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# Collusion, efficiency, and dominant strategies



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## ABSTRACT

Green and Laffont proved that no collusion-resilient dominant-strategy mechanism, whose strategies consist of individual valuations, guarantees efficiency in multi-unit auctions. Chen and Micali bypassed this impossibility by slightly enlarging the strategy spaces, yet assuming knowledge of the maximum value a player may have for a copy of the good, and the ability of imposing high fines on the players. For unrestricted combinatorial auctions, efficiency in collusion-resilient dominant strategies has remained open, with or without the above two assumptions. We fully generalize the notion of a collusion-resilient dominant-strategy mechanism by allowing for arbitrary strategy spaces; construct one such mechanism for multi-unit auctions, without relying on the above two assumptions; and prove that no such mechanism exists for unrestricted combinatorial auctions, with or without any additional assumptions. Our results hold when the mechanism does not know who colludes with whom, and players in the same coalition can perfectly coordinate their strategies.

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## 1. Introduction

Collusion is a major problem in mechanism design, and in auctions in particular. Dominant-strategy mechanisms assure, under the most minimal of rationality assumptions, that the players will choose the desired strategies, but not in the presence of collusion. When the players act independently, in unrestricted combinatorial auctions the VCG mechanism guarantees efficiency in dominant strategies (Vickrey, 1961; Clarke, 1971; Groves, 1973). Yet, Ausubel and Milgrom (2006) show that just two collusive players can destroy the efficiency of the VCG mechanism.<sup>3,4</sup> It is therefore legitimate to ask whether efficiency in unrestricted combinatorial auctions can be guaranteed in “collusion-resilient dominant strategies”.

In a classical paper, Green and Laffont (1979) put forward one such notion, *coalition incentive compatibility*. In their collusion model,

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<sup>3</sup> They may focus on revenue, but their example also shows that efficiency is destroyed.

<sup>4</sup> Chen and Micali (2012) show that a collusive group in the VCG mechanism might not have a dominant strategy subprofile, and thus, depending on their beliefs about the other players' strategies, efficiency may be destroyed.

- (a) Coalition members act so as to maximize the sum of their individual utilities;
- (b) The members of each coalition can perfectly coordinate their actions; and
- (c) Coalitions are secret and non-overlapping.

Note that the Green–Laffont collusion model is quite unrestricted. For example, the players of a coalition can make side-payments and enter binding contracts with each other. A mechanism need not know who colludes with whom, and a player need not know anything about the players outside his coalition. Non-overlapping coalitions is a restriction, but a natural one when the focus is on dominant-strategy implementation.<sup>5</sup>

In their collusion model, Green and Laffont prove that coalition-incentive-compatible mechanisms are unable to implement many social choice functions of interest. In particular, no efficient multi-unit auction mechanism satisfies their notion. However, while their collusion model is unrestricted, their notion of dominant-strategy collusion resilience is not, and their result applies only to mechanisms in which a player can only report a single valuation.

In the Green–Laffont collusion model, [Chen and Micali \(2012\)](#) propose a more general notion of a collusion-resilient dominant-strategy mechanism. Crucially, they allow a player to report not only a single valuation, but also a set of other players: allegedly, the set of his colluders. (As we shall see, we heavily rely on their insight in our first theorem.) They prove that one such mechanism guarantees efficiency in multi-unit auctions, where  $m$  identical copies of the same good need to be allocated. Their mechanism never has negative revenue. However, two drawbacks limit its applicability. Namely, their mechanism is assumed to

- (1) know an upper bound,  $V$ , to the players' possible values for a copy of the good, and
- (2) be able to impose fines greater than  $2mV$ , even when all bids are low.

Both requirements are problematic. The first because no upper bound for the players' possible values for a copy of the good may exist, let alone be known to the mechanism. The second because, to ensure that it exceeds any *possible* value a player may have for a copy of the good,  $V$  might be astronomically high, and  $2mV$  even higher. Therefore, a player may not have sufficient funds to pay such a fine. If it is not credible that the envisaged fine is enforceable, then the whole dominant-strategy structure of their mechanism collapses.

The possibility of guaranteeing efficiency in unrestricted combinatorial auctions by means of a non-negative-revenue mechanism, whether practical or not, has remained totally open.

In this paper, we adopt the Green–Laffont collusion model, but propose a fully general notion of collusion resilience in dominant strategies. As before, we demand that each coalition  $C$  (including a singleton coalition) has a best strategy subprofile, no matter what the players outside  $C$  may do. What differentiates our approach from the prior ones is that we do not envisage any restrictions on the strategy spaces of the players. In essence, we take the fundamental insight of Chen and Micali (i.e., that enlarging the strategy spaces may yield collusion resiliency) to the hilt, and allow for *arbitrary* strategy spaces. This extension has significant implications on whether or not efficiency is achievable in collusion-resilient dominant strategies. Specifically, we prove two theorems.

**Theorem 1** shows how to modify the mechanism of Chen and Micali for multi-unit auctions, with non-increasing marginal valuations, so as to guarantee efficiency (in the Green–Laffont collusion model) without the two problematic assumptions discussed above. Although inspired by their mechanism, ours need not know any bound on the players' valuations and need not impose on any player fines larger than the values reported by that player.

Our mechanism,  $\mathcal{M}$ , presupposes the 'decriminalization' of collusion. Indeed, when all players, collusive or not, use the best strategies available to them,  $\mathcal{M}$  may deduce significant information about who colludes with whom. The criminalization of collusion, however, is not theoretically necessary, if one can design mechanisms that are collusion-resilient.

**Theorem 2** shows that no collusion-resilient dominant-strategy mechanism can guarantee efficiency in unrestricted combinatorial auctions in the Green–Laffont collusion model. Our impossibility result is of course unrelated to the possibility result of [Chen and Micali \(2012\)](#), and also requires a proof technique totally different from that of Green and Laffont. Indeed, our impossibility proof applies to all possible mechanisms, no matter what their strategy space might be. Moreover, it holds even for auctions with only two goods and three players.

Let us also mention that weaker notions of collusion resilience have been previously considered in the literature. As put forward by [Schummer \(2000\)](#), *bribe proofness* requires that no two players can benefit by colluding together in a particular manner. The notion of *c-truthfulness*, due to [Goldberg and Hartline \(2005\)](#), assumes that each coalition has at most  $c$  members. Other notions of collusion resiliency envisage that the players are incapable of making side payments; see [Maskin \(1979\)](#), [Moulin and Peleg \(1982\)](#), [Suh \(1996\)](#), [Moulin \(1999\)](#), [Barbera and Jackson \(1995\)](#), [Jain and Vazirani \(2001\)](#), [Moulin and Shenker \(2001\)](#), [Pal and Tardos \(2003\)](#). [Laffont and Martimort \(2000\)](#) and [Che and Kim \(2006\)](#) also study collusion resilience using equilibrium-based solution concepts. *Collusion leveraging*, as put forward by [Chen et al. \(2010\)](#), aims at leveraging the players' knowledge about the payoff types of their opponents.

<sup>5</sup> Indeed, a player belonging to two distinct coalitions may find it impossible to simultaneously maximize the utility of both of them. Should this case occur, his preferences would be unspecified.

## 2. Auctions

**Auction contexts.** In an auction, a player's type is also called a *valuation*.

In a *multi-good auction*, a valuation is a function mapping the empty subset to 0, and every other subset of the goods to a non-negative real number.

The set of players is  $N = \{1, 2, \dots, n\}$ , the number of goods is  $m$ , the set of all possible valuations of a player  $i$  is  $\Theta_i$ , and the true valuation of player  $i$  is  $\theta_i^*$ . An outcome  $\omega$  is a pair  $(A, P)$ , where  $A = (A_0, \dots, A_n)$  is a vector of subsets of the goods and  $P$  is a profile of numbers. Vector  $A$  is referred to as the allocation of  $\omega$  and must be such that  $A_i \cap A_j = \emptyset$  whenever  $i \neq j$ . Component  $A_0$  represents the set of unallocated goods, and, for  $i > 0$ ,  $A_i$  represents the subset of the goods allocated to player  $i$ . Each  $P_i$  represents the price paid by player  $i$ . For each player  $i$ ,  $i$ 's utility function  $u_i$  maps a valuation  $\theta_i$  and an outcome  $\omega = (A, P)$  to  $u_i(\theta_i, \omega) \triangleq \theta_i(A_i) - P_i$ . For brevity, when the true valuation of a player  $i$  is clear, we may write  $u_i(\omega)$  instead of  $u_i(\theta_i^*, \omega)$ . An allocation  $A$  is *efficient* if  $\sum_i \theta_i^*(A_i) \geq \sum_i \theta_i^*(A'_i)$  for all allocations  $A'$ .

In a *multi-unit auction*, all  $m$  items for sale are identical copies of the same good. As traditionally done, starting with Vickrey (1961), we assume that valuations have non-increasing marginals.<sup>6</sup> We will represent a valuation of a player  $i$  by a vector of non-negative real numbers,  $t_i = t_i^{(1)}, \dots, t_i^{(m)}$ , such that  $t_i^{(1)} \geq \dots \geq t_i^{(m)} \geq 0$ . That is,  $t_i^{(j)}$  represents  $i$ 's marginal value for a  $j$ -th copy of the good. An allocation  $A$  is a sequence of  $n + 1$  non-negative values whose sum is  $m$ ,  $A = A_0, A_1, \dots, A_n$ , where  $A_0$  is the number of unallocated copies and, for  $i > 0$ ,  $A_i$  is the number of copies allocated to player  $i$ . An outcome  $\omega$  is a pair  $(A, P)$ , where  $A$  is an allocation and  $P$  a profile of prices. The utility of a player  $i$  with valuation  $t_i$  for an outcome  $\omega = (A, P)$  is  $\sum_{j=1}^{A_i} t_i^{(j)} - P_i$ .<sup>7</sup> All other notions and notations for multi-good auctions (such as efficiency and  $u_i(\omega)$ ) automatically extend to multi-unit ones.

**Auction mechanisms.** As for any other mechanism, an auction mechanism  $M$  specifies:

- For each player  $i$ , the set  $S_i$  of pure strategies available to  $i$ , and
- a function (traditionally also denoted by  $M$ ) mapping each strategy profile in  $S = S_1 \times \dots \times S_n$  to an outcome  $(A, P)$ .

For  $s \in S$ , we denote by  $M(s)$  the outcome generated by  $M$  and by  $u_i(M(s))$  the corresponding utility of player  $i$ . If  $M$  is probabilistic,  $M(s)$  is a distribution over outcomes, and  $u_i(M(s))$  is the corresponding expected utility of  $i$ . If the underlying mechanism  $M$  is clear, we may write  $u_i(s)$  instead of  $u_i(M(s))$ .

**Additional notation.** Let  $i$  be a player and  $A$  a subset of players. Then,

- For every profile  $x$ ,  $x_A$  is the subprofile obtained by restricting  $x$  to  $A$ .
- $S_A$  and  $\Theta_A$  respectively are the Cartesian products  $\prod_{i \in A} S_i$  and  $\prod_{i \in A} \Theta_i$ .
- $-A$  is the set  $N \setminus A$  and  $-i$  is the set  $- \{i\}$ .

## 3. Collusive rationality

A partition of a set  $T$  is a collection of subsets  $T_1, \dots, T_k$  of  $T$  such that  $\cup_i T_i = T$  and  $T_i \cap T_j = \emptyset$  whenever  $i \neq j$ .

**The Green–Laffont model.** Before the mechanism is executed (but possibly after the mechanism has been announced), the players are free to form an arbitrary partition of  $N$ . The formed partition, the *collusive partition*, is denoted by  $\mathbb{C}$ . A set in  $\mathbb{C}$  is called a *coalition*. A player  $i$  is *independent* if  $\{i\} \in \mathbb{C}$ .

If  $C \in \mathbb{C}$ , then  $C$  is common knowledge among its members, but a player in  $C$  need not have any additional information about  $\mathbb{C}$ . The mechanism has no information about  $\mathbb{C}$ .

Focusing on auctions, for each subset  $C$  of the players, the *collective utility function* of  $C$ ,  $u_C$ , maps a valuation subprofile  $\theta_C \in \Theta_C$  and an outcome  $\omega$  to  $\sum_{i \in C} u_i(\theta_i, \omega)$ . When the valuation subprofile  $\theta_C$  under consideration is clear, we may write  $u_C(\omega)$  instead of  $u_C(\theta_C, \omega)$ . If  $C$  is a coalition, then its players can perfectly coordinate their actions,<sup>8</sup> and act so as to maximize  $u_C$ .

**Collusive solution concepts.** A pure strategy subprofile  $s_A \in S_A$  is (*weakly*) *dominant* for a subset of players  $A$  if  $u_A(s_A, s_{-A}) = \max_{s'_A \in S_A} u_A(s'_A, s_{-A})$  for all  $s_{-A} \in S_{-A}$ .

A mechanism is *collusive dominant-strategy* if for every subset of players  $A$  and every true valuation subprofile  $\theta_A^* \in \Theta_A$ , there exists a pure strategy subprofile  $d_A \in S_A$  that is dominant for  $A$ .

<sup>6</sup> Besides being natural and traditional, this assumption is also crucial to rule out the already discussed destructive phenomena of Ausubel and Milgrom (2006), and to allow the existence of collusion-resilient multi-unit auction mechanisms like the that of Chen and Micali (2012) and ours.

<sup>7</sup> If  $A_i = 0$ , then, according to usual conventions,  $\sum_{j=1}^{A_i} t_i^{(j)}$  is 0.

<sup>8</sup> E.g., the members of a coalition can enter binding agreements and make side payments to each other.

A collusive dominant strategy mechanism is (ex-post) *coalitionally rational* if for every subset of players  $A$ , every true-valuation subprofile  $\theta_A^* \in \Theta_A$ , and every strategy subprofile  $s_{-A} \in S_{-A}$ , the collective utility of  $A$  is non-negative under  $d_A$ . That is,  $u_A(d_A, s_{-A}) \geq 0$ .

**Efficiency in collusive dominant-strategies.** A collusive dominant-strategy auction mechanism  $M$  is *efficient* if, for every collusive partition  $\mathbb{C} = C_1 \cup C_2 \cup \dots \cup C_k$  and every true valuation profile  $\theta^*$ , the outcome  $M(d_{C_1}, \dots, d_{C_n})$  is an efficient allocation for  $\theta^*$ .

#### 4. A practical collusive dominant-strategy mechanism for multi-unit auctions

Our multi-unit auction mechanism modifies the one of [Vickrey \(1961\)](#).

##### 4.1. The standard Vickrey mechanism

In the mechanism of [Vickrey \(1961\)](#) for multi-unit auctions, a strategy consists of reporting a single valuation. Given a profile  $t$  of reported valuations, the mechanism constructs a sequence of  $n \cdot m$  “value-owner” pairs  $\{(t_i^{(k)}, i) : i \in N, k = 1, \dots, m\}$ , ordered in decreasing order with respect to the first (“value”) component. We call the first  $m$  pairs in the sequence the “winning pairs” and all other ones the “losing pairs”. For each player  $i$ , we let  $m_i$  be the number of winning pairs with owner  $i$ . The mechanism allocates  $m_i$  copies of the good to  $i$ , identifies the first  $m_i$  losing pairs whose owner is not  $i$ , and charges  $i$  the sum of the values of these pairs.

The Vickrey mechanism is efficient in dominant strategies in absence of collusion, but not otherwise ([Chen and Micali, 2012](#)).

##### 4.2. The intuition behind our mechanism $\mathcal{M}$

To describe our mechanism  $\mathcal{M}$ , we find it useful to present first an ‘auxiliary’ mechanism  $\mathcal{M}'$  guaranteeing efficiency under a solution concept that is very strong, but weaker than dominant strategies (that is, dominance after a single round of removal of dominated strategies). We then show how to modify  $\mathcal{M}'$  to obtain our collusive dominant-strategy, collusively rational, and efficient mechanism  $\mathcal{M}$ .

**Mechanism  $\mathcal{M}'$ .** In  $\mathcal{M}'$ , each player reports a sequence of  $m$  “value-beneficiary” pairs,  $(v_1, b_1), \dots, (v_m, b_m)$ , where  $(v_c, b_c) \in \mathbb{R} \times N$  signifies that the bidding player is willing to pay an amount  $v_c$  in order for player  $b_c$  to receive an additional copy of the good. For instance, if  $n = 5$ ,  $m = 3$  and player 2 reports  $(\$10, 3)$ ,  $(\$8, 2)$ ,  $(\$4, 3)$ , then player 2 declares that he is willing to pay to the mechanism \$10 for a copy of the good to be awarded to player 3, \$8 for a copy to be awarded to himself, and \$4 for an additional copy to be given to player 3.

Given all these reports,  $\mathcal{M}'$  transforms each “value-beneficiary” pair into a “value-beneficiary-owner” triplet. If, as in the example above, player 2 reports  $(\$4, 3)$  as one of his pairs, then  $\mathcal{M}'$  constructs  $(\$4, 3, 2)$  as the corresponding triplet. After that,  $\mathcal{M}'$  orders all  $n \cdot m$  triplets in decreasing order by their value component, breaking ties by the beneficiary component (in any fixed order), and then by the owner component (in any fixed order). We call the first  $m$  triplets in the sequence the “winning triplets” and all others the “losing triplets”. For each player  $i$ , let  $m_i$  be the number of winning triplets with beneficiary  $i$ , and let  $m'_i$  be the number of winning triplets with owner  $i$ . Then,  $\mathcal{M}'$  allocates  $m_i$  copies of the good to  $i$ , identifies the first  $m'_i$  losing triplets whose owner is not  $i$ , and charges  $i$  the sum of the value components of these triplets.

**An intermediate solution concept.** As  $\mathcal{M}'$  is not our final mechanism, we do not define the solution concept under which  $\mathcal{M}'$  can be proved to be efficient, nor do we discuss such a proof. We simply provide some intuition in a very informal manner.

Assume for a moment that each player only bids value-beneficiary pairs whose beneficiary belongs to his own coalition. Then, it can be proven that the following strategy subprofile is dominant for each coalition  $C$ : one member of  $C$ ,  $i_C^*$ , bids the  $m$  pairs  $(v_1, b_1), \dots, (v_m, b_m)$ , corresponding to the  $m$  highest marginal valuations of players in  $C$ , and all other members of  $C$  bid pairs with value 0. We refer to such a report as a “smart strategy subprofile”. Clearly, when all coalitions choose smart strategy subprofiles, the allocation returned by  $\mathcal{M}'$  is efficient.

However, if some player  $j \notin C$  bids a pair  $(v, b)$  where  $b \in C$ , then the above strategy subprofile need not be dominant for  $C$ . For instance, suppose there are two copies of the same good, and two players, 1 and 2, both independent and both valuing \$10 a first copy and \$5 a second one. Assume that player 1 truthfully bids the sequence  $(\$10, 1)$ ,  $(\$5, 1)$ , but player 2, for whatever reason, bids  $(\$8, 1)$ ,  $(\$7, 2)$ . Under this bid profile, player 1 gets both goods (which of course is not an efficient allocation) and pays \$7, so that his utility is \$8. Player 1 would be better off, however, bidding  $(\$5, 1)$ ,  $(\$0, 1)$ , so as to get one copy for a price of zero and have a net utility of \$10. When player 1 is truthful, player 2 gets no goods, pays \$5, and his utility is  $-\$5$ . Notice that reporting  $(\$8, 1)$ ,  $(\$7, 2)$  is, for player 2, dominated by reporting  $(\$8, 2)$ ,  $(\$7, 2)$ . More generally, in  $\mathcal{M}'$ , it is always best for a player to name only beneficiaries in his own coalition.

In sum, reporting a smart strategy subprofile is certainly “smart” but not quite dominant.<sup>9</sup>

**From  $\mathcal{M}'$  to  $\mathcal{M}$ .** We modify  $\mathcal{M}'$  by having each player  $i$  bid either (1) a “representative” player  $j \in N$ , or (2) a sequence of  $m$  “value-beneficiary” pairs  $(v_1, b_1), \dots, (v_m, b_m)$ . In the first case, only the representative  $j$  has permission to declare a value-beneficiary pair with  $i$  as a beneficiary, while in the second case no other player has permission to declare a pair with  $i$  as a beneficiary. If any player declares a value-beneficiary pair with a disallowed beneficiary, that pair is discarded by the mechanism.

We must ensure, however, that  $i$  never has incentive to name a player outside of his collusive group as a representative. Namely: if any player  $k$  reports a pair with beneficiary  $i$ , and if  $k$  is not  $i$ ’s declared representative, then the mechanism not only discards the pair but also forces  $k$  to pay to  $i$  an amount of money equal to the value of the discarded pair. With this modification,  $i$  will never have incentive to name an outside player  $k$  as his representative:  $i$  would be better off discarding  $k$ ’s reported pair, receiving the money from  $k$ , and (if desired) reporting the pair himself. We formalize the implementation of this idea in Section 4.3.

#### 4.3. Our mechanism $\mathcal{M}$

**Strategies.** Our mechanism is of normal form. Every player  $i$ , simultaneously with his opponents, reports a strategy  $s_i \in N \cup (\mathbb{R}_+ \times N)^m$ . That is, for every strategy  $s_i$  of player  $i$ , either

- $s_i$  is a player,  $s_i \in N$ , or
- $s_i$  is a sequence of  $m$  pairs,  $s_i = (v_1, b_1), \dots, (v_m, b_m) \in (\mathbb{R}_+ \times N)^m$ .

A player  $i$  reporting  $s_i \in N$  is called *passive*. In this case, we refer to  $s_i$  as “ $i$ ’s representative”, denoted by  $\text{Rep}(i)$ .

A player  $i$  reporting  $s_i = (v_1, b_1), \dots, (v_m, b_m) \in (N \times \mathbb{R}_+)^m$  is called *active*. In this case we refer to each  $(v_j, b_j)$  as a value-beneficiary pair, with value  $v_j$  and beneficiary  $b_j$ .

**Choosing the outcome.** When the players report a strategy profile  $s$ , the mechanism chooses to allocate the copies of the goods and the price charged to each player by means of the following steps:

1. Initialize  $L$  to be an empty list of “value-beneficiary-owner” triplets.
2. For each active player  $i$  and value-beneficiary pair  $(v, b)$  reported by  $i$ :
  - (a) If either (i)  $b = i$  or (ii)  $b$  is passive and  $\text{Rep}(b) = i$ , then append the value-beneficiary-owner triplet  $(v, b, i)$  to  $L$ .
  - (b) Otherwise,  $i$  makes a payment of  $v$  to  $b$ .  
(Call  $R_i$  the net payment of  $i$  after all executions of step 2b.)
3. Sort the value-beneficiary-owner triplets of  $L$  in decreasing order by their value component. (If needed, break ties first by beneficiary, and then by owner, in lexicographic order.)  
(Call the first  $m$  triplets “winning” and the remaining triplets “losing”).
4. Each player  $j$  receives  $m_j$  copies of the good, where  $m_j$  is the number of winning triplets with beneficiary  $j$ .<sup>10</sup>
5. For each active player  $i$ :
  - (a) Let  $m'_i$  be the number of winning triplets with owner  $i$ .
  - (b) Identify the first  $m'_i$  losing triplets in  $L$  whose owner is not  $i$  and (in addition to any payments from step 2b) have player  $i$  pay to the mechanism an amount  $Q_i$  equal to the sum of the value components of these triplets.<sup>11</sup>

Note that, in  $\mathcal{M}$ , the final price paid by a player  $i$  is  $P_i = R_i + Q_i$ . (As usual, if  $P_i$  is positive, then  $i$  disburses money, and otherwise he receives money.)

#### 4.4. Analysis

**Theorem 1.** Mechanism  $\mathcal{M}$  is collusive dominant-strategy, coalitionally rational, and efficient.

**Proof.** Consider an arbitrary collusive group  $A$ . We begin with three claims:

**Claim 1.** Arbitrarily fix a strategy subprofile  $s_A$  for the players in  $A$  and a strategy subprofile  $s_{-A}$  for the other players. Then there exists a strategy subprofile  $\tilde{s}_A \in S_A$  such that

<sup>9</sup> Indeed, if every coalition  $C'$  eliminates all strategy subprofiles that are dominated for  $C'$ , then for every coalition  $C$ , a smart strategy subprofile is dominant for  $C$  with respect to all surviving strategies.

<sup>10</sup> If  $L$  contains fewer than  $m$  winning triplets, the remaining goods are unallocated.

<sup>11</sup> If there are fewer than  $m'_i$  such triplets, instead charge  $i$  the sum of the values of all such triplets.



- (a)  $u_A(\tilde{s}_A, s_{-A}) \geq u_A(s_A, s_{-A})$ , and
- (b) for all  $i \in A$ ,
  - if  $i$  is passive in  $\tilde{s}_i$ , then  $\tilde{s}_i \in A$ 
    - (i.e.,  $i$  chooses his representative in his own coalition)
  - if  $i$  is active in  $\tilde{s}_i$ , then the beneficiaries of all pairs in  $\tilde{s}_i$  are in  $A$ .

**Proof of Claim 1.** For all  $i \in A$ , we construct  $\tilde{s}_i$  by modifying the original strategy  $s_i$  according to the following (exhaustive) three cases.

1. Player  $i$  is active in  $s_i$  and  $(v_1, j_1), \dots, (v_m, j_m)$  are his value-beneficiary pairs. Then player  $i$  is active in  $\tilde{s}_i$ . He bids all pairs  $(v_l, j_l)$  from  $s_i$  for which  $j_l \in A$ . For the remaining pairs, if any, he bids  $(0, i)$ .
2. Player  $i$  is passive in  $s_i$  but nominates a player  $j \notin A$  as his representative. Then player  $i$  is active in  $\tilde{s}_i$  and chooses his  $m$  value-beneficiary pairs as follows:
  - If  $j$  is passive in  $s_{-A}$ , then  $\tilde{s}_i$  consists of  $m$  pairs  $(0, i)$ .
  - If  $j$  is active in  $s_{-A}$ , letting  $(v_1, i), \dots, (v_k, i)$  denote all the value-beneficiary pairs of  $s_j$  with beneficiary  $i$ , set  $\tilde{s}_i$  to consist of the  $k$  pairs  $(v_1, i), \dots, (v_k, i)$  and the  $m - k$  pairs  $(0, i)$ .
3. Player  $i$  is passive in  $s_i$  and nominates a player  $j \in A$  as his representative. Then  $\tilde{s}_i = s_i$ .

Let us first show that each player in  $A$  receives at least as many copies of the good under  $(\tilde{s}_A, s_{-A})$  as he does under  $(s_A, s_{-A})$ . Indeed, we shall argue that the triplets with beneficiary in  $A$  inserted into  $L$  under these two strategy profiles differ only in their owner components, with the exception that, under  $(\tilde{s}_A, s_{-A})$ , additional triplets may be inserted into  $L$  with value 0 and beneficiary in  $A$ . This holds because of the following observations:

- (Case 1 above) For any  $i \in A$  who is active in  $s_i$ , we have only removed his pairs  $(v, j)$  when  $j \notin A$ . Furthermore, for any pair  $(v, j)$  with  $j \in A$ , this pair “survives” into  $L$  under  $s$  only if either  $j = i$  or  $j$  is passive with  $\text{Rep}(j) = i$ . In both of these cases, the pair will also survive into  $L$  under  $(\tilde{s}_A, s_{-A})$ .
- (Case 2 above) For every  $i \in A$  who is passive with  $\text{Rep}(i) = j \notin A$  in  $s_i$ , there are up to  $m$  triplets  $(v_1, i, j), \dots, (v_k, i, j)$  which are inserted into  $L$  under  $(s_A, s_{-A})$  but not under  $(\tilde{s}_A, s_{-A})$ . The triplets  $(v_1, i, i), \dots, (v_k, i, i)$  are, however, inserted into  $L$  under  $(\tilde{s}_A, s_{-A})$  but not under  $(s_A, s_{-A})$ .
- (Case 3 above) For every  $i \in A$  who is passive with  $\text{Rep}(i) = j \in A$  in  $s_i$ , there are two simple subcases. First, if  $j$  is passive in  $s_j$ , then there will be no triplets with beneficiary  $i$  added to  $L$  under either  $(\tilde{s}_A, s_{-A})$  or  $(s_A, s_{-A})$ . If instead  $j$  is active in  $s_j$ , then the triplets  $(v_k, i, j)$  which are inserted into  $L$  under  $(\tilde{s}_A, s_{-A})$  are precisely the same as the triplets  $(v_k, i, j)$  inserted into  $L$  under  $(s_A, s_{-A})$ .

Therefore, changing from  $(s_A, s_{-A})$  to  $(\tilde{s}_A, s_{-A})$  affects only the owner component of triplets inserted into  $L$  with beneficiary in  $A$ , possibly appends additional triplets with value 0, and might remove triplets with beneficiary outside  $A$ . Since  $\mathcal{M}$  breaks ties according to the beneficiary components of triplets before the owner components, every winning triplet under  $(s_A, s_{-A})$  has a corresponding winning triplet under  $(\tilde{s}_A, s_{-A})$ .

Moreover, we claim that the net amount charged to  $A$  (i.e.,  $\sum_{i \in A} P_i$ ) under  $(\tilde{s}_A, s_{-A})$  is no more than the net amount charged to  $A$  under  $(s_A, s_{-A})$ . Indeed, if a player  $i \in A$  is passive in  $s_i$  with  $\text{Rep}(i) \in A$ , then  $R_i$  is the same and  $Q_i = 0$  in both scenarios. If  $i$  is passive in  $s_i$  with  $\text{Rep}(i) \notin A$ , then the amount he is newly charged in step 5b of  $\mathcal{M}$  is no more than the additional amount he receives in step 2b. That is, in the original scenario  $Q_i$  was 0, and in the new scenario may have become positive, yet the increase in  $Q_i$  is no more than the decrease in  $R_i$ . Thus, it is in  $i$ 's interest to “discard” triplets with owner outside of  $A$ , receive payment equal to the discarded triplets' values, and declare the appropriate pairs himself.

Finally, if  $i$  is active in  $s_i$ , then  $R_i$  either stays the same or decreases in the second scenario, while  $Q_i$  may increase or decrease. However, denoting by  $A'$  the subset of the coalition  $A$  who are active in  $s_A$ , a simple case analysis shows that  $\sum_{j \in A'} Q_j$  does not increase in  $(\tilde{s}_A, s_{-A})$ . The only crucial case to consider is when, for some  $i, j \in A'$ ,  $\ell \in A$ , and  $k \notin A$ , a triple  $(x, k, i)$  was winning in  $(s_A, s_{-A})$ , but after replacing  $(x, k, i)$  with  $(0, i, i)$  a previously losing triple  $(x', \ell, j)$  now wins in  $(\tilde{s}_A, s_{-A})$ . In this case,  $Q_j$  increases by at most  $x'$ , while  $Q_i$  decreases by at least  $x'$ .  $\square$

**Claim 2.** Arbitrarily fix a strategy subprofile  $s_{-A}$  for  $-A$ . Let  $\tilde{s}_A$  be a strategy subprofile for  $A$  satisfying property (b) of Claim 1, namely:

- For  $i \in A$ , if  $i$  is passive in  $\tilde{s}_i$ , then  $\tilde{s}_i \in A$ .
- For  $i \in A$ , if  $i$  is active in  $\tilde{s}_i$ , then the beneficiaries of all pairs in  $\tilde{s}_i$  are in  $A$ .

Then there exists a subprofile  $s'_A \in S_A$  such that (i) a single player in  $A$ ,  $i'$ , is active, (ii) all other  $j \in A$  are passive and  $s'_j = i'$  (i.e.  $\text{Rep}(j) = i$ ), and (iii)  $u_A(s'_A, s_{-A}) \geq u_A(\tilde{s}_A, s_{-A})$ .

**Proof of Claim 2.** Let  $(v_1, b_1, q_1), \dots, (v_k, b_k, q_k)$ , where  $k \leq m$ , be the value-beneficiary-owner triplets which, under  $(\tilde{s}_A, s_{-A})$ , are “winning” in  $L$  and have beneficiary  $b_i \in A$ .

Set  $s'_A$  to be the strategy subprofile where a single  $i' \in A$  bids the pairs  $(v_1, b_1), \dots, (v_k, b_k)$  along with  $m - k$  pairs  $(0, i')$ . Comparing the execution of step 2 of the mechanism under the strategy profiles  $(s'_A, s_{-A})$  and  $(s_A, s_{-A})$ , we note the following differences: (i) the new triplets  $(v_1, b_1, i'), \dots, (v_k, b_k, i')$  and  $m - k$  triplets  $(0, i', i')$  are appended to  $L$  under  $(s'_A, s_{-A})$  but not under  $(s_A, s_{-A})$ , (ii) the triplets  $(v_1, b_1, q_1), \dots, (v_k, b_k, q_k)$  are appended to  $L$  under  $(s_A, s_{-A})$  but not under  $(s'_A, s_{-A})$ , and (iii) the losing triplets with beneficiary in  $A$  which are appended to  $L$  under  $(s_A, s_{-A})$  are not appended under  $(s'_A, s_{-A})$ .

Since for every winning triplet  $(v_l, b_l, q_l)$  under  $(s_A, s_{-A})$  with  $b_l \in A$  there is a corresponding triplet  $(v_l, b_l, i')$  under  $(s'_A, s_{-A})$  differing only in the owner component, and since the ordering on  $L$  breaks ties lexicographically by beneficiary before owner, it is clear that the allocation under  $(s'_A, s_{-A})$  assigns every player in  $A$  at least as many copies of the good as under  $(\tilde{s}_A, s_{-A})$ . (We note  $i'$  might receive additional copies under  $(s'_A, s_{-A})$  if any of the  $m - k$  new triplets  $(0, i', i')$  are winning.) Furthermore, removing non-winning triplets from  $L$  does not affect the final allocation and does not increase the price  $Q_j$  charged to any player  $j$  in step 5b.

It is clear that the amount  $Q_{i'}$  charged to  $i'$  in step 5b under  $(s'_A, s_{-A})$  is no more than the net amount  $\sum_{j \in A} Q_j$  charged to all of  $A$  in step 5b under  $(\tilde{s}_A, s_{-A})$ . Indeed, by having a single player declare all of the pairs with beneficiary in  $A$ , more of the high-value triplets are skipped when computing the price in step 5b. In short, the subprofile  $s'_A$  avoids the scenario where a triplet owned by a player in  $A$  causes the price paid by a different player in  $A$  to increase. We note that any payments  $R_j$  received by players  $j \in A$  from step 2b are the same under the two strategy profiles.  $\square$

**Claim 3.** Let  $d_A$  be a strategy subprofile for  $A$  of the following form:

- A single player  $i^* \in A$  is active. He declares the  $m$  triplets  $(v_1, j_1), \dots, (v_m, j_m)$  corresponding to the  $m$  highest marginal valuations amongst players in  $A$ .<sup>12</sup>
- All other players  $j \in A$  are passive and announce  $d_j = \text{Rep}(j) = i^*$ .

Then  $d_A$  is a best response of  $A$  against every  $s_{-A} \in S_{-A}$ .

**Proof of Claim 3.** By Claim 1 and Claim 2, it suffices to show that  $u_A(d_A, s_{-A}) \geq u_A(s'_A, s_{-A})$ , where  $s'_A$  is any strategy subprofile such that a single  $i' \in A$  is active and all other  $j \in A$  declare  $\text{Rep}(j) = i'$ .

We notice that under all such strategies  $s'_A$ , the player  $i'$  owns all the triplets in  $L$  which have beneficiary in  $A$ . The only remaining relevant features of the strategy subprofile are the pairs which  $i'$  bids. From the perspective of  $i'$ , he is playing a standard Vickrey auction of a single good of limited supply, where his marginal valuations are the maximum of the marginal valuations in  $A$ . Thus, truthfully declaring the  $m$  highest marginal valuations is best for  $i'$ . A formal proof of this fact is nearly identical to the proof that the standard Vickrey auction is truthful. Therefore, if  $i^* = i'$  then we are done. For the case where  $i^* \neq i'$ , one only need to notice that all strategy subprofiles of the form described in Claim 3 result in the same outcome for  $A$ .  $\square$

Finally, notice that, for every profile of true valuations and every collusive partition  $\mathbb{C} = A_1 \cup A_2 \cup \dots \cup A_k$ , when all coalitions  $A_i$  play the dominant subprofile  $d_A$  as in Claim 3, the resulting allocation is efficient. This holds since, under this strategy profile, one representative from each  $A_i$  bids truthfully on behalf of the  $m$  highest marginal valuations of  $A_i$ , no pairs are discarded in step 2b of  $\mathcal{M}$ , and the triplets with the  $m$  highest valuations are winners.  $\square$

In the mechanism  $\mathcal{M}$ , choosing a strategy sub-profile  $s_C$ , dominant for a coalition  $C$ , is trivial if the valuations of the members of  $C$  are common knowledge among  $C$ . Note, however, that  $\mathcal{M}$  does not require such common knowledge. It only requires that the  $m$  value-beneficiary pairs announced by the representative of a coalition  $C$  are indeed ‘optimal’ for  $C$ .<sup>13</sup>

Also note that  $\mathcal{M}$  has some form of robustness. Informally speaking, when the players are not perfectly rational, but are sufficiently rational to avoid fines and to have the representative of each coalition  $C$  report  $m$  value-beneficiary pairs that are sufficiently good for  $C$ , then  $\mathcal{M}$  generates a sufficiently good social welfare.

Further note that  $\mathcal{M}$  does not depend on the way in which the members of  $C$  choose their representative, nor on the way in which they choose to “split their proceeds” (issues important in the collusion literature). This independence, of course, makes Theorem 1 only stronger. The only requirement is that the  $m$  value-beneficiary pairs announced by every coalition  $C$  (via its representative) are indeed ‘optimal for  $C$ ’.

<sup>12</sup> For example, suppose  $A = \{1, 2\}$ ,  $m = 3$ , player 1's value of obtaining a first, second, and third copy of the good are \$10, \$6 and \$2, respectively, and player 2's values of obtaining an additional copy of the good are \$9, \$7, and \$5. Then the active player would bid  $(\$10, 1)$ ,  $(\$9, 2)$ , and  $(\$7, 2)$ . In the event of a tie among the marginal valuations,  $i^*$  can break the tie lexicographically according to the beneficiary.

<sup>13</sup> In fact, this value-beneficiary information need not be known within  $C$  a priori. For instance, it may be computed via some sort of iterative process within  $C$ , where some member proposes an initial set of  $m$  pairs, which is then “locally corrected” by some other member, over and over again, until no more corrections are made.



We remark that it is only for conceptual simplicity that our mechanism  $\mathcal{M}$  has the representative of a coalition  $C$  make all payments for the copies of the good received by all members of  $C$ . (This simplification clearly leverages the ability that coalition members have, in our model, to enter all kinds of binding contracts, including contracts about money transfers.) We could modify our mechanism – and its proof – so that each member of a coalition pays directly for the copies he receives in a way similar to Vickrey mechanism. We wish to stress, however, that this modification does not imply that we should disallow a coalition's ability to contract internal money transfers. Indeed, the meaningfulness of [Theorem 1](#) (and [Theorem 2](#)) is enhanced when no restrictions are imposed on the abilities of members of a coalition, so long as they act to maximize the sum of their individual utilities.

Finally, note that mechanism  $\mathcal{M}$  is collusive dominant-strategy whether or not one allows contractual post-auction transfer of copies of the good among members of the same coalition. If one allowed such transfers, then  $\mathcal{M}$  would also have additional collusive dominant strategies. For instance, a distinguished player in a coalition could bid the coalition's top marginal valuations always naming himself as the only beneficiary, while every other member  $i$  in the coalition bids  $(0, i)$  for each copy of the good; and after the auction the distinguished player distributes the goods optimally amongst the members of his coalition. (With such transfers, even the original Vickrey mechanism would be efficient in collusive dominant strategies.)

## 5. A simple generalization of the revelation principle

In the next section we prove that no coalitionally rational and collusive dominant-strategy mechanism exists for unrestricted combinatorial auctions. Our proof applies to mechanisms with arbitrary strategy sets. (Already for our multi-unit auction mechanism  $\mathcal{M}$ , the pure strategies of a player  $i$  do not coincide with his set of possible valuations,  $\Theta_i$ .) To simplify our analysis we argue, very much in the spirit of the revelation principle, that we need to consider only mechanisms with very specific strategy sets. Namely,

**Definition 1.** A collusive dominant-strategy mechanism  $M$  is *hyper-truthful* if:

- For each player  $i$ ,  $S_i = \{(A, \theta_A) : A \subseteq N, i \in A \text{ and } \theta_A \in \Theta_A\}$  and
- For every  $A \subseteq N$ , the strategy subprofile  $t_A$  where every  $j \in A$  selects strategy  $(A, \theta_A^*) \in S_j$  is dominant for  $A$ .

In every collusive dominant-strategy mechanism, the optimal strategy subprofile for a set  $A$  of players depends only on the set  $A$  and the true type subprofile  $\theta_A^*$  of the group, and not on the strategies or collusive structure of the outside players. Therefore, hyper-truthful mechanisms suffice in view of the following fact, whose proof is nearly identical to that of the revelation principle of [Myerson \(1979\)](#).

**Fact 1.** *If there exists an efficient, coalitionally rational, collusive dominant-strategy auction mechanism  $M$ , then (for the same number of players and number of goods) there also exists an efficient, coalitionally rational, hyper-truthful mechanism  $M'$ .*

Roughly, the proof of [Fact 1](#) is as follows: Let  $M$  be efficient, coalitionally rational, and collusive dominant-strategy. We will construct  $M'$ . For each player  $i$ , define the function  $f_i$  mapping strategies of  $M'$  (which are of the form  $\{(A, \theta_A) : A \subseteq N, i \in A \text{ and } \theta_A \in \Theta_A\}$ ) to strategies for player  $i$  in  $M$  by the rule that  $f_i(A, \theta_A)$  is the  $i$ -th component of the pure strategy profile  $d_A$  which is dominant in  $M'$  for a collusive group  $A$  with their true valuations  $\theta_A$ . The outcome of mechanism  $M'$  under strategy profile  $\prod_i (A_i, \theta_{A_i})$  is defined to be the outcome of  $M$  under strategy profile  $\prod_i f_i(A_i, \theta_{A_i})$ . When all players of  $M'$  report truthfully, the outcome is identical to the outcome of  $M$  when each coalition  $A$  selects the dominant strategy subprofile  $d_A$ , and thus is efficient. Suppose for the sake of contradiction that there exists some  $A \subseteq N$  and  $\theta_A^* \in \Theta_A$  such that the strategy subprofile  $\prod_{i \in A} (A, \theta_A^*)$  in mechanism  $M'$  was not a dominant subprofile for a collusive group  $A$  with true type  $\theta_A^*$ . Then there exists a strategy subprofile  $\prod_{j \notin A} (A_j, \theta_{A_j})$  for mechanism  $M'$  such that the strategy subprofile  $\prod_{i \in A} (A, \theta_A^*)$  is not a best response. However, this means that  $\prod_{i \in A} f_i(A, \theta_A^*)$  is not a best response in  $M$  for the collusive group against the subprofile  $\prod_{j \notin A} f_j(A_j, \theta_{A_j})$ , which contradicts the assumption that  $\prod_{i \in A} f_i(A, \theta_A^*)$  is a dominant subprofile in  $M$ .

For example, let us show how our mechanism  $\mathcal{M}$  can be transformed into a hyper-truthful mechanism  $\mathcal{M}^*$ . In an execution of  $\mathcal{M}^*$ , denote by  $(A_i, \theta_{A_i})$  the strategy selected by each player  $i$ . Then  $\mathcal{M}^*$  works by simulating an execution of  $\mathcal{M}$  as follows: If  $i$  is the lexicographically first player in  $A_i$ , then he is made active and made to bid the  $m$  highest marginal valuations in  $\theta_{A_i}$ . Otherwise,  $i$  is made passive and made to declare his representative to be the lexicographically first player in  $A_i$ .

**Remark.** [Fact 1](#) is not, strictly speaking, a corollary of the revelation principle. This is so because a player's information about his own coalition is not an “original type”. Indeed, the players partition themselves arbitrarily into collusive groups after a mechanism is announced. In principle, for one mechanism the players might want to collude in pairs, while for another mechanism they might want to collude in triples, and so forth. Nevertheless, the proof of the above fact follows from applying the revelation principle “conditionally on the collusive partition”.

## 6. An impossibility result for combinatorial auctions

**Theorem 2.** *In combinatorial auctions with at least 3 players and 2 goods, no collusive dominant-strategy mechanism can be both coalitionally rational and efficient if either 1) no upper bound exists on the maximum possible valuations or 2) the mechanism never has negative revenue.*

**Proof.**<sup>14</sup> In light of Fact 1, it suffices to prove our thesis for mechanisms which guarantee efficiency in collusively dominant hyper-truthful strategies.

We proceed by contradiction. Assume the existence of an efficient, coalitionally rational, dominant-strategy hyper-truthful mechanism  $\mathcal{M}$  for combinatorial auctions with 3 players and 2 goods. We derive a contradiction by showing that, when no upper-bound exists for a player's value for a subset of the goods, then  $\mathcal{M}$  must pay an infinite amount of money to the players when they report a special valuation profile. More precisely, for every  $x > 0$ , as long as every player may value  $5x$  a subset of the goods, then  $\mathcal{M}$  must return revenue  $\leq -x$  whenever all players report that they belong to the same coalition and that they all value 0 every subset of the goods. (This would already be a contradiction for the case of bounded valuations, if the mechanism were required never to lose money.)

We obtain this contradiction via a sequence of 7 scenarios. In the first scenario, all players are independent and the true context and the bids are as follows:

Truth					Bids		
Player	Coalition	Type			Player	Declared coalition	Type subprofile
		{a}	{b}	{a, b}			
1	{1}	5x	0	5x	1	{1}	(5x, 0, 5x)
2	{2}	0	5x	5x	2	{2}	(0, 5x, 5x)
3	{3}	0	0	2x	3	{3}	(0, 0, 2x)

Scenario 1

Faced with these bids,  $\mathcal{M}$  must allocate good  $a$  to player 1, good  $b$  to player 2, and nothing to player 3, since it is efficient. Let us now argue that the prices charged in Scenario 1, which we denote  $P_1^{(1)}$ ,  $P_2^{(1)}$ , and  $P_3^{(1)}$ , are  $\leq 0$ . We note first that  $P_3^{(1)} \leq 0$  follows immediately from individual rationality, since player 3 receives no goods. Suppose now that  $P_1^{(1)} = \epsilon > 0$ . Then player 1 would have incentive to change his bid to  $\{1\}$ ,  $(\epsilon/2, 0, \epsilon/2)$ . In fact, by changing his strategy in this manner, while the bids of the other two players are unchanged, he would still be allocated the good (because the so modified bid profile could have been truthful and  $\mathcal{M}$  is efficient), but would be charged at most  $\epsilon/2$  (because the so modified bid profile could have been truthful and  $\mathcal{M}$  is individual rational). This contradicts the fact that bidding truthfully is a dominant strategy for player 1 in Scenario 1. Therefore,  $P_1^{(1)} \leq 0$ . A similar argument shows that  $P_2^{(1)} \leq 0$ .

Consider now the following scenario, borrowed from Ausubel and Milgrom (2006):

Truth					Bids		
Player	Coalition	Type			Player	Declared coalition	Type subprofile
		{a}	{b}	{a, b}			
1	{1, 2}	x/2	x/2	x/2	1	{1}	(5x, 0, 5x)
2	{1, 2}	x/2	x/2	x/2	2	{2}	(0, 5x, 5x)
3	{3}	0	0	2x	3	{3}	(0, 0, 2x)

Scenario 2

That is, players 1 and 2 are collusive but not truthful, so that the bids are identical to those of Scenario 1. Accordingly,  $\mathcal{M}$  returns in Scenario 2 the same outcome as in Scenario 1. In particular, in Scenario 2 the collective utility of coalition  $\{1, 2\}$  is at least  $x$ . Consider now the following scenario:

Truth					Bids		
Player	Coalition	Type			Player	Declared coalition	Type subprofile
		{a}	{b}	{a, b}			
1	{1, 2}	x/2	x/2	x/2	1	{1, 2}	$(\frac{x}{2}, \frac{x}{2}, \frac{x}{2})$
2	{1, 2}	x/2	x/2	x/2	2	{1, 2}	$(\frac{x}{2}, \frac{x}{2}, \frac{x}{2})$
3	{3}	0	0	2x	3	{3}	(0, 0, 2x)

Scenario 3

<sup>14</sup> If  $n > 3$  or  $m > 2$ , we can extend the proof below by making the extra players and goods “irrelevant”. That is, we can consider identical examples where only the first three players ever have (or declare) non-zero valuations and where no player ever has (or declares) any value for goods beyond the first two.

The true context of Scenario 3 is identical to that of Scenario 2, but now players 1 and 2 are also truthful. Accordingly, since  $\mathcal{M}$  is hyper-truthful, it must produce an outcome in which the collective utility of coalition  $\{1, 2\}$  is at least  $x$ . (Else, players 1 and 2 could increase their collective utility by bidding as in Scenario 2.) Furthermore, because  $\mathcal{M}$  must return an efficient allocation when the bids are truthful, it must allocate both goods to player 3. Thus, to ensure that the collective utility of  $\{1, 2\}$  is at least  $x$ ,  $\mathcal{M}$  must pay to players 1 and 2 a total of at least  $x$  (i.e.,  $P_1^{(3)} + P_2^{(3)} \leq -x$ ). We now consider Scenario 4:

Truth					Bids		
Player	Coalition	Type			Player	Declared coalition	Type subprofile
		$\{a\}$	$\{b\}$	$\{a, b\}$			
1	$\{1, 2\}$	0	0	0	1	$\{1, 2\}$	$(\frac{x}{2}, \frac{x}{2}, \frac{x}{2})$
2	$\{1, 2\}$	0	0	0	2	$\{1, 2\}$	$(\frac{x}{2}, \frac{x}{2}, \frac{x}{2})$
3	$\{3\}$	0	0	$2x$	3	$\{3\}$	$(0, 0, 2x)$

Scenario 4

Since the bids of Scenario 4 are identical to those of Scenario 3,  $\mathcal{M}$  must return the same outcome. That is, the coalition  $\{1, 2\}$  obtains no items but receives a total payment of at least  $x$ . We now consider Scenario 5:

Truth					Bids		
Player	Coalition	Type			Player	Declared coalition	Type subprofile
		$\{a\}$	$\{b\}$	$\{a, b\}$			
1	$\{1, 2\}$	0	0	0	1	$\{1, 2\}$	$(0, 0, 0)$
2	$\{1, 2\}$	0	0	0	2	$\{1, 2\}$	$(0, 0, 0)$
3	$\{3\}$	0	0	$2x$	3	$\{3\}$	$(0, 0, 2x)$

Scenario 5

Under the bids of Scenario 5, since  $\mathcal{M}$  is efficient, both goods must be allocation to player 3. Furthermore, since  $\mathcal{M}$  is hyper-truthful, the coalition  $\{1, 2\}$  must collectively receive at least as much utility as in Scenario 4, and thus must collectively receive a net payment of at least  $x$ . Finally, player 3 could only receive money from  $\mathcal{M}$  (that is,  $P_3^{(5)} \leq 0$ ), since otherwise (analogously to the argument above) he would have incentive to lower his bid and still receive the goods. We now consider Scenario 6:

Truth					Bids		
Player	Coalition	Type			Player	Declared coalition	Type subprofile
		$\{a\}$	$\{b\}$	$\{a, b\}$			
1	$\{1, 2, 3\}$	0	0	0	1	$\{1, 2\}$	$(0, 0, 0)$
2	$\{1, 2, 3\}$	0	0	0	2	$\{1, 2\}$	$(0, 0, 0)$
3	$\{1, 2, 3\}$	0	0	0	3	$\{3\}$	$(0, 0, 2x)$

Scenario 6

The bids of scenario 6 are identical to those of scenario 5. Thus, the collective utility of coalition  $\{1, 2, 3\}$  is  $-P_1^{(5)} - P_2^{(5)} - P_3^{(5)}$ , which is at least  $x$ . Finally, we consider Scenario 7:

Truth					Bids		
Player	Coalition	Type			Player	Declared coalition	Type subprofile
		$\{a\}$	$\{b\}$	$\{a, b\}$			
1	$\{1, 2, 3\}$	0	0	0	1	$\{1, 2, 3\}$	$(0, 0, 0)^3$
2	$\{1, 2, 3\}$	0	0	0	2	$\{1, 2, 3\}$	$(0, 0, 0)^3$
3	$\{1, 2, 3\}$	0	0	0	3	$\{1, 2, 3\}$	$(0, 0, 0)^3$

Scenario 7

Since  $\mathcal{M}$  is dominant-strategy hyper-truthful, the coalition  $\{1, 2, 3\}$  must obtain at least as much net utility in Scenario 7 as they do in Scenario 6. Thus, the three players together must receive a net payment of at least  $x$ .  $\square$

**Remark.** Theorem 2 holds because the mechanism must work properly no matter how high a player's value for a subset of the goods may be. However, if (1) an upper bound  $V$  to every possible value that a player may have for a subset of the goods exists, and (2) negative revenue is not a problem, then there is an hyper-truthful, collusively rational, and collusive

dominant-strategy mechanism that guarantees efficiency in unrestricted combinatorial auctions by “generating very negative revenue”.<sup>15</sup>

Note that the proof of our impossibility result for general combinatorial auctions still holds even in the setting where players within a coalition have the ability to re-distribute the goods among themselves after the auction.<sup>16</sup>

## 7. Final remarks

As collusion is a major problem in mechanism design, it is useful to develop strong notions of collusion resilience. It is also useful to construct mechanisms that, like  $\mathcal{M}$  for multi-unit auctions, guarantee such resilience in dominant strategies without additional assumptions.

The fact that no such mechanisms exist for unrestricted combinatorial auctions is a ‘fact of life’ (but such combinatorial auctions are also problematic in many other ways). This fact too, however, is useful to know.

In sum, it is important to understand which social choice functions are implementable in a collusion-resilient way (and hopefully without any special assumptions) and which are not so implementable. For example, it is important to study how much revenue can be guaranteed in a collusion-resilient way.

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<sup>15</sup> Very informally, such a mechanism  $\mathcal{M}'$  can be constructed by modifying the one of [Chen and Micali \(2012\)](#) as follows.

If the reports of all players are consistent (i.e., all declared coalitions and valuations subprofiles match), then  $\mathcal{M}'$  (1) gives a reward of  $kV$  to each player reporting to belong to a coalition of size  $k$ , and (2) “conceptually coalesces each coalition  $C$  to a single player, conceptually runs the VCG mechanism with these coalesced players so as to compute the subset of goods  $A_C$  and price  $P_C$  of each coalition  $C$ , charges  $P_C$  to – say – the lexicographically first player of  $C$ , and finally uses the declared valuation subprofile of  $C$  to allocate efficiently, and *ad personam*, the goods in  $A_C$  within  $C$ ”.

If, instead, a player  $i$  claims to be colluding with  $j$  but either (a)  $j$ ’s declared coalition differs from the coalition declared by  $i$ , or (b)  $i$  declares a different valuation function for  $j$  than  $j$  declares for himself, then  $\mathcal{M}'$  imposes to each such player  $i$  a fine of  $2n^2V$  payable to the mechanism and a huge fine payable to  $j$ , and no goods are allocated.

<sup>16</sup> If one allowed, however, players from different coalitions to transfer goods after the auction, then it would be unclear what it means for an auction to guarantee an efficient allocation.