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Dynamic Pricing for Heterogeneous Time-Sensitive Customers

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A core problem in the area of revenue management is pricing goods in the presence of strategic customers. We study this problem when customers are heterogeneous with respect to their initial valuations for the good and their time sensitivities, i.e., the customers differ in both their initial valuations and the rates at which their initial valuation decreases with a delay in the purchase. We characterize the optimal mechanism for selling durable goods in such environments and show that delayed allocation and dynamic pricing can be effective screening tools for maximizing firm profit. We also investigate the impact of production and holding costs on the optimal mechanism.

1. Introduction

Dynamic pricing is increasingly prevalent in many industries. One of the main advantages of dynamic pricing is that it helps mitigate the risk associated with demand uncertainty (see, for instance, Aviv and Pazgal 2008 and Cachon and Swinney 2011). In this paper, we show that dynamic pricing (DP) can play an important role in differentiating between customers over time, even in the absence of demand uncertainty. In many settings, especially in fashion and electronic retail, a customer's willingness to pay (or valuation) for a product is time-sensitive and decreases over time. In these situations, customers are not only different in terms of their initial willingness to pay for these products when they are first introduced to the market, but they are also different in terms of how rapidly they lose interest in these products. Thus, we may have customers who initially value the product at a high level but as time progresses, they lose interest in the product completely. We may also have customers who initially value the product at a low level, but still remain interested in the product as time progresses. That is, the willingness to pay of the lower valuation customers may diminish at a lower rate relative to that of the higher valuation customers. This phenomenon is illustrated in Figure 1.

In this paper, we show that when a firm sells to customers who have heterogeneously decreasing valuations, the firm can achieve significant benefits by incorporating dynamic

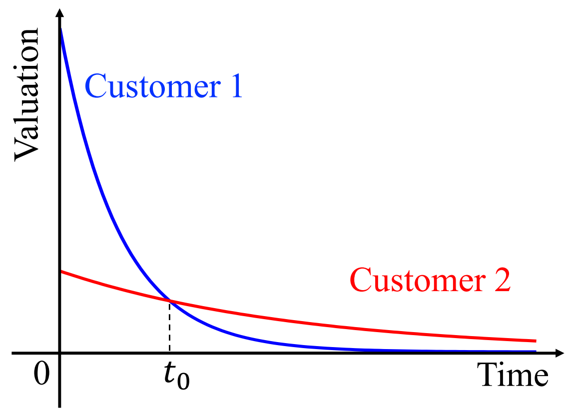


Figure 1 At time 0, customer 1 has a higher value than customer 2. But, his value decreases faster than customer 2, and beyond t_0, customer 2 has a higher value.

pricing, even in the absence of demand uncertainty. This is not the case if customers were homogeneous in their valuation decay rate. In that case, in the absence of demand uncertainty, the firm's optimal pricing strategy would be to post a fixed price, and dynamic pricing would have no benefit. When customer valuations decrease at different rates, the ranking of customers (in terms of their valuations) changes over time (as in Figure 1). This allows a firm to generate more profit by revising its initial price to target customers who currently have higher valuations even though they initially had lower valuations.

Formally, we characterize a profit-optimal selling mechanism for a firm with customers who have heterogeneous valuations that decrease in a heterogeneous fashion. We assume that the firm knows the total demand[1] and the customer valuation distribution, but does not know the precise valuation of each individual customer. The firm acquires or produces the units that it would like to sell to the customers, and it does so prior to the start of the selling period. The firm may incur production costs to procure the units and holding costs to hold units in inventory until they are sold. In our setting, the firm commits to a price trajectory, and the customers are strategic in selecting the best time to purchase so as to maximize their individual net utility. We assume that customers with higher initial valuation also have a higher rate of valuation decrease. To the best of our knowledge, this setting has not been studied in the literature.

We next describe the main characteristics of this optimal mechanism ignoring the production and holding costs, i.e., when the firm has no capacity constraints.[2] The optimal

[1] We investigate the impact of this assumption in Section 5.3.

[2] The impact of inventory constraints is studied in Appendix B.

mechanism consists of the firm posting a series of decreasing prices, which essentially divides the customers into three groups based on their initial valuations. The first group comprises all customers with initial valuations above a threshold (*high-type* customers) who purchase the product immediately. The second group consists of all customers with initial valuations below a smaller second threshold (*low-type* customers). For these customers, the posted prices are designed in a way to *extract their entire surplus*. Finally, the third group consists of customers with valuations between the two thresholds (*medium-type* customers) who do not purchase the product immediately but purchase before the low-type customers, and obtain a positive net utility, or surplus.

The low-type customers in our mechanism play an important role in contrast to what occurs in fixed pricing. In a fixed pricing policy, all customers with valuations above the price would immediately purchase, and those with valuations below the price would not purchase. However, in our optimal mechanism, the low-type customers purchase the product after some delay and the firm is able to extract their entire surplus. In the absence of production and holding costs, the firm sells the product to *all* customers in this fashion. Selling to all the customers can not only increase social welfare but can also generate significant additional profit. For instance, we show that a firm can increase its profit by approximately 23% by employing the optimal mechanism (relative to fixed pricing) when the initial valuation distribution is uniform and the valuation decay rates are proportional to initial valuations; in fact, more than three quarters of this increase is obtained by selling to the low-type customers.

We show that our main results and insights extend to more general settings. Namely, we investigate the impact of the length of the horizon, the production costs, and the holding costs on the optimal selling mechanism.[3]

We establish our main results for a setting with no restrictions on the length of the selling horizon. Then, we generalize these results to the case in which the time horizon is exogenously fixed. We show that when the length of the time horizon is small, similar to the optimal unrestricted mechanism, the high-type customers purchase the item at time zero, and medium-type customers delay their purchase but make a purchase before the

[3] We also investigate the impact of the inventory constraints on the structure of the optimal mechanism. We show that the production costs and inventory constraints affect the optimal mechanism in a similar fashion. Because of this, in the introduction, we only discuss how the production costs influence the optimal mechanism.

end of the time horizon. Finally, the low-type customers are the bargain hunters. These customers get the item at the end of the time horizon at the lowest price. We also present an approximately-optimal mechanism when the length of the time horizon is "large." This mechanism resembles the features of the optimal unrestricted mechanism. Furthermore, we show that as the length of the horizon increases, the profit of this mechanism approaches the optimal profit rapidly.

We observe that both production and holding costs motivate the firm to reduce the length of the selling period but in different ways. Interestingly, we find that the optimal selling mechanism is rather robust to the production cost. In particular, in the presence of a production cost, the optimal mechanism naturally introduces a cut-off on the customer valuations so that the firm sells only to customers with initial valuations higher than this cut-off. However, all such customers who purchase the product do so at the same time as in the baseline setting (with no production costs). Thus, production costs only change the price of purchase, and not the time of purchase.

We find that the optimal mechanism is more sensitive to holding costs. These costs motivate the firm to price in a manner so that customers are incentivized to make their purchases earlier (than in the baseline case). We find that depending on the holding cost, the optimal mechanism takes three forms. If the holding cost is larger than a threshold, then the firm determines it is too expensive to carry the product and simply posts a fixed price so that all customers who purchase the product do so immediately. If the holding cost is moderate (below the previous threshold and above another lower threshold), then the firm benefits from DP but cannot extract the entire surplus of customers with low initial valuations. If the holding cost is below the lower threshold, then the structure of the optimal mechanism is similar to that of the baseline optimal mechanism. There are three distinct groups of customers and the firm can extract the entire surplus of customers with low valuations. Overall, the value of DP decreases with increasing production and holding costs. These results are presented in the appendix.

Finally, we summarize our main *technical contribution*. In our setting, one of the hurdles in characterizing the optimal selling mechanism is the lack of consistent customer ranking based on customer types. As a result, satisfying the individual rationality and incentive compatibility constraints is challenging.[4] Note that when there is a consistent

[4] To characterize the optimal mechanism, using the revelation principle, it suffices to focus only on mechanisms in which customers have an incentive to participate, that is, the individual rationality constraints hold, and customers are

ranking of customers, individual rationality constraints are binding for the lowest customer type that the firm would like to sell the product to, and the mechanism is incentive-compatible if the allocation rule is monotone in the customer type. In contrast, in our setting, the individual rationality constraint is binding for a group of customers with low initial valuation. Furthermore, the monotonicity of the time of purchase in the initial valuation does not guarantee that the mechanism is incentive-compatible. To characterize the optimal mechanism, we first establish necessary and sufficient conditions to thus have an incentive-compatible mechanism. One of these conditions resembles the traditional *envelope condition* (Myerson 1981), which ensures that the mechanism is locally incentive-compatible. The other condition, called *interval condition*, ensures that the mechanism is globally incentive-compatible. We first relax the problem by ignoring the interval condition and characterizing a profit-optimal mechanism that satisfies the individual rationality constraints and envelope condition. Then, by establishing several additional properties of this mechanism, we show that the mechanism indeed satisfies the interval condition, and thus is optimal.

Related Work

Our work is related to the growing literature on pricing mechanisms for customers who strategically time their purchases. There is also an extensive literature on dynamic pricing with myopic customers (see for example Lazear 1984, Wang 1993, Gallego and Van Ryzin 1994, Feng and Gallego 1995, Bitran and Mondschein 1997, Federgruen and Heching 1999, and Talluri and Van Ryzin 2004). We do not provide a summary of this line of literature here, but we refer the reader to excellent surveys by Bitran and Caldentey (2003), Chan et al. (2004), and Shen and Su (2007).

Coase (1972) is one of the first papers to study pricing for strategic customers. Coase conjectured that when a firm sells a durable good to patient and strategic customers and cannot commit to a sequence of posted prices, then the prices would converge to the production cost. Later Stokey (1979), Gul et al. (1986), and Besanko and Winston (1990) found that with commitment, posting a decreasing sequence of prices is optimal. In particular, Stokey (1979) showed that when production cost declines over time, posting a decreasing sequence of prices results in higher profit for the firm. However, when the

willing to reveal their private information to the mechanism designer, that is, the incentive compatibility constraints hold (see Myerson 1981).

production cost is zero, DP is not beneficial. In contrast, we show that with heterogeneous decay rates, DP can improve profit even if the production cost is zero.

In the context of revenue management, several papers show that DP can increase firm profit when demand is uncertain (see, for example, Su 2007, Aviv and Pazgal 2008, Elmaghraby et al. 2008, Araman and Caldentey 2009, Cachon and Swinney 2011, Aviv et al. 2015, and Yu et al. 2015). Specifically, Aviv and Pazgal (2008) studied a model in which a firm sells a limited inventory of a product in two periods to an unknown number of strategic customers who are heterogeneous in their valuations and time of arrival. They showed that when the level of heterogeneity in customers' valuation increases, the benefit of customer segmentation using pricing decreases. Conversely, in this work, we show that as the level of heterogeneity in customers' decay rates increases, the firm can better differentiate customers and generate more profit.

One important factor that differentiates our work from the aforementioned research is that in our work, demand uncertainty is not a key driver of DP. That is, even in the absence of demand uncertainty, DP increases profit significantly. Furthermore, in the aforementioned papers, the firm uses the customers' fear of rationing to extract more profit from strategic customers (see also Liu and Van Ryzin 2008 and Bansal and Maglaras 2009), but in our work, customers do not face such a risk. In fact, when the production and holding costs are zero, all the customers purchase the product.

Other papers have examined intertemporal pricing with new consumers arriving in every period. Conlisk et al. (1984), Besbes and Lobel (2015), and Chen and Shi (2016) showed that when customers arrive over time, the firm's optimal strategy is to use a cyclic pricing policy. Borgs et al. 2014 studied how to set prices to extract profit while guaranteeing service availability to all paying customers arriving and departing at different times. See Board and Skrzypacz (2016) and Garrett (2011) for other papers that study pricing with heterogeneous arrivals. In these papers, the firm can gain from DP since it can differentiate customers based on their arrival times. In contrast, in our work, all of the customers are in the market when the sales starts and they strategically optimize their time of purchase. That is, we attempt to isolate and capture the impact of heterogeneity of valuation decay rates on the optimal DP policy, absent any other considerations.[5]

[5] We further discuss this assumption in Section 2.

One of the closest papers in the literature to ours is that by Chen and Farias (2015), who studied the design of dynamic selling mechanisms when valuations of the customers decay over time at possibly different rates and the customers incur monitoring costs. They proposed an *approximately profit-optimal* pricing policy that incentivizes customers to purchase the product *immediately* rather than waiting to get the product later. In contrast, we design an optimal pricing policy in a setting in which customers with higher initial valuations are more time-sensitive than those with lower initial valuations.

Chen and Farias (2016) built on the results in Chen and Farias (2015) by imposing a constraint on the structure of customer's disutility from waiting. In particular, they assume that the customer's disutility is a non-decreasing and concave function of his valuation. Under this assumption, they show that posting a fixed price is asymptotically optimal. This is in contrast with our results where we show dynamic pricing can be significantly beneficial. Put differently, our work complements Chen and Farias (2016) by showing that dynamic pricing is beneficial when the customer's disutility is not concave.

Another closely related work is by Chen and Shi (2016). This paper studied joint pricing and inventory management for a setting where customers suffer from delay disutility if they postpone their purchases and *wait* for the product to get delivered. In their model, unlike our setting, customers' valuation does not decrease with time if customers purchase the item later. However, customers incur waiting costs when the product gets delivered with a delay. They show that without production and holding costs, adopting dynamic pricing is not profitable. In contrast to our work, they find that the key driver of dynamic pricing in their setting is inventory-related costs rather than heterogeneity in delay disutility.

Our work also relates to the growing body of research on dynamic mechanism design; see Bergemann and Said (2011) for a survey. There, the firm offers a direct mechanism that allocates the products over time as a function of customers' reports of their private valuations. See Akan et al. (2009), Kakade et al. (2013), Pavan et al. (2014), Battaglini and Lamba (2012), Boleslavsky and Said (2013), Golrezaei and Nazerzadeh (2016), and Lobel and Xiao (2013) for recent results on designing optimal dynamic mechanisms. In these papers, the buyer's value changes with time, as the buyer receives new private information over time. In contrast, in our work, the buyer value changes with time because of gradual loss of interest in the product. The paper closest to ours within this literature is Akan et al. (2009), where customers are heterogeneous in their valuation distribution and in how fast they learn their true value. Akan et al. (2009) showed that when high-type customers (such as business travelers) learn their valuation slower, relative to low-type customers (such as leisure travelers), in the optimal mechanism, the firm sequentially screens customers by offering them a menu of expiring refund contracts.

Organization of Paper

The remainder of this paper is organized as follows: In Section 2, we formally introduce our model. Section 3 performs some preliminary analysis by introducing direct mechanisms in this setting and the conditions needed for these to be incentive-compatible. In Section 4, we characterize our key structural results and the optimal mechanism in the absence of production and holding costs. In Section 5, we show how our results extend when the simplifying assumptions with respect to the length of time horizon, production cost, and uncertainty in market are relaxed (we discuss the case of holding costs and inventory constraints in Appendices A and B). We conclude in Section 6.

2. Model

We consider a firm that sells multiple units of an item (product) to a mass of customers over a sales period of duration T time units. The firm produces and stores all units just prior to the start of the sale period. The cost for producing each unit is c, and the holding cost to store each unit is h per unit time. For convenience, we focus on the case in which the sales time horizon is unbounded, i.e., $T = \infty$, the production and holding costs are zero, i.e., $c = h = 0$, and the firm does not face any inventory constraints. This simpler version of the model allows us to understand and highlight the key trade-offs. We then discuss how the results generalize in Section 5.

The firm's goal is to implement a selling mechanism to maximize its profit. At time 0, the firm declares and commits to a price trajectory $\mathfrak{p}(t)$, $t \geq 0$. Given the pre-announced prices, customers decide whether and when to purchase the item. Each customer is assumed to be infinitesimal and demands a single unit of the item. The valuation of a customer at time t is $V(\theta, t)$ where $V : \mathbb{R}^2 \to \mathbb{R}$ and θ is the customer type. To capture customer heterogeneity both on their initial valuation and time sensitivity in a tractable manner, we consider a multiplicative model $V(\theta, t) = \theta e^{-g(\theta)t}$. Here, $V(\theta, 0) = \theta$ is the initial valuation, and g represents the (exponential) rate of decay of the initial valuation.

We note that in our model, we assume that customer decay rate, $g(\theta)$, is the same across all the customers with the same type, and as a result, consumer private information can be represented by a one-dimensional signal. This assumption simplifies our problem, as multidimensional mechanism design problem has shown to be very challenging to solve (Briest et al. 2010, Chawla et al. 2010, and Alaei et al. 2012). There are papers that have made some progress in designing optimal/suboptimal multidimensional mechanisms by imposing some assumptions; see, for example, Cai et al. (2012), Chen and Farias (2015, 2016), and Hart and Nisan (2017).

Our focus is on cases in which there is some structure on this decay rate g. In particular, g should be positive so that valuations decay over time. We further assume that g is strictly increasing. This implies that customers with higher initial valuations lose interest in the item much faster than customers with lower initial valuations. Furthermore, the monotonicity of g implies that for any $\theta_1 \neq \theta_2 > 0$, there exists a unique intersection point $\tau > 0$ that solves $V(\theta_1, \tau) = V(\theta_2, \tau)$.[6] That is, any two valuation curves cross each other exactly once.[7] Because of this, the customers are not ranked in a persistent manner over time; see Figure 1. The fact that a persistent ranking for customers does not exist makes the problem of designing an optimal mechanism challenging. Furthermore, as we will show later, it allows the firm to extract more profit by revisiting its prices over time.

We make two additional assumptions on g that give us analytical tractability: we assume that g is log-concave and further that $\theta g'(\theta)$ is increasing. The log-concavity assumption implies that $\frac{g'(x)}{g(x)}$ is decreasing.[8] We will further discuss the log-concavity assumption in Section 4. The condition that $\theta g'(\theta)$ is increasing ensures that the time of allocation in the optimal mechanism is decreasing in θ. This condition will be further discussed in Section 4. Roughly speaking, this condition, which is satisfied by all the convex and increasing function $g(\cdot)$, implies that function $g(\cdot)$ cannot be "too concave". Note that $\theta g'(\theta)$ is increasing if $g''(\theta) \geq -\frac{g'(\theta)}{\theta}$.

We would like to add that our valuation function satisfies the following assumptions that have been made in the seminal work by Stokey (1979): (i) $\partial_1 V(\theta, t) \leq 0$ and (ii) for any

[6] Equation $V(\theta_1, \tau) = V(\theta_2, \tau)$ gives us $\tau = \frac{\log(\theta_2) - \log(\theta_1)}{g(\theta_2) - g(\theta_1)}$.

[7] In Section 5.4, we use an example to discuss situations in which this assumption may not hold.

[8] Every positive concave function is log-concave. However, the reverse does not necessarily hold (Boyd and Vandenberghe 2004).

$t \le \frac{1}{g'(\theta)\theta}$, $\partial_2 V(\theta,t) \le 0$ and $\partial_{1,2} V(\theta,t) \le 0$. As it becomes clear later, $\frac{1}{g'(\theta)\theta}$ is the maximum delay in purchase that customers of type θ experience in the optimal mechanism. Further, $\partial_i V(\theta,t)$, $i = 1, 2$, is the derivative of the valuation function $V(\theta,t)$ w.r.t. its i-th argument, and $\partial_{1,2} V(\theta,t) = \frac{\partial^2 V(\theta,t)}{\partial\theta\partial t}$.

To summarize, we make the following assumptions on g:

ASSUMPTION 1. *For any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, function $g(\theta)$ is differentiable, positive, strictly increasing, and log-concave. Furthermore, $\theta g'(\theta)$ is increasing in θ.*

In our model, the valuation function V, which includes function g, is known to the firm and customers. However, the customer type is the customer's private information, and these types are independently drawn from a known distribution F with probability density function f, where $F : [\underline{\Theta}, \bar{\Theta}] \to [0,1]$ and $\underline{\Theta} \ge 0$. The negative inverse hazard rate associated with distribution F is denoted by $\alpha : [\underline{\Theta}, \bar{\Theta}] \to \mathbb{R}$, and is defined as $\alpha(x) = -\frac{1-F(x)}{f(x)}$. Throughout the paper, we make the following assumption, which implies α is non-decreasing.

ASSUMPTION 2. *The type distribution F has a non-decreasing hazard rate. That is, $\alpha(\cdot)$ is non-decreasing.*

This is a common assumption in the literature and is satisfied by several common distributions such as uniform, exponential, gamma, etc.

In our model, all customers are present in the market at time 0 and exit after making a purchase. That is, customers can make a purchase at any time $t \ge 0$. We would like to point out that this model is very common in the literature of dynamic pricing. Stokey (1979) was one of the first papers that adopted such a model. Later, such a model has been used in a series of work; see for example Besanko and Winston (1990), Elmaghraby et al. (2008), Levin et al. (2009), Liu and Van Ryzin (2008), Dasu and Tong (2010), Liu and Van Ryzin (2011), Cachon and Swinney (2011), and Aviv et al. (2015). That is, in all aforementioned papers as well as our work, the customers are not heterogeneous in their time of arrival.[9] This allows us to focus on the impact of the customer heterogeneity in their valuations and valuation decay rates. We note that this is a good model for electronic products such as iPhones and iPods. For these products, potential customers wait for a new product launch

[9] There have been some other papers in the literature that consider such heterogeneity; see for example Su (2007) and Aviv and Pazgal (2008).

and a new launch is often announced publicly. Because of these, it is fair to assume that all the potential customers are in the market when the product is launched.

Customers are fully strategic about whether and when they purchase the item from the firm. Specifically, each customer either does not purchase the item, or purchases a unit in the period in which his utility gets maximized. Customers are risk neutral, and utility of a customer with type θ who purchases the item at time t at price p is $V(\theta,t)-p$. Furthermore, all customers are present during the entire time horizon. Then, given prices $\mathbf{p}=\{p_t : t\geq 0\}$, a customer with type θ purchases a unit of the item at time $\mathfrak{t}^*(\theta) := \arg\max_{\tau\geq 0}\{V(\theta,\tau)-p_\tau\}$ if $V(\theta,\mathfrak{t}^*(\theta))-p_{\mathfrak{t}^*(\theta)}\geq 0$, and she does not purchase otherwise. Here, p_t is the price for the item at time t.

We consider a deterministic baseline model where the firm knows the total mass of customers, i.e., the market size. The assumption of deterministic demand is justified when the number of customers is large and fairly predictable. This modeling choice allows us to study the impact of strategic customers and decay in customer valuation, but it deliberately removes the element of uncertainty from the model. That is, we seek to understand if the firm gains from DP when there is no demand uncertainty. In Section 5.3, we relax this assumption and show that our results can hold even with uncertainty in the market size.

3. Direct Mechanisms and Optimality

To characterize a profit-maximizing (optimal) selling mechanism, by the revelation principle, we focus on direct incentive-compatible and individually rational mechanisms where customers first report their type and then the mechanism determines their payment and time of allocation.

More precisely, any direct mechanism $\mathcal{M}$ consists of a tuple $(\mathfrak{t},\varsigma,\mathfrak{p})$, where $\mathfrak{p}:[\underline{\Theta},\bar{\Theta}]\to\mathbb{R}$ is a transfer scheme and $(\mathfrak{t},\varsigma):[\underline{\Theta},\bar{\Theta}]\to\mathbb{R}\times\{0,1\}$ is an allocation rule. That is, $\mathfrak{p}(\theta)$ and $\mathfrak{t}(\theta)$ are respectively the price for a unit of the item and time of purchase for a customer with type θ.[10] Further, $\varsigma(\theta)=1$ when the customer of type θ purchases the item, and is zero otherwise. One can assume that $\mathfrak{t}(\theta)=\infty$ when $\varsigma(\theta)=0$.

We note that the mechanism design theory enables us to focus on specifying the allocation rule policy rather than the pricing rule. In fact, once the allocation rule is determined, by the revenue equivalence theorem, we can characterize the pricing rule. Due to this

[10] Note that the allocation rule $\mathfrak{t}$ for a customer is only a function of the customer type, and does not depend on the type of other customers, because in our model, each customer is infinitesimal and there is no inventory constraint.

property, mechanism design approach has been widely used in the literature of dynamic pricing; see, for example, Gershkov and Moldovanu (2012), Chen and Farias (2015, 2016), and Board and Skrzypacz (2016). We further note that as it becomes more clear later, we use direct mechanism to propose optimal *posted price* mechanisms, which are widely used in practice.

We start by defining incentive compatibility and individual rationality. Let $u(\theta, \hat{\theta})$ be the expected utility of a customer with type θ when she reports $\hat{\theta}$. That is,

$$u(\theta, \hat{\theta}) = \varsigma(\hat{\theta})(V(\theta, \mathfrak{t}(\hat{\theta})) - \mathfrak{p}(\hat{\theta})) .$$

Then, mechanism $\mathcal{M}$ is incentive-compatible (IC) if for each customer with type $\theta \in [\underline{\Theta}, \bar{\Theta}]$, truthfulness is a best response, that is, $u(\theta, \hat{\theta}) \leq u(\theta, \theta)$. Roughly speaking, in IC mechanisms, no customer wants to deviate from the truthful strategy.

We can now define the individual rationality constraints for the mechanism. IC mechanisms are individually rational (IR) if for each customer with type θ, his utility under the truthful strategy is nonnegative, i.e., for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, we have $u(\theta, \theta) \geq 0$.

The following lemma presents the necessary and sufficient conditions under which a mechanism is IC.

LEMMA 1 **(Necessary and Sufficient Conditions for IC)**. *Consider mechanism $\mathcal{M}$ with allocation rule $(\mathfrak{t}(\cdot), \varsigma(\cdot))$. Then, the mechanism is IC if and only if both conditions, stated below, are satisfied.*

- *Envelope Condition: For any $\theta, \hat{\theta} \in [\underline{\Theta}, \bar{\Theta}]$,*

$$u(\theta, \theta) - u(\hat{\theta}, \hat{\theta}) = \int_{z=\hat{\theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)} (1 - g'(z)\mathfrak{t}(z)z) dz . \tag{1}$$

- *Interval Condition: For any $\hat{\theta} < \theta$,*

$$\int_{z=\hat{\theta}}^{\theta} \varsigma(\hat{\theta}) e^{-g(z)\mathfrak{t}(\hat{\theta})} (1 - g'(z)\mathfrak{t}(\hat{\theta})z) dz \leq u(\theta, \theta) - u(\hat{\theta}, \hat{\theta}) \leq \int_{z=\hat{\theta}}^{\theta} \varsigma(\theta) e^{-g(z)\mathfrak{t}(\theta)} (1 - g'(z)\mathfrak{t}(\theta)z) dz . \tag{2}$$

All the proofs are provided in the appendix.

Lemma 1 is analogous to the characterization of incentive compatibility in standard static settings, where an envelope condition and monotonicity of allocation rule are used to characterize incentive compatibility (see Myerson (1981)). The envelope condition above is a standard one and can be rewritten as $u(\theta,\theta) - u(\hat{\theta},\hat{\theta}) = \int_{z=\hat{\theta}}^{\theta} \varsigma(z)\partial_1 V(z,\mathfrak{t}(z))dz$, where $\partial_1 V(\theta,t) = \frac{\partial V(\theta,t)}{\partial \theta}$. But, the interval conditions, which can written as

$$\int_{z=\hat{\theta}}^{\theta} \varsigma(\hat{\theta})\partial_1 V(z,\mathfrak{t}(\hat{\theta}))dz \ \le\ \int_{z=\hat{\theta}}^{\theta} \varsigma(z)\partial_1 V(z,\mathfrak{t}(z))dz \ \le\ \int_{z=\hat{\theta}}^{\theta} \varsigma(\theta)\partial_1 V(z,\mathfrak{t}(\theta))dz\ ,$$

replace the monotonicity conditions. The interval conditions compare the utility obtained by the truthful strategy (middle term in Eq. (2)) with untruthful strategies (the rightmost and leftmost terms in Eq. (2)).

We are now ready to characterize the firm's profit under any IC mechanism. Note that the profit of a mechanism $\mathcal{M}$ from a customer of type θ is the customer's payment minus the production and holding costs, which can be written as $\varsigma(\theta)(\mathfrak{p}(\theta) - c - h\mathfrak{t}(\theta))$. Thus, the expected payment is given by $\int_{\theta=\underline{\Theta}}^{\bar{\Theta}} f(\theta)\varsigma(\theta)(\mathfrak{p}(\theta) - c - h\mathfrak{t}(\theta))d\theta = \mathrm{E}[\varsigma(\theta)\cdot(\mathfrak{p}(\theta) - c - h\mathfrak{t}(\theta))]$. Note that throughout the manuscript, unless stated otherwise, all expectations are with respect to customer type θ. Then, the total profit of the mechanism is the market size times the expected profit from selling the item to one customer. Considering that the market size is constant, that is, demand is deterministic, the total profit of the mechanism is maximized if we maximize the expected profit from a single customer.

An IC and IR mechanism is optimal if it maximizes the expected profit among all IC and IR mechanisms.

The following lemma characterizes the firm's profit in any IC mechanism $\mathcal{M}$.

LEMMA 2 **(Profit of IC Mechanisms)**. *In any IC mechanism, the expected firm profit from a single customer is given by*

$$\mathrm{E}\Big[\varsigma(\theta)\Big(e^{-g(\theta)\mathfrak{t}(\theta)}\Big(\theta + \alpha(\theta)\big(1 - g'(\theta)\mathfrak{t}(\theta)\theta\big)\Big) - h\mathfrak{t}(\theta) - c\Big) - u(\underline{\Theta},\underline{\Theta})\Big]\,, \tag{3}$$

where the expectation is taken with respect to the customer type θ.

Lemma 2 suggests that in order to optimize profit, the optimal mechanism should maximize virtual profit, that is, $\mathrm{E}\Big[\varsigma(\theta)\Big(e^{-g(\theta)\mathfrak{t}(\theta)}\big(\theta + \alpha(\theta)\big(1 - g'(\theta)\mathfrak{t}(\theta)\theta\big)\big) - h\mathfrak{t}(\theta) - c\Big) - u(\underline{\Theta},\underline{\Theta})\Big]$, and pick a transfer scheme that makes it both IC and IR. Throughout the paper, we refer to $\Big(e^{-g(\theta)t}\big(\theta + \alpha(\theta)\big(1 - g'(\theta)t\theta\big)\big) - ht - c\Big)$ as virtual value/profit of a customer with type θ at time t.

In the next section, we present an optimal mechanism for the case when both production and holding costs are zero. We discuss generalizations of this in Section 5.

4. Optimal Mechanism

We begin with characterizing an optimal mechanism when both the production and holding costs are zero. By Lemma 2, an optimal mechanism should maximize virtual profit subject to IC and IR constraints. Given that $c = h = 0$, the expected virtual profit is given by

$$\mathrm{E}\Big[\varsigma(\theta)\Big(e^{-g(\theta)\mathfrak{t}(\theta)}\big(\theta+\alpha(\theta)\big(1-g'(\theta)\mathfrak{t}(\theta)\theta\big)\big)\Big)-u(\underline{\Theta},\underline{\Theta})\Big] := \mathrm{E}\left[\varsigma(\theta)R(\theta,\mathfrak{t}(\theta))-u(\underline{\Theta},\underline{\Theta})\right], \tag{4}$$

where $R(\theta,t) = e^{-g(\theta)t}\big(\theta+\alpha(\theta)(1-g'(\theta)t\theta)\big)$ is the virtual value of a customer of type θ at time t. Note that the initial virtual value, i.e., $R(\theta,0)=\theta+\alpha(\theta)$, is equal to the virtual value in a standard static setting (c.f. Myerson 1981).

To characterize the optimal mechanism, we need to solve the following optimization problem.

$$\max_{\{u(\underline{\Theta},\underline{\Theta})\geq 0,\ (\mathfrak{t},\varsigma,\mathfrak{p})\}} \left\{\mathrm{E}\left[\varsigma(\theta)R(\theta,\mathfrak{t}(\theta))-u(\underline{\Theta},\underline{\Theta})\right]\right\}$$
$$s.t.\ \ \text{IC and IR constraints}. \tag{OPT}$$

Solving the above optimization problem is rather involved because we are maximizing over the allocation and payment functions $\mathfrak{t}(\cdot)$, $\varsigma(\cdot)$, and $\mathfrak{p}(\cdot)$. For this reason, we characterize the optimal solution of the above equation in two steps. Recall that by Lemma 1, satisfying the IC constraints is equivalent to satisfying the envelope and interval conditions. In the first step, we relax the problem by ignoring the interval conditions. That is, in the first step, we only focus on satisfying the envelope conditions and IR constraints. In particular, we consider the following relaxed problem:

$$\max_{\{u(\underline{\Theta},\underline{\Theta})\geq 0,\ (\mathfrak{t},\varsigma)\}} \mathrm{E}\left[\varsigma(\theta)R(\theta,\mathfrak{t}(\theta))-u(\underline{\Theta},\underline{\Theta})\big\}\right]$$
$$\text{s.t.}\ \ u(\theta,\theta) \ = \ u(\underline{\Theta},\underline{\Theta})+\int_{\underline{\Theta}}^{\theta}\varsigma(z)e^{-g(z)\mathfrak{t}(z)}(1-g'(z)\mathfrak{t}(z)z)dz \ \geq \ 0\,, \quad \theta\in[\underline{\Theta},\bar{\Theta}]\,. \tag{RELAXED}$$

Note that the constraint follows from the envelope conditions. In the second step, we show that the solution to the relaxed problem satisfies the interval conditions. This implies that the optimal solution of Problem RELAXED is also an optimal solution of Problem OPT.

Next, we characterize the optimal solution of the relaxed problem. We need one more definition to present our result. Let

$$\mathfrak{t}_f(\theta) \;=\; \frac{1}{\alpha(\theta)g'(\theta)} + \frac{1}{g(\theta)} + \frac{1}{g'(\theta)\theta} \tag{5}$$

be the solution to the first-order (necessary) optimality conditions. That is, $\mathfrak{t}_f(\theta) = \arg\max_{t\geq 0} R(\theta,t)$ and solves $\partial_1 R(\theta,\mathfrak{t}_f(\theta)) = 0$.Then, we have the following result.

Lemma 3 **(Optimal Solution of Problem relaxed)**. *If Assumptions 1 and 2 hold, then in an optimal solution of Problem* relaxed, $u(\underline{\Theta},\underline{\Theta}) = 0$, $\varsigma(\theta) = 1$, $\theta \in [\underline{\Theta},\bar{\Theta}]$, *and the allocation time, denoted by* $\mathfrak{t}_g$, *is given by*

$$\mathfrak{t}_g(\theta) = \begin{cases} 0 & \text{if } \theta \geq \theta_H \quad \textit{High-type;} \\ \mathfrak{t}_f(\theta) & \text{if } \theta \in [\theta_L,\theta_H] \quad \textit{Medium-type;} \\ \frac{1}{g'(\theta)\theta} & \text{if } \theta \in [\underline{\Theta},\theta_L] \quad \textit{Low-type,} \end{cases} \tag{6}$$

where $\mathfrak{t}_f(\theta)$ *is defined in Eq. (5),* θ_H *solves* $\theta_H + \alpha(\theta_H) + \frac{\alpha(\theta_H)g'(\theta_H)\theta_H}{g(\theta_H)} = 0$, *and* θ_L *solves* $g(\theta_L) + \alpha(\theta_L)g'(\theta_L) = 0$.

As we will show in the proof of Lemma 3, the log-concavity of $g(\cdot)$ implies that $\Big(1 - \mathfrak{t}_g(\theta)g'(\theta)\theta\Big) \geq 0$ for any θ. This allows us to conclude that the described mechanism is IR; that is $u(\theta,\theta) \geq 0$. This also leads to monotonicity of $u(\theta,\theta)$ in θ.

The following theorem shows that the optimal solution of the relaxed problem, characterized in Lemma 3, fulfills the interval conditions, and thus is optimal.

Theorem 1 **(Optimal Mechanism)**. *Suppose that Assumptions 1 and 2 hold. Then, the optimal mechanism sells to customers of type* $\theta \in [\underline{\Theta},\bar{\Theta}]$ *at time* $\mathfrak{t}_g(\theta)$, *defined in Eq. (6), and at price* $\mathfrak{p}(\theta) = \theta e^{-g(\theta)\mathfrak{t}_g(\theta)} - \int_{\underline{\Theta}}^{\theta} e^{-g(z)\mathfrak{t}_g(z)}\big(1 - \mathfrak{t}_g(z)g'(z)z\big)dz$. *In addition, in the optimal mechanism,* $\varsigma(\theta) = 1$ *for any* $\theta \in [\underline{\Theta},\bar{\Theta}]$.

We next discuss the main insights of Theorem 1. We first note that the firm sells the item to all customers. In addition, as we will prove in Lemma 7, the purchase time of customers $\mathfrak{t}_g(\theta)$ is decreasing in customer type; that is, customers with lower initial valuation purchase the item later than customers with higher initial valuation.

We note that the optimal mechanism divides the customers into three groups: high-type, medium-type, and low-type. The high-type customers who have high initial valuation ($\theta \geq$

θ_H) purchase the item immediately. The low-type customers who have low initial valuation ($\theta \le \theta_L$) delay their time of purchase. We note that the purchase time of high-type and low-type customers does not depend on the distribution of the customer type, F, directly. The time of purchase of these customers depend on F only through thresholds θ_H and θ_L.

We also observe that low-type customers with type θ purchase the item at time $\frac{1}{g'(\theta)\theta}$ and more importantly, get zero utility. To understand why, note that customers with type $\theta < \theta_L$ pay

$$\mathfrak{p}(\theta) \; = \; V(\theta, \mathfrak{t}_g(\theta)) - \int_{\underline{\Theta}}^{\theta} e^{-g(z)\mathfrak{t}_g(z)}\big(1 - \mathfrak{t}_g(z) g'(z) z\big) dz \; = \; V(\theta, \mathfrak{t}_g(\theta))\,,$$

where the second equality holds because $\mathfrak{t}_g(z) = \frac{1}{g'(z)z}$ for any $z \le \theta_L$. Note that this is in contrast with the traditional static mechanism design. In the static mechanism design, customers whose type is high enough get the product and enjoy a positive surplus, whereas low-type customers do not get the product at all. In fact, there is typically one customer type on the boundary that gets zero utility after purchasing the product. However, in our setting, there exists a group of customers who purchase the item and obtain zero utility.

The medium-type customers who have medium initial valuation $\theta \in [\theta_L, \theta_H]$ do not purchase the item immediately. However, unlike the low-type customers, these customers enjoy a positive utility.

The optimal mechanism presented in Theorem 1 highlights the fact that the firm benefits from the positive correlation between the valuation decay rate and the initial valuation by adopting DP and delaying allocation. In fact, the extra profit that the firm makes comes partly from the low-type customers from whom the firm extracts their entire surplus.

We would also like to point out that the optimal mechanism of Theorem 1 has an equivalent dynamic pricing interpretation. Specifically, it recommends the firm to post prices $\mathfrak{p}(\cdot)$, and consequently, customers with type θ will find it incentive compatible to purchase at time $\mathfrak{t}_g(\theta)$.

Comparison to a Model with Homogeneous Valuation Decay Rate: Assume that customer valuation decay rate is homogeneous, that is, $V(\theta, t) = \theta e^{-\beta t}$ or equivalently $g(\theta) = \beta$ where $\beta \ge 0$ is a constant. Note that with homogeneous decay rates, the valuation curves do not cross each other. In this case, the optimal mechanism posts a fixed price of θ_0, where θ_0 solves $\theta_0 + \alpha(\theta_0) = 0$. Thus, customers with type greater than θ_0 purchase the item at time zero, and in contrast with the optimal mechanism under heterogeneous valuation decay, customers with type $\theta < \theta_0$ do not purchase the item at all. The reason is that, with homogeneous decay rates, these customers have negative virtual value throughout the time horizon. Thus, the firm is not willing to sell the item to these customers.

To shed more light on Theorem 1, we study the following example.

EXAMPLE 1. Assume that θ follows the uniform distribution in the range of $[0,1]$; that is, $\theta \sim U(0,1)$. Here, we present the optimal mechanism when $g(\theta) = \theta^a$, $a \geq 0$, and $h = c = 0$. It is clear that when $a = 0$, the decay rates are homogeneous across customers. Further, as a increases, the customers' decay rates become more heterogeneous. For this valuation function, thresholds θ_H and θ_L are respectively $\frac{a+1}{a+2}$ and $\frac{a}{a+1}$. Note that both θ_H and θ_L are increasing in a and converge to 1 as a grows without bound. Thresholds θ_H and θ_L are depicted in Figure 2a. We observe that as a increases, the high-type and medium-type regions shrink while the low-type region expands. Recall that low-type customers get zero utility. Thus, when heterogeneity among customers increases, that is, a increases, the firm can extract the entire surplus of more customers. As a result, as depicted in Figure 2b, the firm's profit grows when the decay rate of customers gets more heterogeneous.

Figure 2b shows the profit gain of the optimal mechanism relative to the fixed price (FP) policy. Note that the profit-maximizing FP policy posts a price of $\theta_0 = \frac{1}{2}$ at time zero. We observe that as a increases, by employing DP, the firm increases its profit by more than 23% and 90% at $a = 1$ and 10, respectively. The reason is that by increasing a, the valuation decay rates become more heterogeneous, which, in turn, increases the value of differentiating customers via DP.

Figure 2b also shows that the social welfare of the customers and the firm increases when a increases. Note that for allocation time $\mathfrak{t}_g(\cdot)$, the social welfare equals $\mathrm{E}\left[\theta e^{-g(\theta)\mathfrak{t}_g(\theta)}\right]$, where $\theta e^{-g(\theta)\mathfrak{t}_g(\theta)}$ is the valuation of a customer with type θ who purchases the item at time $\mathfrak{t}_g(\theta)$. Observe that, for any value of a, DP outperforms FP in terms of obtained social welfare. However, the customer's surplus (utility)[11] under DP is lower than that under FP. This is so because by adopting DP, the firm can extract high profit from the customers, see the profit gain of the DP in Figure 2b.

[11] By Lemma 1, the expected customers' surplus (utility) is $\mathrm{E}[u(\theta)] = \mathrm{E}\left[\int_{\underline{\Theta}}^{\theta} e^{-g(z)\mathfrak{t}_g(z)}\big(1 - \mathfrak{t}_g(z)g'(z)z\big)dz\right]$.

Next, we discuss the purchase time, payment, and utility of customers. Given that $g(\theta) = \theta^a$, the purchase time in the optimal mechanism is given by

$$\mathsf{t}_g(\theta) = \begin{cases} 0 & \text{if } \theta \geq \frac{a+1}{a+2}; \\ \frac{(a+2)\theta-(a+1)}{a(\theta-1)\theta^a} & \text{if } \theta \in [\frac{a}{a+1}, \frac{a+1}{a+2}]; \\ \frac{1}{a\theta^a} & \text{if } \theta \in [0, \frac{a}{a+1}] \ . \end{cases} \tag{7}$$

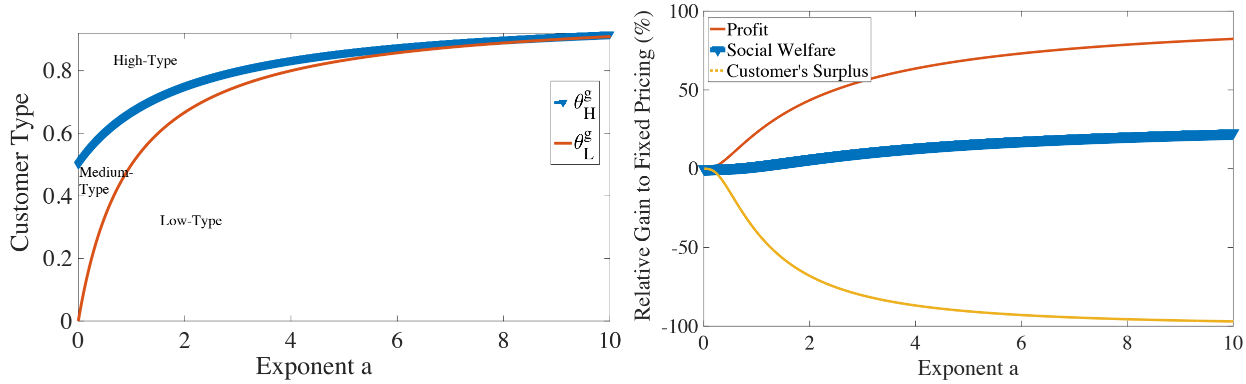
The purchase time is shown in Figure 3a for $a = 0, 0.5$, and 1. We note that $a = 0$ corresponds to homogeneous decay rates. Recall that under homogeneous decay rates, a FP policy is optimal. The figure shows that under FP ($a = 0$), there is a one-time sale where only customers with a type greater than $\theta_0 = \frac{1}{2}$ purchase the item immediately. However, this is no longer the case under DP when $a > 0$. Under DP, the purchase time, $\mathsf{t}_g(\theta)$, is decreasing in customer type; see also Eq. (7). We also observe that when a increases, the purchase time of low-type and high-type customers increases, whereas the purchase time of other customers does not vary remarkably.

Figures 3b and 3c, respectively, show the payment and utility of customers as a function of their types in the optimal mechanism. In Figure 3b, we observe that the payment of customers under DP ($a > 0$) increases as their type increases. High-type customers pay more when the firm uses DP rather than FP. However, this may not be the case for the medium-type customers. This group of customers delay their purchase and their valuation at the purchase time is not as high as their initial valuation. Therefore, the firm may reduce their payments. Furthermore, the DP policy enables the firm to extract more profit from low-type customers. Figure 3b also shows that customer payments are not monotone in a.

Figure 3c shows that under both DP and FP policies, the utility of customers, $u(\theta, \theta)$, is an increasing function of θ. But, the customers earn higher utility under the FP policy. Moreover, the utility of customers decreases when a increases. The reason is that when a is large, the firm can better differentiate the customers in order to extract more profit from them.

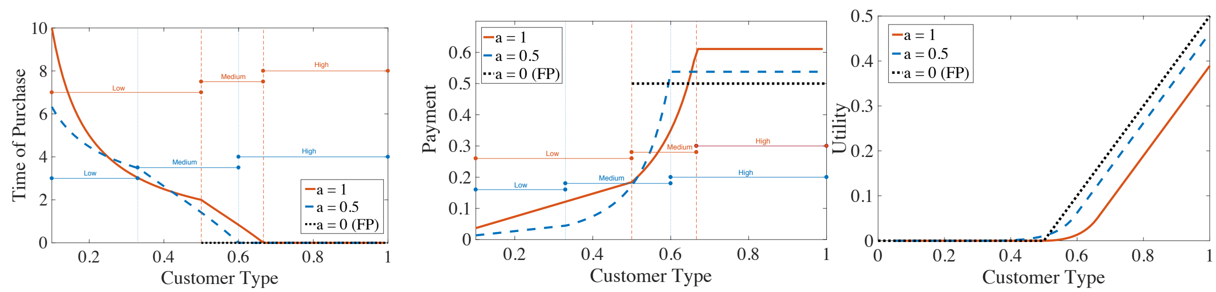
□

Example 1 shows that the DP policy earns significantly more profit than the FP policy. Motivated by this, we present a lower bound on the profit gain of DP over FP in Appendix E.1 when $g(\theta) = \theta^a$. We derive the lower bound on the profit gain by characterizing the extra profit the firm extracts from the low-type customers. The bound implies that for



(a) (b)

Figure 2 **(a) The thresholds θ_H and θ_L as a function of the exponent a for the mechanism described in Theorem 1; (b) Relative gain of profit, social welfare, and customer's surplus of DP (relative to the FP policy) in percentage, as a function of the exponent a. The customer type $\theta \sim U(0,1)$, $V(\theta,t) = \theta e^{-g(\theta)t}$, $g(\theta) = \theta^a$, and $h = c = 0$.**



(a) Time of purchase (b) Payment (c) Utility

Figure 3 **The time of purchase (a), payment (b), and utility of customers (c) in the optimal mechanism described in Theorem 1. Customer type $\theta \sim U(0,1)$, $V(\theta,t) = \theta e^{-g(\theta)t}$, $g(\theta) = \theta^a$, and $h = c = 0$.**

the setting in Example 1, the DP policy earns at least $50 \cdot e^{-\frac{1}{a}}$ percent more profit than FP. Thus, the profit gain of the DP for $a = 0.5, 1, 1.5, 2$, is at least 6.8, 18.4, 25.7, and 30.3 percent, respectively. Another interpretation of this result is if the firm ignores the heterogeneity in decay rates and follows the optimal mechanism under the homogeneous model, it will suffer from at least $50 \cdot e^{-\frac{1}{a}}$ percent profit loss.

5. Extensions

In this section, we relax some of the simplifying assumptions we made for our analysis in the previous section. Namely, in Section 5.1, we consider the case of a finite selling horizon. In Section 5.2, we consider the case of non-zero production costs. Finally, in Section 5.3, we consider the case in which there is uncertainty in the number of customers of each type.

In Section 5.4, we use an example to discuss situations in which customers with higher initial type do not necessarily have higher decay rates. We consider additional extensions of non-zero holding costs and inventory constraints in Appendices A and B, respectively.

5.1. Finite Time Horizon

In Section 4, we characterized the optimal mechanism when the length of the time horizon T is infinite. In practice, because of seasonality and changing the popular trends, the length of the time horizon can be finite and exogenous. Here, we seek to understand how an exogenous time horizon impacts the structure of the optimal mechanism.

In the following, we first present an optimal mechanism when the length of the time horizon is small, specifically, the case $T \le \frac{1}{g'(\theta_L)\theta_L}$ where θ_L solves $g(\theta_L) + \alpha(\theta_L)g'(\theta_L) = 0$. We will show that the optimal mechanism bears a resemblance to the mechanism presented in Section 4 in which the length of the time horizon is ∞. We then focus on the case when the length of the time horizon $T > \frac{1}{g'(\theta_L)\theta_L}$. Motivated by the structure of the optimal mechanism in Theorem 1, we present an approximately optimal mechanism. We show that the gap between the profit of the our mechanism and that of the optimal mechanism converges to zero as T increases.

Proposition 1 **(Small Time Horizon)**. *If Assumptions 1 and 2 hold, the production and holding costs are zero, and the length of the time horizon* $T \le \frac{1}{g'(\theta_L)\theta_L}$*, then the optimal mechanism sells one unit of the item to customers with type* $\theta \ge \theta_L^T$ *at time*

$$\mathfrak{t}_T(\theta) = \begin{cases} 0 & \text{if } \theta \ge \theta_H \\ \mathfrak{t}_f(\theta) & \text{if } \theta \in [\theta_H^T, \theta_H] \\ T & \text{if } \theta \in [\theta_L^T, \theta_H^T] \end{cases} \tag{8}$$

and at price $\mathfrak{p}(\theta) = \theta e^{-g(\theta)\mathfrak{t}_T(\theta)} - \int_{\theta_L^T}^{\theta} e^{-g(z)\mathfrak{t}_T(z)}\big(1 - \mathfrak{t}_T(z)g'(z)z\big)dz$*. Here,* θ_H^T *solves* $\mathfrak{t}_f(\theta_H^T) = T$*,* θ_L^T *solves* $R(\theta_L^T, T) = 0$*,* $\mathfrak{t}_f(\theta)$ *and* $R(\theta, t)$ *are defined in Eq. (5) and Eq. (4), respectively, and* θ_H *and* θ_L *are defined in Lemma 3. Further, for any* $\theta < \theta_L^T$*,* $\varsigma(\theta) = 0$ *and* $\mathfrak{p}(\theta) = \infty$*, and for* $\theta \ge \theta_L^T$*,* $\varsigma(\theta) = 1$*.*

Observe that similar to the mechanism in Section 4, high-type customers, i.e., those with type $\theta \ge \theta_H$, get the item at time zero. Medium-type customers, i.e., those with type $\theta \in (\theta_H^T, \theta_H)$, get the item at the solution to the first order condition (FOC), i.e., $\mathfrak{t}_f(\theta)$. Note that at type θ_H^T, the FOC solution is T.[12] Then, low-type customers, i.e., those with type $\theta \in [\theta_L^T, \theta_H^T]$ get the item at time T, where θ_L^T is the smallest type that the firm would like to sell the item to at time T. We point out that low-type customers are bargain hunters: They wait till the end of the time horizon to get a good deal. Finally, customers with type $\theta < \theta_L^T$ do not get the item at all.

We now focus on the case when the length of the time horizon $T > \frac{1}{g'(\theta_L)\theta_L}$. We will propose a mechanism, denoted by $\mathcal{M}_T$, that only sells to customers with type greater than θ_L^T at the following time

$$\mathfrak{t}_T(\theta) = \begin{cases} 0 & \text{if } \theta \ge \theta_H \\ \mathfrak{t}_f(\theta) & \text{if } \theta \in [\theta_L, \theta_H] \\ \frac{1}{g'(\theta)\theta} & \text{if } \theta \in [\theta_H^T, \theta_L] \\ T & \text{if } \theta \in [\theta_L^T, \theta_H^T] \end{cases} \tag{9}$$

and charges the customers with type $\theta \ge \theta_L^T$, $\mathfrak{p}(\theta) = \theta e^{-g(\theta)\mathfrak{t}_T(\theta)} - \int_{\theta_L^T}^{\theta} e^{-g(z)\mathfrak{t}_T(z)}\big(1 - \mathfrak{t}_T(z)g'(z)z\big)dz$. Here, with a slight abuse of notation, we use θ_H^T to refer to the solution of $\frac{1}{g'(\theta_H^T)\theta_H^T} = T$ and we use θ_L^T to denote the solution to $R(\theta_L^T, T) = 0$. Note that the allocation rule of mechanism $\mathcal{M}_T$ is very similar to the allocation rule of the mechanism in Theorem 1. However, in mechanism $\mathcal{M}_T$, we ensure that the sale ends at time T and customers that have a very low type, i.e., those with type $\theta < \theta_L^T$ do not purchase the item. For these customers, the firm gets a negative virtual value regardless of their time of purchase, i.e., $R(\theta, t) < 0$ for any $\theta < \theta_L^T$ and $t \in [0, T]$.

We note that when the selling horizon is small, the optimal mechanism posts the prices in such a way that the customers who purchase the item do so at time $\mathfrak{t}_T(\theta) = \min(\max(0, \mathfrak{t}_f(\theta)), T)$. However, when the selling horizon is large, one cannot design prices in a manner to incentivize customers of type θ to purchase the item at time $\min(\max(0, t_f(\theta)), T)$. Similar to the optimal mechanism in Theorem 1 where $T = \infty$, the time of purchase of low-type customers should be distorted to make the selling mechanism truthful. Recall that in Theorem 1, low-type customers of type θ purchase the item at time $\frac{1}{g'(\theta)\theta}$. While such a distortion does not hurt the seller's profit when $T = \infty$, it can slightly

[12] In fact, the assumption that $T \le \frac{1}{g'(\theta_L)\theta_L}$ ensures that $\theta_H^T \ge \theta_L$. To see why observe that $\mathfrak{t}_f(\theta_L) = \frac{1}{g'(\theta_L)\theta_L}$, and by Lemma 7, $\mathfrak{t}_f(\theta)$ is decreasing in θ. This implies that $\mathfrak{t}_f(\theta) \le \frac{1}{g'(\theta_L)\theta_L}$ for any $\theta \ge \theta_L$. Thus, when $T \le \frac{1}{g'(\theta_L)\theta_L}$, θ_H^T that solves $\mathfrak{t}_f(\theta_H^T) = T$ is greater than θ_L.

reduce the seller's profit when T is large but finite. This is the case because with a finite selling horizon, the seller cannot afford to sell to all customers. The following proposition shows that mechanism $\mathcal{M}_T$ is approximately optimal.

PROPOSITION 2 **(Large Time Horizon)**. *Suppose that Assumptions 1 and 2 hold, and the production and holding costs are zero, and the length of the time horizon* $T > \frac{1}{g'(\theta_L)\theta_L}$. *Then,*

$$\mathsf{Rev}_{opt} - \mathsf{Rev}_{\mathcal{M}_T} \ \leq \ \theta_L^T \exp\left(-\frac{g(\theta_L^T)}{g'(\theta_L^T)\theta_L^T}\right) F(\theta_L^T)\,,$$

where Rev_{opt} *and* $\mathsf{Rev}_{\mathcal{M}_T}$ *are the optimal profit and profit of mechanism* $\mathcal{M}_T$*, respectively.*

To get a better understanding of the bound in Proposition 2, let us assume that $g(\theta) = \theta^a$ and $\theta \sim U(0,1)$, where $a > 0$. Then, $\mathsf{Rev}_{opt} - \mathsf{Rev}_{\mathcal{M}_T} \leq (\theta_L^T)^2 e^{-\frac{1}{a}}$. As we show in the proof of the proposition, $1 - Tg'(\theta_L^T)\theta_L^T \geq 0$. This gives us an upper bound on θ_L^T. In particular, we get $\theta_L^T \leq \frac{1}{(Ta)^{1/a}}$. By applying this in our bound, we yield

$$\mathsf{Rev}_{opt} - \mathsf{Rev}_{\mathcal{M}_T} \leq \frac{1}{(Ta)^{2/a}} e^{-\frac{1}{a}} = O\left(\frac{1}{T^{2/a}}\right).$$

Thus, when $a = 0.5, 1$, and 1.5, the maximum profit loss of mechanism $\mathcal{M}_T$ converges to zero at the rates of $\frac{1}{T^4}$, $\frac{1}{T^2}$, and $\frac{1}{T^{4/3}}$, respectively.

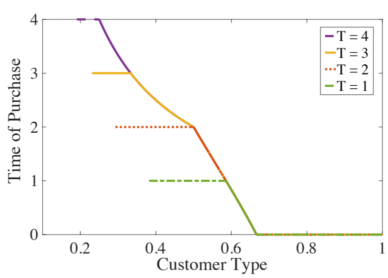
In the following, we revisit Example 1 to evaluate mechanism $\mathcal{M}_T$. Note that when $T \leq \frac{1}{\theta_L g'(\theta_L)}$, we also refer to the mechanism in Proposition 1 as mechanism $\mathcal{M}_T$. Then, $\mathcal{M}_T$ is optimal when $T \leq \frac{1}{\theta_L g'(\theta_L)}$.

EXAMPLE 2 (REVISITING EXAMPLE 1: FINITE TIME HORIZON). We evaluate mechanism $\mathcal{M}_T$ for the setting in Example 1. The allocation and payment rules of this mechanism are depicted in Figures 4a and 4b, respectively, with $T = 1, 2, 3$, 4, and $g(\theta) = \theta$. Recall that mechanism $\mathcal{M}_T$ is optimal when $T \leq \frac{1}{g'(\theta_L)\theta_L} = 2$. We observe that as T decreases, more customers wait until the end of the time horizon to avail the lower price. In addition, as T decreases, the high-type customers' payment decreases, whereas that of the low-type customers increases. Overall, the spread in the customer payments decreases when the firm needs to end the sale earlier.

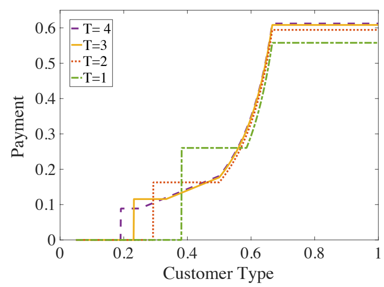
Figure 4c illustrates the maximum profit loss of this mechanism as a percentage of an upper bound on the optimal profit when $g(\theta) = \theta^a$ with $a = 0.5, 1$, and 1.5. For the upper bound, we use the tighter bound, which is provided in the proof of Proposition 2; see Eq. (22). In particular, when $T > \frac{1}{g'(\theta_L)\theta_L} = \frac{1}{a(\theta_L)^a}$, we have

$$\mathsf{Rev}_{opt} \leq \mathsf{Rev}_{\mathcal{M}_T} + \int_{\underline{\Theta}}^{\theta_L^T} e^{-g(z)T}\left(z - (1 - g'(z)Tz)\frac{g(z)}{g'(z)}\right) f(z)dz\,,$$

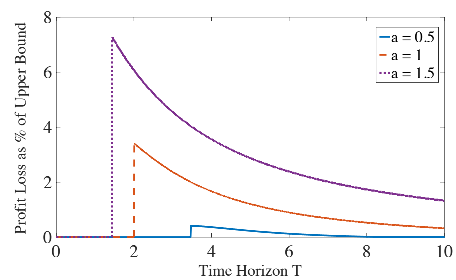
where $\theta_L = \frac{a}{1+a}$. For $T < \frac{1}{a(\theta_L)^a}$, we have $\mathsf{Rev}_{opt} = \mathsf{Rev}_{\mathcal{M}_T}$. Figure 4c illustrates the upper bound on the profit loss of mechanism $\mathcal{M}_T$ (in percentage), i.e., $100 \cdot \frac{\mathsf{gap}(T)}{\mathsf{gap}(T)+\mathrm{Rev}_{\mathcal{M}_T}}$, where $\mathsf{gap}(T) = \int_{\underline{\Theta}}^{\theta_L^T} e^{-g(z)T}\left(z - (1 - g'(z)Tz)\frac{g(z)}{g'(z)}\right) f(z)dz$. We observe that the upper bound on the profit loss decreases as a gets smaller. In addition, the upper bound decreases as T increases, and it gets maximized at $T = \frac{1}{a(\theta_L)^a}$. However, the upper bound does not exceed 0.4%, 3.3%, and 7.2% when $a = 0.5, 1$, and 1.5. respectively. The upper bound is maximized at $T = \frac{1}{a(\theta_L)^a}$ and the jump in the plot occurs at $T = \frac{1}{a(\theta_L)^a}$ because the upper bound on the optimal profit is not tight when T is close to $\frac{1}{a(\theta_L)^a}$. We note that despite the fact that the bound is not tight, the profit loss of mechanism $\mathcal{M}_T$ is insignificant.



(a) Time of purchase



(b) Payment



(c) Profit loss

Figure 4 **The time of purchase (a) and payment of customers (b) in mechanism $\mathcal{M}_T$ when $g(\theta) = \theta$, $h = c = 0$, and $\theta \sim U(0,1)$. The profit loss of the mechanism $\mathcal{M}_T$ (c) as a percentage of the upper bound when $g(\theta) = \theta^a$ with $a = 0.5, 1$, and 1.5.**

□

5.2. Production Costs

In this section, we present an optimal mechanism when the production cost $c \geq 0$, and the holding cost h is zero. We show that when the firm faces production costs, it ends the sale sooner, compared to when the production cost is zero. In particular, the production cost introduces a cut-off such that customers whose type θ is greater than the cut-off purchase at time $\mathsf{t}_g(\theta)$, and other customers do not purchase the item at all. That is, a positive production cost does not change the time of allocation of customers who purchase the item.

Defining θ_c as the smallest value that solves $R\big(\theta_c, \mathfrak{t}_g(\theta_c)\big) = c$, we present the main result of this section. Here, $R\big(\theta, t\big)$ and $\mathfrak{t}_g(\cdot)$ are defined in Equations (4) and (6), respectively.

Theorem 2 **(Production Costs)**. *If Assumptions 1 and 2 hold, the production cost $c \in [0, \bar{\Theta}]$, and the holding cost $h = 0$, then the optimal mechanism only sells to customers with type $\theta \geq \theta_c$ at time $\mathfrak{t}_g(\theta)$, given in Eq. (6), and at price $\mathfrak{p}(\theta) = \theta e^{-g(\theta)\mathfrak{t}_g(\theta)} - \int_{\theta_c}^{\theta} e^{-g(z)\mathfrak{t}_g(z)}\big(1 - \mathfrak{t}_g(z)g'(z)z\big)dz$. Furthermore, for $\theta < \theta_c$, $\varsigma(\theta) = 0$ and $\mathfrak{p}(\theta) = \infty$, and for $\theta \geq \theta_c$, $\varsigma(\theta) = 1$.*

In Theorem 2, we assume that the production cost c is less than $\bar{\Theta}$, as the firm has no incentive to produce and sell the items when the production cost is greater than the maximum valuation of customers $\bar{\Theta}$.

The main idea of the proof is to show that the virtual value of a customer with type θ at time $\mathfrak{t}_g(\theta)$, that is, $R\big(\theta, \mathfrak{t}_g(\theta)\big)$, is increasing in θ. Then, provided that $R\big(\theta_c, \mathfrak{t}_g(\theta_c)\big) - c = 0$, we have $R\big(\theta, \mathfrak{t}_g(\theta)\big) - c < 0$ for any $\theta < \theta_c$. This implies that the firm would rather not sell the item to customers with type $\theta < \theta_c$.

Theorem 2 suggests that the production cost will not change the allocation time of customers with type $\theta \geq \theta_c$; rather, it only changes the payment rule such that the lower-type customers are not willing to purchase the item. In other words, the payment rule is designed to enforce a cut-off, θ_c, in the allocation rule. For more insight into Theorem 2, we revisit Example 1.

Example 3 (Revisiting Example 1: Production Costs). Consider the same setting in Example 1. Figure 5a illustrates the cut-off θ_c as a function of the production cost when $g(\theta) = \theta^a$ and $a = 0, 0.5, 1$. Here, with $a = 0$, the optimal mechanism can be implemented via an FP policy. For any $a > 0$, FP policies are no longer optimal. Figure 5a compares the threshold θ_c with that in the FP policy. Note that in the FP policy, the threshold $\theta_{c,f}$ solves $R(\theta_{c,f}, t = 0) = c$. We observe that the threshold is smaller than in the DP policy for $a = 0.5, 1$, which suggests that the DP policy sells to more customers than the FP policy. In addition, the cut-off is decreasing in exponent a. That is, the firm is willing to sell to lower-type customers when a gets larger and differentiating customers gets easier. However, when the production cost increases, the difference in the cut-off gets smaller and converges to that of the FP policy.

Figure 5b shows the payment of the customers in the optimal mechanism when the production cost is $0.1, 0.2, 0.3$, and 0.4, and $g(\theta) = \theta$. Note that in the figures, the cut-offs are depicted via dashed vertical lines. Figure 5b shows that when the production cost increases, the firm increases the payment of the customers who purchase the item. By increasing the payment, the firm can enforce the cut-off and ensure that customers with type less than θ_c do not purchase the item. Note that as the production cost increases from 0.1 to 0.2, the payments of customers with type greater than $\theta_{0.2}$, the cut-off at $c = 0.2$, remain the same. The reason for this is that all the extra customers who make a purchase at the production cost of 0.1, i.e., those with type $\theta \in [\theta_{0.1}, \theta_{0.2}]$, buy at time $\frac{1}{g'(\theta)\theta}$ and receive zero utility (see the time of purchase of the optimal mechanism presented in Theorem 1). Therefore, selling to these customers does not impact the utility and payment of other customers. Recall that by Lemma 1, in any IC mechanism with the allocation rule $\mathfrak{t}(\cdot)$, the utility of a customer with type θ is $u(\theta) = u(\underline{\Theta}) + \int_{z=\underline{\Theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)}(1 - g'(z)\mathfrak{t}(z)z)dz$, and here $(1 - g'(z)\mathfrak{t}(z)z) = 0$ for any $z \in [\theta_{0.1}, \theta_{0.2}]$.

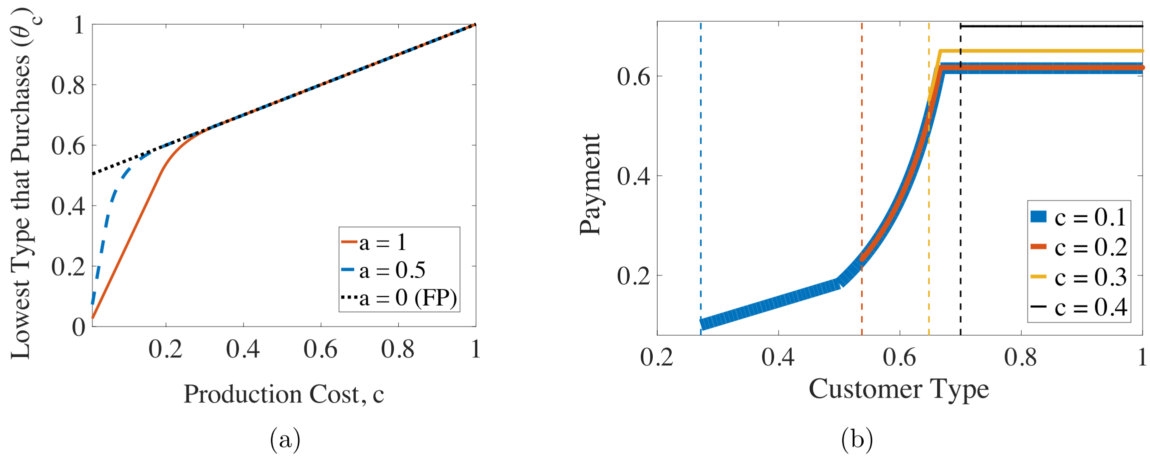


Figure 5 **(a) Cut-off θ_c as a function of the production cost c in the optimal mechanism with $g(\theta) = \theta^a$ and $a = 0, 0.5, 1$; (b) The payment of customers as a function of their type with $c = 0.1, 0.2, 0.3$, and 0.4 and $g(\theta) = \theta$. Customer type $\theta \sim U(0,1)$ and the holding cost $h = 0$.**

□

5.3. Uncertainty in the Market

In this section, we investigate the impact of uncertainty in the market. We consider a market of finite size with customers belonging to a set of discrete number of possible types (there may be infinite types). There is uncertainty both in the total market size and the number of customers of each type.

In particular, we compare the profit of the firm under two scenarios. In the first scenario, the firm is not aware of the exact market size and only knows the expected number of customers, i.e., the average market size. In the second scenario, the firm is fully aware of the market size and based on this knowledge, it designs an optimal selling mechanism. We show that the profit of the firm under these two scenarios is very close to each other. Specifically, the difference between these two profits converges to zero as the market size grows without bound.

We use M to denote the total number of customers, where M is drawn from distribution $\mathcal{D}$. We use $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$ for any $K > 0$ to denote the set of all possible customer types. We say that a customer is of type k when his type is θ_k. Then, conditional on M, the number of customers of each type, denoted $\mathbf{m} = \{m_1, m_2, \ldots, m_K\}$, is drawn from a multinomial distribution with M trials and probabilities $\mathbf{q} = \{q_1, q_2, \ldots, q_K\}$, where q_k is the probability that a customer is of type $k \in [K] := \{1, \ldots, K\}$.

We consider the following two scenarios:

Scenario 1 (Unknown Market Size): In this scenario, the firm is not aware of the market size M and exact number of customers of each type, $\{m_k : k = 1, \ldots, K\}$. The firm is only aware of the distribution $\mathcal{D}$ and the probabilities $\{q_k : k = 1, \ldots, K\}$. Let $\mathsf{Rev}^1_{opt}(M)$ denote the expected profit that firm can obtain from a single customer under scenario 1, conditional on the market size realization being M. We refer to $\mathsf{Rev}^1_{opt}(M)$ as the normalized profit under scenario 1. This profit can be calculated by solving the following optimization problem:

$$\max_{\{(t_k, p_k, \varsigma_k),\ k \in [K]\}} \sum_{k \in [K]} \varsigma_k p_k q_k \qquad (\mathsf{Rev})$$

$$s.t. \quad \varsigma_k(\theta_k e^{-g(\theta_k)t_k} - p_k) \ \geq \ \varsigma_j(\theta_k e^{-g(\theta_k)t_j} - p_j), \quad j, k \in [K], \quad (\text{IC})$$

$$\varsigma_k(\theta_k e^{-g(\theta_k)t_k} - p_k) \ \geq \ 0, \quad k \in [K]. \quad (\text{IR})$$

Here, p_k and t_k are the payment and time of allocation of customer of type $k \in [K]$, respectively. Finally, $\varsigma_k = 1$ if the customer of type k purchases the item and is zero otherwise. The first set of constraints ensures that customers do not have an incentive to be untruthful and the second set of constraints guarantees that the utility of the customers under truthful strategy is nonnegative. For convenience, we denote the optimal solution to (Rev) by $\mathsf{Rev}(\mathbf{q})$, and we note that $\mathsf{Rev}^1_{opt}(M) = \mathsf{Rev}_{opt}(\mathbf{q})$, and is a constant independent of M. Thus, in the following, we denote $\mathsf{Rev}^1_{opt}(M)$ by Rev^1_{opt}.

Scenario 2 (Known Market Size): In this scenario, the firm is fully aware of both the market size M and the number of customers of each type, $\{m_k : k = 1, \ldots, K\}$. Then, defining $\tilde{\mathbf{q}} = \{\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_K\}$, where $\tilde{q}_k = m_k/M$, the optimal profit from a single customer in this scenario, denoted by $\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}, M)$, equals $\mathsf{Rev}(\tilde{\mathbf{q}})$. We note that given $\tilde{\mathbf{q}}$, $\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}, M)$ is independent of M. Hence, we denote $\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}, M)$ by $\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}})$.

In the following proposition, we show that $\mathrm{E}[\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}) \mid M] = \mathsf{Rev}^1_{opt}$ where the expectation is w.r.t. $\tilde{\mathbf{q}}$. We further show that for large market sizes, the gap between the normalized profits in the two scenarios converges to zero.

Proposition 3 **(Uncertainty in the Market)**. *The expected normalized profits in the two scenarios (with known market size and unknown market size) are equal, i.e.,* $\mathsf{Rev}^1_{opt} = \mathrm{E}[\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}) \mid M]$. *In addition, for market size distributions such that $M \geq n$ a.s., for some n, we obtain*

$$\mathrm{E}\left[\left|\mathsf{Rev}^1_{opt} - \mathrm{E}[\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}) \mid M]\right|\right] \leq \bar{\Theta}\sqrt{\frac{\log n}{2n}} + \frac{\bar{\Theta}}{n} \ , \tag{10}$$

where $\bar{\Theta} = \max_{k\in[K]} \theta_k$ and the inner and outer expectations is with respect to $\tilde{\mathbf{q}}$ and market size M, respectively.

By Proposition 3, the gap between the expected normalized profits in the two scenarios converges to zero as n, the lower bound on M, increases. Furthermore, the bound in Proposition 3 is not a function of the number of types K and holds for any value of K. Thus, for large markets, the normalized profits in the two scenarios are quite close to each other. Proposition 3 can be extended to show that the gap between total profits is also small (sub-linear) for a sequence of systems with increasing market sizes, in which each of the individual profits increases in a linear fashion.

5.4. Generalizing Customer Valuation Model

In this paper, we assume that customers with higher initial valuations have higher valuation decay rates compared with customers with lower initial valuations. We briefly discuss the impact of relaxing this assumption using an example.

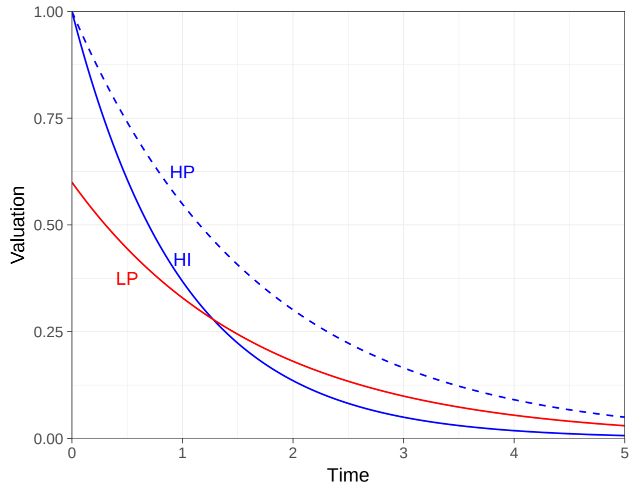


Figure 6 **Example with three customer types: LP represents low initial valuation patient customers, HP represents high initial valuation patient customers, and HI represents high initial valuation impatient customers.**

Consider three customer types: low initial value and patient (LP), high initial value and impatient (HI), and high initial value and patient (HP). Customer valuations decay over time as $V(ab,t) = v_a e^{-\delta_b t}$, where $a \in \{\text{L,H}\}$ and $b \in \{\text{I,P}\}$. We set $v_L = 0.6$, $v_H = 1$, $\delta_P = v_L$, and $\delta_I = v_H$; see Figure 6. We denote the intersection point of HI and LP valuation curves by $\tau = 1.28$. Further, we set the mass of LP-customers at unity, the mass of HI-customers at $1-\gamma$ and that of HP-customers at γ for some $0 \le \gamma \le 1$.

Notice that at $\gamma = 0$, this model reduces to a discretized version of our original model with high initial valuation customers being more impatient than low initial valuation customers. Further, at $\gamma = 0$, the optimal mechanism comprises two price points with HI-customers purchasing at $t = 0$ at price $p_1 = 0.93$, and LP-customers purchasing at $t = 0.82$ at price $p_2 = 0.37$.[13] In this mechanism, LP-customers receive zero net utility, whereas HI-customers receive a positive surplus.

Now if we consider the case of $\gamma > 0$, then the firm has no means of separating HP customers from both HI and LP customers. If the firm sets price p_1 at $t = 0$ and price $p_2 = V(\text{LP}, t_2)$ for some $t_2 \le \tau$ such that it can sell to HI and LP customers at time 0 and t_2, respectively, then HP customers strictly prefer to purchase at time t_2 than to purchase at time 0. Thus, this model becomes effectively equivalent to one in which the LP-customer

[13] We note that to characterize the optimal mechanism, it suffices to consider the mechanisms that offer p_1 at $t = 0$, and $p_2 = V(\text{LP}, t_2)$ at some $t_2 \le \tau$ such that the mechanism incentivizes HI-customers to purchase at $t = 0$ and LP-customers to purchase at $t = t_2$. That is, to obtain the optimal mechanism, we only need to optimize on p_1 and t_2.

mass equals $(1+\gamma)$. Consequently, as γ increases, the optimal mechanism changes such that p_2 increases (with the corresponding time of purchase also decreasing so that LP customers receive zero net utility), and p_1 decreases (with a time of purchase zero). We find that at $\gamma = 0.3$, the optimal mechanisms sets $p_1 = p_2 = v_L$ and sells to all customers at $t = 0$. This example provides some intuition as to how one can tackle situations in which Assumption 1 does not hold.

6. Conclusion

Dynamic pricing is a common practice in many industries and has proven to be an effective tool in mitigating the negative impact of demand uncertainty. This work contributes to the literature by showing that dynamic pricing can have significant benefit even in the absence of demand uncertainty. Specifically, we show that when customers' valuations are time-sensitive and decay at different rates, the firm can increase its profit by implementing dynamic pricing even when the firm knows the overall market size with certainty. We find that the heterogeneity in customer time sensitivities allows the firm to differentiate between customers more effectively. In addition to extracting more profits, this differentiation also increases product allocations so that customers with low initial valuations can also procure the product. In this fashion, we show that a firm can successfully embrace a seemingly unfavorable scenario of decaying customer valuations to improve its profit and customers' welfare.

In our setting, the customers' strategic nature is an essential ingredient required to extract the aforementioned benefits. The fact that each customer is forward-looking and times his purchase to obtain the best value for himself is what allows the firm to set a price trajectory that can effectively differentiate between the customers over time. In this sense, our work also illustrates how dynamic pricing can be beneficial when customers are forward-looking.

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Appendix

This appendix is organized as follows. In Appendix A, we generalize our optimal mechanism to include positive holding costs. Appendix B is dedicated to the optimal mechanism with inventory constraints. Appendix C proves Theorem 1. Appendix D proves the results in Section 5. Appendix E proves the results in Sections 3 and 4, and also establishes an additional supporting result. Appendix F proves Theorem 3 (of Appendix A). Supporting lemmas in the proof of Theorems 1 and 3 are proved in Appendices G and H, respectively.

Appendix A: Holding Costs

In this section, we characterize an optimal mechanism for a setting with positive holding cost. Similar to production costs, holding costs motivate the firm to end the sale sooner. However, with a positive holding cost, the firm incentivizes customers to purchase the item earlier, as carrying the items in inventory is costly. This is in contrast with the optimal mechanism with a positive production cost. There, the purchase time for all customers who make a purchase remain the same, whereas with a positive holding cost, customers are incentivized to purchase the item sooner. To simplify the exposition, here, we focus on an exponential valuation function $V(\theta,t)=\theta e^{-\theta t}$; that is, we assume that $g(\theta)=\theta$.[14]

Before presenting the optimal mechanism with a positive holding cost, to get intuition on the impact of the holding cost, we revisit Example 1 when $h>0$ and $g(\theta)=\theta$.

Example 4 (Revisiting Example 1: Holding Costs). We present the optimal mechanism for the setting in Example 1 when the holding cost $h>0$, the production cost $c=0$, and $g(\theta)=\theta$. Figure 7a shows how the optimal mechanism divides customers into different regions. A precise definition of the boundaries of these regions will be given later in Eqs. (11) and (13).

Observe that when the holding cost is small ($h<H_l:=0.04$), there are four regions: high-type, medium-type, low-type, and no-allocation. We later define H_l in Eq. (11) for any type distribution F. While customers in the high-type region get the item immediately, customers in the low-type and medium-type regions delay their purchase time. Moreover, customers in the no-allocation region do not purchase the item at all. These customers and customers in the low-type region get zero utility. We note that as the holding cost increases, the low-type region shrinks, whereas other regions grow. This pattern continues until the holding cost hits H_l. At $h=H_l$, the low-type region vanishes and there will be only three regions: high-type, medium-type, and no-allocation. As we increase the holding cost further from H_l to $H_h:=0.25$, the high-type region gets larger while the medium-type region gets smaller; see the definition of H_h in Eq. (11) for any type distribution F. In fact, at $h=H_h$, the medium-type region disappears. Finally, for $h\geq H_h$, only two regions remain: those of high-type and no-allocation. That is, when the holding cost is high enough, the firm posts a fixed price, which only incentivizes the high-type customers to purchase the item at time zero.

Figure 7b shows the customers' surplus, social welfare, and profit gain of DP in percentage (relative to the FP policy) as a function of the holding cost. We note that the FP policy does not change as the holding cost

[14] All the results can be generalized to $g(\theta)=\beta\theta$ where $\beta>0$ is a constant.

varies. For any value of the holding cost, the FP policy posts a price of $\theta_0 = \frac{1}{2}$. Thus, the social welfare of the FP policy equals $\mathrm{E}[\theta \times \mathbb{1}\{\theta \geq \theta_0\}]$. The social welfare of the DP policy is the expected value of customers at the time of purchase $\mathfrak{t}_h(\cdot)$ minus the holding cost, that is, $\mathrm{E}[(\theta e^{-\theta \mathfrak{t}_h(\theta)} - h\mathfrak{t}_h(\theta)) \times \mathbb{1}\{\theta \geq \underline{\theta}_h\}]$ where $\underline{\theta}_h$ is the lowest type that purchases the item. We formally define $\mathfrak{t}_h(\theta)$ and the cut-off $\underline{\theta}_h$ in Eqs. (13) and (11), respectively.

Interestingly, social welfare is not monotone in the holding cost. At first glance, we expect social welfare to decrease when the holding cost gets larger, but this is only the case when the holding cost is not too large. For larger holding cost values, social welfare increases in h. To understand why, note that by increasing the holding cost, the firm incentivizes customers to purchase earlier as holding the items is costly.This, in turn, enhances the social welfare as the net value increases for customers at their time of purchase. Furthermore, we observe that when the holding cost is not too large, the social welfare of the optimal DP mechanism is greater than that of FP. Thus, for small holding cost values, DP not only increases the firm's profit but also the social welfare.

Figure 7b shows that DP outperforms FP by a higher percentage when the holding cost is small, because a smaller holding cost allows the firm to lower prices and further delay the time of allocation to customers. This enables the firm to earn more profit from the customers. Due to the same reason, the customer surplus is increasing in the holding cost.

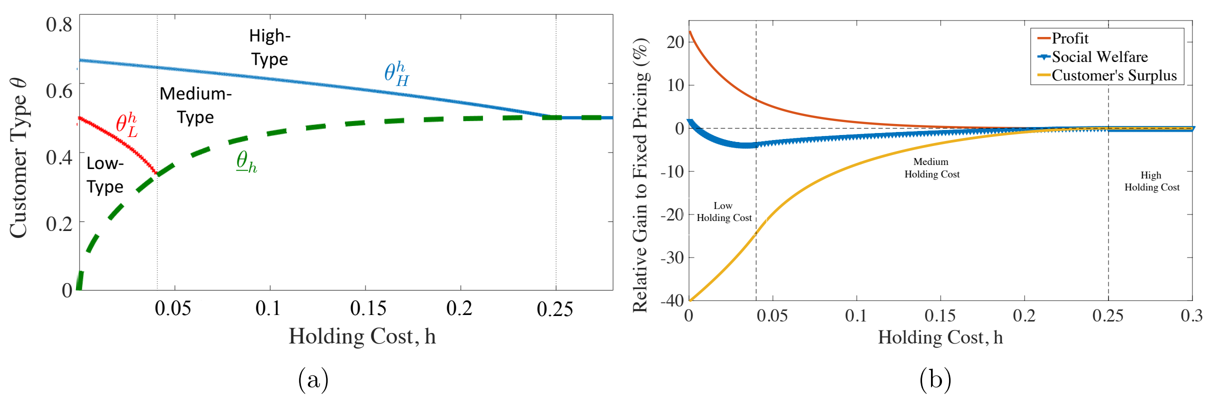

Figure 7 **(a) Structure of the optimal mechanism as a function of the holding cost h; (b) Customer surplus, social welfare, and profit gain of the optimal mechanism (relative to the FP policy) in percentage as a function of the holding cost. The customer type $\theta \sim U(0,1)$, $V(\theta,t) = \theta e^{-\theta t}$, and the production cost $c = 0$.**

□

Example 4 illustrates how the holding cost influences the structure of the optimal mechanism. Next, we formalize these observations by presenting the optimal mechanism with a positive holding cost. We will show that the optimal mechanism only sells to customers with initial valuation $\theta \geq \underline{\theta}_h$. That is, $\varsigma(\theta) = 1$ when $\theta \geq \underline{\theta}_h$ and zero otherwise. Here, the cut-off $\underline{\theta}_h$ depends on the holding cost, h, and is given by

$$\underline{\theta}_h := \begin{cases} \max\{\underline{\theta}_L, \underline{\Theta}\} & \text{if } h < H_l \quad \text{Low Holding Cost;} \\ \underline{\theta}_M & \text{if } h \in [H_l, H_h] \quad \text{Medium Holding Cost;} \\ \theta_H & \text{if } h > H_h \quad \text{High Holding Cost.} \end{cases} \tag{11}$$

Here, $H_l = (\tilde{\theta})^2 e^{-1}$ and $H_h = \theta_L^2 = \theta_0^2$ where $\tilde{\theta}$ solves $2\tilde{\theta} + \alpha(\tilde{\theta}) = 0$ and θ_L is defined in Lemma 3. In particular, when $g(\theta) = \theta$, θ_L solves $\theta_L + \alpha(\theta_L) = 0$; that is $\theta_L = \theta_0$. Noting that $\tilde{\theta} \le \theta_0$, it is easy to observe that $H_l < H_h$. We refer to the holding cost as low and medium when $h < H_l$ and $h \in [H_l, H_h]$, respectively, and we refer to it as high when $h > H_h$. For low holding costs $h \le H_l$, the cut-off $\underline{\theta}_h = \underline{\theta}_L$, where $\underline{\theta}_L$ solves $e^{-1}(\underline{\theta}_L)^2 = h$. Observe that at $h = H_l$, the cut-off $\underline{\theta}_L = \tilde{\theta}$, and at $h = 0$, we have $\underline{\theta}_L = 0$.

When the holding cost is medium, only customers with initial valuation $\theta \ge \underline{\theta}_M$ purchase the item where $\underline{\theta}_M$ solves $R(\underline{\theta}_M, \mathfrak{t}_f(\underline{\theta}_M)) - h\mathfrak{t}_f(\underline{\theta}_M) = 0$, $R(\theta, t)$ is defined in Eq. (4), and $\mathfrak{t}_f(\theta)$ is the FOC solution. We note that when $g(\theta) = \theta$, we have

$$R(\theta, t) = e^{-\theta t}\left(\theta + \alpha(\theta)(1 - t\theta)\right). \tag{12}$$

Further, the FOC solution, $\mathfrak{t}_f(\theta)$, solves $\frac{\partial R(\theta,t) - ht}{\partial t}\big|_{\mathfrak{t}_f(\theta)} = 0$. One can show that when $h = H_l$, $\tilde{\theta}$ solves $R(\tilde{\theta}, \mathfrak{t}_f(\tilde{\theta})) - h\mathfrak{t}_f(\tilde{\theta}) = 0$.

We show that in the optimal mechanism, the purchase time of a customer with type $\theta \ge \underline{\theta}_h$ is

$$\mathfrak{t}_h(\theta) = \begin{cases} 0 & \text{if} \quad \theta \ \ge \ \theta_H^h & \text{High-Type;} \\ \mathfrak{t}_f(\theta) & \text{if} \quad \theta \ \in \ [\theta_L^h, \theta_H^h] & \text{Medium-Type;} \\ \frac{1}{\theta} & \text{if} \quad \theta \ \in \ [\underline{\Theta}, \theta_L^h] & \text{Low-Type.} \end{cases} \tag{13}$$

Note that when $\theta \in [\underline{\Theta}, \theta_L^h]$, the time of purchase is $\frac{1}{g'(\theta)\theta} = \frac{1}{\theta}$. In Eq. (13),

$$\begin{cases} \theta_H^h = \theta_0 & \text{if} \quad h > H_h; \\ \mathfrak{t}_f(\theta_H^h) = 0 & \text{if} \quad h \le H_h. \end{cases} \tag{14}$$

That is, for $h \le H_h$, the FOC solution at $\theta_H^h \in [\theta_0, \theta_H]$ is zero:

$$\frac{\partial (R(\theta_H^h, t) - ht)}{\partial t}\Big|_{t=0} = -\theta_H^h \ (\theta_H^h + 2\alpha(\theta_H^h)) - h = 0 \,. \tag{15}$$

We note that for any $\theta < \theta_H^h$, the FOC solution is negative, and $R(\theta, t) - ht$ is maximized at $t = 0$. We also observe that at $h = 0$ and $h = H_h$, θ_H^h is respectively θ_H and θ_0. Furthermore, θ_H^h is decreasing in h, indicating that as the holding cost increases, more customers purchase the item at time zero.

We now define θ_L^h in Eq. (13):

$$\begin{cases} \theta_L^h = \tilde{\theta} & \text{if} \quad h > H_l; \\ \mathfrak{t}_f(\theta_L^h) = \frac{1}{\theta_L^h} & \text{if} \quad h \le H_l. \end{cases} \tag{16}$$

That is, for $h \le H_l$, $\theta_L^h \in [\tilde{\theta}, \theta_0]$ solves

$$\frac{\partial (R(\theta_L^h, t) - ht)}{\partial t}\Big|_{t=\frac{1}{\theta_L^h}} = -\theta_L^h \ e^{-1}(\theta_L^h + \alpha(\theta_L^h)) - h = 0 \,. \tag{17}$$

We note that at $h = 0$ and $h = H_l$, θ_L^h equals θ_0 and $\tilde{\theta}$, respectively. Furthermore, θ_L^h is decreasing in h. This suggests that as the holding cost increases, the highest customer type who purchases and obtains zero utility decreases. That is, the low-type group gets smaller. Figure 7a in Example 4 shows how θ_L^h and θ_H^h vary as the holding cost h increases.

We now describe the optimal mechanism by consolidating the time of purchase $\mathfrak{t}_h(\cdot)$ and the cut-off $\underline{\theta}_h$. In the optimal mechanism, when the holding cost is low, that is, $h \le H_l$, the firm sells to high-type and medium-type customers and some low-type customers with $\theta \in [\max\{\underline{\Theta}, \underline{\theta}_L\}, \theta_L^h]$. Thus, with low holding costs, the firm can extract the full surplus of some of low-type customers. However, under medium and high holding costs, the firm has no such opportunity, as it must end the sale early due to the high cost of carrying the items. In particular, when the holding cost is medium, $h \in [H_l, H_h]$, the firm only sells to high-type and some of medium-type customers, that is, those with $\theta \in [\underline{\theta}_M, \theta_H^h]$. Finally, when the holding cost is high, the firm does not benefit from the heterogeneity of valuation decay rates, and it simply posts a fixed price of θ_0. Then, only customers with a type greater than $\underline{\theta}_h = \theta_0$ purchase the item at time zero. The following theorem formally characterizes the optimal mechanism.

THEOREM 3 **(Holding Costs)**. *Suppose that $V(\theta, t) = \theta e^{-\theta t}$. If Assumption 2 holds and $\tilde{\theta}$ is the unique solution of $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) = 0$, then the optimal mechanism sells to customers of type $\theta \geq \underline{\theta}_h$ at time $\mathsf{t}_h(\theta)$ and at price $\mathfrak{p}(\theta) = V(\theta, \mathsf{t}_h(\theta)) - \int_{\underline{\theta}_h}^{\theta} e^{-z\mathsf{t}_h(z)}(1 - z\mathsf{t}_h(z))dz$ where $R(\theta, t)$, $\mathsf{t}_h(\theta)$, and $\underline{\theta}_h$ are defined in Equations (12), (13), and (11), respectively. Further, for any $\theta < \underline{\theta}_h$, $\varsigma(\theta) = 0$ and $\mathfrak{p}(\theta) = \infty$, and for $\theta \geq \underline{\theta}_h$, $\varsigma(\theta) = 1$.*

In Theorem 3, we assume that at $h = H_l$, the solution of $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) = 0$ is unique. In Lemma 21 in Appendix F.4, we show that if $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) = 0$ has a unique solution, then the solution of $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) = 0$ is also unique for any $h \in [H_l, H_h]$. We use this assumption to characterize the optimal mechanism when the holding cost is medium and large ($h \geq H_l$). This assumption ensures that $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) \leq 0$ for any $\theta < \underline{\theta}_h = \underline{\theta}_M$. We note that this holds when the virtual value of customers, that is, $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta)$ is increasing in the customer type θ. In this sense, this assumption resembles the standard assumption in the standard mechanism design literature where it is assumed that the virtual value of customers is monotone in their types.

In Appendix F.4, we provide sufficient conditions to satisfy this assumption. We show that if for any $\theta \leq \tilde{\theta}$, $\alpha'(\theta)$ is small enough, then this assumption holds. The aforementioned condition is satisfied for the uniform, exponential, and truncated normal distributions.

Theorem 3 shows that holding costs, similar to production costs, introduces a cut-off $\underline{\theta}_h$. That is, the mechanism only sells the item to customers with type greater than or equal to $\underline{\theta}_h$. However, unlike the production cost, the holding cost changes both the purchase time and the price. Moreover, the thresholds that divide customers into different groups, that is, θ_H^h and θ_L^h, also change with h.

Observe that when $h = 0$, we can recover the optimal mechanism with no holding cost, as presented in Theorem 1. To understand why, note that $\theta_H^0 = \theta_H$, $\theta_L^0 = \theta_L = \theta_0$, and $\underline{\theta}_L^0 = 0$. Furthermore, the FOC solution, $\mathsf{t}_f(\theta) = \mathsf{t}_g(\theta) = \frac{\theta + 2\alpha(\theta)}{\theta\alpha(\theta)}$. Note that when we increase the holding cost from 0 to $\epsilon > 0$ with $\epsilon \approx 0$, the purchase time remains the same for low-type and high-type customers, but medium-type customers purchase the item sooner; see Example 4.

The proof of Theorem 3 is presented in Appendix F.

Appendix B: Inventory Constraints

In this section, we study the impact of the inventory constraints on the structure of the optimal mechanism. Here, the firm has X units of the item at time zero, and would like to sell them to the customers. For convenience and without loss of generality, we normalize the market size to one, and correspondingly, we assume that $X \in [0,1]$; we refer to X as the normalized capacity. Then, given $X \in [0,1]$, the firm can only sell the item to X fraction of the customers. In the following proposition, we show that the firm prefers to sell to customers with type above a threshold.

PROPOSITION 4 **(Inventory Constraints)**. *Suppose that Assumptions 1 and 2 hold, the production and holing costs are zero. Let θ_X solve $\Pr[\theta \geq \theta_X] = X$.*[15] *Then, the optimal mechanism only sells to customers with type $\theta \geq \theta_X$ at time $\mathfrak{t}_g(\theta)$, given in Eq. (6), and at price $\mathfrak{p}(\theta) = \theta e^{-g(\theta)\mathfrak{t}_g(\theta)} - \int_{\theta_X}^{\theta} e^{-g(z)\mathfrak{t}_g(z)}\big(1 - \mathfrak{t}_g(z)g'(z)z\big)dz$. Furthermore, for $\theta < \theta_X$, $\varsigma(\theta) = 0$ and $\mathfrak{p}(\theta) = \infty$, and for $\theta \geq \theta_X$, $\varsigma(\theta) = 1$.*

Observe that the impact of the inventory constraint here is similar to that of the production cost in the sense that the inventory constraint does not change the allocation time of the customers who make a purchase. That is, customers of type $\theta \geq \theta_X$ purchase at the same time as they would have if there were no inventory constraint.

We only provide a sketch of the proof of Proposition 4, because this proof is similar to that of Theorem 2. To establish Proposition 4, we apply the weak duality theorem. Specifically, we characterize an upper bound on the profit of the firm by dualizing the inventory constraint, and we then show that the mechanism in Proposition 4 obtains the upper bound. Thus, it is optimal. To dualize the inventory constraints, we use the Lagrangian multiplier of $\lambda_X = R(\theta_X, \mathfrak{t}_g(\theta_X))$. Then, considering the fact that $R(\theta, \mathfrak{t}_g(\theta))$ is increasing in θ (see the proof of Theorem 2), we have $R(\theta, \mathfrak{t}_g(\theta)) - \lambda_X \geq 0$ for $\theta \geq \theta_X$, and $R(\theta, \mathfrak{t}_g(\theta)) - \lambda_X < 0$ for $\theta < \theta_X$, where $R(\theta, t)$ is the virtual value of customer of type θ at time t, and is defined in Eq. (4). This implies that the firm prefers to sell the item to customers with type $\theta > \theta_X$. Then, by definition of θ_X, we ensure that the inventory constraint is binding, and as a result, by the weak duality theorem, the mechanism in Proposition 4 is optimal.

Appendix C: Proof of Theorem 1

We start with presenting the proof of Lemma 3. That is, we first show that $\mathfrak{t}_g(\cdot)$, given in Eq. (6), and $u(\underline{\Theta}) = 0$ solve Problem RELAXED. Then, we show that this solution satisfies the interval condition. Therefore, it is an optimal solution of Problem OPT.

C.1. Proof of Lemma 3

Throughout the proof, to simplify the notation, we denote $u(\theta, \theta)$ with $u(\theta)$.

The proof has two parts. In the first part, we need to show that $\mathfrak{t}_g(\theta)$ and $u(\underline{\Theta}) = 0$ construct a feasible solution of Problem RELAXED. To this aim, we use our assumption that $g(\cdot)$ is log-concave; that is, $\frac{g'(\theta)}{g(\theta)}$ is decreasing in θ. In the second part, we show that the solution is optimal. To this end, we find an upper bound on the relaxed problem by dualizing the IR constraints. Then we show that $\mathfrak{t}_g(\theta), \varsigma(\theta) = 1$, $\theta \in [\underline{\Theta}, \bar{\Theta}]$, and $u(\underline{\Theta}) = 0$ achieves the upper bound.

First Part: Here, we show that the allocation rule $(\mathfrak{t}_g(\theta, \varsigma(\theta))$ and $u(\underline{\Theta}) = 0$ construct a feasible solution of Problem RELAXED. Here, $\varsigma(\theta) = 1$ for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$. To this aim, we show that $u(\theta) = \int_{\underline{\Theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}_g(z)}(1 -$

[15] If the solution is not unique, we choose the smallest one.

$g'(z)\mathfrak{t}_g(z)z)dz = 0$ for any $\theta \le \theta_L$, and $u(\theta) > 0$ otherwise. The former is easy to verify as $\mathfrak{t}_g(\theta) = \frac{1}{\theta g'(\theta)}$ for $\theta \le \theta_L$. To verify that $u(\theta) > 0$ for any $\theta > \theta_L$, we make use of the following lemma.

LEMMA 4. *For any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, we have $(1 - g'(\theta)\mathfrak{t}_g(\theta)\theta) \ge 0$.*

To show Lemma 4, we use the assumption that $g(\cdot)$ is log-concave; that is, $\frac{g'(\theta)}{g(\theta)}$ is decreasing.

In what follows, we do not show dependency to $\varsigma(\theta)$, as $\varsigma(\theta)$ is one for any θ and we simplify denote the allocation rule by $\mathfrak{t}_g(\cdot)$. Lemma 4 shows that $u(\theta) = \int_{\underline{\Theta}}^{\theta} e^{-g(z)\mathfrak{t}_g(z)}(1 - g'(z)\mathfrak{t}_g(z)z)dz \ge 0$ for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$. Next, we show that $(\mathfrak{t}_g(\cdot), u(\underline{\Theta}) = 0)$ is an optimal solution of Problem RELAXED. In the following, we find an upper bound for the optimal value of Problem RELAXED using the weak duality theorem. Then, we will show that $(\mathfrak{t}_g(\cdot), u(\underline{\Theta}) = 0)$ achieves the upper bound, thus it is optimal.

In the proof, with a slight abuse of notation, we denote the optimal value of Problem RELAXED with RELAXED.

Second Part: Here, we present an upper bound on RELAXED. For any allocation time $\mathfrak{t}(\cdot)$, and Lagrangian function $\lambda : [\underline{\Theta}, \bar{\Theta}] \to \mathbb{R}^+$, we define the following function.

$$L(\mathfrak{t}(\cdot), \lambda(\cdot), u(\underline{\Theta})) \ = \ \mathrm{E}[R(\theta, \mathfrak{t}(\theta))] + \int_{\underline{\Theta}}^{\bar{\Theta}} \lambda(z)u(z)dz - u(\underline{\Theta}) \ ,$$

where $u(\theta) = u(\underline{\Theta}) + \int_{\underline{\Theta}}^{\theta} e^{-g(z)\mathfrak{t}(z)}(1 - g'(z)\mathfrak{t}(z)z)dz$. Then, considering the fact that $\lambda(\theta) \ge 0$, for any $(\mathfrak{t}(\cdot), u(\underline{\Theta}))$ such that $u(\theta) = u(\underline{\Theta}) + \int_{\underline{\Theta}}^{\theta} e^{-g(z)\mathfrak{t}(z)}(1 - g'(z)\mathfrak{t}(z)z)dz \ge 0$, we have we have

$$\mathrm{E}[R(\theta, \mathfrak{t}(\theta)) - u(\underline{\Theta})] \ \le \ L(\mathfrak{t}(\cdot), \lambda(\cdot), u(\underline{\Theta}))$$

Therefore, for any $\lambda : [\underline{\Theta}, \bar{\Theta}] \to \mathbb{R}^+$,

$$\max_{(\mathfrak{t}(\cdot), u(\underline{\Theta})) \in \mathcal{T}} \{\mathrm{E}[R(\theta, \mathfrak{t}(\theta))] - u(\underline{\Theta})\} \ \le \ \max_{(\mathfrak{t}(\cdot), u(\underline{\Theta})) \in \mathcal{T}} \{L(\mathfrak{t}(\cdot), \lambda(\cdot), u(\underline{\Theta}))\} \ , \tag{18}$$

where

$$\mathcal{T} = \left\{ (\mathfrak{t}(\cdot), u(\underline{\Theta})) : u(\underline{\Theta}) \ge 0, \ \mathfrak{t}(\theta) \ge 0, \text{ and } u(\underline{\Theta}) + \int_{\underline{\Theta}}^{\theta} e^{-g(z)\mathfrak{t}(z)}(1 - g'(z)\mathfrak{t}(z)z)dz \ge 0 \text{ for any } \theta \in [\underline{\Theta}, \bar{\Theta}] \right\}$$

is the set of feasible solution. In the following, we will characterize an upper bound for Problem RELAXED by evaluating the r.h.s. of the above equation for a specific Lagrangian function, defined below.

$$\lambda_g(\theta) = \begin{cases} \left(f(\theta)(\frac{g(\theta)}{g'(\theta)} + \alpha(\theta))\right)' & \text{if } \theta \le \theta_L; \\ 0 & \text{otherwise;} \end{cases} \tag{19}$$

LEMMA 5. *If Assumptions 2 and 1 hold, for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, $\lambda_g(\theta) \ge 0$.*

In the proof of Lemma 5, we use our assumption that $g'(\theta) \ge 0$ and $\frac{g'(\theta)}{g(\theta)}$ is decreasing in θ.

Then, the proof is completed by the following claim.

Claim: With a slight abuse of notation, let $(t_\lambda(\cdot), u_\lambda) = \arg\max_{(\mathfrak{t}(\cdot), u(\underline{\Theta})) \in \mathcal{T}} \{L(\mathfrak{t}(\cdot), \lambda_g(\cdot), u(\underline{\Theta}))\}$. Then, $t_\lambda(\theta) = \mathfrak{t}_g(\theta)$ for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$ and $u_\lambda = 0$. Furthermore, $L(\mathfrak{t}_g(\cdot), \lambda_g(\cdot), u(\underline{\Theta}) = 0) = \mathrm{E}[R(\theta, \mathfrak{t}_g(\theta))]$, where the expectation is taken w.r.t. θ.

Proof of the Claim: By Eq. (19),

$$L(\mathfrak{t}(\cdot),\lambda_g(\cdot),u(\underline{\Theta})) = \int_{z=\theta_L}^{\bar{\Theta}} R(z,\mathfrak{t}(z))f(z)dz + \int_{\underline{\Theta}}^{\theta_L}\Big(R(z,\mathfrak{t}(z))f(z)+\lambda_g(z)u(z)\Big)dz - u(\underline{\Theta})\ , \tag{20}$$

where the second term can be rewritten as

$$\begin{aligned}
&\int_{\underline{\Theta}}^{\theta_L} R(z,\mathfrak{t}(z))f(z)dz + \int_{\underline{\Theta}}^{\theta_L} u(z)d\left(f(z)\left(\frac{g(z)}{g'(z)}+\alpha(z)\right)\right). \\
&= \int_{\underline{\Theta}}^{\theta_L} f(z)e^{-g(z)z}\big(z+\alpha(z)(1-g'(z)\mathfrak{t}(z)z)\big)dz \\
&+ u(z)f(z)\left(\frac{g(z)}{g'(z)}+\alpha(z)\right)\Big|_{\underline{\Theta}}^{\theta_L} - \int_{\underline{\Theta}}^{\theta_L} e^{-g(z)z}\big(1-g'(z)\mathfrak{t}(z)z\big)f(z)\left(\frac{g(z)}{g'(z)}+\alpha(z)\right)dz \\
&= \int_{\underline{\Theta}}^{\theta_L} f(z)e^{-g(z)\mathfrak{t}(z)}\left(z-(1-g'(z)t(z)z)\frac{g(z)}{g'(z)}\right)dz - u(\underline{\Theta})f(\underline{\Theta})\left(\frac{g(\underline{\Theta})}{g'(\underline{\Theta})}+\alpha(\underline{\Theta})\right),
\end{aligned} \tag{21}$$

where the second equation follows from Eq. (4) and integrating by part, and the last equation follows from definition of θ_L; that is, $\frac{g(\theta_L)}{g'(\theta_L)}+\alpha(\theta_L) = 0$. By plugging Eq. (21) in Eq. (20), we have

$$\begin{aligned}
L(\mathfrak{t}(\cdot),\lambda_g(\cdot),u(\underline{\Theta})) &= \int_{\theta_L}^{\bar{\Theta}} R(z,\mathfrak{t}(z))f(z)dz \\
&+ \int_{\underline{\Theta}}^{\theta_L} f(z)e^{-g(z)\mathfrak{t}(z)}\left(z-(1-g'(z)\mathfrak{t}(z)z)\frac{g(z)}{g'(z)}\right)dz - u(\underline{\Theta})f(\underline{\Theta})\frac{g(\underline{\Theta})}{g'(\underline{\Theta})}\ .
\end{aligned}$$

Considering that the coefficient of $u(\underline{\Theta})$, i.e., $-f(\underline{\Theta})\frac{g(\underline{\Theta})}{g'(\underline{\Theta})} \le 0$, to maximize the above equation, we set $u(\underline{\Theta}) = 0$. That is, $u_\lambda = 0$. Then,

$$\begin{aligned}
\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta})=0)\in\mathcal{T}}\{L(\mathfrak{t}(\cdot),\lambda_g(\cdot),u(\underline{\Theta})=0)\} \ \le\ & \int_{\theta_L}^{\bar{\Theta}} f(z)\max_{t\ge 0}\big\{R(z,t)\big\}dz \\
&+ \int_{\underline{\Theta}}^{\theta_L} f(z)\max_{t\ge 0}\left\{e^{-g(z)t}\left(z-(1-g'(z)tz)\frac{g(z)}{g'(z)}\right)\right\}dz\ .
\end{aligned}$$

In Lemma 6, stated below, we show that for any $\theta \ge \theta_L$, we get $\arg\max_{t\ge 0}\big\{R(z,t)\big\} = \mathfrak{t}_g(z)$. Then the result follows because for any $z \le \theta_L$, we have

$$\arg\max_{t\ge 0}\left\{e^{-g(z)t}\left(z-(1-g'(z)tz)\frac{g(z)}{g'(z)}\right)\right\} \ =\ \frac{1}{g'(z)z}\ .$$

Lemma 6. *For any $\theta \ge \theta_L$, we have* $\arg\max_{t\ge 0}\{R(\theta,t)\} = \mathfrak{t}_g(\theta)$.

C.2. Optimal Solution of Problem OPT

In this section, we show the optimal solution of the relaxed problem satisfies the interval conditions. This implies the solution given in Theorem 1 is an optimal solution of Problem OPT. Note that the interval conditions hold if for any $\hat{\theta},\theta \in [\underline{\Theta},\bar{\Theta}]$ such that $\hat{\theta} \le \theta$:

$$\begin{aligned}
\int_{\hat{\theta}}^{\theta} A_g(z,\mathfrak{t}_g(\hat{\theta}))dz \ &\le\ \int_{\hat{\theta}}^{\theta} A_g(z,\mathfrak{t}_g(z))dz\ , \\
\int_{\hat{\theta}}^{\theta} A_g(z,\mathfrak{t}_g(z))dz \ &\le\ \int_{\hat{\theta}}^{\theta} A_g(z,\mathfrak{t}_g(\theta))dz\ ,
\end{aligned}$$

where $A_g(z,t) = e^{-g(z)t}(1-g'(z)tz)$. To this aim, we show that for any $z \ge \hat{\theta}$, $A_g(z,\mathfrak{t}_g(\hat{\theta})) \le A_g(z,\mathfrak{t}_g(z))$ and for any $z < \theta$, $A_g(z,\mathfrak{t}_g(z)) \le A_g(z,\mathfrak{t}_g(\theta))$.

We make use of the following lemma.

LEMMA 7. *The allocation rule $\mathfrak{t}_g(\cdot)$, given in Eq. (6), is decreasing.*

We start by showing that $A_g(z, \mathfrak{t}_g(\hat{\theta})) \leq A_g(z, \mathfrak{t}_g(z))$ for any $z \geq \hat{\theta}$. We consider two cases: 1- $(1 - g'(z)\mathfrak{t}_g(\hat{\theta})z) \leq 0$ and 2- $(1 - g'(z)\mathfrak{t}_g(\hat{\theta})z) > 0$.

Assume that $(1 - g'(z)\mathfrak{t}_g(\hat{\theta})z) \leq 0$. Then,

$$e^{-g(z)\mathfrak{t}_g(\hat{\theta})}(1 - g'(z)\mathfrak{t}_g(\hat{\theta})z) \;\leq\; 0 \;\leq\; e^{-g(z)\mathfrak{t}_g(z)}(1 - g'(z)\mathfrak{t}_g(z)z) \;,$$

where the second inequality follows from Lemma 4 given in the proof of Lemma 3. There, we show that for any $z \in [\underline{\Theta}, \bar{\Theta}]$, we have $e^{-g(z)\mathfrak{t}_g(z)}(1 - g'(z)\mathfrak{t}_g(z)z) \geq 0$. The above equation implies that $A_g(z, \mathfrak{t}_g(\hat{\theta})) \leq A_g(z, \mathfrak{t}_g(z))$.

Now, assume that $(1 - g'(z)\mathfrak{t}_g(\hat{\theta})z) > 0$. By Lemma 7, $\mathfrak{t}_g(\cdot)$ is decreasing. This leads to

$$(1 - g'(z)\mathfrak{t}_g(z)z) \;\geq\; (1 - g'(z)\mathfrak{t}_g(\hat{\theta})z) \;\geq\; 0, \quad \text{and} \quad e^{-g(z)\mathfrak{t}_g(z)} \;\geq\; e^{-g(z)\mathfrak{t}_g(\hat{\theta})}.$$

By the above equations, we have $A_g(z, \mathfrak{t}_g(\hat{\theta})) \;\leq\; A_g(z, \mathfrak{t}_g(z))$.

Next, we will verify that $A_g(z, \mathfrak{t}_g(z)) \;\leq\; A_g(z, \mathfrak{t}_g(\theta))$, for $z < \theta$. Since $\mathfrak{t}_g(\cdot)$ is decreasing, for any $z < \theta$, we get

$$0 \;\leq\; (1 - g'(z)\mathfrak{t}_g(z)z) \;\leq\; (1 - g'(z)\mathfrak{t}_g(\theta)z), \quad \text{and} \quad e^{-g(z)\mathfrak{t}_g(z)} \;\leq\; e^{-g(z)\mathfrak{t}_g(\theta)},$$

where the first inequality holds because as we showed in Lemma 3, for any $z \in [\underline{\Theta}, \bar{\Theta}]$, we have $e^{-g(z)\mathfrak{t}_g(z)}(1 - g'(z)\mathfrak{t}(z)z) \geq 0$. The above equations lead to $A_g(z, \mathfrak{t}_g(z)) \leq A_g(z, \mathfrak{t}_g(\theta))$.

Appendix D: Proof of Results in Section 5

D.1. Proof of Proposition 1

To show the result, we use Lemma 2. That is, we maximize the virtual profit subject to IC and IR constraints. We start with ignoring both IC and IR constraints and for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, we characterize $\arg\max_{t \in [0,T]} R(\theta, t)$. We show that for any $\theta \geq \theta_L^T$, $\arg\max_{t \in [0,T]} R(\theta, t) = \mathfrak{t}_T(\theta)$ and for $\theta < \theta_L^T$, $\arg\max_{t \in [0,T]} R(\theta, t) = T$ and $R(\theta, T) < 0$. To complete the proof, we show that the mechanism that only sells the item to customer of type $\theta > \theta_L^T$ at time $\mathfrak{t}_T(\theta)$ and price $\mathfrak{p}(\theta)$, defined in the proposition, is IR and IC. Thus, it is optimal.

First of all, using the proof of Theorem 1, it is easy to show that $\arg\max_{t \in [0,T]} R(\theta, t) = \mathfrak{t}_T(\theta)$ for $\theta < \theta_L^T$. Thus, in the following, we show that for $\theta < \theta_L^T$, $\arg\max_{t \in [0,T]} R(\theta, t) = T$ and $R(\theta, T) < 0$. Observe that for any $\theta < \theta_L^T$, $\mathfrak{t}_f(\theta) = \arg\max_{t \geq 0} R(\theta, t) \geq T$. This follows from the fact that $\mathfrak{t}_f(\theta_H^T) = T$ and $\mathfrak{t}_f(\cdot)$ is decreasing; see Lemma 7. Then, as we show in the proof of Lemma 6, $R(\theta, t)$ has an inverted u-shape in t. This implies that the unique maximum of $R(\theta, t)$, $\mathfrak{t}_f(\theta)$, is greater than T, and as a result, $\arg\max_{t \in [0,T]} R(\theta, t) = T$, for any $\theta < \theta_L^T$. Next, we show that $R(\theta, T) < 0$ when $\theta < \theta_L^T$. To do so, we confirm that $e^{g(\theta)T} R(\theta, T)$ is increasing in θ. Then, the result follows because by definition, $R(\theta_L^T, T) = 0$.

The derivative of $e^{g(\theta)T} R(\theta, T)$ w.r.t. θ is given by

$$\frac{\partial(e^{g(\theta)T} R(\theta, T))}{\partial\theta} \;=\; \frac{\partial(\theta + \alpha(\theta)(1 - g'(\theta)\theta T))}{\partial\theta} \;=\; 1 + \alpha'(\theta)(1 - g'(\theta)\theta T) - \alpha(\theta)(g'(\theta)\theta)' T \;>\; 0\,,$$

where the inequality holds because by Assumption 1, $g'(\theta)\theta$ is increasing. By the monotonicity of $g'(\theta)\theta$ and the fact that $(1 - g'(\theta_L^T)\theta_L^T T) \geq 0$, we have $(1 - g'(\theta)\theta T) > 0$ for any $\theta < \theta_L$.

So far, we characterized $\arg\max_{t \in [0,T]} R(\theta, t)$. To complete the proof, we need to show the mechanism is IR and IC. We start with showing the mechanism is IR. Given the payment rule of the mechanism, the mechanism is IR if $1 - g'(\theta)\theta \mathfrak{t}_T(\theta) \geq 0$ for any $\theta \geq \theta_L^T$. To show this, we consider the following cases:

- Case 1 ($\theta \geq \theta_H$): For this range of θ, $1-g'(\theta)\theta\mathsf{t}_T(\theta)=1>0$.
- Case 2 ($\theta \in [\theta_H^T, \theta_H]$:) For this range of θ, $1-g'(\theta)\theta\mathsf{t}_T(\theta)=1-g'(\theta)\theta\mathsf{t}_f(\theta)$. To show the result, we make use of Lemma 4, where we have that $1-g'(\theta)\theta\mathsf{t}_f(\theta)\geq 0$ for any $\theta\in[\theta_L,\theta_H]$. The result then follows by the fact that $\theta_H^T \geq \theta_L$.
- Case 3 ($\theta \in [\theta_L^T, \theta_H^T]$:) For this range of θ, $1-g'(\theta)\theta\mathsf{t}_T(\theta)=1-g'(\theta)\theta T$. By Case 2 and the fact that $\mathsf{t}_f(\theta_H^T)=T$, we have $1-g'(\theta_H^T)\theta_H^T T\geq 0$. Then, the monotonicity of $g'(\theta)\theta$ implies that $1-g'(\theta)\theta T\geq 0$ for any $\theta\in[\theta_L^T,\theta_H^T]$.

Finally, the mechanism is IC because for any $\theta\geq\theta_L^T$, $1-g'(\theta)\theta\mathsf{t}_T(\theta)\geq 0$ and $\mathsf{t}_T(\cdot)$ is decreasing; see the proof of Theorem 1 for details.

D.2. Proof of Proposition 2

First observe that mechanism $\mathcal{M}_T$ is IR and IC because $\mathsf{t}_T(\cdot)$ is decreasing and $1-g'(\theta)\theta\mathsf{t}_T(\theta)\geq 0$ for any $\theta\geq\theta_L^T$; see the proof of Proposition 1. Next, we show that mechanism $\mathcal{M}_T$ is approximately optimal. To this aim, we dualize the IR constraints to construct an upper bound on the profit of the optimal mechanism and we then compare the profit of mechanism $\mathcal{M}_T$ with the upper bound.

To dualize the IR constraints, we use $\lambda_g(\cdot)$, defined in Eq. (19). Following the proof of Theorem 1, one can show that

$$\begin{aligned}\mathsf{Rev}_{opt}-\mathsf{Rev}_{\mathcal{M}_T} &\leq \int_{\underline{\Theta}}^{\theta_L^T}\max_{t\in[0,T]}\left\{e^{-g(z)t}\left(z-(1-g'(z)tz)\frac{g(z)}{g'(z)}\right)f(z)\right\}dz\\ &\leq \int_{\underline{\Theta}}^{\theta_L^T} e^{-g(z)T}\left(z-(1-g'(z)Tz)\frac{g(z)}{g'(z)}\right)f(z)dz \;\leq\; \int_{\underline{\Theta}}^{\theta_L^T} z e^{-\frac{g(z)}{g'(z)z}} f(z)dz\,, \end{aligned} \tag{22}$$

where the equality follows because $\arg\max_{t\in[0,T]}\left\{e^{-g(z)t}\left(z-(1-g'(z)tz)\frac{g(z)}{g'(z)}\right)\right\}=T$ and the second inequality holds because $\arg\max_{t\geq 0}\left\{e^{-g(z)t}\left(z-(1-g'(z)tz)\frac{g(z)}{g'(z)}\right)\right\}=\frac{1}{g'(z)z}$. To get the desired bound, we show that $z\mapsto ze^{-\frac{g(z)}{g'(z)z}}$ is increasing. This implies that $\mathsf{Rev}_{opt}-\mathsf{Rev}_{\mathcal{M}_T}\leq\theta_L^T\exp\left(-\frac{g(\theta_L^T)}{g'(\theta_L^T)\theta_L^T}\right)F(\theta_L^T)$.

The derivative of $ze^{-\frac{g(z)}{g'(z)z}}$ w.r.t. z is given by

$$e^{-\frac{g(z)}{g'(z)z}}\left(1-z\frac{(g'(z))^2z-(g'(z)z)'g(z)}{(g'(z)z)^2}\right) \;=\; z\frac{(g'(z)z)'g(z)}{(g'(z)z)^2}\geq 0\,,$$

where the inequality holds because by Assumption 1, $g(z)\geq 0$ and $g'(z)z$ is increasing.

D.3. Proof of Theorem 2

We need to find an optimal solution of Problem OPT where the objective function is replaced with $\mathrm{E}\left[\varsigma(\theta)(R(\theta,\mathsf{t}(\theta))-c)-u(\underline{\Theta},\underline{\Theta})\right]$.

The proof of this theorem is similar to that of Theorem 1. We first relax the problem by ignoring the interval conditions. We will show that in an optimal solution of the relaxed problem, $\varsigma(\theta)=1$ for $\theta\geq\theta_c$, and is zero otherwise. Here, θ_c solves $R(\theta_c,\mathsf{t}_g(\theta_c)) = c$. We further show that for customers with type $\theta\geq\theta_c$, the optimal allocation time is $\mathsf{t}_g(\theta)$.

To show the result, we make use of Lemma 8, stated at the end of this section, where we show that $R(\theta,\mathsf{t}_g(\theta))$ is an increasing function of $\theta\in[\underline{\Theta},\bar{\Theta}]$. Then, to complete the proof, we show that the optimal solution of the relaxed problem satisfies the envelope conditions. This part of the proof is similar to that of Theorem 1. Thus, it is omitted.

LEMMA 8. *$R(\theta, \mathsf{t}_g(\theta))$ is increasing in θ where $R(\theta, t)$ and $\mathsf{t}_g(\theta)$ are defined in Equations (4) and (6).*

Proof of Lemma 8 By definition, we have

$$R(\theta, \mathsf{t}_g(\theta)) = \begin{cases} \theta + \alpha(\theta) & \text{if } \theta \geq \theta_H; \\ R(\theta, \mathsf{t}_f(\theta)) & \text{if } \theta \in [\theta_L, \theta_H]; \\ e^{-\frac{g(\theta)}{g'(\theta)\theta}}\theta & \text{if } \theta \in [\underline{\Theta}, \theta_L]; \end{cases}$$

$R(\theta, \mathsf{t}_g(\theta))$ is obviously increasing in θ when $\theta \geq \theta_H$. Furthermore, given that $\theta \leq \theta_L$, then $R(\theta, \mathsf{t}_g(\theta))$ is also increasing. To see why note that for any $\theta \leq \theta_L$,

$$\frac{dR(\theta, \mathsf{t}_g(\theta))}{d\theta} \;=\; \frac{d(e^{-\frac{g(\theta)}{g'(\theta)\theta}}\theta)}{d\theta} \;=\; e^{-\frac{g(\theta)}{g'(\theta)\theta}} \frac{\theta g(\theta)(g'(\theta)\theta)'}{(g'(\theta)\theta)^2} \;\geq\; 0\,,$$

where the inequality holds because $g'(\theta)\theta$ is increasing. Furthermore, observe that $R(\theta, \mathsf{t}_g(\theta))$ is a continuous function of θ because $\mathsf{t}_g(\theta)$ is continuous. Thus, it suffices to show that $R(\theta, \mathsf{t}_g(\theta))$ is increasing in $\theta \in [\theta_L, \theta_H]$.

Recall that $\mathsf{t}_g(\theta) = \mathsf{t}_f(\theta)$ for $\theta \in [\theta_L, \theta_H]$. That is, $\mathsf{t}_g(\theta)$ is the FOC solution. Thus, by the Envelope theorem, the derivative of $R(\theta, \mathsf{t}_f(\theta))$ w.r.t. θ is given by

$$\begin{aligned} \frac{\partial\big(R(\theta, \mathsf{t}_f(\theta))\big)}{\partial\theta} \;&=\; e^{-g(\theta)\mathsf{t}_f(\theta)}\Big(-g'(\theta)\mathsf{t}_f(\theta)\big(\theta + \alpha(\theta)(1 - g'(\theta)\mathsf{t}_f(\theta)\theta)\big) \\ &+\; 1 + \alpha'(\theta)(1 - g'(\theta)\mathsf{t}_f(\theta)\theta)) - \mathsf{t}_f\alpha(\theta)(g'(\theta)\theta)'\Big) \\ &=\; e^{-g(\theta)\mathsf{t}_f(\theta)}\Big((1 - g'(\theta)\mathsf{t}_f(\theta)\theta)\big(\alpha'(\theta) - \alpha(\theta)\big) - \mathsf{t}_f(\theta)\alpha(\theta)(g'(\theta)\theta)'\Big) \;\geq\; 0\,, \end{aligned}$$

where the inequality holds because, as we show in Lemma 4, $(1 - g'(\theta)\mathsf{t}_f(\theta)\theta) \geq 0$ for any $\theta \in [\theta_L, \theta_H]$, and by Assumption 1, $g'(\theta)\theta$ is increasing in θ. □

D.4. Proof of Proposition of 3

Recall that $\mathsf{Rev}^1_{opt} = \mathsf{Rev}(\mathbf{q})$ and $\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}) = \mathsf{Rev}(\tilde{\mathbf{q}})$. Therefore, any feasible solution of Problem $\mathsf{Rev}(\mathbf{q})$ is a feasible solution of Problem $\mathsf{Rev}(\tilde{\mathbf{q}})$ and vice versa. In fact, for any feasible solution of Problem Rev, we have

$$\begin{aligned} \mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}}) \;&=\; \sum_{k\in[K]} \varsigma_k p_k (q_k - \tilde{q}_k) \\ \Rightarrow \quad \mathsf{Rev}(\mathbf{q}) - \mathrm{E}[\mathsf{Rev}(\tilde{\mathbf{q}}) \mid M] &= \sum_{k\in[K]} \varsigma_k p_k \;\mathrm{E}[(q_k - \tilde{q}_k) \mid M] \;=\; 0\,, \end{aligned}$$

where the above equation is the desired result.

Next, we prove claim (10). Let x_{ik} be 1 if customer i is of type k and zero otherwise. Note that for any $i \in [M]$, we have $\sum_{k\in[K]} x_{ik} = 1$. Then, for any feasible solution of Problem $\mathsf{Rev}(\tilde{\mathbf{q}})$, we get

$$\begin{aligned} \mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}}) \;&=\; \sum_{k\in[K]} \varsigma_k p_k (q_k - \tilde{q}_k) \;=\; \sum_{k\in[K]} \varsigma_k p_k \Big(q_k - \frac{1}{M}\sum_{i\in[M]} x_{ik}\Big) \\ &=\; \sum_{k\in[K]} \varsigma_k p_k \Big(\mathrm{E}[x_{ik}] - \frac{1}{M}\sum_{i\in[M]} x_{ik}\Big) \;=\; \sum_{k\in[K]} \varsigma_k \frac{p_k}{M} \sum_{i\in[M]} \Big(\mathrm{E}[x_{ik}] - x_{ik}\Big) \\ &=\; \frac{1}{M}\sum_{i\in[M]}\sum_{k\in[K]} \varsigma_k p_k \big(\mathrm{E}[x_{ik}] - x_{ik}\big)\,. \end{aligned}$$

Define $y_i = \sum_{k\in[K]} \varsigma_k p_k \big(\mathrm{E}[x_{ik}] - x_{ik}\big)$. Note that $\mathrm{E}[y_i] = 0$, $i \in [M]$. For any M, let $\epsilon = \bar{\Theta}\sqrt{\frac{\log(M)}{2M}}$ and define the following event

$$\mathcal{A} = \Big\{\frac{1}{M}\sum_{i\in[M]} y_i - \mathrm{E}[y_i] \geq \epsilon\Big\}$$

We will show that $\Pr\left(\mathcal{A} \mid M\right) \le \frac{1}{M}$. This implies that with high probability $\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}}) \le \bar{\Theta}\sqrt{\frac{\log(M)}{2M}}$. Then, we get

$$
\begin{aligned}
\mathrm{E}\Big[\mathrm{E}\left[|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})|\big|M\right]\Big] \;&=\; \mathrm{E}\Big[\mathrm{E}\left[|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})|\mathbb{1}\{\mathcal{A}\} + |\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})|\mathbb{1}\{\mathcal{A}^c\}\big|M\right]\Big] \\
&\le\; \mathrm{E}\left[\frac{\bar{\Theta}}{M} + \Theta\sqrt{\frac{\log(M)}{2M}}\right] \;\le\; \Theta\sqrt{\frac{\log(n)}{2n}} + \frac{\bar{\Theta}}{n}\,, \qquad (23)
\end{aligned}
$$

where $\mathcal{A}^c$ is the complement of event $\mathcal{A}$. To obtain the first inequality, we used the fact that (i) under event $\mathcal{A}^c$, $|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})| \le \epsilon = \Theta\sqrt{\frac{\log(M)}{2M}}$, (ii) $|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})| \le \bar{\Theta}$, and (iii) $\Pr(\mathcal{A}|M) \le \frac{1}{M}$. Further, the second inequality, which is the desired result, holds because $M \ge n$ a.s.

To complete the proof, we show that $\Pr\left(\mathcal{A} \mid M\right) \le \frac{1}{M}$. To do so, we use the Azuma-Hoeffding inequality:

$$
\Pr\left(\mathcal{A} \mid M\right) \;=\; \Pr\left(\frac{1}{M}\sum_{i\in[M]} y_i - \mathrm{E}[y_i] \ge \epsilon \mid M\right) \;\le\; \exp\left(-\frac{2M\epsilon^2}{\max_{i\in[M]}\ |y_i|^2}\right) \qquad (24)
$$

In the following, we present an upper bound on $\max_{i\in[M]}|y_i|^2$. Let $k' \in [K]$ be the type of customer i; that is $x_{ik'} = 1$ and $x_{ik} = 0$ for $k \in k'$. Then,

$$
\begin{aligned}
y_i \;&=\; \sum_{k\in[K]} \varsigma_k p_k\big(\mathrm{E}[x_{ik}] - x_{ik}\big) \;=\; \sum_{k\in[K]} \varsigma_k p_k q_k - \varsigma_{k'} p_{k'} \\
\Rightarrow \quad |y_i| \;&\le\; \max_{k\in[K]}\ p_k \;\le\; \max_{k\in[K]}\ \theta_k \;=\; \bar{\Theta}\,, \qquad (25)
\end{aligned}
$$

where the last inequality follows from the IR constraints. Applying (25) in (24), we get, $\Pr\left(\mathcal{A}|M\right) \;\le\; \exp\left(-\frac{2M\epsilon^2}{\bar{\Theta}^2}\right) \;=\; \frac{1}{M}$.

Appendix E: Proofs and Additional Result for Sections 3 and 4

Proof of Lemma 1 The proof falls naturally into two parts. In the first part, we show that in an incentive-compatible mechanism conditions in Equations (1) and (2) hold. In the second part, we show that if Equations (1) and (2) hold, the mechanism is IC.

First Part: Consider a customer with type θ that reports $\hat{\theta}$. Without loss of generality, we assume that $\theta \ge \hat{\theta}$. Then, the utility of the customer is given by $u(\theta,\hat{\theta}) = \varsigma(\hat{\theta}).\big(V(\theta,\mathfrak{t}(\hat{\theta})) - \mathfrak{p}(\hat{\theta})\big)$. Incentive compatibility implies that

$$
\begin{aligned}
u(\theta,\theta) - u(\hat{\theta},\hat{\theta}) \;&\le\; u(\theta,\theta) - u(\hat{\theta},\theta) \;=\; \varsigma(\theta).\big(V(\theta,\mathfrak{t}(\theta)) - V(\hat{\theta},\mathfrak{t}(\theta))\big) \\
&=\; \int_{z=\hat{\theta}}^{\theta} \varsigma(\theta)\partial_1 V(z,\mathfrak{t}(\theta))dz \;=\; \int_{z=\hat{\theta}}^{\theta} \varsigma(\theta)e^{-g(z)\mathfrak{t}(\theta)}(1 - g'(z)\mathfrak{t}(\theta)z)dz\ , \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
u(\theta,\theta) - u(\hat{\theta},\hat{\theta}) \;&\ge\; u(\theta,\hat{\theta}) - u(\hat{\theta},\hat{\theta}) \;=\; \varsigma(\hat{\theta}).\big(V(\theta,\mathfrak{t}(\hat{\theta})) - V(\hat{\theta},\mathfrak{t}(\hat{\theta}))\big) \\
&=\; \int_{z=\hat{\theta}}^{\theta} \varsigma(\hat{\theta})\partial_1 V(z,\mathfrak{t}(\hat{\theta}))dz \;=\; \int_{z=\hat{\theta}}^{\theta} \varsigma(\hat{\theta})e^{-g(z)\mathfrak{t}(\hat{\theta})}(1 - g'(z)\mathfrak{t}(\hat{\theta})z)dz\ , \qquad (27)
\end{aligned}
$$

where $\partial_1 V(\theta,t) = \frac{\partial V(\theta,t)}{\partial\theta}$. Then, using the above equations, we have

$$
\begin{aligned}
\frac{\int_{z=\hat{\theta}}^{\theta} \varsigma(\hat{\theta})e^{-g(z)\mathfrak{t}(\hat{\theta})}(1 - g'(z)\mathfrak{t}(\hat{\theta})z)dz}{\theta - \hat{\theta}} &\le \frac{u(\theta,\theta) - u(\hat{\theta},\hat{\theta})}{\theta - \hat{\theta}}\ , \\
\frac{\int_{z=\hat{\theta}}^{\theta} \varsigma(\theta)e^{-g(z)\mathfrak{t}(\theta)}(1 - g'(z)\mathfrak{t}(\theta)z)dz}{\theta - \hat{\theta}} &\ge \frac{u(\theta,\theta) - u(\hat{\theta},\hat{\theta})}{\theta - \hat{\theta}}\ .
\end{aligned}
$$

Finally by taking the limit as $\hat{\theta} \to \theta^-$, we get Eq. (1).[16] Then, by Equations (1), (26), and (27), we get the second condition, given in Eq. (2).

Second Part: Here, we will show that if in a mechanism Equations (1) and (2) hold, the mechanism is IC. By Eq. (1),

$$\begin{aligned} u(\theta,\theta) - u(\hat{\theta},\hat{\theta}) &= \int_{z=\hat{\theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)}(1-g'(z)\mathfrak{t}(z)z)dz \ \geq \ \int_{z=\hat{\theta}}^{\theta} \varsigma(\hat{\theta}) e^{-g(z)\mathfrak{t}(\hat{\theta})}(1-g'(z)\mathfrak{t}(\hat{\theta})z)dz \\ &= \varsigma(\hat{\theta})\left(\theta e^{-g(\theta)\mathfrak{t}(\hat{\theta})} - \hat{\theta} e^{-g(\hat{\theta})\mathfrak{t}(\hat{\theta})}\right) \ = \ u(\theta,\hat{\theta}) - u(\hat{\theta},\hat{\theta}) \ , \end{aligned} \tag{28}$$

where the inequality follows from Eq. (2). The final equation implies that $u(\theta,\theta) \geq u(\theta,\hat{\theta})$. Similarly,

$$\begin{aligned} u(\theta,\theta) - u(\hat{\theta},\hat{\theta}) &= \int_{z=\hat{\theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)}(1-g'(z)\mathfrak{t}(z)z)dz \ \leq \ \int_{z=\hat{\theta}}^{\theta} \varsigma(\theta) e^{-g(z)\mathfrak{t}(\theta)}(1-g'(z)\mathfrak{t}(\theta)z)dz \\ &= \varsigma(\theta)\left(\theta e^{-g(\theta)\mathfrak{t}(\theta)} - \hat{\theta} e^{-g(\hat{\theta})\mathfrak{t}(\theta)}\right) \ = \ u(\theta,\theta) - u(\hat{\theta},\theta) \ , \end{aligned} \tag{29}$$

That is, $u(\hat{\theta},\hat{\theta}) \geq u(\hat{\theta},\theta)$. The above equation along with Eq. (29) imply that the mechanism is IC.

□

Proof of Lemma 2 Consider any IC mechanism. Then, the expected profit of the firm from selling to a single customer is given by

$$\mathrm{E}[\varsigma(\theta)(\mathfrak{p}(\theta) - h\mathfrak{t}(\theta) - c)] \ = \ \mathrm{E}\left[\varsigma(\theta)\left(\theta e^{-g(\theta)\mathfrak{t}(\theta)} - u(\theta,\theta) - h\mathfrak{t}(\theta) - c\right)\right], \tag{30}$$

where the expectation is with respect to the customer type θ. In the following, we compute $\mathrm{E}[u(\theta,\theta)]$. By Lemma 1

$$\begin{aligned} \mathrm{E}[u(\theta,\theta)] &= u(\underline{\Theta},\underline{\Theta}) + \int_{\theta=\underline{\Theta}}^{\bar{\Theta}} dF(\theta) \int_{z=\underline{\Theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)}(1-g'(z)\mathfrak{t}(z)z)dz \\ &= u(\underline{\Theta},\underline{\Theta}) + \int_{z=\underline{\Theta}}^{\bar{\Theta}} \int_{\theta=z}^{\bar{\Theta}} dF(\theta) \varsigma(z) e^{-g(z)\mathfrak{t}(z)}(1-g'(z)\mathfrak{t}(z)z)dz \\ &= u(\underline{\Theta},\underline{\Theta}) + \int_{z=\underline{\Theta}}^{\bar{\Theta}} (1-F(z))\varsigma(z) e^{-g(z)\mathfrak{t}(z)}(1-g'(z)\mathfrak{t}(z)z)dz \\ &= u(\underline{\Theta},\underline{\Theta}) + \mathrm{E}\left[\frac{(1-F(\theta))}{f(\theta)} \varsigma(\theta) e^{-g(\theta)\mathfrak{t}(\theta)}(1-g'(\theta)\mathfrak{t}(\theta)\theta)\right] \ . \end{aligned} \tag{31}$$

By replacing Eq. (31) in Eq. (30), we get the desired result. □

E.1. Lower Bound on the Profit Gain of the Dynamic Pricing Policy

Here, we compare the profit of the optimal mechanism given in Theorem 1 with that of the optimal FP policy when $g(\theta) = \theta^a$, $a \geq 0$. We note that the FP policy is optimal when $a = 0$. Under the FP policy, the firm only sells to customers with type $\theta \geq \theta_0$ at time zero by posting a fixed price of θ_0 where θ_0 solves $\theta_0 + \alpha(\theta_0) \ = \ 0$.

LEMMA 9 **(Lower Bound on the Profit Gain of DP)**. *Suppose that $g(\theta) = \theta^a$. Then, we have*

$$\frac{\mathsf{Rev}_{opt} - \mathsf{Rev}_f}{\mathsf{Rev}_f} \ \geq \ e^{-\frac{1}{a}} \frac{\mathrm{E}\left[\theta \mathbb{1}\{\theta \leq \theta_0\}\right]}{\theta_0(1-F(\theta_0))} \,,$$

where Rev_f and Rev_{opt} are the expected profit of the firm under the FP policy and optimal DP policy, respectively, and θ_0 solves $\theta_0 + \alpha(\theta_0) = 0$.

[16] By Theorem 2 in Milgrom and Segal (2002), to satisfy Eq. (1), $\mathfrak{t}(\cdot)$ is not required to be continues.

Proof of Lemma 9 is given at the end of this section.

Assume that the customer type θ is drawn from the uniform distribution in the range of $[0,1]$; that is, $\theta \sim U(0,1)$. Then, $\theta_0 = \frac{1}{2}$, and $\mathsf{Rev}_f = \frac{1}{4}$. Lemma 9 implies that the firm can increase its profit by more than $100.e^{-\frac{1}{a}}.\frac{\int_0^{\theta_0} x dx}{\theta_0(1-\theta_0)} = 50.e^{-\frac{1}{a}}$ percent by using DP. The profit gain of the DP (in %) for $a = 0.5, 1, 1.5$, and 2 is at least 6.8, 18.4, 25.7, and 30.3, respectively.

Proof of Lemma 9 By Lemma 2, under the FP policy ($a = 0$),

$$\mathsf{Rev}_f \ = \ \mathrm{E}\left[\varsigma(\theta) R(\theta, 0)\right] \ = \ \mathrm{E}\left[(\theta + \alpha(\theta))\mathbb{1}\{\theta \geq \theta_0\}\right] \ , \tag{32}$$

where the last inequality holds because in the FP policy, $\mathfrak{t}(\theta) = 0$ for $\theta \geq \theta_0$ and $\varsigma(\theta) = 1$ only for customers of type $\theta \geq \theta_0$.

Similarly, under the mechanism described in Theorem 1, we have

$$\begin{aligned}\mathsf{Rev}_{opt} \ &= \mathrm{E}\left[R(\theta, \mathfrak{t}_g(\theta))\right] \ = \ \mathrm{E}\left[e^{-g(\theta)\mathfrak{t}_g(\theta)}\left(\theta + \alpha(\theta)(1 - g'(\theta)\mathfrak{t}_g(\theta)\theta)\right)\right] \ = \ \mathrm{E}\left[e^{-\theta^a \mathfrak{t}_g(\theta)}\left(\theta + \alpha(\theta)(1 - a\theta^a \mathfrak{t}_g(\theta))\right)\right] \\ &= \mathrm{E}\Big[(\theta + \alpha(\theta)) \times \mathbb{1}\{\theta \geq \theta_H\} + e^{-\theta^a \mathfrak{t}_g(\theta)}\left(\theta + \alpha(\theta)(1 - a\theta^a \mathfrak{t}_g(\theta))\right) \times \mathbb{1}\{\theta \in (\theta_L, \theta_H)\} + e^{-\frac{1}{a}}\theta \times \mathbb{1}\{\theta \leq \theta_L\}\Big],\end{aligned}$$

where the second equation holds because $g(\theta) = \theta^a$ and the third equation follows from the time of purchase in the optimal DP policy, i.e., $\mathfrak{t}_g(\cdot)$, which is given in Eq. (6).

We consider the following two cases.

- $\theta_L \leq \theta_0$: We start with rewriting Rev_{opt} as follows.

$$\begin{aligned}\mathsf{Rev}_{opt} \ = \ \mathrm{E}\Big[&(\theta + \alpha(\theta)) \times \mathbb{1}\{\theta \geq \theta_H\} + \ e^{-\theta^a \mathfrak{t}_g(\theta)}\left(\theta + \alpha(\theta)(1 - a\theta^a \mathfrak{t}_g(\theta))\right) \times \mathbb{1}\{\theta \in (\theta_0, \theta_H)\} \\ &+ e^{-\theta^a \mathfrak{t}_g(\theta)}\left(\theta + \alpha(\theta)(1 - a\theta^a \mathfrak{t}_g(\theta))\right) \times \mathbb{1}\{\theta \in (\theta_L, \theta_0)\} + \ e^{-\frac{1}{a}}\theta \times \mathbb{1}\{\theta \leq \theta_L\}\Big].\end{aligned}$$

In the above equation, we broke down the middle term of Rev_{opt} into two terms. Considering that $\theta \in (\theta_L, \theta_H)$, $\mathfrak{t}_g(\theta)$ is the FOC solution, i.e., $\mathfrak{t}_g(\theta) = \arg\max_{t \geq 0}\{R(\theta, t)\}$, we get

$$\begin{aligned}\mathsf{Rev}_{opt} \ &\geq \ \mathrm{E}\Big[(\theta + \alpha(\theta)) \times \mathbb{1}\{\theta \geq \theta_H\} + (\theta + \alpha(\theta)) \times \mathbb{1}\{\theta \in (\theta_0, \theta_H)\} \\ &\quad + e^{-\frac{1}{a}}\theta \times \mathbb{1}\{\theta \in (\theta_L, \theta_0)\} + \ e^{-\frac{1}{a}}\theta \times \mathbb{1}\{\theta \leq \theta_L\}\Big] \ , \\ &= \ \mathrm{E}\Big[(\theta + \alpha(\theta)) \times \mathbb{1}\{\theta > \theta_0\} + e^{-\frac{1}{a}}\theta \times \mathbb{1}\{\theta \leq \theta_0\}\Big]\end{aligned}$$

By the above equation and Eq. (32), we get $\mathsf{Rev}_{opt} - \mathsf{Rev}_f \ \geq \ e^{-\frac{1}{a}}\mathrm{E}\left[\theta \times \mathbb{1}\{\theta \leq \theta_0\}\right]$. Then the result follows because $\mathsf{Rev}_f = \theta_0(1 - F(\theta_0))$.

- $\theta_L > \theta_0$: Since for $\theta \in (\theta_L, \theta_H)$, $\mathfrak{t}_g(\theta)$ is the FOC solution, we have

$$\mathsf{Rev}_{opt} \ \geq \ \mathrm{E}\Big[(\theta + \alpha(\theta)) \times \mathbb{1}\{\theta > \theta_L\} + e^{-\frac{1}{a}}\theta \times \mathbb{1}\{\theta \leq \theta_L\}\Big]$$

This leads to

$$\mathsf{Rev}_{opt} - \mathsf{Rev}_f \ \geq \ e^{-\frac{1}{a}}\mathrm{E}\left[(\theta e^{-\frac{1}{a}} - \theta - \alpha(\theta)) \times \mathbb{1}\{\theta \in (\theta_0, \theta_L]\} + \theta e^{-\frac{1}{a}}\mathbb{1}\{\theta \leq \theta_0\}\right]$$

To complete the proof, we show that for any $\theta \in (\theta_0, \theta_L]$, $(\theta e^{-\frac{1}{a}} - \theta - \alpha(\theta)) \geq 0$. This gives us $\mathsf{Rev}_{opt} - \mathsf{Rev}_f \ \geq \ e^{-\frac{1}{a}}\mathrm{E}\left[\theta \times \mathbb{1}\{\theta \leq \theta_0\}\right]$, which is the desired result.

To show $(\theta e^{-\frac{1}{a}} - \theta - \alpha(\theta)) \geq 0$ for any $\theta \in (\theta_0, \theta_L]$, we will verify that (i) $(\theta e^{-\frac{1}{a}} - \theta - \alpha(\theta)) \geq 0$ at $\theta = \theta_L$, and (ii) $(\theta e^{-\frac{1}{a}} - \theta - \alpha(\theta))$ is decreasing in θ. By definition, $\mathfrak{t}_g(\cdot)$ is continuous at $\theta = \theta_L$. This implies $\mathfrak{t}_g(\theta) = \frac{1}{a\theta^a}$ when $\theta = \theta_L$. Then, considering the fact that $\mathfrak{t}_g(\theta_L)$ is the FOC solution, we have $R(\theta_L, \mathfrak{t}_g(\theta_L)) \geq R(\theta_L, 0)$. Thus, we get $\theta e^{-\frac{1}{a}} - \theta - \alpha(\theta) \geq 0$ at $\theta = \theta_L$. Finally, $\theta e^{-\frac{1}{a}} - \theta - \alpha(\theta)$ is decreasing in θ because $e^{-\frac{1}{a}} - 1 \leq 0$ and $\alpha(\cdot)$ is increasing.

□

Appendix F: Proof of Theorem 3 of Appendix A

The proof of Theorem is divided into three lemmas: Lemma 10, 11, and 12. In Lemma 10, 11, and 12, we characterize the optimal mechanism for when the holding cost is low, medium, and high, respectively.

To characterize the optimal mechanism, by Lemma 2, we should solve the following optimization problem.

$$\begin{aligned} \max_{\{u(\underline{\Theta},\underline{\Theta})\geq 0,\ (\mathfrak{t},\varsigma,\mathfrak{p})\}} \quad & \mathrm{E}\Big[\varsigma(\theta)(R(\theta,\mathfrak{t}(\theta)) - h\mathfrak{t}(\theta))\Big] - u(\underline{\Theta},\underline{\Theta}) \\ \text{s.t.} \quad & u(\theta,\theta) \ \geq \ u(\theta,\hat{\theta}) \qquad \theta,\hat{\theta} \in [\underline{\Theta},\bar{\Theta}] \qquad \text{(IC)} \\ & u(\theta,\theta) \ \geq \ 0 \qquad \theta \in [\underline{\Theta},\bar{\Theta}] \qquad \text{(IR)} \end{aligned} \qquad \text{(OPT-H)}$$

Here, the objective function is the virtual profit and $R(\theta, \mathfrak{t}(\theta))$ is defined in Eq. (12). The first and second sets of constraints ensure that the mechanism is IC and IR, respectively.

Lemma 10 **(Low Holding Cost)**. *If Assumption 2 holds, the valuation function $V(\theta,t) = \theta e^{-\theta t}$, and the holding cost $h \leq H_l$, then the optimal mechanism sells to the customer of type $\theta \geq \max\{\underline{\theta}_L, \underline{\Theta}\}$ at time $\mathfrak{t}_h(\theta)$ and at price $\mathfrak{p}(\theta) = V(\theta, \mathfrak{t}_h(\theta)) - \int_{\max\{\underline{\theta}_L,\underline{\Theta}\}}^{\theta} e^{-\mathfrak{t}_h(z)z}(1 - \mathfrak{t}_h(z)z)dz$ where H_l, $\mathfrak{t}_h(\cdot)$, and $\underline{\theta}_L$ are defined in Equations (11) and (13). Further, for $\theta < \max\{\underline{\theta}_L, \underline{\Theta}\}$, $\mathfrak{p}(\theta) = \infty$ and $\varsigma(\theta) = 0$, and for $\theta \geq \max\{\underline{\theta}_L, \underline{\Theta}\}$, $\varsigma(\theta) = 1$.*

The proof of Lemma 10 is given in Appendix F.1.

Lemma 11 **(Medium Holding Cost)**. *If Assumption 2 holds, the valuation function $V(\theta,t) = \theta e^{-\theta t}$, the holding cost $h \in [H_l, H_h]$, and $R(\theta, \mathfrak{t}_f(\theta)) - h\mathfrak{t}_f(\theta) = 0$ has a unique solution, then the optimal mechanism sells to the customer of type $\theta \geq \underline{\theta}_M$ at time $\mathfrak{t}_h(\theta)$ and at price $\mathfrak{p}(\theta) = V(\theta, \mathfrak{t}_h(\theta)) - \int_{\underline{\theta}_M}^{\theta} e^{-\mathfrak{t}_h(z)z}(1 - \mathfrak{t}_h(z)z)dz$ where $R(\theta,t)$ is defined in Eq. (12) and H_l, H_h, $\underline{\theta}_M$, $\mathfrak{t}_h(\cdot)$, and the FOC solution $\mathfrak{t}_f(\cdot)$ are defined in Equations (11) and (13). Further, for $\theta < \underline{\theta}_M$, $\mathfrak{p}(\theta) = \infty$ and $\varsigma(\theta) = 0$, and for $\theta \geq \underline{\theta}_M$, $\varsigma(\theta) = 1$.*

The assumption in Lemma 11 is discussed in Appendix F.4, and the proof of Lemma 11 is provided in Appendix F.2.

Lemma 12 **(High Holding Cost)**. *If Assumption 2 holds, the valuation function $V(\theta,t) = \theta e^{-\theta t}$, the holding cost $h \geq H_h$, and $R(\theta, \mathfrak{t}_f(\theta)) - H_h\mathfrak{t}_f(\theta) = 0$ has a unique solution, then the optimal mechanism sells to customers with type $\theta \geq \theta_0$ at time zero and at price $\mathfrak{p}(\theta) = \theta_0$ where θ_0 solves $\theta_0 + \alpha(\theta_0) = 0$, $R(\theta,t)$ is defined in Eq. (12), and H_l, H_h, and the FOC solution $\mathfrak{t}_f(\cdot)$ are defined in Eq. (11). Further, for $\theta < \theta_0$, $\mathfrak{p}(\theta) = \infty$ and $\varsigma(\theta) = 0$, and for $\theta \geq \theta_0$, $\varsigma(\theta) = 1$.*

The proof is given in Appendix F.3.

F.1. Optimal Mechanism for a Low Holding Cost

In this section, we present the proof of Lemma 10. Throughout the proof, for convenience, we assume that $\underline{\theta}_L \geq \underline{\Theta}$. We need to show that the time of allocation in the optimal mechanism is given by

$$\mathfrak{t}^*(\theta) := \begin{cases} \mathfrak{t}_h(\theta) & \text{if } \theta \geq \max\{\underline{\theta}_L, \underline{\Theta}\}; \\ \infty & \text{if } \theta < \max\{\underline{\theta}_L, \underline{\Theta}\} \end{cases} = \begin{cases} 0 & \text{if } \theta \geq \theta_H^h; \\ \mathfrak{t}_f(\theta) & \text{if } \theta \in [\theta_L^h, \theta_H^h]; \\ \frac{1}{\theta} & \text{if } \theta \in [\underline{\theta}_L, \theta_L^h]; \\ \infty & \text{if } \theta < \underline{\theta}_L. \end{cases} \quad (33)$$

Note that $\mathfrak{t}^*(\theta) = \infty$ implies that the mechanism does not allocate the item to customers with type θ; that is $\varsigma(\theta) = 0$.

To characterize the optimal mechanism, by Lemma 2, we need to solve the optimization Problem OPT-H. That is, we need to maximize the expected virtual profit subject to IR and IC constrains. Lemma 1 shows that a mechanism is IC if and only if the interval and envelope conditions hold. In the following, we relax Problem OPT-H and only consider the IR and envelope conditions. We then show that the solution of the relaxed problem also satisfies the interval condition. Thus, it is optimal.

The relaxed problem can be formulated as follows.

$$\max_{\{u(\underline{\Theta},\underline{\Theta})\geq 0,\ (\mathfrak{t},\varsigma)\}} \mathrm{E}\left[\varsigma(\theta)(R(\theta,\mathfrak{t}(\theta)) - h\mathfrak{t}(\theta))\right] - u(\underline{\Theta},\underline{\Theta})$$

$$\text{s.t.} \quad u(\theta,\theta) \ = \ u(\underline{\Theta},\underline{\Theta}) + \int_{\underline{\Theta}}^{\theta} \varsigma(z) e^{-z\mathfrak{t}(z)}(1-\mathfrak{t}(z)z)dz \ \geq \ 0 \quad \text{for} \quad \theta \in [\underline{\Theta}, \bar{\Theta}] \ , \quad \text{(IR)} \quad \text{(OPT-H-R)}$$

where the maximization is taken over the purchase time $\mathfrak{t}(\theta)$ and utility of a customer with type $\underline{\Theta}$, i.e., $u(\underline{\Theta},\underline{\Theta})$. Here, $R(\theta,t)$ is the virtual value of customer of type θ at time t, and is defined in Eq. (12).

The following lemma characterizes the optimal solution of the relaxed problem.

LEMMA 13. *Suppose that $V(\theta) = \theta e^{-\theta t}$. Then, if Assumptions 2 hold and the holding cost $h \leq H_l$, in an optimal solution of Problem* OPT-H-R*, $u(\underline{\Theta},\underline{\Theta}) = 0$, the optimal allocation rule is $\mathfrak{t}^*(\cdot)$ where $\mathfrak{t}^*(\cdot)$ is defined in Eq. (33).*

The proof is provided in Appendix F.1.1. In the proof, we first show that $\mathfrak{t}^*(\cdot)$ is a feasible solution of the relaxed problem. Then, we show that it is optimal.

To verify that $\mathfrak{t}^*(\cdot)$ is an optimal solution of Problem OPT-H, we show that the interval condition specified in Lemma 1 is fulfilled. That is, for any $\hat{\theta}, \theta \in [\underline{\Theta}, \bar{\Theta}]$ such that $\hat{\theta} \leq \theta$,

$$\int_{\hat{\theta}}^{\theta} A(z, \mathfrak{t}^*(\hat{\theta}))dz \ \leq \ \int_{\hat{\theta}}^{\theta} A(z, \mathfrak{t}^*(z))dz \ ,$$
$$\int_{\hat{\theta}}^{\theta} A(z, \mathfrak{t}^*(z))dz \ \leq \ \int_{\hat{\theta}}^{\theta} A(z, \mathfrak{t}^*(\theta))d \ ,$$

where $A(z,t) = \partial_1 V(z,t) = e^{-zt}(1-zt)$. Note that $A(z,t) = 0$ when t goes to infinity. Thus, for any $z < \underline{\theta}_L$ and $t \geq 0$, we have $A(z, \mathfrak{t}^*(z)) = \varsigma(z)A(z,t)$. Further, for any $z \geq \underline{\theta}_L$, we have $A(z, \mathfrak{t}^*(z)) = \varsigma(z)A(z, \mathfrak{t}_h(z))$. Thus, showing the above equations is equivalent to verifying the interval conditions in Lemma 1. In addition, note that $\int_{\hat{\theta}}^{\theta} A(z, \mathfrak{t}^*(z))dz = u(\theta,\theta) - u(\hat{\theta},\hat{\theta})$. To this aim, we show that for any $z \geq \hat{\theta}$, $A(z, \mathfrak{t}^*(\hat{\theta})) \leq A(z, \mathfrak{t}^*(z))$ and for any $z \leq \theta$, $A(z, \mathfrak{t}^*(z)) \leq A(z, \mathfrak{t}^*(\theta))$.

We will make use of the following preliminary results.

LEMMA 14. *The FOC solution $\mathsf{t}_f(\theta)$, defined in Eq. (11), is a decreasing function of θ as long as $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) \geq 0$. In addition, for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, $0 \leq A(\theta, \mathsf{t}_h(\theta)) \leq 1$ where $A(z,t) = e^{-zt}(1-zt)$.*

LEMMA 15. *For any $h \geq 0$, $R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)$ is increasing in $\theta \geq \underline{\theta}_h$. Furthermore, $R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta) \geq 0$ for any $\theta \geq \underline{\theta}_h$.*

Unless stated otherwise, the proof of all technical lemmas is given in Appendix H.

By Lemma 15, $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) \geq 0$ for any $\theta \in [\theta_L^h, \theta_H^h]$. This and Lemma 14 imply that $\mathsf{t}_h(\theta) = \mathsf{t}_f(\theta)$ is decreasing for any $\theta \in [\theta_L^h, \theta_H^h]$. Then, considering the fact that $\mathsf{t}_h(\theta) = 0$ for $\theta \geq \theta_H^h$, $\mathsf{t}_h(\theta) = \frac{1}{\theta}$ for $\theta \in [\underline{\theta}_L, \theta_L^h]$, $\mathsf{t}_h(\theta_H^h) = 0$, and $\mathsf{t}_h(\theta_L^h) = \frac{1}{\theta_L^h}$, we can conclude that $\mathsf{t}_h(\theta)$ is decreasing in $\theta \geq \underline{\theta}_L$.

Now, we are ready to show that the interval conditions are satisfied.

We first note that when $\hat{\theta} \leq \underline{\theta}_L$, it easy to show that for any $z \geq \hat{\theta}$, $A(z, \mathsf{t}^*(\hat{\theta})) \leq A(z, \mathsf{t}^*(z))$. This holds because $A(z, \mathsf{t}^*(\hat{\theta})) = 0$ and as shown in Lemma 14, $A(z, \mathsf{t}^*(z)) \geq 0$. In addition, when $\theta \leq \underline{\theta}_L$, we have $A(z, \mathsf{t}^*(z)) \leq A(z, \mathsf{t}^*(\theta))$ for any $z \leq \theta$. This follows from the fact that both $A(z, \mathsf{t}^*(z))$ and $A(z, \mathsf{t}^*(\theta))$ are both zero.

Next, we assume that both θ and $\hat{\theta}$ are greater than $\underline{\theta}_L$. Recall that for $\theta \geq \underline{\theta}_L$, $\mathsf{t}^*(\theta) = \mathsf{t}_h(\theta)$. We start with showing $A(z, \mathsf{t}_h(\hat{\theta})) \leq A(z, \mathsf{t}_h(z))$, $z \geq \hat{\theta}$. We consider two cases: 1- $(1 - \mathsf{t}_h(\hat{\theta})z) \leq 0$ and 2- $(1 - \mathsf{t}_h(\hat{\theta})z) > 0$. Assume that $(1 - \mathsf{t}_h(\hat{\theta})z) \leq 0$. Then, we have

$$e^{-z\mathsf{t}_h(\hat{\theta})}(1 - \mathsf{t}_h(\hat{\theta})z) \;\leq\; 0 \;\leq\; e^{-z\mathsf{t}_h(z)}(1 - \mathsf{t}_h(z)z) \;,$$

where the second inequality follows from Lemma 14 where we show that $A(z, \mathsf{t}_h(z)) = e^{-z\mathsf{t}_h(z)}(1 - \mathsf{t}_h(z)z) \geq 0$. By the above equation, we get $A(z, \mathsf{t}_h(\hat{\theta})) \leq A(z, \mathsf{t}_h(z))$.

Now, assume that $(1 - \mathsf{t}_h(\hat{\theta})z) > 0$. Then, considering the fact that $\mathsf{t}_h(\cdot)$ is decreasing, for any $z \geq \hat{\theta}$, we have $(1 - \mathsf{t}_h(z)z) \geq (1 - \mathsf{t}_h(\hat{\theta})z)$, and $e^{-\mathsf{t}_h(z)z} \geq e^{-\mathsf{t}_h(\hat{\theta})z}$. By multiplying these two equations, we get $A(z, \mathsf{t}_h(\hat{\theta})) \leq A(z, \mathsf{t}_h(z))$.

Next, we will verify that $A(z, \mathsf{t}_h(z)) \leq A(z, \mathsf{t}_h(\theta))$. Given that $\mathsf{t}_h(\cdot)$ is decreasing, for any $z \geq \theta$, we have

$$0 \leq (1 - \mathsf{t}_h(z)z) \leq (1 - \mathsf{t}_h(\theta)z), \quad \text{and} \quad e^{-\mathsf{t}_h(z)z} \leq e^{-\mathsf{t}_h(\theta)z} \;,$$

where the first inequality follows from Lemma 14 where we show $A(z, \mathsf{t}_h(z)) = e^{-z\mathsf{t}_h(z)}(1 - \mathsf{t}_h(z)z) \geq 0$. By multiplying these two equations, we have $A(z, \mathsf{t}_h(z)) \leq A(z, \mathsf{t}_h(\theta))$.

F.1.1. Proof of Lemma 13 Here, with some abuse of notation, we denote $\mathsf{t}^*(\cdot)$ with $\mathsf{t}_h(\cdot)$. Recall that $\mathsf{t}^*(\theta) = \mathsf{t}_h(\theta)$ when $\theta \geq \underline{\theta}_L$ and is ∞ otherwise. In addition, for simplicity, we denote $u(\theta, \theta)$ by $u(\theta)$.

The proof has two parts. In the first part, we show that the solution given in Lemma 13 is a feasible solution of Problem OPT-H-R. In the second part, we verify that this solution is an optimal solution of this problem.

Feasibility: To show that $\mathsf{t}_h(\cdot)$ is a feasible solution of Problem OPT-H-R, we will verify that $u(\theta) \geq 0$ for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$. For any $\theta \leq \theta_L^h$, it is easy to verify that $u(\theta) = u(\underline{\Theta}) = 0$. Thus, we only need to show that $u(\theta) \geq 0$ for any $\theta \geq \theta_L^h$. To prove that $u(\theta) \geq 0$ for $\theta \geq \theta_L^h$, we make use of Lemma 14 where we show that $e^{-\mathsf{t}_h(\theta)\theta}(1 - \mathsf{t}_h(\theta)\theta) \geq 0$. This implies that $u(\theta) \;=\; \int_{\underline{\Theta}}^{\theta} e^{-\mathsf{t}_h(z)z}(1 - \mathsf{t}_h(z)z)dz \;\geq\; 0$

Optimality: Here, we will show that the solution given in Lemma 13, is an optimal solution of Problem OPT-H-R. To this end, we find an upper bound for the optimal value of Problem OPT-H-R by dualizing the IR constraints. Then, we will show that the solution given in Lemma 13 achieves the upper bound and thus is optimal.

Upper Bound of OPT-H-R*:* For any purchase time $\mathfrak{t}(\cdot)$ and Lagrangian function $\lambda:[\underline{\Theta},\bar{\Theta}]\to\mathbb{R}^+$, we define the following function.

$$L_h(\mathfrak{t}(\cdot),\lambda(\cdot),u(\underline{\Theta}))=\mathrm{E}[R(\theta,\mathfrak{t}(\theta))-h\mathfrak{t}(\theta)-u(\underline{\Theta})]+\int_{\underline{\Theta}}^{\bar{\Theta}}\lambda(z)u(z)dz\ ,$$

where $u(z)=\int_{\underline{\Theta}}^{z}e^{-\mathfrak{t}(\theta)\theta}(1-\mathfrak{t}(\theta)\theta)d\theta+u(\underline{\Theta})$, and R is defined in Eq. (12). Note that $\mathrm{E}[\varsigma(\theta)(R(\theta,\mathfrak{t}(\theta))-h\mathfrak{t}(\theta))-u(\underline{\Theta})]$ is the objective function of Problem OPT-H-R. However, here we remove $\varsigma(\theta)$ and instead, we assume that $R(\theta,\mathfrak{t}(\theta))-h\mathfrak{t}(\theta)=0$ when $\mathfrak{t}(\theta)=\infty$. Recall that given that $\mathfrak{t}_h(\theta)=\infty$, we have $\varsigma(\theta)=0$.

Then, considering the fact that $\lambda(\cdot)\geq 0$, for any $(\mathfrak{t}(\cdot),u(\underline{\Theta}))$ such that $u(\theta)=u(\underline{\Theta})+\int_{\underline{\Theta}}^{\theta}e^{-z\mathfrak{t}(z)}(1-z\mathfrak{t}(z))\geq 0$, we have

$$L_h(\mathfrak{t}(\cdot),\lambda(\cdot),u(\underline{\Theta}))\ \geq\ \mathrm{E}[R(\theta,\mathfrak{t}(\theta))-h\mathfrak{t}(\theta)-u(\underline{\Theta})]$$

One can think of $\lambda(\theta)$ as a dual variable for the IR constraints. Therefore, for any $\lambda:[\underline{\Theta},\bar{\Theta}]\to\mathbb{R}^+$,

$$\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{\mathrm{E}[R(\theta,\mathfrak{t}(\theta))-h\mathfrak{t}(\theta)-u(\underline{\Theta})]\}\ \leq\ \max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{L_h(\mathfrak{t}(\cdot),\lambda(\cdot),u(\underline{\Theta}))\}\ , \tag{34}$$

where

$$\mathcal{T}=\Big\{(\mathfrak{t}(\cdot),u(\underline{\Theta})):u(\underline{\Theta})\geq 0,\ \mathfrak{t}(\theta)\geq 0,\ \text{and}\ u(\underline{\Theta})+\textstyle\int_{\underline{\Theta}}^{\theta}e^{-z\mathfrak{t}(z)}(1-z\mathfrak{t}(z))\geq 0\ \text{for any}\ \theta\in[\underline{\Theta},\bar{\Theta}]\Big\}$$

is the set of feasible solutions. In the following, we will characterize an upper bound for $\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{\mathrm{E}[R(\theta,\mathfrak{t}(\theta))-h\mathfrak{t}(\theta)-u(\underline{\Theta})]\}$ by considering a specific Lagrangian function, defined below.

$$\lambda_h(\theta)=\begin{cases}0 & \text{if }\theta>\theta_L^h;\\ \Big(f(\theta)(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}})\Big)' & \text{if }\theta\in[\underline{\theta}_L,\theta_L^h];\\ \Big(f(\theta)(2\theta+\alpha(\theta))\Big)' & \text{if }\theta\in[\underline{\Theta},\underline{\theta}_L];\end{cases} \tag{35}$$

where $\big(f(\theta)(2\theta+\alpha(\theta))\big)'$ and $\big(f(\theta)(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}})\big)'$ are respectively the derivative of $\big(f(\theta)(2\theta+\alpha(\theta))\big)$ and $\big(f(\theta)(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}})\big)$ with respect to θ. The following lemma establishes that $\lambda_h(\theta)\geq 0$.

LEMMA 16. *When $h\leq H_l$, for any $\theta\in[\underline{\Theta},\bar{\Theta}]$, $\lambda_h(\theta)$, defined in Eq. (35), is nonnegative.*

The following claim shows that $(\mathfrak{t}_h(\cdot),u(\underline{\Theta})=0)$ is an optimal solution of Problem OPT-H-R.

Claim: With a slight abuse of notation, let

$$(t_\lambda(\cdot),u_\lambda)=\arg\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{L_h(\mathfrak{t}(\cdot),\lambda_h(\cdot),u(\underline{\Theta}))\}\ .$$

Then, $t_\lambda(\cdot)=\mathfrak{t}_h(\theta)$ for any $\theta\in[\underline{\Theta},\bar{\Theta}]$ and $u_\lambda=0$. Furthermore, $L_h(\mathfrak{t}_h(\cdot),\lambda_h(\cdot),u_\lambda)=\mathrm{E}[R(\theta,\mathfrak{t}_h(\theta))-h\mathfrak{t}_h(\theta)-u_\lambda]$.

Proof of the Claim: By definition, $\lambda_h(\theta)\ =\ 0$ for $\theta>\theta_L^h$. Thus, we get

$$\begin{aligned}L_h(\mathfrak{t}(\cdot),\lambda_h(\cdot),u(\underline{\Theta}))=&\int_{z=\theta_L^h}^{\bar{\Theta}}\big(R(z,\mathfrak{t}(z))-h\mathfrak{t}(z)\big)f(z)dz\\&+\int_{z=\underline{\Theta}}^{\theta_L^h}\Big(\big(R(z,\mathfrak{t}(z))-h\mathfrak{t}(z)\big)f(z)+\lambda_h(z)u(z)\Big)dz-u(\underline{\Theta})\ .\end{aligned} \tag{36}$$

From definition of $\lambda_h(\cdot)$, the last two terms of Eq. (36) can be written as

$$\int_{\underline{\theta}_L}^{\theta_L^h} \big(R(z,\mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z)dz + \int_{\underline{\theta}_L}^{\theta_L^h} u(z)d\left(f(z)(z+\alpha(z)+\frac{h}{ze^{-1}})\right)$$

$$\int_{\underline{\Theta}}^{\underline{\theta}_L} \Big(\big(R(z,\mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z)dz + \int_{\underline{\Theta}}^{\underline{\theta}_L} u(z)d\Big(f(z)(2z+\alpha(z))\Big) - u(\underline{\Theta}) \ . \tag{37}$$

We first focus on the first two terms where $z \in [\underline{\theta}_L, \theta_L^h]$. By integrating by part and using the definition of R, the first two terms can be rewritten as

$$\int_{\underline{\theta}_L}^{\theta_L^h} \big(e^{-\mathfrak{t}(z)z}\big(z+\alpha(z)(1-\mathfrak{t}(z)z\big) - h\mathfrak{t}(z)\big) f(z)dz$$

$$+ u(z)f(z)(z+\alpha(z)+\frac{h}{ze^{-1}})\Big|_{\underline{\theta}_L}^{\theta_L^h} - \int_{\underline{\theta}_L}^{\theta_L^h} e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big) f(z)\left(z+\alpha(z)+\frac{h}{ze^{-1}}\right)dz \ .$$

In the above equation, we use the fact that $\frac{du(z)}{dz} = e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big)$. Then, by definition of θ_L^h, i.e., the fact that $(\theta_L^h + \alpha(\theta_L^h) + \frac{h}{\theta_L^h e^{-1}}) = 0$, the above equation is simplified as

$$- u(\underline{\theta}_L)f(\underline{\theta}_L)(\underline{\theta}_L + \alpha(\underline{\theta}_L) + \frac{h}{\underline{\theta}_L e^{-1}})$$

$$+ \int_{\underline{\theta}_L}^{\theta_L^h} f(z)\left(e^{-\mathfrak{t}(z)z}z^2\mathfrak{t}(z) - h\mathfrak{t}(z) - e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big)\frac{h}{ze^{-1}}\right)dz \ . \tag{38}$$

Now, we focus on the last three terms of Eq. (37). Again, by integrating by part and using definition of R, the last two terms of Eq. (37) can be rewritten as

$$\int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\left(e^{-\mathfrak{t}(z)z}\big(z+\alpha(z)(1-\mathfrak{t}(z)z)\big) - h\mathfrak{t}(z)\right)dz$$

$$+ u(z)f(z)(2z+\alpha(z))\Big|_{\underline{\Theta}}^{\underline{\theta}_L} - \int_{\underline{\Theta}}^{\underline{\theta}_L} e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big) f(z)\left(2z+\alpha(z)\right)dz - u(\underline{\Theta})$$

$$= u(\underline{\theta}_L)f(\underline{\theta}_L)(2\underline{\theta}_L + \alpha(\underline{\theta}_L)) + u(\underline{\Theta})\Big(-1 - f(\underline{\Theta})(2\underline{\Theta}+\alpha(\underline{\Theta}))\Big)$$

$$+ \int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\left(ze^{-\mathfrak{t}(z)z}(-1+2\mathfrak{t}(z)z) - h\mathfrak{t}(z)\right)dz \ . \tag{39}$$

Note that the coefficient of $u(\underline{\Theta})$, i.e., $\big(-1 - f(\underline{\Theta})(2\underline{\Theta}+\alpha(\underline{\Theta}))\big)$, can be simplified as $-2\underline{\Theta}f(\underline{\Theta}) \le 0$. By plugging Equations (38) and (39) into Eq. (36), and by using definition of $\underline{\theta}_L$, we get

$$L_h(\mathfrak{t}(\cdot), \lambda_h(\cdot), u(\underline{\Theta})) = \int_{\theta_L^h}^{\bar{\Theta}} \big(R(z,\mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z)dz$$

$$+ \int_{\underline{\theta}_L}^{\theta_L^h} f(z)\left(e^{-\mathfrak{t}(z)z}z^2\mathfrak{t}(z) - h\mathfrak{t}(z) - e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big)\frac{h}{ze^{-1}}\right)dz$$

$$+ \int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\left(ze^{-\mathfrak{t}(z)z}(-1+2\mathfrak{t}(z)z) - h\mathfrak{t}(z)\right)dz - 2\underline{\Theta}f(\underline{\Theta})u(\underline{\Theta}) \ .$$

First of all, since the coefficient of $u(\underline{\Theta})$ is negative, to maximize the above equation, we need to set $u(\underline{\Theta})$ to zero. That is, $u_\lambda = 0$. Then, $\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{L_h(\mathfrak{t}(\cdot), \lambda_h(\cdot), u(\underline{\Theta}))\}$ can be upper-bounded as follows

$$\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{L_h(\mathfrak{t}(\cdot), \lambda_h(\cdot), u(\underline{\Theta})=0)\} \le \int_{\theta_L^h}^{\bar{\Theta}} f(z)\max_{t\ge 0}\big\{R(z,t) - ht\big\}dz$$

$$+ \int_{\underline{\theta}_L}^{\theta_L^h} f(z)\max_{t\ge 0}\left\{\left(e^{-tz}z^2t - ht - e^{-tz}\big(1-tz\big)\frac{h}{ze^{-1}}\right)\right\}dz$$

$$+ \int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\max\left\{\max_{t\ge 0}\{(ze^{-tz}(-1+2tz) - ht)\}, 0\right\}dz \ . \tag{40}$$

We take advantage of the following lemma to simplify the first term of the above equation.

LEMMA 17. *If Assumption 2 holds and the holding cost $h \le H_l$, then for any $z \ge \theta_L^h$, we have $\arg\max_{t\ge 0}\{R(z,t)-ht\} = \mathfrak{t}_h(z)$, where $R(\theta,t)$ is defined in Eq. (12).*

Note that the optimal solution characterized in Lemma 17 is the maximum of the FOC solution and zero. We now simplify the second term of Eq. (40). It is easy to verify that for any $z \in [\underline{\theta}_L, \theta_L^h]$, we have

$$\arg\max_{t\ge 0}\left\{\left(e^{-tz}z^2t - ht - e^{-tz}(1-tz)\frac{h}{ze^{-1}}\right)\right\} = \frac{1}{z} = \mathfrak{t}_h(z) . \tag{41}$$

Finally, the following lemma characterizes an optimal solution of the third term of Eq. (40).

LEMMA 18. *If Assumption 2 holds and the holding cost $h \le H_l$, for any $z \le \underline{\theta}_L$, we have*

$$\max_{t\ge 0}\{(ze^{-tz}(-1+2tz)-ht)\} \le 0 .$$

Lemmas 17, 18, and Eq. (41) show that $t_\lambda(\theta) = \mathfrak{t}_h(\theta)$ and $u_\lambda = 0$. Then, the proof is completed by observing that $L_h(\mathfrak{t}_h(\cdot), \lambda_h(\cdot), 0) = \mathrm{E}[R(\theta, \mathfrak{t}_h(\theta)) - h\mathfrak{t}_h(\theta)]$.

F.2. Optimal Mechanism for a Medium Holding Cost

Here, we present the proof for Lemma 11. We show that in the optimal solution of Problem OPT-H, the time of purchase is given by

$$\mathfrak{t}^*(\theta) = \begin{cases} \mathfrak{t}_h(\theta) & \text{if } \theta \ge \underline{\theta}_M; \\ \infty & \text{if } \theta < \underline{\theta}_M \end{cases} = \begin{cases} 0 & \text{if } \theta \ge \theta_H^h, \\ \mathfrak{t}_f(\theta) & \text{if } \theta \in [\underline{\theta}_M, \theta_H^h], \\ \infty & \text{if } \theta < \underline{\theta}_M, \end{cases} \tag{42}$$

and $u(\theta,\theta) = \int_{\underline{\theta}_M}^{\theta} e^{-z\mathfrak{t}_h(z)}(1 - z\mathfrak{t}_h(z))dz$. Here, $\mathfrak{t}^*(\theta) = \infty$ implies that customer with type θ does not purchase the item; that is, $\varsigma(\theta) = 0$.

The proof has three main steps. In the first step, we relax the problem by ignoring both IC and IR constraints and we find an allocation rule that maximizes the virtual profit. Then, we show that the solution of this relaxed problem can construct a mechanism that satisfy the IR and envelope conditions. Finally, we show that the aforementioned solution also satisfies the interval conditions, as a result, it is optimal.

- Maximizing virtual profit without IC and IR constraints: Consider that following optimization problem.

$$\max_{\{\mathfrak{t}(\theta)\ge 0:\ \theta \in [\underline{\Theta}, \bar{\Theta}]\}} \mathrm{E}\left[R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta)\right] , \tag{OPT-H-1}$$

where $R(\theta,t)$ is defined in Eq. (12).[17] The following lemma shows that $\mathfrak{t}^*(\cdot)$, given in Eq. (42), is an optimal solution of Problem OPT-H-1.

LEMMA 19. *The optimal solution of Problem* OPT-H-1 *is given by $\mathfrak{t}^*(\cdot)$ where $\mathfrak{t}^*(\cdot)$ is defined in Eq. (42).*

The proof is similar to the proof of Lemma 17; thus, it is omitted. The main idea of the proof is to show that $R(\theta,t) - ht$ as a function of t has an inverted u-shape. Thus, it obtains its maximum at $\max\{0, \mathfrak{t}_f(\theta)\}$, where $\mathfrak{t}_f(\theta)$ is the FOC solution. Note that to show Lemma 19, we need the assumption that $\underline{\theta}_M$, i.e., the solution of $R(\theta, \mathfrak{t}_f(\theta)) - h\mathfrak{t}_f(\theta) = 0$, is unique. By this assumption, for any $\theta < \underline{\theta}_M$ we get

$$\max_{t\ge 0}\{R(\theta,t) - ht\} = R(\theta, \mathfrak{t}_f(\theta)) - h\mathfrak{t}_f(\theta) < 0 \quad \text{for} \quad \theta < \underline{\theta}_M .$$

This implies that it is optimal not to allocate the item to customers with type $\theta < \underline{\theta}_M$ and set $\varsigma(\theta) = \infty$.

[17] Again, we assume that $R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta) = 0$ when $\mathfrak{t}(\theta) = \infty$.

• Maximizing virtual profit with IR and envelope constraints: Here, we show that the purchase time $\mathfrak{t}^*(\cdot)$ is an optimal solution of Problem OPT-H-R. [18] To this aim, we verify that

$$u(\theta,\theta) \;=\; \int_{\underline{\theta}_M}^{\theta} e^{-\mathfrak{t}_h(z)z}(1-\mathfrak{t}_h(z)z)dz \;\geq\; 0\;.$$

Particularly, we show that for any $\theta \geq \underline{\theta}_M$, $(1-\theta\mathfrak{t}_h(\theta)) \geq 0$. Since $\mathfrak{t}_h(\theta)=0$ for $\theta \geq \theta_H^h$, it suffices to show that $(1-\theta\mathfrak{t}_h(\theta)) \geq 0$ for any $\theta \in [\underline{\theta}_M, \theta_H^h]$.

Lemma 20. *For any $h \in [H_l, H_h]$ and $\theta \in [\underline{\theta}_M, \theta_H^h]$, we have $1-\theta\mathfrak{t}_h(\theta) \geq 0$.*

In the proof, we show that when $h \geq H_l$ and $\theta \geq \tilde{\theta}$, we have $1-\mathfrak{t}_f(\theta)\theta \geq 0$. Then, we show that $\underline{\theta}_M \geq \tilde{\theta}$. This implies that $1-\mathfrak{t}_f(\theta)\theta \geq 0$ for any $\theta \in [\underline{\theta}_M, \theta_H^h]$, which is the desired result.

• Maximizing virtual profit with IR and IC constraints: Here, we need to show that the time of purchase $\mathfrak{t}^*(\cdot)$ and its associated payment, given in Lemma 11, satisfy the interval conditions presented in Lemma 1. This part of the proof is very similar to that of Lemma 10. Thus, we do not repeat it here.

F.3. Optimal Mechanism for a High Holding Cost

In this section, we present the proof of Lemma 12.

In the following, we show that $\max_{t\geq 0}\{R(\theta,t)-ht\} \;=\; R(\theta,0) \;=\; \theta+\alpha(\theta) \;\geq\; 0$ for $\theta \geq \theta_0$, and for any $\theta < \theta_0$, $\max_{t\geq 0}\{R(\theta,t)-ht\} \;<\; 0$ where R is defined in Eq. (12). This implies that in the optimal mechanism, the firm only sells to customers with type $\theta \geq \theta_0$.

We first show that for any $\theta \geq \theta_0$, $\arg\max_{t\geq 0}\{R(\theta,t)-ht\}=0$. To this aim, we will verify that $\frac{\partial(R(\theta,t)-ht)}{\partial t}\big|_{t=0} \leq 0$. This will give us the desired result because as we show in Lemma 17, $R(\theta,t)-ht$ as a function of t has an inverted u-shape. Therefore, if $\frac{\partial(R(\theta,t)-ht)}{\partial t}\big|_{t=0} \leq 0$, we have $\arg\max_{t\geq 0}\{R(\theta,t)-ht\}=0$.

By definition,

$$\frac{\partial\,(R(\theta,t)-ht)}{\partial t} \;=\; -\theta e^{-t\theta}(\theta+\alpha(\theta)(2-\theta t))-h\;,$$

and at $t=0$ and for any $\theta \geq \theta_0$, we have

$$\frac{\partial\,(R(\theta,t)-ht)}{\partial t}\Big|_{t=0} \;=\; -\theta(\theta+2\alpha(\theta))-h \;\leq\; 0\;, \tag{43}$$

where the inequality holds because

$$h \;\geq\; H_h \;=\; \theta_0^2 \;=\; -\theta_0(\theta_0+2\alpha(\theta_0)) \;=\; \max_{\theta\in[\theta_0,\bar{\Theta}]}\{-\theta(\theta+2\alpha(\theta))\}\;.$$

The first equality follows because $\theta_0+\alpha(\theta_0)=0$ and last equality holds because $\arg\max_{\theta\in[\theta_0,\bar{\Theta}]}\{-\theta(\theta+2\alpha(\theta))\}=\theta_0$. To see why the latter holds note that

$$\big(-\theta(\theta+2\alpha(\theta))\big)' \;=\; -2(\theta+\alpha(\theta))-2\theta\alpha'(\theta) \;\leq\; 0\;,$$

where the inequality follows because for any $\theta \geq \theta_0$, we have $(\theta+\alpha(\theta)) \geq 0$.

Next, we will verify that for any $\theta < \theta_0$, $\max_{t\geq 0}\{R(\theta,t)-ht\} \;<\; 0$. Note that it suffices to show that $\max_{t\geq 0}\{R(\theta,t)-H_h t\} \;<\; 0$ considering the fact that $R(\theta,t)-ht$ is decreasing in h.

[18] It is easy to observe that in an optimal solution of Problem OPT-H-R, we need to set $u(\underline{\Theta},\underline{\Theta})$ to zero.

By Eq. (43), at $h = H_h$ we have $\mathsf{t}_f(\theta_0) = 0$, and

$$\max_{t\geq 0}\{R(\theta,t) - H_h t\} = R(\theta_0, \mathsf{t}_f(\theta_0)) - H_h \mathsf{t}_f(\theta_0) = \theta_0 + \alpha(\theta_0) = 0 \ .$$

Then, by our assumption that $R(\theta, \mathsf{t}_f(\theta)) - H_h \mathsf{t}_f(\theta) = 0$ has unique solution, we have

$$R(\theta, \mathsf{t}_f(\theta)) - H_h \mathsf{t}_f(\theta) = \max_{t\geq 0}\{R(\theta,t) - H_h t\} < 0 \quad \text{for any } \theta < \theta_0 \ .$$

F.4. Discussing the Assumption in Theorem 3

In this section, we discuss the assumption in Theorem 3. This assumption requires that the solution of equation $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) = 0$ to be unique, where $R(\theta, t)$ is defined in Eq. (12).

The following lemma shows that for any $h \in [H_l, H_h]$, $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) = 0$ has a unique solution if the solution of $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) = 0$ is unique. In addition, it shows that $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) = 0$ has a unique solution when $\alpha'(\theta)$ is small enough.

LEMMA 21. *If the solution of $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) = 0$ is unique, then, for any $h \in [H_l, H_h]$, $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) = 0$ has a unique solution. Furthermore, the solution of $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) = 0$ is unique if $\alpha'(\theta) \leq \frac{(\sqrt{5}+1)^2}{2} \approx 5.2$ for any $\theta \leq \tilde{\theta}$ where $\tilde{\theta}$ solves $2\tilde{\theta} + \alpha(\tilde{\theta}) = 0$ and H_l and the FOC solution $\mathsf{t}_f(\cdot)$ are defined in Eq. (11).*

The proof of Lemma 21 is given at the end of this section. Note that for the uniform and exponential distributions, we have $\alpha'(\theta) \leq 5.2$. In fact, for the uniform distribution $U(a,b)$, we have $\alpha'(\theta) = 1$ for any $\theta \in [a,b]$ where $a < b$ and $a, b \in \mathbb{R}$. For the exponential distribution with rate $\lambda \geq 0$, $\alpha'(\theta) = 0$ for any $\theta \geq 0$. Furthermore, for a truncated normal distribution with mean μ, standard deviation σ, and cut-off greater than $\mu - \sigma$, we have $\alpha'(\theta) \leq 4.48$ for any $\theta \geq (\mu - \sigma)$. Note that the domain of the truncated normal distribution with cut-off $\mathcal{C}$ is $[\mathcal{C}, \infty)$.

Proof of Lemma 21 First, we show that if the solution of Eq. (44) is unique at $h = H_l$, then this equation has a unique solution for any $h \in [H_l, H_h]$.

$$R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) = 0 \ . \tag{44}$$

By Lemma 11, $\tilde{\theta}$ solves $R(\tilde{\theta}, \mathsf{t}_f(\tilde{\theta})) - H_l \mathsf{t}_f(\tilde{\theta}) = 0$ where $2\tilde{\theta} + \alpha(\tilde{\theta}) = 0$ and $1 - \mathsf{t}_f(\tilde{\theta})\tilde{\theta} = 0$. Then, by our assumption, $\tilde{\theta}$ is the unique solution of Eq. (44) at $h = H_l$. This assumption and the proof of Lemma 20 imply that for any $h > H_l$, any solutions of Eq. (44) satisfy the following property: $1 - \theta \mathsf{t}_f(\theta) \geq 0$.

Next, we use this property to show that for any $h \in [H_l, H_h]$, there is only one solution to Eq. (44). Let θ^* solve Eq. (44). By the Envelope theorem, the derivative of $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta)$ w.r.t. θ at θ^* is given by

$$\begin{aligned}
\frac{\partial\big(R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta)\big)}{\partial \theta}\Big|_{\theta=\theta^*} &= -\mathsf{t}_f(\theta^*) e^{-\mathsf{t}_f(\theta^*)\theta^*}(\theta^* + \alpha(\theta^*)(2 - \mathsf{t}_f(\theta^*)\theta^*)) \\
&+ e^{-\mathsf{t}_f(\theta^*)\theta^*}(1 + \alpha'(\theta^*)(1 - \mathsf{t}_f(\theta^*)\theta^*)) \\
&= h\frac{\mathsf{t}_f(\theta^*)}{\theta^*} + e^{-\mathsf{t}_f(\theta^*)\theta^*}(1 + \alpha'(\theta^*)(1 - \mathsf{t}_f(\theta^*)\theta^*)) > 0 \ ,
\end{aligned} \tag{45}$$

where the second equality follows from the FOC, i.e., $\frac{\partial(R(\theta^*,t) - ht)}{\partial t}\Big|_{\mathsf{t}_f(\theta^*)} = 0$ and the inequality holds because $(1 - \mathsf{t}_f(\theta^*)\theta^*) \geq 0$. By the above equation, the derivative of $R(\theta^*, \mathsf{t}_f(\theta^*)) - h\mathsf{t}_f(\theta^*)$ w.r.t. θ^* is always positive. This implies that Eq. (44) has a unique solution.

Next, we show that at $h = H_l$, the solution of Eq. (44) is unique if for any $\theta \leq \tilde{\theta}$, $\alpha'(\theta) \leq \frac{(\sqrt{5}+1)^2}{2} \approx 5.2$.

We first argue that any $\theta > \tilde{\theta}$ cannot solve Eq. (44). To this end, we use the proof of Lemma 11 where we show $1 - \mathsf{t}_f(\theta)\theta \geq 0$ for any $\theta \geq \tilde{\theta}$. The fact that $1 - \mathsf{t}_f(\theta)\theta \geq 0$ for any $\theta \geq \tilde{\theta}$ implies that $\frac{\partial\left(R(\theta,\mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta)\right)}{\partial\theta} > 0$; see Eq. (45). Then, considering the fact that $R(\tilde{\theta}, \mathsf{t}_f(\tilde{\theta})) - H_l \mathsf{t}_f(\tilde{\theta}) = 0$, we have $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) > 0$ for any $\theta \geq \tilde{\theta}$.

Next, we show that any $\theta < \tilde{\theta}$ cannot solve Eq. (44). Let $\zeta(\theta) = \theta \mathsf{t}_f(\theta)$. For simplicity, we denote $\zeta(\theta)$ by ζ. Then, Eq. (44) at $h = H_l$ can be written as

$$G(\theta, \zeta) \;:=\; \theta e^{-\zeta}(\theta + \alpha(\theta)(1-\zeta)) - H_l \zeta \;=\; 0 \;.$$

We assume, contrary to our result, that there exists $\theta^* < \tilde{\theta}$ that solves Eq. (44). Then, we show that we have $\frac{\partial G}{\partial \theta}\big|_{\theta=\theta^*} > 0$ and $\frac{\partial G}{\partial \theta}\big|_{\theta=\tilde{\theta}} > 0$. This implies that there cannot exist $\theta^* < \tilde{\theta}$ that solves Eq. (44).

We consider the following two cases: i- $1 - \zeta \geq 0$ and ii- $1 - \zeta < 0$.

Case i- By the FOC, we have $\frac{\partial G}{\partial \zeta} = 0$. This implies that

$$\begin{aligned}
\frac{\partial G}{\partial \theta}\Big|_{\theta=\theta^*} \;&=\; e^{-\zeta}(\theta^* + \alpha(\theta^*)(1-\zeta)) + \theta^* e^{-\zeta}(1 + \alpha'(\theta^*)(1-\zeta)) \\
&=\; \frac{H_l \zeta}{\theta^*} + \theta^* e^{-\zeta}(1 + \alpha'(\theta^*)(1-\zeta)) \;\geq\; 0 \;,
\end{aligned} \tag{46}$$

where the second equation holds because $G(\theta^*, \zeta) = 0$ and the inequality holds because $1 - \zeta \geq 0$. Note that the above equation also implies that $\frac{\partial G(\theta,\zeta)}{\partial \theta}\big|_{\theta=\tilde{\theta}} > 0$ considering the fact that at $\theta = \tilde{\theta}$, we have $1 - \zeta \;=\; 1 - \tilde{\theta}\mathsf{t}_f(\tilde{\theta}) \;=\; 0$.

Case ii- Next we focus on the case of $1 - \zeta < 0$. In the following, we show when $1 - \zeta < 0$ and $\alpha'(\theta) \leq \frac{(\sqrt{5}+1)^2}{2} \approx 5.2$ for any $\theta \leq \tilde{\theta}$, we get $\frac{\partial G}{\partial \theta}\big|_{\theta=\theta^*} > 0$. This implies that θ^* does not exist.

By definition,

$$\begin{aligned}
\frac{\partial G}{\partial \theta}\Big|_{\theta=\theta^*} \;&=\; e^{-\zeta}\left(2\theta^* + (\theta^*\alpha'(\theta^*) + \alpha(\theta^*))(1-\zeta)\right) \\
&\geq\; e^{-\zeta}\left(2\theta^* + (\theta^*\alpha'(\theta^*) - 2\theta^*)(1-\zeta))\right) \\
&=\; e^{-\zeta}\theta^*\left(\alpha'(\theta^*)(1-\zeta) + 2\zeta\right) \;.
\end{aligned} \tag{47}$$

The inequality holds because $1 - \zeta < 0$ and $\theta^* \leq \tilde{\theta}$. Note that for any $\theta^* \leq \tilde{\theta}$, $\alpha(\theta^*) \leq -2\theta^*$. To complete the proof, we show that $(\alpha'(\theta^*)(1-\zeta) + 2\zeta) \geq 0$ when $\alpha'(\theta^*) \;\leq\; \frac{(\sqrt{5}+1)^2}{2}$.

First assume that $\alpha'(\theta^*) \leq 2$. Then, we get

$$(\alpha'(\theta^*)(1-\zeta) + 2\zeta) \;\geq\; 2(1-\zeta) + 2\zeta \;=\; 2 \;>\; 0 \;,$$

where the first inequality holds because $1 - \zeta < 0$.

Now, assume that $\alpha'(\theta^*) \in [2, \frac{(\sqrt{5}+1)^2}{2}]$. We make use of the following claim.

Claim: Let $\theta^* < \tilde{\theta}$ solve Eq. (44). Then, $\zeta \;=\; 1 - \mathsf{t}_f(\theta^*)\theta^* \;\leq\; \frac{1+\sqrt{5}}{2}$.

The proof of the claim is given at the end of the proof of this lemma.

Given that $\alpha'(\theta^*) \in [2, \frac{(\sqrt{5}+1)^2}{2}]$, then $\zeta \mapsto (\alpha'(\theta)(1-\zeta)+2\zeta)$ is decreasing. Then, by the claim, we get

$$\alpha'(\theta)(1-\zeta)+2\zeta \;\geq\; \alpha'(\theta)\left(1-\frac{1+\sqrt{5}}{2}\right)+2\left(\frac{1+\sqrt{5}}{2}\right) \;\geq\; 0 \;,$$

where the second inequality holds because $\alpha'(\theta) \;\leq\; \frac{(\sqrt{5}+1)^2}{2} \;\approx\; 5.2$.

Proof of Claim: Since θ^* solves Eq. (44) and $\mathfrak{t}_f(\theta^*)$ is the FOC solution, we get

$$\theta^* e^{-\zeta}(\theta^*+\alpha(\theta^*)(1-\zeta)) \;=\; H_l\zeta \qquad \text{and} \qquad -\theta^* e^{-\zeta}(\theta^*+\alpha(\theta^*)(2-\zeta)) \;=\; H_l \;,$$

where the second equation implies that $\zeta \leq 2$. By dividing these two equations, we get $-\frac{(\theta^*+\alpha(\theta^*)(1-\zeta))}{(\theta^*+\alpha(\theta^*)(2-\zeta))} - \zeta = 0$. This can be simplified as

$$\theta^* + \alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta} \;=\; 0 \;,$$

where for any $\zeta \in [1,2]$, $\zeta \mapsto \frac{1+\zeta-\zeta^2}{1+\zeta}$ is decreasing, $\frac{1+\zeta-\zeta^2}{1+\zeta}$ crosses zero at $\frac{1+\sqrt{5}}{2}$, and at $\zeta=1$, $\frac{1+\zeta-\zeta^2}{1+\zeta}\big|_{\zeta=1} = \frac{1}{2}$. We note that $\theta^*+\alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta}\big|_{\zeta=1} = \theta^* + \frac{1}{2}\alpha(\theta^*) \;<\; 0$ for any $\theta^* < \tilde{\theta}$, and $\theta^*+\alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta}\big|_{\zeta=\frac{1+\sqrt{5}}{2}} \;=\; \theta^* \;\geq\; 0$. Then, we can conclude that ζ that solves $\theta^*+\alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta} \;=\; 0$ should be less than $\frac{1+\sqrt{5}}{2}$.

□

Appendix G: Proof of Supporting Results of Appendix C

G.1. Proof of Lemma 4

It is easy to verify that $(1-g'(\theta)\mathfrak{t}_g(\theta)\theta) = 1$ for any $\theta > \theta_H$, and it is zero for any $\theta \leq \theta_L$. Thus, it suffices to show that $(1-g'(\theta)\mathfrak{t}_g(\theta)\theta) \geq 0$ when $\theta \in [\theta_L, \theta_H]$.

By definition, for any $\theta \in [\theta_L, \theta_H]$, we have

$$(1-g'(\theta)\mathfrak{t}_g(\theta)\theta) \;=\; -\theta\left(\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}\right) \;.$$

Since $\theta > 0$, to show $(1-g'(\theta)\mathfrak{t}_g(\theta)\theta) > 0$, we only need to verify that $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)} \leq 0$. To that end, we show that $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}$ is decreasing in θ. Then by the fact that $\frac{1}{\alpha(\theta_L)}+\frac{g'(\theta_L)}{g(\theta_L)} = 0$, we have $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)} \leq 0$ for any $\theta \in [\theta_L, \theta_H]$.

The derivative of $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}$ w.r.t. θ is given by

$$\frac{-\alpha'(\theta)}{\alpha(\theta)^2} + (\frac{g'(\theta)}{g(\theta)})' \;\leq\; 0 \;.$$

The inequality holds because by Assumption 2, we have $\alpha'(\theta) \geq 0$, and by Assumption 1, $\frac{g'(\theta)}{g(\theta)}$ is decreasing in θ.

G.2. Proof of Lemma 5

To show the result, we will verify that $\lambda_g(\theta) \geq 0$ for any $\theta \leq \theta_L$.

By Eq. (19), for any $\theta \leq \theta_L$,

$$\lambda_g(\theta) \;=\; f'(\theta)\left(\frac{g(\theta)}{g'(\theta)}+\alpha(\theta))\right) + f(\theta)\left(\left(\frac{g(\theta)}{g'(\theta)}\right)' + \alpha'(\theta)\right) \;.$$

We consider the following cases:

1- $f'(\theta) \le 0$: Observe that the first term of $\lambda_g(\theta)$ is nonnegative. This is the case because by Assumption 1, $\frac{g(\theta)}{g'(\theta)} + \alpha(\theta) \le 0$ for any $\theta \le \theta_L$. To see why note that $\frac{g(\theta)}{g'(\theta)} + \alpha(\theta)$ is increasing in θ, and $\frac{g(\theta_L)}{g'(\theta_L)} + \alpha(\theta_L) = 0$. In addition, note that the second term of $\lambda_g(\theta)$, i.e., $f(\theta)\big(\big(\frac{g(\theta)}{g'(\theta)}\big)' + \alpha'(\theta)\big)$, is greater than or equal to zero. This holds because by Assumption 1, $\left(\frac{g(\theta)}{g'(\theta)}\right)' \ge 0$.

1- $f'(\theta) \ge 0$: Since $g'(\theta) \ge 0$, we have

$$\lambda_g(\theta) \ \ge \ f'(\theta)\alpha(\theta) + f(\theta)\alpha'(\theta) \ = \ (f(\theta)\alpha(\theta))' \ \ge \ 0 \ .$$

The last inequality holds because $f(\theta)\alpha(\theta) = F(\theta) - 1$ is increasing in θ.

G.3. Proof of Lemma 6

Here, we show that for any $\theta \ge \theta_L$, $\arg\max_{t \ge 0} \{R(\theta,t)\} = \mathfrak{t}_g(\theta)$. To this end, we show that the objective function, i.e., $R(\theta,t)$, has an inverted u-shape in t. Then, we show that for $\theta \ge \theta_H$, $R(\theta,t)$ achieves its maximum at $t = 0$, and for $\theta \in [\theta_L, \theta_H]$, $R(\theta,t)$ gets maximized at $\mathfrak{t}_g(\theta)$, where $\mathfrak{t}_g(\theta)$ solves the FOC; that is, $\mathfrak{t}_g(\theta) = \arg\max_t R(\theta,t)$ for $\theta \in [\theta_L, \theta_H]$.

$R(\theta,t)$ Has an Inverted U-shape: Let $t_0 = \frac{g(\theta)\theta + 2\alpha(\theta)g'(\theta)\theta + \alpha(\theta)g(\theta)}{\alpha(\theta)g(\theta)g'(\theta)\theta}$. We will show that $R(\theta,t)$ is concave for any $t \le t_0$ and is convex otherwise. We further show that $R(\theta,t)$ is decreasing for any $t \ge t_0$ and is increasing at $t = -\infty$. This implies that $R(\theta,t)$ has an inverted u-shape and $\arg\max_t R(\theta,t) < t_0$.

The second derivative of $R(\theta,t)$ w.r.t. t is given by

$$\frac{\partial^2 R(\theta,t)}{\partial^2 t} \ = \ -g(\theta)e^{-g(\theta)t}(-g(\theta)\theta - \alpha(\theta)g(\theta)(1 - g'(\theta)\theta t) - 2\alpha(\theta)g'(\theta)\theta)$$

It is easy to observe that the second derivative is negative for any $t \le t_0$, and positive otherwise. This implies that the objective function is concave for any $t \le t_0$.

Next, we discuss the first derivative of $R(\theta,t)$ w.r.t. t. By definition,

$$\frac{\partial R(\theta,t)}{\partial t} \ = \ e^{-g(\theta)t}\big(-g(\theta)\theta - \alpha(\theta)g(\theta)(1 - g'(\theta)\theta t) - \alpha(\theta)g'(\theta)\theta\big). \tag{48}$$

This leads to

$$\frac{\partial R(\theta,t)}{\partial t}\Big|_{t=t_0} \ = \ e^{-g(\theta)t_0}\big(\alpha(\theta)g'(\theta)\theta\big) \le 0$$

Then, considering the fact that $\lim_{t\to\infty} \frac{\partial R(\theta,t)}{\partial t} = 0$ and $R(\theta,t)$ is convex for any $t \ge t_0$, we can conclude that $\frac{\partial R(\theta,t)}{\partial t} \le 0$ for any $t \ge t_0$. The proof of this part is completed by observing $\lim_{t\to-\infty} \frac{\partial R(\theta,t)}{\partial t} > 0$.

So far, we established that $R(\theta,t)$ has an inverted u-shape in t. This implies that the unique maximizer of the $R(\theta,t)$, i.e., $\arg\max_t R(\theta,t)$ solves the FOC. Let us call the unique maximizer, the FOC solution and denote it by $\mathfrak{t}_f(\theta)$. We will show that for any $\theta \ge \theta_H$, the FOC solution is negative. Then, by the fact that $R(\theta,t)$ has an inverted u-shape, we can conclude that for any $\theta \le \theta_H$, $R(\theta,t)$ gets maximized at $\mathfrak{t}_g(\theta) = 0$. Similarly, one can show that the FOC solution is positive for any $\theta \in [\theta_L, \theta_H]$.

By Eq. (48), the FOC solution, which solves $\frac{\partial R(\theta,t)}{\partial t} = 0$, is equal to $\mathfrak{t}_f(\theta) = \frac{\alpha(\theta)g(\theta) + g(\theta)\theta + \alpha(\theta)g'(\theta)\theta}{\alpha(\theta)g(\theta)g'(\theta)\theta}$. Since $g'(\theta) \ge 0$, the FOC solution is negative if

$$g'(\theta)\mathfrak{t}_f(\theta) \ = \ \frac{1}{\alpha(\theta)} + \frac{g'(\theta)}{g(\theta)} + \frac{1}{\theta} \ \le \ 0 \ .$$

By Assumptions 1 and 2, $\frac{1}{\alpha(\theta)} + \frac{g'(\theta)}{g(\theta)} + \frac{1}{\theta}$ is decreasing in θ. Then, considering this and the fact that $g'(\theta_H)\mathfrak{t}_f(\theta_H) = 0$, we have $g'(\theta)\mathfrak{t}_f(\theta) \le 0$ for any $\theta \ge \theta_H$. This implies that $\mathfrak{t}_f(\theta) \le 0$ for any $\theta \ge \theta_H$.

G.4. Proof of Lemma 7

First of all, it is easy to observe that $\mathsf{t}_g(\theta)$ is continuous and by definition of $\mathsf{t}_g(\cdot)$ and Assumption 1, $\mathsf{t}_g(\theta)$ is decreasing for $\theta \geq \theta_H$ and $\theta < \theta_L$. Considering this, in the following we will show that $\mathsf{t}_f(\theta)$ is decreasing when $\theta \in [\theta_L, \theta_H]$. Recall that for this range of θ, $\mathsf{t}_f(\theta)$ is the FOC solution.

By Eq. (48), the FOC solution, $\mathsf{t}_f(\theta)$ solves $Q(\theta, \mathsf{t}_f(\theta)) = 0$, where

$$Q(\theta, t) \; = \; \theta + \alpha(\theta)\Big(1 - g'(\theta)\theta t + \frac{g'(\theta)}{g(\theta)}\theta\Big). \tag{49}$$

This implies that $\partial_1 Q(\theta, \mathsf{t}_f(\theta)) + \partial_2 Q(\theta, \mathsf{t}_f(\theta))\frac{d\mathsf{t}_f(\theta)}{d\theta} = 0$, where $\partial_i Q(\theta, \mathsf{t}_f(\theta))$, $i = 1, 2$, is the derivative of Q w.r.t. to its i^{th} argument. This leads to

$$\frac{d\mathsf{t}_f(\theta)}{d\theta} \; = \; -\frac{\partial_1 Q(\theta, \mathsf{t}_f(\theta))}{\partial_2 Q(\theta, \mathsf{t}_f(\theta))} \; = \; \frac{1 + \alpha'(\theta)\Big(1 - g'(\theta)\theta \mathsf{t}_f(\theta) + \frac{g'(\theta)}{g(\theta)}\theta\Big) + \alpha(\theta)\partial_1 H(\theta, \mathsf{t}_f(\theta))}{g'(\theta)\theta\alpha(\theta)} \tag{50}$$

where $H(\theta, t) = 1 - g'(\theta)\theta t + \frac{g'(\theta)}{g(\theta)}\theta$. Next, we will show that $\partial_1 H(\theta, \mathsf{t}_f(\theta)) \leq 0$. This confirms that $\frac{d\mathsf{t}_f(\theta)}{d\theta} \leq 0$. This is so because $\alpha'(\theta) \geq 0$ and $\big(1 - g'(\theta)\theta \mathsf{t}_f(\theta) + \frac{g'(\theta)}{g(\theta)}\theta\big) = -\frac{\theta}{\alpha(\theta)} \geq 0$.

We consider the following cases:

- $\theta + \alpha(\theta) \geq 0$: By definition,

$$\partial_1 H(\theta, \mathsf{t}_f(\theta)) = -(g'(\theta)\theta)'\mathsf{t}_f(\theta) + \left(\frac{g'(\theta)}{g(\theta)}\right)'\theta = -\left(\frac{g'(\theta)}{g(\theta)}\theta g(\theta)\right)'\mathsf{t}_f(\theta) + \left(\frac{g'(\theta)}{g(\theta)}\right)'\theta$$
$$= -\left(\left(\frac{g'(\theta)}{g(\theta)}\right)'\theta g(\theta) + (\theta g(\theta))'\left(\frac{g'(\theta)}{g(\theta)}\right)\right)\mathsf{t}_f(\theta) + \left(\frac{g'(\theta)}{g(\theta)}\right)'\theta$$
$$= -\left(\frac{g'(\theta)}{g(\theta)}\right)'\theta(\mathsf{t}_f(\theta)g(\theta) - 1) - \left((\theta g(\theta))'\left(\frac{g'(\theta)}{g(\theta)}\right)\right)\mathsf{t}_f(\theta)$$

To show $\partial_1 H(\theta, \mathsf{t}_f(\theta)) \leq 0$, it suffices to verify that $(\mathsf{t}_f(\theta)g(\theta) - 1) \leq 0$. To see why note that $\frac{g'(\theta)}{g(\theta)}$ is decreasing and $g(\theta)$ is increasing in θ. By Eq. (5), we have

$$\mathsf{t}_f(\theta)g(\theta) - 1 \; = \; \frac{g'(\theta)}{g(\theta)}\left(\frac{\theta + \alpha(\theta)}{\alpha(\theta)\theta}\right) \; \leq \; 0$$

The inequality, which is the desired result, holds because $\theta + \alpha(\theta) \geq 0$.

- $\theta + \alpha(\theta) < 0$: By definition, $H(\theta, t) = 1 - g'(\theta)\theta\big(t - \frac{1}{g(\theta)}\big)$. Then, by taking derivative w.r.t. θ, we have

$$\partial_1 H(\theta, \mathsf{t}_f(\theta)) = -(g'(\theta)\theta)'\big(\mathsf{t}_f(\theta) - \frac{1}{g(\theta)}\big) - \frac{(g'(\theta))^2\theta}{g^2(\theta)}$$

To show $\partial_1 H(\theta, \mathsf{t}_f(\theta)) \leq 0$, it suffices to show the first term, i.e., $-(g'(\theta)\theta)'\big(\mathsf{t}_f(\theta) - \frac{1}{g(\theta)}\big)$, is negative. By Assumption 1, $(g'(\theta)\theta)$ is increasing. Thus, we only need to verify $\big(\mathsf{t}_f(\theta) - \frac{1}{g(\theta)}\big) \geq 0$. By Eq. (5),

$$\mathsf{t}_f(\theta) - \frac{1}{g(\theta)} \; = \; \frac{1}{g'(\theta)}\left(\frac{\theta + \alpha(\theta)}{\alpha(\theta)\theta}\right) \; \geq \; 0,$$

where the inequality holds because $\theta + \alpha(\theta) < 0$.

Appendix H: Proof of Supporting Results of Appendix F

H.1. Proof of Lemma 14

- **$\mathsf{t}_f(\theta)$ is decreasing in θ:** Since $\mathsf{t}_f(\theta)$ is the FOC solution, we have

$$\frac{\partial(R(\theta,t)-ht)}{\partial t}\Big|_{t=\mathsf{t}_f(\theta)} = -\theta e^{-\mathsf{t}_f(\theta)\theta}(\theta+\alpha(\theta)(2-\theta\mathsf{t}_f(\theta))) - h \;=\; 0 \;.$$

Define $W(\theta,t) := \frac{\partial(R(\theta,t)-ht)}{\partial t} = -\theta e^{-t\theta}(\theta+\alpha(\theta)(2-\theta t)) - h$. Then, the FOC implies that $W(\theta,\mathsf{t}_f(\theta)) = 0$. Thus,

$$\frac{\partial \mathsf{t}_f(\theta)}{\partial\theta} = -\frac{W_\theta(\theta,\mathsf{t}_f(\theta))}{W_t(\theta,\mathsf{t}_f(\theta))}\;,$$

where $W_\theta(\theta,\mathsf{t}_f(\theta)) = \frac{\partial W(\theta,t)}{\partial\theta}\big|_{t=\mathsf{t}_f(\theta)}$ and $W_t(\theta,\mathsf{t}_f(\theta)) = \frac{\partial W(\theta,t)}{\partial t}\big|_{t=\mathsf{t}_f(\theta)}$. Throughout the proof, for simplicity, we denote $\mathsf{t}_f(\theta)$ by t. In the following, we will show that both $W_\theta(\theta,t)$ and $W_t(\theta,t)$ are non-positive. This implies that $\frac{\partial \mathsf{t}_f(\theta)}{\partial\theta} \le 0$.

By definition, we get

$$\begin{aligned} W_\theta(\theta,t) &= -(1-t\theta)e^{-t\theta}\big(\theta+\alpha(\theta)(2-t\theta)\big) - \theta e^{-t\theta}(1+\alpha'(\theta)(2-t\theta)-t\alpha(\theta)) \\ &= (1-t\theta)\frac{h}{\theta} - \theta e^{-t\theta}(1+\alpha'(\theta)(2-t\theta)-t\alpha(\theta))\;, \end{aligned}$$

where the second equality follows because $W(\theta,t) \;=\; 0$. Again, by the fact that $W(\theta,t) = 0$, we can replace $-\theta e^{-t\theta}$ by $\frac{h}{(\theta+\alpha(\theta)(2-\theta t))}$. Then,

$$\begin{aligned} W_\theta &= (1-t\theta)\frac{h}{\theta} + \frac{h}{\theta+\alpha(\theta)(2-t\theta)} - \theta e^{-t\theta}(\alpha'(\theta)(2-t\theta)-t\alpha(\theta)) \\ &= h(2-t\theta)\frac{\theta+\alpha(\theta)(1-t\theta)}{\theta\big(\theta+\alpha(\theta)(2-t\theta)\big)} - \theta e^{-t\theta}(\alpha'(\theta)(2-t\theta)-t\alpha(\theta)) \;\le\; 0\;. \end{aligned} \tag{51}$$

The inequality holds because by the FOC condition, i.e., $W(\theta,t) \;=\; 0$, we have $2-t \ge 0$ and $\big(\theta+\alpha(\theta)(2-t\theta)\big) \;\le\; 0$, and by our assumption that $R(\theta,t)-ht \ge 0$, we have $\theta+\alpha(\theta)(1-t\theta) \ge 0$. Note that since $R(\theta,t)-ht \;=\; e^{-t\theta}\big(\theta+\alpha(\theta)(1-t\theta)\big) - ht \ge 0$, we get $\theta+\alpha(\theta)(1-t\theta) \ge 0$.

Next, we show that $W_t(\theta,t) \le 0$. By definition,

$$W_t(\theta,t) =^2 \theta^2 e^{-t\theta}\big(\theta+\alpha(\theta)(2-t\theta)\big) +^2 \theta^2\alpha(\theta)e^{-t\theta} \;\le\; 0\;,$$

where the inequality holds because by the FOC condition, $\big(\theta+\alpha(\theta)(2-t\theta)\big) \le 0$. The above equation along with Eq. (51) imply that $\mathsf{t}_f(\cdot)$ is decreasing.

- $\boldsymbol{A(\theta,\mathsf{t}_h(\theta)) \in [0,1]}$**:** Note that $A(\theta,\mathsf{t}_h(\theta)) = 0$ for $\theta \le \theta_L^h$ and is 1 for $\theta \ge \theta_H^h$. Thus, it suffices to show that $A(\theta,\mathsf{t}_h(\theta)) \in [0,1]$ for any $\theta \in [\theta_L^h,\theta_H^h]$; see Lemma 22.

LEMMA 22. *When $h \le H_l$, then $(1-\mathsf{t}_h(\theta)\theta) \;\ge\; 0$ for any $\theta \in [\theta_L^h,\theta_H^h]$.*

H.1.1. Proof of Lemma 22 Here, we show that set $\{\theta : \theta > \theta_L^h,$ and $1 - \mathsf{t}_f(\theta)\theta = 0\}$ is empty. That is, there does not exists any $\theta > \theta_L^h$ with $1 - \mathsf{t}_f(\theta)\theta = 0$. Then, by the fact that $1 - \mathsf{t}_f(\theta_H^h)\theta_H^h = 1$ and $1 - \mathsf{t}_f(\theta_L^h)\theta_L^h = 0$, we have $1 - \theta\mathsf{t}_f(\theta) \geq 0$ for any $\theta \in [\theta_L^h, \theta_H^h]$.

Assume, contrary to our result, that there exists $\theta^* > \theta_L^h$ that solves $1 - \mathsf{t}_f(\theta^*)\theta^* = 0$. Then, we show that this cannot happen.

Let $\theta \in \{\theta_L^h, \theta^*\}$. Since $\mathsf{t}_f(\theta)$ is the FOC solution, we have $\frac{\partial(R(\theta,t)-ht)}{\partial t}\Big|_{t=\mathsf{t}_f(\theta)} = 0$. This condition can be rewritten as

$$W(\theta, \zeta, h) := -\theta e^{-\zeta}(\theta + \alpha(\theta)(2-\zeta)) - h = 0 ,$$

where $\zeta = \theta\mathsf{t}_f(\theta)$. In the following, we will show that for $\theta \in \{\theta_L^h, \theta^*\}$, we have $\frac{\partial\zeta}{\partial\theta} = -\frac{W_\theta}{W_\zeta} \leq 0$, where $W_\theta := \frac{\partial W(\theta,\zeta,h)}{\partial\theta}$ and $W_\zeta := \frac{\partial W(\theta,\zeta,h)}{\partial\zeta}$. This implies that there does not exists $\theta^* > \theta_L^h$ that solves $1 - \mathsf{t}_f(\theta^*)\theta^* = 0$.

To show $\frac{\partial\zeta}{\partial\theta} \leq 0$, we will verify that $W_\theta \leq 0$ and $W_\zeta \leq 0$. By definition,

$$W_\zeta = \theta e^{-\zeta}\big(\theta + \alpha(\theta)(2-\zeta)\big) + \theta\alpha(\theta)e^{-\zeta} \leq 0 ,$$

where the inequality follows from the FOC, i.e., the fact that $W(\theta, \zeta, h) = 0$. To make it more clear, by the FOC, $(\theta + \alpha(\theta)(2-\zeta)) < 0$ and as a result, $W_\zeta \leq 0$.

Next, we show that $W_\theta \leq 0$ for $\theta \in \{\theta_L^h, \theta^*\}$. By definition,

$$W_\theta = -e^{-\zeta}\left(2\theta + \big(\alpha(\theta) + \theta\alpha'(\theta)\big)(2-\zeta)\right).$$

By the fact that for $\theta \in \{\theta_L^h, \theta^*\}$, we have $1 - \theta\mathsf{t}_f(\theta) = 0$, and thus $\zeta = 1$. This shows that

$$W_\theta = -e^{-1}\left(2\theta + \alpha(\theta) + \theta\alpha'(\theta)\right) \leq 0 ,$$

where the inequality holds because $\theta_L^h \geq \tilde{\theta}$ and as a result $2\theta + \alpha(\theta) \geq 0$ for $\theta \in \{\theta_L^h, \theta^*\}$. Recall that $\theta^* > \theta_L^h$.

H.2. Proof of Lemma 15

We show the result for $h \leq H_l$ where H_l is defined in Eq. (11). A similar argument holds for $h > H_l$.

By definition, for any $h \leq H_l$, we have

$$R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta) = \begin{cases} \theta + \alpha(\theta) & \text{if } \theta \geq \theta_H^h; \\ R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) & \text{if } \theta \in [\theta_L^h, \theta_H^h]; \\ e^{-1}\theta - \frac{h}{\theta} & \text{if } \theta \in [\underline{\theta}_L, \theta_L^h]; \end{cases}$$

$R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)$ is obviously increasing when and $\theta \geq \theta_H^h$ and $\theta \leq \theta_L^h$. Furthermore, $R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)$ is a continuous function of θ because $\mathsf{t}_h(\theta)$ is continuous. Thus, it suffices to show that $R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)$ is increasing in $\theta \in [\theta_L^h, \theta_H^h]$.

Recall that $\mathsf{t}_h(\theta) = \mathsf{t}_f(\theta)$ for $\theta \in [\theta_L^h, \theta_H^h]$. That is, $\mathsf{t}_h(\theta)$ is the FOC solution. Thus, by the Envelope theorem, the derivative of $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta)$ w.r.t. θ is given by

$$\begin{aligned} \frac{\partial\big(R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta)\big)}{\partial\theta} &= -\mathsf{t}_f(\theta)e^{-\mathsf{t}_f(\theta)\theta}(\theta + \alpha(\theta)(2 - \mathsf{t}_f(\theta)\theta)) \\ &+ e^{-\mathsf{t}_f(\theta)\theta}(1 + \alpha'(\theta)(1 - \mathsf{t}_f(\theta)\theta)) \\ &= h\frac{\mathsf{t}_f(\theta)}{\theta} + e^{-\mathsf{t}_f(\theta)\theta}(1 + \alpha'(\theta)(1 - \mathsf{t}_f(\theta)\theta)) \geq 0 , \end{aligned}$$

where the inequality holds because, as we show in Lemma 14, $1-\mathsf{t}_f(\theta)\theta \ge 0$ for any $\theta \in [\theta_L^h, \theta_H^h]$, and the second equality follows from the FOC, i.e., by the fact that

$$\frac{\partial R(\theta,t)}{\partial t}\Big|_{t=\mathsf{t}_f(\theta)} - h \;=\; -\theta e^{-\mathsf{t}_f(\theta)\theta}(\theta+\alpha(\theta)(2-\theta\mathsf{t}_f(\theta))) - h \;=\; 0\,.$$

Finally, since $R(\underline{\theta}_L, \mathsf{t}_h(\underline{\theta}_L)) - h\mathsf{t}_h(\underline{\theta}_L) \;=\; 0$, we have $R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta) \ge 0$ for $\theta \ge \underline{\theta}_L$; see definition of $\underline{\theta}_L$ in Eq. (11).

H.3. Proof of Lemma 16

The proof is naturally divided into two parts. In the first part, we show that $\lambda_h(\theta) \ge 0$ for any $\theta < \underline{\theta}_L$ and in the second part, we show that $\lambda_h(\theta) \ge 0$ for any $\theta \in [\underline{\theta}_L, \theta_L^h]$.

First Part: By Eq. (35), for any $\theta \le \underline{\theta}_L$, we have

$$\lambda_h(\theta) \;=\; f'(\theta)\,(2\theta+\alpha(\theta)) + f(\theta)\,(2+\alpha'(\theta))\;. \tag{52}$$

We note that by definition, we have $e^{-1}(\underline{\theta}_L)^2 = h$. Thus, given that $h \le H_l = \tilde{\theta}^2 e^{-1}$, we have $\underline{\theta}_L \le \tilde{\theta}$. This implies that for any $\theta \le \underline{\theta}_L$, we have $2\theta + \alpha(\theta) \le 0$. Then, if $f'(\theta) \le 0$, we have $\lambda_h(\theta) \ge 0$. Now, assume that $f'(\theta) > 0$. Then,

$$\lambda_h(\theta) \;\ge\; f'(\theta)\alpha(\theta) + f(\theta)\alpha'(\theta) \;=\; (f(\theta)\alpha(\theta))' = (F(\theta)-1)' \ge 0\,, \tag{53}$$

where the first inequality holds because $f'(\theta) \ge 0$.

Second Part: By definition, for any $\theta \in [\underline{\theta}_L, \theta_L^h]$, we have

$$\begin{aligned}
\lambda_h(\theta) \;&=\; f'(\theta)\left(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}}\right) + f(\theta)\left(1+\alpha'(\theta)-\frac{h}{\theta^2 e^{-1}}\right)\\
&\ge\; f'(\theta)\left(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}}\right) + f(\theta)\left(1+\alpha'(\theta)-\frac{h}{(\underline{\theta}_L)^2 e^{-1}}\right)\\
&=\; f'(\theta)\left(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}}\right) + f(\theta)\alpha'(\theta)\;,
\end{aligned}$$

where the inequality holds because $\theta \ge \underline{\theta}_L$, and the last equation follows from definition of $\underline{\theta}_L$. We consider the following two cases.

Case i: $f'(\theta) \le 0$: To show $\lambda_h(\theta) \ge 0$, we use the fact that for $\theta \ge \underline{\theta}_L$, function $\theta \mapsto \theta + \alpha(\theta) + \frac{h}{\theta e^{-1}}$ is increasing in θ. Then, considering the fact that $\theta_L^h + \alpha(\theta_L^h) + \frac{h}{\theta_L^h e^{-1}} \;=\; 0$, we have $\left(\theta + \alpha(\theta) + \frac{h}{\theta e^{-1}}\right) \le 0$ for $\theta \in [\underline{\theta}_L, \theta_L^h]$. This implies that $\lambda_h(\theta) \ge 0$ when $f'(\theta) \le 0$.

The derivative of $\theta + \alpha(\theta) + \frac{h}{\theta e^{-1}}$ w.r.t. θ is given by

$$1+\alpha'(\theta) - \frac{h}{\theta^2 e^{-1}} \;\ge\; 1+\alpha'(\theta) - \frac{h}{(\underline{\theta}_L)^2 e^{-1}} \;=\; \alpha'(\theta) \;\ge\; 0\;,$$

where the first inequality holds because $\theta \ge \underline{\theta}_L$, and the second inequality follows from the definition of $\underline{\theta}_L$.

Case ii: $f'(\theta) > 0$: In this case, we have

$$\lambda_h(\theta) \;\ge\; f'(\theta)\alpha(\theta) + f(\theta)\alpha'(\theta) \;=\; (f(\theta)\alpha(\theta))' \;=\; (F(\theta)-1)' \;\ge\; 0\;.$$

The last inequality completes the proof.

H.4. Proof of Lemma 17

Here, we will show that for any θ, the objective function, $R(\theta,t)-ht$ is a unimodular function of t and achieves its maximum at the FOC solution, denoted by $\mathfrak{t}_f(\cdot)$. Then, we show that $\arg\max_{t\geq 0}\{R(\theta,t)-ht\} = \max\{\mathfrak{t}_f(\theta),0\} = \mathfrak{t}_h(\theta)$.

To show that the objective function is unimodular, we will make the following observations: 1- The derivative of the objective function w.r.t. t at $t=\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$ is negative, at $t=-\infty$ is ∞, and at $t=\infty$ is negative. 2- For any $t\leq \frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$, the objective function is a concave function of t, and for any $t>\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$, the objective function is a convex function of t. These two observations imply that for any given θ, $R(\theta,t)$ is a unimodular function of t, and achieves its maximum at $t<\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$.

First Part: The derivative of the objective function with respect to t is given by

$$\frac{\partial R(\theta,t)}{\partial t} - h \;=\; -\theta e^{-t\theta}(\theta+\alpha(\theta)(2-\theta t)) - h\ . \tag{54}$$

Note that as t approaches $-\infty$, the derivative of the objective function with respect to t converges to ∞. Furthermore, as t converges to ∞, the derivative goes to $-h$. In addition, one can easily show that the derivative is negative at $t=\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$.

Second Part: The second derivative of the objective function with respect to t is given by

$$(\theta)^2 e^{-t\theta}(\theta+\alpha(\theta)(3-\theta t))\ . \tag{55}$$

It is easy to observe that the second derivative is negative for any $t<\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$, and is nonnegative otherwise. This implies that the objective function is concave for any $t\leq\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$ and it is convex for any $t>\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$.

So far, we established that $R(\theta,t)-ht$ is a unimodular function of t and achieves its maximum at the FOC solution, denoted by $\mathfrak{t}_f(\cdot)$. By Lemma 14, the FOC solution is decreasing in θ. This and the fact that $\mathfrak{t}_f(\theta_H^h) = 0$ lead to $\max\{\mathfrak{t}_f(\theta),0\}=0$ for any $\theta\geq\theta_H^h$ and $\max\{\mathfrak{t}_f(\theta),0\}=\mathfrak{t}_f(\theta)=\mathfrak{t}_h(\theta)$ for any $\theta\in[\theta_L^h,\theta_H^h]$.

H.5. Proof of Lemma 18

Let $G(z,t) := ze^{-tz}(-1+2tz)-ht$. We show that for any $z\leq\underline{\theta}_L$, we have $\max_{t\geq 0}\{G(z,t)\}\leq 0$. First observe that $G(z,t=0)=-z\leq 0$ and $G(z,t=\infty)=-\infty$. Then, to show that $\max_{t\geq 0}\{G(z,t)\}\leq 0$, we will verify that $G(z,t)\leq 0$ at the FOC solution, i.e., t that solves

$$\frac{\partial G(z,t)}{\partial t} \;=\; e^{-tz}z^2(3-2tz)-h \;=\; 0\ .$$

We denote the FOC solution by $t_F(z)$, and we show that $G(z,t_F(z))\leq 0$.

To this aim, we show that i- $\frac{\partial G(z,t_F(z))}{\partial z}\geq 0$ when $(-1+2t_F(z)z)\geq 0$, ii- $zt_F(z)$ is increasing in z, and iii- $G(\underline{\theta}_L,t_F(\underline{\theta}_L))=0$. The fact that $zt_F(z)$ is increasing in z implies either $(-1+2t_F(z)z)\geq 0$ for any $z\leq\underline{\theta}_L$, $(-1+2t_F(z)z)\leq 0$ for any $z\leq\underline{\theta}_L$, or there exists $\hat{z}\in[\underline{\Theta},\underline{\theta}_L)$ such that $(-1+2t_F(z)z)>0$ for any $z>\hat{z}$ and $(-1+2t_F(z)z)\leq 0$. We will focus on the third case, as the proof for this case encompasses that of the other two cases.

First we show that $G(z,t_F(z))$ for any $z>\hat{z}$. Since $G(\underline{\theta}_L,t_F(\underline{\theta}_L))=0$ and $\frac{\partial G(z,t_F(z))}{\partial z}\geq 0$ when $(-1+2t_F(z)z)\geq 0$, for any $z>\hat{z}$, we have $G(z,t_F(z))\leq G(\underline{\theta}_L,t_F(\underline{\theta}_L))=0$, which is the desired result. Furthermore, for any $z\leq\hat{z}$, $G(z,t_F(z))\leq 0$ as for this range of z, we have $(-1+2t_F(z)z)\leq 0$.

- Claim i: $\frac{\partial G(z,t_F(z))}{\partial z} \leq 0$ when $(-1+2t_F(z)z) > 0$. By the envelope theorem, we get

$$\frac{\partial G(z,t_F(z))}{\partial z} \;=\; e^{-t_F(z)z}\left(-1+t_F(z)z\big(5-2t_F(z)z\big)\right) \;\geq\; 0\;,$$

where the inequality holds because $(-1+2t_F(z)z) > 0$ and by the FOC $(3-2t_F(z)z) \geq 0$. To see why note that $x \mapsto -1+x\big(5-2x\big)$ is positive when $x \in [\frac{1}{2}, \frac{3}{2}]$.

- Claim ii: $z \mapsto (zt_F(z))$ is an increasing function. Define $\zeta = zt_F(z)$. By the FOC, we have $W(z,\zeta) := e^{-\zeta}z^2(3-2\zeta) - h \;=\; 0$. Then,

$$\frac{\partial \zeta}{\partial z} \;=\; -\frac{\frac{\partial W(z,\zeta)}{\partial z}}{\frac{\partial W(z,\zeta)}{\partial \zeta}} \;=\; \frac{e^{-\zeta}z^2(5-2\zeta)}{2ze^{-\zeta}(3-2\zeta)} \;\geq\; 0\;,$$

where the inequality holds because by the FOC $3-2\zeta \geq 0$.

- Claim iii: $G(\underline{\theta}_L, t_F(\underline{\theta}_L)) = 0$. Note that $t_F(\underline{\theta}_L) = \frac{1}{\underline{\theta}_L}$ and as a result,

$$G(\underline{\theta}_L, t_F(\underline{\theta}_L)) \;=\; \underline{\theta}_L e^{-1} - \frac{h}{\underline{\theta}_L} \;=\; 0\;,$$

where the last equation follows from definition of $\underline{\theta}_L$.

H.6. Proof of Lemma 20

The proof has two parts. In the first part, we show that when $h \geq H_l$ and $\theta \geq \tilde{\theta}$, we have $1 - \mathfrak{t}_f(\theta)\theta \geq 0$. Then, in the second part of the proof, we show that $\underline{\theta}_M \geq \tilde{\theta}$. This implies that $1 - \mathfrak{t}_f(\theta)\theta \geq 0$ for any $\theta \in [\underline{\theta}_M, \theta_H^h]$, which is the desired result.

First Part: Here, we show that any solution of $1 - \mathfrak{t}_f(\theta)\theta = 0$, denoted by θ^*, is less than equal to $\tilde{\theta}$. Let $\bar{\theta}^*$ be the maximum of such solution; that is $\bar{\theta}^* = \max\{\theta : 1 - \mathfrak{t}_f(\theta)\theta = 0\}$. Then, considering the fact that $\bar{\theta}^* \leq \tilde{\theta}$, $1 - \theta_H^h \mathfrak{t}_f(\theta_H^h) = 1$, and $1 - \bar{\theta}^* \mathfrak{t}_f(\bar{\theta}^*) = 0$, we can conclude that $1 - \theta\mathfrak{t}_f(\theta) \;>\; 0$ for any $\theta \in [\tilde{\theta}, \theta_H^h]$.

Suppose, contrary to our claim, that there exists $\theta^* > \tilde{\theta}$ that solves $1 - \mathfrak{t}_f(\theta^*)\theta^* = 0$. By the FOC, we have

$$\frac{\partial R(\theta^*, t)}{\partial t}\Big|_{t=\mathfrak{t}_f(\theta^*)} \;=\; -\theta^* e^{-\theta^* \mathfrak{t}_f(\theta^*)}(\theta^* + \alpha(\theta^*)(2 - \theta^*\mathfrak{t}_f(\theta^*))) - h \;=\; 0$$

Since θ^* solves $1 - \mathfrak{t}_f(\theta^*)\theta^* = 0$, we get

$$\frac{\partial R(\theta^*, t)}{\partial t}\Big|_{t=\mathfrak{t}_f(\theta^*)} \;=\; -\theta^* e^{-1}(\theta^* + \alpha(\theta^*)) \;-\; h\;. \tag{56}$$

We note that $\theta^* \mapsto -\theta^* e^{-1}(\theta^* + \alpha(\theta^*))$ is decreasing in θ^*. This holds because

$$\frac{d(-\theta^*(\theta^* + \alpha(\theta^*)))}{d\theta^*} \;=\; -(2\theta^* + \alpha(\theta^*)) - \theta^*\alpha'(\theta^*) \;\leq\; 0\;,$$

where the inequality follows because $\theta^* > \tilde{\theta}$. This implies that

$$\max_{\theta^* \geq \tilde{\theta}}\{-\theta^* e^{-1}(\theta^* + \alpha(\theta^*))\} \;=\; -\tilde{\theta}e^{-1}(\tilde{\theta} + \alpha(\tilde{\theta})) \;=\; \tilde{\theta}^2 e^{-1} \;=\; H_l\;.$$

Then, by Eq. (56), we can conclude that when $h > H_l$, there does not exists any $\theta^* > \tilde{\theta}$ such that $1 - \mathfrak{t}_f(\theta^*)\theta^* = 0$.

Second Part: Here, we show that $\underline{\theta}_M \geq \tilde{\theta}$. To this aim, we show that $\frac{\partial \underline{\theta}_M}{\partial h} \geq 0$ when $1 - \mathfrak{t}_f(\underline{\theta}_M)\underline{\theta}_M \geq 0$. This verifies that $\underline{\theta}_M$ increases as we increase h from H_l. The reason is that at $h = H_l$, we have $\underline{\theta}_M = \tilde{\theta}$ and $1 - \mathfrak{t}_f(\underline{\theta}_M)\underline{\theta}_M = 0$. This implies at $h = H_l$, when h is increased, we have $\underline{\theta}_M \geq \tilde{\theta}$. Then, by the first part of the lemma, we know that $1-\mathfrak{t}_f(\underline{\theta}_M)\underline{\theta}_M \geq 0$ when we increase h. This allows us to repeat this procedure to show that $\frac{\partial \underline{\theta}_M}{\partial h} \geq 0$ for any $h \geq H_l$.

Let $\theta = \underline{\theta}_M$ and $\zeta = \mathfrak{t}_f(\theta)\theta$. Then, by definition, we have

$$G(\theta,\zeta,h) := \theta e^{-\zeta}(\theta+\alpha(\theta)(1-\zeta)) - h\zeta \ = \ 0 \ ,$$
$$W(\theta,\zeta,h) := -\theta e^{-\zeta}(\theta+\alpha(\theta)(2-\zeta)) - h \ = \ 0 \ .$$

The first equation follows from the fact that at $\theta = \underline{\theta}_M$, $R(\underline{\theta}_M, \mathfrak{t}_f(\underline{\theta}_M)) - h\mathfrak{t}_f(\underline{\theta}_M) = 0$ and the second equation follows from the FOC, i.e., $\frac{\partial (R(\theta,t)-ht)}{\partial t}\Big|_{t=\mathfrak{t}_f(\theta)} = 0$. In the following, we show that $\frac{\partial \theta}{\partial h} \geq 0$ when $1-\zeta \geq 0$.

The aforementioned equations imply that

$$\frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial h} + \frac{\partial G}{\partial \zeta}\frac{\partial \zeta}{\partial h} = \zeta \ ,$$
$$\frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial h} + \frac{\partial W}{d\zeta}\frac{\partial \zeta}{\partial h} = 1 \ .$$

This leads to

$$\frac{\partial \theta}{\partial h} \ = \ \frac{\begin{vmatrix} \zeta & \frac{\partial G}{\partial \zeta} \\ 1 & \frac{\partial W}{\partial \zeta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{d\theta} & \frac{\partial G}{d\zeta} \\ \frac{\partial W}{d\theta} & \frac{\partial W}{d\zeta} \end{vmatrix}} = \frac{\zeta \frac{\partial W}{\partial \zeta} - \frac{\partial G}{\partial \zeta}}{\frac{\partial G}{\partial \theta}\frac{\partial W}{\partial \zeta} - \frac{\partial G}{\partial \zeta}\frac{\partial W}{\partial \theta}} \ ,$$

It is easy to observe that $\frac{\partial G}{\partial \zeta} \ = \ W(\theta,\zeta,h) \ = \ 0$. Thus, $\frac{\partial \theta}{\partial h} \ = \ \frac{\zeta}{\frac{\partial G}{\partial \theta}}$. In the following, we will show that $\frac{\partial \theta}{\partial h} \geq 0$ by verifying $\frac{\partial G}{\partial \theta} \geq 0$. By definition,

$$\frac{\partial G}{\partial \theta} \ = \ e^{-\zeta}(\theta+\alpha(\theta)(1-\zeta)) + \theta e^{-\zeta}(1+\alpha'(\theta)(1-\zeta)) \ .$$

We note that the first term, i.e., $(\theta+\alpha(\theta)(1-\zeta))$, is nonnegative because $G(\theta,\zeta,h) = \theta e^{-\zeta}(\theta+\alpha(\theta)(1-\zeta)) - h\zeta \ = \ 0$. In addition, the second term is positive as $1-\zeta \geq 0$. This gives us $\frac{\partial G}{\partial \theta} \geq 0$ and thus $\frac{\partial \theta}{\partial h} \geq 0$.