June 17, 2019

MODULI OF POLARIZED CALABI-YAU PAIRS

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Compactifying the moduli space of Calabi-Yau varieties is a challenging problem. For a family of K3 surfaces over a punctured disc, Kulikov [Kul77] discovered that, after a base change, there are degenerations with trivial canonical class, but these are usually reducible and there are infinitely many non-isomorphic ones.

The same general framework holds in higher dimensions as well. There are degenerations with trivial canonical class and semi-log-canonical singularities, but usually infinitely many non-isomorphic ones. It does not seem possible to choose one in a sensible and functorial way, unless one imposes extra structures. In increasingly general forms these claims were proved in [KSB88, Ale96, BCHM10, HX13, NX16, KNX18].

A general approach to obtain unique degenerations was first explored by Alexeev [Ale02] for Abelian varieties and Hacking [Hac04] for plane curves. Instead of Calabi-Yau varieties one needs to work with pairs $(X, \epsilon H)$, where X is a Calabi-Yau variety, H is an ample divisor on X and $0 < \epsilon \ll 1$. Note that H is a divisor, not a linear equivalence class or a cohomology class. However, in many cases there is a distinguished choice of the divisor H, and then this approach is especially natural and useful. This happens in the papers [Ale02, Hac04, DeV19, AET19].

For maximal generality, we consider *polarized Calabi-Yau pairs.* These consist of a *Calabi-Yau pair* (that is, a semi-log-canonical pair (X, Δ), as in [Kol13b, Chap.5], where $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to 0), plus an ample $\mathbb{Q}$-Cartier divisor H on X such that $(X, \Delta + \epsilon H)$ is also semi-log-canonical for $0 < \epsilon \ll 1$.

Polarized Calabi-Yau pairs have a natural moduli space, denoted by **PCY**, see Paragraph 4. By construction **PCY** is locally of finite type, but it has infinitely many connected components since we did not even fix the dimension of X. Thus the best one could hope for in general is the following.

Conjecture 1. *The connected components of* **PCY** *are projective.*

Note that this somewhat goes against the conjectures of Reid [Rei87], but the two are not inconsistent since the polarized deformations considered here are not the same as the non-polarized deformations studied in [Rei87].

The aim of this note is to prove a weaker statement, which is however usually sufficient in all concrete situations. This theorem seems to have been known to several people, but it was listed as an open question in some recent preprints, for example in [DeV19, AET19]. Thus it may be worthwhile to write down the precise statement of the general result and its proof. (In [DeV19, AET19] the emphasis is on describing particular irreducible components of **PCY** in concrete terms, so none of the main results of these papers are effected by our theorem.)

Theorem 2. *The irreducible components of* **PCY** *are projective.*

First we need to fix the definitions of the relevant moduli problems. From now on we work over a field k of characteristic 0.

3 (Moduli of stable pairs)**.** (See [Kol13a, Kol17] for details.)

A *pair* (X, Δ) consist of a reduced, pure dimensional variety X and an effective $\mathbb{Q}$-divisor Δ, none of whose irreducible components is contained in $\operatorname{Sing} X$. For moduli purposes it is best to write $\Delta = cD$, where D is a $\mathbb{Z}$-divisor.

A pair (X, Δ) is called *locally stable* if it is semi-log-canonical, see [Kol13b, Chap.5]. (X, Δ) is *stable* if, in addition, X is projective and $K_X + \Delta$ is an ample $\mathbb{Q}$-divisor.

The general definition of *locally stable* and *stable* morphisms is somewhat complicated. However, if S is normal, then the following works, see [Kol17, Chap.3].

A morphism $f : (X, \Delta) \to S$ is *locally stable* if $f : X \to S$ is flat, $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and all fibers are locally stable. (Implicitly, this includes the condition that the fibers should make sense, that is, the restriction of Δ to any fiber should make sense. The later holds iff Δ is $\mathbb{Q}$-Cartier at the generic points of $X_s \cap \operatorname{Supp} \Delta$ for every $s \in S$. There are several technical issues with this when S is not normal, but these become crucial only when we pass from one irreducible component of the moduli space to another. Thus these are not relevant for our current purposes. See [Kol17, Chap.4] for a discussion.)

A morphism is *stable* if, in addition, f is projective and $K_{X/S} + \Delta$ is f-ample.

The main theorem is that, at least in characteristic 0, there is a coarse moduli space **SP**, which is separated and satisfies the valuative criterion of properness.

Once the existence of **SP** is shown, the boundedness results of [Ale94, HMX18] imply that the connected components of **SP** are proper, and then [Kol90, Fuj18, KP17] imply that they are projective.

4 (Moduli of polarized Calabi-Yau pairs)**.** Working in the most general setting, a *Calabi-Yau pair* is a proper, semi-log-canonical pair (X, Δ) where $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to 0. We write $\Delta = cD$, where D is a $\mathbb{Z}$-divisor and consider the constant c as fixed in our moduli problem.

For example, following [Hac04], when we work with the moduli of hypersurfaces D of degree $d \geq n+1$ in $\mathbb{P}^n$, then we think of the objects as Calabi-Yau pairs

$$\left(X, \Delta = \tfrac{n+1}{d} \cdot D\right),$$

where D is allowed to be reducible and can even have components with multiplicity ≥ 2 for $d \geq 2(n+1)$.

A *polarized Calabi-Yau pair* consists of a Calabi-Yau pair (X, Δ) plus an ample $\mathbb{Q}$-Cartier divisor H such that $(X, \Delta + \epsilon H)$ is semi-log-canonical for $0 < \epsilon \ll 1$. The latter holds iff H does not contain any of the semi-log-canonical centers of (X, Δ), see [Kol13b, 2.5 and 2.13].

A *stable family* of polarized Calabi-Yau pairs over a normal base scheme S consists of a flat, proper morphism $f : X \to S$, a $\mathbb{Q}$-divisor Δ on X and a $\mathbb{Q}$-Cartier divisor H such that $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and all fibers (X_s, Δ_s, H_s) are polarized Calabi-Yau pairs.

Let us now fix a rational $0 < \epsilon < 1$ and consider those polarized Calabi-Yau pairs for which $(X, \Delta + \epsilon H)$ is semi-log-canonical. Then $(X, \Delta + \epsilon H)$ is a stable pair. The corresponding objects form an open subset

$$\mathbf{PCY}_\epsilon \subset \mathbf{SP},$$

that gives the moduli space of those polarized Calabi-Yau pairs for which $(X, \Delta + \epsilon H)$ is semi-log-canonical.

If we pick a smaller $0 < \epsilon_2 < \epsilon_1$ then the sets $\mathbf{PCY}_{\epsilon_1}$ and $\mathbf{PCY}_{\epsilon_2}$ are actually disjoint (since we decreased the self-intersection of $K_X + \Delta + \epsilon H$), but sending $(X, \Delta, \epsilon_1 H)$ to $(X, \Delta, \epsilon_2 H)$ defines an open embedding

$$j(\epsilon_1, \epsilon_2) : \mathbf{PCY}_{\epsilon_1} \hookrightarrow \mathbf{PCY}_{\epsilon_2}.$$

As $\epsilon \to 0$, the directed union of these embeddings gives the *moduli space of polarized Calabi-Yau pairs.* We denote it by $\mathbf{PCY}$.

5 (Difficulties of the traditional approach to Theorem 2)**.** Assume for simplicity that $\Delta = 0$.

Working with one Calabi-Yau variety X, we take $(X, \epsilon H)$ with ϵ small enough, its precise value is not important. However, the value of ϵ becomes crucial in families.

Consider a family $f : (X, \epsilon H) \to S$ in $\mathbf{SP}$ whose generic fiber is in $\mathbf{PCY}_\epsilon$. That is, $f : X \to S$ is a flat, projective morphism, $K_{X/S} + \epsilon H$ is f-ample and $K_{X/S}$ is trivial on the generic fiber. Three problems can happen if we want to change ϵ.

- If H is not $\mathbb{Q}$-Cartier then any change in ϵ results in a family that is not allowed in our moduli theory.
- Decreasing ϵ may result in a family $f : (X, \eta H) \to S$ for which $K_{X/S} + \eta H$ is not f-ample.

While both are known to happen for some values of ϵ, standard conjectures of the theory of minimal models suggest that if we start with any family $f : (X, \epsilon H) \to S$ and gradually decrease the value of ϵ, then, after finitely many contractions and flips we should get a new family $f^m : (X^m, \epsilon_0 H^m) \to S$ where neither of the above problems occur for any further decrease of ϵ_0.

This should give a very satisfactory answer for any given family, but there is one more problem.

- The value of ϵ_0 may need to get arbitrarily small, depending on the family we start with, even for families with the same generic fiber.

The latter is usually referred to as a boundedness question of the corresponding moduli problem. In our case the general boundedness results of [Ale94, HMX18] do not apply since the underlying varieties are Calabi-Yau and the value of ϵ is not fixed.

We solve the first 2 problems by first running a carefully chosen auxiliary MMP as in [HX13]. Then we note that any irreducible component of $\mathbf{PCY}$ is covered by a single universal family, so we evade the third problem as well.

6 (Proof of Theorem 2)**.** We prove that the irreducible components of $\mathbf{PCY}$ are proper. Then the general results of [Kol90, Fuj18, KP17] imply that they are projective.

Let $\mathbf{M}$ be an irreducible component of $\mathbf{PCY}$ with generic point g_M. Then there is a finite extension of $K \supset k(g_M)$ such that we have a polarized Calabi-Yau pair (X_K, Δ_K, H_K) over K that corresponds to g_M. We prove in Lemma 7 that for $0 < \epsilon \ll 1$ there is a projective variety S such that $k(S)$ is a finite extension of K, and a stable family of polarized Calabi-Yau pairs

$$f_S : \bigl(X_S, \Delta_S + \epsilon H_S\bigr) \to S,$$

such that over the generic point we recover $(X_{k(S)}, \Delta_{k(S)}, H_{k(S)})$.

If this holds then consider the moduli map $S \to \mathbf{PCY}$. Its image contains g_M and it is proper since S is projective. Thus $\mathbf{M}$, which is the closure of g_M in $\mathbf{PCY}$, is proper. □

Lemma 7. *Let K/k be a function field and (X_K, Δ_K, H_K) a polarized Calabi-Yau pair over K. Then there is a projective variety S such that $k(S)/K$ is finite and a stable family of polarized Calabi-Yau pairs*

$$f_S : \big(X_S, \Delta_S + \epsilon H_S\big) \to S$$

extending $(X_K, \Delta_K + \epsilon H_K) \times_K k(S)$ for some for $0 < \epsilon \ll 1$.

Proof. Assume first that X_K is normal and geometrically irreducible. Choose a log resolution $\pi_K : (Y_K, \Delta_K^Y + H_K^Y) \to (X_K, \Delta_K + H_K)$ such that

$$K_{Y_K} + \Delta_K^Y \sim_{\mathbb{Q}} \pi_K^*\big(K_{X_K} + \Delta_K\big) \quad \text{and} \quad H_K^Y = \pi_K^* H_K.$$

We can extend it to a simultaneous log resolution

$$(Y_{S_1}, \Delta_{S_1}^Y + H_{S_1}^Y) \to (X_{S_1}, \Delta_{S_1} + H_{S_1})$$

over some affine variety S_1 such that $k(S_1) \cong K$. By [AK00, Thm.0.3 and Sec.8.2] there is a projective, generically finite, dominant morphism $\pi : S_2 \to S_1$ and a compactification $S_2 \hookrightarrow S$ such that the pull-back $\big(Y_{S_1}, \operatorname{Supp}(\Delta_{S_1}^Y + H_{S_1}^Y)\big) \times_{S_1} S_2$ extends to a locally stable morphism

$$g_S : \big(Y_S, \operatorname{Supp}(\Delta_S^Y + H_S^Y)\big) \to S,$$

where S is smooth and Y_S has only quotient (hence $\mathbb{Q}$-factorial) singularities. Note that every log canonical center of $\big(Y_S, \operatorname{Supp}(\Delta_S^Y + H_S^Y)\big)$ dominates S.

Due to the presence of the quotient singularities, we can not guarantee that $\big(Y_S, \operatorname{Supp}(\Delta_S^Y + H_S^Y)\big)$ be dlt. However, it has *qdlt singularities* (that is quotients of dlt singularities) as discussed in [dFKX17, Sec.5].

Write $\Delta_S^Y = \Theta_S^+ - \Theta_S^-$ as the difference of effective divisors without common irreducible components. By construction $(X_{k(S)}, \Delta_{k(S)})$ is a good minimal model of $(Y_{k(S)}, \Theta_{k(S)}^+)$.

Let L_S^Y be a general, sufficiently ample divisor on Y_S. Then, for $0 < \eta \ll 1$,

(1) $L_S^Y + H_S^Y$ is relatively ample,
(2) $\big(Y_S, \Theta_S^+ + \eta(L_S^Y + H_S^Y)\big)$ is qdlt and
(3) $g_S : \big(Y_S, \Theta_S^+ + \eta(L_S^Y + H_S^Y)\big) \to S$ locally stable.

By [HX13, 1.1] and [HMX18, 2.9] the relative minimal model program with scaling of $L_S^Y + H_S^Y$ for $g_S : (Y_S, \Theta_S^+) \to S$ terminates with

$$g_S^m : \big(Y_S^m, \Delta_S^m + \epsilon(L_S^m + H_S^m)\big) \to S$$

for some $0 < \epsilon \le \eta$ such that $K_{Y_S^m/S} + \Delta_S^m$ is g_S^m-semiample. (Note that $K_{Y_S/S} + \Theta_S^+$ is $\mathbb{Q}$-linearly equivalent to Θ_S^- on the generic fiber. Thus the above MMP contracts $\operatorname{Supp}\Theta_S^-$, so the images of Θ_S^+ and of Δ_S^Y agree on Y_S^m. This is why we change notation back to Δ_S^m.)

A relative minimal model of a projective, locally stable morphism might not be locally stable, but this holds if the base space is smooth by [KNX18, Cor.10]. Thus

$$g_S^m : \big(Y_S^m, \Delta_S^m + \epsilon(L_S^m + H_S^m)\big) \to S \quad \text{is locally stable.}$$

The log canonical class $K_{Y_S^m/S} + \Delta_S^m$ is semiample and of Kodaira dimension 0 on the generic fiber, hence it is relatively $\mathbb{Q}$-linearly trivial. Thus $g_S^m : (Y_S^m, \Delta_S^m) \to S$

is a locally stable family of Calabi-Yau pairs. Since Y_S is $\mathbb{Q}$-factorial, so is Y_S^m. In particular, H_S^m is $\mathbb{Q}$-Cartier and so

$$g_S^m : \big(Y_S^m, \Delta_S^m + \epsilon H_S^m\big) \to S \quad \text{is also locally stable.}$$

Since H_S^m is relatively big, by [HX13, 1.1] $(Y_S^m, \Delta_S^m + \epsilon H_S^m) \to S$ has a relative canonical model

$$f_S : \big(X_S, \Delta_S + \epsilon H_S\big) \to S.$$

As before, [KNX18, Cor.10] guarantees that f_S is stable. Since $K_{Y_S^m/S} + \Delta_S^m$ is relatively $\mathbb{Q}$-linearly trivial, the same holds for $K_{X_S/S} + \Delta_S$. Thus

$$f_S : \big(X_S, \Delta_S + \epsilon H_S\big) \to S$$

is a family of polarized Calabi-Yau pairs. This completes the proof when X_K is normal.

If X_K is not normal, let $(X_K^n, \Delta_K^n + \epsilon H_K^n)$ denote its normalization. The previous step, applied to each irreducible component, gives $f_S^n : (X_S^n, \Delta_S^n + \epsilon H_S^n) \to S$. Finally the gluing theory of [Kol16] and [Kol13b, Chap.5] applies and we get $f_S : (X_S, \Delta_S + \epsilon H_S) \to S$. See [Kol17, Sec.2.4] for details. □

Acknowledgments. We thank V. Alexeev and K. DeVleming for helpful conversations. Partial financial support was provided by the NSF under grant numbers DMS-1362960 to JK and DMS-1901849 to CX. Both authors also received support from the grant DMS-1440140 while in residence at MSRI during the Spring 2019 semester.

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