, 43, 8]

$$h^{1,1}(X) = r + T + 2 \quad (2.22)$$

$$h^{2,1}(X) = H_0 - 1 = 272 + V - 29T - H_{\text{charged}} \quad (2.23)$$

Here, $T = h^{1,1}(B) - 1$ is the number of tensor multiplets in the 6D theory; r is the rank of the 6D gauge group and V is the number of vector multiplets in the 6D theory, while H_0 and H_{charged} refer to the number of 6D matter hypermultiplets that are neutral/charged with respect to the *Cartan subalgebra* of the gauge group G . The relation (2.22) is essentially the Shioda-Tate-Wazir formula [44]. The equality (2.23) follows from the gravitational anomaly cancellation condition in 6D supergravity, $H - V = 273 - 29T$, which corresponds to a topological relation on the Calabi-Yau side that has been verified for most matter representations with known nongeometric counterparts [7, 10]. The nonabelian part of the gauge group G can be read off from the Kodaira types of the singularities in the elliptic fibration according to Table 1 (up to the discrete part, which does not affect the Hodge numbers and that we do not compute in detail here).

One use of these conditions is to compute the shifts in Hodge numbers for a given tuning of an enhanced gauge group on a given divisor or set of divisors. In many of the local situations we consider here, we can directly compute the shift in the Hodge number $h^{2,1}$ by determining the number of complex degrees of freedom (neutral scalar fields) that must be fixed in the Weierstrass model to realize the desired tuning. In other cases, where we do not have a local model, we can use (2.23) to compute $h^{2,1} = H_0 - 1$ for a tuning based simply on the spectrum of the theory. Note that $h^{1,1}$ follows simply from the gauge group and number of tensors, and does not depend upon the detailed matter spectrum. One subtlety is that in cases where a tuned group can be broken to a smaller group without decreasing the rank, in particular for $G_2 \rightarrow SU(3)$, $F_4 \rightarrow SO(8)$, and $SO(2N+1) \rightarrow SO(2N)$, the charged fields under the larger group that are uncharged under the smaller group of equal rank (and which do not carry charge under any other group) still contribute to H_0 and $h^{2,1}(X)$ as neutral multiplets even from the larger group as they are uncharged under the Cartan subalgebra [2]², so that the Hodge numbers of the Calabi-Yau do not change in such a breaking. This phenomenon will be treated in more detail elsewhere [45]. Note that a somewhat related situation in the $\mathcal{N} = 2$ 4D context is discussed in [46]. In these situations, in the low-energy theory the additional vector fields in the larger nonabelian group cancel in the anomaly conditions with the remaining charged fields in a charged multiplet; *e.g.*, for $\mathfrak{g}_2 \rightarrow \mathfrak{su}(3)$, a $\mathbf{7} \rightarrow \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1}$, the

²Thanks to Y. Wang for discussions on this point

$\mathbf{1}$ acts as a neutral scalar, and the $\mathbf{3} + \bar{\mathbf{3}}$ cancel the additional six vector bosons in \mathfrak{g}_2 . This is relevant for many of the tunings discussed here. For clarity, when performing tunings we compute explicitly the shift in the number of completely uncharged hypermultiplets H_u ; in all cases except tunings of $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{so}(2N+1)$ these correspond precisely to a shift in $h^{2,1}$, while in the case of \mathfrak{g}_2 , *etc.*, the shift in $h^{2,1}$ should be that associated with the equal-rank tuning with $\mathfrak{su}(3)$ *etc.* so the shift in Hodge numbers can be determined by considering the related model that is reached after a rank-preserving breaking. In the latter cases, where $H_u \neq H_0$, we denote the shift in H_u in brackets $[\Delta H_u]$ to indicate this distinction.

In cases where we have a global toric model, there is a direct relationship between H_0 and the number W of Weierstrass moduli given by the toric monomials in M that describe f, g . This relationship is given by

$$H_0 = W - w_{\text{aut}} + N_{-2}, \quad (2.24)$$

where N_{-2} is the number of -2 curves (on which the discriminant does not identically vanish), and w_{aut} is the number of automorphisms of the base, given by 2 for a generic base with no toric curves of self-intersection 0 or greater and adding $n + 1$ for every toric curve of self-intersection $n \geq 0$. This formula allows us to directly compute the shift in $h^{2,1}$ even in local toric models by computing the local change in this quantity. There is one further additional subtlety [15], which is that certain combinations of -2 curves form degenerate elliptic curves; such configurations have an effective value of N_{-2} that must be decreased by 1. We encounter this subtlety in §5.2.

One of the goals of this paper is to continue to develop a systematic set of tools for classifying elliptic Calabi-Yau threefolds through F-theory. This might seem like the reverse of the logical order: to apply F theory, one needs to know about (elliptically fibered) Calabi-Yaus. But there are still many unanswered questions about Calabi-Yau threefolds in general; for example, it is still unknown whether there are a finite or infinite number of topological types of non-elliptic Calabi-Yau threefolds. Some evidence suggests [47, 43, 48, 49, 17, 50] that, particularly for large Hodge numbers, a large fraction of Calabi-Yau threefolds and fourfolds that can be realized using known construction methods are elliptically fibered. Since the number of elliptic Calabi-Yau threefolds is finite this suggests that the number of Calabi-Yau threefolds may in general be finite, and that understanding and classifying elliptic Calabi-Yau threefolds may give insights into the general structure of Calabi-Yau manifolds. As an example of how the methods developed here can be used in classification of elliptic Calabi-Yau threefolds, in §10.1 we identify several large classes of known Calabi-Yau threefolds in the Kreuzer-Skarke database as tunings of generic elliptic fibrations over allowed bases.

In the context of classification of Calabi-Yau threefolds, there is an additional point that should be brought out. Our classification is essentially one of Weierstrass models, which contain various Kodaira singularity types. While any elliptic Calabi-Yau threefold has a corresponding Weierstrass model, the Weierstrass models for any theory with a nontrivial Kodaira singularity type, corresponding to a nonabelian gauge group in the low-energy 6D F-theory model, have singular total spaces. The singularities in the total space must be

resolved to get a smooth Calabi-Yau threefold. This resolution at the level of codimension one singularities maps essentially to Kodaira’s original classification of singularities. Resolutions at codimension two, however, are much more subtle, and in many cases a singular Weierstrass model can have multiple distinct resolutions at codimension two, corresponding to different Calabi-Yau threefolds with the same Hodge numbers but different triple intersection numbers. There has been quite a bit of work in recent years on these codimension two resolutions in the F-theory context [33, 51–56], but there is as yet no complete and systematic description of what elliptic Calabi-Yau threefolds can be related to a given Weierstrass model. For the purposes of classifying 6D F-theory models this distinction is irrelevant, but it would be important in any systematic attempt to completely classify all smooth elliptic Calabi-Yau threefolds.

3 Outline of results

The following three sections represent the core of this work. In them, we present and derive a set of fairly simple rules that can be used to determine which gauge symmetries and matter representations are allowed, given the local geometric data of a set of one or more intersecting divisors within a complex base surface appropriate for F-theory. For each tuning over the local divisor geometry, we compare the constraints given from low-energy consistency conditions to the possibility of an explicit F-theory construction. Before diving into details, we pause to delineate our results and outline our methods and strategy.

The setting of our analysis is 6D F-theory, *i.e.* F-theory compactified on a Calabi-Yau threefold that results from an elliptic fibration with section over a two (complex)-dimensional base manifold B . We focus on local combinations of effective divisors (curves) in B . We focus particularly on smooth rational (genus 0) curves that intersect pairwise in a single point. These cases are particularly amenable to study: we can analyze them locally using toric methods, they are the only divisor combinations needed to tune elliptic Calabi-Yau threefolds that arise as hypersurfaces in toric varieties as studied in [57], and they are the only configurations needed for analyzing 6D SCFTs. For these combinations of curves, we carry out a thorough analysis using both the field theory (anomaly) approach and a local geometric approach for explicit construction of Weierstrass and/or Tate models. In these cases we can confirm that, with a few notable exceptions that we highlight, the anomaly constraints match perfectly with the set of configurations that is allowed in a local Weierstrass model. In addition to these cases where we have both local geometry and field theory control of the configuration, we also consider more briefly more general configurations needed to complete the classification of tunings over a generic base, including higher genus curves (§4.4), exotic matter representations that can arise for non-generic tunings on smooth curves or tunings on singular curves (§7), and tuning of abelian gauge symmetries, which requires global structure through the Mordell-Weil group (§8).

The results of our analysis could be applied in a variety of ways. Most simply, they provide a toolkit for easily developing a broad range of examples of 6D F-theory supergravity models and corresponding elliptic Calabi-Yau threefolds and/or 6D SCFTs; given a base geometry

one can construct a set of tuned models with any particular desired properties subject to constraints imposed by the base geometry. More generally, these results can be used in a systematic classification of 6D supergravity models or SCFTs. A complete list of toric bases that support 6D supergravity models was computed in [14]. The results presented here in principle give the local information needed to construct all possible tunings on toric curves over these bases, which could be used to compare with the Kreuzer-Skarke database [58] to give an interpretation of many of the constructions in that large dataset in terms of elliptic fibrations and to identify those examples of Calabi-Yau threefold that are *not* elliptically fibered. The broader set of constraints described here for more general tunings in principle gives the basic components needed for a systematic classification of all tunings, including on non-toric curves over generic bases. Combined with the systematic classification of bases [16], this provides a framework for the complete classification of all Weierstrass models for elliptic Calabi-Yau threefolds. A more detailed description of how such an algorithm would proceed is given in §9. Note that in this more general context, and even to some extent in the more restricted toric context, our rules really only provide a superset of the set of allowed tunings. The local rules that we provide, even when supported by local explicit Weierstrass constructions, must be checked for a global tuning for compatibility by explicit construction of a global Weierstrass model that satisfies all the conditions needed for the tuning. While we expect that at least in the toric context, local rules are essentially adequate for determining the set of allowed tunings, in a more global context this is less clear. For toric bases, there is an explicit description of the Weierstrass model in terms of monomials [14], so that, at least for tuning over toric divisors in toric bases, the technology for producing a global Weierstrass model is available. For more general bases, or tunings over non-toric curves, a concise and effective approach for tuning Weierstrass models is not at present known to the authors.

Within this setting, we summarize the results of the following sections. These results can be summarized in terms of the following data: given a base-independent local collection of divisors $\{D_i\}$ with given genera and self- and mutual intersections, we determine a list $\mathcal{L}[\{D_i\}]$ of the possible gauge symmetries over these divisors, along with the matter representations and shifts in the Hodge numbers $(\Delta h^{1,1}, \Delta h^{2,1})$ between the generic and tuned models.

- Section 4 analyzes tunings $\mathcal{L}[\Sigma]$ for isolated divisors Σ with $-12 \leq \Sigma \cdot \Sigma$. Curves of self-intersection below -12 cannot arise in valid F-theory bases, and no tuning is possible over any curve with self intersection below -6 . Local models are used to describe all tunings on genus 0 curves, and tunings on higher genus curves are constrained through anomalies.
- Section 5 determines $\mathcal{L}[C]$ for NHCs C that consist of strings of multiple intersecting divisors. Explicitly, these are the multi-curve NHCs $(-3, -2, -2)$, $(-2, -3, -2)$, $(-3, -2)$, and clusters of -2 curves of arbitrary size. (There are in practice bounds on the size and complexity of such -2 clusters that can appear F-theory SUGRA bases, some of which we discuss here). Local toric models are used for the NHCs with -3 curves, and

a simple “convexity” feature is used to classify tunings over -2 clusters, the validity of which is checked in Tate models in §6.

- Section 6 analyzes multiple intersecting curves beyond the NHCs. We show that there are only five combinations of gauge algebras (or families thereof) that can be tuned on intersecting pairs of divisors, and analyze the constraints on these combinations using local (largely Tate) methods. We also consider constraints on tunings of multiple branes intersecting a single brane, both when the single brane carries a gauge group and when it does not. The latter case includes the “ E_8 rule” [18] governing what gauge groups can be realized on divisors that intersect a -1 curve, which we generalize to include tunings, and a similar but weaker rule for curves of self-intersection zero.
- Section 7 gives some further rules that apply for tuning exotic matter representations with a finer tuning that leaves the gauge group (and $h^{1,1}(X)$) invariant while modifying the matter content. The underlying F-theory geometry and corresponding mathematical structure of non-Tate Weierstrass models is only partially understood at this point so this set of results may be incomplete.
- Section 8 gives a guide to tuning abelian gauge factors over a given base. While much is known and we can make some clear statements about tunings and constraints, this is also a rapidly evolving area of research and this set of results may also be improved by further progress in understanding such models.

4 Classification I: isolated curves

In this section we consider all possible tunings of enhanced groups on individual divisors in the base. In general, a divisor in the base is a curve Σ of genus g and self-intersection $\Sigma \cdot \Sigma = n$. In this section we concentrate on generic tunings of a given gauge group, which means that the curve Σ is generally smooth, and supports only certain generic types of matter. For example, for $\mathfrak{su}(N)$ a generic Weierstrass tuning on a genus 0 curve will give matter only in the fundamental (N) and antisymmetric ($N(N-1)/2$) representations; when the genus g is nonzero, there are also g adjoint (N^2-1) matter fields. Further tunings that keep the gauge group fixed but enhance the matter content are discussed in §7.

For each type of curve and gauge group we consider both anomaly constraints and the explicit tuning through the Weierstrass model of the gauge group. We focus primarily on rational (genus 0) curves. For individual rational curves we find an almost perfect matching between those tunings that are allowed by anomaly cancellation and what can be realized explicitly in F-theory Weierstrass models. For curves of negative self-intersection that can occur in non-Higgsable clusters, and for exceptional algebras tuned on arbitrary rational curves, we compute the Hodge shifts explicitly in Weierstrass models and confirm the match with anomaly conditions. For curves of self-intersection -2 and above supporting the classical

series $\mathfrak{su}(N)$, $\mathfrak{sp}(N)$, and $\mathfrak{so}(N)$, we use the Tate method to construct Weierstrass models explicitly, and anomaly cancellation to predict the Hodge number shifts.

A summary of the allowed tunings on isolated genus 0 curves and associated Hodge shifts are presented in Table 4. These tunings listed are all those that may be allowed by anomaly cancellation. The details of the analysis for these cases are presented in §4.1-§4.3. In §4.4 we use anomaly cancellation to predict the possible tunings and Hodge number shifts for tunings on higher genus curves, though we do not compute these explicitly using local models as the local toric methodology we use here is not applicable in those cases. The upshot of this analysis is that for genus 0 curves, virtually everything in Table 4 that is allowed by anomaly cancellation can be realized explicitly in Weierstrass models, with the exception of some large $SU(N)$ tunings on curves of self-intersection -2 or greater, as discussed explicitly in §4.3.

\mathfrak{g}	matter	$(\Delta h^{1,1}, \Delta H_u)$
$\mathfrak{su}(2)$	$(6n + 16)\mathbf{2}$	$(1, -12n - 29)$
$\mathfrak{su}(N)$	$((8 - N)n + 16)\mathbf{N}$ $+(n + 2)\frac{N(N-1)}{2}$	$(N - 1, -(\frac{15N-N^2}{2})n - (15N + 1))$
$\mathfrak{sp}(N/2)$	$((8 - N)n + 16)\mathbf{N}$ $+(n + 1)\left(\frac{N(N-1)}{2} - 1\right)$	$\left(N/2, \left[-(\frac{15N-N^2-2}{2})n - \left(\frac{31N-N^2}{2}\right)\right]\right)$
$\mathfrak{so}(N)$	$(n + (N - 4))\mathbf{N}$ $+(n + 4)2^{5-\lfloor(N+1)/2\rfloor}\mathbf{S}$	$\left(\lfloor N/2 \rfloor - r_*, -h_* - (N + 16)n - (\frac{N^2-7N+128}{2})\right)$
\mathfrak{g}_2	$(3n + 10)\mathbf{7}$	$(2 - r_*, [-7(3n + 8) - h_*])$
\mathfrak{f}_4	$(n + 5)\mathbf{26}$	$(4 - r_*, [-26(n + 3) - h_*])$
\mathfrak{e}_6	$(n + 6)\mathbf{27}$	$(6 - r_*, (78 - h_*) - 27(n + 6))$
\mathfrak{e}_7	$(4 + n/2)\mathbf{56}$	$(7 - r_*, (133 - h_*) - 28(n + 8))$

Table 4. Possible tunings of gauge groups with generic matter on a curve Σ of self-intersection n , together with matter and shifts in Hodge numbers, computed from anomaly cancellation conditions. For algebras that can be obtained from Higgsing chains from \mathfrak{e}_8 , these matter contents were previously computed in [31]. r_*, h_* are the rank and difference $V - H_{\text{charged}}$ of any non-Higgsable gauge factor that may exist on the curve before tuning. Note that tunings are impossible when the formula for the multiplicity of representations yields a negative number or a fraction. (Multiplicities of $\frac{1}{2}$ are allowed when the representation in question is self-conjugate.) N for $\mathfrak{sp}(N/2)$ is assumed to be even. H_u is total number of uncharged scalars. In cases that admit rank-preserving breaking the Hodge numbers are given by those of the model after the breaking; brackets indicate cases where $H_u \neq H_0$.

To make the method of analysis completely transparent, we carry out an explicit computation of the possible tunings on a -3 curve using both anomaly and Weierstrass methods in §4.1. This example demonstrates how such anomaly calculations and local toric calculations are done in practice. It also serves to highlight the non-trivial agreement between these completely distinct methods: both at the gross level of which algebras can be tuned, but also at the detailed level of Hodge number shifts. The corresponding calculations for curves of

self-intersection -4 and below, as well as the multiple-curve non-Higgsable clusters, can be found in Appendix C.

Before proceeding with the example calculation, we should note that the core of this section's results, Table 4, can be found to a large extent in a corresponding table in [31]. Our version differs from that in [31] in two respects: we include algebras that are not subalgebras of \mathfrak{e}_8 , and we also include shifts in Hodge numbers that result from implementing these tunings. This extra information is essential in order to use these tunings as an organizational tool to search through the set of elliptically fibered threefolds. Finally, our analysis of local Weierstrass models allows us to determine that virtually all of these configurations that are allowed by anomaly analysis are actually realizable locally in F-theory, with some specific possible exceptions that we highlight.

Following the extended example of tunings on a -3 curve, we give a general analysis of tunings on curves of self-intersection -2 and above; these cases can be uniformly described in a single framework. In these sections and in the Appendix, we discuss all possible tunings except for $\mathfrak{so}(N)$, because these tunings are particularly delicate. $\mathfrak{so}(N)$ tunings are separately described in §4.2.

4.1 Extended example: tunings on a -3 curve

Let us begin with an extended example that will illustrate many of the features of the following computations. On an isolated -3 curve, the minimal gauge algebra is $\mathfrak{su}(3)$, which can be enlarged as

$$\begin{aligned} \mathfrak{g} = \mathfrak{su}(3) &\longrightarrow \mathfrak{g}_2 \longrightarrow \mathfrak{so}(7) \longrightarrow \mathfrak{so}(8) \longrightarrow \mathfrak{f}_4 \longrightarrow \mathfrak{e}_6 \longrightarrow \mathfrak{e}_7 \\ (f, g) = (2, 2) &\longrightarrow \{ \quad (2, 3) \quad \} \longrightarrow \{ (3, 4) \} \longrightarrow (3, 5) \end{aligned} \quad (4.1)$$

The middle three and subsequent two gauge algebras are distinguished by monodromy of the singularity, as per the Kodaira classification; we will describe this in detail below. These tuned algebras and their associated matter all fall in a Higgsing chain from \mathfrak{e}_7 . The complete set of tunings *a priori* allowed on a -3 curve also includes $\mathfrak{so}(N)$ for $8 < N \leq 12$, but these will be discussed in the following section.

4.1.1 Spectrum and Hodge shifts from anomaly cancellation

First we will perform an anomaly calculation; then we will discuss a local toric model (essentially \mathbb{F}_3 with the $+3$ curve removed) on which we can implement these tunings. A tabulation of the relevant anomaly coefficients A_R, B_R, C_R and λ values is given in Appendix B Taking

the “C” condition, we find:

$$\begin{aligned}
\Sigma \cdot \Sigma &= \frac{\lambda^2}{3} \left(\sum_R C_R - C_{Adj} \right) \\
-3 &= \frac{1}{3} \left(\sum_R C_R - 9 \right) \\
0 &= \sum_R C_R
\end{aligned} \tag{4.2}$$

Since all coefficients $C_R > 0$ for $\mathfrak{su}(3)$ (which follows from the definition of C_R and the absence of a quartic Casimir), this implies that no matter transforms under this gauge group; there is only the vector multiplet in the adjoint **8**. This in turn implies that the presence of this gauge algebra contributes to the quantity H_0 (2.23) by an amount $h_* = V - H_{\text{charged}}$ and $r_* = 2$ (this algebra’s rank) to $h^{1,1}$. Since the gauge algebra $\mathfrak{su}(3)$ (with no matter) corresponds to the generic elliptic fibration over a -3 curve (*i.e.*, -3 is an NHC), we conclude that all shifts between the generic case and a tuned case in the Hodge numbers $(\Delta h^{1,1}, \Delta h^{2,1} \sim \Delta H_0) = (\Delta r, \Delta(V - H_{\text{charged}}))$ must be calculated as $(\Delta h^{1,1}, \Delta H_0) = (r_{\text{tuned}} - 2, V_{\text{tuned}} - H_{\text{charged,tuned}} - 8)$, as denoted in Table 4.

With this most generic case in mind, let us calculate the corresponding quantities for \mathfrak{g}_2 . Assuming only fundamental matter³, with a multiplicity N_f , anomaly calculation gives

$$\begin{aligned}
\Sigma \cdot \Sigma &= \frac{\lambda^2}{3} \left(\sum_R C_R - C_{Adj} \right) \\
-3 &= \frac{4}{3} \left(\frac{N_f}{4} - \frac{5}{2} \right) \\
N_f &= 1
\end{aligned} \tag{4.3}$$

The contribution to H_u is (recall that the adjoint of \mathfrak{g}_2 has dimension 14 and the fundamental has dimension 7):

$$\Delta H_u = 14 - 7 - 8 = +7 - 8 = [-1] \tag{4.4}$$

In other words, implementing this tuning decreases H_u by one in comparison to the generic case. Note that, as mentioned in §2.5, one of the charged scalars in the **7** of \mathfrak{g}_2 will really act as a neutral scalar for purposes of computing $h^{2,1}(X)$, since it can be used to break the gauge group without reducing rank. We continue to treat this scalar as charged, without contributing to H_u , here and in the rest of the paper, but this caveat should be kept in mind for all \mathfrak{g}_2 , \mathfrak{f}_4 and $\mathfrak{so}(2n+1)$ tunings, and is indicated by the notation $[-1]$.

³This is the generic matter type expected for \mathfrak{g}_2 . More generally, other C coefficients are $\geq 5/2$ and therefore the presence of even one hypermultiplet in one of these non-fundamental representations makes it impossible to satisfy the C condition on any negative self-intersection curve.

For $\mathfrak{so}(7)$, $C_{Adj} = 3$, which implies [41] that the only relevant representations on negative self-intersection curves are $\mathbf{7}_f$ and $\mathbf{8}_s$. Since $C_f = 0$ and $C_s = \frac{3}{8}$, we have

$$\begin{aligned}\Sigma \cdot \Sigma &= \frac{\lambda}{3} \left(\sum_R C_R - C_{Adj} \right) \\ -3 &= \frac{4}{3} \left(N_s \frac{3}{8} - 3 \right) \\ N_s &= 2\end{aligned}\tag{4.5}$$

One can then use the ‘‘A’’ condition to demonstrate⁴ that $N_f = 0$:

$$\begin{aligned}K \cdot \Sigma &= \frac{\lambda}{6} \left(A_{adj} - \sum_R A_R \right) \\ 1 &= \frac{1}{3} (5 - N_s - N_f) \\ 3 &= (5 - 2 - N_f) \\ N_f &= 0\end{aligned}\tag{4.6}$$

With knowledge of the representation content in hand, we can compute the change in H_u :

$$\Delta H_u = \Delta(V - H) = 21 - 2 \times 8 - 8 = -3.\tag{4.7}$$

Note that the absence of fundamental matter in this case means that there is no rank-preserving breaking $\mathfrak{so}(7) \rightarrow \mathfrak{g}_2$, so that the shift in $H_u = H_0$ is not denoted in brackets. A similar calculation for $\mathfrak{so}(8)$ yields $N_s = 2$, $N_f = 1$, hence

$$\Delta H_u = \Delta(V - H) = 28 - 3 \times 8 - 8 = +4 - 8 = -4\tag{4.8}$$

Proceeding to \mathfrak{f}_4 , we find again that only fundamental matter is possible on a (-3) -curve, and

$$\begin{aligned}\Sigma \cdot \Sigma &= \frac{\lambda}{3} \left(\sum_R C_R - C_{Adj} \right) \\ -3 &= \frac{6^2}{3} \left(\frac{N_f}{12} - \frac{5}{12} \right) \\ -3 &= N_f - 5 \\ N_f &= 2\end{aligned}\tag{4.9}$$

Recalling that the dimensions of the fundamental and adjoint are 26 and 52, respectively, we find

$$\Delta H_u = 52 - 2 \times 26 - 8 = [-8]\tag{4.10}$$

⁴In calculating $K \cdot \Sigma$, we use $(K + \Sigma) \cdot \Sigma = 2g - 2 = -2$ for a genus 0 curve (topologically \mathbb{P}^1).

For \mathfrak{e}_6 , we find

$$\begin{aligned}\Sigma \cdot \Sigma &= \frac{\lambda^2}{3} \left(\sum_R C_R - C_{Adj} \right) \\ -3 &= \frac{6^2}{3} \left(\frac{N_f}{12} - \frac{1}{2} \right) \\ N_f &= 3\end{aligned}\tag{4.11}$$

Given that the dimensions of fundamental and adjoint are 27 and 78, respectively,

$$\Delta H_u = 78 - 3 \times 27 - 8 = -3 - 8 = -11\tag{4.12}$$

Enhancing finally to \mathfrak{e}_7 , we find

$$\begin{aligned}\Sigma \cdot \Sigma &= \frac{12^2}{3} \left(\frac{N}{24} - \frac{1}{6} \right) \\ -3 &= 2(N - 4) \\ N &= \frac{5}{2}\end{aligned}\tag{4.13}$$

which is possible because the fundamental **56** of \mathfrak{e}_7 is self-conjugate, and hence admits a half-hypermultiplet in six dimensions. This contributes to H_u in the amount $+133 - \frac{5}{2}56 = -7$, *i.e.* represents a shift of -4 subsequent to tuning an \mathfrak{e}_6 , or in total $\Delta H_u = -15$ from the generic case of $\mathfrak{su}(3)$.

These calculations can be summarized simply as:

$$\begin{array}{ccccccc} \mathfrak{g} & \mathfrak{su}(3) & \mathfrak{g}_2 & \mathfrak{so}(7) & \mathfrak{so}(8) & \mathfrak{f}_4 & \mathfrak{e}_6 & \mathfrak{e}_7 \\ \hline \Delta H_u & 0 & [-1] & -3 & -4 & [-8] & -11 & -15 \end{array}\tag{4.14}$$

4.1.2 Spectrum and Hodge shifts from local geometry

Now we would like to explain this from a more direct geometric viewpoint. We will find that we must be careful to implement the most generic tuning, which (when we consider monodromy) will not always be obtained simply by setting monomial coefficients to zero. We use a local model that can be considered a convenient way to visualize the monomials in f and g (in local coordinates); alternately, our local models are simply concrete ways of generating the full set of monomials consistent with equations 2.11. Torically, the self-intersection number of any toric divisor $\Sigma \leftrightarrow v_i$ corresponding to v_i in the fan can be determined by the formula $v_{i-1} + v_{i+1} = -(\Sigma \cdot \Sigma)v_i$. Therefore, a linear chain of k rational curves with any specified self-intersection numbers may be realized by a toric fan with $k + 2$ rays, which corresponds to a non-compact toric variety. In this example, we need three rays (corresponding to the -3 curve and its neighbors). Without loss of generality, we take this fan to be $(3, -1), (1, 0), (0, 1)$. Using the methods of section 2, we find that the monomials of $-nK$ are determined to lie within (or on the boundary of) a wedge determined by the conditions: $x \geq -n$, $y \geq -n$, and $y \leq n + 3x$. The first condition is automatically satisfied when the latter two are.

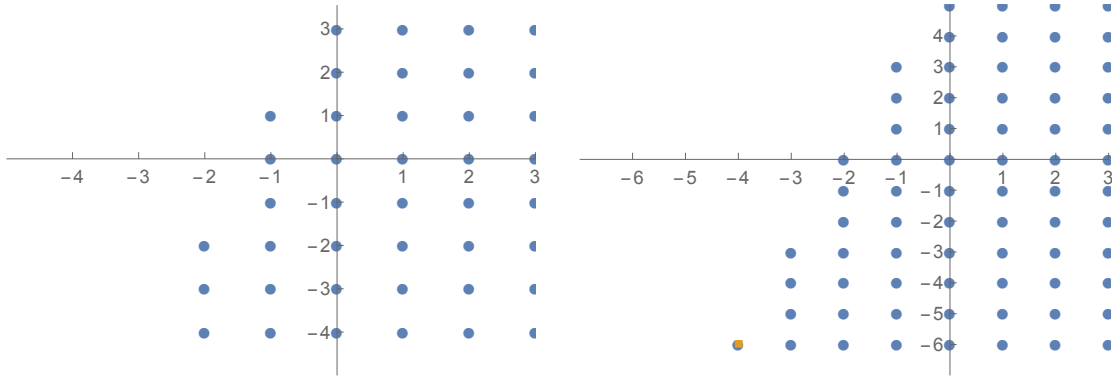


Figure 1. A representation of monomials in $-4K$ (left) and $-6K$ (right) over the local model of a -3 curve Σ and its two neighbors. Both sets of monomials should be considered as extending infinitely in the positive x and y directions. To write these monomials explicitly, we may establish a local coordinate system z, w such that $\Sigma = \{z = 0\}$ (corresponding to the ray $(1, 0)$ of the toric fan) and one of its two neighbors Σ' (corresponding to the ray $(0, 1)$) is $\Sigma' = \{w = 0\}$. Then a monomial $(a, b) \in -kK$ corresponds concretely to $z^{a+k}w^{b+k}$.

In this description of the fan, the -3 curve Σ corresponds to the ray $v = (1, 0)$. We will study the monomials of f ($-4K$) and g ($-6K$) in order to determine the number of degrees of freedom (*i.e.*, complex structures) that must be removed in order to implement a given tuning. We describe the different cases in order:

Case 1: $\mathfrak{su}(3)$: This is the untuned case, apparent from the diagrams. If we put local coordinates z and w such that $\Sigma = \{z = 0\}$ and a neighboring curve (say corresponding to the ray $(0, 1)$) is $\{w = 0\}$, then the monomials in $-kK$ represented by points (a, b) correspond concretely to $z^{a+k}w^{b+k}$. Since the lowest- x monomials in f and g are at -2 and -4 respectively, this implies that the orders (f, g) are $(2, 2)$ on Σ . This places us in Kodaira case *IV*. To determine whether this corresponds to $\mathfrak{su}(3)$ or $\mathfrak{su}(2)$ requires testing a monodromy condition as outlined in *e.g.* [32]. To state this condition, let us expand $f = \sum_i f_i z^i$ and $g = \sum_i g_i z^i$ as Taylor series in z . Explicitly, then, the monodromy condition to check is whether $g_2(w)$ is a perfect square: if it is, the fibration corresponds to an $\mathfrak{su}(3)$; otherwise, it corresponds to $\mathfrak{su}(2)$. In our case, it is clear that $g_2(w)$ (being the constant polynomial in w) is a perfect square. The properties of -3 curve geometry conspire to force us into this usually non-generic branch of the Kodaira type *IV* case. Having established $\mathfrak{su}(3)$ as the base (untuned) case, let us investigate tuned fibrations. Along the way, we will count how many complex degrees of freedom (monomial coefficients) must be fixed in order to tune a given model.

Case 2: \mathfrak{g}_2 : This is the generic case of an $(f, g) = (2, 3)$ type singularity. To implement this tuning, then, all that is required is that g vanish to degree 3, easily accomplished by removing the single monomial in g_2 . Hence, implementing this tuning removes precisely one

degree of freedom

$$\Delta H_u = [-1]$$

as we had concluded earlier using anomaly calculations.

Case 3: $\mathfrak{so}(7)$: We now encounter a more subtle issue of counting. The monodromy conditions that distinguish the three gauge algebras that can accompany a $(2, 3)$ singularity are specified by the factorization properties of the polynomial

$$x^3 + f_2(w)x + g_3(w), \quad (4.15)$$

$$\begin{aligned} x^3 + Ax + B \quad (\text{generic}) &\Rightarrow \mathfrak{g}_2 \\ (x - A)(x^2 + Ax + B) &\Rightarrow \mathfrak{so}(7) \\ (x - A)(x - B)(x + (A + B)) &\Rightarrow \mathfrak{so}(8) \end{aligned} \quad (4.16)$$

The coefficients here are chosen in order to ensure that no quadratic term appears in the total cubic polynomial. To obtain the second condition ($\mathfrak{so}(7)$), we proceed by writing explicitly

$$x^3 + (f_{2,0} + f_{2,1}w + f_{2,2}w^2)x + (g_{3,0} + g_{3,1}w + g_{3,2}w^2 + g_{3,3}w^3). \quad (4.17)$$

This expression uses explicit knowledge of the monomials. Recalling that the order in w of a monomial (a, b) in $-4K$ is $b + 4$, we may read off that the only monomials of f_2 are $\{w^0, w^1, w^2\}$. Similarly, the only monomials available for g_3 are $\{w^0, w^1, w^2, w^3\}$. The seven coefficients above must then be tuned to enforce the appropriate factorization. Expanding the factorized version of the cubic (in x) polynomial, it is clear that we must impose that the coefficient of x be given by $B - A^2$ and that of x^0 given by $-AB$. This can be minimally accomplished by setting to zero the coefficients c and g above. More generally, A and B must be respectively linear and quadratic, with 5 independent degrees of freedom. This represents a loss of two additional degrees of freedom (besides the first, which represented tuning from $\mathfrak{su}(3)$ to \mathfrak{g}_2). Hence

$$\Delta H_u = -3 \quad (4.18)$$

again in accordance with the anomaly results.

Case 4: $\mathfrak{so}(8)$: Consulting the list above, to achieve $\mathfrak{so}(8)$, we must completely factorize the polynomial. Expanding yields the constraints

$$\begin{aligned} a + bw + cw^2 &= -A^2 - AB - B^2 \\ d + ew + fw^2 + gw^3 &= AB(A + B) \end{aligned} \quad (4.19)$$

This requires that now both B and A must be linear in w , so we can for example simply set the f coefficient to zero as well. This removes an additional 1 degree of freedom (beyond the previously removed three) leading to

$$\Delta H_u = -4 \quad (4.20)$$

as expected from anomaly results.

Case 5: \mathfrak{f}_4 : To tune to the $\mathfrak{f}_4/\mathfrak{e}_6$ case, we must enhance the degrees of vanishing of f and g to $(3, 4)$, which requires that we eliminate *all* $(a, b) \in -4K$ with $b \leq -2$ and $(c, d) \in -6K$ with $d \leq -3$. The generic such tuning is an \mathfrak{f}_4 algebra. Inspecting the monomial figure, we find that from the initial (untuned) scenario, this requires us to eliminate the leftmost column of f (3 monomials) and the leftmost two columns of g ($1 + 4$ monomials), so that in total

$$\Delta H_u = [-8] \quad (4.21)$$

as expected from anomaly results.

Case 6: \mathfrak{e}_6 : In this case, the monodromy condition is whether g_4 is a perfect square. Counting up from the left (as before), the available monomials are $\{w^0, w^1, \dots, w^6\}$. We can make this polynomial of degree 6 into a square by restricting it to be of the form

$$g_4(w) = (\alpha + \beta w + \delta w^2 + \epsilon w^3)^2, \quad (4.22)$$

which clearly preserves 4 of the 7 original degrees of freedom. This counting indicates that (as expected) we lose 3 more degrees of freedom in tuning from \mathfrak{f}_4 to \mathfrak{e}_6 , for a total change of

$$\Delta H_u = -11 \quad (4.23)$$

from the original (untuned) $\mathfrak{su}(3)$. Note that this minimal tuning cannot be reached by simply setting to 0 three of the coefficients in g_4 .

Case 7: \mathfrak{e}_7 : Enhancing finally to \mathfrak{e}_7 requires enhancing the degrees of vanishing of (f, g) to $(3, 5)$. Up to this point, we have already enhanced to $(3, 4)$, so it remains only to eliminate the remaining 7 monomials of g_4 , yielding a shift in H_u of -7 in comparison to a tuning of \mathfrak{f}_4 , or a shift of -4 subsequent to tuning \mathfrak{e}_6 , in accordance with anomaly calculations.

We thus have found that explicit computation of Weierstrass models with monomials confirms that all tunings compatible with anomaly cancellation on a -3 curve can be realized in F-theory, with the proper number of degrees of freedom tuned in each case.

4.2 Special case: tuning $\mathfrak{so}(N > 8)$

We have not so far explored the tuning of $\mathfrak{so}(N)$ with $N > 8$, *i.e.* a Kodaira type $I_{m>0}^*$ singularity. Such gauge algebras are non-generic in two ways: 1) they require a vanishing of Δ to order greater than the minimum $\min\{2 \text{ ord}(f), 3 \text{ ord}(g)\}$, and 2) even and odd $\mathfrak{so}(N)$ are distinguished by a subtle monodromy condition [59]. Due to these complications, we have reserved the treatment of these tunings to this section. The results to follow are almost precisely in accord with Table 4, by applying the rule that a tuning is not allowed when the formula for the multiplicity of any of its representations becomes negative or fractional (with fractions of $\frac{1}{2}$ allowed for real representations). However, for clarity, we explicitly list the allowed tunings for -2 , -3 , and -4 curves in Table 5.

Curve	$\mathfrak{so}(N)$	N_f	N_s	$(\Delta h^{1,1}, \Delta H_u)$
-2	7	1	4	$(3, [-18])$
	8	2	4	$(4, -20)$
	9	3	2	$(4, [-23])$
	10	4	2	$(5, -27)$
	11	5	1	$(5, [-32])$
	12	6	1	$(6, -38)$
	13	7	$\frac{1}{2}$	$(6, [-45])$
-3	7	0	1	$(1, -3)$
	8	1	2	$(2, -4)$
	9	2	1	$(2, [-6])$
	10	3	1	$(3, -9)$
	11	4	$\frac{1}{2}$	$(3, [-13])$
	12	5	$\frac{1}{2}$	$(4, -18)$
-4	N	$N - 8$	0	$(\lfloor (N - 8)/2 \rfloor, N(-\frac{N}{2} + \frac{15}{2}) + 28)$

Table 5. Allowed $\mathfrak{so}(N \geq 7)$ representations and associated matter. No $\mathfrak{so}(N)$ tuning is allowed on a curve of self-intersection ≤ -5 . All shifts are from the generic gauge algebras: \emptyset , $\mathfrak{su}(3)$, and $\mathfrak{so}(8)$, respectively.

The situations in which these complications arise on curves of negative self-intersection are quite limited. For instance, it is impossible for $\mathfrak{so}(N)$ to arise on a curve of self-intersection ≤ -5 . This is a straightforward consequence of the NHC classification, which dictates that such curves must at least host algebras of \mathfrak{f}_4 or $\mathfrak{e}_{6,7,8}$. Because no $\mathfrak{so}(N)$ contains any exceptional algebra as a subalgebra, we can conclude that these low self-intersection curves cannot be enhanced to any $\mathfrak{so}(N)$. We focus here on tunings only on individual curves of self-intersection -4 , -3 , and -2 , excluding tunings on clusters of more than one curve. These are the cases relevant for tunings on NHC's. Indeed, it is easy to see, upon consulting the analysis of -2 chains, that higher $\mathfrak{so}(N)$'s are impossible on any chain of more than one -2 curve. Therefore such tunings, if they occur, can do so only on isolated -2 , -3 , or -4 curves. We proceed to analyze each case separately. Tunings of $\mathfrak{so}(N)$ on individual curves of more general self-intersection are treated in the following subsection.

As is our general strategy, we examine each potential tuning both geometrically (in a local toric model) and from the standpoint of anomaly cancellation. We again find that these methods agree. The structure of this section is slightly different from previous ones: we first perform anomaly calculations for curves of self-intersection -4 , -3 , and -2 ; and then perform geometric calculations for these curves. We will see that tunings are limited by the existence of spinor representations—they become too large to satisfy anomaly cancellation. Geometrically, this will manifest as a $(4, 6)$ singularity at a codimension-2 (*i.e.* dimension-0) locus in the base which corresponds to the location of spinor matter. Let us now see how this unfolds explicitly.

4.2.1 Spectrum and Hodge shifts from anomaly cancellation

The anomaly calculation proceeds simply, but reveals an intricate pattern of spinors depending upon the curve self-intersection, matching *e.g.* [31]. First notice that the adjoint representation has $C_{adj} = 3$, and therefore the C condition can be satisfied only for representations with $C \leq 3$. (There are no representations with negative C .) Given $\lambda = 2$ for $\mathfrak{so}(N)$, the C condition takes the form

$$\Sigma \cdot \Sigma = \frac{4}{3} \left(\sum_R C_R - 3 \right) \quad (4.24)$$

Indeed, since we consider only curves with $\Sigma \cdot \Sigma < 0$, the only representations that can cancel the -3 are those with $C < 3$. As discussed in [41], the only such representations are the fundamental ($C = 0$) and the spinor (for $N \leq 13$). Given that the fundamental representation does not contribute at all to this condition, this equation uniquely fixes the number of spinor representations on a given curve. The results are summarized in table 5. It is important in implementing these conditions to recall that the 32 and 64 dimensional spinor representations (of $\mathfrak{so}(11/12)$ and $\mathfrak{so}(13)$) are both self-conjugate. Therefore, half-hypermultiplets can transform in these representations. These hypers are counted with multiplicity $\frac{1}{2}$; if they were to be counted with multiplicity 1, anomaly cancellation for $\mathfrak{so}(11/12)$ would be impossible on a -3 curve and anomaly cancellation for $\mathfrak{so}(13)$ would be impossible on a -2 curve. As an example, on a -3 curve, the C condition reads $-\frac{9}{4} + 3 = \frac{3}{4} = N_s C_s$. For $N = 9, 10$, $C_s = \frac{3}{4}$, so there is one spinor rep. For $N = 11, 12$, $C_s = \frac{3}{2}$ and therefore a half-hyper spinor rep is required. Above this, we can see that $C_s = 3$, and even the smallest amount of spinor matter, a half-hyper, can no longer satisfy anomaly cancellation. As spinors were the only candidates to satisfy these conditions in the first place, we can decisively state that $\mathfrak{so}(N)$ tunings with $N > 12$ are forbidden on -3 curves.

As to the fundamental representations, their numbers can be determined from the A condition

$$K \cdot \Sigma = \frac{1}{3} \left(\sum_R A_R - A_{adj} \right) \quad (4.25)$$

Using $A_s = 2^{\lfloor (N+1)/2 \rfloor - 4}$, $A_{adj} = N - 2$, and $A_f = 1$, we can easily solve for N_f . For instance, on a -3 curve, we have already determined that there is 1 spinor hyper for $\mathfrak{so}(9/10)$ and $\frac{1}{2}$ for $\mathfrak{so}(11/12)$. Since the left hand side of the condition is $K \cdot \Sigma = -1$, we obtain $-3 = N_f + N_s A_s - (N - 2)$. As an example, for $\mathfrak{so}(9)$ on a -3 curve, this gives $N_f = 7 - 2 - 3 = 2$. For $\mathfrak{so}(10)$, the only change is that $A_{adj} = 10 - 2 = 8$ increases by one, hence $N_f = 3$. Similarly, for $N = 11, 12$, although only a half-hyper transforms in the spin representation, the coefficient A_s doubles in comparison to the previous cases. Thus again, the only numerical change is that A_{adj} increases by one as N increases; $N_f = 5, 6$ for $\mathfrak{so}(11/12)$. This one example of tunings on a -3 curve illustrates the general pattern of matter representations on all three curves considered; further calculations are entirely analogous and are therefore omitted.

One final remark is in order: we found that $\mathfrak{so}(13)$ was the largest $\mathfrak{so}(N)$ that anomalies allow on a -2 curve and $\mathfrak{so}(12)$ was the highest allowed on a -3 curve. One might expect this pattern to continue, with *e.g.* $\mathfrak{so}(10)$ the highest possible tuning on a -4 curve. But, at this very lowest self-intersection curve before such tunings become impossible, they regain renewed vigor: all $\mathfrak{so}(N)$'s are tunable on a -4 curve. It is straightforward to verify that the matter in table 5 for a -4 curve satisfies anomaly cancellation. The reason there is no restriction is simply that it was the spin representations that led to a problem before, and on a -4 curve, there are no spin representations: all matter is in the fundamentals.

4.2.2 Spectrum and Hodge shifts from local geometry

Now we check the predictions from anomaly cancellation by constructing, where possible, local models for the allowed $\mathfrak{so}(N)$ tunings and showing how the disallowed tunings fail. These local calculations are subtler than others we have so far encountered. Namely, we must impose a non-generic vanishing of $\text{ord}(\Delta) > \min\{3 \text{ord}(f), 2 \text{ord}(g)\}$. Moreover, there are two distinct monodromy conditions distinguishing $\mathfrak{so}(8+2m)$ from $\mathfrak{so}(7+2m)$ in the I_m^* Kodaira case, one condition each for n even and odd. These conditions are clearly stated in [10, 32]. For our purposes, instead of using these results directly, we note that the monodromy conditions can be summarized succinctly as follows. To be in the generic Kodaira case I_m^* , *i.e.* $\mathfrak{so}(7+2m)$, requires that Δ vanish to order $6+m$. However Δ_{6+m} must not vanish; otherwise we would be in the next highest Kodaira case. All monodromy conditions for I_m^* can be summarized as the requirement that Δ_{6+m} be a perfect square. When this is the case, the resulting gauge algebra is enhanced $\mathfrak{so}(7+2m) \rightarrow \mathfrak{so}(8+2m)$.

For our local models, we use the fan $\{(0,1), (1,0), (n,-1)\}$, where n is (negative) the self-intersection number of the middle curve Σ represented by the vector $(1,0)$ and assumes the values 2, 3, 4. Because a -3 curve is the simplest example that captures the complexity of these tunings, we begin with it, moving then to -2 and finally -4 curves.

On a -3 curve, we will be able to tune up to but not including an $\mathfrak{so}(13)$. Let us see how this is possible. We have already seen how an $\mathfrak{so}(8)$ may be tuned, so let us investigate the first new case: $\mathfrak{so}(9)$, *i.e.* the generic I_1^* singularity. To implement this tuning, we simply require

$$\Delta_6 \propto f_2^3 + g_3^2 = 0 \tag{4.26}$$

In the above we suppress the coefficients of the separate terms (4 and 27, respectively) as they will play no role in determining whether this quantity can be set to zero and (if this is possible) how many degrees of freedom must be fixed to do so. In implementing this condition, we must keep in mind that the orders of f and g must remain 2 and 3, respectively. It is clear that such a tuning (on a smooth divisor) will be possible iff f_2 is a perfect square and g_3 is a perfect cube. Indeed, expanding in a local defining coordinate w for the curve represented by $(0,1)$, we see that $f_n \sim w^{3n-4}$ and $g_n \sim w^{3n-3}$, which we will use repeatedly. In particular, $f_2 \sim w^2$

and $g_3 \sim w^3$,⁵ whence this condition can be satisfied if $f_2 \propto \phi^2$, $g_3 \propto \phi^3$ for an arbitrary linear term $\phi = a + bw$. Note that for cancellation between these terms, the two coefficients a and b are arbitrary but fixed between f and g , and moreover, once an overall coefficient for ϕ^2 is chosen for f_2 , this fixes the overall coefficient of ϕ^3 . Hence there are 2 degrees of freedom remaining, whereas we started with $3 + 4 = 7$ in arbitrary quadratic and cubic polynomials. Therefore, we lose 5 degrees of freedom in this tuning, in precise accordance with the change ΔH_u predicted from the matter determined by anomaly cancellation. (NB: This change is counted from \mathfrak{g}_2 , the generic $(2, 3)$ singularity.) Comparing with the known change of $\Delta H_u = -3$ in tuning $\mathfrak{so}(8)$ from \mathfrak{g}_2 , we see that this tuning represents a loss of 2 additional degrees of freedom. (Or, from the generic non-Higgsable algebra $\mathfrak{su}(3)$, we have $\Delta H_u = -6$). To reassure ourselves that this indeed works, we recall the anomaly calculation: $\mathfrak{su}(3)$ has no hypers, only a vector in the adjoint, so its contribution to H_u (in the terms $V - H_{\text{charged}}$) is $+8$. For $\mathfrak{so}(9)$, we have both vectors and hypermultiplets, so the contribution to $V - H_{\text{charged}}$ is $+36 - 1 \cdot 16 - 2 \cdot 9 = +2$, a loss of 6 degrees of freedom.

To tune an $\mathfrak{so}(10)$ requires implementing a monodromy condition: $g_4 + \frac{1}{3}\phi f_3$ must be a perfect square [32]. This is indeed possible, and since $g_4 \sim w^6$, we lose 3 degrees of freedom in requiring that it take the form $(\sim w^3)^2$. Again, we have a match with anomaly cancellation.

To tune an $\mathfrak{so}(11)$ we now require $\Delta_7 = 0$, namely

$$\begin{aligned} 0 &= \Delta_7 \propto f_3 f_2^2 + g_3 g_4 \\ &= f_3 \phi^4 + \phi^3 g_4 \end{aligned} \tag{4.27}$$

Now this will be possible with $g_4 \propto \phi f_3$, which consumes all the degrees of freedom of g_4 , *i.e.* all 7 (or the 4 that remain after tuning an $\mathfrak{so}(10)$); again, we have agreement with anomaly cancellation calculations.

To tune an $\mathfrak{so}(12)$, we now impose the more complicated monodromy condition that

$$\mu = 4\phi(g_5 + \frac{1}{3}\phi f_4) - f_3^2 \tag{4.28}$$

be a perfect square. This is also possible. We have 10 degrees of freedom in $g_5 \sim w^9$ and 6 in $f_3 \sim w^5$. By requiring $g_5 \propto \phi \tilde{g}_5$, $\tilde{g}_5 \sim w^8$ and $f_3 \propto \phi \tilde{f}_3$, $\tilde{f}_3 \sim w^4$, we can factor out ϕ^2 from μ ; we can then tune \tilde{g}_5 so that μ/ϕ^2 is a perfect square, so that we satisfy the monodromy condition (4.28). This fixes $1 + 4 = 5$ degrees of freedom, consistent with anomaly cancellation.

What goes wrong at the crucial case $\mathfrak{so}(13)$? To implement this tuning, we must set to zero Δ_8 :

$$\begin{aligned} 0 &= \Delta_8 \propto f_3^2 f_2 + f_2^2 f_4 + g_3 g_5 + g_4^2 \\ &= f_3^2 \phi^2 + \phi^4 f_4 + \phi^3 g_5 \end{aligned} \tag{4.29}$$

We combined the first and last terms upon recalling that $g_4 \propto \phi f_3$. For this quantity to be zero, ϕ must divide f_3^2 , which implies in this case that ϕ divides f_3 because ϕ cannot be a

⁵The notation $\sim w^n$ will be used throughout this section to denote a polynomial of degree n in w with arbitrary coefficients.

perfect square. This leads to an unacceptable singularity on the curve $C_\phi = \{\phi = 0\}$! This arises because, now, $\phi^2|f_2$ and $\phi|f_3$, so that f vanishes to order 4 on C_ϕ . Meanwhile, we have already found that $\phi^3|g_3$, and now $\phi^2|g_4$ and $\phi|g_5$. This leads directly to a $(4, 6)$ singularity on C_ϕ .

On a -2 curve, the story is similar but slightly more complicated: we can tune up to but not including $\mathfrak{so}(14)$, in agreement with anomaly cancellation.⁶ Geometrically, we will see that the $\mathfrak{so}(14)$ tuning fails for reasons similar to but slightly different from the failures we have previously encountered. To begin investigating these tunings, note that on this geometry, $f_n, g_n \propto w^{2n}$. On an isolated -2 curve, we have $f_2 \sim w^4, g_3 \sim w^6$, which implies that an I_1^* can be tuned by taking $\phi = a + bw + cw^2$ to be an arbitrary quadratic. In the process, $5 + 7 - 3 = 9$ degrees of freedom are fixed. This is to be interpreted as a change from \mathfrak{g}_2 , the generic $(2, 3)$ gauge algebra; or since $\mathfrak{so}(8)$ can be tuned from \mathfrak{g}_2 by fixing 6 degrees of freedom, this is a change of 3 additional degrees of freedom from $\mathfrak{so}(8)$, consistent with anomaly cancellation calculations. (Recall that anomaly cancellation predicts $\Delta H_u = -14, -20, -23$ for $\mathfrak{g}_2, \mathfrak{so}(8)$, and $\mathfrak{so}(9)$, respectively.) We can continue the analysis exactly as before, and no subtleties arise in counting degrees of freedom. Let us jump, then, to the case of $\mathfrak{so}(13)$, the generic I_3^* singularity. Again we must require

$$0 = \Delta_8 \propto f_3^2 \phi^2 + \phi^4 f_4 + \phi^3 g_5 \quad (4.30)$$

but in this case, ϕ is quadratic and can therefore be chosen to be a perfect square. Hence we can satisfy the required condition $\phi|f_3^2$ while also maintaining $\phi \nmid f_3$. Let us denote $\psi^2 = \phi$. Making this choice, namely fixing a quadratic to be the square of a linear function eliminates one degree of freedom. Also, factoring $f_3 = \psi \bar{f}_3$, for generic degree 5 \bar{f}_3 , we lose one more degree of freedom. Finally, setting $g_5 = \bar{f}_3^2 + \phi f_4$ fixes all 11 degrees of freedom of $g_5 \sim w^{10}$. This completes the required cancellation, fixing $1 + 1 + 11 = 13$ degrees of freedom in the process. This matches with the shift from $\mathfrak{so}(11)$ expected from anomaly cancellation.

It is not possible to tune $\mathfrak{so}(14)$. The appropriate monodromy condition is to require that Δ_9 be a perfect square.

$$\Delta_9 = \psi^3 \left(-\frac{1}{2} \bar{f}_3^3 - 18\psi^2 \bar{f}_3 f_4 + 108\psi^3 (g_6 - f_5 \psi^2) \right) \quad (4.31)$$

As the \bar{f}_3^3 term has lowest order (in ψ), it cannot be cancelled by any other term unless $\psi|\bar{f}_3^3$, hence $\psi|\bar{f}_3$. But now $\psi|g_5 = \bar{f}_3^2 + \psi^2 f_4$, and one can check that our previous constraints have likewise made all lower order terms in f and g divisible by ψ to sufficiently high order that a $(4, 6)$ singularity at $\{\phi = 0\}$ is inevitable, and we conclude that $\mathfrak{so}(14)$ cannot be tuned on a -2 curve. This conclusion was also reached in [64] based on the analysis of [10], and matches the expectation from anomaly cancellation.

On a -4 curve, the discussion is completely analogous, save for one crucial difference: $f_2, g_3 \sim z^0$ are constants, therefore ϕ is a constant. It is always true that a constant divides

⁶Thanks to Nikhil Raghuram and Yuji Tachikawa for illuminating discussions on this point.

higher order terms in Δ , and this condition therefore places no restrictions on the tunings. On an isolated -4 curve, there is no apparent restriction on tuning $\mathfrak{so}(N)$'s. In SUGRA models, of course, large N will eventually cease to be tunable because $h^{2,1}$ is finite and there will not be sufficiently many complex degrees of freedom to implement the tuning. Such failures result from global properties of the base. From an anomaly cancellation standpoint, this failure eventually results from an inability to satisfy gravitational anomaly cancellation. As we discuss in §6.3, a local bound on N can also be imposed by other curves of non-positive self intersection that intersect a -4 curve supporting an $\mathfrak{so}(N)$, though such bounds are not fully understood in the low-energy theory. In SCFT's, there is no reason to expect that the series of $\mathfrak{so}(N)$ tunings will ever terminate at any N .

4.3 Tuning on rational curves of self-intersection $n \geq -2$

In this section we consider the possible tunings on an isolated curve of self-intersection $n \geq -2$. For such curves, it is straightforward to use anomaly analysis along the lines of the preceding section to confirm that in general the generic matter spectrum is that given by Table 4 for each of the groups listed. Since in some cases the set of possible tunings is unbounded given only local constraints, a case-by-case analysis is impossible. Fortunately, beginning at $n = -2$, we can systematically organize the computation easily as a function of n . For the exceptional Lie algebras ($\mathfrak{e}, \mathfrak{f}, \mathfrak{g}$) we can check that the tunings are possible using Weierstrass, and explicitly check the Hodge numbers. For the classical Lie algebras ($\mathfrak{sp}, \mathfrak{su}$) we use Tate form. For $\mathfrak{so}(N)$ the analysis closely parallels that of the previous section, and we compare the Tate and Weierstrass perspectives.

4.3.1 Tuning exceptional algebras on $n \geq -2$ curves

For simplicity we begin with -1 curves, then generalize. From the local toric analysis, we have an expansion of f, g in polynomials f_k, g_k in a local coordinate w , where $\deg(f_k) = 4 + k$, $\deg(g_k) = 6 + k$. The number of independent monomials in f_0, f_1, \dots and g_0, g_1, \dots are thus 5, 6, \dots and 7, 8, \dots respectively.

To tune an \mathfrak{e}_7 on the -1 curve, we must set $f_0 = f_1 = f_2 = g_0 = \dots g_4 = 0$. This can clearly be done by setting 63 independent monomials to vanish. Thus, we can tune the \mathfrak{e}_7 and we confirm the Hodge number shift in Table 4. A similar computation allows us to tune \mathfrak{f}_4 by the same tuning but leaving g_4 generic (11 monomials), giving the correct Hodge shift of 52. For \mathfrak{e}_6 we have the monodromy condition that g_4 is a perfect square, so we get the correct shift by 57. Finally, for \mathfrak{g}_2 we leave f_2, g_3 generic ($7 + 10 = 17$ monomials), for a correct Hodge shift of 35. We have the usual caveat regarding the \mathfrak{f}_4 and \mathfrak{g}_2 and rank-preserving breaking.

We can generalize this analysis by noting that on a curve of self-intersection $n \geq -2$, the local expansion gives

$$\deg(f_k) = 8 + n(4 - k), \quad \deg(g_k) = 12 + n(6 - k). \quad (4.32)$$

Tuning any of the exceptional algebras on any such curve then is possible, since the degree is nonnegative for $k \leq 4, 6$ for f, g respectively. The number of Weierstrass monomials that must

be tuned can easily be computed in each case and checked to match with Table 4. In this comparison it is important to recall from (2.24) that each -2 curve contributes an additional neutral scalar field, while each curve of self-intersection $n \geq 0$ contributes $n + 1$ to the number of automorphisms, effectively removing Weierstrass moduli from the neutral scalar count. For example, for \mathfrak{e}_7 the number of monomials tuned is

$$((9 + 4n) + \cdots + (9 + 2n)) + ((13 + 6n) + \cdots + (13 + 2n)) = 92 + 29n. \quad (4.33)$$

The number of neutral scalars removed by this tuning is then

$$92 + 29n - (n + 1) = 91 + 28n, \quad (4.34)$$

in perfect agreement with the last line of Table 4. The shifts for the other exceptional groups can similarly be computed to match the anomaly prediction.

4.3.2 Tuning $\mathfrak{su}(N)$ and $\mathfrak{sp}(N)$ on $n \geq -2$ curves

We now consider the classical Lie algebras, beginning with $\mathfrak{su}(N)$. These tunings are more subtle for several reasons. First, the tunings involve a cancellation in Δ that is not automatically imposed by vanishing of lower order terms in f, g so the computation of such tunings is more algebraically involved. Second, these tunings can involve terms of arbitrarily high order in Δ, f, g , and can be cut off when higher order terms do not exist in f, g , even in a purely local analysis.

To illustrate the first issue consider the tuning of $\mathfrak{su}(2)$ on a -1 curve. The analysis of the general $\mathfrak{su}(N)$ tuning through Weierstrass was considered in [33]. For $\mathfrak{su}(2)$, this tuning involves setting $f_0 = -3\phi^2, g_0 = 2\phi^3$ to guarantee the cancellation of Δ_0 , and then solving the condition $\Delta_1 = 0$ for g_1 . On a -1 curve this amounts to replacing the $5 + 7 + 8 = 20$ monomials in f_0, f_1, g_1 with 3 monomials in a quadratic ϕ . The shift in H_u is therefore by 17, in agreement with Table 4. As N increases, the explicit tuning of the Weierstrass model in this way becomes increasingly complicated. For $N \geq 6$, there are multiple branches, including those with non-generic matter contents, even for smooth curves; we return to this in §7. A systematic procedure for tuning $\mathfrak{su}(N)$ for arbitrary N through explicit algebraic manipulations of the Weierstrass model is not known. Thus, in these cases rather than attempting to explicitly compute the Weierstrass model to all orders we simply use the Tate approach described in §2.1.

Already from Table 4, we can see that as n increases, the bound of allowed values on N so that the number of fundamental representations is nonnegative decreases. We then wish to determine which values of N can be realized using the Tate description and compare with this bound from anomalies. For $n = -2, -1, 0$, there is no bound on N from anomalies. To analyze the Tate forms, we determine the degrees of the coefficients in an expansion $a_i = \sum_k a_{i(k)} z^k$ to be, in parallel to (4.32),

$$\deg(a_{i(k)}) = 2i + n(i - k). \quad (4.35)$$

To tune $\mathfrak{su}(N)$, we must tune in Tate form a_2 to vanish to order 1, a_3 to vanish to order $\lfloor n/2 \rfloor$, etc.. This can clearly be done for any N on $n = -2, -1, 0$ curves, so there is no problem with tuning any of these groups, consistent with the absence of a constraint from anomalies.

Now, however, consider for example a curve of self-intersection $+1$. From anomalies we see that the number of fundamental matter representations is $16 + (8 - N)n = 24 - N$, which becomes negative for $N > 24$. So we want to check which values of $N \leq 24$ admit a Tate tuning of $\mathfrak{su}(N)$. For $n = 1$, the maximum degree possible of the a_i 's is

$$\deg(a_1, a_2, a_3, a_4, a_6) = (3, 6, 9, 12, 18). \quad (4.36)$$

For each a_i , this is the largest value of k such that (4.35) is nonnegative. To tune a Tate $\mathfrak{su}(24)$, we need the a 's to vanish to orders $(0, 1, 12, 12, 24)$. This can be achieved by setting $a_3 = a_6 = 0$ and leaving arbitrary the largest terms in a_4 (*i.e.*, $a_{4(12)}$). So we can tune through Tate an $\mathfrak{su}(24)$. Tuning a higher $\mathfrak{su}(N)$ would require the vanishing also of a_4 to all orders, which would produce a singular Weierstrass model with $\Delta = 0$ everywhere, consistent with the anomaly constraint. This is not the end of the story, however. To tune a Tate $\mathfrak{su}(23)$ requires the a 's to vanish to orders $(0, 1, 11, 12, 23)$. But since there is no order 11 term in a_3 or order 23 term in a_6 , this drives the Tate model automatically to $\mathfrak{su}(24)$. Thus, there is no Tate tuning of $\mathfrak{su}(23)$ on a $+1$ curve. Similarly, there is no Tate tuning of $\mathfrak{su}(21)$, although $\mathfrak{su}(22)$ may be tuned; and $\mathfrak{su}(19)$ can also be tuned without obstruction. Precisely this same pattern was encountered in [33] when an attempt was made to tune these groups directly in the Weierstrass model over a $+1$ curve, although in that context a particular simplification was made and there was no complete proof that there was no more complicated construction of these algebras. The upshot, however, is that on a curve of self-intersection $+1$, there is a slight discrepancy between the anomaly constraints and what we have been able to explicitly tune through Tate or Weierstrass. We have an almost exact agreement, but the gauge algebras $\mathfrak{su}(21)$ and $\mathfrak{su}(23)$ lie in the “swampland” of models that seem consistent from low-energy conditions but cannot at this time be realized in any known version of string theory.

We can perform a similar analysis for the $\mathfrak{su}(N)$ groups on other curves of positive self intersection; the results of this analysis are tabulated in Table 6. Several other curve types have similarly missing $\mathfrak{su}(N)$ groups in the Tate analysis. For $+2, +3$ curves it is impossible to tune $\mathfrak{su}(15), \mathfrak{su}(13)$ respectively using the Tate form, an explicit attempt to construct Weierstrass models showed a similar obstruction (with some simplifying assumptions made) in [8]. It is interesting to note that in most of the cases where the Tate analysis does not provide an $\mathfrak{su}(N-1)$ but does allow for tuning an algebra $\mathfrak{su}(N)$, the $\mathfrak{su}(N)$ theory always has either zero or one hypermultiplets in the fundamental \mathbf{N} representation, so that there is no direct Higgsing to $\mathfrak{su}(N-1)$. One might think that in the two cases ($n = 1, N = 22$ and $n = 7, N = 10$) there should be two fundamentals, so that the theory might be Higgsable to the missing model. In the $\mathfrak{su}(22)$ case, for example however, this tuning also forces⁷ an additional $\mathfrak{su}(2)$ to arise in some cases on a curve that intersects Σ ; this would absorb the two fundamentals into a

⁷Thanks to Nikhil Raghuram for pointing this out.

n	anomaly bound on N	Tate realizations	swamp
+1	24	$\dots, 20, 22, 24$	21, 23
+2	16	$\dots, 14, 16$	15
+3	13	$\dots, 12$	13
+4	12	$\dots, 10, 12$	11
+5	11	$\dots, 10$	11
+6	10	$\dots, 10$.
+7, +8	10	$\dots, 8, 10$	9
+9, \dots , +16	9	$\dots, 8$	9
> +16	8	$\dots, 8$.

Table 6. Table of $\mathfrak{su}(N)$ algebras that can be realized on a curve of positive self-intersection $+n$ using Tate, compared to bounds from anomaly cancellation. The last column are cases in the “swampland”.

single bifundamental, so that there may generally be no direct Higgsing to $\mathfrak{su}(21)$. It may also be relevant that in the explicit tuning of $\mathfrak{su}(24)$ on a $+1$ curve in [33], the resulting gauge group had a global discrete quotient, so that the precise gauge group is $SU(24)/\mathbb{Z}_2$, not $SU(24)$. In any case, it seems likely that there is no Weierstrass model corresponding to these configurations that cannot be constructed using Tate, and this would give a self-consistent picture with the other results in this paper, but we do not have a complete proof of that statement.

For tunings of $\mathfrak{sp}(N)$, the story is similar but simpler. Anomaly cancellation shows that $\mathfrak{sp}(N)$ can only be tuned on curves of self-intersection $n \geq -1$. From the Kodaira conditions it is immediately clear that $\mathfrak{sp}(N)$ cannot be tuned on a curve of self-intersection -3 or below. For a curve of self-intersection -2 , the monodromy condition that distinguishes $\mathfrak{sp}(N)$ from $\mathfrak{su}(N)$ automatically produces an $\mathfrak{su}(N)$ group, since the condition is that $f_0 = \phi^2$ where ϕ itself is a perfect square, and since f_0 is a constant on a -2 curve, it is always a perfect square. Just as for $\mathfrak{su}(N)$ we can use Tate to determine when $\mathfrak{sp}(N)$ can be tuned on a given curve of self-intersection $n \geq -1$. In this case there are no inconsistencies between anomaly conditions and the tuning possibilities; the swampland in this case is empty, and all possibilities in Table 4 that have nonnegative matter content are allowed.

4.3.3 Tuning $\mathfrak{so}(N)$ on $n \geq -2$ curves

Finally, we consider $\mathfrak{so}(N)$ on curves of self-intersection $n \geq -2$. Complementing the analysis of §4.2, we see what the Tate analysis has to say about these cases. It is straightforward to check that there is no problem with tuning up to $\mathfrak{su}(12)$ using Tate for a local analysis around any curve of self-intersection $n \geq -2$. We simply cancel according to the rules in Table 2 and we get Weierstrass models that provide the desired group. The Tate procedure breaks down, however, at $\mathfrak{so}(13)$. To tune this algebra the a ’s must be taken to vanish to orders

(1, 1, 3, 4, 6). Taking the Tate form

$$y^2 + z\tilde{a}_1yx + z^3\tilde{a}_3y = x^3 + z\tilde{a}_2x^2 + z^4\tilde{a}_4x + z^6\tilde{a}_6, \quad (4.37)$$

and converting to Weierstrass form we find that \tilde{a}_2 divides all coefficients in f and g up to f_4, g_6 . This is the ϕ that played a key role in the analysis of §4.2. Unless $\phi = \tilde{a}_2$ is a constant, the Tate tuning of $\mathfrak{so}(13)$ and beyond gives a problematic Weierstrass model. For -4 curves alone, ϕ is a constant, so the Tate form breaks down for all other curves. Note that one might try to set ϕ to a constant, even though it has monomials of higher order. This leads to a problem at the coordinate value $w = \infty$ on the curve where the group is tuned.

Analysis of the anomaly equations and the properties of the $\mathfrak{so}(N)$ spinor representations as discussed above indicates that the anomaly conditions are satisfied for $\mathfrak{so}(13)$ on a curve of self-intersection n if and only if n is even. While the Tate analysis is problematic in these cases, the Weierstrass analysis of §4.2 easily generalizes to arbitrary $n \geq -4$. As long as the degree of ϕ is even, which occurs when the degree of f_2 is a multiple of four, we can decompose $\phi = \psi^2$ and find a Weierstrass solution for $\mathfrak{so}(13)$. This occurs precisely when n is even, so the Weierstrass analysis shows that all $\mathfrak{so}(N)$ gauge groups with $N \leq 13$ allowed by anomaly cancellation can be tuned on a single smooth rational curve in a local analysis. In the same way that $\mathfrak{so}(14)$ develops a (4, 6) singularity on a -2 curve as described in §4.2, the same occurs on any curve of self-intersection $-2 + 4m, m > 0$.

The only remaining situation is $\mathfrak{so}(N)$ on an isolated -4 curve where $N > 13$. In this case, there is no constraint from anomalies as the number of fundamental matter fields is $N - 8$ and there are no spinors. Similarly, there is no constraint from Tate for any N . So in this case, everything allowed from anomalies can be tuned in F-theory.

This completes our analysis of tunings of all gauge groups on rational curves on the base using only constraints from the local geometry.

4.4 Higher genus curves

In the discussion in this section so far we have focused on curves of genus 0. For tuning toric curves on toric bases, or for 6D SCFT's, this is all that is necessary. For tuning more general curves on either toric or non-toric bases for general F-theory supergravity models, however, we must consider tuning gauge groups on curves of higher genus. For example, we could tune a gauge group on a cubic on the base \mathbb{P}^2 ; such a curve has genus one.

For a smooth curve of genus g , the matter content includes g matter hypermultiplets in the adjoint representation of the group, and the rest of the matter content is determined accordingly from the anomaly cancellation condition. The generic matter content and Hodge number shifts for tunings over a curve of general genus g are given in Table 7. Unlike the genus 0 cases, where we have performed explicit local analyses in each case (except those of large N for the classical groups), in this Table we have simply given the results expected from anomaly cancellation. In each case, the matter content is uniquely determined from the anomaly cancellation conditions (2.13–2.17) with $\Sigma \cdot \Sigma = n$ and $(K + \Sigma) \cdot \Sigma = 2g - 2$, given the

constraint that only the adjoint and generic matter types (*e.g.* the fundamental and two-index antisymmetric representations for $\mathfrak{su}(N)$) arise.

\mathfrak{g}	matter	$\Delta(h^{1,1}, H_u)$
$\mathfrak{su}(2)$	$(6n + 16 - 16g)\mathbf{2} + (g)\mathbf{3}$	$(1, -12n - 29(1 - g))$
$\mathfrak{su}(3)$	$(6n + 18 - 18g)\mathbf{3} + (g)\mathbf{8}$	$(2, -18n - 46(1 - g))$
$\mathfrak{su}(N)$	$((8 - N)n + 16(1 - g))\mathbf{N}$ $+ (n + 2 - 2g)\frac{\mathbf{N}(\mathbf{N}-1)}{2} + (g)(\mathbf{N}^2 - 1)$	$(N - 1, -(\frac{15N-N^2}{2})n - (15N + 1)(1 - g))$
$\mathfrak{sp}(N/2)$	$((8 - N)n + 16(1 - g))\mathbf{N}$ $+ (n + 1 - g)\left(\frac{\mathbf{N}(\mathbf{N}-1)}{2} - 1\right) + (g)\frac{\mathbf{N}(\mathbf{N}+1)}{2}$	$\left(N/2, \left[-(\frac{15N-N^2-2}{2})n - \left(\frac{31N-N^2}{2}\right)(1 - g)\right]\right)$
$\mathfrak{so}(N)$	$(n + (N - 4)(1 - g))\mathbf{N}$ $+ (n + 4 - 4g)2^{5-\lfloor(N+1)/2\rfloor}\mathbf{S} + (g)\frac{\mathbf{N}(\mathbf{N}-1)}{2}$	$\left(\lfloor N/2 \rfloor, -(N + 16)n - (\frac{N^2-7N+128}{2})(1 - g)\right)$
\mathfrak{g}_2	$(3n + 10 - 10g)\mathbf{7} + (g)\mathbf{14}$	$(2, [-7(3n + 8 - 8g)])$
\mathfrak{f}_4	$(n + 5 - 5g)\mathbf{26} + (g)\mathbf{52}$	$(4, [-26(n + 3 - 3g)])$
\mathfrak{e}_6	$(n + 6 - 6g)\mathbf{27} + (g)\mathbf{78}$	$(6, -27n - 84(1 - g))$
\mathfrak{e}_7	$(4 - 4g + n/2)\mathbf{56} + (g)\mathbf{133}$	$(7, -28n - 91(1 - g))$

Table 7. Possible tunings on a curve Σ of genus g and self-intersection n , together with matter and shifts in Hodge numbers. Note that $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ are listed separately; the antisymmetric in the case of $\mathfrak{su}(2)$ is a singlet, and does not contribute to the Hodge number shift, so this case differs slightly from the general $\mathfrak{su}(N)$ formula. The $\mathfrak{su}(3)$ case, which also lacks a quartic Casimir, is also listed explicitly for convenience.

5 Classification II: multiple-curve clusters

It is useful to break our analysis of allowed tuned gauge symmetries into tunings on isolated curves and on multiple-curve clusters. First, these multiple-curve NHCs already arise in regular patterns in bases; and second, the possible tunings on multiple-curve NHCs are very tightly constrained and thus represent a very small subset of combinations of *a priori* allowed tunings on each curve within these clusters. For clusters containing a -3 curve, some possibilities that satisfy anomaly cancellation (and all rules to be discussed later) are not possible; these therefore deserve special attention and cannot be treated except as individual cases. For -2 chains, on the other hand, a very distinctive “critical” structure appears, which is also best highlighted by examining this class individually.

5.1 The clusters $(-3, -2, -2)$, $(-2, -3, -2)$, and $(-3, -2)$

Multiple-curve NHCs containing a -3 curve present examples of an interesting phenomenon: although the calculations proceed similarly to the above (and can be found in appendix C), we will pause to highlight this phenomenon. Simple anomaly cancellation and geometry-based arguments both immediately show that the NHC $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$ cannot be enhanced to more

cluster	\mathfrak{g}	$(\Delta h^{1,1}, \Delta H_u)$	matter
(-3,-2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	(0,0)	$(\mathbf{7}, \frac{1}{2}\mathbf{2}) + \frac{1}{2}\mathbf{2}$
	$\mathfrak{so}(7) \oplus \mathfrak{su}(2)$	(1,-1)	$(\mathbf{8}_s, \frac{1}{2}\mathbf{2}) + \mathbf{8}_s$
(-3,-2,-2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	(0,0)	$(\mathbf{7}, \frac{1}{2}\mathbf{2}) + \frac{1}{2}\mathbf{2}$
(-2,-3,-2)	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	(0,0)	$(\frac{1}{2}\mathbf{2}, \mathbf{8}_s, \cdot) + (\cdot, \mathbf{8}_s, \frac{1}{2}\mathbf{2})$

Table 8. *A priori* possible tuned gauge algebras, together with matter and Hodge shifts, on the NHCs with multiple divisors. (Tunings on -2 curves can be found in the separate Table 9 in the following subsection.)

than $\mathfrak{so}(8) \oplus \mathfrak{su}(2)$. From the geometry point of view, this restriction arises because the next Kodaira singularity type beyond $\mathfrak{so}(8)$ is \mathfrak{f}_4 , which would lead to a $(4,6)$ singularity at the intersection between the -2 and -3 curves. (A more detailed analysis shows that an attempt to enhance $\mathfrak{su}(2)$ to $\mathfrak{su}(3)$ by tuning monodromy will also force a $(4,6)$ singularity.) This leaves two possible enhancements, both of which satisfy anomaly cancellation: $\mathfrak{so}(7) \oplus \mathfrak{g}_2$ and $\mathfrak{so}(8) \oplus \mathfrak{su}(2)$. In no case can $\mathfrak{so}(8)$, however, be realized. The allowed tunings are presented in table 8.

This curious fact was first derived in [38, 19], where it was shown generally that a Kodaira type I_0^* meeting type *IV* can only be consistently implemented when the I_0^* is \mathfrak{g}_2 ; when meeting type *III*, it can only be implemented as a \mathfrak{g}_2 or $\mathfrak{so}(7)$. Our local analysis simply confirms these results while also explicitly constructing local models for those cases that are allowed. Although these facts are mysterious from the standpoint of anomaly cancellation, some progress has been made to explain this discrepancy solely in the language of field theory (in particular global symmetries [60]).

5.2 -2 clusters

The final cluster type to consider is a configuration of intersecting -2 curves. We begin by discussing linear chains of -2 curves connected pairwise by simple intersections, focusing on tunings of $\mathfrak{su}(N)$ gauge algebras. We then comment on -2 clusters with more general structure, and discuss the small number of specific possible tunings with larger gauge algebras.

Several interesting phenomena arise in the study of tunings on clusters of -2 curves. It seems that as far as tunings are concerned, -2 is a “critical” value of the self-intersection number; first of all, these curves and clusters are the lowest in self-intersection number to admit a null tuning. They form the only unbounded family of NHC’s for 6D F-theory models, at least insofar as they arise in non-compact toric bases for F-theory compactifications. Second, tunings of $\mathfrak{su}(N)$ on -2 chains are also critical, in that precisely *all* matter transforming under a given $\mathfrak{su}(N)$ can be shared with neighboring $\mathfrak{su}(N)$ ’s.⁸ Finally, certain combinations of -2 curves can form degenerate elliptic curves associated with an elliptic curve in the base itself.

⁸When extending our analysis to non-compact bases, it is also of interest that -2 curves present the main ingredient in constructing the “end-points” crucial to the study of SCFT’s in [19].

cluster	\mathfrak{g}	$(\Delta h^{1,1}, \Delta H_u)$
22	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	$(3, [-12])$
	$\mathfrak{so}(7) \oplus \mathfrak{su}(2)$	$(4, [-15])$
222	$\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$	$(4, [-8])$
	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	$(5, [-12])$
	$\mathfrak{g}_2 \oplus \mathfrak{su}(2) \oplus \cdot$	$(3, [-11])$
2222	$\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2) \oplus \cdot$	$(4, [-8])$
22222	$\cdot \oplus \mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2) \oplus \cdot$	$(4, [-8])$
$2_1 \cdots 2_{k>5}$	(Only $\mathfrak{su}(n)$'s; see §5.2)	see (5.5)

Table 9. Table of possible tunings on -2 chains. (Chains are listed with self-intersections sign-reversed.) Because matter is very similar between these cases, we do not list it explicitly, preferring to display the shift in Hodge numbers resulting from that matter. For convenience, we summarize the relevant matter content here: $4 \times \mathbf{2}$ for $\mathfrak{su}(2)$, $4 \times \mathbf{7}$ for \mathfrak{g}_2 , $4 \times \mathbf{8}_s + \mathbf{7}$ for $\mathfrak{so}(7)$ and $2 \times \mathbf{8}_s + 2 \times \mathbf{8}_c + 2 \times \mathbf{8}_f$ for $\mathfrak{so}(8)$, where the $\mathbf{8}_s, \mathbf{8}_c$ matter representations are functionally equivalent. $\mathfrak{su}(2)$ shares a half-hypermultiplet with all groups but itself, where it shares a whole hyper; with \mathfrak{so} 's, it is the spinor representation which is shared.

We will proceed to analyze these chains both by local models, general geometric arguments, and anomaly cancellation arguments. The results of this analysis are summarized in table 9.

5.2.1 Linear -2 chains

Rather than using local toric models, in this section we primarily use a simple local feature of the tuning over -2 curves to simplify the analysis. This feature, which can be understood both geometrically and from anomaly cancellation, gives a simple picture of the structure of -2 cluster tuning that avoids detailed technical analysis. The conclusions of this simple analysis can then be checked using local models, which we do in part later in this section and in part in the following section. To see the basic structure of tuning over -2 curves, recall that the Zariski analysis leads to inequality 2.10, which constrains the minimum order of a section of $-nK$ given its orders on neighboring curves. The feature of interest appears when this formula is applied to -2 curves. Indeed, letting k denote the order of vanishing of any section (of *any* $-nK$) on a -2 curve Σ and k_R and k_L denote the orders of vanishing of the section on the neighbors of Σ , 2.10 becomes

$$k \geq \frac{k_L + k_R}{2} \quad (5.1)$$

This feature and some of its consequences was used in [17], and was dubbed the “convexity condition” in [19]. More generally, if the -2 curve Σ intersects j other curves $\Sigma_i, i = 1, \dots, j$, then the order of vanishing on Σ satisfies $k \geq (\sum_i k_i)/2$. The consequence for $\mathfrak{su}(N)$ tunings on a -2 curve connected to a set of other -2 curves with tuned gauge algebras $\mathfrak{su}(M_i)$ is that

$$2N \geq \sum_i M_i. \quad (5.2)$$

This condition follows immediately from anomaly cancellation, since at every intersection there is a hypermultiplet of shared matter in the $(\mathbf{N}, \mathbf{M}_i)$ representation, and Σ only carries $2N$ matter fields in the fundamental representation. Thus, this simple convexity condition naturally captures the constraints of anomaly cancellation.

Comparison to harmonic functions yields some immediate insight. For instance, on a closed or infinite chain of -2 curves, a $(0, 0, n)$ tuning on any divisor forces a $(0, 0, n)$ tuning on every divisor. More generally, imagine that a curve Σ supports a tuned $\mathfrak{su}(n)$ gauge algebra, associated with vanishing orders of (f, g, Δ) of $(0, 0, n)$. Now consider a linear chain of k (-2) -curves connected in sequence to Σ , with curves labeled by $\Sigma_1, \dots, \Sigma_k$, with $\Sigma_k = \Sigma$. The order of vanishing of Δ on D_l then satisfies $n_l \geq \lceil n \frac{l}{k} \rceil$. We can recover from this rule the infinite case. Note that in some cases the inequality cannot be saturated.

The local rule (5.2) gives a clear bound on possible tunings of -2 curves combined in an arbitrary cluster. In the following section we prove using Tate tunings that, at least at the local level of pairwise intersections, every tuning of a combination of $\mathfrak{su}(N)$ algebras on intersecting divisors that is allowed from (5.2) can be realized through a Weierstrass construction, so at least locally there is a perfect match between the constraints of the low-energy field theory and F-theory. Here we proceed simply using (5.2) to make some observations about possible tunings of $\mathfrak{su}(N)$ combinations on -2 clusters.

The local rule (5.2) has simple consequences for tunings over any linear chain of -2 curves. In particular, on a linear chain of -2 curves Σ_i , the sequence of gauge algebras $\mathfrak{su}(N_i)$ must be convex, with each N_i greater or equal to the average of the adjoining N_{i-1}, N_{i+1} . This constraint gives a systematic framework for analyzing local tunings on any linear chain of -2 curves. Note, however, that the set of possible tunings even on a single isolated -2 is *a priori* infinite, when no further constraints from neighboring divisors are taking into account. The finite bound on possible tunings of curves of self-intersection -2 or above is discussed in the following section and §9, in the context of intersection with other curves in the base. For 6D supergravity models with a compact base, the actual number of possible tunings is always finite, while for 6D SCFT's the family of possible tunings is infinite. Similarly, for 6D SCFT's there is no bound on the number of possible -2 curves that can be combined in a chain, while for compact supergravity models there is a finite bound.

Given this structure, we can simply classify all $\mathfrak{su}(N)$ tunings over clusters of -2 curves. The tunings allowed are precisely those that satisfy (5.2). If we have a set of l curves Σ_i that carry gauge algebras $\mathfrak{su}(N_i)$, with intersection numbers $I_{ij} \in \{0, 1\}$, then the total gauge algebra is

$$\oplus_i \mathfrak{su}(N_i), \quad (5.3)$$

the matter content is

$$\sum_{i,j: I_{ij}=1} (\mathbf{N}_i, \mathbf{N}_j), \quad (5.4)$$

and the shift in Hodge numbers is

$$\Delta(h^{1,1}, h^{2,1}) = \left(\sum_i (N_i - 1), \sum_i (-N_i^2 - 1) + \sum_{i,j: I_{ij}=1} (N_i N_j) \right). \quad (5.5)$$

The “critical” nature of -2 chains is particularly apparent when the inequality (5.2) is saturated. In this case, all of the $2N$ fundamental matter fields on Σ are involved in bifundamental matter fields. An interesting feature of this is that there is an almost perfect cancellation between the number of vector and hyper multiplets. In particular, for a closed chain of -2 curves, with $\mathfrak{su}(N)$ tuned on each, we have a contribution to $H_{\text{charged}} - V$ of precisely 1 for each -2 curve. This interesting possibility is discussed further in a related set of circumstances in the following subsection.

5.2.2 Nonlinear -2 clusters

We can use (5.2) to describe $\mathfrak{su}(N)$ tunings on more complicated configurations of -2 curves, which may include branching or loops. Remarkably, this simple averaging rule strongly constrains the kinds of clusters that can support tunings, revealing a potentially very interesting structure.

First, consider a -2 curve Σ that is connected to c linear chains of -2 curves of length $l_i - 1$. Assume that $\mathfrak{su}(N)$ is tuned on Σ . Then from the above analysis, each of the curves connected to Σ from the linear chains must support at least a gauge factor $\mathfrak{su}(\lceil N(l_i - 1)/l_i \rceil)$. From (5.2), however, the sum of the resulting M_i has an upper bound of $2N$. This immediately bounds the types of chains that can be connected to Σ . A chain of length 1 contributes at least $N/2$ to $\sum_i M_i$, a chain of length 2 contributes at least $2N/3$, *etc.*. Thus, we can have at most four chains connected to Σ , and this is possible only for chains of length one. If we have 3 chains, it is straightforward to check that the allowed lengths are $(1, 1, l - 1)$ for arbitrary l , $(1, 2, l - 1)$ for $l \leq 6$, $(1, 3, 3)$, and $(2, 2, 2)$. Thinking of these configurations as Dynkin diagrams, the extremal cases in this enumeration correspond precisely with the classification of degenerate elliptic curves associated with affine Dynkin diagrams \hat{D}_4 , \hat{D}_{l+1} , and \hat{E}_n [29]. Examples of these degenerate elliptic curves are illustrated in Figure 2. All of these nontrivial -2 curve configurations can be realized, for example on rational elliptic surfaces [61, 62]. Specific examples of such realizations were encountered in the classification of \mathbb{C}^* bases in [15].

In the extremal cases, we have a situation where combinations of $\mathfrak{su}(N)$ can be tuned on these divisors with a contribution to $H_{\text{charged}} - V$ that is independent of N . For example, on the $(2, 2, 2)$ configuration corresponding to \hat{E}_6 , we can tune a gauge algebra $\mathfrak{su}(3N)$ on Σ , $\mathfrak{su}(2N)$ on the components of the chains that intersect Σ , and $\mathfrak{su}(N)$ on the terminal links in the chains. This presents an apparent puzzle, since in a compact base only a finite number of tunings are possible and we would expect higher-rank tuning to require more moduli.

Weierstrass models for the extremal -2 clusters can be analyzed using the methodology used in [15]. For example, for the case of a -2 curve Σ intersecting four other -2 curves, in

local coordinates where the other curves intersect Σ at $w = 0, 1, 2, \infty$, the generic Weierstrass model takes the form

$$f = f_0 + f_2 \zeta z^2 + f_4 \zeta^2 z^4 + \dots \quad (5.6)$$

$$g = g_0 + g_2 \zeta z^2 + g_4 \zeta^2 z^4 + \dots, \quad (5.7)$$

where $\zeta = w(w-1)(w-2)$. We can set $\Delta = \mathcal{O}(z^2)$ by tuning g_0 to cancel in the leading term. This gives a gauge algebra of $\mathfrak{su}(2)$ on Σ and no gauge algebra on the other curves. The shift in H_0 is then given by $\Delta H_0 = V - H_{\text{charged}} = -5$. This appears surprising as we have only tuned one modulus, but keeping careful track of extra moduli from -2 curves this is correct; once we have done this tuning, the discriminant identically vanishes on the four additional curves, so they are no longer counted as contributing to N_{-2} as discussed in §2.5. We can further tune g_2 so that the next term in the discriminant vanishes. This then gives an $\mathfrak{su}(4)$ on Σ and an $\mathfrak{su}(2)$ on the other four curves. We now have $V - H_{\text{charged}} = -5$ again, but we have nonetheless tuned a modulus. Repeating this, we use one modulus each time we increase the algebra on Σ by $N \rightarrow N+1$. This represents an apparent disagreement between the moduli needed for Weierstrass tuning and anomaly cancellation.

Some insight can be gleaned into what is transpiring in these situations by observing that these -2 configurations are essentially degenerate genus one curves that satisfy $\Sigma \cdot \Sigma = -K \cdot \Sigma = 0$. On a smooth curve of this type, the only matter would be a single adjoint representation of $\mathfrak{su}(N)$, giving $H_{\text{charged}} - V = 1$, independent of N . When the smooth genus one curve degenerates into a combination of -2 curves, the resulting configuration of $\mathfrak{su}(N)$ groups is precisely that realized on the extremal -2 clusters, with no matter in the fundamental, and bifundamental matter at the intersection points. Indeed, the multiplicity of N that is tuned on each -2 curve for each extremal cluster associated with a degenerate elliptic curve is precisely the proper multiplicity to give the elliptic curve. Thus, this can be thought of in each of these cases as tuning an $\mathfrak{su}(N)$ on the elliptic curve and taking a degenerate limit.

The tuning of a single degree of freedom for each increase in N can be understood as the motion of a single seven-brane in the transverse direction to the genus one curve Σ . The fact that N can only be tuned to a certain maximum value for the smooth genus one curve Σ with $\Sigma \cdot \Sigma = 0$ follows from the fact that $\Delta = -12K$; there is always some maximum N such that $-12K - N\Sigma$ is effective. In a compact base with an extremal -2 cluster, this corresponds to the fact that the Weierstrass expansion terminates and only a finite number of tunings are possible. For example, tuning two of these clusters, with the sets of four additional -2 curves in each cluster connected pairwise by -1 curves gives a base studied in [15], which has the unusual feature of supporting a generic $U(1)$ factor associated with a higher rank Mordell-Weil group. This structure may be a clue to the anomalous anomaly behavior. In this compact base, f, g in (5.6) only go out to order z^8, z^{12} respectively, giving a bound on the gauge group that can be tuned.

A closed loop of -2 curves fits into this framework as the affine \hat{A}_{l-1} Dynkin diagram/degenerate elliptic curve. We comment on this further in §6.5. Note also that in

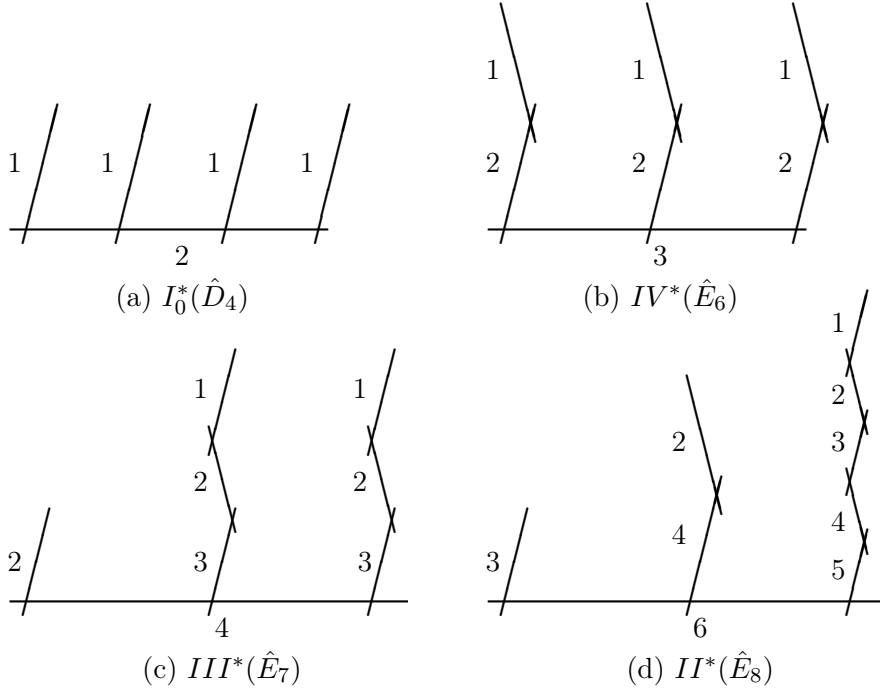


Figure 2. Some configurations of -2 curves associated with Kodaira-type surface singularities associated with degenerate elliptic fibers. The numbers given are the weightings needed to give an elliptic curve with vanishing self-intersection. Labels correspond to Kodaira singularity type and associated Dynkin diagram.

some situations these degenerate genus one curve configurations of -2 curves can be blown up further, giving more complicated configurations with related properties [63]; in general such configurations, once blown up, have curves of self-intersection -5 or below, and do not admit infinite tunings. Blowing up a loop of -2 curves to form a loop of alternating -1 , -4 curves is an exception, as mentioned in §6.5.

5.2.3 Tuning other groups on -2 curve clusters: special cases

So far all tunings we have discussed on clusters with multiple -2 curves have involved only $\mathfrak{su}(N)$. It happens that only a handful of other algebras can be tuned on -2 clusters. For instance, $\mathfrak{sp}(N)$ cannot be tuned on any ≤ -1 curve. Also, it is straightforward to see that the \mathfrak{f} and \mathfrak{e} algebras cannot arise on multiple -2 curve clusters. (Simply observe that a $(3, 4)$ singularity on one -2 curve must force at least a $(2, 2)$ singularity on any intersecting -2 curve, hence these exotic algebras would share matter. In section 6.1, we discuss why this is always impossible both for geometric and anomaly-cancellation reasons.) This analysis leaves only the case I_n^* .

In this subsection we go through the explicit analysis of the various cases of -2 curve clusters that support algebras other than $\mathfrak{su}(2)$, looking at Weierstrass models and the corresponding anomaly conditions. While some aspects of this analysis are essentially covered by the rules of tunings on intersecting brane combinations in the following section, it is worth emphasizing one subtlety, which is related to the fact that certain algebras such as $\mathfrak{su}(2)$ can be tuned in several different ways, either as a type I_2 or as a type III or IV . We have for the most part not emphasized this distinction as it is not relevant for minimal tunings in most

cases, and is not easy to understand in terms of the low-energy theory, however it is relevant in these cases, which serve as an illustration of how these distinctions are relevant in special cases.

In more detail, the Kodaira cases to consider are combinations of I_0^* tunings with $\mathfrak{su}(2)$ tuned as either type *III* or *IV*. Let us see why this is. To understand why $I_{\geq 1}^*$ cannot be tuned, suffice it to say that geometrically, it forces a $(4, 6)$ singularity at the intersection point with any other -2 curve. Although this could be demonstrated directly, we note here that it will follow from our geometric arguments that even $\mathfrak{so}(8)$ cannot be tuned on any -2 cluster; the monomials in f and g that must be set to zero to achieve an $\mathfrak{so}(8)$ are a strict subset of those which must be set to zero in tuning $\mathfrak{so}(9)$. From the anomaly cancellation standpoint, this statement is perfectly consistent, because we expect all matter to be 8 half-hypermultiplets of an $\mathfrak{su}(2)$ forced on an adjacent curve, and therefore the dimension of the matter shared with $\mathfrak{su}(2)$ cannot exceed 8, but of course would be for $\mathfrak{so}(\geq 9)$. In fact, it is impossible for an I_0^* to appear next to anything but $\mathfrak{su}(2)$. This is because an I_0^* $(2, 3)$ singularity forces at least a $(1, 2)$ singularity on any intersecting -2 curve. Attempting to tune $\mathfrak{su}(3)$ with a *IV* singularity requires that g_2 be a perfect square. However, a toric analysis for two intersecting -2 curves reveals that g_2 has a lowest order term of order w^3 , which cannot be the lowest order term of a perfect square. Eliminating this monomial directly produces a $(4, 6)$ singularity at the intersection point. This conclusion is also reasonable from the field theory point of view: the fundamental of $\mathfrak{su}(3)$ is not self-conjugate, and therefore the matter shared with it can be at most 6-dimensional.

In summary, the only tunings on -2 curve clusters that contain algebras other than $\mathfrak{su}(N)$ are combinations of I_0^* and *III/IV* $\mathfrak{su}(2)$'s. In fact, the averaging rule implies that tunings containing one I_0^* component may only occur in chains with ≤ 5 curves. (On larger clusters, the averaging rule implies that the $(2, 3, 6)$ singularity would persist to at least the nearest neighbor, immediately yielding a $(4, 6)$ singularity at the intersection point.) Therefore, our task is to classify the allowed combinations of $\mathfrak{su}(2)$, \mathfrak{g}_2 , $\mathfrak{so}(7)$, and $\mathfrak{so}(8)$. We will find that only \mathfrak{g}_2 and $\mathfrak{so}(7)$ can be realized, but not $\mathfrak{so}(8)$.⁹ As discussed above, an I_0^* tuning necessarily forces at least a *III* singularity on an intersecting -2 curve, so we will not encounter any isolated I_0^* tunings. (This geometrical constraint is not yet well characterized in terms of the low-energy field theory; see §6.3)

Let us now classify these tunings: namely, a single \mathfrak{g}_2 , $\mathfrak{so}(7)$, or $\mathfrak{so}(8)$ together with its neighbors, which must be $\mathfrak{su}(2)$'s. This must occur on a chain of length ≤ 5 . We will construct these models in order of increasing length of chain, starting on a configuration (Σ_1, Σ_2) of two intersecting -2 curves, and progressing to a chain $(\Sigma_1, \dots, \Sigma_5)$ of five -2 curves in a linear configuration. For each of these four configurations, we tune a \mathfrak{g}_2 , then attempt to enhance to $\mathfrak{so}(7)$ or $\mathfrak{so}(8)$. Because we will immediately find that $\mathfrak{so}(8)$ cannot be tuned, we will not consider it on any configuration but the first. (Configurations with additional -2 curves have

⁹This matches with the low-energy constraint from global symmetries [60], as discussed in more detail in §6.3.

strictly fewer monomials, so any obstruction to tuning on smaller configurations will apply trivially to larger configurations as well.) To set notation, we will use $\{z = 0\}$ to be a local defining equation for the curve Σ on which the type I_0^* singularity is tuned; any intersecting curve of interest we will take to be defined as $\{w = 0\}$. We will consider $f = \sum_{i,j} f_{i,j} z^i w^j$ and similarly for g ; when discussing orders of vanishing on *e.g.* $\Sigma = \{z = 0\}$, we will use terms $f_{i,\cdot}$ and $g_{i,\cdot}$ with one blank subscript to refer to functions of w in an implicit expansion $f = \sum_i f_{i,\cdot}(w) z^i$. Similarly $f_{\cdot,j}$ refers to an implicit expansion $f = \sum_i f_{\cdot,j}(z) w^j$. While we only discuss linear chains here explicitly, a similar analysis governs the tuning of an I_0^* factor on a small branched -2 cluster; details are left to the reader.

Case 1 (a): \mathfrak{g}_2 on $(-2, -2)$. To implement the tuning \mathfrak{g}_2 on Σ_1 , we must impose $(\text{ord}_{\Sigma_1} f, \text{ord}_{\Sigma_1} g) = (2, 3)$, so for f we eliminate $1 + 2$ degrees of freedom in setting $f_{0,\cdot}$ and $f_{1,\cdot}$ to zero, while for g we eliminate $1 + 2 + 4$ degrees of freedom in setting $g_{0,\cdot}$, $g_{1,\cdot}$, and $g_{2,\cdot}$ to zero. Now let us examine $f_{2,\cdot} = \sum_{i=1}^4 f_{2,i} w^i$ and $g_{3,\cdot} = \sum_{i=2}^6 g_{3,i} z^i$. By inspection, the geometry does not force a non-generic factorization, so we have indeed tuned \mathfrak{g}_2 , and not one of the other I_0^* cases. The removal of these monomials forces $(\text{ord}_{\Sigma_2} f, \text{ord}_{\Sigma_2} g) = (1, 2)$, so that we obtain an $\mathfrak{su}(2)$ neighbor. As a check, note that implementing this tuning fixes 10 monomials as well as two -2 curve moduli, which yields a shift in H_u of -12 , consistent with anomaly cancellation. No further complications have arisen in this instance.

(b): $\mathfrak{so}(7)$ Enhancing this tuning to $\mathfrak{so}(7)$, we impose

$$f_2 = B - A^2 \tag{5.8}$$

$$g_3 = -AB \tag{5.9}$$

which is most generically achieved by setting $A = A_1 w + A_2 w^2$ and $B = B_1 w + B_2 w^2 + B_3 w^3 + B_4 w^4$, for a total loss of $9 - 6 = 3$ degrees of freedom. Note that all coefficients $f_{2,i}$ and $g_{3,i}$ will generally remain nonzero. This implies that the orders of (f, g) on Σ_2 remain $(1, 2)$, so that the neighboring $\mathfrak{su}(2)$ remains type *III*. It is worth mentioning that these calculations agree with the anomaly calculations of shifts in H_u when one tunes the $\mathfrak{g}_2 \oplus \mathfrak{su}(2) \rightarrow \mathfrak{so}(7) \oplus \mathfrak{su}(2)$ combination, provided that it is one of the four $\mathbf{8}_s$ representations of $\mathfrak{so}(7)$ that is also charged as a fundamental under $\mathfrak{su}(2)$ —not the $\mathbf{7}_f$. This bifundamental matter is consistent with the process of Higgsing back to \mathfrak{g}_2 as it leaves the $\mathbf{7}$ unharmed to play the role of the $\mathbf{7}$ fundamental of \mathfrak{g}_2 . Moreover, this matches with the global symmetry analysis of [60], which indicates that the $\mathbf{8}_s$ representation must be shared instead of the $\mathbf{7}$ representation.

(c): $\mathfrak{so}(8)$: Forbidden This tuning is impossible, as mentioned above; hence all $\mathfrak{so}(8)$ tunings will be impossible on larger -2 chains. To see this in this context, recall that to enhance the tuning to $\mathfrak{so}(8)$, we require the more stringent factorization condition

$$f_2 = AB - (A + B)^2 \tag{5.10}$$

$$g_3 = AB(A + B) \tag{5.11}$$

which is most generically achieved by setting $A = A_1 w + A_2 w^2$, $B = B_1 w + B_2 w^2$. Notice that this removes all monomials $z^a w^b$ in g with $b \leq 3$, hence $a + b \geq 6$. Therefore the order of

g at the intersection point between the -2 curves is at least 6. Notice as well that this tuning removes the order z^2 term in f . Combining these facts, this tuning attempt would yield a $(4, 6)$ singularity where the -2 curves meet.

Case 2 (a,i): \mathfrak{g}_2 on Σ_1 of $(-2, -2, -2)$. In this cluster we may tune either on the first curve Σ_1 or the middle curve Σ_2 . The former presents distinct differences, which we will discuss first, after which we will move on to discuss the tuning on Σ_2 , which proceeds analogously to the tuning on $(-2, -2)$.

Let us now implement the tuning on Σ_1 . The additional complication will be the possibility of a forced gauge algebra on the final curve Σ_3 . On Σ_2 , the effect is a type *IV* singularity with g_2 consisting of a single monomial. In fact, this monomial is z^3 in a defining coordinate z for Σ_2 , so we will never encounter the issue of an $\mathfrak{su}(3)$ on Σ_2 . On Σ_3 , there is only an *II* type singularity, which does not produce a gauge algebra. This tuning has proceeded without obstruction.

(a,ii): $\mathfrak{so}(7)$: Forbidden Since $f_{2,\cdot} = f_{2,1}w + f_{2,2}w^2$ and $g_{3,\cdot} = g_{3,2}w^2 + g_{3,3}w^3 + g_{3,4}w^4$, the required factorization condition is satisfied by the choice $A = A_1w$, $B = B_1w + B_2w^2$. Note that this requires the w^4 term in $g_{3,\cdot}$ to vanish, in addition to imposing a relation among the remaining coefficients. We lose 2 degrees of freedom, as expected from the point of view of anomaly calculations above. This factorization constraint removes the single monomial of in g that is order 2 over Σ_3 , leading to a *III* singularity on Σ_3 . An $\mathfrak{su}(2)$ on Σ_3 would have to share more matter than it carries in the first place; therefore this enhancement, as well as subsequent ones, are inconsistent. In more geometric terms, a $(4, 6)$ singularity appears at $\Sigma_1 \cdot \Sigma_2$.

(b,i) \mathfrak{g}_2 on Σ_2 Attempting to tune I_0^* on the middle curve Σ_2 proceeds without difficulty, in complete analogy to tuning on the Σ_1 in the configuration $(-2, -2)$ of case 1. Implementing a tuning of \mathfrak{g}_2 on the middle curve Σ_2 , we investigate the forced tunings on its neighbors Σ_1 and Σ_3 . (By symmetry, it suffices to consider only Σ_1 , which incurs an $(f, g) = (1, 2)$ type *III* singularity.) One can confirm that there is generic factorization on Σ_2 , so we are indeed in the \mathfrak{g}_2 case.

(b,ii): $\mathfrak{so}(7)$ Since (on Σ_2) $f_{2,\cdot} = f_{2,1}w + f_{2,2}w^2 + f_{2,3}w^3$ and $g_{3,\cdot} = g_{3,2}w^2 + g_{3,3}w^3 + g_{3,4}w^4$, the relevant factorization condition can be achieved with $A = A_1w$, $B = B_1w + B_2w^2 + B_3w^3$, for a loss of 2 degrees of freedom. This is consistent with anomaly calculations, and moreover the singularities on the adjacent curves remain type *III*.

Case 3 (a,i): \mathfrak{g}_2 on Σ_1 of $(-2, -2, -2, -2)$ The novel contribution to this cluster is the possibility of tuning on the initial curve Σ_1 . However, this is impossible. The pathology of this attempted tuning is visible even without investigating monomials. A $(2, 3, 6)$ singularity on Σ_1 will, by the averaging rule, immediately produce at least a $(2, 3, 6)$ singularity on Σ_2 , leading to an unacceptable $(4, 6)$ singularity at $\Sigma_1 \cdot \Sigma_2$. Indeed, this same logic shows that *no* tuning of I_0^* is possible on a chain of ≥ 6 -2 curves, as there is no curve in this chain that with have fewer than 3 additional -2 curves to one side. We reemphasize: even \mathfrak{g}_2 cannot be tuned here.

(b,i): \mathfrak{g}_2 on Σ_2 of $(-2, -2, -2, -2)$. In this case, we will tune on Σ_2 , which is quite anal-

ogous to tuning on Σ_1 of the configuration $(-2, -2, -2)$. We find no additional restrictions, but the presence of the -2 curve Σ_1 leads to the presence of another (type *III*) $\mathfrak{su}(2)$ neighbor. Examining first the effect on Σ_1 , a \mathfrak{g}_2 produces the expected *III* or $(f, g) = (1, 2)$ singularity. On Σ_3 , the effect is a type *IV* singularity with $g_{\cdot,2}$ consisting of a single monomial, and on Σ_4 , there is only a type *II* type singularity, as in case (a, i) above.

(b,ii): $\mathfrak{so}(7)$: Forbidden This is already assured from the analysis of case 2, as we have merely added another -2 curve, which can only add constraints.

Case 4 (a): \mathfrak{g}_2 on $(-2, -2, -2, -2)$. The previous analysis already shows that we cannot tune an I_0^* singularity anywhere but the middle curve; otherwise, there would be a string of ≥ 3 -2 curves to one side of the I_0^* , which would force a $(4, 6)$ singularity. Thus, our goal is simply to verify that a \mathfrak{g}_2 can be tuned on the middle curve Σ_3 , in precise analogy to tuning on Σ_2 of case 3. We already know that tuning $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$ tunings are forbidden in this context, because this case is obtained from the previous one by an additional blowup at the endpoint -1 curve of the local model. The task, then, is simply to verify that the generic I_0^* \mathfrak{g}_2 singularity can be consistently imposed. By investigating the monomials, one can implement this tuning, finding that on Σ_3 , $f_{2,\cdot} = f_{2,2}w^2$ and $g_{3,\cdot} = g_{3,2}w^2 + g_{3,3}w^3 + g_{3,4}w^4$ (in a defining coordinate w for Σ_2), which implies that no factorization is generically forced, so this tuning belongs in the \mathfrak{g}_2 subcase of I_0^* tunings, as desired. Moreover, on Σ_2 , there is a forced *IV* singularity, for which $g_{\cdot,2} = g_{3,2}z^3$, yielding $\mathfrak{su}(2)$. Similarly for Σ_4 . This ensures an $\mathfrak{su}(2)$ is adjacent to the tuned \mathfrak{g}_2 on either side. As to $\Sigma_{1,5}$, each carries a type *II* singularity—in other words, no tuned algebra. This ensures that such a \mathfrak{g}_2 tuning can in fact be realized, completing the desired classification.

6 Classification III: connecting curves and clusters

At this point, we have investigated tunings over individual curves or non-Higgsable clusters. To go further, we would like to determine constraints on what groups can be tuned over intersecting divisors. In particular, low-energy anomaly cancellation conditions and corresponding F-theory geometric conditions impose clear constraints on what groups can be tuned over intersecting divisors. At this point we are unaware of any specific constraints on global models that go beyond conditions that can be expressed in terms of gauge groups tuned on a single divisor Σ and its immediate neighbors (*i.e.*, divisors intersecting Σ). Thus, it may be that determining local constraints on such configurations may be sufficient to determine the full set of global tunings that is possible. We do not attempt to prove the completeness of local conditions here, but focus in this section on various conditions that constrain tunings that are possible on multiple intersecting curves.

In §6.1 we give a simple set of arguments that show that there are only 5 (families of) pairs of gauge groups that can be tuned on a pair of intersecting divisors. In §6.2, we determine constraints on these families in terms of the self-intersections of the curves involved and the group types. In §6.3 we consider a more general set of constraints on a curve Σ that intersects

\mathfrak{g}_n	\mathfrak{g}_m	anomaly constraints on n, m	matter	ΔH_u
$\mathfrak{su}(N)$	$\mathfrak{su}(M)$	$\lfloor N/2 \rfloor \leq 8 + (4 - \lfloor M/2 \rfloor)m, \lfloor M/2 \rfloor \leq 8 + (4 - \lfloor N/2 \rfloor)n$	(\mathbf{N}, \mathbf{M})	$+NM$
$\mathfrak{sp}(j)$	$\mathfrak{sp}(k)$	$j \leq 8 + (4 - k)m, k \leq 8 + (4 - j)n$	$(\mathbf{2j}, \mathbf{2k})$	$+4jk$
$\mathfrak{su}(N)$	$\mathfrak{sp}(k)$	$\lfloor N/2 \rfloor \leq 8 + (4 - k)m, k \leq 8 + (4 - \lfloor N/2 \rfloor)n$	$(\mathbf{N}, \mathbf{2k})$	$+2Nk$
$\mathfrak{so}(N)$	$\mathfrak{sp}(k)$	$k \leq n + N - 4, N \leq 32 + (16 - 4k)m$	$\frac{1}{2}(\mathbf{N}, \mathbf{2k})$	$+Nk$
		$N = 7 : k \leq 8 + 2n, 2 \leq 8 + (4 - k)m$	$\frac{1}{2}(\mathbf{8_s}, \mathbf{2k})$	$+8k$
\mathfrak{g}_2	$\mathfrak{sp}(k)$	$k \leq 3n + 10, 7 \leq 32 + (16 - 4k)m$	$\frac{1}{2}(\mathbf{7}, \mathbf{2k})$	$+7k$

Table 10. Algebras of product groups that can be tuned on a pair of intersecting curves of self-intersection n, m , and constraints from anomaly conditions. Further constraints from Tate tuning conditions are discussed in text. Shift to Hodge numbers is relative to the shift of the individual group tunings.

with two or more other curves supporting gauge groups, including generalizations of the E_8 rule for curves Σ that do not themselves support a gauge group.

In this analysis we continue to focus on divisors with single pairwise intersections. A few comments on more general intersection possibilities are made in §6.5

6.1 Types of groups on intersecting divisors

We begin by giving some simple arguments that rule out all but five possible combinations of (families of) pairs of algebras supported on divisors Σ_1, Σ_2 that intersect at a single point. The allowed combinations, determined from anomaly cancellation conditions, are listed in Table 10.

From the field theory point of view, the possibilities of the groups that are tuned is constrained from the anomaly equation 2.18

$$\Sigma_1 \cdot \Sigma_2 = \lambda_1 \lambda_2 \sum_{R_{1,2}} A_{R_1} A_{R_2} x_{R_{1,2}} \quad (6.1)$$

In words, the shared matter, weighted with the product of its A coefficients and its multiplicity, must equal $\Sigma_1 \cdot \Sigma_2$ (which is one or zero in the cases we studied here).

From this low-energy constraint it is clear that bi-charged matter is quite difficult to achieve for any algebras other than $\mathfrak{su}(N)$ and $\mathfrak{sp}(N)$. For these algebras $\lambda = 1$. For \mathfrak{g}_2 and $\mathfrak{so}(N)$, $\lambda = 2$. For all the other algebras, $\lambda > 2$. For all matter representations that appear in generic F-theory models, and all known matter representations that can arise in F-theory, the coefficients A_R are integers. We assume that this is generally the case though we have no completely general proof. Thus, we can only have $\Sigma_1 \cdot \Sigma_2 = 1$ when both factors are either $\mathfrak{su}(N)$ or $\mathfrak{sp}(N)$ and $A_1 = A_2 = x = 1$, such as for a situation where there is a full matter hypermultiplet in the bifundamental representation, or when one factor is \mathfrak{g}_2 or $\mathfrak{so}(N)$ and the other is $\mathfrak{su}(N)$ or $\mathfrak{sp}(N)$ and we have a half-hypermultiplet in the bifundamental representation. (Note that the $\mathfrak{so}(N)$ fundamental can be replaced by a spinor when $N = 7, 8$ and the anomaly conditions are unchanged.) While the fundamental $2N$ of $\mathfrak{sp}(N)$ is self-conjugate (pseudoreal), only for the special case $SU(2) = Sp(1)$ is the fundamental of

$\mathfrak{su}(n)$ self-conjugate. Thus, field theory considerations seem immediately to restrict to the 5 possibilities in Table 10.

We can show directly in F-theory using a local monomial analysis that indeed the five possibilities in Table 10 are the only combinations of algebras on intersecting divisors that admit a tuning in the Weierstrass model. We begin by showing that \mathfrak{f}_4 cannot live on a curve that intersects another curve supporting any nontrivial algebra. To begin, let us label the curves of \mathfrak{f}_4 and its neighbor \mathfrak{g} as Σ_1 and Σ_2 respectively. Now notice that \mathfrak{f}_4 corresponds to a $(3, 4)$ singularity. If a non-trivial algebra other than one in the I_n series appeared on Σ_2 , it would have to be at least a $(1, 2)$ singularity—immediately leading to a $(4, 6)$ singularity at $\Sigma_1 \cdot \Sigma_2$. Thus, such tunings are inconsistent. It remains only to prove that a $I_{\geq 2}$ algebra cannot appear on Σ_2 , for which it will be sufficient to rule out just I_2 , *i.e.* an $\mathfrak{su}(2)$ tuned by $(0, 0, 2)$. By thinking in toric monomials, it is easy to see why this is inconsistent with a $(3, 4, 8)$ tuning on Σ_1 . Let w be a local coordinate on Σ_1 such that $\Sigma_2 = \{w = 0\}$, and vice versa for z and $\Sigma_1 = \{z = 0\}$. Then we can expand f , g , and Δ in Taylor series in z and w . By hypothesis, f contains no monomials with powers of z lower than z^3 , and g contains none with powers lower than z^4 . To tune Δ to order 2 on Σ_2 , we must require the vanishing of both Δ_0 and Δ_1 in the expansion $\Delta = \sum_{i=0} \Delta_i w^i$. For $\Delta_0 \sim f_0^3 + g_0^2$, this implies $f_0 \propto \phi^2$ and $g_0 \propto \phi^3$ for some expression ϕ . Immediately we see that f_0 must be a perfect square, which forces us to exclude the monomial $w^0 z^3$; moreover, for g_0 to be a perfect cube, the coefficients in g of both $w^0 z^4$ and $w^0 z^5$ must be zero.

Now let us impose the constraint $\Delta_1 \sim f_0^2 f_1 + g_0 g_1 = 0$. The lowest order (in z) term in $f_0^2 f_1$ is now z^{11} ($4 + 4 + 3$) whereas the lowest order term in $g_0 g_1$ is z^{10} ($6 + 4$). The required cancellation between these terms can only occur, then, if g_1 vanishes to order at least 5 instead of 4. But now this putative tuning is in serious trouble. We have removed from f the single monomial $w^0 z^3$ with $m + n < 4$ (where $w^m z^n$). Moreover, we have removed from g the three monomials with $m + n < 6$. This guarantees a $(4, 6)$ singularity at $\Sigma_1 \cdot \Sigma_2$, as we expected to find. This proves that no gauge algebra can be tuned on a divisor intersecting a divisor carrying a singularity of order $(3, 4)$ or higher.

A similar analysis shows that divisors intersecting other divisors carrying the groups with algebras $\mathfrak{so}(n), \mathfrak{g}_2$ can only have $\mathfrak{sp}(n)$ algebras. If we assume Σ_1 carries a gauge algebra with (f, g) vanishing to orders $(2, 3)$, we similarly analyze f_0, g_0 at leading orders in z , *etc.*. A second curve Σ_2 intersecting Σ_1 cannot carry a $(2, 3)$ singularity or we immediately have a $(4, 6)$ singularity at the intersection point. We thus need only consider gauge algebras $\mathfrak{su}(n), \mathfrak{sp}(n)$ on Σ_2 . We consider I_n type singularities. As above, we have $f_0 \propto \phi^2, g_0 \propto \phi^3$. For an $\mathfrak{su}(n)$ algebra the split condition dictates that we must have ϕ a perfect square $\phi = \phi_0^2$. But then $z|\phi_0, z^4|f_0$ and similarly $z^6|g_0$. To tune an $\mathfrak{su}(3)$ we have [33] $f_1 \sim \phi_0 \psi_1, g_1 \sim \phi f_1, g_2 \sim \psi_1^2 + \phi f_2$, and since g_2 and ϕf_2 scale at least as $z^3, z^4, z^2|\psi_1$, which means $z^3|f_1, z^5|g_1, z^4|g_2$, so we get a $(4, 6)$ singularity at the intersection. A similar effort to tune an $\mathfrak{su}(3)$ through a type IV singularity would give a term z^3 in g_2 , which implies that g_2 is not a perfect square so the monodromy gives an $\mathfrak{su}(2)$ algebra on any curve with a type IV singularity intersecting a singularity of order $(2, 3)$.

This completes the demonstration that the only possible pairs of nontrivial algebras that can be realized on intersecting curves are those in Table 10. Note that the analysis here was independent of the dimension of the base, so the same result holds for 4D F-theory compactifications.

6.2 Constraining groups on intersecting divisors

We now consider the possible combinations of gauge groups that can actually be realized for the five possible pairings from Table 10. In each case we compare the constraints from anomaly cancellation to a local Tate analysis, as was done for single curves in §4.3. We take the self-intersections of the two curves to be $\Sigma_1 \cdot \Sigma_1 = n$, $\Sigma_2 \cdot \Sigma_2 = m$, and we are assuming that $\Sigma_1 \cdot \Sigma_2 = 1$. We consider the various cases in turn, indicating potential swampland contributions in each case.

$\mathfrak{sp}(\mathbf{j}) \oplus \mathfrak{sp}(\mathbf{k})$ (no swamp):

We begin with the case $Sp(j = N/2) \times Sp(k = M/2)$, where the analysis is simplest. In this case, we expect a single (full) bifundamental hypermultiplet in the $(\mathbf{N}, \mathbf{M}) = (\mathbf{2j}, \mathbf{2k})$ representation. The number of fundamentals on each of the two curves is, from Table 4, $16 + (8 - 2j)n$, $16 + (8 - 2k)m$ respectively. We therefore have the constraints from anomaly cancellation

$$\begin{aligned} j &\leq 8 + (4 - k)m, \\ k &\leq 8 + (4 - j)n. \end{aligned} \tag{6.2}$$

Here, the self-intersections satisfy $n, m \geq -1$, since $\mathfrak{sp}(k)$ cannot be tuned on a -2 curve.

Now let us consider the Tate model. Tuning $\mathfrak{sp}(j)$ on a curve of self-intersection n requires tuning the a coefficients $(a_1, a_2, a_3, a_4, a_6)$ to vanish to orders $(0, 0, j, j, 2j)$. The weakest constraint comes from the a_4 condition. From (4.35), we see that the degrees of the coefficients in a_4 are $\deg(a_{4(s)}) = 8 + n(4 - s)$. Imposing both constraints, we see that a_4 can be written in terms of monomials $z^s w^t$ subject to the conditions that $t \leq 8 + (4 - s)n$, $s \leq 8 + (4 - t)m$. Tuning the algebras $\mathfrak{sp}(j) \oplus \mathfrak{sp}(k)$ on our curves of self-intersections n, m requires having a monomial in a_4 with degrees $s = j, t = k$, and we see that such a monomial exists if and only if the constraints (6.2) are satisfied. This shows that in a local model, using the Tate construction, all possible $\mathfrak{sp}(j) \oplus \mathfrak{sp}(k)$ algebras consistent with anomaly constraints can be tuned on a pair of intersecting curves.

$\mathfrak{su}(\mathbf{2j}) \oplus \mathfrak{su}(\mathbf{2k})$ (no swamp):

A similar analysis can be carried out in the other cases of Table 10. We next consider $SU(N) \times SU(M)$. If N, M are both even, with $N = 2j, M = 2k$, the tuning is precisely like that of the $Sp(j) \times Sp(k)$ case just considered, except for the tuning of a_2 to first order. The a_2 tuning is always possible, so it cannot affect the conclusion, and so for even N, M everything that is allowed from anomalies can be realized using Tate. Note that for these algebras, we can have in particular $n = m = -2$.

$\mathfrak{su}(\mathbf{2j} + \mathbf{1}) \oplus \mathfrak{su}(\mathbf{2k})$ (apparent swamp):

Next consider the case $N = 2j + 1, M = 2k$. In this case, the constraint from a_4 is just as (6.2) but with j replaced by $j + 1$. But there must also *either* be at least one monomial in a_3 of order at least j in z or a monomial in a_6 of order at least $2j + 1$, or else the symmetry automatically enhances to $SU(N + 1)$. The conditions that must be satisfied are then $(a_4$ and $(a_3$ or $a_6))$, where

$$\begin{aligned} a_4 : \quad j &\leq 7 + (4 - k)m, \quad k \leq 8 + (3 - j)n \\ a_3 : \quad j &\leq 6 + (3 - k)m, \quad k \leq 6 + (3 - j)n \\ a_6 : \quad 2j + 1 &\leq 12 + (6 - 2k)m, \quad 2k \leq 12 + (5 - 2j)n. \end{aligned} \tag{6.3}$$

Some of these conditions imply others. In particular, the a_6 condition on j is always stronger than the a_4 condition on j , and the a_3 condition on k always implies the a_4 condition on k . Nonetheless, the analysis is a bit subtle as different combinations are ruled in or out in different ways. For example, $\mathfrak{su}(3) \oplus \mathfrak{su}(6)$ violates the a_3 condition but satisfies the a_6 condition, while $\mathfrak{su}(9) \oplus \mathfrak{su}(2)$ satisfies the a_3 condition but violates the a_6 condition.

Let us consider some specific cases of even-odd $SU(N) \times SU(M)$. First, we note that when $m = -2$, the a_6 condition on j is weaker than the a_3 condition, and equal to the a_4 condition as well as to the anomaly condition. And when $n = -2$, the a_6 condition is again equivalent to the a_4 condition and the anomaly condition, and all of these are in this case stronger than the a_3 condition. It follows that when $n = m = -2$ a Tate tuning is possible precisely when the anomaly conditions are satisfied, and there is no swampland contribution.

Now, however, we consider the case $n = -1, m = -2$. In this case, the a_6 condition on k is $2k \leq 7 + 2j$; this is stronger than the a_4 constraint and weaker than the a_3 constraint so it must be satisfied for a Tate tuning. But this condition is also stronger than the anomaly cancellation condition $2k \leq 9 + 2j$. For $j = 1$, there is a potential swamp contribution at $k = 5$, and more generally the algebras $\mathfrak{su}(2j + 1) \oplus \mathfrak{su}(2j + 8)$ will be allowed by anomalies but not by Tate. This represents a simple family of cases that either should be shown to be inconsistent in the low-energy theory or realized through Weierstrass if possible. These cases are of particular interest since they are relevant for 6D SCFT's as they can be realized on intersecting $-1, -2$ curves that can be blown down to give a decoupled field theory. The simplest of these examples is $\mathfrak{su}(3) \oplus \mathfrak{su}(10)$ where the $\mathfrak{su}(10)$ has 20 hypermultiplet matter fields in the fundamental representation and $\mathfrak{su}(3)$ has 12 matter fields in the fundamental representation (of which one is technically distinct as it lies in the anti-fundamental, affecting the global symmetry group even though the content is identical as 6D hypermultiplets include a complex degree of freedom in a representation R and a matching complex degree of freedom in the conjugate representation).

Next, consider $m = n = 1$, where two curves of self-intersection 1 are intersecting. In this case the anomaly constraint says that $N + M \leq 24$, and the even-odd Tate constraints impose the condition $N + M \leq 19$. This is similar in spirit to the results of §4.3, and can be related explicitly in some circumstances. For example, on \mathbb{P}^2 a pair of lines supporting gauge groups $SU(N), SU(M)$ can be tuned to be coincident to reach a gauge group $SU(N + M)$ on

n	m	swamp contribution $(N, M) \rightarrow (\mathfrak{su}(N) \oplus \mathfrak{su}(M))$
-2	-2	no swamp
-2	-1	(10, 3), (11, 3), (12, 5), (13, 5), ...
-1	-1	(3, 10), (3, 11), (5, 12), (5, 13), ... (+ \leftrightarrow)
0	0	(2, 15), (3, 14), (3, 15), (3, 16), (4, 15), (5, 14), ... (+ \leftrightarrow)
1	1	(2, 19), (2, 21), (3, 18), (3, 19), (3, 20), (3, 21), ... (+ \leftrightarrow)

Table 11. Some combinations of $\mathfrak{su}(N) \oplus \mathfrak{su}(M)$ algebras that appear to be in the “swampland” as they cannot be tuned on intersecting curves of self-intersection n, m through Tate, but satisfy anomaly cancellation.

a single line. Thus, both the upper bound and the “swamp” of models where $N + M = 21, 23$ are consistent between these pictures.

$\mathfrak{su}(2j+1) \oplus \mathfrak{su}(2k+1)$ (apparent swamp):

Finally, we consider the odd-odd case $N = 2j+1, M = 2k+1$. In this case the constraints are similar to (6.3), with appropriate replacement of $k \rightarrow k+1$ in the a_4 constraint and $2k \rightarrow 2k+1$ in the a_6 constraint. Again we must satisfy a_4 and either a_3 or a_6 . As in the even-odd cases, once again in the special case $n = m = -2$, this set of constraints again leads to no conditions beyond those imposed by anomalies. For other combinations we have further contributions to the potential swampland from tunings that are not possible in Tate. For $n = m = 1$, where the anomaly constraint is $N + M \leq 24$, the odd-odd Tate conditions impose the stronger condition $N + M \leq 20$, so odd-odd combinations with $N + M = 22, 24$ cannot be tuned by Tate.

We thus see that the Tate approach only gives a subset of the $\mathfrak{su}(N) \oplus \mathfrak{su}(M)$ models that anomaly cancellation suggests should be allowed on a pair of intersecting divisors, giving some apparent additional contributions to the “swampland”. The number of cases with no known F-theory construction is relatively large, and it would be nice to understand whether these admit Weierstrass constructions or are somehow inconsistent due to low-energy constraints, or neither.

We summarize some of the apparent swampland contributions where Tate tuning is not possible in Table 11. The fact that there is no swamp when $n = m = -2$ indicates that, at least locally, the convexity condition used in the preceding section is the only constraint on tuning product groups on any -2 cluster that need be considered.

$\mathfrak{su}(N) \oplus \mathfrak{sp}(k)$ (apparent swamp):

We now consider $\mathfrak{su}(N) \oplus \mathfrak{sp}(k)$. From the above analysis, the anomaly and Tate constraints are identical to the case $\mathfrak{su}(N) \oplus \mathfrak{su}(2k)$. When N is even there are no swamp contributions. When N is odd and $n > -2$, there are additional potential swamp contributions.

$\mathfrak{g}_2 \oplus \mathfrak{sp}(k)$ (apparent swamp):

Next consider $\mathfrak{g}_2 \oplus \mathfrak{sp}(k)$. The anomaly constraints dictate $k \leq 3n+10, 7 \leq 32+(16-4k)m$.

The primary constraining Tate coefficient is again a_4 . To have a \mathfrak{g}_2 on the n -curve, a_4 must have a coefficient proportional to z^2 . This imposes the constraint $2 \leq 8 + (4 - k)m$, equivalent to the second anomaly constraint. On the other hand, the Tate constraint on k is $k \leq 8 + 2n$. For $n = -2$ this is equivalent to the anomaly constraint ($k \leq 4$); for larger n , however, the Tate constraint is stronger. There is also a further constraint from the condition that a_6 must have a coefficient proportional to z^3 , or the \mathfrak{g}_2 will be enhanced to $\mathfrak{so}(7)$ or greater. This condition implies that $2k \leq 12 + 3n$, which is substantially stronger than the anomaly conditions. For $n = -2$, we have $k \leq 3$, for $n = 1$ we have $k \leq 4$, for $n = 0$ we have $k \leq 6$, *etc.*. Thus, there is an apparent tuning swampland of models that contains, for example, $\mathfrak{g}_2 \oplus \mathfrak{sp}(4)$ when the \mathfrak{g}_2 is on a -2 curve, and has four fundamental hypermultiplets. This is true in particular when the $\mathfrak{sp}(n)$ factor is on a -1 curve, in which case this should be an SCFT, so this represents another potential contribution to the 6D SCFT swampland. It is not clear whether there can or cannot be a Weierstrass model in this case as Weierstrass and Tate are not necessarily equivalent for $\mathfrak{sp}(4)$, though as mentioned earlier it seems likely that no non-Tate Weierstrass model can be realized for this algebra without involving exotic matter. Further elements of the swampland in this case include $\mathfrak{g}_2 \oplus \mathfrak{sp}(5)$ through $\mathfrak{g}_2 \oplus \mathfrak{sp}(7)$ when the \mathfrak{g}_2 is on a -1 curve, $\mathfrak{g}_2 \oplus \mathfrak{sp}(7)$ through $\mathfrak{g}_2 \oplus \mathfrak{sp}(10)$ when the \mathfrak{g}_2 is on a 0 curve, *etc.*

$\mathfrak{so}(N) \oplus \mathfrak{sp}(k)$ (apparent swamp):

Finally, we turn to the cases $\mathfrak{so}(N) \oplus \mathfrak{sp}(k)$. These are the most delicate cases; we consider the small values of N explicitly. We begin with $\mathfrak{so}(8) \oplus \mathfrak{sp}(k)$. In this case, the three different eight-dimensional representations $\mathbf{8}_v, \mathbf{8}_s, \mathbf{8}_c$ are equivalent under anomalies, and physically related through triality symmetry. Anomaly constraints limit $k \leq 4 + n, 2 \leq 8 + (4 - k)m$. For a Tate tuning the constraints associated with upper bounds on the size of the group are, similar to the \mathfrak{g}_2 case, $2 \leq 8 + (4 - k)m$, again like the second anomaly constraint, and again the constraint $k \leq 8 + 2n$ which now matches the first anomaly constraint. So it seems that the anomaly and Tate constraints are consistent. There is however a subtlety here associated with the monodromy condition for $\mathfrak{so}(8)$. This imposes the condition that the order z^4 term in $a_2^2 - 4a_4$ must vanish [32]. In the special case that the $\mathfrak{sp}(k)$ has $k = 1$, and $n = -2$, the only possible monomial in a_4 of order w is wz^2 . But this cannot be part of a perfect square, so the $\mathfrak{so}(8)$ monodromy condition cannot be satisfied. Thus, in this case the tuning is not possible. Furthermore, in this case even a Weierstrass tuning is not possible. This fact was mentioned in §5.1, and we elaborate further here. It was shown in [38] that $\mathfrak{so}(8) \oplus \mathfrak{su}(2) = \mathfrak{so}(8) \oplus \mathfrak{sp}(1)$ cannot be tuned on any pair of intersecting divisors where the second factor is realized through a Kodaira type III or IV singularity; the argument there was given in the context of threefold bases, but holds for bases of any dimension. The argument given there shows in this context that in 6D, $\mathfrak{so}(8) \oplus \mathfrak{su}(2)$ cannot be tuned on any pair of intersecting divisors where the second divisor is a -2 curve. It was also shown in [19], in the context of 6D SCFTs, that an $\mathfrak{so}(8) \oplus \mathfrak{su}(2)$ cannot be realized on a pair of intersecting $-3, -2$ curves. The upshot of this analysis is that an $\mathfrak{su}(2)$ on any -2 curve cannot intersect an $\mathfrak{so}(8)$ on any divisor (not just

a -3 curve)¹⁰. This apparent element of the tuning swampland was shown to be inconsistent at the level of field theory in [60]; actually, the argument there demonstrated this result only at the superconformal point, but the same result should hold for a general 6D F-theory supergravity model, since locally the -2 curve can generally be contracted to form an SCFT. This is an interesting example of a case where an apparent element of the tuning swampland is removed by realization of a new field theory inconsistency.

We turn now to $\mathfrak{so}(7) \oplus \mathfrak{sp}(k)$. The anomaly constraints are similar but now depend on whether the bi-charged matter is in the spinor ($\mathbf{8}_s$) or fundamental ($\mathbf{7}$) representation. In all cases, the second constraint $2 \leq 8 + (4 - k)m$ agrees between anomalies and Tate. The anomaly conditions for spinor matter are $k \leq 8 + 2n$ and for fundamental matter are $k \leq 3 + n$. Performing a generic Tate tuning, the bound is $k \leq 8 + 2n$. This suggests that the general Tate tuning gives matter must be in the spinor-fundamental representation. This matches with the known examples of the $-2, -3, -2$ non-Higgsable cluster, which carries spinor matter for an $\mathfrak{so}(7)$ on the -3 curve, and the results of [60] that the four $\mathfrak{su}(2)$ fundamental matter fields on an -2 curve transform in the spinor representation of an $\mathfrak{so}(7)$ flavor group. That result implies that for an $\mathfrak{su}(2)$ on a -2 curve there cannot be a bifundamental with $\mathfrak{so}(7)$, but a remaining open question is whether there can be an explicit Weierstrass tuning of an $\mathfrak{so}(7) \oplus \mathfrak{sp}(k)$ algebra on a more general pair of divisors with bifundamental matter, and if this is indeed impossible what the field theory reason is.

The other $\mathfrak{so}(N) \oplus \mathfrak{sp}(k)$ tunings can be analyzed in a similar fashion, though the analysis is simpler since anomalies show that spinors cannot appear at the intersection point. For $\mathfrak{so}(9)$ the Tate condition computed using the Tate form from Table 2 appears to give the bound $k \leq 6 + n$ from a_6 . This cannot be correct, since the anomaly bound gives $k \leq 5 + n$, which seems to allow Tate constructions of models that violate anomaly cancellation. In fact, the maximal case $k = 6 + n$ actually gives $\mathfrak{so}(10)$. This can be understood if we carefully impose the proper additional monodromy condition. In terms of the Tate form of $\mathfrak{so}(9)$ from Table 2, the algebra is actually $\mathfrak{so}(10)$ if $(a_3^2 + 4a_6)/z^4$ is a single monomial when evaluated at $z = 0$, and thus a perfect square, which occurs for the generic Tate form in the current context precisely when $k = 6 + n$. So this Tate condition and the corresponding anomaly condition for $\mathfrak{so}(9)$ match perfectly. The Tate bound on k from m , $4 \leq 12 + (6 - k)m$, is stronger than the anomaly bound, $9 \leq 32 + (16 - 4k)m$, but both are satisfied for all compatible values of m, k from §4.3.2, so there is no swampland.

For $\mathfrak{so}(10)$, we can enforce the monodromy condition that $(a_3^2 + 4a_6)/z^4$ is a perfect square on $z = 0$ by setting a_6 to vanish to order z^5 instead of z^4 . In this class of tunings, the anomaly condition $k \leq 6 + n$ matches the a_3 Tate condition $k \leq 6 + (3 - 2)n$, while the m condition $10 \leq 32 + (16 - 4k)m$ from anomalies is slightly weaker than the Tate condition $2 \leq 6 + (3 - k)m$, leaving in the tuning swampland for example $\mathfrak{so}(10) \oplus \mathfrak{sp}(k)$ for $k = 8, 9$ when $m = +1$ (and necessarily $n \geq 2, 3$). For $\mathfrak{so}(11)$, the Tate condition without considering monodromy is $k \leq 8 + n$, which is again weaker than the anomaly condition $k \leq 7 + n$. The

¹⁰Thanks to Clay Cordova for discussions on this point

discrepancy can again be corrected by the monodromy condition that for Tate $\mathfrak{so}(11)$ as in table 2, we have $\mathfrak{so}(12)$ when $(a_4^2 - 4a_2a_6)/z^6$ is a perfect square on $z = 0$. This monodromy condition is stated in [32] for $\mathfrak{so}(4n+4)$ with $n \geq 3$, but the analysis here indicates that it must also hold at $n = 2$. With this monodromy, the first conditions agree; the other conditions, $11 \leq 32 + (16 - 4k)m$ and $2 \leq 8 + (4 - k)m$ also agree. There is an exact matching between anomaly conditions and Tate conditions for the cases $\mathfrak{so}(12)$.

For tunings of $\mathfrak{so}(N) \oplus \mathfrak{sp}(k)$ where the $\mathfrak{so}(N)$ is on a $m = -4$ curve, there is no *a priori* upper bound on N , and the pattern continues in a similar way as for $\mathfrak{so}(9)$ - $\mathfrak{so}(12)$. For $\mathfrak{so}(N = 4j)$, the a_4 Tate bounds $k \leq 8 + (4 - j)m = N - 8$, $j \leq 8 + (4 - k)m$ precisely match the anomaly bounds. For $\mathfrak{so}(N = 4j + 2)$, the a_3 Tate bound $k \leq 6 + (3 - j)m = N - 8$ matches the anomaly bound and $j \leq 6 + (3 - k)m \Rightarrow N \leq 26 + (12 - 4k)m$ is stronger than the anomaly bound $N \leq 32 + (16 - 4k)m$, leaving a small swampland contribution. Similarly for $\mathfrak{so}(4j + 1), \mathfrak{so}(4j + 3)$ when the proper monodromy conditions are incorporated as for $\mathfrak{so}(9), \mathfrak{so}(11)$. The upshot of this analysis is that $\mathfrak{so}(N) \oplus \mathfrak{sp}(k)$ tunings have a few swampland contributions, but not many.

6.3 Multiple curves intersecting Σ

Having analyzed the combinations of algebras that can be tuned on a pair of intersecting curves, we can consider the more general class of local constraints associated with a single curve Σ that supports a gauge algebra \mathfrak{g} , and which intersects k other curves $\Sigma_i, i = 1, \dots, k$, with each curve having a fixed self-intersection. In principle such geometries can be analyzed using the same methods used in the preceding section for a pair of curves. A more general structure relevant for this analysis is related to the global symmetry of the field theory over the curve Σ . Such global symmetries were recently analyzed in the context of 6D SCFTs in [64]. From the field theory point of view, the global symmetry can be determined by the nature of the matter transforming under \mathfrak{g} . For example, the fundamental representations of $\mathfrak{su}(N)$ are complex, and on a curve carrying M such representations, there is a global symmetry $\mathfrak{su}(M)$ that rotates the representations among themselves. In general, the direct sum of the algebras \mathfrak{g}_i supported on the k curves Σ_i that intersect Σ must be a subalgebra of the global algebra of Σ . The global symmetries for curves of negative self intersection were computed in [64], and these are included in the Tables in the Appendix of information about tunings of groups on curves of fixed self-intersection. A similar computation can be carried out for curves of nonnegative self intersection; indeed, the computations in the preceding section are closely related to the computation of the global symmetry, though for the global symmetry the constraint associated with the curve intersecting the desired curve would be dropped. Note that in [64], only global symmetries associated with generic intersections were incorporated, more generally, for example, there could be a component of the global symmetry group associated with antisymmetric representations of $\mathfrak{su}(N)$, which can appear in more complicated bi-charged matter configurations [35]. Note also that in considering situations where multiple curves intersect a given curve Σ that supports a gauge algebra \mathfrak{g} , the distinction between different realizations of \mathfrak{g} , such as between type I_2 and III, IV realizations of $\mathfrak{su}(2)$,

are important. These distinctions are relevant for instance for the cases in §5.2.3, and are tracked in [64]. A complete analysis of all local rules for a single curve intersecting multiple other curves would need to distinguish these cases.

In general, in the situation where multiple curves Σ_i intersect a single curve Σ supporting a gauge algebra \mathfrak{g} , there are constraints on the gauge algebras that can be tuned over the Σ_i coming from the pairwise constraints determined in the preceding subsection, and a further overall constraint associated with the global symmetry on Σ . Every configuration that satisfies these constraints automatically satisfies the local anomaly conditions. It is natural to expect that perhaps all possibilities compatible with these two conditions can actually be realized in F-theory. In principle this could be investigated systematically for all possible combinations. We do not do this here, but to illustrate the point we consider a couple of specific examples; in one case this hypothesis holds, and in the other case it seems not to and there are further contributions to the swampland. We leave a detailed analyses of all the cases for this story to future work.

Consider in particular the case where Σ has self-intersection n and supports a gauge algebra $\mathfrak{su}(2N)$. The global symmetry in this case associated with the $16 + (8 - 2N)n$ matter fields in the fundamental representation of $\mathfrak{su}(2N)$ is $\mathfrak{su}(16 + (8 - 2N)n)$. In a local model around Σ we have the usual Tate expansion. Let us ask what gauge groups $\mathfrak{su}(2M_i)$ can be tuned on the intersecting divisors Σ_i . We take all the $2N, 2M_i$ to be even for simplicity; as discussed above in the odd cases there will be some anomaly-allowed models that cannot be tuned by Tate. Without imposing any constraints from the self-intersections of the Σ_i , anomaly constraints impose the condition $\sum_i M_i \leq N$. This is also the condition imposed by the global symmetry. Looking at the constraining term a_4 in the Tate expansion, we see that as above $\deg(a_{4(s)}) = 8 + n(4 - s)$. The gauge group on Σ indicates that we set to vanish all coefficients in a_4 up to $s = N$. The leading coefficient is then of degree $8 + n(4 - N)$. To tune gauge groups $\mathfrak{su}(2M_i)$ at points $w = w_i$ on Σ , we must then take $a_{4(N)} = \prod_i (z - z_i)^{M_i}$. This can precisely be done for all sets $\{M_i\}$ that satisfy the condition. Thus, in this case all possible tunings are possible that are compatible with anomaly constraints, which are the same as the tunings obeying the pairwise and global symmetry constraints.

Another interesting class of cases arises when we consider a -1 curve intersecting with two -4 curves. In this case, with a $\mathfrak{sp}(k)$ on the -1 curve, anomaly cancellation and the global symmetry group suggest that it should be possible to tune $\mathfrak{so}(N), \mathfrak{so}(M)$ on the two -4 curves as long as $N + M \leq 16 + 4k$. This does not always seem to be the case, at least with Tate tunings, even when each intersection is pairwise allowed. For example, while $\mathfrak{so}(11) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(9)$ and $\mathfrak{so}(11) \oplus \mathfrak{sp}(2) \oplus \mathfrak{so}(13)$ can be tuned in Tate, $\mathfrak{so}(15) \oplus \mathfrak{sp}(2) \oplus \mathfrak{so}(9)$ cannot. Thus, it seems there is a further component of the tuning swampland associated with cases allowed by the global group and pairwise intersections that cannot be realized as three-divisor tunings.

6.4 No gauge group on Σ (generalizing the “ E_8 rule”)

We can also consider situations where Σ carries no gauge group and intersects a set of other curves Σ_i . Although anomaly cancellation does not give any apparent constraint in such a situation, F-theory geometries are still constrained. An example of this is the E_8 rule that has been mentioned above, which from the SCFT point of view can be viewed as a generalization of the above arguments regarding global symmetries. When a rational curve E is a generic exceptional divisor $E \cdot E = -1$, the analysis of *e.g.* [65] establishes that in the limit in which the curve shrinks to zero size in a non-compact geometry, the resulting SCFT has a global E_8 symmetry. Therefore it is natural to expect that $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \subseteq \mathfrak{e}_8$ for gauge algebras on a pair of curves Σ_1, Σ_2 that intersect Σ , or more generally that the sum of algebras over any set of curves that intersect Σ is contained in \mathfrak{e}_8 . The E_8 rule holds in the case of NHCs, as discussed in Appendix C of [18]; the full set of rules for NHCs that can intersect a -1 curves is given in [13].

It is natural to conjecture that the E_8 rule holds for all tunings on any set of curves Σ_i that intersect a -1 curve that does not support a gauge algebra. A consequence of this would be that any tunings over Σ and the Σ_i that could be Higgsed to break all gauge factors over Σ would also lead to a configuration that satisfies the E_8 rule.

A slightly stronger version of the E_8 rule would be that any tuning allowed by the E_8 rule and anomaly cancellation should be possible. We have used Tate tunings to investigate the validity of the tuned version of the E_8 rule, both in the weaker form and the stronger form. It is straightforward to check, given a pair of algebras $\mathfrak{a}, \mathfrak{b}$, what the consequences are of the Tate tuning of these algebras on a pair of curves intersecting a -1 curve Σ . In analogy with the rule (5.1), from the Zariski decomposition (or a local toric analysis), it follows that if a -1 curve Σ intersects other curves Σ_i on which $a_n \in \mathcal{O}(-nK)$ vanishes to orders k_i , then a_n must vanish to order $k \geq 0$ where

$$k \geq -n + \sum_i k_i. \quad (6.4)$$

Thus, for example, if we try to perform a Tate tuning of $\mathfrak{su}(5)$ on each of two divisors Σ_1, Σ_2 that intersect the -1 curve Σ , since for $\mathfrak{su}(5)$ we have $\text{ord}(a_1, a_2, a_3, a_4, a_6) = (0, 1, 2, 3, 5)$, it follows that $\text{ord}_\Sigma(a_1, a_2, a_3, a_4, a_6) \geq (0, 0, 1, 2, 4)$, which forces an $\mathfrak{su}(3)$ on Σ . In fact, even trying to tune $\mathfrak{su}(5) \oplus \mathfrak{su}(4)$ leads to an $\mathfrak{su}(2)$ on Σ . This suggests that the stronger version of the E_8 rule may fail. As another check on this, we can consider a form of Weierstrass analysis. As discussed previously, the Weierstrass and Tate formulations are equivalent for tunings of $\mathfrak{su}(N)$, $N \leq 5$ on a smooth irreducible divisor. We can write the general Weierstrass form for $\mathfrak{su}(5)$ on such an irreducible divisor in the form [32, 33]

$$f = -\frac{1}{3}\Phi^2 + \frac{1}{2}\phi_0\psi_2\sigma^2 + \tilde{f}\sigma^3 \quad (6.5)$$

$$g = -\frac{1}{3}\Phi f - \frac{1}{27}\Phi^3 + \frac{1}{4}\psi_2^2\sigma^4 + \tilde{g}\sigma^5, \quad (6.6)$$

$$\Phi = \phi_0^2/4 + \tilde{\Phi}\sigma$$

and the resulting discriminant is of the form

$$\Delta = \frac{1}{16}(\tilde{g}\phi_0^6 - \tilde{f}\psi_0^5\psi_2 + \phi_0^4\tilde{\Phi}\psi_2^2)\sigma^5 + \mathcal{O}(\sigma^6). \quad (6.7)$$

We can try using this form for Weierstrass to tune an $\mathfrak{su}(5) \oplus \mathfrak{su}(5)$, by considering this as a single $\mathfrak{su}(5)$ on a reducible divisor. We thus consider the discriminant now in terms of a local Weierstrass analysis on $\Sigma = \{z = 0\}$. The term \tilde{f} multiplies σ^3 , giving a section of $-4K$. From a local toric analysis like those we have been doing, it follows that $\tilde{f}\sigma^3$ (essentially a_4 in the Tate analysis) must vanish at least to order z^2 . Similarly, $\psi_2\sigma^2$ ($\sim a_3$) is a section of $-3K$, which must vanish to at least order z , and $\tilde{g}\sigma^5$ ($\sim a_6$) must vanish to order z^3 . It follows that this form of the Weierstrass model has at least a Kodaira I_2 singularity on Σ that supports an $\mathfrak{su}(2)$ when we tune $\mathfrak{su}(5) \oplus \mathfrak{su}(5)$ on a pair of curves intersecting Σ . While this is not the most general form of Weierstrass model that might be considered¹¹, it illustrates the challenge of this kind of tuning.

This shows that the tuned version of the E_8 rule may fail, in the sense that there are some configurations that this rule apparently would accept from the low-energy point of view, which we do not have a method for realizing in F-theory. We can view this as part of the current swampland, assuming that the justification of the E_8 rule from field theory holds for an arbitrary -1 curve holds, that is that there is always a limit where the curve shrinks to an SCFT with global symmetry E_8 . Similar considerations show that other subgroups of E_8 suffer from similar tuning issues, in particular this occurs for $\mathfrak{su}(9)$. While it is possible, for example, that even though there is no Tate realization of $\mathfrak{su}(9)$ next to a -1 curve carrying no gauge group there may be a non-Tate Weierstrass realization of such an $\mathfrak{su}(9)$, such a realization seems likely to have exotic matter or other unusual features so that there is potentially no standard realization of $\mathfrak{su}(9)$ with generic matter even though it is a subalgebra of \mathfrak{e}_8 . It does seem, on the other hand, that all tunings that go beyond the groups contained in E_8 are disallowed, at least at the level of Tate tunings. A summary of this analysis is given in Table 12.

In regard to the failure of tuning $\mathfrak{su}(9)$ using Tate on a divisor intersecting a -1 curve, a further comment is in order. Note that the analysis in [32, 33, 35] gives a systematic description of all $\mathfrak{su}(N)$ tunings for $N < 9$, but that there is as yet no completely systematic description of $\mathfrak{su}(9)$ tunings. In fact, a similar issue has been encountered in tuning an $\mathfrak{su}(9)$ algebra on a divisor to attain a non-generic triple-antisymmetric matter field at a singular point on the divisor that would need to have an \mathfrak{e}_8 enhancement [33, 35]. It may be that the unusual way in which \mathfrak{e}_8 contains $\mathfrak{su}(9)$ as a subgroup may act as some kind of obstacle to F-theory realizations of $\mathfrak{su}(9)$ in contexts where other subgroups of \mathfrak{e}_8 are allowed.

Further investigation of the E_8 rule in the context of tunings, particularly trying to understand whether any of the subgroups that cannot be realized using Tate constructions

¹¹Thanks to David Morrison and Tom Rudelius for discussions related to this point, which has been corrected from v1 of this paper.

Algebras $\mathfrak{a} \oplus \mathfrak{b} \subseteq \mathfrak{e}_8$ that can be tuned in F-theory	$\mathfrak{e}_8 \oplus \cdot, \mathfrak{e}_7 \oplus \mathfrak{su}(2), \mathfrak{e}_6 \oplus \mathfrak{su}(3), \mathfrak{f}_4 \oplus \mathfrak{g}_2, \mathfrak{so}(8) \oplus \mathfrak{so}(8),$ $\mathfrak{so}(8) \oplus \mathfrak{su}(4), \mathfrak{su}(4) \oplus \mathfrak{su}(4), \mathfrak{su}(6) \oplus \mathfrak{su}(2),$ $\mathfrak{su}(5) \oplus \mathfrak{su}(3), \mathfrak{su}(8) \oplus \cdot, \mathfrak{so}(16) \oplus \cdot$
Algebras $\mathfrak{a} \oplus \mathfrak{b} \subseteq \mathfrak{e}_8$ with no known F-theory tuning (“ E_8 (Tate) swamp”)	$\mathfrak{su}(5) \oplus \mathfrak{su}(5), \mathfrak{su}(4) \oplus \mathfrak{su}(5), \mathfrak{su}(3) \oplus \mathfrak{su}(6),$ $\mathfrak{su}(2) \oplus \mathfrak{su}(7), \mathfrak{su}(9) \oplus \cdot$ $\mathfrak{so}(7) \oplus \mathfrak{so}(9), \mathfrak{sp}(2) \oplus \mathfrak{so}(11)$
Algebras $\mathfrak{a} \oplus \mathfrak{b}$ not in \mathfrak{e}_8 that cannot be tuned in F-theory	$\mathfrak{e}_8 \oplus \mathfrak{su}(2), \mathfrak{e}_7 \oplus \mathfrak{su}(3), \mathfrak{e}_6 \oplus \mathfrak{g}_2, \mathfrak{e}_6 \oplus \mathfrak{su}(4), \mathfrak{f}_4 \oplus \mathfrak{su}(4)$ $\mathfrak{su}(4) \oplus \mathfrak{so}(9), \mathfrak{so}(17) \oplus \cdot$

Table 12. Tunings that do and do not satisfy the E_8 rule, which states that any pair of algebras tuned on curves intersecting a -1 curve not supporting an algebra must have a combined algebra that is a subalgebra of \mathfrak{e}_8 . Tunings in the swamp are only tested using Tate, and may admit Weierstrass realizations. Algebras not in \mathfrak{e}_8 that cannot be tuned in F-theory are largely checked using both Tate and Weierstrass. This list is not comprehensive but illustrates the general picture.

have a Weierstrass construction, either with generic or exotic matter, may provide fruitful insight into the connection of F-theory and low-energy supergravity theories.

We can also ask about the analogue of the E_8 rule for curves of higher self-intersection. We consider here the situation of a 0-curve Σ intersecting two or more other curves Σ_i . For all the exceptional groups (including $\mathfrak{g}_2, \mathfrak{f}_4$), it is clear that any pair of groups can be tuned on a pair of divisors Σ_1, Σ_2 , so for example we can tune $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ on the two divisors Σ_i . There are, however, constraints on what classical groups can be tuned on the Σ_i . Most simply, it is clear from the Tate construction, since $\deg a_{4(s)} = 8$, that using Tate to produce any combination of algebras $\mathfrak{su}(2M_i)$ is only possible if $\sum_i M_i \leq 8$, so that the total algebra is always a subalgebra of $\mathfrak{su}(16)$. Similarly, an \mathfrak{e}_8 tuned on one side can be combined through Tate with an $\mathfrak{su}(8)$ on the other side, but not with $\mathfrak{su}(N)$, $N > 8$. We can also tune $\mathfrak{so}(32)$ on one side, or *e.g.* $\mathfrak{e}_8 \oplus \mathfrak{so}(16)$. It is tempting to speculate that the consistency condition is related to the weight lattice being a sublattice of one of the even self-dual dimension 16 lattices $\mathfrak{e}_8 \oplus \mathfrak{e}_8, \text{spin}(32)/\mathbb{Z}_2$. It is also possible that an $\mathfrak{su}(17)$ algebra may be realizable using a Weierstrass construction [66]; in any case, it seems that the rank of the algebra realized must be 16 or less.

Note that there are some analogues of the E_8 rule for -2 curves, as discussed in §5, which depend upon details of the geometry that are not easily understood from the low-energy point of view. For example, when a -2 curve Σ is connected to two other -2 curves, it is not possible to tune an $\mathfrak{su}(2)$ on one of them and not on Σ . This phenomenon is not currently understood from the low-energy point of view.

Although one could imagine an extension of the E_8 rule and the corresponding rule for a 0-curve to a curve of positive self-intersection, the primary constraints on gauge groups tuned on other divisors intersecting such a curve seem to come from the connection between the other curves and the remaining geometry. We leave further investigation of such non field-theoretic constraints for further work.

6.5 More general intersection structures

We have focused here on situations where multiple curves intersect a single curve Σ each at a single point. More complicated possibilities can arise geometrically. For example, the curve Σ can intersect itself at one or more points, or acquire a more complicated singularity. In such situations, a gauge group on Σ will either require an adjoint representation (when the tuning is in Tate form on a singular divisor) or a more exotic “higher genus” matter representation when the tuning is in non-Tate form; such configurations were discussed in §7.

Another interesting situation can arise when two curves Σ, Σ' intersect at multiple points. In principle such geometries can be analyzed using similar methods to those used here. We point out, however, one case of particular interest. If two -2 curves intersect at two points, or more generally if k -2 curves intersect mutually pairwise in a loop, then if we were to be able to tune an $SU(N)$ group on each curve there would be bifundamental matter on each pair of curves, and the shift in Hodge number $h^{2,1}(X)$ would be the same for every N . This would appear to give rise to an infinite family of theories with a finite tuning. This example, along with a handful of other similar situations, was encountered in the low-energy theory in [67], and shown to be impossible in any supergravity theory with $T < 9$ in [41, 9]. Closed loops of this kind are also encountered in the context of F-theory realizations of little string theories [68]. From the F-theory point of view the possibility of an infinite family of tunings is incompatible with the proof of finiteness for Weierstrass models in [9]. We know, however, that such -2 curve configurations are possible¹², for example in certain rational elliptic surfaces that can act as F-theory bases. In fact, these configurations are another example of degenerate elliptic curves, like those discussed in §5.2.2, but in this case associated with the affine Dynkin diagram \hat{A}_{k-1} . As in those cases, we expect that the tuning of a single modulus will increase N by one, and that there is an upper bound on N associated with the maximum value such that $-12K - N\Sigma$ is effective. Note, however, that from the low-energy point of view this constraint is not understood. This issue is discussed further in §11.

Note that similar closed cycles of curves can occur for alternating $-4, -1$ sequences, with alternating $SO(2k), Sp(k)$ gauge groups. These can be thought of as arising from blowing up points between every pair of -2 curves, and again correspond to degenerate genus one curves with $\Sigma \cdot \Sigma = K \cdot \Sigma = 0$, and the explanation for finite tuning is again similar.

7 Tuning exotic matter

Up to this point we have focused on classifying the gauge groups that can be tuned over the various effective divisors in the base through tuning codimension one singularities in the Weierstrass model. In many circumstances, the gauge group content and associated Green-Schwarz terms, which are fixed by the divisors supporting the gauge group, uniquely determine the matter content of the theory. In other cases, however, there is some freedom in tuning codimension two singularities that can realize different *anomaly-equivalent* matter representations

¹²Thanks to Yinan Wang for discussions on this point

that can be realized in different F-theory models associated with distinct Calabi-Yau threefold geometries. The existence of anomaly-equivalent matter representations for certain gauge groups was noted in [31, 69, 33, 10], and explicit Weierstrass models for some non-generic matter representations were constructed in [33–36]. Tuning a non-generic matter representation without changing the gauge group in general involves passing through a superconformal fixed point (SCFT) [35]. At the level of the Calabi-Yau threefold, such a transition leaves the Hodge number $h^{1,1}(X)$ invariant (since the rank of the gauge group and the base are unchanged), but generally decreases $h^{2,1}(X)$ as generic matter representations are exchanged for a more exotic singularity associated with non-generic matter.

A systematic approach to classifying possible exotic matter representations that may arise was developed in [70, 33]. Associated with each representation R of a gauge group G is an integer

$$g_R = \frac{\lambda}{12} (2\lambda C_{\mathbf{R}} + B_{\mathbf{R}} - A_{\mathbf{R}}) . \quad (7.1)$$

The number g_R corresponds to the arithmetic genus of a singularity needed in a curve C to support the representation R , in all cases with known Weierstrass realizations; it was argued in [70] that this relationship should hold for all representations. For example, antisymmetric k -index representations of $\mathfrak{su}(N)$ have $g_R = 0$ and can be realized on smooth curves, while the symmetric k -index representation of $\mathfrak{su}(2)$ has $g_k = k(k-1)$, and for $k=3$ a half-hypermultiplet of this representation is realized on a triple point singularity in a base curve having arithmetic genus 3 [36].

The exotic codimension two tunings of exotic matter on a single gauge factor that have been shown explicitly to be possible through construction of Weierstrass models are listed in Table 13, along with the corresponding shifts in the Hodge number $h^{2,1}(X)$. The change in $h^{2,1}(X)$ corresponds to the number of uncharged hypermultiplets that enter into the corresponding matter transition, *i.e.* to the number of moduli that must be tuned to effect the transition. This list may not be complete; it is possible that other exotic matter representations may be tuned through appropriate Weierstrass models. This subject is currently an active area of research. Nonetheless, if there are one or more other exotic representations possible that are not listed in the table, for a given tuning of gauge groups from codimension one singularities on a given base, the generic matter content is finite. Each divisor supporting a gauge group has a finite genus, and there are a finite number of states in each of the generic representations. In each specific case, anomalies in principle constrain the number of possible transitions to exotic matter to a finite set, so that the evaluation of a superset of the set of possible codimension two tunings is a finite process; the remaining uncertainty is whether each of those models with exotic matter not listed in Table 13 has an actual realization in F-theory. At the time of writing this is not determined for all possible matter representations, but further progress on this in some cases will be reported elsewhere. Note that certain related exotic multi-charged representations have also been identified [35], such as the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $\mathfrak{su}(2)$, the $(\mathbf{6}, \mathbf{2})$ of $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$, *etc.*, which can arise from Higgsing of the exotic matter listed in Table 13; while not listed explicitly in the table, such multi-charged exotic matter

\mathfrak{g}	R	g_R	initial matter	tuned matter	$\Delta h^{2,1}$
$\mathfrak{su}(N)$	$N(N+1)/2$	1	$(N^2 - 1) \oplus \mathbf{1}$	$(N(N+1)/2) \oplus (N(N-1)/2)$	-1
$\mathfrak{su}(2)$	4	3	$3 \times \mathbf{3} \oplus 7 \times \mathbf{1}$	$\frac{1}{2}\mathbf{4} \oplus 7 \times \mathbf{2}$	-7
$\mathfrak{su}(6)$	20	0	$\mathbf{15} \oplus \mathbf{1}$	$\frac{1}{2}\mathbf{20} \oplus \mathbf{6}$	-1
$\mathfrak{su}(7)$	35	0	$3 \times \mathbf{21} \oplus 7 \times \mathbf{1}$	$\mathbf{35} \oplus 5 \times \mathbf{7}$	-7
$\mathfrak{su}(8)$	56	0	$4 \times \mathbf{28} \oplus 16 \times \mathbf{1}$	$\mathbf{56} \oplus 9 \times \mathbf{8}$	-16
$\mathfrak{sp}(3)$	14'	0	$\mathbf{14} \oplus 2 \times \mathbf{1}$	$\frac{1}{2}\mathbf{14}' \oplus \frac{3}{2}\mathbf{6}$	-2
$\mathfrak{sp}(4)$	48	0	$4 \times \mathbf{27} \oplus 20 \times \mathbf{1}$	$\mathbf{48} \oplus 10 \times \mathbf{8}$	-20

Table 13. Known exotic matter representations that can be realized by tuning codimension two singularities over a divisor in the base, including the tuning of Weierstrass moduli that impose singularities in a generically smooth divisor. Half hypermultiplets are indicated explicitly in the transitions; when the result of the transition is a half-hypermultiplet, the genus given is that of the half-hyper. Hodge number shifts are indicated in each case. Other exotic matter representations may be possible that have not yet been realized explicitly through Weierstrass models. Note that the first example listed, of transitions from an adjoint to symmetric + antisymmetric matter, has only been explicitly realized so far in Weierstrass models for $SU(3)$.

should also be considered in possible tunings.

A further issue is that in some cases there are combinations of exotic and conventional matter multiplets that appear from low-energy anomaly cancellation considerations to be possible but that cannot be realized in F-theory. Thus, certain transitions that appear to be possible may be obstructed in F-theory. As an example, at least for the method of constructing Weierstrass models developed in [35], an $\mathfrak{su}(8)$ theory with some $\mathbf{56}$ multiplets must also have at least one $\mathbf{28}$ multiplet. This gives another class of situations where the finite enumeration of tunings gives a superset of the set of allowed Weierstrass models, of which some may not represent consistent F-theory constructions.

8 Tuning abelian gauge factors

In the analysis so far we have focused on tuning nonabelian gauge factors, which are determined by the Kodaira singularity types in the elliptic fibration over each divisor in the base. Abelian gauge factors are much more subtle, as they arise from nonlocal structure that is captured by the Mordell-Weil group of an elliptic fibration [2]. There has been substantial work in recent years on abelian factors in F-theory, which we do not attempt to review here. While there are still open questions related to abelian constructions, particularly those of high rank, the general understanding of these structures has progressed to the point that a systematic approach can be taken to organizing the tuning of abelian factors in F-theory models over a generic base. We describe here how this can be approached in the context of the general tuning framework of this paper, beginning with a single abelian factor and then considering multiple abelian factors and discrete abelian groups.

8.1 Single abelian factors

A general form for a Weierstrass model with rank one Mordell-Weil group, corresponding to a single U(1) factor, was described by Morrison and Park [71]. Over a generic base, such a Weierstrass model takes the form

$$y^2 = x^3 + (e_1 e_3 - \frac{1}{3} e_2^2 - b^2 e_0) x + (-e_0 e_3^2 + \frac{1}{3} e_1 e_2 e_3 - \frac{2}{27} e_2^3 + \frac{2}{3} b^2 e_0 e_2 - \frac{1}{4} b^2 e_1^2). \quad (8.1)$$

Here, b is a section of a line bundle $\mathcal{O}(L)$, where L is effective, and e_i are sections of line bundles $\mathcal{O}((i-4)K + (i-2)L)$, where K is the canonical class of the base. This provides a general approach to tuning a Weierstrass model with a single U(1) factor. One chooses the divisor class L , and then curves b, e_i in the corresponding classes, which define the Weierstrass model.

A simple conceptual way of understanding this construction and the associated spectrum comes from the observation, developed in [71, 72] (see also [73, 74]), that when $b \rightarrow 0$ (8.1) becomes the generic form of a Weierstrass model from the SU(2) Tate form, where e_3 is the divisor supporting the SU(2) gauge group. The divisor class $[e_3] = -K + L$ always has positive genus, since L is effective. Thus, the resulting SU(2) group has some adjoint representations. The U(1) model (8.1) can be found as the Higgsing of the SU(2) model on an adjoint representation, and has the corresponding spectrum. While in some situations the enhanced SU(2) leads to a singular F-theory model, the spectrum can still be analyzed consistently from this point of view.

Thus, the generic tuning of a U(1) factor on an arbitrary base can be carried out by choosing a curve e_3 of genus $g > 0$. From Table 7, and the usual rules of Higgsing an SU(2) to a U(1), we see that the resulting matter spectrum consists of

$$\text{generic } U(1) : \quad \text{matter} = (6n + 16 - 16g)(\pm 1) + (g - 1)(\pm 2) \quad (8.2)$$

where $n = [e_3] \cdot [e_3]$ is the self-intersection of the curve e_3 . Since the Higgsing introduces one additional modulus for each uncharged scalar, the change in Hodge numbers for this tuning is g less than that for the SU(2) model:

$$\text{generic } U(1) : \quad \Delta(h^{1,1}, h^{2,1}) = (1, -12n + 30(g - 1) + 1). \quad (8.3)$$

As the simplest example, choosing e_3 to be a cubic on \mathbb{P}^2 with $n = 9$ gives $g = 1$, and a matter content of 108 fields of charge ± 1 under the U(1), while the Hodge numbers are (3, 166). Note that here the Hodge number $h^{2,1}(X)$ is determined from the low-energy theory using the rules of Higgsing and anomaly cancellation; directly computing the number of independent moduli in the Weierstrass model (8.1) is tricky due to possible redundancies, and has not yet been carried out in general, to the best knowledge of the authors.

This approach allows for the tuning of a generic U(1) on an arbitrary base. The spectrum will become more complicated when e_3 intersects other divisors that carry gauge groups, and must be analyzed in a parallel fashion to other intersecting divisors that each carry nonabelian

gauge factors. A recent systematic analysis of this type, for example, was carried out in [75]. When the $U(1)$ derives from the Higgsing of an $SU(2)$, however, the matter follows directly from the Higgsing process and can be tracked in the low-energy theory. Note also that the curve e_3 can be reducible, in which case the corresponding $SU(2)$ model will arise on a product of irreducible factors, with bifundamental matter in place of adjoint matter; such situations are discussed in some detail in [34].

The $U(1)$ factors tuned in this way will have only the generic types of $U(1)$ matter, associated with charges $q = \pm 1, \pm 2$. It is known, however, that $U(1)$ models with higher charges such as $q = \pm 3$ can be found, see *e.g.* [73]. As described in [36], such $U(1)$ models can be described as arising from the Higgsing of $SU(2)$ models with exotic matter, such as in the three-index symmetric matter representation. From the point of view of the unHiggsed nonabelian model, the $SU(2)$ factor on e_3 can acquire exotic matter through a transition such as the one described in the second line of Table 13 where 3 adjoints of $SU(2)$ and seven neutral fields are exchanged for a half-hypermultiplet in the $\mathbf{4}$ representation and seven fundamentals, with a shift in $h^{2,1}$ of -7. If the resulting theory has at least one further adjoint, on which the $SU(2)$ is then Higgsed, this gives a $U(1)$ factor with charge $q = \pm 3$. By systematically constructing all $SU(2)$ models with exotic matter we thus should in principle be able to construct all $U(1)$ models with higher charges. There are a number of questions here that are still open for further work, however. We summarize the situation briefly. As discussed in [72], when the $U(1)$ is unHiggsed to $SU(2)$ it may introduce (4, 6) singularities at codimension two or even at codimension one. Thus, a systematic analysis of all $U(1)$ models with exotic matter through Higgsing might require constructing classes of singular $SU(2)$ models. A more direct approach would be to consider how the $SU(2)$ transition to exotic matter is inherited in the $U(1)$ theory, where it should correspond to a direct transition of $U(1)$ models exchanging standard matter types for $q = \pm 3$ or higher matter. From the general analysis of [35], such abelian transitions should pass through superconformal fixed points of the theory. A direct construction of these transitions in the $U(1)$ theory has not yet been completed. It is also not known in principle whether all $U(1)$ models with charge 3 matter can be constructed in this fashion through Higgsing of models with the same or higher rank nonabelian symmetry. Further work is thus needed to complete the classification of non-generic F-theory models with a single $U(1)$ factor and higher charged matter over a given base. One way to frame this question is in the context of 6D string universality [23]; from the low-energy point of view, for a given gauge group and associated divisor class, we can classify the finite set of matter representations that are in principle allowed. For a theory with abelian factors, some progress was made in classifying the allowed spectra in [40, 76, 77], but a complete classification has not been made. The open question is whether in all cases with abelian factors, all anomaly-allowed matter representations can be realized by explicit Weierstrass models in F-theory. In particular, for generic $SU(2)$ models with $\mathbf{4}$ or higher matter or the corresponding Higgsed $U(1)$ models with matter of charges $q = \pm 3$ or higher, while some F-theory examples have been constructed others have not, and the universality question is still open.

8.2 Multiple $U(1)$'s

As the rank of the abelian group rises, the algebraic complexity of explicit constructions increases substantially. A general rank two $U(1) \times U(1)$ Weierstrass model was constructed in [34] (less general $U(1)^2$ models were constructed in [78–83]). The general $U(1)^2$ model can be understood in a similar fashion to the $U(1)$ models just described, in terms of unHiggsing to a nonabelian model. In general, there is a divisor class associated with each $U(1)$ factor; most generally, these divisor classes can be reducible, and there may be some overlap between the curves supporting the resulting $SU(2)$ factors, in which case the rank is increased to two over the common divisor. Specifically, if we denote by AC, BC the curves on which the “horizontal” $U(1)$ divisors become vertical, with C the common factor, as described in [34], the unHiggsed gauge group will be $SU(2) \times SU(2) \times SU(3)$, with the factors supported on curves A, B, C , each of which may be further reducible in which case the gauge group acquires multiple factors accordingly. The charges of the general $U(1) \times U(1)$ model can then be understood simply from the Higgsing of the appropriate nonabelian model, in parallel to the construction described above for a single $U(1)$ factor from the unHiggsed $SU(2)$ spectrum. The details become somewhat more complicated, and we do not go into them explicitly here (see [34]), but the tuning and full spectra of a generic $U(1) \times U(1)$ model can in principle be described using this analysis.

As in the case of a single $U(1)$, the generic $U(1) \times U(1)$ spectrum described here consists of only the most generic types of charged matter under the two abelian factors. Note, however, that this includes matter charges associated with a symmetric representation of $SU(3)$, which can be realized for example, from the Higgsing of an $SU(3)$ on a singular curve with double point singularities and a non-Tate model with a symmetric representation and at least one additional adjoint, as constructed in [34, 35]. For more exotic matter representations, there is as yet no general understanding, and many of the open questions described above, such as the explicit realization of abelian matter transitions, and the existence and Higgsing of appropriate rank two nonabelian models with exotic matter, are relevant here as well.

Constructing models with more $U(1)$'s becomes progressively more difficult. One class of $U(1)^3$ models was constructed in [84], though these models do not capture many of the spectra that could result from Higgsing nonabelian rank 3 models, and are certainly not general. The construction of a completely general abelian model with $U(1)^k$ where $k > 2$ is still an open problem. Nonetheless, from the point of view of geometry and field theory a general approach was outlined in [34] that in principle gives an approach to the tunings that gives what should be a superset of the set of allowed possibilities, in the spirit of this paper. The idea is that each $U(1)$ factor should come from a divisor C_i , and these divisors can be reducible, with separate components in principle for each subset of the divisors, generalizing the AC, BC rank two construction described above. To proceed then, we consider all possible divisor combinations that can support an unHiggsed rank k nonabelian group. The possible rank k abelian group constructions, and the corresponding charges, can then be determined from Higgsing the corresponding nonabelian model. In each case, the specific spectrum and

anomaly cancellation conditions allow us to compute the potential shift in Hodge numbers. This gives in principle a finite list of possibilities that would need to be checked for the existence of an explicit Weierstrass model. In practice, the fact that no generic way is known to implement Higgsing in the Weierstrass context makes the explicit check impossible with current technology, even for generic types of matter. Here we also in principle would need to deal with exotic matter contents.

Another approach to constructing higher-rank abelian models proceeds through constructing fibrations with particular special fiber types that automatically enhance the Mordell-Weil rank, see *e.g.* [85, 73]. It is not clear, however, how this approach can be used in the systematic construction of models, particularly through the perspective of tuned Weierstrass models as we have considered here. Nonetheless, this approach may provide a useful alternative perspective on the systematic construction of higher-rank abelian theories.

To summarize, for rank one $U(1)$ models with generic matter, we have a systematic approach to constructing all tunings. When considering more exotic matter or higher rank nonabelian groups, we have a systematic algorithm for constructing a finite set of possibilities along with the Hodge numbers of the elliptic Calabi-Yau threefold, but the technology does not yet exist to explicitly check all possibilities. Note that there is also as yet no proof that the general higher rank Higgsing strategy will give all possible higher-rank abelian spectra, unlike for rank one and two with generic matter, where the results of [71, 34] represent general constructions, all of which are compatible with the Higgsing approach.

8.3 Discrete abelian gauge factors

Finally, we describe briefly the possibility of systematic tunings of discrete abelian gauge factors. Such discrete factors, and corresponding matter, have been the subject of substantial recent work [72, 86–93]. As described in [72], one systematic way to approach discrete abelian factors is through the Higgsing of continuous abelian $U(1)$ factors on states of higher charge. For example, Higgsing a $U(1)$ theory with matter of charge ± 1 on a field of charge $+2$ gives a theory with a discrete abelian \mathbb{Z}_2 symmetry and matter of charge 1 under the \mathbb{Z}_2 . In the context of (8.1), this Higgsing can be realized by transforming b^2 into a generic section e_4 of the line bundle $\mathcal{O}(2L)$. This gives an explicit approach to constructing the simplest class of discrete abelian gauge models, those with \mathbb{Z}_2 gauge group and charges 1. On a generic base, choosing e_3 to be a curve of genus g and self-intersection n , the resulting spectrum and Hodge shifts should be

$$\text{generic } \mathbb{Z}_2 : \text{matter} = (6n + 16 - 16g)(\pm 1), \quad \Delta(h^{1,1}, h^{2,1}) = (0, -12n + 32(g - 1)). \quad (8.4)$$

A construction of a theory with a discrete abelian \mathbb{Z}_3 symmetry from a $U(1)$ factor with matter of charge $q = \pm 3$ was given in [91]. While constructions of models with more complicated groups and/or matter over generic bases have not been given explicitly in full generality, we can follow the same approach as used for the general $U(1)^k$ models to construct a class of potential Hodge numbers and spectra that should be a superset of the set of allowed F-theory

possibilities. Basically, for each possible $U(1)^k$ model, we consider the Higgsings on charged matter that leave a residual discrete gauge group. In addition to the caveats discussed above for the $U(1)^k$ models, there are also the issues that the $U(1)^k$ model may in principle be singular even if the model with the discrete symmetry is not, and that in principle there may be allowed models with discrete symmetries that cannot be lifted to $U(1)^k$ models. These are all good open questions for further research that would need to be resolved to complete the classification process in this direction.

9 A tuning algorithm

We now describe a general algorithm that, given any explicit choice of base B , produces a finite list of possibilities for tuned Weierstrass models. In the most concrete case of toric bases and tunings over toric divisors, this algorithm can be carried out in an explicit way to enumerate and check all possibilities. More generally, the algorithm will produce a superset of possible tunings, for which explicit realizations as Weierstrass models must be confirmed. We begin by describing the algorithm in a step-by-step fashion. We then summarize the outstanding issues related to this algorithm.

9.1 The algorithm

i) Choose a base

We begin by picking a complex surface base that supports an elliptically fibered Calabi-Yau threefold. As summarized earlier, from the work of Grassi [21] and the minimal model program, this surface must be a blow-up of \mathbb{P}^2 or a Hirzebruch surface $\mathbb{F}_m, m \leq 12$. The Enriques surface can also be used as a base, but the canonical class is trivial up to torsion, so f, g do not seem to have interesting tunings. The important data on the base that must be given includes the Mori cone of effective divisors and the intersection form.

ii) Tune nonabelian groups with generic matter on effective divisors of self-intersection ≤ -1

The set of effective irreducible curves of negative self-intersection forms a connected set. This can be seen inductively: The statement certainly holds for all the Hirzebruch surfaces \mathbb{F}_m ; for $m > 0$, \mathbb{F}_m contains a single curve of negative self-intersection, and for $m = 0$ there are no such curves. And any point p in a Hirzebruch surface, or any blow-up thereof, either lies on a curve of negative self-intersection, or on a fiber of the original Hirzebruch with self-intersection 0, so blowing up p gives another base with the desired property. (In the latter case, the fiber becomes a -1 curve after the blow-up, which intersects the negative self-intersection curve on the original \mathbb{F}_m .)

Furthermore, at least one curve of negative self-intersection in any base containing such curves will intersect an effective curve of self-intersection 0. This can be seen by taking, for example the original $-m$ curve on any blow-up of $\mathbb{F}_m, m > 0$, and noting that any base other than \mathbb{P}^2 and F_0 can be seen as a blow-up of $\mathbb{F}_m, m > 0$.

Together, these statements and our analysis of §4, §6.2 are sufficient to prove that in principle there are a finite number of possible tunings on all curves of negative self-intersection as long as the Mori cone contains a finite number of generators. We can proceed by starting with a negative self-intersection curve Σ that intersects a 0-curve, construct the finite set of possible tunings over Σ , *etc.*, and then proceed by constructing tunings over curves that intersect that curve, *etc.*, checking consistency with previous curves at each stage. This shows in principle that there is a finite algorithm for constructing all tunings over curves of negative self-intersection given a finitely generated Mori cone.

Note that the Mori cone contains a finite number of curves of self-intersection -2 or below. In practice, we can proceed effectively by using the results of Section 4, 5 to construct all possible tunings of nonabelian gauge groups on individual effective divisors and non-Higgsable clusters represented by curves of self-intersection -2 or below as units in the algorithm. The connection with -1 curves and at least one 0-curve are needed in principle to bound the infinite families that otherwise could be tuned on chains of -2 curves or alternating $-4, -1, -4, -1, \dots$ chains, and are also useful in practice to bound the exponential complexity that would be encountered by independently tuning the clusters without consideration of their connections.

Note also that in some unusual cases like dP_9 , the Mori cone has an infinite number of generators, associated with an infinite number of (-1) -curves. This algorithm appears inadequate in such cases, however in all cases that we are aware of of this type, nothing can be tuned on the infinite family of -1 curves due to a low number of available moduli in $h^{2,1}$. This issue is discussed further below.

iii) **Tune nonabelian groups on the remaining effective divisors**

We now consider tunings on remaining effective curves of non-negative self-intersection. We restrict attention to cases where the number of generators of the Mori cone is finite, and there are a finite number of effective curves with genus below any fixed bound; we discuss below situations where the number of Mori generators is infinite. The effective curves on which gauge groups can be tuned are generally quite constrained. From the analysis of §6.1, no gauge group can be tuned on any divisor that intersects a curve on which an algebra of \mathfrak{f}_4 or above is supported. Thus, the non-negative curves on which we are allowed to tune are perpendicular to all such curves, in particular perpendicular to all curves of self-intersection -5 or below. This acts as a powerful constraint, particularly for bases with large $h^{1,1}(B)$, which can only arise in the presence of many curves that have large non-Higgsable gauge factors. An example is given in §10.2. Restricting attention to curves C in the subspace with $C \cdot D = 0$ for all D of self-intersection $D \cdot D \leq -5$, the genus grows as $g = 1 + (K + C) \cdot C/2$, which increases rapidly with the self-intersection of C . Practically, this rapidly bounds the set of curves on which tunings are possible. While we do not give here an explicit algorithm for efficiently enumerating these curves, in general, the finiteness of the number of tunings follows from an argument given in [9], which uses the Hilbert Basis Theorem to show that the number of distinct strata of tuning in the moduli space of Weierstrass models is finite. We now can in principle consider all possible tunings of the divisors that admit tunings (using

Tables 4 and 7), and constrain using the rules that govern connected divisors described in Section 6. This gives a finite list of possible nonabelian gauge factors tuned on divisors in the base, which by construction satisfy the 6D anomaly cancellation conditions.

iv) **Tune abelian gauge factors**

We can use the methods described in Section 8 to identify the set of possible abelian models and spectra that could in principle be realized from the Higgsing of additional nonabelian gauge factors on effective divisors.

v) **Tune exotic matter**

Finally, we can, in any specific case, identify a finite number of possible tunings to anomaly-equivalent exotic matter content, as described in Section 7. Table 13 gives the set of possible such transitions that have been explicitly identified in Weierstrass models. This gives a finite set of possible matter contents for a given nonabelian gauge content, though as discussed earlier not all of these may have Weierstrass realizations. Analogous transitions can be carried out for abelian factors, either through the corresponding unHiggsed nonabelian theory, or in principle directly through abelian matter transitions.

9.2 Open questions related to the classification algorithm

Here we summarize places where the algorithm encounters issues that are not yet resolved. Each of these is an interesting open research problem. Note that for tunings of nonabelian gauge groups with generic matter over toric divisors in toric bases, there are no outstanding issues, and the algorithm can in principle be carried out for all bases and tunings.

- **Base issues**

The algorithm described here requires that the cone of effective divisors on the base have a finite number of generators. This is not the case for some special cases of bases such as the 9th del Pezzo surface dP_9 . The algorithm described here would not work for such bases. While the algorithm described here can be carried out for any specific base with a finitely generated cone of effective divisors, the full program of classifying all elliptic Calabi-Yau threefolds also requires classifying the set of allowed bases. In [16], all non-toric bases that support elliptic Calabi-Yau threefolds with $h^{2,1} \geq 150$ were constructed (these all have finitely generated Mori cones, so the algorithm here could be applied without encountering this problem in classifying all tuned elliptic Calabi-Yau threefolds with $h^{2,1} \geq 150$). To continue the algorithm used there to arbitrarily low Hodge numbers would require resolving several issues in addition to the finite cone issue. In particular, that algorithm used the intersection structure of classes in the Mori cone. In some cases at lower Hodge numbers this does not uniquely fix the intersection structure of effective divisors on the base; for example, one must distinguish cases where 3 curves intersect one another in pairs from the case where all three intersect at a point.

- **Apparent infinite families**

As mentioned above, in some bases such as dP_9 the Mori cone has an infinite number of -1 curves. Our algorithm would appear to break down in such situations. Because the number of tunings is proven to be finite, however, there cannot be any tunings on an infinite family of distinct curves. Thus, it seems that the finite number of moduli in any given case must limit the possibilities so that there are nonetheless a finite number of tunings. For example, for the base dP_9 we have $H = 273 - 29 \cdot 9 = 12$, giving insufficient moduli to tune even an $SU(2)$ on a -1 curve, so in fact the number of tunings here is finite even though the number of -1 curves is infinite.

We have also not given a completely rigorous proof and explicit algorithm for enumerating the set of curves of self-intersection 0 or above on a base with a finitely generated Mori cone. While we believe that this is in principle possible, and in explicit examples seems straightforward, a more general analysis and explicit algorithm relevant for non-toric bases would be desirable.

• Explicit Weierstrass tunings

In the work here we have carried out local analyses that ensure the existence of Weierstrass models for any of the local tunings over individual curves or clusters of curves of self-intersection -2 or below, except some cases of large rank classical groups or complicated -2 curve structures. Beyond these cases, we have used anomaly cancellation conditions to determine a superset of the set of allowed models for tunings over general local configurations of arbitrary curves, with Tate models used to produce most allowed constructions in the case of intersecting rational curves. An explicit implementation of the algorithm would need to confirm the existence of Weierstrass models to determine which models in the superset admit explicit constructions. We are not aware of any known exceptions to the existence of Weierstrass models other than those discussed explicitly here, but we cannot rule them out for example when considering multiple intersecting curves supporting nonabelian gauge groups, or gauge groups on higher genus curves.

• Exotic matter representations

We have listed in Table 13 the set of non-generic matter representations that have been found identified in the literature through explicit Weierstrass models. Even for these matter representations, it is not clear whether all combinations of fields that satisfy anomaly cancellation can be realized in F-theory constructions. It is also not known whether there are other exotic matter representations that may admit realizations in F-theory. We have also assumed that all exotic matter representations can be realized through a transition from an anomaly-equivalent set of generic representations; this statement is not proven.

• Codimension two resolutions

As mentioned in §2.5, while our algorithm in principle could hope to classify the complete finite set of Weierstrass models over a given base, there is a further challenge in finding all resolutions of the Weierstrass model to a smooth elliptic Calabi-Yau threefold. Despite the recent work on codimension two resolutions in the F-theory context [33, 51–56], there is as yet no general understanding or systematic procedure for describing such resolutions, particularly

in the context of the exotic matter representations just mentioned where the curve in the base supporting a nontrivial Kodaira singularity is itself singular. While the number of distinct Weierstrass models must be finite by the argument of [9], to the best of our knowledge there is no argument known that the number of distinct resolutions of codimension two singularities in a given Weierstrass model is finite¹³, so a complete classification of elliptic Calabi-Yau threefolds would require further progress in this direction.

- **Abelian gauge groups**

We have outlined an approach to constructing a superset of the set of possible abelian models over a given base. For a single $U(1)$ and generic (charge 1, 2) matter, this can be done very explicitly using the Morrison-Park form [71] and Higgsing of $SU(2)$ models on higher genus curves. For two $U(1)$ factors and generic matter this can in principle similarly be done following the analysis of [34], though we have not gone through the details of the possibilities here. For more $U(1)$ factors, while the approach described here and in [34] can in principle give a finite set of abelian models through Higgsing nonabelian models, which should represent a superset of the set of allowed possibilities, there is no general construction known of the explicit multiple $U(1)$ models. For non-generic $U(1)$ charged matter, again while in principle a finite list of possibilities compatible with anomaly cancellation can be made, explicit Weierstrass constructions beyond those of charge 3 matter in [36] are not known. In principle, exotic matter transitions could be classified directly in terms of the abelian spectrum, though this has not yet been done. For discrete abelian groups, again in principle a finite set of possibilities can be constructed by Higgsing the abelian models, but explicit Weierstrass constructions are not known beyond the generic \mathbb{Z}_2 models and some \mathbb{Z}_3 models mentioned above.

Most of the complications and issues that arise in confirming the existence of Weierstrass models for complicated gauge-matter combinations arise only as the Hodge number $h^{2,1}(X)$ becomes small. None of these issues were relevant in the classification of Weierstrass models for elliptic Calabi-Yau threefolds with $h^{2,1} \geq 350$ in [17], and we expect that one could go quite a bit further down in $h^{2,1}$ before encountering a problem with the systematic classification that would require substantially new insights into any of these problems.

We also emphasize that in principle, there is no obstruction to carrying out this algorithm for arbitrary toric constructions, with nonabelian gauge tunings only over toric divisors.

10 Examples

In this section we give some examples of applications of the methods developed and described here. In each case, the goal is not to be comprehensive, but to illustrate the utility of the methodology developed in this paper and to suggest directions for more comprehensive future work.

¹³Thanks to D. Morrison for discussions on this point

10.1 Example: two classes of tuned elliptic fibrations in Kreuzer-Skarke

The rules that we have established so far must in particular be satisfied by any Calabi-Yau elliptic fibration over toric surfaces. We have a complete set of rules that list the allowed tunings on isolated toric curves; on multiple-curve NHCs; and on clusters either neighboring or separated by a -1 curve. In each case we have provided a formula for the shift in Hodge numbers of the resulting threefold in comparison to a general elliptically fibered Calabi-Yau over the same base. It is perhaps useful at this point to see how these rules simplify practical computations.

To this end, we use our rules to explore two classes of tuned fibrations within the Kreuzer-Skarke database [58], which contains all Calabi-Yau threefolds that can be realized as hypersurfaces in toric varieties associated with reflexive 4D polytopes. In the future, these rules may be useful to perform a more exhaustive study of all tuned elliptic fibrations over toric (or more general) surfaces. For purposes of illustration, we will consider the following simple classes of fibrations over toric bases as classified in [14]:

- tunings of \mathfrak{e}_6 , \mathfrak{e}_7 over -4 curves.¹⁴
- tunings of $\mathfrak{su}(2)$ over -2 curves.

Proceeding to the first example, whenever a base contains a -4 curve, we enhance its generic $\mathfrak{so}(8)$ to \mathfrak{e}_6 and \mathfrak{e}_7 when possible. By the E_8 rule, this will be possible whenever the neighboring clusters (separated by -1 curves) support at most $\mathfrak{su}(3)$ or $\mathfrak{su}(2)$ algebras, respectively. For instance, the -4 curve in the sequence $(\dots, -3, -2, -1, -4, -1, -2, -3, \dots)$ will admit enhancement to \mathfrak{e}_7 , whereas the -4 curve in the sequence $(\dots, -1, -3, -1, -4, -1, -3, -1, \dots)$ will admit an enhancement only to \mathfrak{e}_6 , and no enhancement is possible on $(\dots, -2, -3, -1, -4, -1, -3, -2, -1, \dots)$, since the -3 curve of a $(-3, -2)$ NHC supports the algebra \mathfrak{g}_2 and $\mathfrak{g}_2 \oplus \mathfrak{e}_6 \not\subseteq \mathfrak{e}_8$.

Implementing this program yields the following results, as plotted in the diagram below. There are 1,906 distinct Hodge numbers of generic fibrations over bases that support these tunings, and 1,562 distinct Hodge numbers of tuned fibrations over these bases. The diagram is a scatterplot of both sets of Hodge numbers.

Let us now consider our second example, namely tunings of $\mathfrak{su}(2)$ on -2 curves in toric bases. As we saw above, chains of -2 curves have simple properties with respect to tunings. For $\mathfrak{su}(2)$, the allowed tunings are precisely controlled by the averaging rule: a $\mathfrak{su}(2)$ can be tuned on any divisor in a -2 chain, but once a second $\mathfrak{su}(2)$ is tuned on a different curve, all curves in between are forced to carry $\mathfrak{su}(2)$ s as well (at least). Therefore, to find all these tunings, we sweep all toric bases and identify -2 chains. For each base, for each -2 chain, we choose a starting and ending point (which could coincide) for the tuned $\mathfrak{su}(2)$ s. The total set of such tunings on a given base is found by activating all independent combinations of such tunings on the different -2 chains of the base. Since we are here interested in a coarse

¹⁴It is likely that a non-rank-enhancing tuning of $\mathfrak{so}(8)$ to \mathfrak{f}_4 does not yield a distinct Calabi-Yau but rather merely specializes to a subspace of the original's moduli space.

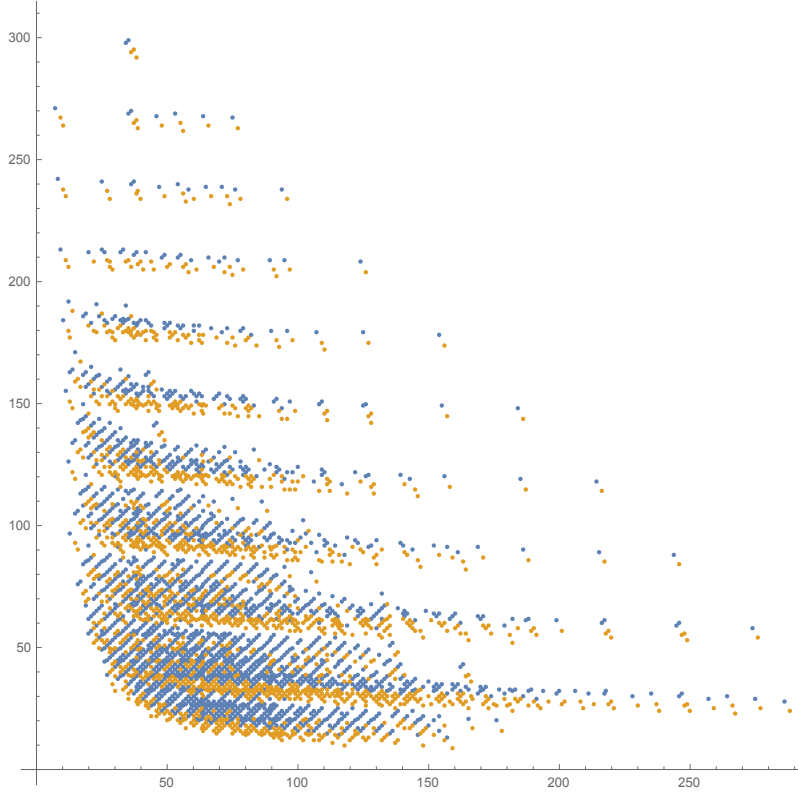


Figure 3. [Color online.] Tunings of \mathfrak{e}_6 and \mathfrak{e}_7 on -4 curves. Blue dots mark Hodge numbers of untuned models over toric bases where the $\mathfrak{so}(8)$ gauge symmetry on a -4 curve can be enhanced to at least an \mathfrak{e}_6 . Orange dots mark Hodge numbers of tuned models over the same bases (without distinguishing between \mathfrak{e}_6 and \mathfrak{e}_7 . Here the y axis $h^{2,1}$ is plotted versus the x axis $h^{1,1}$.

classification of tuned manifolds by their Hodge numbers, in this case there is a shift by $\Delta(h^{1,1}, h^{2,1}) = (+l, -(l+4))$ for each tuned group of $\mathfrak{su}(2)$ s of length l .

Searching for tunable -2 curves, we find 8,517 distinct Hodge numbers of bases on which tunings are possible, resulting in 4,537 distinct tuned Hodge numbers. In this example and in the above, all Hodge numbers are in the Kreuzer-Skarke database, strongly suggesting that these tuned elliptic fibrations represent different constructions of the models in this database. This construction is a rather simple and direct way to see that at least some manifolds with these Hodge numbers are elliptically fibered. The Hodge numbers of generic fibrations over bases with -2 chains, together with the Hodge numbers of resulting tunings, are graphed in the scatterplot.

10.2 Example: tunings on \mathbb{F}_{12}

As an example of the general tuning algorithm, we consider tuned Weierstrass models over the toric base \mathbb{F}_{12} , the twelfth Hirzebruch surface. This provides an illustration both of the general

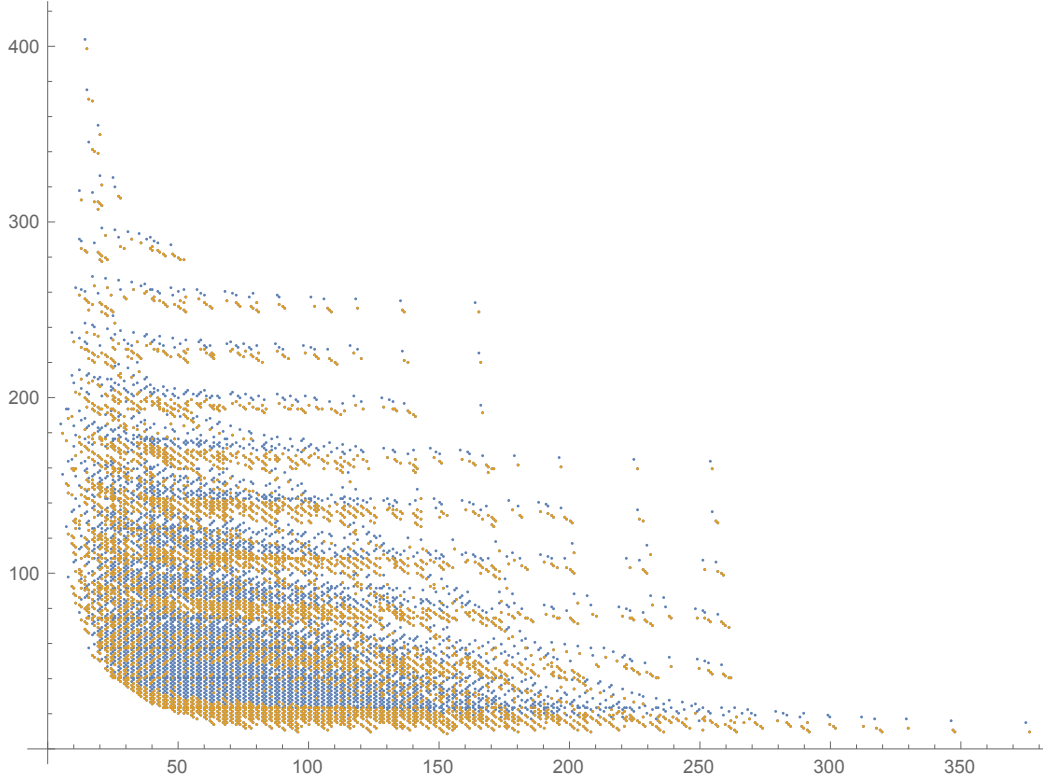


Figure 4. [Color online.] Tunings of $\mathfrak{su}(2)$ on (chains of) -2 curves. As for the previous example, blue dots mark Hodge numbers of untuned models over toric bases with -2 curves that can support a $\mathfrak{su}(2)$ gauge symmetry. Orange dots mark Hodge numbers of tuned models over the same bases. Here the y axis $h^{2,1}$ is plotted versus the x axis $h^{1,1}$.

principles of tunings and of some specific issues of interest. We do not strive for completeness in this listing, but give a broad class of tunings to illustrate the ideas and methodology involved.

The Hirzebruch surface \mathbb{F}_{12} has a cone of effective divisors generated by the curves S and F , where $S \cdot S = -12$, $S \cdot F = 0$, $F \cdot F = 0$. The toric divisors are S, F, \tilde{S}, F where $\tilde{S} = S + 12F$ has self-intersection $\tilde{S} \cdot \tilde{S} = +12$. The curve S is a non-Higgsable cluster supporting a gauge group E_8 . No curve intersecting S can carry a gauge group, since this would produce matter charged under the E_8 and a $(4, 6)$ singularity. Thus, the only curves on which we can tune nonabelian gauge factors are multiples of \tilde{S} , $k\tilde{S}$. This simplifies the classification significantly. The generic elliptic fibration over \mathbb{F}_{12} has Hodge numbers $(11, 491)$ [43], so this is our starting point.

First, we consider tuning single nonabelian gauge factors on \tilde{S} . This curve has self-intersection $n = +12$ and genus $g = 0$. Since f, g can be described torically as functions in a local coordinate z of degrees $4, 7$, where $\tilde{S} = \{z = 0\}$, with at least one monomial at each order, we can immediately tune the gauge group types $\mathfrak{su}(2), \mathfrak{e}_7$ that are enforced simply by the order of vanishing of f, g . The resulting tunings are tabulated in Table 14. It is also

G	curve(s)	matter	$(h^{1,1}(X), h^{2,1}(X))$
.	.	.	(11, 491)
$\mathfrak{su}(2)$ (T)	\tilde{S}	$88 \times \mathbf{2}$	(12, 318)
$\mathfrak{su}(3)$ (T)	\tilde{S}	$90 \times \mathbf{3}$	(13, 229)
$\mathfrak{su}(4)$ (T)	\tilde{S}	$64 \times \mathbf{4} + 14 \times \mathbf{6}$	(14, 166)
$\mathfrak{su}(5)$ (T)	\tilde{S}	$52 \times \mathbf{5} + 14 \times \mathbf{10}$	(15, 115)
$\mathfrak{su}(6)$ (T, $W_{(r>0)}$)	\tilde{S}	$(40 + r) \times \mathbf{6} + (14 - r) \times \mathbf{15}$	(16, $76 - r$)
$\mathfrak{su}(7)$ (T, $W_{(r>0)}$)	\tilde{S}	$(28 + 5r) \times \mathbf{7} + (14 - 3r) \times \mathbf{21}$	(17, $49 - 7r$)
$\mathfrak{su}(8)$ (H)	\tilde{S}	$25 \times \mathbf{8} + 10 \times \mathbf{28} + 1 \times \mathbf{56}$	(18, 18)
$\mathfrak{so}(8)$ (T)	\tilde{S}	$16 \times (\mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c)$	(15, 135)
$\mathfrak{so}(10)$ (T)	\tilde{S}	$18 \times \mathbf{10} + 16 \times \mathbf{16}$	(16, 101)
$\mathfrak{so}(12)$ (T)	\tilde{S}	$20 \times \mathbf{12} + 8 \times \mathbf{32}$	(17, 51)
\mathfrak{e}_6 (T)	\tilde{S}	$18 \times \mathbf{27}$	(17, 83)
\mathfrak{e}_7 (T)	\tilde{S}	$10 \times \mathbf{56}$	(18, 64)
$\mathfrak{su}(2) \times \mathfrak{su}(2)$	\tilde{S}, \tilde{S}	$64 \times (\mathbf{1}, \mathbf{2}) + 64 \times (\mathbf{2}, \mathbf{1}) + 12 \times (\mathbf{2}, \mathbf{2})$	(13, 193)
...
$\mathfrak{su}(4) \times \mathfrak{su}(4)$ (?)	\tilde{S}, \tilde{S}	$16 \times (\mathbf{1}, \mathbf{4}) + 16 \times (\mathbf{4}, \mathbf{1}) + 12 \times (\mathbf{4}, \mathbf{4})$ $+ 14 \times (\mathbf{6}, \mathbf{1}) + 14 \times (\mathbf{1}, \mathbf{6})$	(17, 3)
$\mathfrak{su}(2)$	$2\tilde{S}$	$128 \times \mathbf{2} + 11 \times \mathbf{3}$	(12, 205)
$\mathfrak{su}(3)$	$2\tilde{S}$	$118 \times \mathbf{3} + 11 \times \mathbf{8}$	(13, 87)
$\mathfrak{su}(4)$ (?)	$2\tilde{S}$	$32 \times \mathbf{4} + 28 \times \mathbf{6} + 11 \times \mathbf{15}$	(14, 45)
$\mathfrak{su}(2)$	$3\tilde{S}$	$120 \times \mathbf{2} + 34 \times \mathbf{3}$	(12, 152)
$\mathfrak{su}(3)$ (?)	$3\tilde{S}$	$54 \times \mathbf{3} + 34 \times \mathbf{8}$	(13, 65)
$\mathfrak{su}(2)$ (?)	$4\tilde{S}$	$64 \times \mathbf{2} + 69 \times \mathbf{3}$	(12, 159)
$U(1)$ (W)	$[2\tilde{S}]$	$128 \times (\pm \mathbf{1}) + 10 \times (\pm \mathbf{2})$	(12, 216)
$U(1)$ (W)	$[3\tilde{S}]$	$120 \times (\pm \mathbf{1}) + 33 \times (\pm \mathbf{2})$	(12, 186)
\mathbb{Z}_2 (W)	$[2\tilde{S}]$	$256 \times \mathbf{1}$	(11, 235)
\mathbb{Z}_2 (W)	$[3\tilde{S}]$	$240 \times \mathbf{1}$	(11, 251)

Table 14. Some tunings of elliptic Calabi-Yau threefolds and corresponding F-theory models on the base $B = \mathbb{F}_{12}$. Gauge group described is tuned on a multiple of \tilde{S} , additional non-Higgsable factor of E_8 from S is dropped. Curve for $U(1)$ models is corresponding value of $[e_3]$ supporting associated unHiggsed $SU(2)$. Tunings with rank-equivalent descriptions (*e.g.* \mathfrak{g}_2) not included in list. All tunings listed satisfy anomaly conditions and have no known inconsistency with F-theory. Those denoted by (T, W, H) have explicit tunings using Tate, Weierstrass, or heterotic descriptions. Those denoted by (?) not only have no explicit F-theory description but are particularly suspicious as possible swampland residents. Models without a denotation can all be understood as Higgsings of other models with known Tate or Weierstrass realizations.

straightforward to confirm from a direct toric monomial analysis that we can tune the orders of vanishing of f, g with a proper choice of monodromy conditions to get any of the other gauge groups with algebras $\mathfrak{su}(3), \mathfrak{g}_2, \mathfrak{so}(7), \mathfrak{so}(8), \mathfrak{f}_4, \mathfrak{e}_6$. This illustrates the general methods

of §4.3 in a specific context.

The classical groups with algebras $\mathfrak{su}(N)$, $\mathfrak{sp}(N)$, $\mathfrak{so}(N)$ must be considered separately. We focus attention particularly on the $SU(N)$ groups. For $\mathfrak{su}(N)$, $N = 6, 7$ there are exotic matter contents that can be tuned using Weierstrass as described in §7 [35]; these are included in the Table. For $SU(N)$ and $Sp(k)$, from the matter spectrum in Table 4 it is clear that at $N = 2k = 10$ there is a problem as the number of fundamental representations becomes negative. In fact, a Tate analysis indicates that tuning $\mathfrak{su}(8)$ eliminates the single monomial of a_6 that has order ≤ 5 on the -12 curve, forcing a non-minimal singularity there. This is an example of the constraint discussed in §6.4 that a 0-curve with an \mathfrak{e}_8 on one side cannot have an $\mathfrak{su}(N)$ with $N \geq 8$ on the other side tuned using Tate. This is an example of a tuning in the swampland, which looks consistent from the low-energy point of view but may not be allowed in F-theory. Note that the groups that we can tune in this way are subgroups of E_8 , matching with the expectation that any tuning on this base should have a heterotic dual, with the resulting gauge group realized from an E_8 bundle over K3 with instanton number 24. An explicit construction was given in [35] for non-Tate Weierstrass tunings of $\mathfrak{su}(8)$ with r matter fields in the triple-antisymmetric **(56)** representation; there it was argued that the only heterotic dual to a tuning on the $+12$ curve of \mathbb{F}_{12} has $r = 1$. The other Weierstrass tunings thus must have singularities¹⁵ when restricted to the compact base \mathbb{F}_{12} . From the low-energy point of view it seems quite obscure why an $E_8 \times SU(8)$ theory constructed in this way would be inconsistent with $r = 0$ but consistent with $r = 1$. Understanding this kind of question better and in more generality is an interesting area for further research.

Since \tilde{S} is a non-rigid divisor ($n \geq 0$), we can tune multiple independent gauge factors on different curves Σ_i in this divisor class, which will then intersect one another with $\Sigma_i \cdot \Sigma_j = 12$. For example, tuning two $SU(2)$ factors on such curves gives a model with gauge group $(E_8 \times) SU(2) \times SU(2)$, where there are 12 bifundamental fields in the **(2, 2)** representation. From the spectrum and anomaly constraints we would expect to be able to tune various product groups. It is fairly straightforward to see using Tate and the toric picture that it should be possible to tune any product of groups $SU(N_1) \times \cdots \times SU(N_k)$ with $\sum_i N_i < 8$, though we do not pursue the details here. Cases with $\sum_i N_i = 8$ are less clear and would be interesting to explore in more detail. Note that these product group configurations can be understood from Higgsing on k -index antisymmetric representations of the $\mathfrak{su}(N)$ models on \tilde{S} (see, *e.g.*, [35] for an explicit discussion of several examples of such Higgsing in Weierstrass and dual heterotic descriptions). For example, Higgsing $\mathfrak{su}(6)$ on a two-index antisymmetric **(15)** representation gives a model with gauge algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(4)$, while Higgsing on a three-index antisymmetric **(20)** gives a model with gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(4)$. The $\mathfrak{su}(4) \oplus \mathfrak{su}(4)$ model that is allowed by anomalies would in principle come from an anomaly-allowed model with an $\mathfrak{su}(8)$ gauge algebra and a 4-index antisymmetric matter field, but constructing a Weierstrass model for such configurations seems problematic [35]. Any model that can be realized in field theory from the Higgsing of a model with a good F-theory description must also have a good

¹⁵Thanks to N. Raghuram for confirming this in the case $r = 0$.

F-theory description, even if the explicit Weierstrass construction is difficult to identify. Thus, the product groups with $\sum_i N_i < 8$ are also naturally understood as Higgsings of the models with a single $\mathfrak{su}(N)$ algebra on \tilde{S} .

Now let us consider tunings on the curve $\Sigma = 2\tilde{S}$. This curve has self-intersection $n = 48$, and from $-K \cdot \Sigma = 2(-K \cdot \tilde{S}) = 28$, we have $g = 11$. From Table 7, we see that anomalies suggest that we can tune such models for $SU(2)$, $SU(3)$, and $SU(4)$. Roughly, we are tuning on a curve described by a quadratic in the coordinate z that vanishes on \tilde{S} , so as for $\mathfrak{su}(8)$ on \tilde{S} , the $\mathfrak{su}(4)$ algebra on $2\tilde{S}$ may be problematic. The $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ models can be understood in field theory from the Higgsing of the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ models described above on a bifundamental field [35], and thus must exist in F-theory. The $\mathfrak{su}(4)$ model on $2\tilde{S}$ is also unclear from this point of view.

Similarly, we can consider tuning $SU(2)$ and $SU(3)$ models on $3\tilde{S}$ ($n = 108, g = 34$) and $SU(2)$ on $4\tilde{S}$ ($n = 192, g = 69$). We have not attempted to explicitly construct Weierstrass models for these theories; the first of these should arise from Higgsing of an $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ model on $\tilde{S} + \tilde{S} + \tilde{S}$, which should in turn come from a Higgsing of $\mathfrak{su}(6)$ on \tilde{S} , while the others are likely problematic though acceptable from anomaly cancellation.

From these realizations of $SU(2)$ and other nonabelian groups on higher genus curves we can also proceed to implement the construction of $U(1)$ models as discussed in §8. As discussed in the main body of the text, $U(1)$ models and their spectra can be realized from Higgsings of nonabelian models over the same base (which can be allowed to have some singularities that are removed in the Higgsing). In the simplest cases, we achieve the $U(1)$ model by Higgsing an $SU(2)$ with generic matter; these can be explicitly realized using the Morrison-Park form [71]. In this case, we can consider the $SU(2)$ realizations on $k\tilde{S}$ for $k = 2, 3$. These give $U(1)$ theories with various spectra, as listed in the table. Many further $U(1)$ models with explicit Weierstrass models could be constructed with $q = 3$ charges through matter transitions in the unHiggsed $SU(2)$ theory to $SU(2)$ models with 4 matter, as discussed in [36]. and the superset of all such models could be constructed by transferring an arbitrary number of groups of 3 adjoints into 4's, though Weierstrass models for all these are not known. Furthermore, many $U(1) \times U(1)$ models with generic matter spectra could be constructed using the methods of [34], and the hypothetical superset of all $U(1)^k$ models may be constructable at the level of spectra by considering all Higgsings of nonabelian models including those constructed above, though we have neither a method for explicitly constructing Weierstrass models in these cases, nor a proof that this exhausts all possibilities for $k > 2$. We do not explore these considerations further here.

We conclude by constructing the model with discrete \mathbb{Z}_2 gauge group with what seems to be the largest value of $h^{2,1}$. This follows from taking the $k = 2$ generic $U(1)$ model above and Higgsing on a field of charge 2, following [72]. Many more models with discrete gauge groups and various charges could be constructed, some explicitly in Weierstrass, and a larger superset by considering all Higgsings on non-unit charges of abelian $U(1)^k$ models.

11 Conclusions and Outlook

In this paper we have reported on progress towards a complete description of the set of Weierstrass tunings over a given complex surface B that supports elliptic Calabi-Yau threefolds. These Weierstrass tunings can be used to classify elliptic Calabi-Yau threefolds and to study F-theory supergravity and SCFT models in six dimensions. In particular, for a given base B the results accumulated here give a set of constraints on the set of possible tunings over B , which give a finite superset of the finite set of consistent tunings. While we have not completely solved the tuning classification problem, we have framed the structure of the problem, developed many of the components needed for a full solution, and identified a few remaining components that need a more complete analysis for a full understanding.

The tools developed in this work can be used in a number of ways, including generating examples of elliptic Calabi-Yau threefolds and F-theory models with particular features of interest, the classification of elliptic Calabi-Yau threefolds and corresponding 6D supergravity theories, and exploration of the “swampland” of 6D theories that seem consistent but cannot be realized explicitly in F-theory. In this concluding section we summarize the specific results of this paper as well as a set of further issues to be addressed, and we discuss the implications for the 6D swampland and the potential extension of this kind of analysis to 4D F-theory models.

11.1 Summary of results

This progress extends previous work in the following ways:

- We have completely classified local tunings of arbitrary gauge groups with generic matter on a single rational curve, and shown in all cases except $\mathfrak{su}(N)$ that the tunings allowed by anomaly cancellation can be realized in explicit local Weierstrass/Tate models; for $\mathfrak{su}(N)$ we have found Tate models for almost all cases, with a few exceptions at large odd N and divisors of positive self-intersection for which Tate models are impossible and no Weierstrass models are known.
- We have completely classified local tunings in the same way over all multiple-curve non-Higgsable clusters.
- We have classified allowed local tunings on a pair of intersecting rational curves, and shown that a large fraction of anomaly-allowed tunings can be realized by Tate or Weierstrass models, but we have also identified quite a few exceptions.
- We have identified some specific configurations, such as $\mathfrak{su}(10) \oplus \mathfrak{su}(3)$ and $\mathfrak{g}_2 \oplus \mathfrak{sp}(4)$ on a pair of intersecting curves of self-intersections -2 and -1 , that are allowed by anomalies but cannot be realized through a Tate construction in F-theory. These represent a component of the “swampland” both for supergravity theories and for SCFTs, and must be explained if the assertion of F-theory universality for 6D SCFT’s is to be

proven. We have found a substantially larger number of configurations in the supergravity swampland that do not have low-energy field theory descriptions through an SCFT with gravity decoupled as they involve curves of nonnegative self-intersection.

- We have identified extremal configurations of -2 curves, associated with degenerate elliptic curves satisfying $\Sigma \cdot \Sigma = -K \cdot \Sigma = 0$, as loci that in F-theory admit a finite number of $\mathfrak{su}(N)$ tunings though low-energy consistency does not constrain N in any known way.
- We have investigated the validity of the “ E_8 ” rule [18], which constrains the gauge factors that can be tuned on divisors that intersect a -1 curve carrying no gauge group, in the context of tuning enhanced gauge symmetries. We found that the E_8 rule is a necessary condition for tunings to be possible, but may not be a sufficient condition as some combinations of gauge groups that live within E_8 cannot be realized using Tate. We also identified an analogous rule for curves of self-intersection 0.
- Combining the preceding results gives a complete set of tools that can in principle produce the finite set of all possible tunings over toric curves in toric bases. Work in this direction is in progress [94]. In the toric case, each prospective tuning can be checked for global consistency in a global Weierstrass model using toric methods.
- These tools, in the context of 6D SCFT’s, give a systematic description of tunings of an SCFT in terms of a Weierstrass or Tate model on the set of contracted curves, complementing the analysis of [19]. In particular, this work goes beyond that reported in [19] in that we systematically construct explicit Weierstrass models for the configurations allowed by anomaly constraints, and identify some new configurations that do not admit Tate tunings and do not have known or straightforward Weierstrass models, yet which satisfy low-energy consistency conditions. These tunings can also be applied in the closely related context of F-theory realizations of little string theories [68].
- We have used anomaly cancellation to classify the set of possible tunings over curves of arbitrary genus that are acceptable from low-energy considerations.
- We have computed explicitly the Hodge number shifts for the elliptic Calabi-Yau threefold for all the preceding tunings.
- We have provided geometric proofs of strong constraints on local combinations of allowed tunings, matching constraints from anomaly considerations. In particular, we have shown that the only possible pairs of gauge group factors that can arise on intersecting divisors, *and hence the only combinations of gauge groups that can share matter in any low-energy theory arising from F-theory*, have one of the five combinations of algebras listed in Table 10 (or arise as a product subgroup of one of the allowed realized individual or product groups after an appropriate breaking).

- We have given a general procedure for classifying allowed tunings of non-generic matter and abelian gauge fields, which will give a finite set of tunings allowed over any given base, and which should be a superset of the complete set that can be explicitly realized in F-theory.

11.2 The 6D $\mathcal{N} = 1$ “tuning” swampland

In general, one of the goals of this work is to narrow down the “swampland” of models that seem consistent from low-energy considerations but that lack UV descriptions in string/F-theory [23]. For 6D supergravity models, this problem can be broken into two parts: first, the matching of completely Higgsed 6D supergravity models to F-theory constructions, and second the matching of all possible gauge enhancements through tuning/unHiggsing in the F-theory and supergravity models. There are still substantial outstanding questions related to the first part; in particular, we do not have a proof that a low-energy model with a BPS dyonic string of Dirac self-charge -3 or below implies the presence of a non-Higgsable gauge field, while F-theory implies this condition. In this paper we address the second part of the question: given a completely Higgsed 6D supergravity theory with an F-theory realization we ask whether all possible unHiggsings of the 6D SUGRA theory that are consistent with anomaly cancellation can be realized as tunings of the corresponding F-theory model. By comparing field theory and F-theory geometric analysis of various local combinations of gauge groups over different curve types, we have shown that in almost all cases, F-theory reproduces precisely the set of gauge groups and matter through tunings that are allowed by anomaly cancellation conditions and other low-energy consistency constraints. We have also, however, identified some situations where field theory and F-theory are not in agreement, or seem from our current understanding not to be in agreement. We list these here.

Tunings on a single divisor For local tunings of generic matter types over a single rational curve, we found that virtually everything that is allowed by anomaly cancellation has an explicit Tate or Weierstrass realization. The only class of exceptions were the tunings of large-rank $\mathfrak{su}(N)$ algebras listed in Table 6. For those cases, Tate models are not possible. In some examples such as $\mathfrak{su}(21)$ and $\mathfrak{su}(23)$ on a +1 curve, a straightforward approach to Weierstrass models also fails; although we have not proven rigorously that a Weierstrass realization is impossible this seems likely to be true as other known non-Tate Weierstrass models realize exotic matter. These swampland examples may have a low-energy inconsistency, may be realizable through exotic Weierstrass models or may be stuck in the swampland. The large $\mathfrak{su}(N)$ tunings constitute the complete set of single-divisor swampland examples encountered in this work.

Tunings on a pair of divisors For local tunings of generic matter over a pair of rational curves that intersect at a single point, we found a larger class of instances of models that are allowed through anomalies but not through Tate constructions. In addition to a couple of known examples such as $\mathfrak{so}(8) \oplus \mathfrak{su}(2)$ when the $\mathfrak{su}(2)$ is on a -2 curves (which is known to have field theory inconsistencies [60]), we found other examples of algebras $\mathfrak{su}(N) \oplus \mathfrak{sp}(k)$,

$\mathfrak{su}(N) \oplus \mathfrak{su}(M)$, $\mathfrak{g}_2 \oplus \mathfrak{sp}(k)$, and $\mathfrak{so}(N) \oplus \mathfrak{sp}(k)$ that are acceptable according to anomaly cancellation but do not have Tate realizations. A simple example is $\mathfrak{su}(2j+1) \oplus \mathfrak{su}(2j+8)$ on a pair of curves of self-intersection $-1, -2$. A further list of examples of tunings on two intersecting curves without Tate forms is given in §6.2. Like the examples on a single rational curve, we do not have a proof that Weierstrass models cannot be found for any of these cases, though we do not expect a non-Tate Weierstrass model in any of these cases with generic matter content.

Tunings on degenerate elliptic curves As discussed in §5.2.2, §6.5, there are some local combinations of divisors, for example a sequence of two or more -2 curves mutually intersecting in a loop, which naively admit an infinite number of gauge group tunings with a finite number of moduli needed for the tuning. These correspond to low-energy 6D supergravity theories with $T \geq 9$ with no apparent inconsistency. We have identified these configurations in F-theory as degenerate elliptic curves satisfying $\Sigma \cdot \Sigma = -K \cdot \Sigma = 0$. From the F-theory point of view $\mathfrak{su}(N)$ gauge groups can be tuned on such curves with N taking values only up to a specific bound associated with the constraint $\Delta = -12K$. As discussed in [9], however, there is no low-energy understanding at this time of this “Kodaira constraint,” so that for effective cones containing such -2 curve configurations there is effectively an infinite swampland. This is an example of the more general issue that adding a gauge group with only adjoint matter (essentially an $\mathcal{N} = (1, 1)$ multiplet) does not affect anomaly conditions, and can be limited in F-theory but not in the low-energy theory.

Exotic matter We have listed in §7 some exotic matter content for which there are known constructions. Other types of matter appear to be allowed by the anomaly constraints, but are at this point lacking Weierstrass constructions. The resolution of this part of the tuning swampland will be addressed further elsewhere.

Constraints from divisors without gauge factors

Constraints such as the E_8 rule that involve divisors that do not carry gauge factors are not currently understood from 6D supergravity, though they can be partly explained in SCFT limits. We have identified some possible exceptions to the sufficiency of E_8 rule for determining allowed tunings; for example, $\mathfrak{su}(5) \oplus \mathfrak{su}(5)$ and $\mathfrak{su}(9) \oplus \cdot$ cannot be tuned using the Tate procedure on a pair of divisors that intersect a -1 curve without forcing a gauge factor on the -1 curve. Such examples suggest some new low-energy consistency condition in both supergravity and SCFT or a novel construction of non-Tate Weierstrass models with generic matter for certain groups. We have also sketched out the analogue of the E_8 rule for divisors of self-intersection 0. These kinds of constraints are somewhat similar to the constraint that, for example, a low-energy theory containing a BPS dyonic string of Dirac self-charge -3 , which would correspond in F-theory to a -3 curve in the base, needs to carry a gauge group $\mathfrak{su}(3)$ in this case. A related set of issues is the distinction between type III, IV , and I_2, I_3 realizations of $\mathfrak{su}(2), \mathfrak{su}(3)$, which have slightly different rules for intersections but are not easily distinguished in the low-energy theory. Understanding the E_8 rule and these other

related conditions from low-energy considerations is an important part of the outstanding problem of clearing the 6D swampland.

Abelian gauge factors We have outlined an algorithm, following [34], which in principle gives a superset of the set of possible F-theory models with abelian gauge field content. This algorithm is based on Higgsing of nonabelian gauge factors with adjoint matter. Proving that all abelian F-theory models can be constructed in this fashion, and matching precisely with low-energy anomaly constraints, particularly for higher-rank abelian groups, remains an outstanding research problem.

11.3 Tate vs. Weierstrass

One interesting question that arises in attempting to do generic tunings is the extent to which Tate models can produce the full set of possible tunings. It is known that there are Weierstrass tunings of $\mathfrak{su}(N)$ with $N = 6, 7, 8$ that cannot be realized through Tate [32, 33, 35], though these are associated with exotic matter (*e.g.*, in the three-index antisymmetric representation). We have also identified cases of $\mathfrak{so}(N)$ with $N = 13$ on curves of self-intersection n with n even but not $n = -4$, where Weierstrass models can be realized but Tate cannot. It is known that Weierstrass and Tate tunings of $\mathfrak{su}(N)$ are equivalent for $N < 6$, and are believed to be more generally equivalent when exotic matter representations are not included. It therefore seems possible that many of the tunings found here on a single curve or a pair of curves that do not have Tate realizations also do not have Weierstrass realizations with generic matter content and represent elements of the swampland.¹⁶ But identifying precisely the set of cases where Tate and Weierstrass forms are not equally valid, is a remaining task that needs to be completed to clear out this part of the F-theory swamp.

11.4 4D F-theory models

The focus of this paper has been on 6D supergravity theories described by F-theory compactifications on complex surfaces. An analogous set of constructions give 4D $\mathcal{N} = 1$ supergravity theories from F-theory compactifications on complex threefolds. While the 4D case is much less well understood, and the connection between the underlying F-theory geometry and low-energy physics is made more complicated by fluxes, D-brane world-volume degrees of freedom, a nontrivial superpotential, and a weaker set of anomaly constraints, the basic principles of tuning that we have developed here are essentially the same in 4D. For threefold bases, divisors that have a local toric description can again be analyzed torically, and we can write Weierstrass and Tate models for the kinds of gauge groups and matter that can be tuned.

Perhaps the clearest result of this paper that immediately generalizes to 4D F-theory constructions is the constraint derived in §6.1 that limits the possible products of gauge groups on intersecting divisors to only the five (families of) algebra pairings listed in Table 10.

¹⁶(Added in v2:) David Morrison has pointed out that some models with groups that cannot be realized using Tate can have Weierstrass realizations when there is a discrete Mordell-Weil group associated with a quotient of the usual simply-connected simple group associated with a given algebra.

This constraint is also valid for tunings in 4D F-theory models, with the same microscopic derivation from the Weierstrass analysis. A consequence of this result is that we have shown in general that any $\mathcal{N} = 1$ low-energy theory of supergravity coming from F-theory can only have matter charged under multiple gauge factors when the factors are either among those in Table 10, or both factors come from the breaking of a larger single group (or product group) such as \mathfrak{e}_8 that is realized in F-theory.

More generally, the methods summarized in §2.2 in association with (2.11), which were developed in [95, 38], can be used to identify the non-Higgsable gauge group and local tunings on any local combination of divisors on a base threefold, with explicit Weierstrass and Tate constructions for local toric geometries. For single divisors, the set of possible tunings will follow a similar pattern to that found here in §4. The non-Higgsable clusters over single toric divisors were analyzed (in the context of \mathbb{P}^1 bundles) in [95, 96], and the finite set of possible tunings over such combinations of divisors can be constructed using the same methods as those used here and is again finite in most cases. A similar analysis is also possible for divisors with a positive or less negative normal bundle. For example, in analogy with a $+1$ curve, it is straightforward to confirm that any of the exceptional gauge algebras can be tuned on a divisor with the geometry of $D = \mathbb{P}^2$ with a normal bundle of $+H$, and that a Tate form for $Sp(k)$ and $SU(2k)$ can be realized in a toric model, for $k \leq 16$, in analogy with the bound of 12 for the same tunings on the $+1$ curve [33].

Also for multiple intersecting divisors, a similar analysis can be carried out.

The difficult part of generalizing the analysis of this paper to 4D is the absence of strong low-energy constraints for 4D $\mathcal{N} = 1$ supergravity models. While in 6D, as we have shown here, the set of constraints imposed by low-energy anomaly cancellation conditions is almost precisely equivalent to the constraints imposed by Weierstrass tuning, in 4D the known low-energy constraints are much weaker, so the apparent swampland is much larger. Whether this is an indication that F-theory describes a much smaller part of the space of consistent 4D supergravity theories, or we are simply lacking insight into 4D low-energy consistency conditions, is an important open question for further research (see [97] for some initial investigations in this direction).

11.5 Outstanding questions

In this paper we have made progress towards a complete classification of allowed tunings for Weierstrass models over a given base. A desirable final goal of this program would be a complete set of local constraints (in terms only of gauge algebras, matter representations, and the self-intersection matrix of the base) such that a fibration exists that produces a Weierstrass model for a Calabi-Yau threefold and corresponding 6D supergravity model *if and only if* that fibration satisfies all of the local constraints. Here we summarize some questions that still need to be addressed to complete the classification and to match Weierstrass tunings in 6D F-theory models to low-energy supergravity theories.

- The remaining local configurations in the “tuning swampland” summarized in §11.2

should hopefully be able to be identified either as allowed by as-yet-unknown Weierstrass tunings, or as inconsistent in UV-complete quantum 6D supergravity theories.

- We have addressed in this paper local constraints associated with the tunings of gauge groups and matter over a single divisor corresponding to a curve in the base and the set of other curves intersecting that divisor. We do not know that every model that satisfies all local constraints of this type is globally consistent, though we do not know of any counterexamples. It would be desirable to either prove that local constraints are sufficient or identify conflicts that can arise nonlocally.
- We have only checked Weierstrass/Tate realizations for rational curves with local toric descriptions. It would be desirable to expand the methodology to higher-genus curves without a local toric description.
- Tuned gauge factors such as \mathfrak{g}_2 that can be broken without decreasing rank give additional contributions to $h^{2,1}(X)$ from the associated charged matter fields that are uncharged under the smaller group. This should be understood better geometrically, and may also be related to the issue of tuning $\mathfrak{su}(N)$ factors on degenerate elliptic curves; in both cases the essential issue is the addition of a $(1,1)$ multiplet with gauge bosons and hypermultiplets that precisely cancel anomaly constraints. The connection between this phenomenon and the topology of the elliptic Calabi-Yau threefold should be better understood, and will be described further elsewhere [45].
- The classification of exotic matter representations, particularly those realized by gauge groups supported on singular divisors in the base, must be completed. In particular, the method of analysis in §7 assumed that any exotic matter configuration can be realized as a tuning of a generic matter configuration — that is, that there are no non-Higgsable exotic matter configurations possible. While we believe that this is true we do not have a rigorous proof of this statement.
- It needs to be shown whether the approach used here of constructing abelian gauge factors from Higgsing of nonabelian tuned gauge factors is able to produce all abelian gauge structures of arbitrary rank; even if this is possible, a systematic understanding of how this can be implemented for higher-rank gauge groups and what singularities are possible in the nonabelian enhanced model for a consistent Higgsed abelian theory must be better understood.

A further set of questions, which fit into this general framework but which go beyond the goal of classifying all tunings over a given base surface, include the following:

- We have not addressed the question of different resolutions of the Kodaira singularities, which are not relevant for the low-energy 6D physics, but would need to be addressed in general for a complete classification of smooth elliptic Calabi-Yau threefolds, as discussed in §2.5.

- While substantial progress has been made towards the complete classification of non-toric bases that support elliptic Calabi-Yau threefolds [16], technical issues remain to be solved for a complete classification of all allowed bases for $h^{2,1}(X) < 150$.
- The complete elimination of the 6D swampland would require progress on relating apparently consistent completely Higgsed low-energy models (such as those with -3 dyonic strings but no gauge group) to F-theory constructions and/or developing new constraints on low-energy 6D supergravity theories.
- As discussed in the previous subsection, much work remains to be done to generalize this story to 4D F-theory constructions.

The partial progress presented in this paper, then, should be viewed as both a set of tools for Weierstrass constructions and as a framing of the remaining challenges and an invitation to meet them. It is becoming increasingly clear that the sets of elliptically fibered Calabi-Yau threefolds, associated Weierstrass models, and 6D supergravity theories are tightly controlled, richly structured, and closely related. Moreover, as discussed in §2.5, elliptically fibered Calabi-Yau manifolds may represent a very large fraction of the total number of Calabi-Yau varieties in any dimension, so that the analysis of elliptic Calabi-Yau spaces may give insight into the more general properties of Calabi-Yau manifolds. Following this general line of inquiry will doubtless reveal many other physical and geometric insights yet undiscovered.

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A Tabulated results

Ultimately, one of the principal utilities of our results is to enable easy calculations. In practical calculations, it is convenient to have explicit lists of all *a priori* allowed tunings. Therefore, in this appendix, we unpack the formulas for Hodge shifts in terms of self-intersection number n (and possibly a group parameter N), re-packaging them in tables that explicitly list all allowed tunings on a given self-intersection number curve or given cluster. We give a table for each isolated curve or multi-curve cluster with negative self-intersection number. These tables for isolated curves are presented first, followed by the tables concerning multi-curve clusters. For each curve or cluster, the data listed include: algebra, matter representations, Hodge shifts

from the generic fibration, and finally the global symmetry group as determined in [64]. This last piece of information is reproduced here because it provides a field theory constraint on which algebras can be tuned on intersecting divisors as discussed in section §6.2.

One final note for using these tables: instead of displaying changes in $h^{2,1}$, we display changes in H_u , the number of neutral hypermultiplets of the theory. For rank-preserving tunings (in which $h^{1,1}$ does not change), the actual Hodge numbers of the Calabi-Yau threefold do not change, and the Calabi-Yau threefolds may be related [45]. Thus, from the point of view of Hodge numbers, the Hodge numbers of the threefold for the theory with the larger group are equal to those with that reached after rank-preserving breaking. Nonetheless, it seems that the resulting physical theories are distinct, so from the standpoint of studying 6D SUGRA and SCFT, these tuned models must be included. Brackets $[\cdot]$ are placed around shifts ΔH_u that cannot be equated to $\Delta h^{2,1}$; such models do not contribute distinct Hodge number combinations.

\mathfrak{g}	matter	$(\Delta h^{1,1}, \Delta H_u)$	global symmetry algebra(s)
$\mathfrak{su}(2)$	$10 \times \mathbf{2}$	$(1, -17)$	$\mathfrak{so}(20) \dots$
$\mathfrak{su}(3)$	$12 \times \mathbf{3}$	$(2, -28)$	$\mathfrak{su}(11) \dots$
$\mathfrak{su}(N)$	$(8 + N) \times \mathbf{N} + \frac{\mathbf{N}(\mathbf{N}-1)}{2}$	$(N - 1, -\frac{15N+N^2}{2} - 1)$	$\mathfrak{su}(8 + N)$
$\mathfrak{sp}(N/2)$	$(8 + N) \times \mathbf{N}$	$(\frac{N}{2}, -8N - N^2 - 1)$	$\mathfrak{so}(16 + 2N) \dots$
$\mathfrak{so}(7)$	$2 \times \mathbf{7} + 6 \times S$	$(3, [-41])$	$\mathfrak{sp}(6) \oplus \mathfrak{sp}(2)$
$\mathfrak{so}(8)$	$3 \times (\mathbf{8}_f + \mathbf{8}_s + \mathbf{8}_s)$	$(4, -44)$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(3) \oplus \mathfrak{sp}(1)^{\oplus 3}$
$\mathfrak{so}(9)$	$4 \times \mathbf{9} + 3 \times S$	$(4, [-48])$	$\mathfrak{sp}(4)$
$\mathfrak{so}(10)$	$5 \times \mathbf{10} + 3 \times S$	$(5, -53)$	$\mathfrak{sp}(5)$
$\mathfrak{so}(11)$	$6 \times \mathbf{11} + \frac{3}{2} \times S$	$(5, [-59])$	$\mathfrak{sp}(6)$
$\mathfrak{so}(12)$	$7 \times \mathbf{12} + \frac{3}{2} \times S$	$(6, -66)$	$\mathfrak{sp}(7)$
\mathfrak{g}_2	$7 \times \mathbf{7}$	$(2, [-35])$	$\mathfrak{sp}(7)$
\mathfrak{f}_4	$4 \times \mathbf{26}$	$(4, [-52])$	-
\mathfrak{e}_6	$5 \times \mathbf{27}$	$(6, -57)$	-
\mathfrak{e}_7	$\frac{7}{2} \times \mathbf{56}$	$(7, -63)$	-

Table 15. Table of all tunings on an isolated -1 curve. The ellipses “...” indicate that $\mathfrak{su}(2)$, $\mathfrak{su}(3)$, and the $\mathfrak{sp}(N)$ series may have different global symmetry algebras other than those listed here depending on the details of their tuning; see [64]. We have simply listed global symmetry for the most generic tuning.

We should emphasize: all of the information in this section is in principle contained in the body of the paper, *e.g.* Table 4. At first glance, it may appear that the tables presented here *do* have more information, namely the information about when certain series can no longer be consistently tuned. However, it is possible to read this off from the original tables as well: given a general formula for the matter multiplicities of an algebra \mathfrak{g} in terms of self-intersection number n (and possibly a group parameter N), a group will be impossible to tune whenever one of the following occurs: the formula predicts either negative multiplicity

g	matter	$(\Delta h^{1,1}, \Delta H_u)$	global symmetry algebra(s)
$\mathfrak{su}(2)$	$4 \times \mathbf{2}$	$(1, -5)$	$\mathfrak{su}(4)$ (I_2); $\mathfrak{so}(7)_s$ (III/IV)
$\mathfrak{su}(3)$	$6 \times \mathbf{3}$	$(2, -10)$	$\mathfrak{su}(6)$...
$\mathfrak{su}(N)$	$2N \times \mathbf{N}$	$(N-1, -N^2-1)$	$\mathfrak{su}(2N)$
$\mathfrak{so}(7)$	$\mathbf{7} + 4 \times \mathbf{S}$	$(3, [-18])$	$\mathfrak{sp}(4) \oplus \mathfrak{sp}(1)$
$\mathfrak{so}(8)$	$2 \times (\mathbf{8}_f + \mathbf{8}_f + \mathbf{8}_s)$	$(4, -20)$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)^{\oplus 2}$
$\mathfrak{so}(9)$	$3 \times \mathbf{9} + 2 \times S$	$(4, [-23])$	$\mathfrak{sp}(3)$
$\mathfrak{so}(10)$	$4 \times \mathbf{10} + 2 \times S$	$(5, -27)$	$\mathfrak{sp}(4)$
$\mathfrak{so}(11)$	$5 \times \mathbf{11} + S$	$(5, [-32])$	$\mathfrak{sp}(5)$
$\mathfrak{so}(12)$	$6 \times \mathbf{12} + S$	$(6, -38)$	$\mathfrak{sp}(6)$
$\mathfrak{so}(13)$	$7 \times \mathbf{13} + \frac{1}{2} \times S$	$(6, [-46])$	$\mathfrak{sp}(7)$
\mathfrak{g}_2	$4 \times \mathbf{7}$	$(2, [-14])$	$\mathfrak{sp}(4)$
\mathfrak{f}_4	$3 \times \mathbf{26}$	$(4, [-26])$	-
\mathfrak{e}_6	$4 \times \mathbf{27}$	$(6, -30)$	-
\mathfrak{e}_7	$3 \times \mathbf{56}$	$(7, -35)$	-

Table 16. Table of all tunings on an isolated -2 curve. We have explicitly included two global symmetry algebras for $\mathfrak{su}(2)$, depending on whether it is tuned as an I_2 or III/IV singularity type. Again, an ellipsis “...” denotes that there are other symmetry algebras for $\mathfrak{su}(3)$ when it is tuned in a non-generic way.[64]

g	matter	$(\Delta h^{1,1}, \Delta H_u)$	global symmetry algebra
$\mathfrak{su}(3)$	\emptyset	$(0, 0)$	-
\mathfrak{g}_2	$\mathbf{7}$	$(0, -1)$	$\mathfrak{sp}(1)$
$\mathfrak{so}(7)$	$2 \times S$	$(1, -3)$	$\mathfrak{sp}(2)$
$\mathfrak{so}(8)$	$\mathbf{8}_f + \mathbf{8}_s + \mathbf{8}_c$	$(2, -4)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$
$\mathfrak{so}(9)$	$2 \times \mathbf{9} + S$	$(2, [-6])$	$\mathfrak{sp}(2)$
$\mathfrak{so}(10)$	$3 \times \mathbf{10} + S$	$(3, -9)$	$\mathfrak{sp}(3)$
$\mathfrak{so}(11)$	$4 \times \mathbf{11} + \frac{1}{2} \times S$	$(3, [-13])$	$\mathfrak{sp}(4)$
$\mathfrak{so}(12)$	$5 \times \mathbf{12} + \frac{1}{2} \times S$	$(4, -18)$	$\mathfrak{sp}(5)$
\mathfrak{f}_4	$2 \times \mathbf{26}$	$(2, [-8])$	-
\mathfrak{e}_6	$3 \times \mathbf{27}$	$(4, -11)$	-
\mathfrak{e}_7	$\frac{5}{2} \times \mathbf{52}$	$(5, -15)$	-

Table 17. Table of all tunings on an isolated -3 curve.

representations or fractional representations (This last requires some care, since $\frac{1}{2}$ -multiplicity representations may occur when the representation is self-conjugate, in which case this denotes a half-hypermultiplet.). This discussion also makes it clear that while the information in this appendix is already contained throughout the body of the paper, unpacking it requires some work. Hence, the motivation to collect these expressions more explicitly here.

\mathfrak{g}	matter	$(\Delta h^{1,1}, \Delta H_u)$	global symmetry algebra
$\mathfrak{so}(8)$	\emptyset	$(0, 0)$	-
$\mathfrak{so}(N > 8)$	$(N - 8) \times \mathbf{N}$	$(\lfloor (N - 8)/2, -N \frac{N-15}{2} + 28)$	$\mathfrak{sp}(N - 8)$
\mathfrak{f}_4	$\mathbf{26}$	$(0, [-2])$	-
\mathfrak{e}_6	$2 \times \mathbf{27}$	$(2, -4)$	-
\mathfrak{e}_7	$2 \times \mathbf{52}$	$(3, -7)$	-

Table 18. Table of all tunings on an isolated -4 curve.

\mathfrak{g}	matter	$(\Delta h^{1,1}, \Delta H_u)$
\mathfrak{f}_4	$\mathbf{26}$	$(0, 0)$
\mathfrak{e}_6	$\mathbf{27}$	$(2, -1)$
\mathfrak{e}_7	$\frac{3}{2} \times \mathbf{52}$	$(3, -3)$

Table 19. Table of all tunings on an isolated -5 curve. All matter has trivial global symmetry algebra.

\mathfrak{g}	matter	$(\Delta h^{1,1}, \Delta H_u)$
\mathfrak{e}_6	\emptyset	$(0, 0)$
\mathfrak{e}_7	$\mathbf{52}$	$(1, -1)$

Table 20. Table of all tunings on an isolated -6 curve. All matter has trivial global symmetry algebra.

cluster	\mathfrak{g}	$(\Delta h^{1,1}, \Delta H_u)$
22	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	$(3, [-12])$
	$\mathfrak{so}(7) \oplus \mathfrak{su}(2)$	$(4, [-15])$
222	$\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$	$(4, [-8])$
	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	$(5, [-12])$
	$\mathfrak{g}_2 \oplus \mathfrak{su}(2) \oplus \cdot$	$(3, [-11])$
2222	$\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2) \oplus \cdot$	$(4, [-8])$
22222	$\cdot \oplus \mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2) \oplus \cdot$	$(4, [-8])$
$2_1 \cdots 2_k$	(Only $\mathfrak{su}(n)$'s; see §5.2)	see (5.5)

Table 21. Table of possible tunings on -2 chains. (Chains are listed with self-intersections sign-reversed.) Because matter is very similar between these cases, we do not list it explicitly, preferring to display the shift in Hodge numbers resulting from that matter. For convenience, we summarize the relevant matter content here: $\mathbf{42}$ for $\mathfrak{su}(2)$, $\mathbf{47}$ for \mathfrak{g}_2 , $\mathbf{48_s} + \mathbf{7}$ for $\mathfrak{so}(7)$ and $\mathbf{48_s} + \mathbf{28_f}$ for $\mathfrak{so}(8)$. $\mathfrak{su}(2)$ shares a half-hypermultiplet with all groups but itself, where it shares a whole hyper; with \mathfrak{so} 's, it is the spinor representation which is shared.

Also note that the two final tables in this section, which pertain to multiple-curve clusters, are reproduced here for convenience; the identical tables also appear in the body of the paper.

cluster	\mathfrak{g}	$ (\Delta h^{1,1}, \Delta H_u)$	matter
(-3,-2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	(0,0)	$(\mathbf{7}, \frac{1}{2}\mathbf{2}) + \frac{1}{2}\mathbf{2}$
	$\mathfrak{so}(7) \oplus \mathfrak{su}(2)$	(1,-1)	$(\mathbf{8}_s, \frac{1}{2}\mathbf{2}) + \mathbf{8}_s$
(-3,-2,-2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	(0,0)	$(\mathbf{7}, \frac{1}{2}\mathbf{2}) + \frac{1}{2}\mathbf{2}$
(-2,-3,-2)	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	(0,0)	$(\frac{1}{2}\mathbf{2}, \mathbf{8}_s, \cdot) + (\cdot, \mathbf{8}_s, \frac{1}{2}\mathbf{2})$

Table 22. Table of possible tuned gauge algebras, together with matter and Hodge shifts, on the NHCs with multiple divisors.

Group	representation R	dimension	A_R	B_R	C_R
$SU(N)$	\square	N	1	1	0
	Adjoint	$N^2 - 1$	$2N$	$2N$	6
	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\frac{N(N-1)}{2}$	$N - 2$	$N - 8$	3
	$\begin{smallmatrix} \square & \square \end{smallmatrix}$	$\frac{N(N+1)}{2}$	$N + 2$	$N + 8$	3
	$\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$	$\frac{N(N-1)(N-2)}{6}$	$\frac{N^2-5N+6}{2}$	$\frac{N^2-17N+54}{2}$	$3N - 12$
	$\begin{smallmatrix} \square & \square & \square \end{smallmatrix}$	$\frac{N(N+1)(N+2)}{6}$	$\frac{N^2+5N+6}{2}$	$\frac{N^2+17N+54}{2}$	$3N + 12$
$SO(N)$	\square	N	1	1	0
	$\begin{smallmatrix} \square \\ \square \end{smallmatrix} = \text{Adjoint}$	$\frac{N(N-1)}{2}$	$N - 2$	$N - 8$	3
	$\begin{smallmatrix} \square & \square \end{smallmatrix}$	$\frac{N(N+1)}{2}$	$N + 2$	$N + 8$	3
	S	$2 \lfloor (N-1)/2 \rfloor$	$2 \lfloor (N+1)/2 \rfloor - 4$	$-2 \lfloor (N+1)/2 \rfloor - 5$	$3 \cdot 2 \lfloor (N+1)/2 \rfloor - 7$
$Sp(\frac{N}{2})$	\square	N	1	1	0
	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\frac{N(N-1)}{2} - 1$	$N - 2$	$N - 8$	3
	$\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$	$\frac{N(N-4)(N+1)}{6}$	$\frac{N^2-5N+6}{2}$	$\frac{N^2-17N+54}{2}$	$3N - 12$
	$\begin{smallmatrix} \square & \square \end{smallmatrix} = \text{Adjoint}$	$\frac{N(N+1)}{2}$	$N + 2$	$N + 8$	3

Table 23. Values of the group-theoretic coefficients A_R, B_R, C_R for some representations of $SU(N)$, $SO(N)$ and $Sp(N/2)$. Also note that we do not distinguish between S_{\pm} spin representations of $SO(N)$ for even N , as these representations have identical group theory coefficients. For $SU(2)$ and $SU(3)$, there is no quartic Casimir; $B_R = 0$ for all representations, and $C_R^{(SU(2,3))} = C_R + B_R/2$ in terms of the values given in the table.

B Tabulations of group theory coefficients

In this appendix, we present tables of the coefficients A, B, and C, which appear in anomaly cancellation conditions in 6D. All these coefficients have been calculated elsewhere, but as the existing calculations and results are somewhat scattered throughout the literature, we collect these results here for ease of reference. Many of these coefficients were originally derived in [40]; additional classical group coefficients are reproduced from [41], and normalization coefficients λ are defined as in [9].

Group	representation R	dimension	A_R	C_R
G_2	\square	7	1	$\frac{1}{4}$
	Adjoint	14	4	$\frac{5}{2}$
F_4	\square	26	1	$\frac{1}{12}$
	Adjoint	52	3	$\frac{5}{12}$
E_6	\square	27	1	$\frac{1}{12}$
	Adjoint	78	4	$\frac{1}{2}$
E_7	\square	56	1	$\frac{1}{24}$
	Adjoint	133	3	$\frac{1}{6}$
E_8	$\square = \text{Adjoint}$	248	1	$\frac{1}{100}$

Table 24. Group theoretic coefficients A_R and C_R for the exceptional groups. Note that B_R is not included as it vanishes for all exceptional groups.

Group	$\text{SU}(N)$	$\text{Sp}(N)$	$\text{SO}(N)$	G_2	F_4	E_6	E_7	E_8
λ	1	1	2	2	6	6	12	60

Table 25. Group theoretic normalization constants λ for all simple Lie groups.

C Complete HC Calculations

Here we construct a local model of each tuning possible on an NHC. We perform both anomaly calculations and calculations in local geometry to show which tunings are allowed and which cannot be realized. For each tuning, we find the matter representations using anomaly cancellation arguments, and calculate ΔH_u using anomaly cancellation as well as local geometry. The results of this section, as well as other results of this paper, are summarized in appendix refsec:results.

C.1 The Cluster (-3,-2)

An appropriate local toric model has the fan $\{v_i\}_{i=0}^4 = \{(3,1), (1,0), (0,1), (-1,2)\}$. Let us now consider the base (untuned) case and check that it corresponds to $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$ on $\{-3, -2\}$, as has already been derived as part of the NHC classification. Although we have already given several anomaly calculations, it is useful to follow the anomaly calculations in these cases, because the “A” condition allows one to determine the shared matter, and highlights an interesting feature of $\mathfrak{su}(2)$ shared matter. This is a feature of these calculations that cannot be seen in clusters besides those containing multiple divisors.

The “C” calculations straightforwardly yield $N_f = 1$ for \mathfrak{g}_2 on a (-3) -curve, as we already saw when discussing the (-3) -cluster, above. Similarly, for an $\mathfrak{su}(2)$ on a (-2) -curve, we

obtain

$$\begin{aligned}
\Sigma \cdot \Sigma &= \frac{\lambda^2}{3} \left(\sum_R C_R - C_{Adj} \right) \\
-2 &= \frac{1}{3} \left(\frac{N}{2} - 8 \right) \\
N &= 4
\end{aligned} \tag{C.1}$$

fundamentals. The “A” condition of bifundamental matter is more interesting in this case, because it dictates

$$\xi_i \cdot \xi_j = \lambda_i \lambda_j \sum_R A_R^i A_R^j x_R^{ij} \tag{C.2}$$

Recall, we are considering $\xi_i \cdot \xi_j = 1$ for neighbors and is zero otherwise. The sum on the right hand side is over all shared representations between the two gauge groups, which (consistent with the string theory description) can occur only between divisors that intersect. The A_i are all integer-valued (and must be positive), the x_{ij} denote the multiplicities of the representations, and the λ are group theory coefficients introduced earlier. The key fact in this case is that, whereas $\lambda = 1$ for $\mathfrak{su}(2)$, $\lambda = 2$ for \mathfrak{g}_2 , $\mathfrak{so}(7)$, and $\mathfrak{so}(8)$ (λ is always the same for different gauge algebras distinguished only by monodromy). Since $A = 1$ by definition for fundamental representations, the equation now reads $1 = 2x$, so it is crucial that x , the multiplicity of the shared hypermultiplet, can be $x = \frac{1}{2}$. This is the case because the fundamental representation of $\mathfrak{su}(2)$ is self-conjugate, and in six dimensions, a half-hypermultiplet can be shared. Indeed, organizing these representations as explicitly as possible, we have

$$\frac{\mathfrak{g}_2 \oplus \mathfrak{su}(2)}{(7, \frac{1}{2}\mathbf{2})} \tag{C.3}$$

and it is clear that if $\mathfrak{su}(2)$ could not share half-hypermultiplets, then the bifundamental would imply 7 (as opposed to $3\frac{1}{2}$ shared fundamentals of $\mathfrak{su}(2)$, which would be a contradiction because $\mathfrak{su}(2)$ only has 4 fundamentals total. This same feature in principle, from anomaly cancellations alone, would allow $\mathfrak{su}(2)$ to remain adjacent to $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$, with 7 and 8 dimensional fundamental representations, as well as 8 dimensional spinor representations. Returning to the main calculation for $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$, we have a total contribution to H_u of $14 + 3 - \frac{1}{2}7 \times 2 - \frac{1}{2} \times 2 = +9$. Thus, to calculate shifts from this generic case, we must subtract 9. This is the base, untuned value against which the further tunings are compared.

Similar anomaly calculations for $\mathfrak{so}(7) \oplus \mathfrak{su}(2)$ yield matter $N_s = 2$ spinors and $N_f = 0$ fundamentals, giving a contribution to H_u of $+21 + 3 - \frac{1}{2}8 \times 2 - 8 = +8$; in other words, a shift of $\delta H_u = 8 - 9 = -1$.

Likewise, for $\mathfrak{so}(8) \oplus \mathfrak{su}(2)$, we obtain $N_s = 2$, $N_f = 1$, for a contribution of $+28 + 3 - \frac{1}{2}8 \times 2 - 2 \times 8 = +7$; in other words, this would be a further shift of -1 from $\mathfrak{so}(7)$, or a

total shift of -2 from the untuned case. From anomaly cancellations alone, this configuration appears to be allowed.

Let us now match these results with those of the local geometric model. By inspecting the model, we see that (f, g) vanish to orders $(2, 3)$ on the -3 curve Σ_1 corresponding to v_1 , and to orders $(1, 2)$ on the curve C_2 corresponding to v_2 . Moreover, expanding in a local coordinate w that defines $\{w = 0\} = \Sigma_2$ and z such that $\{z = 0\} = \Sigma_1$, let us define f_i and g_i by the expansion $f = \sum_i f_i z^i$, $g = \sum_i g_i z^i$. Then $f_2 = f_{1,1}w + f_{1,2}w^2$ and $g_3 = g_{3,2}w^2 + g_{3,3}w^3$. Hence we immediately see that on Σ_2 we are in Kodaira case *III*, an $\mathfrak{su}(2)$, whereas on Σ_1 we are in case I_0^* with generic $g_3 \neq 0$; hence this curve carries \mathfrak{g}_2 . So far, this just confirms that our local model reproduces the known gauge algebras of this NHC.

Proceeding to the tunings, we implement $\mathfrak{so}(7)$ by imposing the appropriate condition

$$\begin{aligned} x^3 + f_2 x + g_3 &= (x - A)(x^2 + Ax + B) \\ x^3 + (f_{1,1}w + f_{1,2}w^2)x + (g_{3,2}w^2 + g_{3,3}w^3) &= x^3 + (B - A^2)x - AB \end{aligned} \quad (\text{C.4})$$

This immediately implies that A must be proportional to w ; with this restriction, B can be of the form $B_1 w + B_2 w^2$, so we lose only one degree of freedom. This is in accordance with the anomaly calculation.

Proceeding to $\mathfrak{so}(8)$, we find that this tuning is impossible. It requires the factorization

$$\begin{aligned} x^3 + (f_{1,1}w + f_{1,2}w^2)x + (g_{3,2}w^2 + g_{3,3}w^3) &= (x - A)(x - B)(x + (A + B)) \\ &= x^3 + (AB - (A + B)^2)x + AB(A + B) \end{aligned} \quad (\text{C.5})$$

which requires now that $B \propto w$ have no quadratic term; hence we would lose one further degree of freedom, in perfect agreement with anomaly cancellation. However, let us ask what form f and g now take. Indeed, $f \propto w^2 z^2 + z^3(w^2 + \dots)$ and $g \propto w^3 z^3 + z^4(w^2 + \dots) + \dots$, which implies that at the intersection point $\Sigma_1 \cdot \Sigma_2 = \{z = w = 0\}$, (f, g) vanish to orders $(4, 6)$. Hence the $\mathfrak{so}(8)$ tuning is not allowed. This is another example of the result discussed in the main text that global symmetries prevent such a gauge group from intersecting with an $\mathfrak{su}(2)$ on a -2 curve realized through a Kodaira type III or IV singularity.

C.2 The Cluster $(-3, -2, -2)$

One might naively expect that the previous analysis would extend without modification to this cluster. However we will see that even enhancement to $\mathfrak{so}(7)$ is impossible, *i.e.* no monodromy at all is allowed for the type I_0^* singularity that generically gives rise to \mathfrak{g}_2 . To see this, again explicitly construct the local model, which can be simply obtained from the previous model by adding the vector $v_5 = (-2, 3)$ to the fan. This modifies the monomials in such a way that the generic orders of (f, g) have orders $(2, 2)$ on Σ_2 and $(2, 3)$ on Σ_1 . Moreover, making the same expansion of f and g in powers of z as above, we have

$$\begin{aligned} f_2 &= f_{2,2}w^2 \\ g_3 &= g_{3,2}w^2 + g_{3,3}w^3 \end{aligned} \quad (\text{C.6})$$

Notice that the only change insofar as we are concerned from the cluster $(-3, -2)$ is that f_2 now has no linear term. In fact, this prevents any tuning at all. To see this, we will attempt to implement the monodromy condition for $\mathfrak{so}(7)$, the most modest enhancement. The failure of this tuning will imply the failure of all potential higher tunings.

Indeed, $\mathfrak{so}(7)$ requires $f_2 = B - A^2$ and $g_3 = -AB$, whence B must be linear and therefore g_2 must be purely cubic in w . But this implies that the total f and g have the lowest order terms $f \propto w^2 z^2 + \dots$ and $g \propto w^3 z^3 + w^2 z^4 + \dots$ where in each case ellipses indicate higher order terms. In other words, we again have a $(4, 6)$ singularity at the intersection $\Sigma_1 \cdot \Sigma_2 = \{z = w = 0\}$.

To summarize, no tunings are allowed on this cluster.

C.3 The Cluster $(-2, -3, -2)$

Our local model will be a portion of \mathbb{F}_3 blown up four times, with the overall divisor structure $(+1, -2, -1, -2, -3, -2, -1, -2)$, which is to be cyclically identified, as always. By omitting all but 5 rays in the fan (the middle three of which correspond to the sequence $(-2, -3, -2)$, we obtain a local model of this geometry.

The anomaly calculation proceeds as above, for the cluster $(-3, -2)$. The only difference is that now the original (untuned) gauge algebra on the -3 curve is $\mathfrak{su}(7)$, which must as above have matter $2 \times \mathbf{8}_s$. Therefore *all* matter is shared, in the form

$$\frac{\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)}{\begin{pmatrix} \frac{1}{2}\mathbf{2} & \mathbf{8}_s \\ \mathbf{8}_s & \frac{1}{2}\mathbf{2} \end{pmatrix}} \quad (\text{C.7})$$

This yields a contribution to H_u of $+21 + 3 + 3 - 2 \times 8 = +11$. This configuration is enhanced to $\mathfrak{su}(2) \oplus \mathfrak{so}(8) \oplus \mathfrak{su}(2)$, which is identical in matter except that the $\mathfrak{so}(8)$ now carries one additional multiplet in the fundamental $\mathbf{8}_f$, for a contribution to H_u of $+28 + 3 + 3 - 3 \times 8 = +10$; in other words, $\Delta H_u = -1$ upon performing this tuning. We will find that these tunings are not possible in the following monomial analysis. Although consistent with anomaly cancellation, these tunings suffer from a field theory inconsistency identified in [60].

The monomial calculations first confirm the untuned gauge / matter content: consulting the figures, it is clear that there are no monomials for f_0 or for $g_{i \leq 1}$ on the divisor $v_1 = (1, 0)$. Similarly for v_3 . These two divisors, adjacent to the (-3) -curve (v_4) are the (-2) -curves on which $\mathfrak{su}(2)$'s are forced. Similarly, on the middle curve v_4 , one directly sees that the degrees of vanishing on v_4 are $(f, g) = (2, 4)$. This falls into the I_0^* case. In order to distinguish monodromies, we first read off the available monomials for f_2 ($\{w^1\}$) and for g_3 ($\{\}$). The polynomial to investigate then takes the form

$$\begin{aligned} x^3 + f_{2,1}wx + 0 &= x^3 + (B - A^2) + AB \\ \text{or} \quad &= x^2 + (AB - (A + B)^2)x + AB(A + B) \end{aligned} \quad (\text{C.8})$$

corresponding to $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$ respectively. In order for the first equation to hold, the constant term requires that either A or B be equal to zero. Investigating the x^1 term, we

must choose $A = 0$, while B can be proportional to w^1 . Hence we begin with one degree of freedom (in f_2), and end with one. This implies that the $\mathfrak{so}(7)$ is indeed the minimal gauge group. In the second case of tuning an $\mathfrak{so}(8)$, it is clear that this is only possible when $A = B = 0$, resulting in the loss of one degree of freedom. This is in accordance with the anomaly calculation. However, we must pause to examine the reality check allowed by the local toric model. This factorization can only be satisfied with $(f, g) = (3, 4)$, hence at the point of intersection of -3 with either -2 , we have a total vanishing of order $(f, g) = (4, 6)$, so this tuning cannot be achieved. In other words, the Non-Higgsable Cluster $(-2, -3, -2)$ is completely rigid: it admits no tunings.

C.4 The Cluster (-4)

Our model is \mathbb{F}_4 (with $+4$ curve removed), and the analysis proceeds along nearly identical lines to that of \mathbb{F}_3 . For instance, the conditions defining the monomials for f and g are identical save for the one modification: the slope of the line bounding the top of the triangle is now $-\frac{1}{4}$. (It intercepts the $b = 0$ axis at $n = 4, 6$ still for f and g , respectively.)

From the anomaly point of view, the initial (forced) gauge algebra is $\mathfrak{so}(8)$; one way to see this is to note that no group of lesser rank can satisfy all anomaly cancellation conditions on a curve of self-intersection ≤ -3 . (For example a $\mathfrak{su}(N \geq 4)$ has adjoint $C_{Adj} = 6$, which means that on a $-n$ curve, $-3n = \sum_R N_R C_R - 6$. Since $C_R \geq 0$ for all representations R , it is clearly impossible to satisfy this equation on a $-n$ curve for $n \geq 3$.) Investigating the anomaly conditions for $\mathfrak{so}(8)$ reveals that they are satisfied with no matter, leading to a contribution to H_u of $+28$ vectors (in its adjoint). Enhancement to \mathfrak{f}_4 is accompanied by the appearance of 1 fundamental hypermultiplet, for a contribution of $+52 - 26 = +26$ to H_u , or a change of $\Delta H_u = -2$. Finally, enhancement to \mathfrak{e}_6 is accompanied by 2 fundamentals, for a contribution to H_u of $+78 - 2 \times 27 = +24$, *i.e.* $\Delta H_u = -4$ from the generic fibration.

In the monomial counting picture, we have the following explanation: the untuned version is $\mathfrak{so}(8)$ because the slope of the triangle's upper boundary ($-\frac{1}{4}$) implies that from $(6, 0)$ the boundary rises to a maximum of height at $(-6, 3)$; this is the unique monomial in g_3 , and its first component is even, which implies that $g_3 = w^0$ is a perfect square. It is clear that to increase the order of f and g from $(2, 3)$ to $(3, 4)$, only one monomial from each of $-4K$ and $-6K$ need be removed; we lose 2 degrees of freedom. Enhancing to \mathfrak{e}_6 requires imposing the monodromy condition that g_4 be a perfect square. The available monomials are $\{w^0, w^1, w^2, w^3, w^4\}$, so we may impose the condition that this be a perfect square by setting it equal to the square of a generic quadratic. This restricts to a three dimensional subspace of the original 5 parameter space; in other words, we lose 2 more degrees of freedom beyond the \mathfrak{f}_4 tuning. To tune to \mathfrak{e}_7 requires that we enhance the order of (f, g) from $(2, 3)$ to $(3, 5)$ in other words eliminating the 1 monomial of f_2 as well as all $1 + 5$ monomials of g_3 and g_4 ; so we shift by -7 in H_u from the untuned $\mathfrak{so}(8)$, or by -3 subsequent to a tuning to \mathfrak{e}_6 . This is all in accordance with the anomaly results.

C.5 The Cluster (-5)

We begin with the base (untuned) case: \mathfrak{f}_4 , which can be enhanced exactly twice, to \mathfrak{e}_6 and further to \mathfrak{e}_7 . Anomaly calculations yield no matter for the \mathfrak{f}_4 (so it contributes the dimension of its adjoint, $+52$, to H_u), whereas for \mathfrak{e}_6 we find 1 fundamental hypermultiplet, yielding a contribution of $+78 - 27 = +51$ to H_u ; *i.e.* $\Delta H_u = 51 - 52 = -1$. A final enhancement to \mathfrak{e}_7 reveals $1\frac{1}{2}$ hypermultiplets (the fundamental also enjoys the self-conjugate property as for $\mathfrak{su}(2)$'s), which yields a contribution of $+133 - \frac{3}{2}56 = +49$ to H_u , *i.e.* $\Delta H_u = -3$ from the generic fibration.

A monomial analysis confirms this. Examining the local model, we find $f_{i \leq 2}$ and $g_{i \leq 3}$ have no monomials, hence f and g are forced to vanish to degree at least $(3, 4)$ on Σ . We also see that g_4 is the span of $\{w^0, w^1, w^2\}$, so that generically there is no factorization. To tune to \mathfrak{e}_6 , we need only set this quadratic to be the square of a general linear function, thereby losing one degree of freedom. This is in accordance with anomaly results. In order to enhance to \mathfrak{e}_7 , we need only increase the order of g from 4 to 5, *i.e.* to eliminate all 3 monomials in g_4 . This represents a shift of -3 from the original (untuned) \mathfrak{f}_4 , or a shift of -2 subsequent to tuning an \mathfrak{e}_6 , also in agreement with our anomaly calculations.

C.6 The Cluster (-6)

The local model is \mathbb{F}_6 (with the $+6$ curve removed). The initial gauge algebra is \mathfrak{e}_6 , which has no matter, and hence contributes $V = 78$ adjoint vectors to the count of H_u ; this can be confirmed by investigating the “C” condition on a -6 curve. The analagous calculation for \mathfrak{e}_7 reveals 1 fundamental hypermultiplet, which leads to a contribution of $133 - 56 = 77$ to H_u , which leads to a shift $\Delta H_u = -1$.

A monomial analysis confirms these results: the upper boundary of the triangle of monomials now has slope $-\frac{1}{6}$, which implies there is only one monomial in g_4 . This must be removed in order to obtain a degree of vanishing of $(f, g) = (3, 5)$ so that the resulting algebra will be \mathfrak{e}_7 ; hence we indeed lose just one degree of freedom.

C.7 The Clusters (-7), (-8), and (-12)

The algebras of these clusters cannot be enhanced. At this point, it bears mentioning that we have never enhanced a cluster to \mathfrak{e}_8 . Indeed, we are interested in tunings which do not change the base geometry, *i.e.* require no blowups of the base alone. However, an \mathfrak{e}_8 on any curve Σ other than a -12 curve will necessitate blowups in the base. The reason is straightforward: to tune \mathfrak{e}_8 , f and g must be of order 4 and 5. Yet f and g restricted to Σ are polynomials, and will generically have isolated zeroes. Such points will lead to $(4, 6)$ (non-minimal) singularities, which require blowups of the base. In fact, the number of blowups required on a curve Σ with a tuned \mathfrak{e}_8 is always equal to that required to bring the self-intersection of Σ to -12 . (It is not difficult to confirm this. In fact, it is zeroes of g_5 that lead to these singularities. Using equation 2.11, we see that $\deg(g_5) = 12 + n$, where n is the self-intersection of the curve on which \mathfrak{e}_8 appears. Hence g_5 has the correct number of zeroes to bring the self-intersection to

–12 after blowing up.) Without loss of generality then, we can simply restrict to tunings of \mathfrak{e}_8 only on existing –12 curves.

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