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As Published: 10.24033/ast.1074

Publisher: Societe Mathematique de France

Persistent URL: https://hdl.handle.net/1721.1/136200

Version: Original manuscript: author's manuscript prior to formal peer review

Quantum Groups and Quantum Cohomology

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Abstract

In this paper, we study the classical and quantum equivariant cohomology of Nakajima quiver varieties for a general quiver Q. Using a geometric R-matrix formalism, we construct a Hopf algebra Y_Q, the Yangian of Q, acting on the cohomology of these varieties, and show several results about their basic structure theory. We prove a formula for quantum multiplication by divisors in terms of this Yangian action. The quantum connection can be identified with the trigonometric Casimir connection for Y_Q; equivalently, the divisor operators correspond to certain elements of Baxter subalgebras of Y_Q. A key role is played by geometric shift operators which can be identified with the quantum KZ difference connection.

In the second part, we give an extended example of the general theory for moduli spaces of sheaves on $\mathbb{C}^2$, framed at infinity. Here, the Yangian action is analyzed explicitly in terms of a free field realization; the corresponding R-matrix is closely related to the reflection operator in Liouville field theory. We show that divisor operators generate the quantum ring, which is identified with the full Baxter subalgebras. As a corollary of our construction, we obtain an action of the W-algebra $\mathcal{W}\big(\mathfrak{gl}(r)\big)$ on the equivariant cohomology of rank r moduli spaces, which implies certain conjectures of Alday, Gaiotto, and Tachikawa.

Contents

Chapter 1

Introduction

1.1 Fundamental structures and conjectures

1.1.1

This paper is about the equivariant quantum cohomology of Nakajima quiver varieties [84, 87]. We see it as part of a larger project [8] which connects equivariant quantum cohomology of symplectic resolutions with their quantizations and derived autoequivalences. These connections, however, will not be discussed here.

Here we develop a general structural theory for quantum cohomology of Nakajima quiver varieties associated to an arbitrary quiver Q. We formulate our answer in terms of a certain Hopf algebra Y_Q, called the Yangian of Q, which acts on the cohomology of Nakajima quiver varieties.

The construction of Y_Q and an analysis of its basic structure theory is another objective of this paper and occupies the bulk of its first half. In the case when Q has no loops, this construction is related to work of Varagnolo [121] and Nakajima [89], who construct a certain subalgebra of Y_Q via generators and relations. In this paper, we give an alternative approach which we will describe shortly.

In the second half of the paper, we work out explicitly what our theory means for the quiver with one vertex and one loop. In other words, we work out explicitly the quantum cohomology of the moduli spaces $\mathcal{M}(r,n)$ of framed rank r torsion free sheaves on $\mathbb{C}^2$, generalizing the previous work [103, 73] on the Hilbert schemes of points.

1.1.2

Let X be a smooth quasi-projective variety with an action of a reductive group G. Quantum cohomology is a commutative associative deformation of ordinary multiplication in equivariant cohomology $H^{\cdot}_{\mathsf{G}}(X)$ defined by

$$(\gamma_1 * \gamma_2, \gamma_3) = \sum_{\beta > 0} q^\beta \langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta \tag{1.1}$$

where $(\gamma_1, \gamma_2) = \int_X \gamma_1 \cup \gamma_2$ is the standard bilinear form on $H^{\cdot}_{\mathsf{G}}(X)$, β ranges over the cone of effective classes in $H_2(X, \mathbb{Z})$, q^β denotes the corresponding element of the semigroup algebra of the effective cone, and

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta \in H^{\cdot}_{\mathsf{G}}(\mathsf{pt}, \mathbb{Q})$$

is the virtual count of rational curves of degree β meeting cycles Poincaré dual to $\gamma_1, \gamma_2, \gamma_3$. See e.g. [20, 54] for an introduction.

As defined by (1.1), the structure constants of quantum multiplication are formal power series in q^β. However, one conjectures that for all *equivariant symplectic resolutions*, and Nakajima quiver varieties in particular, the series in (1.1) represents a rational function of q^β. We will prove a slightly weaker statement below. Thus we get a family of commutative associative multiplications on $H^{\cdot}_{\mathsf{G}}(X)$.

Note that working in equivariant cohomology is crucial as all nonequivariant counts $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta$ vanish for trivial reasons for $\beta \neq 0$.

1.1.3

A basic property of quantum multiplication is that

$$1 * \gamma = \gamma \,, \quad \forall \gamma \in H^{\cdot}_{\mathsf{G}}(X)\,. \tag{1.2}$$

For any structure of a commutative associative algebra with unit on a vector space H, the operators of multiplication form a maximal commutative subalgebra of $\mathrm{End}(H)$.

In particular, the operators of quantum multiplication, for different values of the quantum parameters q, form a $b_2(X)$-dimensional family of maximal commutative subalgebras in the algebra that they all generate. For brevity, we call these subalgebras the algebras of quantum multiplication. For $q = 0$, they specialize to the algebra of classical multiplication in $H^{\cdot}_{\mathsf{G}}(X)$.

Not much is known or conjectured about this algebraic structure for general X. For Nakajima quiver varieties, by contrast, one expects the following very strong link with much-studied structures in representation theory and mathematical physics.

1.1.4

The Nakajima quiver varieties $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ with parameters

$$\mathsf{v},\mathsf{w} \in \mathbb{N}^I\,, \quad \theta \in \mathbb{R}^I\,, \quad \zeta \in \mathbb{C}^I$$

are associated to a quiver Q with the vertex set I. The quiver Q may have loops and multiple edges. Nakajima varieties have large groups G of automorphism that preserve (or scale, for $\zeta = 0$) their natural symplectic form[1]. By construction, the space

$$H(\mathsf{w}) = \bigoplus_{\mathsf{v}} H^{\cdot}_{\mathsf{G}}(\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}))$$

will be a module over the Yangian Y_Q. By construction, operators of cup product by characteristic classes of universal bundles form a commutative subalgebra in Y_Q.

1.1.5

The algebras Y_Q generalize Yangians of simple finite-dimensional Lie algebras, as defined by Drinfeld [27]. Their origins lie in the theory of quantum integrable systems, see e.g. [34, 55, 64, 112] for an introduction.

A powerful correspondence between quantum integrable systems and moduli of vacua in supersymmetric gauge theories (of which Nakajima varieties are examples) was proposed in the work of Nekrasov and Shatashvili [96, 97, 98]. In particular, quantum group actions on their cohomology or K-theory constructed by Varagnolo and Nakajima fit into this framework.

For us, the main prediction of Nekrasov and Shatashvili is a conjectural identification of algebras of quantum multiplication with *Baxter subalgebras*[2] in the Yangian Y_Q.

[1] Note the quantum product is trivial unless $\zeta = 0$ because all curve contributions are proportional to the weight $\hbar$ of the symplectic form.

[2] Also known as *Bethe subalgebras*.

1.1.6

Independently, Bezrukavnikov conjectured a relation between the monodromy of the quantum differential equation, see (1.15) below, and autoequivalences of $D^b \operatorname{Coh}_{\mathsf{G}} X$ for symplectic resolutions X, see Section 1.6.2. This was inspired, in part, by the work of T. Bridgeland [13, 14], see also [4].

Towards the end of the special 2007/08 year at IAS, it was realized this conjecture is naturally a composition of two more basic ones. The first, which is proven in this paper for Nakajima varieties, identifies the quantum differential equation with the *trigonometric Casimir connection* for a certain Lie algebra $\mathfrak{g}_Q$. A related conjecture about quantum cohomology of Laumon spaces was made in [37].

For finite-dimensional Lie algebras, trigonometric Casimir connections were defined and studied by Toledano Laredo in [120]. As explained there, they are very closely related to the Yangians of the same Lie algebras. This links the conjectural frameworks of Nekrasov-Shatashvili and Bezrukavnikov. The trigonometric Casimir connection generalizes the rational Casimir connection studied in [44, 80, 119] and also by C. De Concini (unpublished).

After this, the second step of Bezrukavnikov's conjecture becomes a geometric description of the monodromy of trigonometric Casimir connections. This could be viewed as a natural extension of the monodromy conjecture made in [120].

1.1.7

It appears the ideas of both Nekrasov-Shatashvili and Bezrukavnikov may apply more generally than just for symplectic resolutions. For example, Laumon spaces discussed in [37] have a natural Poisson structure which is not symplectic.

Similarly, the most general moduli of vacua considered by Nekrasov and Shatashvili fail all key property of Nakajima varieties: they may not be smooth, not symplectic, and not resolutions of singularities.

In this paper, we use the existence of a symplectic form and of a proper map to an affine variety in an essential way. It is would be very interesting to make our constructions work in greater generality.

1.2 Baxter subalgebras and quantum multiplication

1.2.1

The construction of Y_Q and the notion of a Baxter subalgebra are best explained in the original language of quantum inverse scattering method. The main ingredient there is an *R-matrix*, that is, a collection of vector spaces F_i and operator-valued functions

$$R_{F_i,F_j}(u) \in \mathrm{End}(F_i \otimes F_j) \tag{1.3}$$

which satisfy the *Yang-Baxter equation*

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u)\,, \tag{1.4}$$

as operators in $F_i \otimes F_j \otimes F_k$. Here

$$R_{12} = R_{F_i,F_j} \otimes 1_{F_k} \in \mathrm{End}(F_i \otimes F_j \otimes F_k)\,,$$

et cetera. In principle, the argument u could be taken from an arbitrary abelian group; the case $u \in \mathbb{C}$ corresponds to Yangians.

For $m \in \mathrm{End}(F)$ and all $W \in \{F_i\}$, consider the operators

$$T_F(m,u) = \mathrm{tr}_F\,(m \otimes 1)\, R_{F,W}(u) \in \mathrm{End}(W)\,,$$

where the trace is taken over the first tensor factor. In the formalism of Faddeev, Reshetikhin, and Takhtajan [35], these operators generate the Yangian Y associated to R.

1.2.2

Let $\mathfrak{G} \subset \prod GL(F_i)$ be the centralizer of all R-matrices and take $g \in \mathfrak{G}$. It follows at once from the Yang-Baxter equation and invertibility of R that

$$[T_{F_1}(g,u_1), T_{F_2}(g,u_2)] = 0\,. \tag{1.5}$$

A pictorial proof of this is given in Figure 1.1. This means the operators $T_F(g,u)$, for fixed $g \in \mathfrak{G}$ and all $F \in \{F_i\}$, $u \in \mathbb{C}$ generate a commutative subalgebra of the Yangian. This is what is called a Baxter (or Bethe) subalgebra.

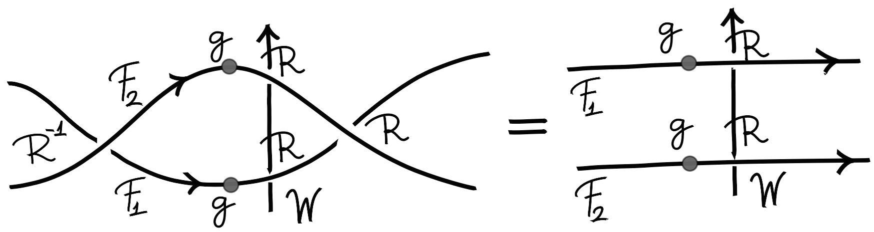

Figure 1.1: From the YB equation and $[g \otimes g, R] = 0$ *we deduce that* R_{F_2,F_1} *conjugates* $g_{F_2} R_{F_2,W} g_{F_1} R_{F_1,W}$ *to the product in the opposite order. Taking the trace over* $F_2 \otimes F_1$ *gives* (1.5).

1.2.3

Assuming for simplicity that $\mathfrak{G}$ is connected, a natural parameter set for Baxter subalgebras is a maximal torus

$$\mathfrak{H} \subset \mathfrak{G}/\text{Centralizer}(\mathsf{Y}) .$$

It may be compactified to $\overline{\mathfrak{H}} \supset \mathfrak{H}$ by considering limits of Baxter subalgebras as g degenerates. To connect with quantum cohomology, we need a map

$$\mathfrak{H} \to H^2(X, \mathbb{C})/2\pi i\, H^2(X, \mathbb{Z}) , \tag{1.6}$$

that extends to

$$\overline{\mathfrak{H}} \to \text{Kähler moduli space of } X .$$

1.2.4

There is a small, but essential detail in this identification, namely a shift of origin,

$$\mathfrak{H} \ni 1 \mapsto \pi i \boldsymbol{\kappa}_X \in H^2(X, \mathbb{C})/2\pi i\, H^2(X, \mathbb{Z})$$

for a certain class

$$\boldsymbol{\kappa}_X \in H^2(X, \mathbb{Z}/2)$$

that we call the canonical theta characteristic.

When $X = T^*Y$ then $\boldsymbol{\kappa}_X$ is the pull-back of the canonical class K_Y to X. Nakajima varieties are cotangent bundles only in sense of stacks, but still $\boldsymbol{\kappa}_X$ is well-defined, see Section 2.2.8.

1.2.5

It is very convenient to incorporate the shift

$$q^\beta \mapsto (-1)^{(\beta,\boldsymbol{\kappa})} q^\beta \tag{1.7}$$

into the definition of the quantum product. We call it the *modified* quantum product.

With this modification, we can use the map

$$H_2(X,\mathbb{Z}) \ni q^\beta \mapsto e^{(\beta,\cdot)} \in \mathfrak{H}^\wedge , \tag{1.8}$$

dual to (1.6), to identify operators $T_F(g,u)$ with operators of quantum multiplication. Note that a trace over an auxiliary space is an element in the group algebra $\mathbb{C}[\mathfrak{H}^\wedge]$, or its completion if the auxiliary space is infinite-dimensional.

1.2.6

To turn this into a practical description of the quantum product, one needs an R-matrix construction of the Yangian Y_Q.

The main geometric idea is simple and uses the embedding

$$\bigsqcup_{\mathsf{v}_1+\mathsf{v}_2=\mathsf{w}} \mathcal{M}_{\theta,\zeta}(\mathsf{v}_1,\mathsf{w}_1) \times \mathcal{M}_{\theta,\zeta}(\mathsf{v}_2,\mathsf{w}_2) \hookrightarrow \mathcal{M}_{\theta,\zeta}(\mathsf{v}_1+\mathsf{v}_2,\mathsf{w}) \tag{1.9}$$

as a fixed point set of a $\mathbb{C}^\times$-action. This embedding is, of course, well-known and played a central role in the work of M. Varagnolo and E. Vasserot [122, 123], H. Nakajima [90], and A. Malkin [71]. See in particular the paper [91] for further developments in this direction, closely related to our construction.

1.2.7

Suppose a torus A acts on a holomorphic symplectic variety X preserving the symplectic form. Then under fairly general hypotheses listed in Chapter 3, one can define a collection of maps, called *stable envelopes*,

$$\mathrm{Stab}_{\mathfrak{C}} : H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A}) \to H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X)$$

parameterized by certain chambers $\mathfrak{C}$ in Lie(A). Here G_A denotes the centralizer of A in G. Stable envelopes enjoy a number of remarkable geometric properties, see Chapters 3 and 4.

For $\mathsf{A} = \mathbb{C}^\times$ with fixed points (1.9), there are just two chambers $\pm\mathfrak{C}$ and one defines

$$R(u) = \mathrm{Stab}_{-\mathfrak{C}}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}}$$

where $u \in \mathbb{C} = \mathrm{Lie}\,\mathsf{A}$ is the equivariant parameter for A. The Yang-Baxter equation and other expected properties of R-matrices follow easily from general properties of stable envelopes. Thus, we have R-matrices (1.3) for

$$\{F_i\} = \{H(\mathsf{w})\}_{\mathsf{w}\in\mathbb{N}^I} .$$

See Chapter 5 for a precise definition of the corresponding Yangian Y_Q and Chapter 6 for further discussion of its properties.

1.2.8

Our R-matrices have the form

$$R(u) = 1 + \frac{\hbar}{u}\,\mathbf{r} + O(u^{-2})\,,$$

where $\hbar \in H^2_{\mathsf{G}}(\mathsf{pt})$ is the weight of the symplectic form and

$$\mathbf{r} \in S^2\mathfrak{g}_Q\,,$$

is an invariant tensor for a certain Lie algebra $\mathfrak{g}_Q$ which contains the Kac-Moody Lie algebra associated to the quiver Q. In particular, the action of $\mathfrak{g}_Q$ on $H(\mathsf{w})$ generalizes the construction of Nakajima [84, 87]. The action of $\mathfrak{g}_Q$ commutes with R-matrices.

If Q is a quiver of finite type then, modulo center, $\mathfrak{g}_Q$ is the corresponding Kac-Moody Lie algebra, but in general it is larger. For example, it may not be finitely generated like $\mathfrak{g}_Q \cong \widehat{\mathfrak{gl}(1)}$ for the quiver with one vertex and one loop. We expect the assignment

$$Q \mapsto \mathfrak{g}_Q$$

to behave well with respect to the natural operations on quivers. In particular, the results of Section 4.3 relate $\mathfrak{g}_{Q/\Gamma}$ and $\mathfrak{g}_Q^\Gamma$, where

$$Q \to Q' = Q/\Gamma$$

is a covering of quivers corresponding to $\Gamma \subset \pi_1(Q')$. An example of this is the well-known relation between $\widehat{\mathfrak{gl}(1)}$ and infinite Toeplitz matrices.

1.2.9

A maximal torus $\overline{\mathfrak{h}}_Q \subset \mathfrak{g}_Q$ is identified with

$$\overline{\mathfrak{h}}_Q = \mathfrak{h} \oplus \mathfrak{z} , \quad \mathfrak{h}, \mathfrak{z} \cong \mathbb{C}^I , \tag{1.10}$$

where $\mathfrak{h}$ and $\mathfrak{z}$ act on $H^{\cdot}_{\mathsf{G}}(\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}))$ by multiplication by linear functions of v and w, respectively. Note that $\mathfrak{z}$ is central in $\mathfrak{g}_Q$ and Y_Q.

1.2.10

The Lie algebra $\mathfrak{g}_Q$ acts on $H(\mathsf{w})$ by correspondences of the following shape. Let $0 \neq \alpha \in \mathbb{N}^I$ be a dimension vector and choose $\mathsf{w}_0 \in \mathbb{N}^I$ so that $\mathsf{w}_0 \cdot \alpha \neq 0$. For example, one can take $\mathsf{w}_0 = \delta_i$ for $i \in \operatorname{supp} \alpha$.

For all v, w, there is a canonical Lagrangian cycle

$$\mathbf{r}_{\mathsf{v},\mathsf{w},\alpha,\mathsf{w}_0} \subset \mathcal{M}(\mathsf{v}+\alpha,\mathsf{w}) \times \mathcal{M}(\mathsf{v},\mathsf{w}) \times \mathcal{M}(\alpha,\mathsf{w}_0) .$$

One can view this cycle as a correspondence between the second and the first factor in which the third factor is a parameter. This gives a map

$$H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\alpha,\mathsf{w}_0)) \to (\mathfrak{g}_Q)_\alpha , \tag{1.11}$$

which is surjective unless $\alpha < 0$, see Proposition 5.3.4. Here

$$\mathfrak{g}_Q = \overline{\mathfrak{h}}_Q \oplus \bigoplus_\alpha (\mathfrak{g}_Q)_\alpha \tag{1.12}$$

is the root decomposition of $\mathfrak{g}_Q$, that is, the decomposition into the eigenspaces of the adjoint action of $\mathfrak{h}$. Reading the same correspondence $\mathbf{r}_{\mathsf{v},\mathsf{w},\alpha,\mathsf{w}_0}$ in the opposite direction produces operators in $(\mathfrak{g}_Q)_{-\alpha}$.

1.2.11

From the construction of $X = \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ as a quotient by the action of $GL(\mathsf{v}) = \prod_{i\in I} GL(\mathsf{v}_i)$, one has tautological bundles $\mathcal{V}_i$ on X of ranks v_i for $i \in I$. The corresponding map

$$\mathbb{Z}^I \to \operatorname{Pic}(X) \cong H^2(X,\mathbb{Z})$$

given by $\det(\mathcal{V}_i)$, $i \in I$, is expected to be surjective for all v and an isomorphism for v sufficiently large (see Section 1.7 below). Dually, we have

$$H_2(X, \mathbb{Z}) \hookrightarrow \mathfrak{H}^\wedge$$

where $\mathfrak{H} \cong (\mathbb{C}^\times)^I$ is the torus with the Lie algebra $\mathfrak{h}$. Since this matches (1.8), we can state the following precise version of the Nekrasov-Shatashvili principle:

Conjecture 1. *The Baxter subalgebras of Y_Q corresponding to $g \in \mathfrak{H}$ are the algebras of modified equivariant quantum multiplication for Nakajima varieties.*

1.3 Quantum multiplication by divisors

1.3.1

Conjecture 1 may be approached in two steps, the first one being the identification of operators of quantum multiplication by divisors, that is, elements of $H^2(\mathcal{M})$.

The Yangian Y_Q has a grading which after doubling corresponds to cohomological degree. In this paper, we prove the following

Theorem 1.3.1. *The operators of cohomological degree 2 in the Baxter subalgebra are the operators of modified quantum multiplication by elements of $H^2_{\mathsf{G}}(\mathcal{M})_{\text{taut}}$*

Here

$$H^{\cdot}_{\mathsf{G}}(\mathcal{M})_{\text{taut}} \subset H^{\cdot}_{\mathsf{G}}(\mathcal{M}) \tag{1.13}$$

is the subalgebra spanned by the characteristic classes of the tautological bundles. An equality in (1.13) is expected.

1.3.2

Theorem 1.3.1 means the following concrete formula for quantum multiplication by

$$c_1(\lambda) = \sum \lambda_i \, c_1(\mathcal{V}_i)\,.$$

The Lie algebra $\mathfrak{g}_Q$ has an invariant bilinear form such that

$$(\mathfrak{g}_Q)_\alpha \perp (\mathfrak{g}_Q)_\beta \,, \quad \alpha+\beta \neq 0\,,$$

see Theorem 5.3.11. Abusing notation, we denote by

$$\{e_\alpha\} \subset (\mathfrak{g}_Q)_\alpha, \{e_{-\alpha}\} \subset (\mathfrak{g}_Q)_{-\alpha},$$

dual bases of root subspaces. Note the dimensions of the root spaces, known as root multiplicities, are finite by surjectivity in (1.11).

Theorem 1.3.1 is equivalent to the following

Theorem 1.3.2. *We have*

$$c_1(\lambda) *_{\text{modif}} \cdot \; = c_1(\lambda) \cup \cdot \; - \hbar \sum_{\theta\cdot\alpha>0} (\lambda,\alpha)\, \frac{q^\alpha}{1-q^\alpha}\; e_\alpha e_{-\alpha} + \dots\,, \tag{1.14}$$

where modified quantum product means the substitution (1.7)*, the sum is over roots of* $\mathfrak{g}_Q$ *with multiplicity, and dots stand for a multiple of the identity.*

The multiple of the identity left as dots in (1.14) is uniquely fixed from equation (1.2).

The operator $c_1(\lambda)\cup$ lies in the Yangian Y if $\theta>0$ or in a certain completion of the Yangian for general θ, see Section 10.1.1. Changing θ corresponds to flops of Nakajima varieties and formula (1.14) has the expected flop-covariance.

One can compare (1.14) to the more abstract structural statement for quantum multiplication by divisors derived in [12].

1.3.3

For $\lambda \in H^2_{\mathsf{G}}(X)$ consider the operators

$$\nabla_\lambda = \frac{d}{d\lambda} - \lambda\, * \tag{1.15}$$

acting in $H^{\boldsymbol{\cdot}}_{\mathsf{G}}(X)\otimes\mathbb{Q}(q^\beta)$ by

$$\frac{d}{d\lambda}\, q^\beta = (\lambda,\beta)\, q^\beta\,.$$

Note that $\frac{d}{d\lambda} = 0$ if λ is purely equivariant, that is, λ comes from $H^2_{\mathsf{G}}(\mathsf{pt})$. It is known very generally that

$$[\nabla_\lambda, \nabla_\mu] = 0$$

for all $\lambda, \mu \in H^2_{\mathsf{G}}(X)$. Hence any section of the projection $H^2_{\mathsf{G}}(X) \to H^2(X)$ defines a flat connection on a trivial $H^2_{\mathsf{G}}(X)$-bundle over $H^2(X)$. This connection is known as the quantum differential equation or Dubrovin connection.

Formula (1.14) precisely means that the quantum differential equation for Nakajima varieties is a trigonometric Casimir connection in the sense of [120]. To be precise, we prove this for $H^2(X)_{\text{taut}}$, which is expected to be the whole $H^2(X)$.

1.3.4

Conjecture 1 would be implied by the affirmative answer to the following

Question 1. *Do the operators* (1.14) *have a simple joint spectrum ? Equivalently, is quantum cohomology of Nakajima varieties generated by tautological divisors ?*

In this paper we treat the following special case.

Theorem 1.3.3. *The quantum cohomology of the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$ is generated by the divisor.*

These moduli spaces are Nakajima varieties associated the quiver of with one vertex and one loop. Our proof of Theorem 1.3.3 is based on an explicit representation of quantum multiplication by divisor in terms of Heisenberg operators.

1.4 Shift operators and qKZ

1.4.1

For simplicity, let us replace the group G by its maximal torus T. By construction, the elements of $H^{\cdot}_{\mathsf{T}}(X) \otimes \mathbb{Q}(q^\beta)$ are functions on

$$\mathfrak{t} \times H^2(X)\,,$$

where $\mathfrak{t} = \operatorname{Lie} \mathsf{T}$. The operators (1.15) define a flat connection along the $H^2(X)$-directions. In fact, this is a part of a flat difference-differential connection, in which the difference part corresponds to the lattice

$$\operatorname{Cochar}(\mathsf{T}) \subset \mathfrak{t}\,.$$

The corresponding operators

$$\mathsf{S}(\sigma) \in \operatorname{End} H^{\bullet}_{\mathsf{T}}(X) \otimes \mathbb{Q}[[q^{\beta}]]$$

are known as *shift operators* because they shift the values of the equivariant parameters in ∇_λ. They are constructed geometrically as follows.

1.4.2

Let

$$\sigma : \mathbb{C}^{\times} \to \mathsf{T}$$

be a cocharacter of T. To it, one associates a nontrivial X-bundle p

$$\begin{array}{ccc} X & \hookrightarrow & X^{\sim} \\ & & \big\downarrow{\scriptstyle p} \\ & & \mathbb{P}^1 \end{array}$$

over $\mathbb{P}^1$, see Chapter 8. By definition, rational curves in $X^{\sim}$ that map to the base $\mathbb{P}^1$ with degree 1 are the σ-twisted rational curves in X. Their enumerative geometry is closely related to the Gromov-Witten theory of X. In particular, the shift operator $\mathsf{S}(\sigma)$ is constructed from the virtual count of twisted 2-pointed rational curves with marked points in $p^{-1}(0), p^{-1}(\infty) \cong X$, see Section 8.1.7.

The flatness condition

$$\left[\nabla_\lambda, e^{-\frac{d}{d\sigma}}\, \mathsf{S}(\sigma)\right] = 0$$

is the $\varepsilon = 1$ specialization of Proposition 8.2.1. Here $e^{\frac{d}{d\sigma}}$ is the translation by $\sigma \in \mathfrak{t}$.

1.4.3

The key step in our proof of Theorem 1.3.2 is an explicit computation of the shift operators $\mathsf{S}(\sigma)$ for certain special cocharacters σ.

An action of $\mathbb{C}^\times$ on a symplectic resolution X is called *minuscule* if $H^0(\mathcal{O}_X)$ is generated by functions of weight $0, \pm 1$. One easily shows, see Section 2.6, that the $\mathbb{C}^\times$-action from (1.9) is minuscule. For minuscule σ, the operators $\mathsf{S}(\sigma)$ may be computed in term of R-matrices as follows.

1.4.4

A σ-fixed point $x \in X^\sigma$ defines a section ζ_x of p. The classes of these sections

$$[\zeta_x] \in H_2(X^\sim, \mathbb{Z})$$

lie in a single $H_2(X, \mathbb{Z})$-coset. Thus, up-to an overall multiple, q^ζ is a well-defined function from the set of components of X^σ to the group algebra of $H_2(X, \mathbb{Z})$. In fact, for Nakajima varieties, there is a preferred way to fix the ambiguity, see Section 9.1.5.

Recall the stable envelope maps

$$\mathrm{Stab}_\pm : H^{\bullet}_{\mathsf{T}}(X^\sigma) \to H^{\bullet}_{\mathsf{T}}(X)$$

and their ratio $R_\sigma = \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+$. Define

$$\nabla^\sigma_\lambda = \mathrm{Stab}_+^{-1} \circ \nabla_\lambda \circ \mathrm{Stab}_+ \,.$$

Theorem 9.3.1 in Section 9.3 is equivalent to the following

Theorem 1.4.1. *For minuscule σ, ∇^σ_λ commutes with the difference connection*

$$\Psi(t+\sigma) = (-1)^{(\zeta, \boldsymbol{\kappa}_X)}\, q^\zeta \, R_\sigma \, \Psi(t) \tag{1.16}$$

where we consider $\Psi \in H^{\bullet}_{\mathsf{T}}(X^\sigma) \otimes \mathbb{Q}(q^\beta)$ as a function of $t \in \mathfrak{t}$.

Here $\boldsymbol{\kappa}_X$ is the canonical theta characteristic discussed in Section 1.2.4 .

1.4.5

In the case of (1.9), we have

$$q^\zeta = q^{\mathsf{v}_1} = q^{\mathsf{v}} \otimes 1$$

where q^{v} lies in the torus $\mathfrak{H}$ with Lie algebra $\mathfrak{h}$. We thus recognize in (1.16) the quantum Knizhnik-Zamolodchikov equation of Frenkel and Reshetikhin, see [47].

1.4.6

It follows from Theorem 1.4.1 that the operator $(-1)^{(\zeta,\boldsymbol{\kappa}_X)}\, q^\zeta\, R_\sigma$ commutes with operators of quantum multiplication for minuscule σ. This plays a key role in the proof of Theorem 1.3.2. In other words, we determine the quantum connection ∇_λ through the commuting difference connection.

For this to work, it is important to relate Nakajima varieties with different framing vectors w as in (1.9). For instance, quantum cohomology of the moduli spaces of framed torsion free sheaves on $\mathbb{C}^2$ is a object of significant geometric interest, see below. From our perspective, it is easier to determine it for general rank then just in the special case of Hilbert schemes.

1.5 Yangian of $\widehat{\mathfrak{gl}(1)}$ and instanton moduli

1.5.1

In the second half of the paper, we make the general theory explicit in the case of the quiver Q with one vertex and one loop. Denote

$$r = \mathsf{w}_1\,, \quad n = \mathsf{v}_1\,.$$

The corresponding Nakajima variety

$$\mathcal{M}(r,n) = \mathcal{M}_{1,0}(\mathsf{v},\mathsf{w})$$

is the moduli space of framed rank r torsion-free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ with $c_2(\mathcal{F}) = n$. A framing of a sheaf $\mathcal{F}$, by definition, is a choice of an isomorphism

$$\phi : \mathcal{F}\big|_{L_\infty} \to \mathcal{O}_{L_\infty}^{\oplus r}$$

where $L_\infty \subset \mathbb{P}^2$ is a fixed line. Usually, the line L_∞ is viewed as the line at infinity of $\mathbb{C}^2 \subset \mathbb{P}^2$. The group

$$\mathsf{G} = GL(2) \times GL(r)$$

acts naturally on $\mathcal{M}(r,n)$, the first factor acting on $\mathbb{C}^2$ while the second acts by changing the framing.

See, for example, [88] for an introduction to the geometry of $\mathcal{M}(r,n)$. It plays an important role in Donaldson theory [26] and in mathematical approaches to supersymmetric quantum gauge theories, particularly in the work of Nekrasov [93]. By a theorem of Donaldson, a dense open subset of $\mathcal{M}(r,n)$, $r>1$, that parameterizes locally free sheaves is the moduli space of framed $U(r)$-instantons of charge n.

1.5.2

For $r=1$, $\mathcal{M}(r,n)$ becomes the Hilbert scheme of points, the quantum cohomology of which was determined in [103], a result that found applications to the enumerative theories of curves in threefolds [74].

Theorem 1.3.2 gives a new proof of this result and extends it to higher rank. We expect it to play a role in the higher rank Donaldson-Thomas theory of threefolds. In fact, higher rank DT theory of threefolds was one of the main motivations for the present work.

1.5.3

In Chapter 12 we relate the Lie algebra $\mathfrak{g}_Q$ to the Heisenberg algebra $\widehat{\mathfrak{gl}(1)}$ that acts on the cohomology of $\mathcal{M}(r,n)$ by the work of Nakajima [86], Grojnowski [51], and Baranovsky [6].

To be precise, for an arbitrary quiver we discuss two versions of the Yangian: the Yangian Y mentioned above and another, more economical, algebra $\mathbb{Y}$ which we call the *core Yangian*. They correspond to different normalization of R-matrices: those for Y fix vacuum vectors while those for $\mathbb{Y}$ scale them by certain Γ-factors, see Section 6.1.10.

For $\mathcal{M}(r,n)$, $\widehat{\mathfrak{gl}(1)}\subset\mathbb{Y}$, while $\mathfrak{g}_Q\subset\mathsf{Y}$ is the quotient of $\widehat{\mathfrak{gl}(1)}$ by the constant loops $\mathfrak{gl}(1)\subset\widehat{\mathfrak{gl}(1)}$.

1.5.4

Recall that Nakajima's Heisenberg algebra acts irreducibly on the cohomology $H(1)$ of

$$\mathcal{M}(1)=\bigsqcup_n \mathcal{M}(1,n)\,,$$

and this identifies $H(1)$ with the standard Fock space of one free boson. Stable envelopes give a map

$$H(1)^{\otimes r}\to H(r)\,,$$

which makes it possible to describe $H(r)$, and the Yangian action on it, in terms of r free bosons.

In this way, one recovers and generalizes many familiar objects of conformal field theory. For example, the Yangian of $\widehat{\mathfrak{gl}(1)}$ contains the Virasoro algebra in the Feigin-Fuchs representation.

The quantum integrable system given by the classical, that is $q = 0$, product in cohomology, is a certain generalization of the second-quantized trigonometric quantum Calogero-Sutherland system to r interacting bosonic fields, see Section 14.2. More generally, a connection between the quantum, that is $q \neq 0$, product in cohomology and a quantum intermediate long-wave equation is explored in [94].

1.5.5

In the literature, one can find many different ways to construct and study algebras that may be called a Yangian of $\widehat{\mathfrak{gl}(1)}$, see for example [29, 31, 41, 36, 68, 79, 108, 109]. Perhaps one of the advantages of our approach is that our $\mathsf{Y}(\widehat{\mathfrak{gl}(1)})$ is obtained by a general procedure, applicable to an arbitrary quiver.

1.5.6

For us, R-matrices are the main objects of study and those for $\mathcal{M}(r,n)$ turn out to be related to very interesting operators in CFT. Namely, in Section 14.3 we relate the R-matrix for $\mathsf{Y}(\widehat{\mathfrak{gl}(1)})$ to the reflection operator in Liouville theory. As far as we know, the Yang-Baxter equation satisfied by R has not been previously explored in the conformal field theory context.

Recall that Theorems 1.3.1 and 1.3.3 identify the algebra of quantum multiplication for $\mathcal{M}(r,n)$ as a Baxter subalgebra in $\mathsf{Y}(\widehat{\mathfrak{gl}(1)})$. The identification of R gives a mechanical procedure to write the corresponding commuting operators in terms of free bosons.

1.5.7

During the workshop at the Simons Center in January 2010, we were asked by Nakajima and Tachikawa whether our theory can help with some of the questions raised in the work of Alday, Gaiotto, and Tachikawa [2].

The connection is, indeed, very strong and some simple applications are immediate. For example, it is easy to describe the image of $\mathsf{Y}(\widehat{\mathfrak{gl}(1)})$ in its representation on $H(1)^{\otimes r}$ in terms of the vertex algebra $\mathcal{W}\big(\mathfrak{gl}(r)\big)$. This is discussed in Section 19.2. We anticipate many further applications in this direction. Similar results have recently been obtained by Schiffman-Vasserot [109].

Although applications to the conjectures of [2] appear at the end of the paper, they require very little of the preceding machinery. In particular, this is about purely classical cohomology of $\mathcal{M}(r,n)$, quantum products play no role.

Classical limits of the formula from which this discussion with Nakajima and Tachikawa started were later independently found in [28] and also [113].

1.6 Further directions

We conclude this Introduction with a brief discussion of some natural directions in which one can pursue the results of this paper.

1.6.1 K-theory

In [89], Nakajima constructs an action of $\mathcal{U}_q(\widehat{\mathfrak{g}_{\mathrm{KM}}})$ on the equivariant K-theory of quiver varieties. Here $\mathfrak{g}_{\mathrm{KM}}$ is a Kac-Moody Lie algebra and $\mathcal{U}_q(\widehat{\mathfrak{g}_{\mathrm{KM}}})$ is the quantized universal enveloping of the loop Lie algebra of $\mathfrak{g}_{\mathrm{KM}}$. These algebras are defined by explicit generators and relations, see [89].

A natural extension of the present work to K-theory would produce a larger Hopf algebra $\mathcal{U}_q(\widehat{\mathfrak{g}_Q})$, defined in the style of [35] and acting naturally on $K_{\mathsf{G}}(\mathcal{M}_Q)$. At least for quiver varieties, one can construct a K-theoretic analog of stable envelopes, which we expect to be the key ingredient for such project.

For the Jordan quiver, the K-theoretic R-matrix was computed in [101]. As expected, it is closely related to the results of [42, 108].

1.6.2 Monodromy of QDE and categorification

The quantum differential equation 1.15 defines a connection ∇ with regular singularities on the Kähler moduli space $\overline{\mathfrak{H}}$ of $\mathcal{M}_Q$, which is a compactification of the torus $\mathfrak{H} \cong (\mathbb{C}^\times)^I$. Consider the regular points

$$\mathfrak{H}_{\text{reg}} = \left\{ q \in \mathfrak{H} \,\middle|\, \forall \alpha \; q^\alpha \neq 1 \right\}$$

of this connection. The monodromy of ∇ defines a homomorphism

$$B = \pi_1(\mathfrak{H}_{\text{reg}}) \to \overline{\mathsf{Y}(\mathfrak{g}_Q)}\,,$$

where bar denotes a certain completion.

A generalization of the Toledano Laredo's monodromy conjecture for trigonometric Casimir connections [120, 48] identifies B with what should be called the quantum Weyl group of $\mathcal{U}_q(\widehat{\mathfrak{g}_Q})$. It was further conjectured by Bezrukavnikov that this action of B lifts to

$$B \to \operatorname{Aut} D^b \operatorname{Coh}_{\mathsf{G}} \mathcal{M}_Q\,.$$

This is known in a handful of cases, in particular for Hilbert schemes of points of $\mathbb{C}^2$, see [9]. Perhaps a categorical version of stable envelopes, obtained from the parabolic induction functors for quantizations of Nakajima varieties, is the proper technical tool to attack these problems.

1.6.3 Higher rank Donaldson-Thomas theory

The quantum cohomology of Hilbert scheme of points of a symplectic surface S is closely related to the Donaldson-Thomas theory of threefolds fibered in S over a curve. In particular, in the case of A_n surfaces, this point of view lead to an explicit description of DT invariants of toric threefolds [74].

For higher rank sheaves on ADE surfaces, there is again a close connection with DT theory, via Diaconescu's work on ADHM-sheaves [24], see also [18]. Arguments parallel to those in this paper should give an effective determination of the virtual invariants of the moduli of ADHM sheaves on a smooth projective curve in terms of our operators of quantum multiplication.

Using a Beilinson-type construction, as in section 7 of [24], the ADHM moduli spaces can be identified with a certain moduli space of higher-rank framed complexes on ADE-fibrations over curves.

For general quivers, Theorem 1.3.2 implies an identification (up to a scalar function) between the small J-function and I-function in these geometries (as defined in [18]),without any change of variables required.

1.6.4 Hilbert Schemes of points of general surfaces

For a general surface S, quantum cohomology of the Hilbert schemes of points and DT theory of S-fibrations will diverge and we expect the latter to have a better structure. However, we expect the classical cohomology $\mathrm{Hilb}(S)$ to be described as a $q = 0$ Baxter subalgebra for a certain R-matrix. In fact, this R-matrix should be the reflection operator R associated in Section 13.4 to the Frobenius algebra $\mathbb{H} = H^{\cdot}_{\mathsf{G}}(S)$. This is a joint project with Vivek Shende and one of its potential goals could be a better structural understanding of some of the many mysterious universal generating series in the theory through representation theory of Yangians.

1.6.5 K-theoretic DT theory

Perhaps one of the most challenging projects for the future would be to upgrade the connection with DT theory of 3-folds to the level of K-theory. K-theoretic DT invariants are a subject of interest in both mathematics and theoretical physics, due to their M-theoretic interpretation [95] and their connection to the motivic DT invariants [75].

1.7 Update

This work reflects what we knew in 2010, with some improvements to exposition made during 2010–12. As we revise it in the early 2017, it seems necessary to add a certain bare minimum of references to subsequent developments, in particular, in connection with directions for further researched outlined above. We decided to limit all such updates to this section.

A survey of the progress since 2012 may be found in [99, 100]. In particular, lectures [99] explain the extension of the present work to equivariant K-theory, including application to K-theoretic Donaldson-Thomas theory. In K-theory, the quantum differential equations studied here become quantum *difference* equation. Those were determined in [102] for all Nakajima varieties.

The monodromy problem for the quantum difference equations was analyzed in [1]. This analysis may be directly linked to Bezrukavnikov's quantization in characteristic $p \gg 0$, to the monodromy conjectures above [9], and to the categorical stable envelopes [53].

K. McGerty and T. Nevins proved in [77] that the equivariant cohomology of Nakajima varieties is generated by characteristic classes of tautological bundles (which is a property also called *Kirwan surjectivity*). At several points, e.g. in Section 1.2.11 or in the statement of Theorem 1.3.1 we had to work our way around Kirwan surjectivity which was not known at the time. The results of [77] make these workarounds unnecessary.

The goals stated in Section 1.6.4 were achieved by N. Arbesfeld [5].

1.8 Acknowledgments

1.8.1

It has taken us a few years to complete this project and, in the process, we received a great deal of help from many people.

At the beginning, the initial motivation came from unpublished conjectures made by Nikita Nekrasov and Samson Shatashvili, on the one hand, and Roma Bezrukavnikov, on the other, and we are grateful to them for sharing their insight with us. Both of us are novices in geometric representation theory and have learned a great deal from conversations with Victor Ginzburg and Hiraku Nakajima. We also thank Sasha Braverman, Pavel Etingof, Valerio Toledano Laredo, and other members of the FRG group who have helped us crystallize many of the ideas here.

We had many discussions with Eric Vasserot and Olivier Schiffmann, whose work [108, 109] has several parallel aspects, in particular, in applications to [2]. We thank Edik Frenkel, Davide Gaiotto, Victor Kac, and Yuji Tachikawa for sharing their knowledge of vertex algebras with us.

Some of our formulas were rediscovered in the literature on the AGT conjectures. In particular, our free-field formulas for cup product by $c_1(\mathcal{O}(1))$ for $\mathcal{M}(r,n)$ were also found in [28]. It is a pleasure to thank Vincent Pasquier and Didina Serban for very interesting discussions.

Additionally, we are grateful to Sabin Cautis, Sachin Gautam, Johan de Jong, David Kazhdan, Dmitry Kaledin, Michael McBreen, Andrei Negut, Rahul Pandharipande, Nikolai Reshetikhin, Vivek Shende, Daniel Shenfeld, Catharina Stroppel, and Zhiwei Yun for many related conversations.

We feel deeply indebted to the anonymous referee for his dedication to the very arduous task of working through these pages. Many places in our narrative have significantly gained in clarity thanks to his attentive and thoughtful

remarks.

1.8.2

As stated at the beginning, this paper is a part of a larger joint project with Bezrukavnikov, Braverman, Etingof, Finkelberg, Toledano Laredo, Losev, and others. This larger project will surely have a nonempty intersection with the ongoing work of Braden, Licata, Proudfoot, Webster, and their collaborators, although we don't know whether quantum cohomology currently plays a role in what they do. Eventually, we hope quantum cohomology will be an important part of the unified geometric and representation-theoretic study of equivariant symplectic resolutions.

1.8.3

We thank NSF for supporting our research. DM has been partially supported by a Clay Research Fellowship.

We thank Simons Center for Geometry and Physics for being the place where many of our results were first presented or written down. Another important venue where these results were presented and discussed was the 2010 Midrasha Mathematicae in Jerusalem. We thank its organizers for the invitation and hospitality.

Part I

General Theory

Chapter 2

Nakajima varieties

In this chapter, we recall definitions and basic facts on the geometry of Nakajima quiver varieties. There is a large literature on the subject, although most of what we need can be found in the original references [84, 87] and papers of Crawley-Boevey [21, 22]. We also explain some results on natural group actions on Nakajima quiver varieties.

2.1 Definition

2.1.1

Let Q be a quiver, i.e. an oriented multigraph, with finite vertex set I. We allow loops and multiple edges in Q. The quiver data is simply the adjacency matrix

$$Q = (q_{ij})_{i,j \in I}$$

where

$$q_{ij} = \left|\{\text{edges from } i \text{ to } j\}\right| .$$

For what follows, we can assume that multiple edges have the same orientation in Q. We also consider quivers $\overline{Q}$ and $\vec{Q}$ with vertex set given by the union $I \sqcup \overline{I}$ of two copies of the set I and with adjacency matrices

$$\overline{Q} = \begin{pmatrix} Q + Q^T & \text{id} \\ \text{id} & 0 \end{pmatrix} , \quad \vec{Q} = \begin{pmatrix} Q & 0 \\ \text{id} & 0 \end{pmatrix} .$$

2.1.2

A representation of a quiver is an assignment of a coordinate vector space to each vertex and of a linear map to each arrow. The dimension of a representation is an element of $\mathbb{N}^I$, where $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$.

For $\mathsf{v}, \mathsf{w} \in \mathbb{N}^I$, denote by $\mathrm{Rep}_{\overline{Q}}(\mathsf{v}, \mathsf{w})$ the space of representations of the quiver $\overline{Q}$ of dimensions v_i for $i \in I$ and w_i for $i \in \overline{I}$. Using the trace pairing, we can write

$$\mathrm{Rep}_{\overline{Q}} = \mathrm{Rep}_{\vec{Q}} \oplus \left(\mathrm{Rep}_{\vec{Q}}\right)^*, \tag{2.1}$$

which gives this linear space a symplectic form ω. This symplectic form is preserved by the action of

$$G_{\mathsf{v}} = \prod GL(\mathsf{v}_i)\,, \quad G_{\mathsf{w}} = \prod GL(\mathsf{w}_i)\,.$$

We can also define an action of the group

$$\prod_i Sp(2q_{ii}) \prod_{i \neq j} GL(q_{ij})\,.$$

as follows. Given a vertex i, loops at this vertex contribute a factor

$$\mathrm{End}(\mathbb{C}^{\mathsf{v}_i})^{\oplus q_{ii}} \oplus \text{its dual} \cong \mathrm{End}(\mathbb{C}^{\mathsf{v}_i}) \otimes \mathbb{C}^{2q_{ii}}\,,$$

to $\mathrm{Rep}_{\overline{Q}}$ where the symplectic form is induced by the symmetric trace pairing on the first factor and the standard symplectic form on the second. The factor $Sp_2(2q_{ii})$ acts naturally on the second factor. Similarly, given distinct vertices i, j, the contribution of edges between these vertices is naturally identified with

$$(\mathrm{Hom}(\mathbb{C}^{\mathsf{v}_i}, \mathbb{C}^{\mathsf{v}_j}) \otimes \mathbb{C}^{q_{ij}}) \oplus \text{its dual}$$

and the factor $GL(q_{ij})$ acts in the natural way. By construction, these groups also preserves the symplectic form ω.

2.1.3

The symplectic form ω is scaled by the action of $\mathbb{C}^\times$ scaling the second summand in (2.1). We denote by $\hbar$ its $\mathbb{C}^\times$-weight. When there are other $\mathbb{C}^\times$'s around, we denote this one by $\mathbb{C}^\times_\hbar$.

We set

$$G_{\text{edge}} = \prod_i Sp(2q_{ii}) \prod_{i \neq j} GL(q_{ij}) \times \mathbb{C}^\times_\hbar\,.$$

As we shall see, this group will act uniformly on all families of quiver varieties associated to Q.

2.1.4 Weight convention

In this paper, we embed group weights into Lie algebra weights. For example, we will also use $\hbar$ to denote the generator of the equivariant cohomology of $\mathbb{C}^\times_\hbar$.

2.1.5

Sometimes it is convenient to consider, following Crawley-Boevey, representations of the quiver Q_∞ with vertex set $I \sqcup \{\infty\}$ and adjacency matrix

$$Q_\infty = \begin{pmatrix} Q + Q^T & \mathsf{w} \\ \mathsf{w}^T & 0 \end{pmatrix} . \tag{2.2}$$

Note that we have a natural identification

$$\mathrm{Rep}_{\overline{Q}}(\mathsf{v}, \mathsf{w}) \cong \mathrm{Rep}_{Q_\infty}((\mathsf{v}, 1)).$$

Furthermore, this isomorphism is equivariant with the natural action of the groups above. For the action of G_w on the right-hand side, we define an edge group G^∞_{edge} analogously to the last section, and it contains a copy of both G_w and G_{edge}.

2.1.6

Consider the moment map

$$\mu : \mathrm{Rep}_{\overline{Q}}(\mathsf{v}, \mathsf{w}) \to \mathfrak{g}_\mathsf{v}^* ,$$

for the action of G_v, where $\mathfrak{g}_\mathsf{v} = \mathrm{Lie}\, G_\mathsf{v}$. Denote by

$$\mathfrak{z}_\mathsf{v} = [\mathfrak{g}_\mathsf{v}, \mathfrak{g}_\mathsf{v}]^\perp \subset \mathfrak{g}_\mathsf{v}^* ,$$

the fixed points of coadjoint action. If we identify $\mathfrak{g}_\mathsf{v}^*$ with $\mathfrak{g}_\mathsf{v}$ via the trace pairing, $\mathfrak{z}_\mathsf{v}$ corresponds to scalar matrices, i.e. a copy of $\mathbb{C}$ for every $i \in \mathbf{I}$ such that $\mathsf{v}_i \neq 0$. We consider the preimage

$$\mathfrak{Z} = \mu^{-1}(\mathfrak{z}_\mathsf{v}).$$

In general, this may be reducible and nonreduced.

2.1.7

Note for any $x \in \mathrm{Rep}_{Q_\infty}$, its stabilizers in G_{v} is the quotient of units by scalars for some associative algebra over $\mathbb{C}$. Hence the G_{v}-stabilizer of x is finite if and only if it is trivial.

2.1.8

Given $\theta \in \mathbb{Z}^I$, it defines a character of G_{v} by the convention

$$(g_i) \mapsto \prod (\det g_i)^{\theta_i} \in \mathbb{C}^\times .$$

We define

$$\begin{aligned} \widetilde{\mathcal{M}}_\theta &= \mathfrak{Z} /\!\!/_\theta \, G_{\mathsf{v}} \,, \\ &= \operatorname{Proj} \bigoplus_{n \geqslant 0} \mathbb{C}\,[\mathfrak{Z}]_{n\theta} \,, \end{aligned}$$

where the subscript $n\theta$ denotes the corresponding G_{v}-isotypic component. The map μ descends to a map

$$\widetilde{\mu} : \widetilde{\mathcal{M}}_\theta \to \mathfrak{z}_{\mathsf{v}} \,.$$

Definition 2.1.1. A Nakajima quiver variety is a fiber of this map:

$$\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) = \widetilde{\mu}^{-1}(\zeta)\,, \quad \zeta \in \mathfrak{z}_{\mathsf{v}} \,.$$

2.2 Basic properties

2.2.1

The following result is proven in [84]

Proposition 2.2.1. *For any Q, v, and w there exits a finite set $\{\alpha_i\} \subset \mathbb{N}^I$ such that $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ contains a strictly semistable point only if*

$$\alpha_i \cdot \theta = \alpha_i \cdot \zeta = 0$$

for some i.

These hyperplanes are closely related to the roots of the Lie algebra $\mathfrak{g}_Q$ that will be associated to the quiver Q in Section 5.3. One corollary of this proposition is that, for θ in the complement of these hyperplanes, the natural map

$$\widetilde{\mu} : \widetilde{\mathcal{M}}_\theta \to \mathfrak{z}_{\mathsf{v}}$$

is smooth, although it is possible that the domain is empty.

We also state the following result, which is well-known. Since we do not use it in the paper, it can be safely skipped. However, we sketch its proof briefly.

Proposition 2.2.2. *If there exists a free G_{v}-orbit contained in $\mathfrak{Z}$, then $\widetilde{\mu}$ is surjective for all values of θ. The generic fiber is smooth and affine.*

Proof. If there exists a free orbit, then the moment map μ is smooth at any point of this orbit and, in particular, the image of $\mathfrak{Z}$ contains a dense, Zariski-open set U of $\mathfrak{z}_{\mathsf{v}}$. By Theorem 1.2 of [21], after further shrinking, the entire fiber of any point of U consists of simple representations of Q_∞. These are θ-stable for all stability conditions θ; consequently, the GIT quotient for any choice of θ equals the categorical quotient of the fiber, which is affine. This proves the second statement.

For the first statement, we use the definition of quiver varieties via hyperkahler reduction, as in [84]. Let U_{v} denote the maximal compact subgroup of G_{v}. If we take the hyperkahler moment map, then the image of the locus of free U_{v}-orbits contains $\mathbb{R}^I \times U \subset \mathbb{R}^I \times \mathfrak{z}_{\mathsf{v}}$. Since it is stable with respect to multiplication by unit quaternions, it contains $\{\theta\} \times \mathfrak{z}_{\mathsf{v}}$ for any suitably generic θ. Consequently, $\widetilde{\mu}$ is surjective for general θ. Finally, if θ lies on a wall on the space of stability conditions,there is a factorization

$$\widetilde{\mathcal{M}}_{\theta'} \to \widetilde{\mathcal{M}}_\theta \to \mathfrak{z}_{\mathsf{v}}$$

where θ' is a nearby stability condition. We can assume $\widetilde{\mu}$ is surjective for θ' which implies $\widetilde{\mu}$ is surjective for all θ. □

In this paper, we are mainly interested in the case where θ is generic in the sense of Proposition 2.2.1 and when $\zeta = 0$. We say $\theta > 0$ if $\theta_i > 0$ for all i. This condition implies that θ is generic in the above sense, for arbitrary quiver Q and dimension vectors v, w.

2.2.2 Group actions

By construction, the group

$$\mathsf{G} = \begin{cases} G_{\mathsf{w}} \times G_{\text{edge}}\,, & \zeta = 0\,, \\ G_{\mathsf{w}} \times \prod_i Sp(2q_{ii}) \prod_{i \neq j} GL(q_{ij})\,, & \zeta \neq 0 \end{cases} \tag{2.3}$$

acts on $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$. The larger group also acts on $\widetilde{\mathcal{M}}_\theta$ and the map

$$\widetilde{\mu} : \widetilde{\mathcal{M}}_\theta \to \mathfrak{z}_{\mathsf{v}} \otimes \hbar^{-1}$$

is G-equivariant.

The action of G is not faithful on $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$. The center $Z(G_{\mathsf{v}})$ of G_{v} has a natural map

$$\rho_Q : Z(G_{\mathsf{v}}) \to G_{\text{edge}}\,.$$

There is also a map

$$\tau_Q : \mathrm{Ker}(\rho_Q) \to G_{\mathsf{w}}$$

given by constants acting by multiplication on $\mathbb{C}^{\mathsf{w}_i}$.

The images of these maps act trivially on $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$, and we could work with the corresponding quotient groups

$$G'_{\text{edge}} = G_{\text{edge}}/\mathrm{Im}(\rho_Q)\,, \quad G'_{\mathsf{w}} = G_{\mathsf{w}}/\mathrm{Im}(\tau_Q)$$

and their product G'.

However, it is sometimes convenient to work with the larger group G since the tautological bundles considered shortly admit a natural G-equivariant structure. In practice, most of the geometric calculations and constructions considered later (e.g. R-matrices, quantum operators) will naturally take values in G'-equivariant cohomology.

2.2.3 Symplectic resolutions

By construction, Nakajima varieties have an algebraic Poisson structure which is symplectic on their smooth locus. The group G preserves this symplectic form when $\zeta \neq 0$ and scales it by the character $\hbar$ when $\zeta = 0$.

Furthermore, they come with a projective map

$$\pi : \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) \to \mathcal{M}_{0,\zeta}(\mathsf{v},\mathsf{w}) = \mathrm{Spec}\,\mathbb{C}[\mu^{-1}(\zeta)]^{G_{\mathsf{v}}}$$

to an affine algebraic variety.

Although π is not always birational, it follows from section 10.3 of [84] that it is birational onto its image. In particular, for θ generic in the sense of Proposition 2.2.1, $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ is an equivariant symplectic resolution. When $\zeta = 0$ it carries a natural torus action that scales ω and is an example of the general theory considered, for example, in [60].

2.2.4 Tautological bundles

As G_{v}-quotients, Nakajima varieties have tautological bundles $\mathcal{V}_i$ of ranks v_i, $i \in I$, associated to representations

$$G_{\mathsf{v}} \to GL(\mathbb{C}^{\mathsf{v}_i})\,.$$

For uniformity, we consider the (topologically trivial) bundles $\mathcal{W}_i$, $i \in I$, of ranks w_i on a similar footing. Since these bundles carry a representation of G_{w}, their equivariant Chern classes capture the framing weights.

2.2.5 Equivariant lifts

The matrix elements of the matrices

$$Q + Q^T\,, \quad \overline{Q}$$

are dimensions of vector spaces which naturally carry representations of G, essentially by the definition of the group G. As a result, we have a natural lift of $Q + Q^T$ and $\overline{Q}$ to matrices with values in the representation ring $K_{\mathsf{G}}(\mathsf{pt})$. Recall from Section 2.1.4 that we embed group weights into Lie algebra weights. Here we treat $\hbar$ etc. as elements of $K_{\mathsf{G}}(\mathsf{pt})$.

If we endow $K_{\mathsf{G}}(\mathsf{pt})$ with the involution given by taking duals, the Hermitian transpose of $\overline{Q}$ satisfies the relation

$$\left(\overline{Q}\right)^* = \hbar \otimes \overline{Q}\,. \tag{2.4}$$

where $\hbar$ denotes the character of G associated to $\mathbb{C}^{\times}_{\hbar}$.

The Cartan matrix of Q admits an equivariant lift

$$\mathsf{C} = 1 + \hbar^{-1} - (Q + Q^T).$$

We also set

$$\overline{\mathsf{C}} = \begin{pmatrix} -\mathsf{C} & \hbar^{-1} \\ 1 & 0 \end{pmatrix},$$

and define the Hermitian forms

$$\begin{aligned} (\mathsf{v}, \mathsf{v}')_Q &= \mathsf{v}^* \, \mathsf{C} \, \mathsf{v}' \\ ((\mathsf{v}, \mathsf{w}), (\mathsf{v}', \mathsf{w}'))_{\overline{Q}} &= (\mathsf{v}, \mathsf{w})^* \, \overline{\mathsf{C}} \, (\mathsf{v}', \mathsf{w}') \, . \end{aligned} \tag{2.5}$$

for $\mathsf{v}, \mathsf{w}, \dots \in K_{\mathsf{G}}(\mathsf{pt})^I$.

Given an arbitrary G-variety X and

$$\mathsf{v}, \mathsf{w}, \mathsf{v}', \mathsf{w}' \in K_{\mathsf{G}}(X)^I \,,$$

the forms (2.5) still make sense and takes values in $K_{\mathsf{G}}(X)$. Of course, very often, one takes just the nonequivariant specialization of (2.5).

2.2.6 Tangent bundle

Given θ generic, if $\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w})$ is nonempty, its dimension is given by

$$\dim \mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w}) = \|(\mathsf{v}, \mathsf{w})\|^2_{\overline{Q}} \,,$$

with respect to the nonequivariant version of (2.5). Using the equivariant lifts described above, we can identify the K-theory class of the tangent bundle as follows.

Lemma 2.2.3. *For θ generic, we have the identification*

$$T\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w}) = \|(\mathcal{V}, \mathcal{W})\|^2_{\overline{Q}} \,, \tag{2.6}$$

in $K_{\mathsf{G}}(\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w}))$, where

$$\mathcal{V}, \mathcal{W} \in K_{\mathsf{G}}(\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w}))^I$$

are vectors of tautological bundles.

Proof. On the affine space of representations of $\overline{Q}$, the tangent bundle is given by

$$T_{\mathrm{Rep}_{\overline{Q}}(\mathsf{v}, \mathsf{w})} = (\mathcal{V}, \mathcal{W})^* \, \overline{Q} \, (\mathcal{V}, \mathcal{W}).$$

Since the moment map is submersive, the tangent bundle on $\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w})$ is obtained by subtracting off

$$\mathfrak{g}_{\mathsf{v}}^* \otimes \hbar^{-1} - \mathfrak{g}_{\mathsf{v}}$$

which gives the result. □

2.2.7 Splitting of tangent bundle

Using the orientation of Q, we can define a virtual bundle

$$T^{\mathbf{1/2}} = \sum_{i,j} (Q_{i,j} - \delta_{i,j}) \operatorname{Hom}(\mathcal{V}_i, \mathcal{V}_j) + \sum_i \operatorname{Hom}(\mathcal{W}_i, \mathcal{V}_i) \in K(\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w})).$$

If $H \subset \mathsf{G}$ denotes the subgroup preserving the decomposition (2.1), then the expression lifts to $K_H(\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w}))$ where it satisfies the identity

$$T\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w}) = T^{\mathbf{1/2}} + \hbar^{-1} \otimes \left(T^{\mathbf{1/2}}\right)^{\vee} \tag{2.7}$$

Nakajima varieties may be viewed as open substacks of the cotangent stacks stacks

$$\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w}) \approx T^* \left(\operatorname{Rep}_{\vec{Q}} / G_{\mathsf{v}}\right)$$

and the virtual bundle $T^{\mathbf{1/2}}$ is the pullback of the tangent bundle from the base in this sense.

2.2.8 Theta characteristic

One notes that

$$\boldsymbol{\kappa}_{\mathcal{M}} = c_1 \left(T^{\mathbf{1/2}}\right) \bmod 2 \quad \in H^2(\mathcal{M}, \mathbb{Z}/2) \tag{2.8}$$

is independent of the orientation of Q. We call it the canonical *theta characteristic* of $\mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w})$. It will be responsible for signs in the formulas for quantum multiplication.

2.2.9

Alternatively, Nakajima varieties may be defined using representation of the quiver Q_∞ and parameters

$$\widehat{\zeta}_\infty = -\sum_{i \in I} \mathsf{v}_i \, \widehat{\zeta}_i \,.$$

This is because diagonal scalars in $\prod_{i \in I \sqcup \{\infty\}} GL(\mathsf{v}_i)$ act trivially on representations of Q_∞.

2.2.10

Note that for θ generic,

$$\mathcal{M}_{\theta,\zeta}(\mathsf{v},0) = \varnothing \tag{2.9}$$

because when $\mathsf{w} = 0$ the action of G_v cannot be free.

2.3 Torus-fixed points

In this section, unless stated explicitly, we assume throughout that θ is generic in the sense of Proposition 2.2.1, so $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ is in particular a smooth holomorphic symplectic variety.

2.3.1

Let

$$\mathsf{A} \subset \mathrm{Ker}\hbar \subset G_{\mathrm{edge}} \times G_\mathsf{w} \tag{2.10}$$

be a torus. Since A preserves ω, its fixed locus $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})^\mathsf{A}$ is a smooth holomorphic symplectic variety. In fact, it is a union of product of smaller Nakajima varieties, which can be seen as follows.

2.3.2

Take $x \in \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})^\mathsf{A}$ and let $X \in \mathrm{Rep}_{\overline{Q}}(\mathsf{v},\mathsf{w})$ be a point above it. The subgroup

$$G^x \subset G_\mathsf{v} \times G_{\mathrm{edge}} \times G_\mathsf{w}$$

such that

$$1 \to G_\mathsf{v} \to G^x \to \mathsf{A} \to 1$$

acts on the orbit of X. Since the G_v action is free, we get a map $G^x \to G_\mathsf{v}$ that splits the above sequence. This gives homomorphisms

$$\mathsf{A} \xrightarrow{\phi} G_\mathsf{v} \times G_{\mathrm{edge}} \times G_\mathsf{w} \to \mathsf{A} \tag{2.11}$$

with identity composition and such that X is fixed by $\phi(\mathsf{A})$.

2.3.3

A homomorphism ϕ is equivalent to a lift of v, w, and Q to vectors and matrices with values in $K_{\mathsf{A}}(\mathsf{pt})$, consistent with the embedding (2.10). To this, one associates a new quiver Q_ϕ as follows. We set

$$I_\phi = I \times \mathsf{A}^\wedge$$

where $\mathsf{A}^\wedge$ is the character group of A, and

$$(Q_\phi)_{(i,\lambda),(j,\nu)} = \text{coefficient of } \nu/\lambda \text{ in } Q_{ij} \, ,$$

where $\lambda, \nu \in \mathsf{A}^\wedge$. This is an infinite quiver with a free action of the group $\mathsf{A}^\wedge$ by automorphisms. We take dimension vectors

$$(\mathsf{v}_\phi)_{(i,\lambda)} = \text{coefficient of } \lambda \text{ in } \mathsf{v}_i$$

and similarly for w_ϕ. These have finite support, which may be disconnected. Clearly, representations of quivers factor over connected components of supports. Finally,

$$G_{\mathsf{v}_\phi} = (G_\mathsf{v})^{\phi(\mathsf{A})} \subset G_\mathsf{v}$$

and this defines the pull-back $(\theta_\phi, \zeta_\phi)$ of (θ, ζ).

2.3.4

We consider two lifts ϕ_1 and ϕ_2 in (2.11) equivalent if they define the same action of A on $\operatorname{Rep}_{\overline{Q}}$.

Proposition 2.3.1. *We have*

$$\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})^{\mathsf{A}} = \bigsqcup_{\phi/\sim} \mathcal{M}_\phi \, ,$$

where $\mathcal{M}_\phi$ is the Nakajima variety associated to the quiver Q_ϕ and the data $\mathsf{v}_\phi, \mathsf{w}_\phi, \theta_\phi, \zeta_\phi$ above.

Proof. It is clear that

$$\operatorname{Rep}_{\overline{Q}}(\mathsf{v},\mathsf{w})^{\phi(\mathsf{A})} = \operatorname{Rep}_{\overline{Q}_\phi}(\mathsf{v}_\phi,\mathsf{w}_\phi) \, .$$

The moment map μ takes this fixed locus to

$$(\mathfrak{g}_\mathsf{v}^*)^{\phi(\mathsf{A})} = \mathfrak{g}_{\mathsf{v}_\phi}^*$$

and coincides with μ_ϕ. It remains to check that

$$\theta\text{-stability} \Leftrightarrow \theta_\phi\text{-stability}\,.$$

The $\Rightarrow$ implication is trivial. The set of all θ-destabilizing subrepresentations is a projective variety with an action of A. If nonempty, it has an A-fixed point which is a θ_ϕ-destabilizing subrepresentation. □

2.3.5

As a first example, take A to be the maximal torus of G'_{edge}. Recall that G'_{edge} is largest quotient of G_{edge} that acts nontrivially. We have

$$\mathsf{A}^\wedge = H_1(Q,\mathbb{Z})$$

and

$$Q_\phi \to Q_\phi/\mathsf{A}^\wedge \cong Q$$

is the universal abelian cover of Q. In particular, for any Q, Q_ϕ is a quiver without loops at vertices.

2.3.6

The restriction of the tangent bundle of $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ to the A-fixed locus is given by the same formula (2.6), but interpreted in the A-equivariant K-theory via the map ϕ.

Expanding (2.6) in characters of A, one expresses the A-eigensubundles in the normal bundle to $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})^{\mathsf{A}}$ in terms of the tautological bundles of $\mathcal{M}_\phi$.

2.3.7

Because the splitting (2.7) is equivariant with respect to all group actions, we have

$$c_1(N_\pm) \bmod 2 = \boldsymbol{\kappa}_{\mathcal{M}} + \boldsymbol{\kappa}_{\mathcal{M}^{\mathsf{A}}} \tag{2.12}$$

in $H^2(\mathcal{M}^{\mathsf{A}},\mathbb{Z}/2)$ for any torus A that preserves the symplectic form.

2.4 Tensor product of Nakajima varieties

2.4.1

For this paper, the main example of the above fixed-point construction arises as follows.

Take a decomposition

$$\mathsf{w} = \mathsf{w}' + \mathsf{w}''$$

and define

$$\mathsf{A} \cong \mathbb{C}^\times \subset G_\mathsf{w}$$

as the subgroup that scales the first term in

$$\mathbb{C}^{\mathsf{w}_i} = \mathbb{C}^{\mathsf{w}'_i} \oplus \mathbb{C}^{\mathsf{w}''_i}\,, \quad i \in I\,, \tag{2.13}$$

with weight 1. In other words, we take

$$\mathsf{w} = z\,\mathsf{w}' + \mathsf{w}'' \in K_{\mathbb{C}^\times}(\mathsf{pt})^I$$

where z is the defining representation. Then the fixed points are precisely

$$\bigsqcup_{\mathsf{v}'+\mathsf{v}''=\mathsf{v}} \mathcal{M}_{\theta,\zeta}(\mathsf{v}',\mathsf{w}') \times \mathcal{M}_{\theta,\zeta}(\mathsf{v}'',\mathsf{w}'') \hookrightarrow \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) \tag{2.14}$$

as in (1.9). Indeed, the fixed points in (2.14) correspond to

$$\mathsf{v} = z\,\mathsf{v}' + \mathsf{v}''$$

and all other ones are empty because of (2.9).

The embedding (2.14) will play a key role in this paper and we call it *tensor product* of Nakajima varieties. See Section 5.1 for a discussion of this term.

2.4.2

For a tensor product of Nakajima varieties, the normal bundle to the fixed locus is

$$N = zN_+ \oplus z^{-1}N_-$$

where $z^{\pm 1}$ is the torus weight,

$$N_- = \sum \mathrm{Hom}(\mathcal{W}'_i, \mathcal{V}''_i) + \sum \mathrm{Hom}(\mathcal{V}'_i, \mathcal{W}''_i) \otimes \hbar^{-1} - \sum \mathsf{C}_{ij} \mathrm{Hom}(\mathcal{V}'_i, \mathcal{V}''_j) \tag{2.15}$$

in the K-theory of the fixed locus, where C_{ij} denotes the equivariant Cartan matrix and

$$N_+ = \hbar^{-1} \otimes N_-^\vee \,.$$

2.5 Slices

2.5.1

Recall the affine quotient

$$\mathcal{M}_{0,\zeta} = \mu^{-1}(\zeta)/G_{\mathsf{v}} \,.$$

Its closed points are the closed G_{v}-orbits in $\mu^{-1}(\zeta) \subset \mathrm{Rep}_{\overline{Q}}$, and those correspond to isomorphism classes of semisimple representations of $\overline{Q}$ or Q_∞.

The natural map

$$\pi : \mathcal{M}_{\theta,\zeta} \to \mathcal{M}_{0,\zeta} \,. \tag{2.16}$$

takes a θ-semistable representation to its semisimplification, see Proposition 3.20 in [87].

2.5.2

Given $X \in \mathcal{M}_{0,\zeta}$, it natural to study $\pi^{-1}(X)$, bearing in mind that it may be empty. Following Nakajima, see Section 6 in [84], $\pi^{-1}(X)$ may be described as $(\pi')^{-1}(0)$ for a different quiver Q'. Here $0 \in \mathcal{M}'_{0,0}$ is the zero representation.

See Proposition 3.2.2 in [89] and Section 4 in [22] for the proof of the following

Theorem 2.5.1 ([84, 89, 22])**.** *For any $X \in \mathcal{M}_{0,\zeta}(\mathsf{v},\mathsf{w})$ there exist a quiver Q' and dimension vectors $(\mathsf{v}',\mathsf{w}')$ such that:*

- *an analytic neighborhood U of X in $\mathcal{M}_{0,\zeta}(\mathsf{v},\mathsf{w})$ is isomorphic to an analytic neighborhood U' of 0 in $\mathcal{M}'_{0,0}(\mathsf{v}',\mathsf{w}') \times \mathbb{C}^k$ and*

- *this isomorphism may be lifted to an isomorphism Σ_X between $(\pi')^{-1}(U')$ and $\pi^{-1}(U)$ that preserves the fibers of π.*

These isomorphism are equivariant with respect to the stabilizer $\mathsf{G}' \subset \mathsf{G}$ of the representation X.

We call the maps Σ_X slices and for brevity write them as rational maps

$$\Sigma_X : \mathcal{M}'(\mathsf{v}', \mathsf{w}') \times \mathbb{C}^k \dashrightarrow \mathcal{M}(\mathsf{v}, \mathsf{w})$$

even though this is not what is claimed in Theorem 2.5.1. The integer k that appears here is the difference in dimensions, see also (2.18) below.

2.5.3

The data $Q', \mathsf{v}', \mathsf{w}'$ are constructed as follows. As a representation of Q_∞, X has a unique decomposition

$$X = X_\infty \oplus \bigoplus_{i \in I'} X_i^{\oplus \mathsf{v}'_i}$$

into nonisomorphic simples X_i with multiplicities v'_i. We denote by

$$\mathsf{d}(X)_{ij} = (\dim X_j)_i \,, \quad i \in I \sqcup \{\infty\}, j \in I' \sqcup \{\infty\} \,,$$

the matrix of their dimension vectors. The subgroup

$$GL(1) \times G_{\mathsf{v}'} \subset GL(1) \times G_{\mathsf{v}}$$

is the stabilizer of $X \in \mathrm{Rep}_{Q_\infty}$ and the matrix $\mathsf{d}(X)$ describes its subgroup conjugacy class.

The representation X_∞ is distinguished from the rest by

$$\mathsf{d}(X)_{\infty,\infty} = 1$$

and then

$$\mathsf{d}(X)_{\infty,j} = 0 \,, \quad j \in I' \,,$$

because $(\dim X)_\infty = 1$.

2.5.4

By definition, $I' \sqcup \{\infty\}$ is the vertex set for the new quiver Q'_∞ and v' is the new dimension vector. We use the matrix

$$\mathsf{d} : \mathbb{Z}^{I' \sqcup \{\infty\}} \to \mathbb{Z}^{I \sqcup \{\infty\}}$$

to transfer the other quiver data to $I' \sqcup \{\infty\}$. For example, we set

$$\widehat{\zeta}' = \mathsf{d}^T \widehat{\zeta}\,.$$

It follows that

$$\zeta' = 0\,, \quad \mathsf{v}' \cdot \theta' = 0\,,$$

because $\mathsf{v} = \mathsf{d}(X) \cdot \mathsf{v}'$ and

$$\sum_{i \in I \sqcup \{\infty\}} \zeta_i \, \mathsf{d}(X)_{i,j} = 0\,, \quad \forall j\,,$$

by the moment map equation.

2.5.5

The adjacency matrix of Q'_∞, and in particular, the new framing vector w' is found from the formula

$$(a,b)_{Q'_\infty} = (\mathsf{d}(X)\, a, \mathsf{d}(X)\, b)_{Q_\infty}\,, \tag{2.17}$$

see (2.2) for the the matrix of this quadratic form.

In the course of the proof, one uses reductivity to write

$$\mathfrak{g}_{\mathsf{v}}^* = \mathfrak{g}_{\mathsf{v}'}^* \oplus \mathfrak{g}_{\mathsf{v}'}^{\perp}$$

and identifies $d\mu^{-1}(\mathfrak{g}_{\mathsf{v}'}^*) = (\mathfrak{g}_{\mathsf{v}} \cdot X)^{\angle}$ and

$$\mathrm{Rep}_{Q'_\infty} \cong (\mathfrak{g}_{\mathsf{v}} \cdot X)^{\angle} \Big/ \mathfrak{g}_{\mathsf{v}} \cdot X$$

as $G_{\mathsf{v}'} \times \mathsf{G}'$-modules, where $\angle$ denotes the symplectic perpendicular. This leads to (2.17).

2.5.6

Note that Q'_∞ may have loops at the distinguished vertex ∞, in fact

$$\#\{\text{loops at } \infty\} = k = \|(\dim X_\infty, \mathsf{w})\|^2_Q \tag{2.18}$$

where k is the number from Theorem 2.5.1. These loops contribute a vector space factor to $\mathrm{Rep}_{Q'_\infty}$ because $\mathsf{v}'_\infty = 1$. Note that (2.18) also describes this vector space as a G'-module.

2.5.7

The following is immediate:

Proposition 2.5.2. *If X_∞ is the only nonzero representation in X then Q' is isomorphic to the subquiver of Q formed by the support of $\mathsf{v}' = \mathsf{v} - \dim X_\infty$ and*

$$\mathsf{w}' = \mathsf{w} - \hbar\, \mathsf{C} \dim X_\infty\,. \tag{2.19}$$

This also covers the trivial case when $X = 0$ and $\mathsf{d}(X)_{ij} = \delta_{ij}$.

2.5.8 Example

Consider the A_n-quiver, that is, that is the quiver with

$$\mathsf{C} = \begin{pmatrix} 1+\hbar^{-1} & -\hbar^{-1} & & \\ -1 & 1+\hbar^{-1} & -\hbar^{-1} & \\ & \ddots & \ddots & \ddots \\ & & -1 & 1+\hbar^{-1} \end{pmatrix}.$$

We fix $1 \leqslant i < j \leqslant n$ and take

$$\mathsf{w} = \hbar\, a\, \delta_i + a\delta_j\,,$$

where a is a weight of G_{w}. For

$$\dim X_\infty = a \sum_{k=i}^{j} \delta_k$$

there is a torus fixed representation X_∞ with such dimension. It takes the framing vector at the jth vertex, applies the arrow in Q to it $(j-i)$ times, and sends it to the framing vector at the ith vertex. Note that the final map in $\mathrm{Hom}(V_i, W_i)$ has torus weight $\hbar^{\otimes -1}$ and the framing weight $\hbar a$ compensates for this.

If the other X_i's are zero, we get

$$\mathsf{w}' = a\,\delta_{i-1} + \hbar\, a\, \delta_{j+1}$$

from formula (2.19).

2.5.9 Example

Take the quiver with one vertex and one loop, for which C is a 1×1 matrix

$$\mathsf{C} = (1 - t_1)(1 - t_2)\,, \quad t_1 \otimes t_2 = \hbar\,,$$

where t_1 and t_2 are the weights of G_{edge}. For

$$\mathsf{w} = a + a\, t_1^{-n}\, t_2^{-1}\,,$$

there is a torus-fixed representation X_∞ with

$$\dim X_\infty = a(1 + t_1^{-1} + \cdots + t_1^{1-n})\,.$$

Just like in the previous example, it takes a framing vector of weight a and applies the t_1-arrow to it $(n-1)$-times (the weights have go change by t_1^{-1} every time to compensate for the t_1 weight of the arrow). We find

$$\mathsf{w}' = at_1^{-n} + at_2^{-1}\,.$$

This and the previous example are special cases of slices considered in Section 6.2.

2.5.10

Equivariance in Theorem 2.5.1 means that slices commute with taking fixed points. That is, if $\mathsf{A}' \subset \mathsf{G}'$ is a torus preserving the symplectic form then

$$(\Sigma_X)^{\mathsf{A}'} : \left(\mathcal{M}'(\mathsf{v}', \mathsf{w}') \times \mathbb{C}^k\right)^{\mathsf{A}'} \dashrightarrow \mathcal{M}(\mathsf{v}, \mathsf{w})^{\mathsf{A}'}$$

is an isomorphism of open subsets of quiver varieties (the fixed points are quiver varieties by Proposition 2.3.1).

In particular, slices are compatible with tensor products, in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}(\mathsf{w}_0+\mathsf{w}') \times \mathbb{C}^{\cdots} & \overset{\Sigma_X}{\dashrightarrow} & \mathcal{M}(\mathsf{w}_0+\mathsf{w}) \\ \uparrow & & \uparrow \\ \mathcal{M}(\mathsf{w}_0) \times \mathcal{M}(\mathsf{w}') \times \mathbb{C}^{\cdots} & \overset{1\times\Sigma_X}{\dashrightarrow} & \mathcal{M}(\mathsf{w}_0) \times \mathcal{M}(\mathsf{w}) \end{array} \tag{2.20}$$

where the vertical arrows are inclusions of fixed points and the representation X is padded by zeros as necessary.

2.6 Minuscule coweights

2.6.1

Let X be an algebraic variety. We call an action

$$\sigma : \mathbb{C}^\times \to \mathrm{Aut}(X)$$

minuscule, if the algebra $H^0(X, \mathcal{O}_X)$ is generated by functions of σ-weight $\{-1, 0, 1\}$. Equivalently, there is an equivariant embedding

$$X_0 = \operatorname{Spec} H^0(X, \mathcal{O}_X) \hookrightarrow V$$

where V is a linear representation of σ with weights in $\{-1, 0, 1\}$. This notion will play a crucial role below.

2.6.2

Proposition 2.6.1. *The $\mathbb{C}^\times$-action corresponding to the tensor product of Nakajima varieties is minuscule.*

Proof. It is enough to prove that

$$\mathbb{C}\,[\mathfrak{Z}]^{G_\mathsf{v}}$$

is generated by the functions of σ-weight in $\{0, \pm 1\}$. Since G_v is reductive, the natural map

$$\mathbb{C}[\mathrm{Rep}_{\overline{Q}}]^{G_\mathsf{v}} \to \mathbb{C}\,[\mathfrak{Z}]^{G_\mathsf{v}}$$

is surjective.

By the first fundamental theorem of invariant theory, see for example Section 9.5 in [125], the G_{v}-invariants are generated by all possible contraction of tensorial indices. Concretely this means either functions of the form

$$\operatorname{tr} P_1 P_2 \cdots P_k$$

where $P_1, P_2, \ldots, P_k$ is a closed chain of edges of $\overline{Q}$ starting and ending at a v-vertex, or any matrix coefficient of

$$P_1 P_2 \cdots P_k$$

where $P_1, P_2, \ldots, P_k$ is a chain of edges going from one w-vertex to another. Clearly, the σ-weights of all these functions are in $\{0, \pm 1\}$. □

Chapter 3

Stable envelopes

Let a torus A act on a nonsingular quasiprojective algebraic variety X and let $\iota : X^{\mathsf{A}} \to X$ denote the inclusion of the fixed locus. We have a natural map

$$\iota^* : H^{\cdot}_{\mathsf{A}}(X) \to H^{\cdot}_{\mathsf{A}}(X^{\mathsf{A}})$$

of degree 0. Our goal in this section is to construct a reasonably canonical map in the other direction

$$\mathrm{Stab}_{\mathfrak{C}} : H^{\cdot}_{\mathsf{A}}(X^{\mathsf{A}}) \to H^{\cdot}_{\mathsf{A}}(X)$$

that takes middle degree to middle degree. We will call $\mathrm{Stab}_{\mathfrak{C}}(\gamma)$ the *stable envelope* of γ. The main ingredients in its construction will be:

- an A-invariant holomorphic symplectic form ω on X,
- a choice of a certain chamber $\mathfrak{C} \subset \mathfrak{a} = \mathrm{Lie}(\mathsf{A})$.

Stable envelopes appear to be useful in a broader context than strictly required for the purposes of the present paper. We therefore discuss them in that greater generality. For symplectic resolutions, a much simpler approach may be used, as we explain in Section 3.7. In many examples, we expect the stable envelopes to specialize to well-known constructions.

We begin by explaining various conventions we use and recalling several basic constructions.

3.1 Assumptions and conventions

3.1.1 Assumptions on X

We assume that X is a nonsingular algebraic variety and $\omega \in H^0(\Omega^2 X)$ is a holomorphic symplectic form on X. In addition, we require a proper map

$$\pi : X \to X_0 \tag{3.1}$$

to an affine variety X_0.

3.1.2 Group actions

We denote by

$$\mathsf{A} \subset \mathsf{T} \subset \mathsf{G} \to \mathrm{Aut}(X)$$

a pair of tori $\mathsf{A} \subset \mathsf{T}$ in some reductive group G acting on X. We denote by $\mathfrak{a} \subset \mathfrak{t} \subset \mathfrak{g}$ the corresponding Lie algebras. We assume:

- $\omega \subset H^0(\Omega^2_X)$ is an eigenvector of G, fixed by A;
- the proper map π is G-equivariant;
- X is a formal T-variety.

See [50] for a discussion of formality. In particular, it implies $H^{\cdot}_{\mathsf{T}}(X)$ is free as a module over $H^{\cdot}_{\mathsf{T}}(\mathsf{pt})$. While this condition is convenient, we expect it can be removed with a little care.

We denote by

$$\hbar \in \mathfrak{g}^* ,$$

the G-weight of ω. By our assumption, A is in the kernel of $\hbar$.

Example 3.1.1. For $X = \mathcal{M}(r, n)$, we take

$$\mathsf{G} = GL(2) \times GL(r)$$

where the first factor acts on $\mathbb{P}^2$ keeping the line at infinity, while the second factor acts by changing the framing. We take T to be the maximal torus of G and $\mathsf{A} = \mathsf{T} \cap GL(r)$. The proper map π is the map to the Uhlenbeck moduli space.

Example 3.1.2. More generally, for $X = \mathcal{M}_{\theta,0}(\mathsf{v},\mathsf{w})$ with θ generic, we take G as defined in section 2.2.2 and T its maximal torus. The proper map π is the map

$$\pi : \mathcal{M}_{\theta,0}(\mathsf{v},\mathsf{w}) \to \mathcal{M}_{0,0}(\mathsf{v},\mathsf{w}).$$

Given a decomposition

$$\mathsf{w} = \sum_{i=1}^{r} \mathsf{w}^{(i)}\,,$$

we obtain a homomorphism

$$\mathsf{A} = \{(z_1,\dots,z_r)\} \to G_{\mathsf{w}}$$

given by $\mathsf{w} = \sum_{i=1}^{r} \mathsf{w}^{(i)}\, z_i$ as in Section 2.3.

3.1.3 Signs and adjoints

The varieties X we will encounter in the paper have no odd cohomology, although the following discussion may be easily modified to include odd cohomology.

When X^{T} is proper, integration over X

$$\gamma \mapsto \int_X \gamma \in \mathbb{Q}(\mathfrak{t})$$

may be defined as an equivariant residue, making $H = H^{\bullet}_{\mathsf{T}}(X)$ a commutative Frobenius algebra over $\mathbb{Q}(\mathfrak{t})$. In fact, it will prove very convenient to introduce the following sign twist in the Frobenius trace τ

$$\tau(\gamma) = (-1)^{\frac{1}{2}\dim X} \int_X \gamma\,.$$

Recall that X is holomorphic symplectic, so $\dim X$ is even. For example, if $X = T^*Y$ and $[Y]$ is the class of the zero section, then

$$\tau\left([Y]^2\right) = \chi(Y)\,.$$

In this paper, we define adjoints using τ. Concretely, this means the following. Consider a T-equivariant cycle, i.e. a $\mathbb{Q}$-linear formal combination of invariant subvarieties

$$Z = \sum a_k Z_k \subset \prod_{i=1}^{n} X_i\,.$$

Notice that we have abused notation to write a cycle as a subset of the ambient variety.

Fix a subset $S \subset \{1, \dots, n\}$. Then Z, viewed as a correspondence, defines a operator

$$\Theta_Z : H^{\cdot}_{\mathsf{T}}\left(\prod_{i \in S} X_i\right) \to H^{\cdot}_{\mathsf{T}}\left(\prod_{i \notin S} X_i\right) \otimes \mathbb{Q}(\mathfrak{t}),$$

see Section 3.2.5 for further discussion. For example, Z could be the diagonal $\Delta \subset X \times X$ and then, for $S = \{1\}$, Θ_Δ is the identity map.

Using τ, we may move factors X_i from the source of the map Θ_Z to the target, and back. We call these new operators *adjoint* to Θ_Z and denote them by $(\Theta_Z)^\tau$, to distinguish it from the ordinary permutations of factors. They acquire a sign $(-1)^p$, where

$$p = \frac{1}{2}\sum_{i \in S'} \dim X_i - \frac{1}{2}\sum_{i \in S} \dim X_i\,,$$

and S' is the source index set for the map $(\Theta_Z)^\tau$.

For example, if $S' = \{1, 2\}$ then

$$(\Theta_\Delta)^\tau(\gamma_1 \otimes \gamma_2) = (-1)^{\frac{1}{2}\dim X} \int_X \gamma_1 \cup \gamma_2 = \tau(\gamma_1 \cup \gamma_2) \in \mathbb{Q}(\mathfrak{t})\,.$$

3.2 Basic constructions

3.2.1 Chamber decomposition

The cocharacters

$$\sigma : \mathbb{C}^\times \to \mathsf{A}$$

form a lattice of rank equal to the rank of A. We denote

$$\mathfrak{a}_\mathbb{R} = \operatorname{Cochar}(\mathsf{A}) \otimes_\mathbb{Z} \mathbb{R} \subset \mathfrak{a}\,.$$

Each weight χ of A defines a rational hyperplane in this vector space.

Definition 3.2.1. The torus roots are the A-weights $\{\alpha_i\}$ occurring in the normal bundle to X^A.

The root hyperplanes partition $\mathfrak{a}_\mathbb{R}$ into finitely many (open) chambers

$$\mathfrak{a}_\mathbb{R} \backslash \bigcup \alpha_i^\perp = \bigsqcup \mathfrak{C}_i\,.$$

Example 3.2.2. In Example 3.1.1, we have

$$X^{\mathsf{A}} = \bigsqcup_{n_1+\cdots+n_r=n} \prod \mathrm{Hilb}(\mathbb{C}^2, n_i),$$

the normal weights α are the roots of $GL(r)$

$$\mathfrak{a} \ni \mathrm{diag}(a_1, \ldots, a_r) \mapsto a_i - a_j \,,$$

and the chambers $\mathfrak{C}$ are the usual Weyl chambers.

Example 3.2.3. Similarly, in Example 3.1.2, we have

$$\mathcal{M}_{\theta,0}(\mathsf{v}, \mathsf{w})^{\mathsf{A}} = \bigsqcup_{\mathsf{v}^{(1)}+\cdots+\mathsf{v}^{(r)}=\mathsf{v}} \mathcal{M}\left(\mathsf{v}^{(1)}, \mathsf{w}^{(1)}\right) \times \cdots \times \mathcal{M}\left(\mathsf{v}^{(r)}, \mathsf{w}^{(r)}\right)$$

by Proposition 2.3.1 and the normal weights are again the roots of $GL(r)$.

The stratification of $\mathfrak{a}_{\mathbb{R}}$ by root hyperplanes coincides with the stratification by the dimensions of the fixed-point locus. In particular, if σ does not lie on any hyperplane $\alpha_i^{\perp}$ then $X^{\sigma} = X^{\mathsf{A}}$.

3.2.2 Attracting, or stable, manifolds

Let $\mathfrak{C}$ be a chamber as above. One says that a point $x \in X$ is $\mathfrak{C}$-stable if the limit

$$\lim_{z \to 0} \sigma(z) \cdot x \in X^{\mathsf{A}}$$

exists for one (equivalently, all) cocharacter $\sigma \in \mathfrak{C}$. The value of this limit is independent of the choice of $\sigma \in \mathfrak{C}$. We will denote it by $\lim_{\mathfrak{C}} x$.

Given a subvariety $Y \subset X^{\mathsf{A}}$, we denote by

$$\mathrm{Attr}_{\mathfrak{C}}(Y) = \{x \mid \lim_{\mathfrak{C}}(x) \in Y\}$$

the set of points attracted to Y by the cocharacters in $\mathfrak{C}$. We have the following:

Lemma 3.2.4. *Let Z be a connected component of X^{A}. Then*

$$\lim_{\mathfrak{C}} : \mathrm{Attr}_{\mathfrak{C}}(Z) \to Z$$

is an affine bundle.

Remark 3.2.5. Note this affine bundle is T-equivariant.

Proof. We apply the classical Bialynicki-Birula theorem to a smooth σ-equivariant projective compactification $X \subset \overline{X}$. We get a diagram

$$\begin{array}{ccc} \mathrm{Attr}_{\mathfrak{C}}(Z) & \hookrightarrow & \mathrm{Attr}_{\mathfrak{C}}(\bar{Z}) \\ {\scriptstyle \lim}\downarrow & & \downarrow{\scriptstyle \overline{\lim}} \\ Z & \hookrightarrow & \bar{Z} \end{array}$$

of σ-equivariant maps in which the horizontal arrows are open dense embeddings and $\overline{\lim}$ is an affine bundle. Since σ acts with positive weights on the fibers of $\overline{\lim}$, any nonempty closed subset of the fiber contains the origin. Therefore, lim is also an affine bundle. □

Example 3.2.6. In Example 3.2.2, take $X = \mathcal{M}(2, n)$, $\mathfrak{C} = \{a_1 > a_2\}$, and

$$Z = \left\{ \mathcal{F}_1 \oplus \mathcal{F}_2 \,\middle|\, \mathcal{F}_i \in \mathrm{Hilb}(\mathbb{C}^2, n_i) \right\} .$$

Then $\mathrm{Attr}_{\mathfrak{C}}(Z)$ is a vector bundle with fiber $\mathrm{Ext}^1(\mathcal{F}_2, \mathcal{F}_1(-1))$, where $\mathcal{F}_1(-1)$ means the twist by minus the line at infinity of $\mathbb{P}^2$.

3.2.3 Partial order by attraction

The choice of a chamber $\mathfrak{C}$ determines a partial ordering on the set

$$\mathsf{Fix} = \pi_0(X^{\mathsf{A}})$$

of connected components Z of the fixed locus. This is a transitive closure of the relation

$$\overline{\mathrm{Attr}_{\mathfrak{C}}(Z)} \cap Z' \neq \varnothing \Rightarrow Z \geq Z' .$$

Using a projective compactification as in proof of Lemma 3.2.4, one sees that this is indeed a partial order, that is

$$Z \leq Z' \,\text{and}\, Z' \leq Z \Rightarrow Z = Z' .$$

Lemma 3.2.7. *For any component Z of X^{A} the set*

$$\mathrm{Attr}^f_{\mathfrak{C}}(Z) = \bigsqcup_{Z' \leq Z} \mathrm{Attr}_{\mathfrak{C}}(Z')$$

is closed in X.

We call $\mathrm{Attr}^f_{\mathfrak{C}}(Z)$ the *full attracting set* of Z.

Proof. Consider the map (3.1) and choose an A-equivariant embedding

$$X_0 \hookrightarrow V$$

into a linear representation V of A. Let $V_{\geqslant 0} \subset V$ denote the span of those weight subspaces that are non-negative on $\mathfrak{C}$. We have

$$\pi\left(\overline{\mathrm{Attr}_{\mathfrak{C}}(Z)}\right) \subset X_0 \cap V_{\geqslant 0}$$

for any component $Z \subset X^{\mathsf{A}}$.

Let x lie in the closure of $\mathrm{Attr}_{\mathfrak{C}}(Z)$. Then $\pi(x) \in V_{\geqslant 0}$ and the limit

$$z' = \lim_{\mathfrak{C}} x \in \overline{\mathrm{Attr}_{\mathfrak{C}}(Z)} \cap X^{\mathsf{A}}$$

exists by the properness of π. Denoting by $Z' \in \mathsf{Fix}$ the component that contains z' we see that $Z' \preceq Z$ and so we are done. □

3.2.4 The ample partial order

It will be more convenient to work with a different partial order on Fix which is a priori finer, that is

$$Z \prec Z' \Rightarrow Z < Z'\,,$$

but is much easier to describe.

Let $\sigma \in \mathfrak{C}$ be a cocharacter and let $C \cong \mathbb{P}^1$ be the closure of a σ-orbit. The degree

$$(\lambda, [C]) \in \mathbb{Z}\,, \quad \lambda \in \mathrm{Pic}(X)\,,$$

may be computed by equivariant localization in terms of weights of λ at the fixed points of C. This number must be positive if λ is ample.

We therefore choose any A-linearization of an ample line bundle λ and define

$$Z > Z' \Leftrightarrow \left(\mathrm{weight}\,\lambda\Big|_Z - \mathrm{weight}\,\lambda\Big|_{Z'}\right)\Big|_{\mathfrak{C}} > 0\,, \tag{3.2}$$

where $\mathrm{weight}\,\lambda\Big|_Z \in \mathfrak{a}^*$ is the weight of the A-action on the fiber of λ restricted to fixed point component Z. Note that the ambiguity in the choice of linearization cancels out of (3.2)

See also Section 4.8.5 below for a related discussion.

Example 3.2.8. Recall that, by construction, Nakajima varieties come with a distinguished ample class, namely

$$\theta = \sum \theta_i \, c_1(\mathcal{V}_i) \,.$$

Consider the fixed points of the tensor product action

$$Z_\eta = \mathcal{M}_{\theta,\zeta}(\eta, \mathsf{w}) \times \mathcal{M}_{\theta,\zeta}(\mathsf{v} - \eta, \mathsf{w}') \subset \mathcal{M}_{\theta,\zeta}(\mathsf{v}, \mathsf{w} + \mathsf{w}') \tag{3.3}$$

as in (2.14). By construction,

$$\text{weight}\, c_1(\mathcal{V}_i)\Big|_{Z_\eta} = \eta_i \,.$$

Therefore

$$Z_\eta > Z_{\eta'} \quad \Leftrightarrow \quad \theta \cdot \eta > \theta \cdot \eta' \,. \tag{3.4}$$

In particular, if $\theta_i > 0$ for all i then $Z_\varnothing = Z_0$ is minimal with respect to the ample order.

3.2.5 Lagrangian correspondences

Given a holomorphic symplectic variety M with symplectic form ω, recall that a subvariety $Z \subset M$ is isotropic if the restriction of ω to the smooth locus of L vanishes. It is Lagrangian if it is also middle-dimensional. We say that a cycle is Lagrangian if each component is Lagrangian.

Let Y be another holomorphic symplectic variety on which group G acts with the same weight $\hbar$ of the symplectic form ω_Y. Let

$$L \subset X \times Y$$

be a T-invariant Lagrangian cycle with respect to the form $\omega_X - \omega_Y$. Recall that we use $\subset$ to denote cycles as well as subvarieties.

If L is proper over X, it defines a map

$$\Theta_L : H^{\boldsymbol{\cdot}}_{\mathsf{T}}(Y) \xrightarrow{p_2^*} H^{\boldsymbol{\cdot}}_{\mathsf{T}}(L) \xrightarrow{(p_1)_*} H^{\boldsymbol{\cdot}}_{\mathsf{T}}(X)$$

As an equivariant residue, Θ_L may be defined with a weaker properness assumption: T has to have proper fixed points in the fibers of the push-forward.

See, for example, [23] for a general discussion of operators defined by correspondences. In particular, Θ_L depends only on the class $[L]$ of L in the T-equivariant Borel-Moore homology of $X \times Y$. Also

$$\Theta_{L_1} \circ \Theta_{L_2} = \Theta_{[L_1]\circ[L_2]} \,.$$

Here the convolution $L_1 \circ L_2$ of two cycles is defined by

$$[L_1] \circ [L_2] = (p_{13})_* \, \Delta^*([L_1] \times [L_2])$$

where the maps

$$X \times Y \times Y \times Z \xleftarrow{\Delta} X \times Y \times Z \xrightarrow{p_{13}} X \times Z$$

are the inclusion of the diagonal and the projection, respectively. Here Δ^* denotes Gysin pullback with respect to a regular embedding. When the map p_{13} is proper on the support of $L_1 \times_Y L_2$, its image is isotropic. As a consequence, the convolution $[L_1] \circ [L_2]$ is the cycle class of a T-invariant Lagrangian cycle in $X \times Z$.

3.2.6 Steinberg correspondences

Let $L \subset X \times Y$ be a Lagrangian correspondence as above.

Definition 3.2.9. A Steinberg correspondence is a Lagrangian correspondence

$$L \subset X \times Y$$

as above such that there exist proper equivariant maps

$$X \xrightarrow{\pi_X} V \xleftarrow{\pi_Y} Y$$

to an affine G-variety V such that

$$L \subset X \times_V Y \,.$$

The following easy lemma gives a sufficient condition for Steinberg correspondences to be closed under convolution.

Lemma 3.2.10. *Given Steinberg correspondences*

$$L_1 \subset X \times_{V_1} Y\,, \quad L_2 \subset Y \times_{V_2} Z\,,$$

the convolution $L_1 \circ L_2$ is a Steinberg correspondence if there exists a commutative diagram of equivariant proper maps

$$\begin{array}{ccc} Y & \xrightarrow{\pi_{Y,1}} & V_1 \\ {\scriptstyle \pi_{Y,2}}\downarrow & & \downarrow \\ V_2 & \longrightarrow & V \end{array} \tag{3.5}$$

with V affine.

Proof. Both X and Z map admit proper, equivariant maps to V. It is clear that the assumptions imply

$$L_1 \circ L_2 \subset X \times_V Z.$$

□

We say that two Steinberg correspondences are composable if they satisfy the sufficient condition described above when they share a common factor.

Example 3.2.11. Fix a quiver Q and dimension vectors $\mathsf{v}, \mathsf{v}^{(i)}$ for $i = 1, \ldots, n$, such that $\mathsf{v} \geqslant \sum \mathsf{v}^{(i)}$, and similarly for $\mathsf{w}, \mathsf{w}^{(i)}$. We have a proper map

$$\prod_{i=1}^{n} \mathcal{M}_{\theta,\zeta}\left(\mathsf{v}^{(i)}, \mathsf{w}^{(i)}\right) \to \mathcal{M}_{0,\zeta}\left(\sum \mathsf{v}^{(i)}, \sum \mathsf{w}^{(i)}\right) \to \mathcal{M}_{0,\zeta}(\mathsf{v}, \mathsf{w})$$

where the first map is given by affinization and direct sum, while the second map is given by taking the direct sum with the zero representation. We will only consider proper maps to affine varieties of this form or products of such maps. As a result, if we have two such maps with the same domain, a commutative diagram of the form (3.5) always exists since the two targets can both be included into a still-larger $\mathcal{M}_{0,\zeta}(\mathsf{v}, \mathsf{w})$. Therefore, the associated Steinberg correspondences will always be composable.

Given a possibly disconnected variety X, if we have a collection of composable Steinberg correspondences between components of X, we can consider the subalgebra of $\operatorname{End} H^{\cdot}_{\mathsf{T}}(X)$ that they span. When the context is clear, It will be called the *Steinberg algebra* of X.

3.3 Characterization of stable envelopes

3.3.1 Supports

For the ease of reading formulas, we use restriction signs for the natural restriction maps in equivariant cohomology. Given a closed T-invariant subset $Y \subset X$ and a class $\gamma \in H^{\bullet}_{\mathsf{T}}(X)$ we say that γ is supported on Y if

$$\gamma\Big|_{H^{\bullet}_{\mathsf{T}}(X\setminus Y)} = 0\,.$$

Equivalently, $\operatorname{supp}\gamma \subset Y$ means that the Borel-Moore class $\gamma \cap [X]$ is pushed forward under $Y \hookrightarrow X$.

3.3.2 Polarization

Let $Z \in \mathsf{Fix}$ be a component of X^{A} and let N_Z be the normal bundle to Z in X. Any chamber $\mathfrak{C}$ gives a T-invariant decomposition

$$N_Z = N_+ \oplus N_-$$

into A-weights that are positive and negative on $\mathfrak{C}$, respectively. The symplectic form ω gives

$$(N_+)^\vee = N_- \otimes \hbar \in K_{\mathsf{T}}(Z)\,, \tag{3.6}$$

where $\hbar$ denotes a trivial line bundle with the corresponding action of T.

Because $\hbar$ is trivial on A, the class

$$\varepsilon^2 = (-1)^{(\operatorname{codim} Z)/2} e(N_Z)\Big|_{H^{\bullet}_{\mathsf{A}}(\mathsf{pt})} = \prod \alpha_i^2\,, \tag{3.7}$$

is a perfect square. Here $\pm\alpha_i \in \mathfrak{a}^*$ are the roots that occur in N_Z.

Definition 3.3.1. A choice of a square root ε in (3.7) will be called a polarization of Z in X. The sign in $\pm e(N_-)$ agrees with polarization if $\pm e(N_-)$ restricts to ε in $H^{\bullet}_{\mathsf{A}}(\mathsf{pt})$.

Example 3.3.2. While polarization is a purely formal choice, geometrically natural choices save on signs.

For example, if $X = T^*Y$ with A-action induced from Y, we can take ε to be the product of nonzero A-weights in the fibers of $TX \to TY$.

More generally, let a cocharacter

$$\sigma : \mathbb{C}^\times \to \mathsf{T}$$

be such that $(\hbar, \sigma) = -1$. This generalizes the scaling action of $\mathbb{C}^\times$ in the fibers of T^*Y. Then we can choose the weights in ε as the σ-negative weights in the fiber of N_Z over some chosen $x \in Z^\sigma$.

Example 3.3.3. We have a canonical polarization associated to Nakajima varieties as follows. Recall from Section 2.2.7 that we have a virtual splitting of the tangent bundle

$$T\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) = T^{\mathbf{1/2}} + \hbar^{-1} \otimes \left(T^{\mathbf{1/2}}\right)^\vee .$$

Let ϵ denote the product, weighted by multiplicity, of the nonzero A-weights in the restriction of $(T^{\mathbf{1/2}})^\vee$ to some $x \in Z$.

3.3.3 Degree in A

Since A acts trivially on X^{A}, we have

$$H^\cdot_{\mathsf{T}}(X^{\mathsf{A}}) = H^\cdot_{\mathsf{T}/\mathsf{A}}(X^{\mathsf{A}}) \otimes_{\mathbb{C}[\mathfrak{t}/\mathfrak{a}]} \mathbb{C}[\mathfrak{t}] .$$

While there is no canonical splitting

$$\mathbb{C}[\mathfrak{t}] \cong \mathbb{C}[\mathfrak{t}/\mathfrak{a}] \otimes \mathbb{C}[\mathfrak{a}] \tag{3.8}$$

any such splitting leads to the same increasing filtration of $H^\cdot_{\mathsf{T}}(X^{\mathsf{A}})$ by the degree $\deg_{\mathsf{A}}$ in $\mathbb{C}[\mathfrak{a}]$. Clearly,

$$\operatorname{gr} H^\cdot_{\mathsf{T}}(X^{\mathsf{A}}) = H^\cdot_{\mathsf{T}/\mathsf{A}}(X^{\mathsf{A}}) \otimes \mathbb{C}[\mathfrak{a}] . \tag{3.9}$$

3.3.4 Characterization

Choose a chamber $\mathfrak{C} \subset \mathfrak{a}$ and an polarization ε of X^{A}. The following theorem is the main result of this section.

Theorem 3.3.4. *There exists a unique map of $H^\cdot_{\mathsf{T}}(\mathsf{pt})$-modules*

$$\mathrm{Stab}_{\mathfrak{C},\varepsilon} : H^\cdot_{\mathsf{T}}(X^{\mathsf{A}}) \to H^\cdot_{\mathsf{T}}(X)$$

such that for any $Z \in \mathsf{Fix}$ and any $\gamma \in H^\cdot_{\mathsf{T}/\mathsf{A}}(Z)$, the stable envelope $\Gamma = \mathrm{Stab}_{\mathfrak{C},\sigma}(\gamma)$ satisfies:

(i) $\operatorname{supp}\Gamma \subset \operatorname{Attr}^f_{\mathfrak{C}}(Z)$,

(ii) $\Gamma\big|_Z = \pm e(N_-) \cup \gamma$, *according to polarization,*

(iii) $\deg_{\mathsf{A}} \Gamma\big|_{Z'} < \frac{1}{2} \operatorname{codim} Z'$, *for any* $Z' \prec Z$ *.*

Remark 3.3.5. The chamber and the polarization are independent parameters in the construction of $\operatorname{Stab}_{\mathfrak{C},\varepsilon}$. The former being much more important than the latter, we abbreviate

$$\operatorname{Stab}_{\mathfrak{C}} = \operatorname{Stab}_{\mathfrak{C},\varepsilon} ,$$

once some polarization ε has been specified.

Remark 3.3.6. We will see $\operatorname{Stab}_{\mathfrak{C}}$ is given by a Lagrangian correspondence on $X \times X^{\mathsf{A}}$, and, in particular, it maps middle degree to middle degree.

The existence of $\operatorname{Stab}_{\mathfrak{C}}$ will be proven later. We now prove the uniqueness a map satisfying the conditions of the theorem.

Proof. Let $\gamma \in H^{\cdot}_{\mathsf{T}}(X)$ be supported on a union of attracting sets and satisfy

$$\deg_{\mathsf{A}} \iota^* \gamma < \tfrac{1}{2} \operatorname{codim} Z ,$$

for any embedding $\iota : Z \hookrightarrow X$ of a fixed component. We claim this forces $\gamma = 0$.

Pick a total ordering on Fix refining $\prec$ and choose $Z \in \mathsf{Fix}$ so that γ is supported on $\operatorname{Attr}^f_{\mathfrak{C}}(Z)$. We can factor $\iota = f_3 f_2 f_1$, where

$$Z \xhookrightarrow{f_1} \operatorname{Attr}_{\mathfrak{C}}(Z) \xhookrightarrow{f_2} \operatorname{Attr}^f_{\mathfrak{C}}(Z) \xhookrightarrow{f_3} X .$$

Here f_1 is regular and f_2 is open. The support condition on γ means that

$$\gamma \cap [X] = (f_3)_* \, \alpha$$

for a certain Borel-Moore homology class α. Standard excess intersection arguments then show

$$\iota^*(\gamma) \cap [Z] = e(N_-) \cap f_1^* \, f_2^* \, \alpha .$$

The multiplication by $e(N_-)$ is injective on (3.9) and

$$\deg_{\mathsf{A}} e(N_-) = \tfrac{1}{2} \operatorname{codim} Z .$$

Because this exceeds the degree of the right-hand side, $f_1^* \, f_2^* \, \alpha = 0$. Since f_1^* is an isomorphism, this forces $f_2^* \alpha$ to vanish, meaning that γ is supported on a smaller union of strata. Arguing inductively, we see $\gamma = 0$.

Now if $\Gamma_1, \Gamma_2 \in H^{\cdot}_{\mathsf{T}}(X)$ are two classes satisfying (i)–(iii) then their difference satisfies the hypothesis above, hence vanishes. □

3.4 Lagrangian residues

Let L be an A-invariant Lagrangian and let

$$\iota : Z \hookrightarrow X$$

be an embedding of a component of X^{A}. The form $\iota^*\omega$ is symplectic and so we can talk about isotropic and Lagrangian subvarieties of Z.

Lemma 3.4.1. *$L \cap Z$ is an isotropic subvariety of Z.*

Proof. Let W be an irreducible component of W of $L \cap Z$. For a general point $w \in W$, there exists a sequence of points $x_1, x_2, \dots$ in the smooth locus of L approaching w such that limit of $T_{x_k}L$ exists as $k \to \infty$ and contains the tangent space $T_w W$. This can be seen, for instance, by choosing a Whitney stratification of L for which $L \cap Z$ is a union of strata. Since the symplectic form on Z is the restriction of the symplectic form on X, the lemma follows. □

Now suppose an polarization ε of Z has been chosen.

Lemma 3.4.2. *There is a unique Lagrangian cycle* $\operatorname{Res}_Z L$ *supported on $L \cap Z$ such that*

$$\iota^*[L] = \varepsilon\,[\operatorname{Res}_Z L] + \dots$$

where dots stand for terms of smaller A*-degree.*

Proof. The class $\iota^*[L]$ is supported on a subvariety $L \cap Z$ of dimension at most $\frac{1}{2}\dim Z$. Therefore, its A-degree can be at most

$$\operatorname{codim}_X L - \operatorname{codim}_Z(L \cap Z) \leqslant \tfrac{1}{2}\operatorname{codim}_X Z\,.$$

Assuming $L \cap Z$ is middle-dimensional, denote by $L_1, L_2, \dots$ its Lagrangian irreducible components. We have

$$\iota^*[L] = \sum [L_i] \cdot f_i + \dots$$

where $f_i \in H^{\cdot}_{\mathsf{A}}(\mathsf{pt})$ is a homogeneous polynomial of degree $\frac{1}{2}\operatorname{codim}_X Z$ and dots stand for terms of smaller degree.

In order to calculate f_i, we shrink X to a neighborhood of a smooth generic point of L_i. Furthermore, we can degenerate to the normal cone of Z inside X and restrict to a transverse slice through a generic point of L_i. After these simplifications, the following lemma finishes the proof. □

Lemma 3.4.3. *Let $V = \mathbb{C}^n$ be a vector space equipped with the diagonal action of* A *by characters $\chi_1, \ldots, \chi_n$. Let $X = V \oplus V^\vee$ be the symplectic vector space equipped with the induced action of* A *and suppose we have a Lagrangian* A*-invariant conical subvariety $L \subset X$. Then the residue of $[L]$ at the origin $0 = Z \subset X$ is an integer multiple of $\varepsilon = \prod_{j=1}^n \chi_j$.*

Proof. We embed A in $T = (\mathbb{C}^\times)^{n+1}$, the maximal torus of $Sp(2n) \times \mathbb{C}^\times$, the stabilizer of the line $\mathbb{C}\omega \in \Omega^2(X)$. We can use T to degenerate L via a family of A-invariant conical subvarieties to a T-invariant conical subvariety and calculate the residue for this limit. Since T scales ω, this limit is still Lagrangian. On the other hand, the only such T-invariant subvarieties are unions of Lagrangian coordinate planes. For a Lagrangian coordinate plane, it is clear that the residue is a product of the characters χ_j up to a sign. □

Lemma 3.4.4. *For any* A*-invariant Lagrangian L and any chamber $\mathfrak{C}$, there exists a Lagrangian cycle L' supported on $\mathrm{Attr}^f_{\mathfrak{C}}(Z)$ such that*

$$\deg_{\mathsf{A}} \iota^*([L] - [L']) < \tfrac{1}{2} \operatorname{codim} Z \,.$$

Proof. We can take L' to be the closure of $\lim_{\mathfrak{C}}^{-1}(\pm \operatorname{Res}_Z L)$, counting multiplicity. □

Lemma 3.4.5. *Let $L \subset X$ be an* A*-invariant Lagrangian subvariety supported on $\mathrm{Attr}^f_{\mathfrak{C}}(Z)$. Then there exists a unique Lagrangian cycle L' such that $L' - L$ is supported on $\bigcup_{Z' < Z} \mathrm{Attr}^f_{\mathfrak{C}}(Z')$ and*

$$\deg_{\mathsf{A}} \iota^*_{Z'}[L'] < \tfrac{1}{2} \operatorname{codim} Z'$$

for any $Z' < Z$.

Proof. The existence follows by induction from Lemma 3.4.4. The uniqueness is shown as in Section 3.3. □

In conclusion, we note that if L is T-invariant, then so are all other Lagrangians occurring in the above Lemmas.

3.5 Proof of existence

Consider the (possibly disconnected) T-variety $X \times X^{\mathsf{A}}$ equipped with the antidiagonal symplectic form $(\omega, -\omega|_{X^{\mathsf{A}}})$. We construct $\mathrm{Stab}_{\mathfrak{C}}$ by exhibiting a correspondence between X^{A} and X.

Proposition 3.5.1. *There exists a* T*-invariant Lagrangian cycle Stab* $\mathcal{L}_{\mathfrak{C}}$ *on* $X \times X^{\mathsf{A}}$*, proper over* X*, with the following properties:*

(i) *For any* $Z \in \mathsf{Fix}$*, the restriction of* $\mathcal{L}_{\mathfrak{C}}$ *to* $X \times Z$ *is supported on* $\mathrm{Attr}^f_{\mathfrak{C}}(Z) \times Z$;

(ii) *the restriction of* $[\mathcal{L}_{\mathfrak{C}}]$ *to* $Z \times Z$ *equals* $\pm e(N_-) \cap [\Delta]$*, according to polarization, where* Δ *is the diagonal*;

(iii) *for* $Z' < Z$*, the restriction of* $[\mathcal{L}_{\mathfrak{C}}]$ *to* $Z' \times Z$ *has* A*-degree less than* $\frac{1}{2} \operatorname{codim} Z'$.

This shows the existence of $\mathrm{Stab}_{\mathfrak{C}}$ by taking the map

$$H^{\cdot}_{\mathsf{T}}(X^{\mathsf{A}}) \to H^{\cdot}_{\mathsf{T}}(X)$$

induced by the correspondence $\mathcal{L}_{\mathfrak{C}}$. Properness over X insures this map is well-defined.

Proof. Fix some Z and let $\pm L$ be the closure of the preimage of Δ under the map

$$\mathrm{Attr}_{\mathfrak{C}} Z \times Z \to Z \times Z\,,$$

with sign as above. Then L is a A-invariant Lagrangian supported on $Z \times \mathrm{Attr}^f_{\mathfrak{C}}(Z)$ which satisfies (i) and (ii). Using Lemma 3.4.5, we can modify it on lower strata so that to achieve (iii). Repeating this for all $Z \in \mathsf{Fix}$, we obtain a Lagrangian cycle $\mathcal{L}_{\mathfrak{C}}$ and it remains to check that its support is proper over X.

As in the proof of Lemma 3.2.7, choose a A-equivariant embedding

$$\pi : X_0 \hookrightarrow V$$

into a linear representation of A and let $V_0 \subset V_{\geqslant 0}$ be the subspaces formed by A-invariant and weights positive on $\mathfrak{C}$, respectively. Let

$$\rho : V_{\geqslant 0} \to V_0$$

be the natural projection. Consider the closed set $\pi^{-1}(V_{\geqslant 0}) \subset X$ (this is just the union of all attracting manifolds), along with the morphism

$$\rho \circ \pi : \pi^{-1}(V_{\geqslant 0}) \to V_0\,.$$

By construction, the Lagrangian cycle $\mathcal{L}_{\mathfrak{C}}$ lies in the fiber product

$$\pi^{-1}(V_{\geqslant 0}) \times_{V_0} X^{\mathsf{A}} \subset X \times X^{\mathsf{A}}\,.$$

Indeed, we construct $\mathcal{L}_{\mathfrak{C}}$ by starting with the diagonal $\Delta \subset X^{\mathsf{A}} \times X^{\mathsf{A}}$ and taking attracting manifolds and closures. The fiber product is closed with respect to both these operations.

On the other hand, the projection onto the first factor

$$\pi^{-1}(V_{\geqslant 0}) \times_{V_0} X^{\mathsf{A}} \to X$$

is proper: since the map $\pi : X \to V$ is proper, we can reduce the statement to the claim that

$$V_{\geqslant 0} \times_{V_0} V_0 \to V$$

is proper, which is obvious.

□

We note the following corollary of the proof. It will play an essential role in proving various properness statements later.

Proposition 3.5.2. *Let X_+ denote the union of all attracting manifolds. Then*

$$\mathcal{L}_{\mathfrak{C}} \subset X_+ \times_{X_0} X^{\mathsf{A}}\,.$$

Remark 3.5.3. Suppose $X = T^*Y$ where Y is a smooth projective variety and assume the action of A is induced from an action on Y with isolated fixed points $\{p_k\}$. Then a choice of chamber $\mathfrak{C}$ defines an A-invariant Bialynicki-Birula stratification of Y by locally closed varieties V_{p_k}. In this case, the stable envelope map $\mathrm{Stab}_{\mathfrak{C}}$ defines a collection of Lagrangian cycles on X. These can be identified (up to a sign depending on the polarization) with the characteristic cycles of the constructible sheaves $(j_k)_! \mathbb{Q}_{V_{p_k}}$ where j_k denotes the inclusion into Y. See, in particular, [3] for recent developments in this direction.

3.6 Torus restriction

Let $\mathfrak{C}$ be a chamber and let $\mathfrak{C}' \subset \mathfrak{C}$ be a face of some dimension. Consider

$$\mathfrak{a}' = \mathrm{Span}\, \mathfrak{C}' \subset \mathfrak{a}$$

with associated subtorus A'. The cone $\mathfrak{C}$ projects to a cone in $\mathfrak{a}/\mathfrak{a}'$ that we denote by $\mathfrak{C}/\mathfrak{C}'$.

Let ε be an polarization of $X^{\mathsf{A}} \subset X$. We can factor

$$\varepsilon = \varepsilon' \, \varepsilon''$$

into weights that are zero and nonzero on $\mathfrak{a}'$, respectively. The factors induce an polarization of $X^{\mathsf{A}} \subset X^{\mathsf{A}'}$ and $X^{\mathsf{A}'} \subset X$, respectively. In the following lemma, we take these induced polarizations.

Lemma 3.6.1. *The diagram*

$$\begin{array}{ccc} H^{\cdot}\left(X^{\mathsf{A}}\right) & \xrightarrow{\quad \mathrm{Stab}_{\mathfrak{C}} \quad} & H^{\cdot}(X) \\ & {\scriptstyle \mathrm{Stab}_{\mathfrak{C}/\mathfrak{C}'}} \searrow \qquad \nearrow {\scriptstyle \mathrm{Stab}_{\mathfrak{C}'}} & \\ & H^{\cdot}\left(X^{\mathsf{A}'}\right) & \end{array} \tag{3.10}$$

is commutative.

Proof. This follows from the uniqueness of the stable envelopes. Let $\mathcal{L}_{\mathfrak{C}'}, \mathcal{L}_{\mathfrak{C}/\mathfrak{C}'}$ be the Lagrangian correspondences constructed in Proposition 3.5.1, and consider their convolution

$$[\mathcal{L}_{\mathfrak{C},\mathfrak{C}'}] = [\mathcal{L}_{\mathfrak{C}/\mathfrak{C}'}] \circ [\mathcal{L}_{\mathfrak{C}'}]$$

which defines a Lagrangian cycle class in $X^{\mathsf{A}} \times X$.

If we can show it satisfies the properties in Proposition 3.5.1, then uniqueness of $\mathrm{Stab}_{\mathfrak{C}}$ gives the result. In fact, using the definition of the chamber $\mathfrak{C}/\mathfrak{C}'$, most of the properties are immediate. For example, (iii) follows from the degree constraints of either $\mathcal{L}_{\mathfrak{C}'}$ or $\mathcal{L}_{\mathfrak{C}/\mathfrak{C}'}$. □

3.7 Symplectic resolutions

3.7.1

In this paper, we are mainly interested in equivariant symplectic resolutions,

$$X \to X_0 = \operatorname{Spec} H^0(\mathcal{O}_X)\,,$$

see [60] for a comprehensive discussion. For symplectic resolutions, stable envelopes are easier to construct and enjoy stronger properties.

In addition to Nakajima quiver varieties $\mathcal{M}_{\theta,0}$ for θ generic, examples of symplectic resolutions include $T^*(G/P)$, where $P \subset G$ is a parabolic subgroup.

3.7.2

We begin with the universal deformation of the pair (X, ω)

$$\begin{array}{ccc} X & \xhookrightarrow{\iota_0} & \widetilde{X} \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ [0] & \hookrightarrow & B \cong H^2(X, \mathbb{C}) \,, \end{array} \tag{3.11}$$

in which the period map ϕ associates to a deformation (X', ω') the class of ω' in $H^2(X') = H^2(X)$. This universal deformation may be written down explicitly for Nakajima varieties and in all other examples, see [60] for further discussion.

The deformation (3.11) is G-equivariant, where G acts on the vector space B by the character $\hbar$. Therefore, the group

$$\mathsf{G}_\omega = \operatorname{Ker} \hbar \supset \mathsf{A}$$

acts on each fiber of ϕ.

3.7.3

Suppose we are given a class

$$\alpha^\vee \in H_2(X, \mathbb{Z})$$

that is an effective curve class in some fiber $(X', \omega') \neq (X, \omega)$. Then

$$\int_{\alpha'} \omega' = 0$$

and hence deformations with nonzero holomorphic curve classes belong to a union of hyperplanes in the base B.

Definition 3.7.1. A coroot hyperplane of X is a hyperplane of B along which the deformation of X has nonzero holomorphic curve classes.

Over their complement

$$B^\circ = B \backslash \bigcup_{\text{coroots}} (\alpha^\vee)^\perp$$

the fiber of ϕ is affine. It is an interesting question to find a geometric definition of coroots of X themselves rather than just their associated hyperplanes.

3.7.4

Consider the diagonal

$$\Delta^\circ \subset \widetilde{X}^\circ \times_{B^\circ} \left(\widetilde{X}^\circ\right)^{\mathsf{A}},$$

where $\widetilde{X}^\circ = \phi^{-1}(B^\circ)$. Since the fibers over B° contain no holomorphic cycles, the inclusion

$$\operatorname{Attr}_{\mathfrak{C}} \Delta^\circ \hookrightarrow \widetilde{X}^\circ \times_{B^\circ} \left(\widetilde{X}^\circ\right)^{\mathsf{A}} \tag{3.12}$$

is a closed embedding and defines a family of cycles over B°. We denote by

$$\widetilde{\mathcal{L}}_{\mathfrak{C}} = \overline{\operatorname{Attr}_{\mathfrak{C}} \Delta^\circ}$$

it closure in $\widetilde{X} \times_B \widetilde{X}^{\mathsf{A}}$. In particular, we can take the A-fixed points

$$\begin{array}{ccc} \left(\widetilde{\mathcal{L}}_{\mathfrak{C}}\right)^{\mathsf{A}} & \hookrightarrow & \widetilde{X}^{\mathsf{A}} \times_B \widetilde{X}^{\mathsf{A}} \\ & & \downarrow \phi \\ & & B \end{array}$$

Proposition 3.7.2. *For any $b \in B$, the top-dimensional components of*

$$\left(\widetilde{\mathcal{L}}_{\mathfrak{C}}\right)^{\mathsf{A}} \cap \phi^{-1}(b)$$

are Steinberg correspondences.

Proof. All fibers of ϕ are symplectic resolutions and we can find a universal proper G-equivariant map $\widetilde{\pi}$

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\pi}} & \widetilde{V} \\ \phi \downarrow & & \downarrow \\ B & \xrightarrow{\text{id}} & B \end{array}$$

into a vector bundle $\widetilde{V}$ over B. The torus A acts trivially on B and we denote by $\widetilde{V}_{\geqslant 0}$ the subbundle formed by A-weights that are nonnegative on $\mathfrak{C}$. As in the proof of Proposition 3.5.1, one shows

$$\widetilde{\mathcal{L}}_{\mathfrak{C}} \subset \widetilde{\pi}^{-1}\left(\widetilde{V}_{\geqslant 0}\right) \times_{\widetilde{V}_0} \widetilde{X}^{\mathsf{A}}\,.$$

Therefore

$$\left(\widetilde{\mathcal{L}}_{\mathfrak{C}}\right)^{\mathsf{A}} \subset \widetilde{X}^{\mathsf{A}} \times_{\widetilde{\pi}} \widetilde{X}^{\mathsf{A}}\,.$$

On the other hand, it is known that the ϕ-fibers of

$$\widetilde{X} \times_{\widetilde{\pi}} \widetilde{X} \subset \widetilde{X} \times_B \widetilde{X}$$

are isotropic.[1] Therefore, their intersections with a symplectic subvariety $\widetilde{X}^{\mathsf{A}} \times \widetilde{X}^{\mathsf{A}}$ are at most Lagrangian. Their Steinberg property is clear from the above. □

Remark 3.7.3. This Proposition gives an abundant source of Steinberg correspondences, as we will see below.

Theorem 3.7.4. *The correspondence $\mathcal{L}_{\mathfrak{C}}$ is the specialization of $\widetilde{\mathcal{L}}_{\mathfrak{C}}$ to the central fiber, that is*

$$[\mathcal{L}_{\mathfrak{C}}] = \iota_0^* \left[\, \widetilde{\mathcal{L}}_{\mathfrak{C}} \,\right] \in H^{BM}_{\mathsf{T}}(X \times X^{\mathsf{A}})\,.$$

Proof. It suffices to check the right-hand side satisfies the conditions of Proposition 3.5.1. Properness is shown exactly as in the proof of Proposition 3.5.1. Similarly, conditions (i) and (ii) follow from construction.

To show (iii) we the consider inclusion

$$\iota : Z' \times Z \hookrightarrow \widetilde{X} \times \widetilde{X}^{\mathsf{A}}\,, \quad Z \neq Z'\,,$$

of an off-diagonal component of $X^{\mathsf{A}} \times X^{\mathsf{A}}$. By Proposition 3.7.2

$$\iota^* \left[\, \widetilde{\mathcal{L}}_{\mathfrak{C}} \,\right] = \sum f_i \,[L_i] + \dots\,, \qquad f_i \in H^{\operatorname{codim} Z'}_{\mathsf{T}}(\mathsf{pt})\,,$$

[1] This widely known and used statement may be deduced from the results of Kaledin [61] and Namikawa [92]. Further details may be found in the forthcoming lecture notes of V. Ginzburg on the subject.

where L_i are the Lagrangian components of the intersection and dots stand for terms of smaller A-degree. The required degree bound

$$\deg_{\mathsf{A}} f_i < \frac{1}{2} \operatorname{codim} Z'$$

follows from a much stronger claim: all f_i are divisible by $\hbar$. We state this as a separate result. □

For any X, not necessarily a symplectic resolutions, we can write

$$[\mathcal{L}_{\mathfrak{C}}]\Big|_{X^{\mathsf{A}}\times X^{\mathsf{A}}} = \pm e(N_-) \cup \Delta + \text{ off-diagonal} \tag{3.13}$$

where the second term is a class supported on

$$\bigsqcup_{Z_1 < Z_2} Z_1 \times Z_2\,, \quad Z_i \in \mathsf{Fix}\,.$$

Theorem 3.7.5. *For symplectic resolutions,*

$$[\mathcal{L}_{\mathfrak{C}}]\Big|_{X^{\mathsf{A}}\times X^{\mathsf{A}}} = \pm e(N_-) \cup \Delta \mod \hbar H^{\bullet}_{\mathsf{T}}(X^{\mathsf{A}} \times X^{\mathsf{A}})\,.$$

Proof. Let Z, Z' be two different components of X^{A}. We will show the pull-back of $\widetilde{\mathcal{L}}$ by

$$\iota : Z' \times Z \hookrightarrow \widetilde{X} \times_B \widetilde{X}^{\mathsf{A}}$$

is divisible by $\hbar$, which also completes the proof of the last theorem. We choose a general line $\ell \subset B$ through the origin in the base of the deformation and denote by $\widetilde{X}_\ell$ the restriction of $\widetilde{X}$ to ℓ. We may factor $\iota = \iota_2 \circ \iota_1$ where

$$Z' \times Z \xrightarrow{\iota_1} (\widetilde{X}_\ell)^{\mathsf{A}} \times_\ell (\widetilde{X}_\ell)^{\mathsf{A}} \xrightarrow{\iota_2} \widetilde{X} \times_B \widetilde{X}^{\mathsf{A}}\,.$$

Only the central fiber of $\widetilde{X}_\ell$ contains holomorphic curves. Therefore, if we consider the connected component W of $(\widetilde{X}_\ell)^{\mathsf{A}} \times_\ell (\widetilde{X}_\ell)^{\mathsf{A}}$ containing $Z \times Z'$, the contribution of W to $\iota_2^*\big[\widetilde{\mathcal{L}}\big]$ is supported over the origin, i.e.

$$\operatorname{supp}_W \iota_2^*\big[\widetilde{\mathcal{L}}\big] \subset Z' \times Z.$$

Therefore $\iota^*\big[\widetilde{\mathcal{L}}\big]$ factors through

$$\iota_1^* \circ (\iota_1)_* = \text{multiplication by } \hbar\,.$$

□

Chapter 4

Properties of R-matrices

4.1 Definition and braid relations

4.1.1

We fix some polarization ε and consider the maps

$$\mathrm{Stab}_{\mathfrak{C}} : H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^{\mathsf{A}}) \to H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X)$$

parameterized by the chambers $\mathfrak{C}$. Here G_A is a reductive group which commutes with A and we denote $\mathfrak{g}_\mathsf{A} = \mathrm{Lie}\, \mathsf{G}_\mathsf{A}$.

The maps $\mathrm{Stab}_{\mathfrak{C}}$ become isomorphisms after inverting $e(N_-)$. Therefore we can make the following

Definition 4.1.1.

$$R_{\mathfrak{C}',\mathfrak{C}} = \mathrm{Stab}_{\mathfrak{C}'}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}} \in \mathrm{End}\left(H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^{\mathsf{A}})\right) \otimes \mathbb{Q}\left(\mathfrak{g}_\mathsf{A}\right) .$$

4.1.2 Example

Take $X = T^*\mathbb{P}^1$ with the action of $\mathsf{A} = \mathbb{C}^\times$ induced from $\mathbb{P}^1$. We have

$$X^{\mathsf{A}} = \{0, \infty\}$$

Let u be the A-weight in $T_0\mathbb{P}^1$ and let $\mathbb{C}^\times_\hbar \subset \mathsf{G}_\mathsf{A}$ scale the cotangent fibers with weight $-\hbar$. Let the polarization ε be given by the fibers. Then

$$\mathrm{Stab}_{\mathfrak{C}}(0) = [\mathbb{P}^1] + [F_\infty], \quad \mathrm{Stab}_{\mathfrak{C}}(\infty) = -[F_\infty]$$

for $\mathfrak{C} = \{u > 0\}$ where

$$\begin{aligned} [\mathbb{P}^1] &= \text{zero section}\,, \\ [F_\infty] &= \text{fiber over } \infty \in \mathbb{P}^1\,. \end{aligned}$$

Similarly

$$\mathrm{Stab}_{-\mathfrak{C}}(0) = -[F_0], \quad \mathrm{Stab}_{-\mathfrak{C}}(\infty) = [\mathbb{P}^1] + [F_0]\,.$$

For $\{z_1, z_2\} = \{0, \infty\}$, we have

$$\mathrm{Stab}_{\pm\mathfrak{C}}(z_j)\Big|_{z_i} = \begin{pmatrix} -u-\hbar & 0 \\ -\hbar & u \end{pmatrix}, \begin{pmatrix} -u & -\hbar \\ 0 & u-\hbar \end{pmatrix}.$$

Therefore,

$$R(u) = \frac{1 - \frac{\hbar}{u}\,\mathbf{s}}{1 - \frac{\hbar}{u}} \tag{4.1}$$

where $\mathbf{s}$ is the permutation of 0 and ∞. Up to proportionality, this is Yang's original R-matrix. It is normalized so that $R(u) = 1$ on the invariants of $\mathbf{s}$.

4.1.3

It will be convenient to represent rational functions appearing in $R_{\mathfrak{C}',\mathfrak{C}}$ as formal power series in inverse roots using some splitting (3.8) and

$$\frac{1}{\alpha + x} = \frac{1}{\alpha} - \frac{x}{\alpha^2} + \frac{x^2}{\alpha^3} + \dots\,.$$

Here $\alpha \in \mathfrak{a}^*$ is a root, i.e. a weight appearing in the normal bundle to X^{A}, and x is the $(\mathsf{G}_{\mathsf{A}}/\mathsf{A})$-equivariant Chern root of the corresponding weight subspace of N_-. Since we only inverting $e(N_-)$, all denominators occurring in the R-matrices are of this form.

One should keep in mind that this expansion depends on a splitting (3.8) and reexpand accordingly if the splitting is changed.

For a different polarization, the R-matrices differ by conjugation by a diagonal ± 1 matrix.

4.1.4 Root R-matrices

Evidently, it is enough to consider R-matrices corresponding to a pair of chambers $\mathfrak{C}, \mathfrak{C}'$ separated by a wall $\alpha = 0$. Here α is a root and we may assume that $\alpha(\mathfrak{C}) > 0$. Consider the subtorus $\mathsf{A}_\alpha \subset \mathsf{A}$ with Lie algebra $\mathfrak{a}_\alpha = \operatorname{Ker}\alpha$. We denote

$$X^\alpha = X^{\mathsf{A}_\alpha}\,.$$

For the $\mathsf{A}/\mathsf{A}_\alpha$-action on X^α, there are two chambers, namely $\alpha \gtrless 0$. We take the induced polarization of $X^\mathsf{A} \subset X^\alpha$ and denote by

$$R_\alpha = R_{<0,>0} \in \operatorname{End}(H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A})) \otimes \mathbb{Q}(\mathfrak{g}_\mathsf{A}/\mathfrak{a}_\alpha)$$

the corresponding R-matrix.

From Lemma 3.6.1 we have the following

Corollary 4.1.2. *If $\mathfrak{C}$ and $\mathfrak{C}'$ are separated by a wall $\alpha = 0$ then*

$$R_{\mathfrak{C}',\mathfrak{C}} = R_\alpha\,.$$

We call operators R_α the *root R-matrices.*

4.1.5 R-matrices for Nakajima varieties

Given a quiver Q, vector w, and a generic choice of θ, we define

$$\mathcal{M}(\mathsf{w}) = \bigsqcup_v \mathcal{M}_{\zeta,0}(\mathsf{v},\mathsf{w})\,, \tag{4.2}$$

where we dropped the moment map parameters on the left-hand side for brevity, and define

$$H(\mathsf{w}) = H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\mathsf{w}))\,.$$

Consider a tensor product of Nakajima varieties as in Section 2.4. There are two chambers

$$\mathfrak{C} = \{u > 0\}\,, \quad \mathfrak{C}' = \{u < 0\}\,,$$

where u is the weight of the defining representation of $\mathsf{A} = \{z\} = \mathbb{C}^\times$. We denote

$$R_{\mathsf{w}',\mathsf{w}''}(u) = R_{\mathfrak{C}',\mathfrak{C}} \in \operatorname{End}(H(\mathsf{w}') \otimes H(\mathsf{w}'')) \otimes \mathbb{Q}(u)$$

the corresponding R-matrix.

4.1.6

More generally, a decomposition

$$\mathsf{w} = \sum_{i=1}^{n} \mathsf{w}^{(i)}$$

gives a homomorphism

$$\mathsf{A} = \{(z_1, \dots, z_n)\} \to G_{\mathsf{w}}$$

given by $\mathsf{w} = \sum_{i=1}^{n} \mathsf{w}^{(i)} z_i$ as in Section 2.3. By Proposition 2.3.1

$$\mathcal{M}(\mathsf{w})^{\mathsf{A}} = \mathcal{M}\left(\mathsf{w}^{(1)}\right) \times \cdots \times \mathcal{M}\left(\mathsf{w}^{(n)}\right)$$

and hence

$$H^{\cdot}_{\mathsf{G}_\mathsf{A}}\big(\mathcal{M}(\mathsf{w})^{\mathsf{A}}\big) = H\left(\mathsf{w}^{(1)}\right) \otimes \cdots \otimes H\left(\mathsf{w}^{(n)}\right) .$$

The walls are the roots of $GL(n)$

$$\alpha = a_i - a_j \,, \quad 1 \leqslant i < j \leqslant n \,,$$

and the corresponding fixed loci are of the form

$$\mathcal{M}(\mathsf{w})^{\alpha} = \mathcal{M}\left(\mathsf{w}^{(i)} + \mathsf{w}^{(j)}\right) \times \prod_{k \neq i,j} \mathcal{M}\left(\mathsf{w}^{(k)}\right) ,$$

where $\mathsf{A}/\mathsf{A}_\alpha$ acts only on the first factor. We conclude

$$R_\alpha = R_{\mathsf{w}^{(i)},\mathsf{w}^{(j)}}(a_i - a_j)_{ij}$$

where the subscript means that it operates in the ith and jth tensor factors.

4.1.7 Normalization

From definitions, we have the following

Proposition 4.1.3.

$$R_\alpha = 1 + O\left(\alpha^{-1}\right) , \quad \alpha \to \infty \,.$$

In other words, R_α, as a formal power series in α^{-1} starts with the identity operator.

For symplectic resolution, we deduce from Theorem 3.7.5

Proposition 4.1.4.

$$R_\alpha = 1 + O(\hbar) \,, \quad \hbar \to 0 \,.$$

In other words, R_α acts as identity on $H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^{\mathsf{A}})/\hbar H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^{\mathsf{A}})$.

4.1.8 Braid relations

Let $\mathfrak{F} \subset \mathfrak{C}$ be a codimension 2 facet and let

$$\mathfrak{C} = \mathfrak{C}_0, \mathfrak{C}_1, \dots, \mathfrak{C}_{2n} = \mathfrak{C}$$

be the chambers containing $\mathfrak{F}$ as a facet, in cyclic order around $\mathfrak{F}$.

Proposition 4.1.5.

$$R_{\mathfrak{C}_0,\mathfrak{C}_1} R_{\mathfrak{C}_1,\mathfrak{C}_2} \dots R_{\mathfrak{C}_{2n-1},\mathfrak{C}_{2n}} = 1 \tag{4.3}$$

This relation, too obvious to be called a theorem, is of fundamental importance for much of what follows.

4.1.9 Example

In the setup of Section 4.1.6, take

$$\mathfrak{F} = \{a_1 = a_2 = a_3\} .$$

Then (4.3) gives

$$R_{12}(a_1 - a_2)\, R_{13}(a_1 - a_3)\, R_{23}(a_2 - a_3) = \\ R_{23}(a_2 - a_3)\, R_{13}(a_1 - a_3)\, R_{12}(a_1 - a_2)\,, \tag{4.4}$$

which is the Yang-Baxter equations with a spectral parameter.

4.2 Changing the torus

4.2.1

Suppose we have and inclusion of tori

$$\mathsf{A}_1 \subset \mathsf{A}_2$$

where A_2 preserves the symplectic form. Clearly,

$$\operatorname{roots}(\mathsf{A}_1) = \operatorname{roots}(\mathsf{A}_2)\Big|_{\mathfrak{a}_1} \setminus \{0\}\,,$$

and so every chamber $\mathfrak{C}_1 \subset \mathfrak{a}_1$ is contained in at least one closed chamber $\mathfrak{C}_2 \subset \mathfrak{a}_2$. From Lemma 3.6.1, we deduce the following

Proposition 4.2.1. *Let chambers $\mathfrak{C}_1, \mathfrak{C}_1' \subset \mathfrak{a}_1$ be faces of $\mathfrak{C}_2, \mathfrak{C}_2' \subset \mathfrak{a}_2$, respectively. Then the diagram*

$$\begin{array}{ccc} H^{\cdot}_{\mathsf{G}_{\mathsf{A}_2}}\left(X^{\mathsf{A}_2}\right) & \xrightarrow{\operatorname{Stab}_{\mathfrak{C}_2/\mathfrak{C}_1}} & H^{\cdot}_{\mathsf{G}_{\mathsf{A}_2}}\left(X^{\mathsf{A}_1}\right) \\ {\scriptstyle R_{\mathfrak{C}_2',\mathfrak{C}_2}}\big\downarrow & & \big\downarrow{\scriptstyle R_{\mathfrak{C}_1',\mathfrak{C}_1}} \\ H^{\cdot}_{\mathsf{G}_{\mathsf{A}_2}}\left(X^{\mathsf{A}_2}\right) & \xrightarrow{\operatorname{Stab}_{\mathfrak{C}_2'/\mathfrak{C}_1'}} & H^{\cdot}_{\mathsf{G}_{\mathsf{A}_2}}\left(X^{\mathsf{A}_1}\right) \end{array}$$

is commutative.

Here $\operatorname{Stab}_{\mathfrak{C}_2/\mathfrak{C}_1}$ really means $\operatorname{Stab}_{\mathfrak{C}_2/\mathfrak{C}_{2,1}}$, where $\mathfrak{C}_{2,1} \subset \mathfrak{C}_2$ is the minimal face that contains $\mathfrak{C}_1$.

4.2.2

Note that there could be many walls between $\mathfrak{C}_2$ and $\mathfrak{C}_2'$ even when $\mathfrak{C}_1$ and $\mathfrak{C}_1'$ are adjacent. Thus enlarging the torus leads to factorization of root R-matrices.

4.2.3

In practice, it convenient to reduce to the situation when

$$\dim \mathfrak{a}_1 = 1\,, \quad \dim \mathfrak{a}_2 = 2\,,$$

by restricting to root R-matrices for A_1 and replacing $\mathfrak{a}_2$ by a generic line in $\mathfrak{a}_2/\mathfrak{a}_1$, if necessary. Denoting by (u_1, u_2) the corresponding coordinates in $\mathfrak{a}_2$, we can go between

$$\mathfrak{C}_2 = \{u_1 \gg u_2 > 0\}\,, \quad \mathfrak{C}_2' = \{u_2 > 0 \gg u_1\}$$

by crossing the walls in the decreasing order of u_1/u_2.

4.2.4 Example

We continue with Example 4.1.6 and take

$$\begin{aligned} \mathfrak{a}_1 &= \{(a_1, 0, \ldots, 0)\}\,, \\ \mathfrak{a}_2 &= \mathfrak{a}_1 \oplus \mathbb{C}(0, a_2, \ldots, a_n)\,. \end{aligned}$$

To ensure that $\mathfrak{a}_2/\mathfrak{a}_1$ is generic in $\mathfrak{a}/\mathfrak{a}_1$, it is enough to take

$$a_2 > a_3 > \cdots > a_n \,. \tag{4.5}$$

Then $X^{\mathfrak{a}_2} = X^{\mathfrak{a}}$, while

$$X^{\mathfrak{a}_1} = \mathcal{M}\left(\mathsf{w}^{(1)}\right) \times \mathcal{M}\left(\mathsf{w} - \mathsf{w}^{(1)}\right) \,.$$

In $\mathfrak{a}_1$, we have two chambers

$$\mathfrak{C}_1 = \{a_1 > 0\}\,, \quad \mathfrak{C}_1' = \{0 > a_1\}\,,$$

corresponding to

$$\mathfrak{C}_2 = \{a_1 > a_2 > \cdots > a_n\}\,, \quad \mathfrak{C}_2' = \{a_2 > \cdots > a_n > a_1\}\,.$$

in $\mathfrak{a}_2$. Crossing from $\mathfrak{C}_2$ to $\mathfrak{C}_2'$, we get

$$R_{\mathfrak{C}_1',\mathfrak{C}_1} = R_{1,n}(a_1 - a_n) \cdots R_{1,3}(a_1 - a_3)\, R_{1,2}(a_1 - a_2) \tag{4.6}$$

in the stable basis of $H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^{\mathfrak{a}_1})$ corresponding to the chamber (4.5) in $\mathfrak{a}/\mathfrak{a}_1$. For a different choice of chamber, one reorders the factors accordingly.

4.3 Covers and factorization of R-matrices

4.3.1

It is interesting to elaborate on the factorization considered in Section 4.2 in the following special case. Let Q be a quiver. We take two vertices $i, j \in I$ and

$$\mathsf{w} = a\delta_i + \delta_j$$

where a is a weight of $\mathsf{A}_1 \cong \mathbb{C}^\times$. We have

$$\mathcal{M}(\mathsf{w})^{\mathsf{A}_1} = \mathcal{M}(\delta_i) \times \mathcal{M}(\delta_j)\,.$$

The corresponding R matrix

$$R_{H_i,H_j}(a) \in \mathrm{End}(H_i \otimes H_j) \otimes \mathbb{Q}(a)\,, \quad H_i = H(\delta_i)\,,$$

is one of the main building blocks of the theory.

4.3.2

We take $\mathsf{A}_2/\mathsf{A}_1$ to be the maximal torus of G'_{edge} and denote by

$$\Gamma = (\mathsf{A}_2/\mathsf{A}_1)^\wedge \cong H_1(Q,\mathbb{Z})$$

its character group. As explained in Section 2.3.5

$$\mathcal{M}(\mathsf{w})^{\mathsf{A}_2} = \widetilde{\mathcal{M}}(\delta_i) \times \widetilde{\mathcal{M}}(\delta_j)\,,$$

where $\widetilde{\mathcal{M}}$ are the quiver varieties associated to the universal abelian cover $\widetilde{Q}$ of the quiver Q.

Here we lift vertices of Q to vertices of $\widetilde{Q}$ that correspond to the trivial character of $\mathsf{A}_2/\mathsf{A}_1$. They form a fundamental domain for the action of Γ.

4.3.3

The walls in A_2 that we need to cross are of the form

$$a = \gamma\,, \quad \gamma \in \Gamma\,, \tag{4.7}$$

and the corresponding fixed loci are $\widetilde{\mathcal{M}}(\mathsf{w}_\gamma)$ where

$$\mathsf{w}_\gamma = a\delta_{\gamma i} + \delta_j\,.$$

Recall that Γ acts freely on the vertices of $\widetilde{Q}$ and the $a\delta_{\gamma i}$ term in w means that the corresponding framing arrow goes from a space of weight a to a space of weight γ. On the wall (4.7) these weights match and we get fixed points.

4.3.4

To order the walls (4.7), we pick a generic vector $t \in \mathfrak{a}_2/\mathfrak{a}_1$ and order them in the decreasing order of $\gamma(t)$. Then

$$R_{H_i,H_j}(a) = \overleftarrow{\prod_{\gamma}} \widetilde{R}_{H_{\gamma i},H_j}(a-\gamma) \tag{4.8}$$

in the stable basis corresponding to $\mathfrak{C}_2 \ni t$ and the ordering of the product is such that we cross the wall with the larger value of $\gamma(t)$ first.

Here $\widetilde{R}$ is the R-matrix for the quiver $\widetilde{Q}$ and we use the embedding $\mathsf{A}_2^\wedge \hookrightarrow \mathfrak{a}_2^*$ to write arguments of the R-matrices.

The infinite product (4.8) is locally finite, that is, all but finitely many factors act trivially on any given cohomology group.

4.3.5

The action of Γ on $\widetilde{Q}$ extends to its action on the corresponding Yangian $\widetilde{\mathsf{Y}}$, which will be defined and discussed in Chapter 5. It satisfies

$$\gamma(x)\big|_{H(\mathsf{w})} = x\big|_{H(\gamma^{-1}\mathsf{w})}, \quad x \in \widetilde{\mathsf{Y}},$$

where the action on framing vectors is by

$$\gamma\delta_i = \delta_{\gamma i}.$$

Note that varieties $\widetilde{\mathcal{M}}(\mathsf{w})$ and $\widetilde{\mathcal{M}}(\gamma^{-1}\mathsf{w})$ are naturally isomorphic and the matrix $\widetilde{R}$ is invariant under $\gamma \otimes \gamma$.

Rewriting (4.8) in terms of this action, we obtain the following

Theorem 4.3.1. *We have*

$$R_{H_i,H_j}(a) = \overleftarrow{\prod_{\gamma\in\Gamma}} (\gamma^{-1} \otimes 1) \cdot \widetilde{R}_{H_i,H_j}(a-\gamma) \tag{4.9}$$

in the stable basis for the maximal symplectic torus in G_{edge}, where the ordering of the factors corresponds to choice of a chamber as in Section 4.3.4.

Factorization of this kind play an important role in the theory of quantum groups, see [32].

4.3.6 Example

Let Q be the quiver with one vertex and one loop. Then

$$\widetilde{Q} = A_\infty,$$

on which the group $\Gamma \cong \mathbb{Z}$ acts by shifts. This action naturally extends to an action on

$$\widetilde{\mathsf{Y}} = \mathsf{Y}(\mathfrak{gl}_\infty).$$

The R-matrix in basic representation of $\mathsf{Y}(\mathfrak{gl}_\infty)$ may be found, for example, by fusion of R-matrices for fundamental representations. This gives a certain infinite product formula for the R matrix for Q, which is an object of significant interest.

4.4 Adjoint operators

In this section, we assume X^g is proper for some $g \in \mathsf{G}_\mathsf{A}$. As in Section 3.1.3, this defines the Poincaré pairing

$$(\gamma_1, \gamma_2)_X = \int_X \gamma_1 \cup \gamma_2 \in \mathbb{Q}(\mathfrak{g}_\mathsf{A})$$

on both X and X^A, the sign-twisted trace map τ, and the corresponding adjoints.

In particular, the adjoint $\mathrm{Stab}_\mathfrak{C}^\tau$ of the map $\mathrm{Stab}_\mathfrak{C}$ is given by the correspondence

$$\mathcal{L}_\mathfrak{C}^\tau = (-1)^{\frac{1}{2}\operatorname{codim} X^\mathsf{A}} (\mathcal{L}_\mathfrak{C})_{21} \subset X^\mathsf{A} \times X \,.$$

Here codim : $\mathsf{Fix} \to \mathbb{Z}$ denotes the codimension of a component of X^A and the subscript 21 refers to a permutation of factors.

Note that since $\mathcal{L}_\mathfrak{C}$ is not proper over X^A, equivariant localization is required to define the adjoint as an operator.

Theorem 4.4.1. *For any polarization ε and any chamber $\mathfrak{C}$, we have*

$$\mathrm{Stab}_{-\mathfrak{C}}^\tau \circ \mathrm{Stab}_\mathfrak{C} = 1 \,.$$

Proof. Let $\Delta : X \to X \times X$ be the diagonal map and consider the cycle class

$$C = \Delta^*(\mathcal{L}_{-\mathfrak{C}}^\tau \times \mathcal{L}_\mathfrak{C})$$

on $X^\mathsf{A} \times X \times X^\mathsf{A}$, where we have pulled back along the internal $X \times X$ factor. By construction,

$$\mathrm{Stab}_{-\mathfrak{C}}^\tau \circ \mathrm{Stab}_{-\mathfrak{C}} = (p_{13})_*(C) \tag{4.10}$$

where p_{13} is the projection along the middle factor.

We claim C is proper over $X^\mathsf{A} \times X^\mathsf{A}$. Indeed, as in the proof of Proposition 3.5.1, we have

$$\mathcal{L}_\mathfrak{C} \subset X^\mathsf{A} \times_{V_0} \pi^{-1}(V_{\geqslant 0}) \,.$$

Since $V_{\geqslant 0} \cap V_{\leqslant 0} = V_0$, we conclude

$$C \subset X^\mathsf{A} \times_{V_0} \pi^{-1}(V_0) \times_{V_0} X^\mathsf{A} \,,$$

whence the claim. Therefore, the composition (4.10) is defined in nonlocalized equivariant cohomology and, in particular, has no terms of negative degree in equivariant parameters.

On the other hand, we may compute (4.10) by localization, that is, as a sum of equivariant residues for all triples $(Z_1, Z_2, Z_3) \in \mathsf{Fix}^{\times 3}$. When

$$Z_1 = Z_2 = Z_3 \,,$$

the stable and unstable Euler classes precisely compensate the denominator in the localization formula, giving the diagonal as a result. All other residues have negative A-degree and hence cancel out. □

Corollary 4.4.2. *We have*

$$R_\alpha^\tau = R_\alpha$$

for any root R-matrix R_α*.*

Note R_α is an operator from $H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A})$ to itself, so R_α^τ coincides with the adjoint with respect to the Poincaré pairing.

4.5 Unitarity

4.5.1

In the theory of quantum groups, an R-matrix

$$R(u) \in \operatorname{End}(V \otimes V) \otimes \mathbb{Q}(u)$$

is called unitary if it satisfies

$$R_{21}(u) = R(-u)^{-1} \,, \tag{4.11}$$

where the subscript in $R_{21}(u)$ means that we permute the tensor factors. We will show that R-matrices for Nakajima varieties are unitary.

4.5.2

Consider the following general setup. Let a group of the form

$$\mathsf{G}_\mathsf{A} = \mathsf{A} \times \mathsf{G}'$$

act on X, where A is a torus preserving the symplectic form ω. Define $\phi \in \operatorname{Aut} \mathsf{G}_\mathsf{A}$ by

$$\phi \cdot (a, g') = (a^{-1}, g') \,.$$

It gives a pull-back map $\phi^* \in \operatorname{End} H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X)$ which is a homomorphism of algebras. In particular, ϕ^* is anti-linear over the base ring

$$\phi^*(a\gamma) = -a\,\phi^*(\gamma)\,, \quad a \in \mathfrak{a}\,.$$

In the cohomology of the fixed locus

$$H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A}) = H^{\cdot}_{G'}(X^\mathsf{A}) \otimes \mathbb{Q}[\mathfrak{a}]$$

the action of ϕ^* amounts to $a \mapsto -a$, $a \in \mathfrak{a}$.

4.5.3

Since weights positive on $\mathfrak{C}$ are precisely the weights negative on $-\mathfrak{C}$, the following diagram commutes

$$\begin{array}{ccc} H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A}) & \xrightarrow{\ \mathrm{Stab}_\mathfrak{C}\ } & H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X) \\ {\scriptstyle a\mapsto -a}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A}) & \xrightarrow{\ \mathrm{Stab}_{-\mathfrak{C}}\ } & H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X)\,. \end{array} \tag{4.12}$$

Note that $\mathrm{Stab}_\mathfrak{C}$ is literally the same correspondence as $\mathrm{Stab}_{-\mathfrak{C}}$ for the opposite action.

In particular, for $\mathsf{A} = \mathbb{C}^\times$ we conclude

$$R(-a) = R(a)^{-1}\,. \tag{4.13}$$

4.5.4

For tensor products of Nakajima varieties, we have

$$\mathcal{M}(\mathsf{w}+\mathsf{w}')^\mathsf{A} = \mathcal{M}(\mathsf{w}) \times \mathcal{M}(\mathsf{w}')\,, \quad \mathsf{A} = \mathbb{C}^\times\,,$$

Note, however, from Section 2.4 that the ordering of factors in the product above *depends* on a lift

$$\mathsf{A} \to G_\mathsf{w}$$

and not just on the image of A in G_w modulo the kernel of the action. The two lifts

$$z\mathsf{w} + \mathsf{w}' \quad \text{vs.} \quad \mathsf{w} + z^{-1}\mathsf{w}'$$

where $z \in \mathbb{C}^\times$ give the same action, but different identification of the fixed locus with the product. From (4.13), the corresponding R-matrices are

$$R(u) = R(-u)_{21}^{-1},$$

where $u \in \operatorname{Lie}\mathbb{C}^\times$. We thus obtain the following

Proposition 4.5.1. *The R-matrices for Nakajima varieties are unitary.*

4.6 Action of Steinberg correspondences

We consider the setup of Section 3.2.6. The union of walls for X and Y defines a partition of $\mathfrak{a}$ into chambers and we let $\mathfrak{C}$ be one of those. Let

$$L \subset X \times_V Y$$

be a $\mathsf{G_A}$-invariant Steinberg correspondence.

For any polarization of A-fixed loci, we denote by

$$\bar{\varepsilon} = (-1)^{\operatorname{codim}/2}\,\varepsilon$$

the opposite polarization. Assuming polarizations $\varepsilon_X, \varepsilon_Y$ of $X^{\mathsf{A}}, Y^{\mathsf{A}}$ have been fixed, we take

$$\varepsilon = \varepsilon_X\,\bar{\varepsilon}_Y$$

as a polarization of $X^{\mathsf{A}} \times Y^{\mathsf{A}} \subset X \times Y$. Using it, we define the residue

$$L_{\mathsf{A}} = \operatorname{Res}_{X^{\mathsf{A}} \times Y^{\mathsf{A}}} L \subset X^{\mathsf{A}} \times Y^{\mathsf{A}}$$

as a Lagrangian cycle class supported on L^{A}, see Section 3.4. As a fixed-point set of a Steinberg correspondence, L^{A} is Steinberg and hence so is L_{A}.

Theorem 4.6.1. *The diagram*

$$\begin{array}{ccc} H^{\cdot}_{\mathsf{G_A}}(Y^{\mathsf{A}}) & \xrightarrow{\operatorname{Stab}_{\mathfrak{C}}} & H^{\cdot}_{\mathsf{G_A}}(Y) \\ {\scriptstyle \Theta_{L_{\mathsf{A}}}}\big\downarrow & & \big\downarrow{\scriptstyle \Theta_L} \\ H^{\cdot}_{\mathsf{G_A}}(X^{\mathsf{A}}) & \xrightarrow{\operatorname{Stab}_{\mathfrak{C}}} & H^{\cdot}_{\mathsf{G_A}}(X) \end{array} \tag{4.14}$$

is commutative for every $\mathfrak{C}$. In particular, the Steinberg correspondence $\Theta_{L_{\mathsf{A}}}$ intertwines the R-matrices of X and Y.

For solutions of the Yang-Baxter equation, an important invariant is their algebra of symmetries, that is, the commutant of $R(u)$ for all u. Theorem shows it contains the Steinberg algebra of X for our geometrically constructed R-matrices.

Proof. We fix one chamber $\mathfrak{C}$ and define

$$L' = \mathrm{Stab}^{\tau}_{-\mathfrak{C},\varepsilon_X} \circ \Theta_L \circ \mathrm{Stab}_{\mathfrak{C},\varepsilon_Y} \subset X^{\mathsf{A}} \times Y^{\mathsf{A}} . \tag{4.15}$$

By Theorem 4.4.1, this makes the diagram (4.14) commute for one particular chamber $\mathfrak{C}$, after tensoring with $\mathbb{Q}(\mathfrak{g}_{\mathsf{A}})$.

We claim the pushforward along $X \times Y$ used in the definition of L' is proper. This is shown as in the proof of Theorem 4.4.1. Namely, we may assume V is a linear representation of A. Let

$$(x_0, x, y, y_0) \in X^{\mathsf{A}} \times X \times Y \times Y^{\mathsf{A}}$$

be such that

$$(x, x_0) \in \mathcal{L}^X_{-\mathfrak{C}}, \quad (x, y) \in L , \quad (y, y_0) \in \mathcal{L}^Y_{\mathfrak{C}} .$$

It then follows that x_0, x, y, y_0 map to the same point of $V_0 = V^{\mathsf{A}}$, implying the properness.

Hence L' is well-defined as a nonlocalized cycle class. It is G_{A}-invariant and Lagrangian, being a composition of such classes. It may be computed by equivariant localization with an arbitrary choice of equivariant parameters.

In particular, we may chose the equivariant parameters to be at infinity of $\mathfrak{a}$. Taking into account the signs in adjoints, we have

$$[\mathcal{L}^Y] = \varepsilon_Y\, [\Delta_{Y^{\mathsf{A}}}] + \dots , \quad [\mathcal{L}^X] = \bar{\varepsilon}_X\, [\Delta_{X^{\mathsf{A}}}] + \dots ,$$

where dots stand for terms of smaller A-degree. Therefore, at infinity of $\mathfrak{a}$, only these diagonal terms contribute and thus L' is supported on L^{A}. By our construction,

$$[L]\big|_{[L^{\mathsf{A}}]} = \varepsilon_X\, \bar{\varepsilon}_Y\, \mathrm{Res}_{L^{\mathsf{A}}} L + \dots .$$

We see that polarizations exactly cancel the denominators in localization formula, thus

$$L' = L_{\mathsf{A}} .$$

Since the original choice of $\mathfrak{C}$ was arbitrary, the theorem follows. □

4.7 Vacuum matrix elements

4.7.1

Let $Z \in \mathsf{Fix}$ be minimal with respect to the partial order defined by a chamber $\mathfrak{C}$.

Theorem 4.7.1. *If $Z \in \mathsf{Fix}$ is minimal as above then*

$$(R_{-\mathfrak{C},\mathfrak{C}} \cdot \gamma_1, \gamma_2) = \int_Z \gamma_1 \cup \gamma_2 \cup \frac{e(N_+ \otimes \hbar)}{e(N_+)},$$

where $N_\pm$ are the stable/unstable subbundles of the normal bundle to Z and $\gamma_i \in H^{\cdot}_{\mathsf{G}_\mathsf{A}}(Z)$.

In other words, the corresponding matrix elements of $R_{-\mathfrak{C},\mathfrak{C}}$ equal the operator of classical multiplication by the class

$$\frac{e(N_+ \otimes \hbar)}{e(N_+)} = \frac{e(N_-)}{e(N_- \otimes \hbar)} \in H^{\cdot}_{\mathsf{G}_\mathsf{A}}(Z)_{\text{localized}}$$

Proof. We use Theorem 4.4.1 and equivariant localization. By minimality of Z, the attracting set

$$\text{Attr}_{\mathfrak{C}}(\Delta_Z) \subset X \times Z$$

is closed and hence is the relevant component of $\mathcal{L}_{\mathfrak{C}}$. Further, $Z \times Z$ is the only component of $X^\mathsf{A} \times Z$ that this attracting set intersects. The localization contributions give

$$(-1)^{\text{codim}(Z)/2} \, \frac{e(N_-)^2}{e(N_Z)} = \frac{e(N_+ \otimes \hbar)}{e(N_+)}.$$

□

4.7.2

Here $e(N_\pm)$ are equivariant Euler classes, in the sense that they account for the nontrivial action of A on $e(N_\pm)$. Since A acts trivially on the base Z, we may expand $e(N_\pm)$ in the characteristic classes of the same bundles with trivial A-linearization.

For example, if $\mathsf{A} = \mathbb{C}^\times$ and it acts on N_+ by its defining representation then

$$\frac{e(N_-)}{e(N_- \otimes \hbar)} = 1 + \frac{\hbar}{u}\, \operatorname{rk} N_- + \\ + \frac{\hbar}{u^2}\left(c_1(N_-) + \frac{\hbar}{2} \operatorname{rk} N_-(\operatorname{rk} N_- + 1)\right) + O\left(\frac{1}{u^3}\right), \quad (4.16)$$

where $u \in \mathfrak{a}^*$ is the weight of the defining representation.

4.7.3

For example, consider the tensor product of Nakajima varieties as in Example 3.2.8 in Section 3.2.4. If $\theta > 0$ then the minimal component in (3.3) is

$$Z_\varnothing = \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) \hookrightarrow \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}+\mathsf{w}')\,, \quad (4.17)$$

which corresponds to

$$\eta = 0$$

in (3.3). By formula (2.15), we have

$$N_-\Big|_{Z_\varnothing} = \bigoplus \mathcal{V}_i^{\oplus \mathsf{w}_i}\,. \quad (4.18)$$

Recall that $\mathcal{M}_{\theta,\zeta}(0,\mathsf{w})$ is a point.

4.7.4

In particular, for moduli spaces of framed sheaves, this embedding takes the form

$$\mathcal{M}(r'') \ni \mathcal{F} \mapsto \mathcal{O}^r \oplus \mathcal{F} \in \mathcal{M}(r+r'')\,.$$

Its normal bundle is

$$N_- = \operatorname{Ext}^1_{\mathbb{P}^2}(\mathcal{O}^r, \mathcal{F}(-1)) = H^1_{\mathbb{P}^2}(\mathcal{F}(-1))^{\oplus r}\,.$$

The bundle

$$\operatorname{Taut} = \mathcal{V}_1 = H^1_{\mathbb{P}^2}(\mathcal{F}(-1))$$

is the tautological bundle on the moduli spaces of framed sheaves.

Theorem 4.7.1, combined with (4.6), gives an R-matrix formula for the operators of classical multiplication by characteristic classes of N_-. We will revisit this point below.

4.7.5

For general θ, the component (4.17) is not minimal. We therefore adopt the following terminology.

For all θ, we will call $Z_\varnothing$ the *vacuum* or the lowest weight component. We will call the minimal component the *true vacuum* component. For Nakajima varieties it coincides with $Z_\varnothing$ if $\theta > 0$.

When the vacuum $Z_\varnothing$ is not the true vacuum, the relation between the vacuum matrix elements of the R-matrix and the operators of classical multiplication becomes more complicated. It will be explored in Section 4.9.

4.8 Classical R-matrices

4.8.1

In this section, we assume that X is a symplectic resolution. Recall the root R-matrices and the subtori A_α introduced in Section 4.1.4. From Propositions 4.1.3 and 4.1.4, it follows that

$$R_\alpha = 1 + \frac{\hbar}{\alpha} r_\alpha + O(\alpha^{-2}), \tag{4.19}$$

for a certain operator

$$r_\alpha \in \operatorname{End}(H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A})).$$

Definition 4.8.1. The operator r_α is called the *classical* R-matrix.

Note that r_α does not depend on a choice of a splitting (3.8).

Proposition 4.8.2. *There is a Steinberg correspondence* $\mathbf{r}_\alpha \subset X^\mathsf{A} \times X^\mathsf{A}$ *that defines the operator* r_α.

Proof. Let

$$\mathrm{Stab}_{>0} : H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A}) \to H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\alpha),$$

the map corresponding to the chamber $\alpha > 0$. By Theorem 4.4.1,

$$R_\alpha = \mathrm{Stab}^\tau_{>0} \circ \mathrm{Stab}_{>0}\,.$$

We compute this push-forward by $(\mathsf{A}/\mathsf{A}_\alpha)$-equivariant localization. From Theorem 3.7.5, we can write

$$[\mathrm{Stab}_{>0}]\,|_{X^\mathsf{A}\times X^\mathsf{A}} = \gamma_{\text{diag}} + \hbar\,\gamma_{\text{off-diag}}\,.$$

Further, by Proposition 3.7.2,

$$\gamma_{\text{off-diag}}\Big|_{Z\times Z'} = \alpha^{\frac{1}{2}\operatorname{codim} Z-1}\,[C_{Z,Z'}] + \dots$$

for a certain Steinberg cycle $C_{Z,Z'} \subset Z \times Z'$. Here codimension is computed in X^α and dots stand for terms of smaller degree in α.

It follows that the quadratic in $\gamma_{\text{off-diag}}$ term doesn't contribute to r_α, while terms linear in $\gamma_{\text{off-diag}}$ contribute a Steinberg correspondence. Same is obviously true for the diagonal term. □

4.8.2

Note from the proof of Proposition 4.8.2

$$\mathbf{r}_\alpha = \Big(\sum_{k\in\mathbb{Q}_{>0}} \frac{\operatorname{rk} N^{[k\alpha]}}{k}\Big)\,\Delta + \text{off-diagonal}\,, \tag{4.20}$$

where $N^{[k\alpha]}$ is the A-weight space of the normal bundle to X^{A} with weight $k\alpha$. This is because the diagonal terms only occurs from the diagonal terms in the localization formula, that is, from the expansion of

$$(-1)^{\frac{1}{2}\operatorname{codim}} \frac{e(N_-^\alpha)^2}{e(N_-^\alpha)\,e(N_+^\alpha)} = \frac{e(N_+^\alpha\otimes\hbar)}{e(N_+^\alpha)}\,,$$

as in the proof of Theorem 4.7.1. Here the codimension and the normal bundles are taken in X^α.

4.8.3

In particular, for tensor product of Nakajima varieties the normal bundle to the fixed locus is identified in (2.15). From (4.20), we can then identify the diagonal part of the classical R-matrix

$$\mathbf{r}_{\text{diag}} = \sum \mathsf{w}_i\otimes\mathsf{v}_i + \sum \mathsf{v}_i\otimes\mathsf{w}_i - \sum \mathsf{C}_{ij}\,\mathsf{v}_i\otimes\mathsf{v}_j\,. \tag{4.21}$$

Here v_i denotes the operator of multiplication by $\mathsf{v}_i \in \mathbb{N}$ and so on.

4.8.4

The classical R-matrices satisfy a classical version of the braid relation. Concretely, the terms of degree -2 in a_1, a_2, a_3 in the expansion of (4.4) as $a_i - a_j \to \infty$ give

$$\begin{aligned} [\mathbf{r}_{12}, \mathbf{r}_{13} + \mathbf{r}_{23}] &= 0 \\ [\mathbf{r}_{23}, \mathbf{r}_{12} + \mathbf{r}_{13}] &= 0\,, \end{aligned} \tag{4.22}$$

which is equivalent to the equation

$$[\bar{\mathbf{r}}_{12}, \bar{\mathbf{r}}_{13}] + [\bar{\mathbf{r}}_{12}, \bar{\mathbf{r}}_{23}] + [\bar{\mathbf{r}}_{13}, \bar{\mathbf{r}}_{23}] = 0\,, \quad \bar{\mathbf{r}}_{ij} = \frac{\mathbf{r}_{ij}}{a_i - a_j}\,. \tag{4.23}$$

This is know as the *classical* Yang-Baxter equation with spectral parameter, see e.g. Section 6.3 in [33].

For brevity, we call $\mathbf{r}$ and not $\bar{\mathbf{r}}$, which contains the exact same information, the classical R-matrix. In the conventional terminology [33], $\bar{\mathbf{r}}$ is known as the classical R-matrix for the Yangian.

4.8.5

Our next goal is to show that the off-diagonal terms in (4.20) are additive over the coroot hyperplanes of the symplectic resolution X. This additivity is best stated in the following language.

Define a map

$$\boldsymbol{\mu} : \mathsf{Fix} \to \operatorname{Pic}(X)^* \otimes \mathfrak{a}^*$$

as follows. Fix an A-linearization for a basis $D_1, D_2, \dots$ of $\operatorname{Pic}(X)$ modulo torsion and let

$$\boldsymbol{\mu}(Z)(D) \in \mathfrak{a}^*$$

be the character of A-action in $D\big|_Z$. If D is ample, this is the moment map for the corresponding Fubini-Study symplectic $(1,1)$-form.

A different choice of the linearization changes $\boldsymbol{\mu}$ by a translation. In particular the difference

$$\boldsymbol{\mu}(Z) - \boldsymbol{\mu}(Z') \in \operatorname{Pic}(X)^* \otimes \mathfrak{a}^*$$

is defined uniquely. If C is an irreducible A-invariant curve joining Z and Z' then by localization

$$\boldsymbol{\mu}(Z) - \boldsymbol{\mu}(Z') = [C] \otimes \operatorname{weight}(T_p C)\,, \quad p = C \cap Z\,. \tag{4.24}$$

Here $[C] \in H_2(X, \mathbb{Z})$ defines an element of $\mathrm{Pic}(X)^*$ via the natural pairing

$$(C, D) = \deg D|_C \,.$$

4.8.6

Let $\beta \in H_2(X, \mathbb{Z})$ be an effective class such that $\beta^\perp$ is a coroot hyperplane of X and let X_β be the general fiber over the coroot hyperplane $\beta^\perp$ in (3.11).

For any root α, X_β has its own classical R-matrix $\mathbf{r}_\alpha(X_\beta)$. The closure of $\mathbf{r}_\alpha(X_\beta)$ defines a Steinberg correspondence $\mathbf{r}_{\alpha,\beta}$ in the fibers of

$$\begin{array}{ccc} X^{\mathsf{A}} \times X^{\mathsf{A}} & \hookrightarrow & (\widetilde{X}_\beta)^{\mathsf{A}} \times (\widetilde{X}_\beta)^{\mathsf{A}} \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & \beta^\perp \,. \end{array} \tag{4.25}$$

Here $\widetilde{X}_\beta$ is the restriction of the universal deformation $\widetilde{X}$ to the hyperplane $\beta^\perp$. We have the following

Theorem 4.8.3. *Let Z, Z' be two different components of X^{A}. If*

$$\boldsymbol{\mu}(Z) - \boldsymbol{\mu}(Z') \in \mathbb{Q}\,\beta \otimes \alpha$$

for some β such that $\beta^\perp$ is a coroot hyperplane then

$$\mathbf{r}_\alpha\big|_{Z' \times Z} = \mathbf{r}_{\alpha,\beta}\big|_{Z' \times Z} \,.$$

Otherwise, $\mathbf{r}_\alpha\big|_{Z' \times Z}$ is empty.

Proof. We first note that for $\mathbf{r}_\alpha$ to be nonempty, Z and Z' must lie in the same component of X^α. Therefore, there must exist a chain of A-invariant rational curves with tangent weights proportional to α that joins Z and Z'. From (4.24), we conclude

$$\boldsymbol{\mu}(Z) - \boldsymbol{\mu}(Z') = \gamma \otimes \alpha$$

for some $\gamma \in H_2(X, \mathbb{Q})$.

To simplify the notation, we will assume that

$$\dim B = 2 \,.$$

If $\dim B > 2$, we can pick a general 2-plane in the base B of the universal deformation and restrict $\widetilde{X}$ to it.

We denote by

$$\widetilde{X} \times (\widetilde{X})^{\mathsf{A}} \xleftarrow{\iota_3} \widetilde{Z}' \times \widetilde{Z} \xleftarrow{\iota_2} Z' \times Z$$

the inclusion of an A-fixed component and the fiber over the origin $0 \in B$, respectively. Recall that ϕ denotes the projection to B. We claim

$$\operatorname{supp} \iota_3^* \widetilde{\mathcal{L}} \subset \begin{cases} \phi^{-1}\left(\gamma^\perp\right), & \gamma^\perp \text{ is a coroot hyperplane}, \\ \phi^{-1}(0), & \text{otherwise}, \end{cases} \tag{4.26}$$

where $\widetilde{\mathcal{L}}$ is an in Section 3.7.

Indeed over a general point of a divisor $\beta^\perp \subset B$, β is the only effective cycle in $H_2(X)$. For the support to be nonempty, there must be a chain of curves of class β joining Z and Z', whence

$$\boldsymbol{\mu}(Z) - \boldsymbol{\mu}(Z') = \beta \otimes \delta$$

for some $\delta \in \mathfrak{a}^*$. This implies $\gamma \in \mathbb{Q}\beta$ and and $\delta \in \mathbb{Q}\alpha$.

We can factor the inclusion ι_2 as follows

$$\widetilde{Z}' \times \widetilde{Z} \xleftarrow{\iota_1} \widetilde{Z}'_\beta \times \widetilde{Z}_\beta \xleftarrow{\iota_0} Z' \times Z\,,$$

where $\widetilde{Z}_\beta$ denotes the restriction of $\widetilde{Z}$ to the divisor $\beta^\perp \subset B$. From (4.26), we conclude

$$\iota_3^* \widetilde{\mathcal{L}} = \sum f_i(a)\, \iota_{1*}[L_i] + \dots\,, \quad \deg_{\mathsf{A}} f_i(a) = \tfrac{1}{2} \operatorname{codim} Z' - 1\,, \tag{4.27}$$

where

$$L_i \subset \widetilde{Z}'_\beta \times \widetilde{Z}_\beta$$

are certain Steinberg correspondences and dots stand for classes that are either of smaller A-degree or in the image of ι_{2*}. Note that

$$\iota_2^* \circ \iota_{2*} = \text{multiplication by } \hbar^2,$$

and therefore the dots in (4.27) do not contribute to classical R-matrices. By contrast, the leading term in (4.27) is what goes into the correspondence $\mathbf{r}_{\alpha,\beta}$. This concludes the proof. □

4.9 Diagonal matrix elements of R-matrices

4.9.1

To simplify notation, we assume that $\mathsf{A} \cong \mathbb{C}^\times$ and that the cocharacter $\sigma \in \mathfrak{C}$ gives this isomorphism. Let $\lambda \in \operatorname{Pic}(X)$ be ample and we linearize it so that its weight is trivial on the vacuum components $Z_\varnothing$. We label all other components Z_k of X^{A} by a nonnegative integer k — the weight of λ.

By construction, our R-matrix comes with a a block Gauss decomposition of the form

$$R = \begin{pmatrix} U_{00} & & & \\ U_{10} & U_{11} & & \\ U_{20} & U_{21} & U_{22} & \\ & \ddots & \ddots & \ddots \end{pmatrix}^{-1} \begin{pmatrix} S_{00} & S_{01} & S_{02} & \\ & S_{11} & S_{12} & \\ & & S_{22} & \\ & & & \ddots \end{pmatrix}, \tag{4.28}$$

where the blocks are indexed as above and

$$S, U : H^{\cdot}_{\mathsf{G_A}}(X^{\mathsf{A}}) \to H^{\cdot}_{\mathsf{G_A}}(X^{\mathsf{A}})$$

is given by

$$S, U = \pm u^{-\operatorname{codim}/2} \operatorname{Res} \circ \operatorname{Stab}_{\pm\mathfrak{C}},$$

according to polarization, where

$$\operatorname{Res} : H^{\cdot}_{\mathsf{G_A}}(X) \to H^{\cdot}_{\mathsf{G_A}}(X^{\mathsf{A}})$$

is the restriction map. With this normalization

$$S_{ij} = \delta_{ij} + O(u^{-1}), \quad u \to \infty, \tag{4.29}$$

and similarly for U_{ij}.

4.9.2

Note that (4.28) implies

$$R_{00} = U_{00}^{-1} S_{00} \tag{4.30}$$

which is the content of Theorem 4.7.1. The proof of Theorem 4.7.1 shows

$$\begin{pmatrix} U_{00}^{-1} S_{00} & & & \\ & U_{11}^{-1} S_{11} & & \\ & & U_{22}^{-1} S_{22} & \\ & & & \ddots \end{pmatrix} = \frac{e(N_-)}{e(N_- \otimes \hbar)} \cup \tag{4.31}$$

as operator on $H^{\bullet}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A})$, where N_- is the unstable part of the normal bundle.

4.9.3

Similarly to (4.30), one computes, for example

$$R_{11} = U_{11}^{-1} S_{11} + R_{10}\, S_{00}^{-1}\, U_{00}\, R_{01}\,.$$

In general, the diagonal matrix elements R_{kk} may be computed as follows. Define

$$\widetilde{U}_{ij} = U_{ii}^{-1}\, U_{ij}$$

and equate the (k,i) matrix elements in

$$U\, R = S\,.$$

For $i = 0, \dots, k-1$, we get the following system of block matrix equations

$$\begin{pmatrix} \widetilde{U}_{k0} & \dots & \widetilde{U}_{k,k-1} \end{pmatrix} \square = -\begin{pmatrix} R_{k,0} & \dots & R_{k,k-1} \end{pmatrix}\,, \tag{4.32}$$

where

$$\square = \begin{pmatrix} R_{00} & \dots & R_{0,k-1} \\ \vdots & & \vdots \\ R_{k-1,0} & \dots & R_{k-1,k-1} \end{pmatrix}, \tag{4.33}$$

while for $i = k$, we obtain

$$R_{kk} = U_{kk}^{-1} S_{kk} - \begin{pmatrix} \widetilde{U}_{k0} & \dots & \widetilde{U}_{k,k-1} \end{pmatrix} \begin{pmatrix} R_{0,k} \\ \vdots \\ R_{k-1,k} \end{pmatrix}.$$

Since

$$\square = 1 + O(u^{-1})\,,$$

the square matrix (4.33) is invertible as a series in u^{-1}. This proves the following

Theorem 4.9.1. *We have*

$$R_{kk} = \left.\frac{e(N_-)}{e(N_- \otimes \hbar)}\right|_{Z_k} + \dots$$

where dots stand for a universal noncommutative expression in the coefficients of the $1/u$-expansion of R_{ij}, R_{ji}, $i < j \leqslant k$, and of the operators

$$\left(\frac{e(N_-)}{e(N_- \otimes \hbar)}\right)^{\pm 1}\Bigg|_{Z_i}, \quad i < k\,.$$

These corrections are found from

$$R_{kk} = U_{kk}^{-1} S_{kk} + \begin{pmatrix} R_{k,0} & \dots & R_{k,k-1}\end{pmatrix} \square^{-1} \begin{pmatrix} R_{0,k} \\ \vdots \\ R_{k-1,k}\end{pmatrix}. \tag{4.34}$$

4.9.4

In particular, Theorem 4.9.1 gives a way to relate operators of classical multiplication to vacuum matrix elements of R-matrices in the case then the vacuum is not the true vacuum in the sense of Section 4.7.5.

4.9.5

The relationship in Theorem 4.9.1 simplifies for operators of small cohomological degree because they appear in small coefficients of the $1/u$-expansion. For example, from

$$R_{ij} = O(u^{-1})\,, \quad i \neq j\,,$$

we conclude the following

Proposition 4.9.2.

$$U_{kk}^{-1} S_{kk} = R_{kk} - \sum_{i<k} R_{ki}\, R_{ik} + O(u^{-3})\,.$$

4.9.6

For Nakajima varieties Proposition 4.9.2 means the following. Recall the Example 3.2.8 in Section 3.2.4 and suppose $\theta \not> 0$. Then

$$Z_\eta < Z_\varnothing\,, \quad \theta \cdot \eta < 0\,.$$

Denote

$$H(\mathsf{w})_\eta = H^{\cdot}_{\mathsf{G_A}}(\mathcal{M}_{\theta,\zeta}(\eta, \mathsf{w}))\,.$$

Consider the matrix element $R_{\eta,0}$ of the R-matrix

$$R_{\eta,0} : H(\mathsf{w})_0 \otimes H(\mathsf{w}')_{\mathsf{v}} \longrightarrow H(\mathsf{w})_\eta \otimes H(\mathsf{w}')_{\mathsf{v}-\eta}$$

and the operator $R_{0,\eta}$ going in the opposite direction. Then Proposition 4.9.2 implies

$$\frac{e\left(N_-^{\varnothing}\right)}{e\left(N_-^{\varnothing} \otimes \hbar\right)} = R_{00} - \sum_{\theta\cdot\eta<0} R_{0,\eta} R_{\eta,0} + O(u^{-3}) \tag{4.35}$$

where

$$N_-^{\varnothing} = \bigoplus \mathcal{V}_i^{\oplus \mathsf{w}_i}\,.$$

is the unstable normal bundle to $Z_\varnothing$, as in (4.18).

Observe that in (4.35) the sum is effectively over $\eta \leqslant \mathsf{v}$ simply because $H(\mathsf{w}')_{\mathsf{v}-\eta} = 0$ if $\eta \nleqslant \mathsf{v}$. It is convenient that we don't have to restrict the range of summation explicitly.

4.10 Flops and stable envelopes

4.10.1

Let X be a symplectic resolution and let

$$\begin{array}{ccc} X & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & B\,, \end{array} \tag{4.36}$$

be its deformation. For our present goals, it suffices to take B a generic line in the base of (3.11) in Section 3.7. By definition, a flop of X is another family over the same base B

$$\begin{array}{ccc} X_{\mathrm{flop}} & \hookrightarrow & \widetilde{X}_{\mathrm{flop}} \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & B\,, \end{array}$$

together with an isomorphism

$$\begin{array}{ccc} \widetilde{X}\backslash X & \xrightarrow{\widetilde{F}} & \widetilde{X}_{\mathrm{flop}}\backslash X_{\mathrm{flop}} \\ \downarrow & & \downarrow \\ B\backslash\{0\} & \xrightarrow{\mathrm{id}} & B\backslash\{0\} \end{array}$$

of families over the punctured base. We require $\widetilde{F}$ to:

1) be equivariant with respect to all group actions,

2) preserve the symplectic form,

3) induce identity on the affine quotients.

For symplectic resolutions, 3) implies 2) because it implies the graph of $\widetilde{F}$ is Lagrangian in the product of fibers.

An example is provided by the natural isomorphism

$$\mathcal{M}_{\theta,t\zeta}(\mathsf{v},\mathsf{w}) \cong \mathcal{M}_{\theta',t\zeta}(\mathsf{v},\mathsf{w})$$

where θ, θ' are arbitrary, $t \in B\backslash\{0\} = \mathbb{C}^\times$, and ζ is generic.

4.10.2

The closure of the graph of $\widetilde{F}$ defines a cycle in $\widetilde{X} \times_B \widetilde{X}_{\mathrm{flop}}$, the restriction of which to the origin defines a G-invariant Steinberg correspondence

$$F \subset X_{\mathrm{flop}} \times X \,.$$

For brevity, we denote the induced map

$$F : H^{\boldsymbol{\cdot}}_{\mathsf{G}}(X) \xrightarrow{\sim} H^{\boldsymbol{\cdot}}_{\mathsf{G}}(X_{\mathrm{flop}})$$

by the same letter. This is an isomorphism because both families are topologically trivial.

4.10.3

For example, if Q is the quiver with one vertex and no edges, $(\mathsf{v}, \mathsf{w}) = (1, n)$ then this is the classical Mukai flop of

$$\mathcal{M}_{\theta,0}(\mathsf{v}, \mathsf{w}) = \begin{cases} T^*\mathbb{P}(W^\vee)\,, & \theta > 0\,, \\ T^*\mathbb{P}(W)\,, & \theta < 0\,, \end{cases}$$

where $W \cong \mathbb{C}^n$ is the framing space and $\mathbb{P}(W)$ is the projective space of lines through the origin in W. In this case

$$F = \mathbb{P}(W^\vee) \times \mathbb{P}(W) + T^\perp \text{Universal hyperplane}\,, \tag{4.37}$$

where $T^\perp$ denotes the conormal bundle and $\mathbb{P}(W) \subset T^*\mathbb{P}(W)$ is the zero section. Note this cycle is $GL(W) \times \mathbb{C}^\times$-invariant.

4.10.4

Let $\mathsf{A} \subset \mathsf{G}$ be a torus preserving the symplectic form. Any such torus acts trivially on the base B. Since a flop is an A-equivariant isomorphism over $B \backslash \{0\}$, we have a natural bijection

$$f : \mathsf{Fix}(X) \xrightarrow{\sim} \mathsf{Fix}(X_{\text{flop}})$$

of components of A-fixed loci. By taking fixed points, F induces a certain flop (potentially trivial)

$$F_i \subset Z_{\text{flop}, f(i)} \times Z_i$$

of each component of $Z_i \subset X^{\mathsf{A}}$.

4.10.5

Since flop is a Steinberg correspondence, Theorem 4.6.1 implies the following square commutes for any chamber $\mathfrak{C}$

$$\begin{array}{ccc} H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^{\mathsf{A}}) & \xrightarrow{\text{Stab}_{\mathfrak{C}}} & H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X) \\ \Big\downarrow F_{\mathsf{A}} & & \Big\downarrow F \\ H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^{\mathsf{A}}_{\text{flop}}) & \xrightarrow{\text{Stab}_{\mathfrak{C},\text{flop}}} & H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X_{\text{flop}})\,. \end{array} \tag{4.38}$$

Here the cycle F_{A} is residue of F, it is a Steinberg cycle supported on F^{A} with signs determined by the polarizations of the fixed loci.

Lemma 4.10.1. *The correspondence F_{A} is the flop of X^{A}, up to signs determined by polarization.*

Proof. By construction

$$F\Big|_{Z_{\mathrm{flop},j}\times Z_i} = 0\,, \quad j \neq f(i)\,,$$

in A-equivariant cohomology. Therefore F_{A} vanishes outside the graph of f. On the graph of f, the statement holds by definition. □

4.10.6

In the example of the Mukai flop, consider the Lagrangian subvarieties

$$\sigma_U = T^{\perp}\mathbb{P}(U) \subset T^*\mathbb{P}(W)$$

corresponding to linear subspaces $U \subset W$. In particular, σ_W is the zero section while $\sigma_0 = \varnothing$. From (4.37), one computes

$$F(\sigma_U) = \sigma_{U^\perp} - (-1)^{\dim U}\sigma_{W^\vee}\,. \tag{4.39}$$

The coefficient of $\sigma_{W^\vee}$ is the sum of

$$\sigma_W \cdot \sigma_U = (-1)^{\dim \mathbb{P}(U)}\chi(\mathbb{P}(U)) = (-1)^{\dim U-1}\dim U$$

and the analogous number for a hyperplane section of U.

Let $\mathsf{A} \subset GL(W)$ be a maximal torus with eigenbasis $e_1, \ldots, e_n \in W$ and the corresponding fixed points $x_i = \mathbb{P}(\mathbb{C}e_i) \in \mathbb{P}(W)$. We have

$$\mathrm{Stab}(x_i) = \sigma_{U_i} + \sigma_{U_{i+1}}\,, \quad U_i = \mathrm{Span}(e_i, \ldots, e_n)$$

for some choice of chamber and polarization. We see that

$$F(\mathrm{Stab}(x_i)) = \sigma_{U_i^\perp} + \sigma_{U_{i+1}^\perp}\,,$$

where $U_{n+1}^\perp = W^\vee$. This is the stable basis for A action on $T^*\mathbb{P}(W^\vee)$ for the same chamber and suitable polarization.

The induced bijection of fixed loci is

$$f(x_i) = \mathbb{P}(\mathbb{C}\xi_{n-i+1})$$

where $\{\xi_1, \ldots, x_n\}$ is the dual basis of $W^\vee$.

4.10.7

Different cones in the space of the stability condition θ give different flops of a given Nakajima variety. Among them is the maximal flop

$$F_{\max} \subset \mathcal{M}_{-\theta,\zeta}(\mathsf{v},\mathsf{w}) \times \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$$

that corresponds to the opposite cone of stability conditions. For an arbitrary symplectic resolution X, one similarly expects to have a flop $F_{\max}$ that takes the ample cone of X to its opposite.

We next observe that for any chamber $\mathfrak{C}$, the map

$$\operatorname{Stab}_{\mathfrak{C}} : H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X^\mathsf{A}) \to H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X)$$

is characterized by its behavior near the diagonal and the opposite triangularity of the supports of $\operatorname{Stab}_{\mathfrak{C}}$ and $F_{\max} \operatorname{Stab}_{\mathfrak{C}}$.

Theorem 4.10.2. *The map* $\operatorname{Stab}_{\mathfrak{C}}$ *is uniquely determined by the the conditions* (i)*,* (ii) *in Theorem 3.3.4 together with a symmetric condition for its maximal flop*

$$\operatorname{supp} F_{\max} \circ \operatorname{Stab}_{\mathfrak{C}}(Z_i) \subset \operatorname{Attr}^f_{\mathfrak{C}}(Z_{\mathrm{flop},f(i)})$$

Proof. The above support condition is satisfied by (4.38) and Lemma 4.10.1. Since a maximal flop takes an ample class to minus an ample class while preserving A-weights, we have

$$i > j \quad \Leftrightarrow \quad f(i) < f(j)$$

in the ample partial ordering, for any $\mathfrak{C}$. Thus the supports of $\operatorname{Stab}_{\mathfrak{C}}$ and $F_{\max} \operatorname{Stab}_{\mathfrak{C}}$ are triangular the opposite way. Hence

$$F_{\max}\Big|_{X^\mathsf{A}_{\mathrm{flop}} \times X^\mathsf{A}} = \operatorname{Stab}_{\mathfrak{C},\mathrm{flop}}\Big|_{X^\mathsf{A}_{\mathrm{flop}} \times X^\mathsf{A}_{\mathrm{flop}}} \circ F_{\max,\mathsf{A}} \circ \left(\operatorname{Stab}_{\mathfrak{C}}\Big|_{X^\mathsf{A} \times X^\mathsf{A}}\right)^{-1} \tag{4.40}$$

is a Gauss factorization, and therefore unique. □

4.10.8

We see from (4.40) that flops give a way to package the information about stable envelopes that is somewhat different from R-matrices. This packaging has several convenient features, among them:

- flops are given by Steinberg correspondences, a very economical and geometric data,
- the maximal flop $F_{\max}$ can be factored into a product of flops that cross a single wall in the space of θ's,
- additional constraints on $F_{\max}$ may be deduced from a noncanonical isomorphism
$$\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) \to \mathcal{M}_{-\theta,\zeta}(\mathsf{v},\mathsf{w})$$
that replaces all quiver data by transposed with respect to some chosen bilinear form.

Chapter 5

Yangians

5.1 Tensor products

5.1.1

Let X satisfy the hypotheses of Section 3.1. By definition, we say that X is a tensor product and write

$$X = X_1 \otimes \cdots \otimes X_n$$

if the maximal torus $\mathsf{A} \subset PGL(n)$ acts on X preserving the symplectic form so that

(1) $X^{\mathsf{A}} = X_1 \times \cdots \times X_n$,

(2) the roots of X are the roots α_{ij} of $PGL(n)$,

(3) the corresponding fixed loci are of the form

$$X^{\alpha_{ij}} = X_{ij} \times \prod_{k \neq i,j} X_k$$

We view this definition as provisional; perhaps a better set of axioms will emerge later. Note that neither existence or uniqueness of tensor products is claimed.

If one requires X to have a unique, up to multiple, holomorphic symplectic form, then this rules out trivial nonuniqueness of the form

$$X \mapsto X \times \text{vector representation of } \mathsf{A}\,.$$

5.1.2

In the case of quiver varieties, recall $\mathcal{M}(\mathsf{w})$ from Section 4.1.5. For any decomposition

$$\mathsf{w} = \sum_{i=1}^{n} \mathsf{w}_i$$

into nonzero terms, we have

$$\mathcal{M}(\mathsf{w}) = \bigotimes \mathcal{M}(\mathsf{w}_i)\,,$$

corresponding to the decomposition

$$\mathsf{w} = \sum z_i \, \mathsf{w}_i$$

as in Section 2.4. Here

$$(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n = \mathsf{A}\,.$$

5.1.3

For $X = X_1 \otimes \cdots \otimes X_n$, the construction of Chapters 3 and 4 gives a set of R-matrices

$$R_{ij}(a_i - a_j) \in \operatorname{End}(F_1 \otimes \cdots \otimes F_n) \otimes \mathbb{Q}(\mathsf{t})\,, \quad F_i = H^{\cdot}_{\mathsf{G}_\mathsf{A}}(X_i)$$

satisfying the Yang-Baxter equation (4.4), a familiar setup in quantum integrable systems.

5.1.4

Given an operator

$$R_{12}(a_1 - a_2) \in \operatorname{End}(F_1 \otimes F_2)\,,$$

its matrix elements in F_1 are operators on F_2. Our main interest is the algebra of operators thus obtained for Nakajima varieties. This algebra is an example of a Yangian.

5.2 Construction of Yangians

5.2.1

Yangians are Hopf algebras associated to rational solutions of the Yang-Baxter equation. There are several ways to describe a Yangian. For us, it is the so-called RTT=TTR formalism of [35] that arises naturally. We briefly recall the basics.

For simplicity, we limit the use of the categorical language, even though many construction and properties are best stated in the language of tensor categories, see for example [115].

5.2.2

Let $\Bbbk \supset \mathbb{Q}$ be a commutative ring without zerodivisors. We write

$$\otimes = \otimes_{\Bbbk}\,, \quad \operatorname{End} = \operatorname{End}_{\Bbbk}$$

for brevity. Let $\{F_i\}$ be a collection of free $\Bbbk$-modules and let

$$R_{F_i,F_j}(u) \in \operatorname{End}(F_i \otimes F_j)(u)$$

be collection of operator-valued rational functions of u satisfying the Yang-Baxter equations (1.4). We assume the normalization

$$R(\infty) = 1\,.$$

We also fix $\hbar \in \Bbbk$ that divides $R(u) - 1$. In geometric applications, this will be the weight of the symplectic form.

5.2.3

To this data, one associates a Hopf algebra Y over $\Bbbk$ that acts on

$$F_i(u) \stackrel{\text{def}}{=} F_i \otimes \Bbbk[u]\,. \tag{5.1}$$

and more generally on

$$F_{i_1}(u_1) \otimes \cdots \otimes F_{i_n}(u_n) = F_{i_1} \otimes \cdots \otimes F_{i_n} \otimes \Bbbk[u_1, \ldots, u_n] \tag{5.2}$$

This action commutes with multiplication by the u_i's, so may be viewed as a family of Y-modules indexed by $\mathbb{A}^n_{\Bbbk}$.

5.2.4

While $F_i[u]$ is a more logical notation for (5.1), the use of parentheses is traditional. The variable u in (5.1) is called the evaluation parameter, in reference to the following.

By one of their many definitions, Yangians are Hopf algebra deformations of $\mathcal{U}(\mathfrak{g}[u])$, where $\mathfrak{g}$ is a Lie algebra over $\Bbbk$ and $\mathfrak{g}[u]$ is the Lie algebra of $\mathfrak{g}$-valued polynomials in u. The identity map

$$\mathfrak{g}[u] \to \mathfrak{g} \otimes \Bbbk[u]$$

may be viewed as family of evaluation homomorphisms $\mathfrak{g}[u] \to \mathfrak{g}$ and any $\mathfrak{g}$-module F can be made a $\mathfrak{g}[u]$-module $F(u)$ by pull-back.

5.2.5

A certain care is required if $\operatorname{rk} F_i = \infty$ for some F_i. We will always assume a grading

$$F_i = \bigoplus_{\alpha \in \mathbb{Z}^n} \big(F_i\big)_\alpha$$

such that all graded pieces are $\Bbbk$-modules of finite rank. We further require that $\big(F_i\big)_\alpha \neq 0$ only for α in a translate of a certain nontrivial cone, which we will assume to be $(\mathbb{Z}_{\geqslant 0})^n$ for simplicity.

The R-matrices will always have grading 0. This makes Y a graded algebra and $F_i(u)$, with the grading induced from F_i, a graded module. The coproduct

$$\Delta : \mathsf{Y} \to \mathsf{Y} \widehat{\otimes} \mathsf{Y} \tag{5.3}$$

to be defined below, takes values in the following completed tensor product. By definition,

$$\mathsf{Y} \widehat{\otimes} \mathsf{Y} = \bigoplus_{\alpha} \big(\mathsf{Y} \widehat{\otimes} \mathsf{Y}\big)_\alpha$$

while

$$\sum_\beta y_{\alpha-\beta} \otimes y_\beta \in \big(\mathsf{Y} \widehat{\otimes} \mathsf{Y}\big)_\alpha$$

if β ranges in a translate of $(\mathbb{Z}_{\geqslant 0})^n$. Such infinite sums act naturally on any $F_i(u_1) \otimes F_j(u_2)$. The iterates of Δ make (5.2) tensor products of (5.1) as Y-modules.

5.2.6 Definition

We define Y as the subalgebra

$$\mathsf{Y} \subset \prod_{i_1,\dots,i_n} \mathrm{End}_{\Bbbk[u_1,\dots,u_n]} \left(F_{i_1}(u_1) \otimes \cdots \otimes F_{i_n}(u_n)\right) \tag{5.4}$$

generated by the following operators. Let

$$W = F_1(u_1) \otimes \cdots \otimes F_n(u_n) \tag{5.5}$$

be one of the spaces in (5.4) where, for brevity, we write F_k in place of F_{i_k} to denote some element of the set $\{F_i\}$. Choose an additional $F_0 \in \{F_i\}$ called an *auxiliary* space and define

$$R_{F_0(u),W} = R_{F_0,F_n}(u-u_n) \cdots R_{F_0,F_1}(u-u_1)\,. \tag{5.6}$$

Let

$$m(u) \in F_0 \otimes F_0^\vee \otimes \Bbbk[u]$$

be a polynomial in u with values in operators in F_0 of finite rank. Here

$$F_0^\vee = \mathrm{Hom}_{\Bbbk}(F_0, \Bbbk)$$

is the graded dual module.

Because $m(u)$ has finite rank and $\hbar$ divides $R-1$, the following operator

$$\mathsf{E}(m(u)) = -\frac{1}{\hbar}\,\mathrm{Res}_{u=\infty}\ \mathrm{tr}_{F_0}\, m(u)\, R_{F_0(u),W} \in \mathrm{End}(W) \tag{5.7}$$

is well-defined for all W in (5.5). Since it comes from an expansion of rational functions of $u-u_i$ as $u \to \infty$, it depends polynomially on $u_1,\dots,u_n$. Thus, it defines an element of the right-hand side in (5.4).

By definition, Y is the $\Bbbk$-subalgebra in (5.4) generated by 1 and (5.7) for all F_0 and all $m(u)$. In English, the Yangian Y is the algebra generated by

all coefficients of the $u \to \infty$ expansion of
all matrix coefficients of the operators (5.6) for
all auxilliary spaces F_0.

Additionally, since all nontrivial matrix elements are divisible by $\hbar$, we divide by $\hbar$ in (5.7).

5.2.7

The product in (5.4) includes the the factor $W = \Bbbk$ corresponding to

$$\{i_1, i_2, \ldots, i_n\} = \varnothing\,.$$

This 1-dimensional Y-module is the counit of the Yangian.

5.2.8

After inverting $\hbar$, (5.7) makes sense for any rational function $m(u)$ of u, in particular,

$$\mathsf{E}\left(m\,u^{-k}\right) = \begin{cases} \hbar^{-1}\ \operatorname{tr} m\,, & k = 1\,, \\ 0\,, & k > 1\,. \end{cases} \tag{5.8}$$

While such operators are not in Y, they will play a role in computation of commutation relations (5.12) below.

5.2.9 RTT=TTR equation

By construction, (5.7) extends to a surjection

$$\mathsf{E} : \text{Tensor algebra}\left(\bigoplus F_i \otimes F_i^\vee \otimes \Bbbk[u]\right) \twoheadrightarrow \mathsf{Y} \tag{5.9}$$

The Yang-Baxter equation shows it factors through the quotient by

$$\begin{aligned}&(m_1(u_1) \otimes m_2(u_2)) \cdot R_{F_1 F_2}(u_1 - u_2) - \\ &\quad R_{F_1 F_2}(u_1 - u_2) \cdot (m_2(u_2) \otimes m_1(u_1))\,, \quad m_i(u) \in F_i \otimes F_i^\vee \otimes \Bbbk[u]\,.\end{aligned} \tag{5.10}$$

This is known as the RTT=TTR relation. The letter T being overused in this paper, we substitute it in this context by E.

The quotient of the tensor algebra by (5.10) is of the same size as the symmetric algebra. This is still very big and below we will discuss how to write further relations in Yangians.

5.2.10 Filtration in the Yangian

The Yangian Y is filtered by degree in u, that is, by defining

$$\deg \mathsf{E}(m(u)) = \deg_u m(u)$$

on the generators of the Yangian. We set $\deg_u 1 = 0$.

Equation (5.8) shows this filtration does not extend to the algebra generated by these more general operators. Therefore, one has to be careful in situations where they appear.

Since scalars cancel out of the RTT=TTR equation, it takes the form

$$[\mathsf{E}(m(u)), \mathsf{E}(m'(v))] = \hbar\, \mathsf{E}\left(\left[\frac{\mathbf{r}_{VV}}{u-v}, m(u) \otimes m'(v)\right]\right) + \dots \tag{5.11}$$

where $\mathbf{r}$ is the classical R-matrix

$$R(u) = 1 + \frac{\hbar}{u}\,\mathbf{r} + O(u^{-2})$$

and dots in (5.11) come from the $O(u^{-2})$ term above.

Note that in the right-hand side of (5.11) there are terms of the same degree as in the left-hand side. They come from the expansion

$$\frac{1}{u-v} = \frac{1}{u} + \frac{v}{u^2} + \frac{v^2}{u^3} + \dots$$

and (5.8), giving the right-hand side of the following formula (5.12).

Proposition 5.2.1. *We have*

$$\left[\mathsf{E}(m\, u^i), \mathsf{E}(m'\, u^j)\right] = \mathsf{E}\left((\mathrm{tr} \otimes 1)\, [\mathbf{r}_{VV}, m \otimes m']\, u^{i+j}\right) + \dots \tag{5.12}$$

where dots stand for terms of smaller degree in u.

Proof. Were it not for (5.8), the right-hand side of (5.11) would have smaller total degree in u and v than $\deg_u m(u) + \deg_v m'(v)$.

Each application of (5.8) brings the total degree up by 1. Note, however, that it can be applied only once and with respect to the variable u, because all terms in (5.11) have nonnegative degree in v. Therefore, the dots in (5.11) have total degree at most $\deg_u m(u) + \deg_v m'(v) - 1$ and can be neglected. □

5.2.11

Note the commutation relation (5.12) has the form

$$\left[a\, u^i, b\, u^j\right] = [a, b]\, u^{i+j}\,, \quad a, b \in \mathfrak{g}\,,$$

of the commutation relations in the Lie algebra of polynomials $\mathfrak{g}[u]$ with values in a Lie algebra $\mathfrak{g}$.

In fact, one of our goals is to show that for the Yangian associated to a quiver Q

$$\operatorname{gr} \mathsf{Y} \cong \mathcal{U}(\mathfrak{g}_Q[u])$$

for a certain Lie algebra $\mathfrak{g}_Q$. Here $\operatorname{gr} \mathsf{Y}$ denotes the associated graded of Y for the filtration by degree in u.

5.2.12 Coproduct

The set of W of the form (5.5) is closed with respect to tensor product. There is a corresponding projection

$$\prod_W \operatorname{End} W \to \prod_{W,W'} \operatorname{End}(W \otimes W') \,.$$

By applying this projection to $\mathsf{E}(m(u))$, it is easy to see that it sends Y to the image of the map

$$\mathsf{Y} \widehat{\otimes} \mathsf{Y} \to \prod_{W,W'} \operatorname{End}(W \otimes W') \,. \tag{5.13}$$

The completion is needed because matrix elements of $R_{F_0,F_1\otimes F_2}$ are infinite sums of products of matrix elements of R_{F_0,F_i} when $\dim F_0 = \infty$.

This defines a natural coproduct (5.3) on Y up to an ambiguity arising from the kernel of (5.13). We will prove at the end of this chapter that Y is flat over $\Bbbk$ and that, as a corollary, the map (5.13) is injective so this ambiguity does not arise. In the meantime, we only discuss the coproduct as evaluated on pairs of representations.

The coproduct is not commutative and in general

$$F_1(u_1) \otimes F_2(u_2) \ncong F_2(u_2) \otimes F_1(u_1)$$

as Y-modules. However,

$$F_1(u_1) \otimes F_2(u_2) \otimes_{\Bbbk[u_1,u_2]} \Bbbk(u_1,u_2) \cong F_2(u_2) \otimes F_1(u_1) \otimes_{\Bbbk[u_1,u_2]} \Bbbk(u_1,u_2)$$

with the explicit intertwiner

$$R^\vee = (12)\, R_{F_1,F_2}(u_1 - u_2) \,.$$

This follows at once from the Yang-Baxter equation.

5.2.13 Translation automorphism

All spaces W in (5.5) have an automorphism ς_c that acts by

$$\varsigma_c(u_i) = u_i + c\,, \quad i = 1, 2, \dots\,,$$

on the variables u_i and as identity on F_i's. It preserves Y because it amounts to a reexpansion of $R(u-c)$ in inverse powers of u. We denote the corresponding automorphism of the Yangian also by ς_c.

5.2.14

In the rest of this chapter, we specialize to the case of Nakajima varieties, see Section 5.1.2. We fix a quiver Q and set

$$\begin{aligned} \Bbbk &= H^{\cdot}_{G_{\text{edge}}}(\mathsf{pt}, \mathbb{Q})\,, \\ F_i &= H^{\cdot}_{G_{\text{edge}}}(\mathcal{M}(\delta_i), \mathbb{Q})\,. \end{aligned} \tag{5.14}$$

Here $\mathsf{w} = \delta_i$ is the delta-function at some $i \in \overline{I}$. Note that in this case $G'_{\mathsf{w}} = 1$. The tensor product construction will identify

$$H^{\cdot}_{G_{\mathsf{A}}}(\mathcal{M}(\mathsf{w})^{\mathsf{A}}) = \bigotimes_{i\in I} F_i(u_{i1}) \otimes \cdots \otimes F_i(u_{i\mathsf{w}_i})$$

where $\mathsf{A} \subset G_{\mathsf{w}}$ is a maximal torus and

$$\begin{pmatrix} u_{i1} & & & \\ & u_{i2} & & \\ & & \ddots & \\ & & & u_{i\mathsf{w}_i} \end{pmatrix} \in \mathfrak{gl}(W_i)\,, \quad i \in I\,,$$

are the equivariant parameters for the group G_{w}.

The collection (5.14) can be enlarged by allowing arbitrary dimension vectors w in place of δ_i. This does not change the Yangian Y because, as we will see, Y already injects into the endomorphisms of tensor products of $F_i(u_{ik})$.

5.3 The Lie algebra $\mathfrak{g}_Q$

5.3.1

Let $\mathfrak{g}_Q \subset \mathsf{Y}$ be the span of operator $\mathsf{E}(m_0)$, where m_0 is constant polynomial in u. In other words, $\mathfrak{g}_Q$ is spanned by the matrix elements of the classical R-matrix $\mathbf{r}$. Formula (5.12) shows $\mathfrak{g}_Q$ is a Lie algebra. The following is clear

Proposition 5.3.1. *All elements of $\xi \in \mathfrak{g}_Q$ are primitive, that is,*

$$\Delta\xi = \xi \otimes 1 + 1 \otimes \xi \,,$$

when evaluated on pairs of representations. In particular,

$$\begin{aligned} \left[\Delta\xi, R\right] &= 0 \,, \\ \left[\Delta\xi, \mathbf{r}\right] &= 0 \,, \end{aligned} \tag{5.15}$$

that is, $\mathfrak{g}_Q$ commutes with R-matrices.

We expect that $\mathfrak{g}_Q$ is the Lie algebra of primitive elements of Y.

5.3.2

As defined, $\mathfrak{g}_Q$ is a Lie algebra over $\mathbb{k}$. We expect a natural isomorphism

$$\mathfrak{g}_Q = (\mathfrak{g}_Q)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{k}$$

for a certain Lie algebra over $\mathbb{Q}$. We think the required $\mathbb{Q}$-structure may be constructed using the Decomposition Theorem.

5.3.3

The identity

$$R(u)^{-1} = R(-u)_{12}$$

from Section 4.5 implies the symmetry of $\mathbf{r}$, that is,

$$\mathbf{r}_{W,W'} = \mathbf{r}_{W',W} \,.$$

after identifying $W \otimes W'$ and $W' \otimes W$.

5.3.4

It follows from formula (4.21) that

$$\overline{\mathfrak{h}}_Q \subset \mathfrak{g}_Q$$

where $\overline{\mathfrak{h}}_Q$ acts by linear functions of v and w. Linear functions can be taken with $\Bbbk$-coefficients or $\mathbb{Q}$-coefficients, and this defines $\overline{\mathfrak{h}}_Q$ as $\Bbbk$-submodule with a canonical $\mathbb{Q}$-submodule. All structures in $\overline{\mathfrak{h}}_Q$ are defined over $\mathbb{Q}$.

Recall the quadratic forms (2.5) with values in $K_{\mathsf{G}}(\mathsf{pt})$. Here we evaluate them at $1 \in \mathsf{G}$, in other words, we use the nonequivariant Cartan matrix. The inverse of the nondegenerate form $(\,\cdot\,,\,\cdot\,)_{\overline{Q}}$ from (2.5) defines a bilinear form $(\,\cdot\,,\,\cdot\,)_{\overline{\mathfrak{h}}}$ on $\overline{\mathfrak{h}}_Q$. From (4.21) we conclude

$$\mathbf{r} = \sum_{i \in I \sqcup \overline{I}} h_i \otimes h^i + \dots$$

where

$$(h_i, h^j)_{\overline{\mathfrak{h}}} = \delta_{ij}$$

and dots stand for off-diagonal elements. Note that, with our conventions,

$$\dim \overline{\mathfrak{h}}_Q = 2|I|\,.$$

While this looks unusual from the perpective of finite-dimensional Lie theory (in which Cartan matrices are nondegenerate), this is very convenient and has been used before e.g. in [43].

By construction, off-diagonal elements have a nonzero commutator with $\overline{\mathfrak{h}}_Q$ acting in one of the tensor factors. We deduce the following

Proposition 5.3.2. *$\overline{\mathfrak{h}}_Q$ is a maximal commutative subalgebra of $\mathfrak{g}_Q$.*

5.3.5

For brevity, we write $\overline{\mathfrak{h}} = \overline{\mathfrak{h}}_Q$, $\mathfrak{g} = \mathfrak{g}_Q$. By Proposition 5.3.2, we can write

$$\mathfrak{g} = \overline{\mathfrak{h}} \oplus \bigoplus_{\eta \neq 0} \mathfrak{g}_\eta \tag{5.16}$$

where $\eta \in \mathbb{Z}^I$ and $\mathfrak{g}_\eta$ is spanned by ξ such that

$$\xi : H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\mathsf{w},\mathsf{v})) \to H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\mathsf{w},\mathsf{v}+\eta))\,.$$

The vectors η such that $\mathfrak{g}_\eta \neq 0$ are called the roots of $\mathfrak{g}$. Clearly

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}\,. \tag{5.17}$$

We call a root η positive if $\eta \in \mathbb{N}^I$.

5.3.6

The decomposition (5.16) parallels the root decomposition for Kac-Moody Lie algebras. As for a Kac-Moody Lie algebra, we define the coroot

$$h_\eta = \overline{\mathsf{C}}\,\eta \in \overline{\mathfrak{h}}$$

for every root η. These satisfy

$$(\alpha, \beta)_{\overline{Q}} = \alpha(h_\beta) = (h_\alpha, h_\beta)_{\overline{\mathfrak{h}}}\,. \tag{5.18}$$

Proposition 5.3.3. *Let η be a root and consider the commutator map*

$$\mathfrak{g}_\eta \otimes \mathfrak{g}_{-\eta} \to \overline{\mathfrak{h}}\,.$$

Its image is $\Bbbk\, h_\eta$ and this gives an embedding

$$\mathfrak{g}_\eta \hookrightarrow \mathfrak{g}_{-\eta}^\vee = \operatorname{Hom}(\mathfrak{g}_{-\eta}, \Bbbk)\,.$$

Later we will see that, in fact, this gives an isomorphism $\mathfrak{g}_\eta \cong \mathfrak{g}_{-\eta}^\vee$.

Proof. Take $\xi \in \mathfrak{g}_\eta$ and consider the $(\eta, 0)$-weight space in (5.15). One of the terms is

$$\Big[\xi \otimes 1, \sum h_i \otimes h^i\Big] = -\xi \otimes \sum_i h_i(\eta)\, h^i = -\xi \otimes h_\eta\,.$$

We conclude

$$\big[1 \otimes \xi, \mathbf{r}_{\eta,-\eta}\big] = \xi \otimes h_\eta\,, \tag{5.19}$$

where $\mathbf{r}_{\eta,-\eta}$ denotes the corresponding weight component. Both claims follows from this. □

5.3.7

By construction, $\mathfrak{g}$ comes with modules F_{w} containing vectors $|\mathsf{w}\rangle$ of lowest weight, that is,

$$\mathfrak{g}_\eta \, |\mathsf{w}\rangle = 0 \,, \quad \eta \not> 0 \,. \tag{5.20}$$

Recall that $\eta > 0$ means $\eta \in \mathbb{N}^I$. Also

$$h \, |\mathsf{w}\rangle = \mathsf{w}(h) \, |\mathsf{w}\rangle \,, \quad h \in \overline{\mathfrak{h}}$$

and $|\mathsf{w}\rangle$ is the unique, up to multiple, vector of weight w. We denote by $F_{\mathsf{w}}(\eta) \subset F_{\mathsf{w}}$ the subspace of weight $\mathsf{w} + \eta$. The $\mathfrak{g}$-action gives maps

$$\mathfrak{g}_\eta \to F_{\mathsf{w}}(\eta) \,, \quad \mathfrak{g}_{-\eta} \to F_{\mathsf{w}}(\eta)^\vee \tag{5.21}$$

that take $\xi \in \mathfrak{g}_\eta$ to $\xi \, |\mathsf{w}\rangle$ and dually for $\mathfrak{g}_{-\eta}$.

Proposition 5.3.4. *If $\eta \not< 0$ and $\mathsf{w}(h_\eta) \neq 0$ then the maps* (5.21) *are injective.*

Proof. Take $\xi \in \mathfrak{g}_\eta$ and $\xi' \in \mathfrak{g}_{-\eta}$. Then

$$\xi' \, \xi \, |\mathsf{w}\rangle = [\xi', \xi] \, |\mathsf{w}\rangle = \mathsf{w} \left([\xi', \xi] \right) |\mathsf{w}\rangle$$

where the step in the middle follows from (5.20). Now the claim follows from Proposition 5.3.3. □

Corollary 5.3.5. *All roots spaces are $\mathbb{k}$-modules of finite rank.*

Corollary 5.3.6. *All roots are either positive or negative.*

5.3.8

The $(\eta, -\eta)$-weight component of $\mathbf{r}$ defines a map

$$F_{\mathsf{w}}(0) \otimes F_{\mathsf{w}}(\eta) \to F_{\mathsf{w}}(\eta) \otimes F_{\mathsf{w}}(0) \,.$$

Since $F_{\mathsf{w}}(0) \cong \mathbb{k}$, this gives an operator

$$\mathsf{P}_\eta : F_{\mathsf{w}}(\eta) \to F_{\mathsf{w}}(\eta) \,.$$

Proposition 5.3.7.

$$\mathsf{P}_\eta^2 = -\mathsf{w}(h_\eta) \, \mathsf{P}_\eta \,. \tag{5.22}$$

Proof. Follows from considering the map

$$F_{\mathsf{w}}(0) \otimes F_{\mathsf{w}}(0) \otimes F_{\mathsf{w}}(\eta) \to F_{\mathsf{w}}(\eta) \otimes F_{\mathsf{w}}(0) \otimes F_{\mathsf{w}}(0)$$

given by (4.22). □

Proposition 5.3.8. *If $\eta > 0$ and $\mathsf{w}(h_\eta) \neq 0$ then image of* (5.21) *is the image of P_η and $\mathsf{P}_\eta^\vee$, respectively.*

Here $\mathsf{P}_\eta^\vee$ denotes the transpose map between the dual modules.

Proof. Apply both sides of (5.19) to $|\mathsf{w}\rangle \otimes |\mathsf{w}\rangle$. □

Corollary 5.3.9. *The root subspaces $\mathfrak{g}_{\pm\eta}$ are dual projective modules over $\Bbbk$. The classical $\mathbf{r}$-matrix*

$$\mathbf{r}_{\eta,-\eta} \in \mathfrak{g}_\eta \otimes \mathfrak{g}_{-\eta}$$

is the canonical element of this tensor product.

Corollary 5.3.10. *The commutator pairing from Proposition 5.3.3 is perfect.*

5.3.9

We summarize the preceding discussion in the following

Theorem 5.3.11. *All roots of $\mathfrak{g}_Q$ are either positive or negative. All roots spaces are projective $\Bbbk$-modules of finite rank. The Lie algebra $\mathfrak{g}_Q$ has an invariant bilinear form $(\,\cdot\,,\,\cdot\,)_{\mathfrak{g}}$ such that $\mathbf{r}$ is the corresponding invariant tensor. With respect to this form, $\mathfrak{g}_{-\eta} = \mathfrak{g}_\eta^\vee$.*

Since for Nakajima varieties $\Bbbk$ is a polynomial ring, the modules $\mathfrak{g}_\eta$ are free. Consequently, we can choose bases $\{e_\alpha^{(i)}\}$ of the root spaces so that

$$(e_\alpha^{(i)}, e_\beta^{(j)})_{\mathfrak{g}} = \delta_{\alpha,-\beta} \cdot \delta_{i,j}\,.$$

Correspondingly, we write

$$\mathbf{r} = \sum h_i \otimes h^i + \sum_{\alpha \neq 0} \sum_i e_\alpha^{(i)} \otimes e_{-\alpha}^{(i)}\,. \tag{5.23}$$

One should bear in mind, however, that it is the invariant tensor $\mathbf{r}$ that is canonically defined, while choosing bases of root spaces is a matter of convenience.

5.3.10

For future use, we record here the following easy lemma:

Lemma 5.3.12. *For each root $\alpha \neq 0$, the quadratic operator*

$$\sum_i e_\alpha^{(i)} e_{-\alpha}^{(i)}$$

acts via a Steinberg correspondence on each $F_{\mathsf{w}}(\mathsf{v})$.

Proof. Since $[e_\alpha, e_{-\alpha}]$ acts via a scalar, it suffices to prove this for $\alpha > 0$. Choose w_0 such that $h_\alpha(\mathsf{w}_0) \neq 0$. Up to a nonzero scalar, the claim then follows from considering the action of the composition of Steinberg operators

$$\mathbf{r}_{-\alpha,\alpha} \circ \mathbf{r}_{\alpha,-\alpha}$$

on $F_{\mathsf{w}_0}(0) \otimes F_{\mathsf{w}}(\mathsf{v})$. □

5.3.11

We note that the projector P_η has a direct geometric meaning for Nakajima variety. It is given by a Steinberg correspondence

$$\mathsf{P}_\eta \subset \mathcal{M}(\mathsf{w}, \eta) \times \mathcal{M}(\mathsf{w}, \eta)$$

supported on

$$\mathrm{Stab}\left(\mathcal{M}(\mathsf{w}, \eta) \times \mathcal{M}(\mathsf{w}, 0)\right) \cap \mathcal{M}(\mathsf{w}, 0) \times \mathcal{M}(\mathsf{w}, \eta)$$

viewed as A-fixed loci in $\mathcal{M}(2\mathsf{w}, \eta)$.

5.4 Operators of classical multiplication

5.4.1

In the Yangian Y, we have the operators

$$\mathsf{E}\left(|\mathsf{w}\rangle\langle\mathsf{w}|\, u^k\right), \quad \mathsf{w} \in \mathbb{Z}^I, \quad k = 1, 2, 3, \dots, \tag{5.24}$$

where

$$|\mathsf{w}\rangle\langle\mathsf{w}| \in \mathrm{End}\, H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\mathsf{w}))$$

is the orthogonal projector onto the vacuum. Recall from Figure 1.1 that for any g such that

$$[g \otimes g, R(u)] = 0$$

the operators

$$\operatorname{tr}_{F_0}(g \otimes 1) R_{F_0,W}(u) \in \operatorname{End}(W) \otimes \mathbb{Q}(u)$$

commute for all W and all values of u as a consequence of the Yang-Baxter equation. In particular, for $g = |\mathsf{w}\rangle\langle\mathsf{w}|$ this shows the operators (5.24) commute.

5.4.2

If $\theta > 0$, the vector $|\mathsf{w}\rangle$ is the true vacuum in the sense of Section 4.7.3. This implies that the operators (5.24) are operators of cup product by certain characteristic classes of the virtual bundle

$$(1 - \hbar) \otimes N_- = (1 - \hbar) \otimes \sum \mathsf{w}_i \, \mathcal{V}_i$$

where N_- is the negative part of the normal bundle to the embedding

$$\mathcal{M}(\mathsf{w}'') \hookrightarrow \mathcal{M}(\mathsf{w} + \mathsf{w}'') \,.$$

In particular, this gives another reason why these operators commute.

It is also clear that the operators (5.24) generate all characteristic classes of $\mathcal{V}_i$ in the case $\theta > 0$.

5.4.3

For general θ, the relation between the operators (5.24) and the operators of classical multiplication may determined along the lines of Theorem 4.9.1. Since the general expression in Theorem 4.9.1 is rather complicated and requires working in a certain completion of the Yangian, we will not do it here.

For the operators of classical multiplication by divisors, which is what we need for the proof of the main result of the paper, the case of general θ will be considered in Section 10.1.

5.4.4

In Proposition 5.5.3 below we will see the Yangian also contains the operators of multiplication by characteristic classes of the bundles $\mathcal{W}_i$.

These bundles are trivial but carry nontrivial group action, so this gives

$$\varprojlim_{\mathsf{w}} H^{\cdot}_{G_{\mathsf{w}}}(\mathsf{pt}) \hookrightarrow \text{center}(\mathsf{Y})\,.$$

5.4.5

We call the subalgebra

$$\mathsf{Classical} \subset \mathsf{Y} \subset \prod_{\mathsf{v},\mathsf{w}} \text{End}\, H^{\cdot}_{\mathsf{G}}(\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}))\,. \tag{5.25}$$

generated by the characteristic classes of $\{\mathcal{V}_i, \mathcal{W}_i\}$ the *algebra of classical multiplication*. Recall we assume that $\theta > 0$, otherwise a certain completion of the Yangian is required.

As already discussed, the algebra of classical multiplication is expected[1] to surject onto all operators of cup product in each factor of (5.25). The following weaker statement will be sufficient for our purposes. Recall that $\mathfrak{t}$ denotes the Lie algebra of a maximal torus in G.

Proposition 5.4.1. *After tensoring with $\mathbb{Q}(\mathfrak{t})$, the algebra of classical multiplication surjects onto all operators of cup products in each factor of* (5.25).

Proof. There is a $\mathbb{C}^\times$ action on $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ that scales all quiver data by the same scalar. After tensoring with $\mathbb{Q}(\mathfrak{t})$, we may replace the cohomology of $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$ by the cohomology of $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})^{\mathbb{C}^\times}$. The structure sheaf of the

$$\text{Diagonal} \subset \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) \times \mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})$$

may be resolved by tautotological bundles $\mathcal{V}_i$, see [87]. Since $\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w})^{\mathbb{C}^\times}$ is compact, it shows that its cohomology is spanned by characteristic classes of tautological bundles. □

[1]This has now been established in [77].

5.5 The structure of the Yangian

5.5.1

In this section we assume $\theta > 0$ for simplicity. Our goal here is the following

Theorem 5.5.1. *The Yangian is generated by the Lie algebra $\mathfrak{g}_Q$ and the operators of classical multiplication. We have*

$$\operatorname{gr} \mathsf{Y} \cong \mathcal{U}(\mathfrak{g}_Q[u])$$

with respect to the filtration by degree in u.

In the course of the proof, it will be convenient to choose a splitting of

$$\mathsf{E} : \bigoplus F_i \otimes F_i^\vee \to \mathfrak{g}_Q \to 0$$

which exists because $\mathfrak{g}_Q$ is a projective $\Bbbk$-module. We will write $\xi = \mathsf{E}(\xi)$ using such splitting. A concrete splitting may be constructed using the projectors P_η from Section 5.3.8.

5.5.2

Proposition 5.5.2. *If $\mathsf{E}(m) = 0$ then*

$$\mathsf{E}(m\, u^k) \in \mathsf{Y}_{<k}$$

with where $\mathsf{Y}_{<k} \subset \mathsf{Y}$ is the corresponding filtration subspace.

Proof. Since $k = 0$ this is a tautology, we take $k > 0$.

The map E is $\overline{\mathfrak{h}}$-equivariant and we can assume that m is an eigenvector of $\overline{\mathfrak{h}}$ of weight μ. If $\mu \neq 0$ then

$$\mu(h)\, \mathsf{E}(mu^k) = \big[\mathsf{E}(h), \mathsf{E}(mu^k)\big] = \big[\mathsf{E}(hu^k), \mathsf{E}(m)\big] + \dots = \dots \tag{5.26}$$

where the step in the middle is based on (5.12).

If $\mu = 0$ then $\mathsf{E}(mu^k)$ is a linear combination of diagonal matrix elements of the R-matrix. Theorem 4.9.1 expresses diagonal matrix elements of the R-matrix in terms of the off-diagonal ones and characteristic classes of N_-.

All terms involving off-diagonal matrix elements in Theorem 4.9.1 have degree $< k$. This is because they are at least quadratic the entries of the R-matrix and there is a degree shift from the expansion

$$R(u) = 1 + \sum_{n \geqslant 0} \frac{R_n}{u^{n+1}}$$

to the filtration in the Yangian: matrix coefficients of R_n belong to $\mathsf{Y}_{\leqslant n}$.

Now consider the characteristic classes of N_-. We have

$$\frac{e(N_-)}{e(N_- \otimes \hbar)} = 1 + \hbar \sum_{n \geqslant 0} \frac{n! \operatorname{ch}_n N_- + \dots}{u^{n+1}} \,,$$

where dots stand for characteristic classes of degree $< n$. In particular, applying this to (2.15), we get

$$\frac{1}{k!} \mathsf{E}(mu^k) = \sum_i (m, \mathsf{w}_i) \operatorname{ch}_k \mathcal{V}_i + \sum_i (m, \mathsf{v}_i) \operatorname{ch}_k \mathcal{W}_i \\ - \sum_{i,j} \mathsf{C}_{i,j} (m, \mathsf{v}_i) \operatorname{ch}_k \mathcal{V}_j + \dots$$

where the pairing with $\mathsf{v}_i, \mathsf{w}_i \in \overline{\mathfrak{h}}$ is the trace pairing and dots stand for elements in $\mathsf{Y}_{<k}$. Note that by induction all characteristic classes of $\mathcal{V}_i$ and $\mathcal{W}_i$ of degree $< k$ are in $\mathsf{Y}_{<k}$.

If $\mathsf{E}(m) = 0$ then $(m, \mathsf{w}_i) = (m, \mathsf{v}_i) = 0$ and this concludes the proof. □

The following is a corollary of the proof.

Proposition 5.5.3. *If $\theta > 0$, all characteristic classes of $\mathcal{V}_i$ and $\mathcal{W}_i$ lie in Y and this inclusion preserves degree.*

The case of general θ may be treated using Theorem 4.9.1. In this case, a certain completion of the Yangian is required.

5.5.3 Proof of Theorem 5.5.1

By Proposition 5.5.2 and (5.12), the operators $\mathsf{E}(\xi u^i)$ for $\xi \in \mathfrak{g}_Q$ generate the Yangian and satisfy the relations in $\mathfrak{g}_Q[u]$ modulo lower degree terms. This gives a surjective map

$$\mathcal{U}(\mathfrak{g}_Q[u]) \to \operatorname{gr} \mathsf{Y} \to 0 \,.$$

Its injectivity may be seen as follows. For any faithful representation of a Lie algebra

$$0 \to \mathfrak{g} \to \operatorname{End}(F)$$

the corresponding representation of the universal enveloping algebra in tensor powers of F

$$0 \to \mathcal{U}\mathfrak{g} \to \bigoplus \operatorname{End}\left(F^{\otimes n}\right)$$

is injective. Since the Yangian is defined as a subalgebra of endomorphisms of tensor products, it remains to check that the map

$$\mathfrak{g}_Q[u] \to \operatorname{gr} \mathsf{Y}$$

is injective, which is elementary. In fact,

$$\mathsf{E}(\xi u^i)\big|_{F(v)} = v^i \, \left(\xi\big|_F\right) + O(v^{i-1})\,, \quad v \to \infty \tag{5.27}$$

where v is the evaluation parameter for the representation $F(v)$ and we identify all $F(v)$ with $F = F(0)$ as linear spaces. Equation (5.27) means that the Yangian degenerates into the loop algebra when all evaluation parameters are very large.

The last claim of the Theorem, the fact the operators of classical multiplication and $\mathfrak{g}_Q$ generate the Yangian follows from (5.26).

5.5.4

As a consequence of the above result, we see that $\operatorname{gr} \mathsf{Y}$ and thus Y are flat as $\Bbbk$-modules. It follows that the map (5.13) is injective. Indeed, using flatness, it suffices to prove injectivity after tensoring with the fraction field K of $\Bbbk$ (which we denote by subscript for brevity). We then have inclusions

$$\mathsf{Y}_K \otimes \mathsf{Y}_K \to \prod_W \operatorname{End} W_K \otimes \prod_{W'} \operatorname{End} W'_K \to \prod_{W,W'} \operatorname{End}(W_K) \otimes \operatorname{End}(W'_K).$$

Injectivity after completion then follows from this case by decomposing the kernel into bi-graded pieces.

As a corollary, the coproduct

$$\Delta : \mathsf{Y} \to \mathsf{Y} \widehat{\otimes} \mathsf{Y}$$

is well-defined.

Chapter 6

Further properties of the Yangian

6.1 The core Yangian

6.1.1

In this section we assume $\theta > 0$. By Proposition 5.5.3, the Yangian Y contains all characteristic classes $\mathrm{ch}_k(\mathcal{W}_i)$ of the bundles $\mathcal{W}_i$. Since $\mathcal{W}_i$ are trivial, $\mathrm{ch}_k(\mathcal{W}_i)$ add little geometric value and it may be desirable to have a smaller algebra $\mathbb{Y}$ that does not contain them. The goal of this section is to define such *core Yangian*

$$\mathbb{Y} \subset \mathsf{Y} \otimes \mathbb{k}\left[\boldsymbol{\delta}^{-1}\right],$$

where $\boldsymbol{\delta} \subset \mathbb{k}$ is a certain equivariant constant that depends on the equivariant Cartan matrix C of the quiver. In particular, if the nonequivariant Cartan matrix is invertible then $\boldsymbol{\delta}^{-1} \in \mathbb{k}$ and $\mathbb{Y} \subset \mathsf{Y}$.

6.1.2

Recall from Theorem 4.9.1 and from the proof of Proposition 5.5.2 and that the characteristic classes of $\mathcal{V}_i$ and $\mathcal{W}_i$ come from the operator of cup product by

$$\frac{e(N_-)}{e(N_- \otimes \hbar)} \in H^{\cdot}_{\mathsf{G}_\mathsf{A}}(\mathcal{M}(\mathsf{w}) \times \mathcal{M}(\mathsf{w}'))$$

that appears in the diagonal matrix elements of the R-matrices. Here

$$\begin{aligned} N_- = &\sum \operatorname{Hom}(\mathcal{W}_i, \mathcal{V}'_i) + \sum \operatorname{Hom}(\mathcal{V}_i, \mathcal{W}'_i) \otimes \hbar^{-1} \\ &- \sum \mathsf{C}_{ij} \operatorname{Hom}(\mathcal{V}_i, \mathcal{V}'_j) \end{aligned} \tag{6.1}$$

in the negative part of the normal bundle to $\mathcal{M}(\mathsf{w}) \times \mathcal{M}(\mathsf{w}')$ inside $\mathcal{M}(\mathsf{w}+\mathsf{w}')$ and C is the equivariant Cartan matrix.

6.1.3

The basic idea for defining $\mathbb{Y}$ is the following. Complete the square in (6.1) as follows

$$N_- = -\sum \mathsf{C}_{ij} \operatorname{Hom}\left(\widehat{\mathcal{V}}_i, \widehat{\mathcal{V}}'_j\right) + \sum \left(\mathsf{C}^{-1}\right)_{ij} \operatorname{Hom}\left(\mathcal{W}_i, \mathcal{W}'_j\right) \otimes \hbar^{-1} \tag{6.2}$$

where

$$\mathsf{C}^{-1} \in \operatorname{Mat}(|I|, K_{\mathsf{G}_\mathsf{A}}(\mathsf{pt})_{\text{localized}})$$

is the inverse of the equivariant Cartan matrix and

$$\widehat{\mathcal{V}} = \mathcal{V} - \hbar^{-1} \otimes \mathsf{C}^{-1} \mathcal{W} \tag{6.3}$$

as vectors in $K_{\mathsf{G}_\mathsf{A}}(\mathcal{M}(\mathsf{w}) \times \mathcal{M}(\mathsf{w}'))^I \otimes K_{\mathsf{G}_\mathsf{A}}(\mathsf{pt})_{\text{localized}}$. In particular, the Chern character

$$\operatorname{ch} \widehat{\mathcal{V}} = \operatorname{ch} \mathcal{V} - e^{-\hbar} (\operatorname{ch} \mathsf{C})^{-1} \cdot \operatorname{ch} \mathcal{W}$$

is defined if C is invertible[1]. However, it may contain terms of negative cohomological degree if the nonequivariant Cartan matrix is not invertible, see below.

The main feature of (6.2) is that its second term is a purely equivariant object and so its Euler class may be taken out as an overall factor from the R-matrix. The diagonal matrix elements of the new R-matrix generate only $\operatorname{ch}_k \widehat{\mathcal{V}}$. This smaller algebra will be the desired core Yangian $\mathbb{Y}$.

We now proceed with the realization of the this plan.

[1] Recall from section 2.1.4 that we embed group weights into Lie algebra weights. While convenient, this could be confusing, especially in the context of Chern character. For example, by this rule, $\operatorname{ch} \hbar = e^{\hbar}$.

6.1.4

Let G be a complex reductive group and $f \in \mathbb{C}(\mathsf{G})$ a rational function on G. We define

$$\mathrm{ch}_k f \in \mathbb{C}(\mathrm{Lie}\,\mathsf{G})$$

by the series expansion

$$\sum_k x^k \ \mathrm{ch}_k f(\xi) = f(\exp(x\xi))\,, \quad \xi \in \mathrm{Lie}\,\mathsf{G}\,, \quad x \in \mathbb{C}\,.$$

This has negative terms if f is not regular at $1 \in \mathsf{G}$.

Functoriality of $\mathrm{ch}_k f$ with respect to homomorphisms $\phi : \mathsf{G}' \to \mathsf{G}$ may fail if $\phi(\mathrm{Lie}\,\mathsf{G}')$ lands in the pole divisor of the Chern character. Because of this, we work in G-equivariant K-theory and cohomology for some fixed group G if the nonequivariant Cartan matrix is not invertible.

For the rest of this section, we fix a group G such that

$$\mathsf{G}_{\mathsf{A}} \supset \mathsf{G} \supset \mathbb{C}^\times_t\,,$$

where $\mathbb{C}^\times_t$ is the group that scales all quiver data by the same number $t \in \mathbb{C}^\times$.

6.1.5

Lemma 6.1.1. *The matrix* C *is invertible in localized* G*-equivariant* K*-theory and*

$$\mathrm{ch}_k\, \mathsf{C}^{-1} = 0\,, \quad k < -2\,.$$

Proof. For the first claim, it suffices to consider the case $\mathsf{G} = \mathbb{C}^\times_t$. Then

$$\mathsf{C} = 1 + t^2 - t\,(Q + Q^T)$$

where Q is the nonequivariant adjacency matrix of the quiver Q. Clearly, this is invertible. As a real symmetric matrix, $Q + Q^T$ is semisimple. This implies C^{-1} has poles of order $\leqslant 2$ for $\mathsf{G} = \mathbb{C}^\times_t$.

For general G, the matrix $e^{\hbar/2}\,\mathrm{ch}\,\mathsf{C}$ is Hermitian when the equivariant parameters lie in the Lie algebra

$$\mathsf{g}_{\mathrm{c}} = \mathrm{Lie}\,\mathsf{G}_{\mathrm{compact}} = \{\xi, \xi^* = -\xi\}$$

of the compact real form of G. Therefore, its eigenvectors and eigenvalues are analytic along any real-analytic arc through the origin in g_{c}. In particular, the orders of the poles of $(\operatorname{ch}\mathsf{C})^{-1}$ along any arc are the orders of vanishing of the eigenvalues of $\operatorname{ch}\mathsf{C}$ along the same arc. The latter are determined by the coefficients of the characteristic polynomial, and, therefore, semicontinuous as a function of the arc. Since they are $\leqslant 2$ for $\mathsf{G} = \mathbb{C}_t^\times$, the Lemma follows. □

6.1.6

We define $\boldsymbol{\delta}$ as the lowest degree term in the expansion

$$\det \operatorname{ch}\mathsf{C} = \boldsymbol{\delta} + \dots .$$

By construction

$$\operatorname{ch}_k \mathsf{C}^{-1} \in \operatorname{Mat}(|I|, H^{\cdot}_{\mathsf{G}}(\mathsf{pt})[\boldsymbol{\delta}^{-1}]) .$$

for all k.

6.1.7

Let Q_d be the quiver with the adjacency matrix $Q + Q^T$, in other words,

$$Q_d = \overline{Q} \backslash \{\text{framing vertices}\} . \tag{6.4}$$

Let $\operatorname{Path}(Q_d)$ denote the path algebra of Q_d and let

$$\Pi(Q_d) = \operatorname{Path}(Q_d) \Big/ \left(\textstyle\sum_{a \in Q_d} [a, a^*]\right)$$

denote the preprojective algebra of Q_d. Here a^* is the arrow in Q_d opposite to an arrow $a \in Q$.

The group G_{edge} acts naturally on $\operatorname{Path}(Q_d)$ and $\Pi(Q_d)$, this action is *dual* to the defining action of G_{edge} on representations of these algebras. In other words, the action of G_{edge} on the generators of $\operatorname{Path}(Q_d)$ is recorded in the matrix $\overline{\mathsf{C}}$. In particular, the natural grading on $\operatorname{Path}(Q_d)$, in which every arrow has degree 1 is given by minus the weight of the $\mathbb{C}_t^\times$-action. All these weight spaces are finite-dimensional.

By construction, $\operatorname{Path}(Q_d)$ has orthogonal idempotents e_i, $i \in I$, namely paths of zero length that start and end at a vertex i. We set

$$\operatorname{Path}(Q_d)_{ij} = e_i \operatorname{Path}(Q_d)\, e_j ,$$

and similarly for $\Pi(Q_d)$. It is known, see for example [72, 30], that

$$\text{character } \Pi(Q_d)_{ij} = \left(\overline{\mathsf{C}}^{-1}\right)_{ji} = \hbar^{-1} \otimes \left(\mathsf{C}^{-1}\right)_{ij} \,, \tag{6.5}$$

provided Q is not a quiver of ADE type. We recall that by our convention C_{ji} records edges going from j to i.

Formula (6.5) provides the following geometric interpretation of the K-theory classes (6.3).

6.1.8

Recall that we assume $\theta > 0$. This means that the natural map of bundles over $\mathcal{M}_{\theta,0}(\mathsf{v},\mathsf{w})$

$$\bigoplus_{j\in I} \mathrm{Path}(Q_d)_{ij} \otimes \mathcal{W}_j \to \mathcal{V}_i$$

is surjective for all $i \in I$. Choose a G-invariant linear map (not algebra homomorphism)

$$s : \Pi(Q_d) \hookrightarrow \mathrm{Path}(Q_d)$$

splitting the canonical surjection in the other direction. The moment map equations for $\mathcal{M}_{\theta,0}(\mathsf{v},\mathsf{w})$ equal the relations in $\Pi(Q_d)$ modulo terms in the image of $\mathcal{W}_j$. Therefore

$$\bigoplus_{j\in I} s\left(\Pi(Q_d)\right)_{ij} \otimes \mathcal{W}_j \to \mathcal{V}_i \to 0\,, \tag{6.6}$$

is still surjective.

The grading by $\mathbb{C}^\times_t$ makes the class of $\Pi(Q_d)_{ij}$ well-defined in completed G-equivariant K-theory. From (6.5), we have the following

Proposition 6.1.2. *If Q is not of ADE type, $\theta > 0$, and G contains $\mathbb{C}^\times_t$, then the G-equivariant K-class of $\widehat{\mathcal{V}}$ is minus the kernel in* (6.6)*.*

There should be a more general statement valid for all quivers and all stability conditions.

6.1.9 Example

Let Q be the quiver with one vertex and one loop, that is the quiver with the adjacency matrix $Q = (1)$. Then

$$\Pi(Q_d) = \mathbb{C}\langle x, y\rangle/(xy - yx) = \mathbb{C}[x, y]\,.$$

The variety

$$\mathcal{M}_{1,0}(n,1) = \mathrm{Hilb}_n(\mathbb{C}^2)$$

is the Hilbert scheme of point of $\mathbb{C}^2$, that is, the moduli space of ideals $I \subset \mathbb{C}[x,y]$ of codimension n. The tautological sequence

$$0 \to I \to \mathbb{C}[x,y] \to \mathbb{C}[x,y]/I \to 0$$

is precisely the sequence

$$0 \to \mathrm{Ker} \to \Pi(Q_d) \to \mathcal{V} \to 0\,.$$

6.1.10

We defined the K-classes that appear in (6.2) and their Chern characters. We now consider the operator

$$\frac{e(N_-)}{e(N_- \otimes \hbar)} = \frac{c(N_-^\vee, u)}{c(N_-^\vee \otimes \hbar^{-1}, u)} \tag{6.7}$$

where

$$c(L,u) = u^{\mathrm{rk}\, L} + c_1(L)\, u^{\mathrm{rk}\, L-1} + \dots \tag{6.8}$$

is the Chern polynomial and the bundle arguments of the Chern polynomials in (6.7) are taken with the trivial action of u.

By definition, we set

$$\log c(L,u) = \sum_k \mathrm{ch}_k\, L\, \ln^{(k)} u\,, \quad \ln^{(k)} u = \left(\tfrac{d}{du}\right)^k \ln u \tag{6.9}$$

for any K-theory class L whose Chern character is defined. Here $\ln^{(-1)} u = u(\ln u - 1)$ etc. This generalizes (6.8) and is the usual ζ-regularization of infinite products given by Γ-functions, see for example [106, 116].

In particular, this defines $e(\widehat{N}_-)/e(\widehat{N}_- \otimes \hbar)$ for

$$\widehat{N}_- = -\sum \mathsf{C}_{ij}\, \mathrm{Hom}\left(\widehat{\mathcal{V}}_i, \widehat{\mathcal{V}}'_j\right)\,.$$

In fact, we will only need it for

$$\widehat{N}_-\Big|_{\mathsf{v}=\mathsf{v}'=0} = -\hbar^{-1} \otimes \sum \left(\mathsf{C}^{-1}\right)_{ij} \mathrm{Hom}\left(\mathcal{W}_i, \mathcal{W}'_j\right)\,. \tag{6.10}$$

We set

$$\Gamma(\mathsf{w},\mathsf{w}') = \left. \frac{e(\widehat{N}_-)}{e(\widehat{N}_- \otimes \hbar)} \right|_{\mathsf{v}=\mathsf{v}'=0}$$

and define the new matrix $\widehat{R}$ as a scalar multiple of the old R-matrix

$$\widehat{R} = \Gamma(\mathsf{w},\mathsf{w}')\, R\,. \tag{6.11}$$

Tautologically, it also satisfies the Yang-Baxter equation.

The old R-matrix was normalized to act by 1 on the vacuum vector, while the new matrix $\widehat{R}$ acts by a certain multivariate Γ-function. An example of $\Gamma(\mathsf{w},\mathsf{w}')$ is given in Section 16.2.1 below. The appearance of Γ-functions in normalization of R-matrices is a well-known phenomenon in the theory of quantum groups, see for example [62]. Here we have yet another angle from which it can be seen.

6.1.11

We modify the definitions of Section 5.2.6 as follows. For W as in (5.5), define

$$\widehat{R}_{F_0(u),W} = \widehat{R}_{F_0,F_n}(u-u_n) \cdots \widehat{R}_{F_0,F_1}(u-u_1)\,.$$

We can write

$$\widehat{R}_{F_0(u),W} = e^{\hbar\gamma_{\text{sing}}}\, \widehat{R}_{F_0(u),W,\text{reg}}$$

where $\widehat{R}_{F_0(u),W,\text{reg}}$ has a $1/u$-expansion and $\hbar\gamma_{\text{sing}}$ is the singular part of the $u \to \infty$ expansion of $\log \widehat{R}_{F_0(u),W}$. In particular, γ_{sing} is a scalar operator.

In fact, Lemma 6.1.1 implies

$$\operatorname{ch}_k \widehat{N} \otimes (1-\hbar) = 0\,, \quad k < -1\,.$$

Therefore

$$\gamma_{\text{sing}} = \mathsf{c}_{-2}\, \ln^{(-1)} u + \mathsf{c}_{-1}\, \ln u \tag{6.12}$$

for certain scalar operators

$$\mathsf{c}_{-2}, \mathsf{c}_{-1} \in \Bbbk[\boldsymbol{\delta}^{-1}][u_1,\dots,u_n]$$

of equivariant degree -2 and -1, respectively. The dependence on u_i comes from

$$\ln^{(-1)}(u-u_i) = \ln^{(-1)}(u) - u_i \ln u + \dots\,,$$

and is at most linear.

Definition 6.1.3. The core Yangian

$$\mathbb{Y} \subset \mathsf{Y} \otimes \Bbbk[\boldsymbol{\delta}^{-1}]$$

is the algebra generated by the matrix coefficients of c_{-2}, c_{-1}, and all coefficients of the $1/u$ expansion of $\widehat{R}_{F_0(u),W,\mathrm{reg}}$. Inside $\mathbb{Y}$ we have a Lie algebra

$$\mathfrak{g}'_Q \subset \mathbb{Y}$$

generated by c_{-2}, c_{-1}, and the u^{-1} coefficient of $\widehat{R}_{F_0(u),W,\mathrm{reg}}$.

Arguing as in Section 5.5 we obtain the following

Theorem 6.1.4. *The core Yangian $\mathbb{Y}$ is generated by $\mathfrak{g}'_Q$ and the operators of cup product by $\mathrm{ch}_k \widehat{\mathcal{V}}_i$ for $k \geqslant 1$ and $i \in I$.*

6.2 Slices and intertwiners

6.2.1

Consider the following setup. It will not be the most general, but will suffice for our purposes and will illustrate the general ideas. Consider $H^{\cdot}_{\mathsf{T}}(\mathcal{M}(\mathsf{w}))$, where $\mathsf{T} \subset G_{\mathsf{w}} \times G_{\mathrm{edge}}$ is a torus and

$$\mathsf{w} = a_i\, \delta_i + a_j\, \delta_j\,.$$

Here δ_i and δ_j are delta functions at some vertices $i, j \in I$ and a_i, a_j are weights of T.

As explained in Section 2.6, the first fundamental theorem of invariant theory gives an embedding of $\mathcal{M}_0(\mathsf{w})$ into a particular vector representation V of T. The weights of this representation correspond to closed paths in (6.4) as well as paths that start and end at vertices in $\{i, j\}$.

6.2.2

Let P be a path of the form

$$j \xrightarrow{P_1} \bullet \xrightarrow{P_2} \bullet \to \cdots \to \bullet \to i$$

where dots represent vertices of Q and P_i are arrows from $\overline{Q}$. The weight of the corresponding G_{v}-invariant function $f_P \in \mathbb{C}[\mathcal{M}_0(\mathsf{w})]$ is computed as follows

$$w_P = -\text{weight}\, f_P = a_i - a_j + \sum t_k$$

where t_k is the weight of the arrow P_k. We *assume* that T is such that

$$w_P \neq -\text{weight}\, f_{P'} \tag{6.13}$$

for any other generator $f_{P'}$ of $\mathbb{C}[\mathcal{M}_0(\mathsf{w})]$. This assumption is satisfied in examples from Sections 2.5.8 and 2.5.9.

Denote $\mathsf{T}' = \operatorname{Ker} w_P$ and let $x_P \in \mathcal{M}_0(\mathsf{w})^{\mathsf{T}'}$ be the unique, up to multiple, nonzero fixed representation. By construction, T scales x_P with weight w_P. By our assumption

$$\mathcal{M}_0(\mathsf{w})^{\mathsf{T}'} = \mathbb{C}x_P\,, \tag{6.14}$$

where $\mathbb{C}x_P$ is the line through x_P.

6.2.3

Let Σ_P denote the slice at x_P

$$\Sigma_P : \mathcal{M}(\mathsf{v}', \mathsf{w}') \times U \dashrightarrow \mathcal{M}(\mathsf{v}, \mathsf{w})\,, \tag{6.15}$$

where

$$\mathsf{v}' = \mathsf{v} - \dim x_P\,, \quad \mathsf{w}' = \mathsf{w} - \hbar \otimes \mathsf{C} \dim x_P \tag{6.16}$$

by Proposition 2.5.2 and

$$U \cong \mathbb{C}^{\dim \mathcal{M}(\mathsf{v},\mathsf{w}) - \dim \mathcal{M}(\mathsf{v}',\mathsf{w}')}$$

is a vector space factor with the T'-character given by (2.18). In particular, restricting to the origin in U we obtain a map

$$\Sigma_{P,0} : \mathcal{M}(\mathsf{v}', \mathsf{w}') \dashrightarrow \mathcal{M}(\mathsf{v}, \mathsf{w})$$

which is regular in the neighborhood of the central fiber of $\mathcal{M}(\mathsf{v}', \mathsf{w}')$ and hence defines a map

$$\Sigma^*_{P,0} : H^{\cdot}_{\mathsf{T}'}(\mathcal{M}(\mathsf{v}, \mathsf{w})) \to H^{\cdot}_{\mathsf{T}'}(\mathcal{M}(\mathsf{v}', \mathsf{w}'))\,.$$

Proposition 6.2.1. *The map $\Sigma^*_{P,0}$ is a $\mathbb{Y}$-intertwiner.*

Proof. Since slice is a Steinberg correspondence, the bottom arrow in the diagram (2.20) intertwines the R-matrices on both sides. The vector space U contributes a scalar factor to the R-matrix, therefore $\Sigma^*_{P,0}$ intertwines R-matrices, up to a multiple. To see that it intertwines $\widehat{R}$-matrices, it suffices to note that

$$\widehat{\mathcal{V}}' = \mathcal{V}' - \hbar^{-1} \otimes \mathsf{C}^{-1}\,\mathcal{W}' = \mathcal{V} - \hbar^{-1} \otimes \mathsf{C}^{-1}\,\mathcal{W} = \widehat{\mathcal{V}}$$

from (6.16) . □

6.2.4

Let $\widetilde{\mathsf{T}}$ be a torus in $G_{\mathsf{w}'} \times G_{\text{edge}}$ that contains T' and a maximal torus $\mathsf{A}' \subset G_{\mathsf{w}'}$. For any chamber $\mathfrak{C} \subset \operatorname{Lie} \mathsf{A}'$, we have a map

$$\operatorname{Stab}_{\mathfrak{C}} : \bigotimes H^{\cdot}_{\widetilde{\mathsf{T}}}(\mathcal{M}(\delta_i))^{\otimes \mathsf{w}'_i} \to H^{\cdot}_{\widetilde{\mathsf{T}}}(\mathcal{M}(\mathsf{w}'))$$

which becomes an isomorphism after tensoring with $\mathbb{Q}(\operatorname{Lie} \widetilde{\mathsf{T}})$ and intertwines the action of both full and core Yangians. The order of tensor factors here is determined by the chamber $\mathfrak{C}$, see Section 4.1.6 .

We denote $\mathbb{K} = \mathbb{Q}(\operatorname{Lie} \mathsf{T}')$ and denote by a'_{kl} the T'-weights in $\mathsf{w}' = \sum a'_{kl}\, \delta_k$.

Proposition 6.2.2. *For any $\mathfrak{C}$, the map $\operatorname{Stab}_{\mathfrak{C}}$ restricts to isomorphism*

$$\bigotimes_{k,l} F_k(a'_{kl}) \otimes \mathbb{K} \xrightarrow{\sim} H^{\cdot}_{\mathsf{T}'}(\mathcal{M}(\mathsf{w}')) \otimes \mathbb{K}$$

of Yangian modules, where F_k are as in (5.14).

Here the evaluation parameters a'_{kl} are as in Section 5.2.14 and the order of tensor factors as before. Note, in particular, the Proposition implies the tensor product on the left gives isomorphic Yangian modules for any ordering of tensor factors.

We begin with the following

Lemma 6.2.3. *The torus T' has a zero weight in U and, therefore, a unique fixed point in $\mathcal{M}_0(\mathsf{w}')$.*

The second claim here follows from the first because of (6.14).

Proof of Proposition 6.2.2. By Theorem 4.4.1, the inverse map is given by $\operatorname{Stab}^{\tau}_{-\mathfrak{C}}$. The lemma shows $\mathcal{M}(\mathsf{w}')^{\mathsf{T}'}$ is proper, therefore $\operatorname{Stab}^{\tau}_{-\mathfrak{C}}$ is well-defined in localized T'-equivariant cohomology. □

6.2.5

Now for $\mathcal{M}(\mathsf{w})$ we want to do the same: first enlarge T to include a maximal torus $\mathsf{A} \cong (\mathbb{C}^\times)^2 \subset G_\mathsf{w}$ and then restrict to T'-equivariant cohomology. For A, there are only two chambers $\mathfrak{C}_>$ and $\mathfrak{C}_<$, corresponding to $a_i \gtrless a_j$. Let

$$F_i(a_i) \otimes F_j(a_j) \xrightarrow{\text{Stab}_>} H^{\boldsymbol{\cdot}}_{\mathsf{T}'}(\mathcal{M}(\mathsf{w})) \xleftarrow{\text{Stab}_<} F_j(a_j) \otimes F_i(a_i)$$

be the corresponding maps.

Proposition 6.2.4. *The map* Stab$_>$ *becomes an isomorphism after tensoring with* $\mathbb{K}$.

Proof. The inverse map is given by $\text{Stab}_<^\tau$. By construction the line (6.14) has weight w_P which is negative on $\mathfrak{C}_<$ and therefore transverse to the images of attracting manifolds. Thus $\text{Stab}_<^\tau$ is well-defined in localized T'-equivariant cohomology. □

Note that the analogous statement for Stab$_<$ fails since the push-forward along $\mathbb{C}\, x_P$ is not defined in T'-equivariant cohomology. We have, however, the following

Proposition 6.2.5. *The operator*

$$(\Sigma_{P,0} \circ \text{Stab}_>)^\tau : H^{\boldsymbol{\cdot}}_{\mathsf{T}'}(\mathcal{M}(\mathsf{w}')) \otimes \mathbb{K} \to F_j(a_j) \otimes F_i(a_i) \otimes \mathbb{K}$$

is a well-defined $\mathbb{Y}$*-intertwiner.*

Proof. The map is well-defined by Lemma 6.2.3 since the image of $\Sigma_{P,0}$ is transverse to $\mathbb{C}x_P$. It is an intertwiner because its transpose is. □

6.2.6

We summarize the preceding discussion as follows. Suppose

$$\mathcal{M}_0(a_i\delta_i + a_j\delta_j)^{\mathsf{T}'} = \mathbb{C}x_P\,,$$

where P is a path that starts at j and ends at i. Define a'_{kl} by the formula

$$\sum a'_{kl}\, \delta_k = a_i\delta_i + a_j\delta_j - \hbar \otimes \mathsf{C}\, \dim x_P\,,$$

where $\dim x_P$ is a vector with values in $K_{\mathsf{T}'}(\mathsf{pt})$.

Theorem 6.2.6. *The slice at x_P gives rise to two $\mathbb{Y}$-intertwiners:*

$$F_i(a_i) \otimes F_j(a_j) \otimes \mathbb{K} \to \bigotimes F_k(a'_{kl}) \otimes \mathbb{K} \tag{6.17}$$

and

$$\bigotimes F_k(a'_{kl}) \otimes \mathbb{K} \to F_j(a_j) \otimes F_i(a_i) \otimes \mathbb{K}\,, \tag{6.18}$$

where the equivariant parameters are specialized to T', $\mathbb{K} = \mathbb{Q}(\operatorname{Lie}\mathsf{T}')$, the order of the $F_k(a_{kl})$-factors is arbitrary in (6.17) *and reverse in* (6.18).

Proof. The first map is given by

$$\mathrm{Stab}^{\tau}_{-\mathfrak{C}} \circ \Sigma^*_{P,0} \circ \mathrm{Stab}_{>}\,,$$

for $\mathfrak{C}$ matching the order of factors. The second map is its transpose. □

6.3 The dual Yangian

6.3.1

We define the dual Yangian Y^* as the algebra generated by the operators

$$\mathsf{E}^*(m^*(v)) = \operatorname{Res}_{v=0}\, \operatorname{tr}_{F_0} m^*(v)\, R_{W,F_0(v)} \in \mathsf{Y}^*\,, \tag{6.19}$$

for all W of the form

$$W = \bigotimes_{i=1}^{n} F_i \otimes \Bbbk(u_i)_\infty\,, \tag{6.20}$$

and

$$m^*(v) \in F_0 \otimes F_0^\vee \otimes v^{-1}\Bbbk[v^{-1}]\,.$$

Here $\Bbbk(u)_\infty$ denotes rational functions of u regular at $u = \infty$.

In English, Y^* is generated by matrix elements of the same matrices $R(u-v)$ but expanded in ascending powers of v. In particular, the operators E^* depend rationally, not polynomially, on the evaluation parameters u_i.

Note that the operators $\mathsf{E}^*\,(m_{-1}v^{-1})$ already give all matrix elements of $R(u)$ and their orbits under shift automorphism span Y^*.

6.3.2

There is a natural pairing between Y and Y^* defined as follows. Let

$$M(u) = m_1(u_1) \otimes \cdots \otimes m_k(u_k)$$

be an element in the domain (5.9) of the map E and, similarly, let

$$M^*(v) = m_1^*(v_1) \otimes \cdots \otimes m_l^*(v_l)$$

lie in the domain of E^*. Let

$$R(u,v) = \overrightarrow{\prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant l}}} R(u_i - v_j)$$

be the corresponding R-matrix where (i,j)th term acts in the spaces with evaluation parameters u_i and v_j and the ordering of the R-matrices is as in (5.6). We define

$$\begin{aligned} \Big(\mathsf{E}(M(u)), \mathsf{E}^*(M^*(v))\Big) &= \left[\frac{1}{u_1 \ldots v_l}\right] \operatorname{tr}_{u,v} \left(M(u) \otimes M^*(v)\right) \, R(u,v) \\ &= \hbar^k \left[\frac{1}{v_1 \ldots v_l}\right] \operatorname{tr}_v \; M^*(v) \, \mathsf{E}(M(u)) \\ &= \left[\frac{1}{u_1 \ldots u_k}\right] \operatorname{tr}_u \; M(u) \, \mathsf{E}^*(M^*(v)) \end{aligned} \tag{6.21}$$

where coefficients are taken in the $u_i \to \infty$, $v_j \to 0$ expansion and the subscripts of traces indicate tensor factors in which they are taken.

6.3.3

As defined, (6.21) is a pairing between the domains of E and E^*. It is clear, however, that the kernels on both sides are exactly the kernels of E and E^*. In other words, we have the following

Proposition 6.3.1.

$$\operatorname{Ker} \mathsf{E} = (\mathsf{Y}^*)^{\perp} \,, \quad \operatorname{Ker} \mathsf{E}^* = (\mathsf{Y})^{\perp} \,.$$

6.3.4

By construction, (6.21) is a Hopf pairing, that is

$$(ab, c) = (a \otimes b, \Delta c)$$

and vice versa, where $(\,\cdot\,,\,\cdot\,)$ is extended to

$$\mathsf{Y}^{\otimes 2} \otimes (\mathsf{Y}^*)^{\otimes 2} \to \Bbbk$$

multiplicatively. Tautologically, this pairing stores the same information as the R-matrices.

6.4 Intertwiners and relations

6.4.1

Let W as in (6.20) be a Y^*-module and let

$$C : W \to W'$$

be a $\Bbbk(u, u')$-linear map, where

$$W' = \bigotimes_{i'=1}^{n'} F_{i'} \otimes \Bbbk(u'_{i'})_\infty \,,$$

be another Y^*-module of the same form. Suppose that for certain values of u and u' the map C becomes a Y^*-intertwiner, that is,

$$[y, C] \in \mathrm{Hom}(W, W') \otimes \mathbf{I}$$

for all $y \in \mathsf{Y}^*$ and a nontrivial ideal

$$\mathbf{I} \subset \Bbbk(u, u')_\infty$$

in the local ring of the point $(u, u') = (\infty, \infty, \ldots, \infty)$.

Note that Y^*-intertwiners are operators that commutes with all R-matrices and, therefore, the same as Y-intertwiners, up to extension of scalars. Intertwiners produce elements in $(\mathsf{Y}^*)^\perp$ and hence relations in Y as follows.

6.4.2

Let $\mathbf{I}^\perp \subset \Bbbk[u,u']$ denote the perpendicular of $\mathbf{I}$ with respect the the residue pairing.

Proposition 6.4.1. *For any $f \in \mathbf{I}^\perp$ and any*

$$m \in \bigotimes F_{i'}^\vee \otimes \bigotimes F_i \,.$$

we have a relation

$$\operatorname{Res}_{u'} \mathsf{E}(fmC) = \operatorname{Res}_u \mathsf{E}(fCm) \tag{6.22}$$

in the Yangian Y.

Here Res_u means taking the coefficient of $(u_1 \cdots u_n)^{-1}$ in the $u_i \to \infty$ expansion. Also note that $m : W' \to W$ is an operator of finite rank, therefore both mC and Cm are in the domain of E.

Note that in the product fC under the E-sign in the left-hand side of (6.22) we should keep only the singular (that is, polynomial) terms in the $u_i \to \infty$ expansion because the residue in (5.7) vanishes for regular terms. Similarly for $u'_j \to \infty$ in the right-hand side of (6.22).

Proof. For any $y \in \mathsf{Y}^*$ we have

$$\operatorname{tr}_W \, m\, C\, y - \operatorname{tr}_{W'} \, C\, m\, y \in \mathbf{I}$$

and therefore

$$\operatorname{Res}_u \operatorname{Res}_{u'} (\operatorname{tr}_W \, f\, m\, C\, y - \operatorname{tr}_{W'} \, f\, C\, m\, y) = 0 \,.$$

This is equivalent to (6.22). □

6.4.3

The whole discussion can be repeated for the core Yangian $\mathbb{Y}$ in place of Y. Since slices produce $\mathbb{Y}$-intertwiners, the following question seems natural.

Question 2. *Do all relations in Yangians come from slices ?*

6.5 Baxter subalgebras and Casimir connection

6.5.1

Recall that $\mathfrak{h} \subset \mathfrak{g}_Q$ acts by linear functions of v and let $\mathfrak{H} \cong (\mathbb{C}^\times)^I$ be the torus with Lie algebra $\mathfrak{h}$. Since $\mathfrak{g}_Q$ commutes with R-matrices, we have

$$[g \otimes g, R(u)] = 0$$

for any $g \in \mathfrak{H}$. Recall from Section 1.2.2 this implies the operators

$$\mathsf{E}_{F_0}(g\, u^k) = \frac{1}{\hbar} \left[\frac{1}{u^{k+1}} \right] \operatorname{tr}_{F_0}(g \otimes 1)\, R_{F_0(u),W} \tag{6.23}$$

commute for all $k = 0, 1, \ldots$ and all auxiliary spaces F_0 for which the trace tr_{F_0} is well defined. This

In general, F_0 is not finite-dimensional and the trace in (6.23) is an infinite sum. However, it is well defined as a formal series in the variable $g \in \mathfrak{H}$ if F_0 satisfies the grading assumption from Section 5.2.5. We denote by

$$q^{\mathsf{v}} \in \Bbbk\mathfrak{H}^\wedge$$

elements of the group $\Bbbk$-algebra of the character group $\mathfrak{H}^\wedge$. These functions of g will be terms in our formal series. Introduce an algebra $\mathsf{Y}[[\mathfrak{H}^\wedge]]$ of formal series in q^{v} with coefficients in Y by

$$\mathsf{Y}[[\mathfrak{H}^\wedge]] = \left\{ \sum_{\mathsf{v} \geqslant \mathsf{v}_0} \mathsf{y}_{\mathsf{v}}\, q^{\mathsf{v}} \right\}.$$

Here $\mathsf{y}_{\mathsf{v}} \in \mathsf{Y}$ and $\mathsf{v} \geqslant \mathsf{v}_0$ means $\mathsf{v} - \mathsf{v}_0 \in \mathbb{Z}^I_{\geqslant 0}$. We have

$$\mathsf{E}_{F_0}(g\, u^k) = \frac{1}{\hbar} \left[\frac{1}{u^{k+1}} \right] \sum_{\mathsf{v}} q^{\mathsf{v}} \operatorname{tr}_{(F_0)_{\mathsf{v}}}(g \otimes 1)\, R_{F_0(u),W} \in \mathsf{Y}[[\mathfrak{H}^\wedge]] \tag{6.24}$$

as a consequence of our grading assumption.

By definition, the subalgebra of $\mathsf{Y}[[\mathfrak{H}^\wedge]]$ generated over $\Bbbk[[\mathfrak{H}^\wedge]]$ by the commuting operators (6.24) is called the *Baxter subalgebra*. It is a formal family of commuting subalgebras of Y.

6.5.2

Baxter subalgebras are graded with respect to the cohomological grading on the Yangian and

$$\deg_{\mathrm{coh}} \mathsf{E}_{F_0}(g\, u^k) = 2k\,.$$

In particular,

$$\big(\text{Baxter subalgebra}\big)_{\text{coh degree } 0} = \mathcal{U}_{\mathbb{Q}}(\overline{\mathfrak{h}})[[\mathfrak{H}^{\wedge}]]\,,$$

where $\overline{\mathfrak{h}} \subset \mathfrak{g}_Q$ acts by linear functions of v and w. Because $\Bbbk$ has nontrivial cohomological grading, the universal enveloping algebra here is over

$$\mathbb{Q} = \big(\Bbbk\big)_{\text{coh degree } 0}\,.$$

Our goal now is to describe the degree 2 part of Baxter subalgebra. It is spanned, over degree 0 part, by equivariant constants and u^{-2} coefficients in (6.23).

6.5.3

Evidently, only diagonal matrix coefficients contribute to the trace in (6.23). The u^{-2}-term of the diagonal matrix coefficients of R-matrices was computed in Proposition 4.9.2. The result can be stated as follows. Let

$$\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}) \times \mathcal{M}_{\theta,\zeta}(\mathsf{v}',\mathsf{w}') \subset \mathcal{M}_{\theta,\zeta}(\mathsf{v}+\mathsf{v}',\mathsf{w}+\mathsf{w}')$$

be a fixed component and let $R_{\mathsf{v},\mathsf{w},\mathsf{v}',\mathsf{w}'}$ be the corresponding diagonal block of the R-matrix. It follows from Proposition 4.9.2 that

$$\frac{1}{\hbar}\left[\frac{1}{u^2}\right] R_{\mathsf{v},\mathsf{w},\mathsf{v}',\mathsf{w}'} = (\mathsf{w} - \mathsf{C}\,\mathsf{v}) \otimes \mathrm{ch}_1\, \mathcal{V}' + \hbar \sum_{\theta\cdot\alpha>0} e_\alpha e_{-\alpha} \otimes e_{-\alpha} e_\alpha + \dots\,, \quad (6.25)$$

where $\mathrm{ch}_1\, \mathcal{V}'$ is a vector of $\mathrm{ch}_1\, \mathcal{V}'_i$, $i \in I$, C is the nonequivariant Cartan matrix, and dots act by a scalar operator in $H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\mathsf{v}',\mathsf{w}'))$.

6.5.4

For F_0 as above define

$$\boldsymbol{\chi}(F_0) \in \mathfrak{h}[[\mathfrak{H}^{\wedge}]]$$

by requiring

$$\operatorname{tr}_{F_0} g\, h_\eta = \eta(\boldsymbol{\chi}(F_0))$$

for all $\eta \in \mathfrak{h}^*$. Here $h_\eta = \overline{\mathsf{C}}\eta \in \overline{\mathfrak{h}}$, see Section 5.3.6. Since $\operatorname{tr}_{F_0} g\, h_\eta$ depends linearly on η, this is well defined. Clearly, $\boldsymbol{\chi}(F_0)$ is linear in the K-theory class of F_0.

Lemma 6.5.1.

$$\operatorname{tr}_{F_0} g\, e_\alpha\, e_{-\alpha} = -\alpha(\boldsymbol{\chi}(F_0))\, \frac{q^\alpha}{1-q^\alpha}\,. \tag{6.26}$$

The rational function in (6.26) is to be expanded in one direction or another, depending on $\alpha \gtrless 0$, to represent an element of $\Bbbk[[\mathfrak{H}^\wedge]]$.

Proof. Using

$$[e_\alpha, e_{-\alpha}] = h_\alpha \tag{6.27}$$

we compute

$$\begin{aligned}\operatorname{tr}_{F_0} g\, e_\alpha\, e_{-\alpha} &= \operatorname{tr}_{F_0} e_{-\alpha}\, g\, e_\alpha \\ &= q^\alpha \operatorname{tr}_{F_0} g\, e_{-\alpha}\, e_\alpha \\ &= q^\alpha \operatorname{tr}_{F_0} g\, e_\alpha\, e_{-\alpha} - q^\alpha \operatorname{tr}_{F_0} g\, h_\alpha\,,\end{aligned}$$

whence the conclusion. □

6.5.5

From Lemma 6.5.1 we deduce the following

Proposition 6.5.2. *We have*

$$\mathsf{E}_{F_0}(gu^2) = \boldsymbol{\chi}(F_0)\cdot \operatorname{ch}_1 \mathcal{V}' - \hbar \sum_{\theta\cdot\alpha>0} \alpha(\boldsymbol{\chi}(F_0))\, \frac{q^\alpha}{1-q^\alpha}\, e_{-\alpha}\, e_\alpha + \dots\,, \tag{6.28}$$

where dots stand for an element of $\mathcal{U}(\overline{\mathfrak{h}})[[\mathfrak{H}^\wedge]]$

By Theorem 10.2.1 below this means that the degree 2 part of Baxter algebra is spanned by operators of quantum multiplication by q-dependent tautological divisors

$$\lambda = \boldsymbol{\chi}(F_0) \in \mathfrak{h}[[\mathfrak{H}^\wedge]]$$

and equivariant constants.

Using formula (10.3), we can rearrange the terms in (6.28) as follows

$$\mathsf{w} \cdot \mathrm{ch}_1 \mathcal{V}' - \hbar \sum_{\theta\cdot\alpha>0} (\alpha, \mathsf{w}) \, \frac{q^\alpha}{1-q^\alpha} \, e_{-\alpha} \, e_\alpha = \\ \mathsf{E}\left(|\mathsf{w}\rangle\langle\mathsf{w}| \, u^2\right) - \hbar \sum_{\theta\cdot\alpha>0} (|\alpha|, \mathsf{w}) \, \frac{q^{|\alpha|}}{1-q^{|\alpha|}} \, e_{-\alpha} e_\alpha + \dots \,, \quad (6.29)$$

where dots stand terms from $\mathcal{U}(\overline{\mathfrak{h}})$ and

$$|\alpha| = \begin{cases} \alpha \,, & \alpha > 0 \,, \\ -\alpha \,, & \alpha < 0 \,. \end{cases}$$

The second line in (6.29) is manifestly an element of $\mathsf{Y}[[\mathfrak{H}\hat{}\,]]$ while the sum over α in the first line converges in a different formal series completion — the one corresponding to the effective cone of $\mathcal{M}_{\theta,0}$.

Chapter 7

Quantum multiplication

7.1 Preliminaries

We first set some notation regarding equivariant Gromov-Witten invariants. Suppose we are given a smooth quasi-projective variety X equipped with the action of a reductive group G. For each effective curve class $\beta \in \mathrm{Eff}(X) \subset H_2(X, \mathbb{Z})$, its associated k-point genus 0 Gromov-Witten invariants are given by integrals

$$\langle \gamma_1, \ldots, \gamma_k \rangle^X_{0,k,\beta} = \int_{[\overline{M}_{0,k}(X,\beta)]^{\mathrm{vir}}} \mathsf{ev}^* \left(\gamma_1 \boxtimes \cdots \boxtimes \gamma_k \right).$$

for $\gamma_i \in H^{\cdot}_{\mathsf{G}}(X, \mathbb{Q})$. Here, the integral is defined over the virtual fundamental class on the moduli space of k-pointed stable maps to X. As always, if X is noncompact (as in our case), the above expression can be defined via equivariant residue. However, since the evaluation maps are proper, operators of quantum multiplication are defined without localization.

7.2 Modified reduced operators

7.2.1

We recall some general results from [12] for the quantum product for any equivariant symplectic resolution

$$X \to X_0 = \operatorname{Spec} H^0(\mathcal{O}_X).$$

Due to the presence of the symplectic form ω, the moduli space of maps carries a *reduced* virtual class in degree one larger than the usual virtual dimension. This reduced class determines the purely quantum contributions to all divisor operators via the relation

$$(\gamma * \gamma_1, \gamma_2) = (\gamma \cup \gamma_1, \gamma_2) + \hbar \sum_{\beta > 0} (\gamma \cdot \beta) q^\beta \int_{\left[\overline{M}_{0,2}(X,\beta)\right]_{\text{vir,red}}} \mathsf{ev}^*(\gamma_1 \boxtimes \gamma_2)\,.$$

Moreover, the pushforward of the reduced virtual fundamental class under the evaluation map

$$\mathsf{ev} : \overline{M}_{0,2}(X,\beta) \to X^2\,,$$

is a $\mathbb{Q}$-linear combination of Steinberg correspondences of $X \times_{X_0} X$. In particular, it does not depend on equivariant parameters. We denote by

$$\mathsf{Q}_{2,\text{red}} \in \operatorname{End} H^{\bullet}_{\mathsf{G}}(X) \otimes \mathbb{Q}[[\operatorname{Eff}(X)]]$$

the purely quantum operator defined by the reduced class

$$(\mathsf{Q}_{2,\text{red}} \cdot \gamma_1, \gamma_2) = \sum_{\beta > 0} q^\beta \int_{\left[\overline{M}_{0,2}(X,\beta)\right]_{\text{vir,red}}} \mathsf{ev}^*(\gamma_1 \boxtimes \gamma_2)\,.$$

This is a correspondence-valued element in the completion of the semigroup algebra of the effective cone of X, each coefficient of which is a Steinberg correspondence for X.

7.2.2

Note that by (1.2)

$$\mathsf{Q}_{2,\text{red}} \cdot 1 = 0\,.$$

This uniquely determines the coefficient of the diagonal in $\mathsf{Q}_{2,\text{red}}$ from the other terms. It will be convenient to work modulo scalar operator contributions to $\mathsf{Q}_{2,\text{red}}$ in this chapter; this relation allows us to fix this indeterminacy.

7.2.3

Given

$$\boldsymbol{\kappa}_X \in H^2(X, \mathbb{Z}/2),$$

we define the *modified* quantum operator $\mathsf{Q}_{2,\mathrm{red},\boldsymbol{\kappa}}$ for X by the substitution

$$q^\beta \mapsto (-1)^{(\kappa_X,\beta)}\, q^\beta\,, \quad \beta \in H_2(X,\mathbb{Z})\,.$$

This is equivalent to changing the origin in the Kähler moduli space $H^2(X,\mathbb{C})/2\pi i H^2(X,\mathbb{Z})$ to $\pi i \kappa_X$.

Let a torus A act on X preserving the symplectic form and let $Y \subset X^{\mathsf{A}}$ be a connected component. Assume we have chosen

$$\boldsymbol{\kappa}_Y \in H^2(Y,\mathbb{Z}/2)\,,$$

such that

$$c_1(N_+) \equiv \boldsymbol{\kappa}_X\big|_Y + \boldsymbol{\kappa}_Y \mod 2$$

where N_+ is the positive part of the normal bundle to Y for some (equivalently, any) choice of the chamber $\mathfrak{C} \subset \mathfrak{a} = \operatorname{Lie}\mathsf{A}$.

For Nakajima varieties, the canonical theta characteristics $\boldsymbol{\kappa}$ were defined in (2.8) and connected to the parity of $c_1(N_+)$ in (2.12).

7.2.4

Our next goal is the following

Theorem 7.2.1. *For X and Y as in Section 7.2.3, the diagram*

$$\begin{array}{ccc} H^{\cdot}_{\mathsf{T}}(Y) & \xrightarrow{\ \mathrm{Stab}_{\mathfrak{C}}\ } & H^{\cdot}_{\mathsf{T}}(X) \\ {\scriptstyle \mathsf{Q}_{2,\mathrm{red},\boldsymbol{\kappa}_Y}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{Q}_{2,\mathrm{red},\boldsymbol{\kappa}_X}} \\ H^{\cdot}_{\mathsf{T}}(Y) & \xleftarrow{\ \mathrm{Stab}^{\tau}_{-\mathfrak{C}}\ } & H^{\cdot}_{\mathsf{T}}(X) \end{array} \tag{7.1}$$

is commutative for any $\mathfrak{C}$ and any polarization, after applying the map

$$\mathbb{Q}[[\mathrm{Eff}(Y)]] \to \mathbb{Q}[[\mathrm{Eff}(X)]]$$

to $\mathsf{Q}_{2,\mathrm{red},\boldsymbol{\kappa}_Y}$ and working modulo scalar operators on $H^{\cdot}_{\mathsf{T}}(Y)$.

Note that the bottom arrow in (7.1) is a priori defined only in localized equivariant cohomology. As a part of the proof, we will see that the composition of the top, right, and the bottom arrows in (7.1) is well-defined without localization.

The proof of this theorem will require the discussion of broken and unbroken curves in equivariant localization. We recall the relevant definitions and results from [103].

7.3 Broken curves

7.3.1

Let $f : C \to X$ be an A-fixed point of $\overline{M}_{0,2}(X, \beta)$ such that the domain C is a chain of rational curves

$$C = C_1 \cup C_2 \cup \cdots \cup C_k\,,$$

with the two marked points p_1, p_2 lying on C_1 and C_k, respectively.

If at every node of C the A-weights of the two branches are opposite and nonzero then we say that f is an *unbroken chain.* We say that f *connects* the points

$$x_0 = f(p_1),\ x_k = f(p_2)$$

of X through the sequence of nodes

$$x_i = f(C_i \cap C_{i+1})\,, \quad i = 1, \dots, k-1\,.$$

Note that all of these points are fixed by A.

More generally, if (C, f) is an A-fixed point of $\overline{M}_{0,2}(X, \beta)$, we say that f is an *unbroken map* if it satisfies one of three conditions:

1. f arises from a map $f : C \to X^{\mathsf{A}}$,
2. f is an unbroken chain, or
3. the domain C is a chain of rational curves

$$C = C_0 \cup C_1 \cup \cdots \cup C_k$$

such that C_0 is contracted by f, the marked points lie on C_0, and the remaining components form an unbroken chain.

Broken maps are A-fixed maps that do not satisfy one of these conditions.

In this last possibility, the contribution of these curves is block-diagonal with respect to A-fixed locus of X, i.e. scalar on each connected component Y, hence we will focus on the unbroken chains in what follows.

7.3.2

We refer the reader to Section 3.8.3 of [103] for the proof of the following.

Theorem 7.3.1 ([103])**.** *Every map in a given connected component of $\overline{M}_{0,2}(X,\beta)^{\mathsf{A}}$ is either broken or unbroken. Only unbroken components contribute to $\mathsf{Q}_{2,\mathrm{red}}$ in A-equivariant localization.*

7.3.3

Let f be an unbroken chain as before and let let $\mathcal{O}(1)$ be a A-linearized ample line bundle on X. We may restrict it to fixed point x_i to get elements of $\mathfrak{a}^*$. We have the following

Lemma 7.3.2. *For an unbroken chain, the points*

$$\mathcal{O}(1)\Big|_{x_i} \in \mathfrak{a}^* , \quad i = 0, \dots, k ,$$

form a monotone sequence of distinct points of a real line.

Proof. We denote this sequence by c_i. Let w denote the (nonzero) A-weight of $T_{p_1}C$. By the unbroken condition, the same weight occurs at all nodes and the weight of $T_{p_2}C$ is $-w$. By localization, the terms of the sequence

$$\frac{c_0 - c_1}{w}, \dots, \frac{c_{k-1} - c_k}{w}$$

are the degrees of $f^*\mathcal{O}(1)$ restricted to C_i, hence positive integers. □

7.3.4

Lemma 7.3.2 is effective in ruling out unbroken loops. More generally, we have the following

Lemma 7.3.3. *There are no A-fixed unbroken chains connecting two points in the same component Y of X^{A}.*

Proof. The A-weight of $\mathcal{O}(1)_y$ is constant for $y \in Y$, which contradicts the fact that points in Lemma 7.3.2 are distinct. □

7.4 Proof of Theorem 7.2.1

7.4.1

Given $\beta \in H_2(X,\mathbb{Z})$ and $\gamma_1, \gamma_2 \in H^{\cdot}_{\mathsf{T}}(Y)$, the statement to prove is

$$\sum_{\beta' \mapsto \beta} (-1)^{(\beta,\kappa_Y)+\frac{1}{2}\dim Y} \langle \gamma_1, \gamma_2 \rangle^Y_{\beta',\mathrm{red}} = \\ (-1)^{(\beta,\kappa_X)+\frac{1}{2}\dim X} \Big\langle \operatorname{Stab}_{\mathfrak{C}}(\gamma_1), \operatorname{Stab}_{-\mathfrak{C}}(\gamma_2) \Big\rangle^X_{\beta,\mathrm{red}} + c_\beta\,, \quad (7.2)$$

where c_β is a constant independent of the insertions γ_1, γ_2.

The sign $(-1)^{\frac{1}{2}\operatorname{codim}_X Y}$ comes from the sign in the definition of the adjoint $\operatorname{Stab}^\tau_{-\mathfrak{C}}$.

7.4.2

Recall that every coefficient of $\mathsf{Q}_{2,\mathrm{red},\boldsymbol{\kappa}_X}$ is given by a Steinberg correspondence. As in the proof of Theorem 4.6.1, this implies the convolution

$$\operatorname{Stab}^\tau_{-\mathfrak{C}} \circ \mathsf{Q}_{2,\mathrm{red},\boldsymbol{\kappa}_X} \circ \operatorname{Stab}_{\mathfrak{C}}\ .$$

is obtained by a proper push-forward. In particular, its coefficients can be determined by any specialization of equivariant parameters.

This means we can compute the RHS of (7.2) by A-equivariant localization, and study its limit after taking the equivariant parameters associated to $\mathfrak{a}$ to infinity, while setting $\hbar = 0$ at the same time.

7.4.3

We only need to consider unbroken components of $\overline{M}_{0,2}(X,\beta)^{\mathsf{A}}$ in equivariant localization.

Since stable envelopes are proportional to fixed points modulo $\hbar$, setting $\hbar = 0$ implies that only components where both marked points map to Y will give nonzero contribution.

If we fix a component of $\overline{M}_{0,2}(X,\beta)^{\mathsf{A}}$ whose elements consist of curves for which both marked points lie on a contracted component attached to an unbroken chain. Since the evaluation map to $Y \times Y$ for this component factors through the diagonal, the contribution of this component will give a scalar operator, so we can ignore it. By Lemma 7.3.3, unbroken chains do not contribute either, so the only contributions come from stable maps which factor through Y. Furthermore, since the localization contribution only depend on the equivariant normal bundle to Y in X, we may replace X by the total space N of the normal bundle.

7.4.4

For a vector bundle

$$p : N \to Y$$

we have the following general result. Suppose A acts on N fiberwise and $N^{\mathsf{A}} = Y$. We decompose

$$N = \bigoplus_{\lambda} N_\lambda$$

according to the characters $\lambda \in \mathsf{A}^\vee \subset \mathfrak{a}^*$.

Given cohomology classes $\gamma_1, \dots, \gamma_k \in H^{\cdot}(Y)$ we want to understand the asymptotic behavior of the Gromov-Witten invariant

$$\langle p^*(\gamma_1), \dots, p^*(\gamma_k) \rangle^N_{\beta,g} \in \mathbb{Q}(\mathfrak{a}^*)$$

defined via equivariant residue, as the variables in $\mathfrak{a}$ approach infinity. Here, $g \geqslant 0$ is the domain genus and $\beta \in H_2(Y, \mathbb{Z}) = H_2(N, Z)$ is the degree of the map.

The residue invariant can be expressed in terms of the Gromov-Witten invariants of Y by adding an Euler class insertion determined by N. The following computation is then a standard application of Riemann-Roch:

Lemma 7.4.1. *We have the asymptotic behavior given by*

$$\langle p^*(\gamma_1), \dots, p^*(\gamma_k) \rangle^N_{\beta,g,k} \sim \prod_{\lambda} \lambda^{-(c_1(N_\lambda),\beta) - \operatorname{rk} N_\lambda (1-g)} \, \langle \gamma_1, \dots, \gamma_k \rangle^Y_{\beta,g,k} \, .$$

7.4.5

We only need the $g = 0$ case of the above lemma. Also

$$N_\lambda = N^\vee_{-\lambda}$$

because of the symplectic form. Therefore, the prefactor in Lemma 7.4.1 becomes

$$(-1)^{(c_1(N_+),\beta)} \big/ \det N \, .$$

Since

$$(\det N_+)^2 / \det N = (-1)^{\frac{1}{2}\operatorname{rk} N} = (-1)^{\frac{1}{2}\operatorname{codim}_X Y}$$

the equality (7.2) follows.

7.5 Additivity

7.5.1

Suppose Y as above factors

$$Y = Y_1 \times Y_2\,,$$

as it is the case in our main example (1.9). Then $\dim H^0(Y, \Omega^2) \geqslant 2$, leading to further constraints on quantum cohomology of Y.

Proposition 7.5.1.

$$\mathsf{Q}^{Y}_{2,\mathrm{red}} = \mathsf{Q}^{Y_1}_{2,\mathrm{red}} \otimes 1 + 1 \otimes \mathsf{Q}^{Y_2}_{2,\mathrm{red}}\,.$$

Proof. Let $\beta = (\beta_1, \beta_2)$ according to

$$H_2(Y, \mathbb{Z}) = H_2(Y_1, \mathbb{Z}) \oplus H_2(Y_2, \mathbb{Z})\,.$$

If $\beta_1 \neq 0$ and $\beta_2 \neq 0$ then the virtual fundamental class may be *doubly reduced*, meaning that the reduced obstruction theory admits a further surjection to a trivial rank 1 bundle. See Section 3.5 in [103]. As a result, the corresponding reduced class vanishes. If $\beta_1 = 0$ or $\beta_2 = 0$, then the curve maps to a point in one of the factors, and the above additivity is obvious. □

7.5.2

Notice that additivity is not the same as primitivity.

In Proposition 7.5.1, we are restricting to diagonal contributions; the off-diagonal terms will still be non-zero.

Chapter 8

Shift operators

8.1 Definition

8.1.1

For any X and any cocharacter

$$\sigma : \mathbb{C}^\times \to \mathsf{G} \subset \mathrm{Aut}(X)$$

we can associate an X-bundle over $\mathbb{P}^1$ as follows

$$X^\sim = (\mathbb{C}^2 \backslash 0) \times X \big/ \mathbb{C}^\times_\sigma \tag{8.1}$$

where $\mathbb{C}^\times_\sigma$ acts on the first factor by scaling and on the second via the homomorphism σ. This is just the classical operation of passing from a principal $\mathbb{C}^\times$-bundle over $\mathbb{P}^1$ to the associated X-bundle.

8.1.2

Since $c_1(X) = 0$, we have[1]

$$c_1\big(X^\sim\big) = (2 + \tfrac{\dim X}{2}(\sigma, \hbar)) \cdot [\mathit{Fiber}] \in H^2(X^\sim, \mathbb{Z}) \tag{8.2}$$

where fiber refers to the natural projection

$$p : X^\sim \to (\mathbb{C}^2 \backslash 0) \big/ \mathbb{C}^\times = \mathbb{P}^1 \,,$$

the inner product is the standard pairing of characters with 1-parameter subgroups.

[1] In the present discussion, one does not need to assume X symplectic. It suffices to assume that the canonical bundle of X is a pure character, which we denote by $\hbar^{\frac{1}{2}\dim X}$.

8.1.3

Let G^σ be the centralizer of the image of σ in G. We define

$$\mathsf{G}^\sim = \mathsf{G}^\sigma \times \mathbb{C}^\times_\varepsilon \subset \operatorname{Aut}(X^\sim) \tag{8.3}$$

where the second factor scales the second (by convention) coordinate of $\mathbb{C}^2$. This group preserves the fibers X_0, X_∞ of p over $0, \infty \in \mathbb{P}^1$. More precisely, it fixes X_0 point-wise, but acts nontrivially on X_∞.

We denote by ε an element of $\operatorname{Lie} \mathbb{C}^\times_\varepsilon$. This is a new equivariant parameter which we have for $X^\sim$.

8.1.4

Any point of X^σ gives a section of p

$$\zeta_x : \mathbb{P}^1 \to \mathbb{P}^1 \times x \subset X^\sim \,, \quad x \in X^\sigma \,. \tag{8.4}$$

The homology class of this section gives an element

$$\zeta \in H^0(X^\sigma, \mathbb{Z}) \otimes H_2(X^\sim, \mathbb{Z}) \,.$$

More formally, for any $D \in H^2(X^\sim, \mathbb{Z})$ we define

$$(D, \zeta) = \operatorname{proj}_* \operatorname{incl}^*(D) \in H^0(X^\sigma) \tag{8.5}$$

where

$$X^\sigma \xleftarrow{\text{proj}} X^\sigma \times \mathbb{P}^1 \xrightarrow{\text{incl}} X^\sim$$

are the natural maps.

8.1.5

We have

$$0 \to H_2(X, \mathbb{Z}) \to H_2(X^\sim, \mathbb{Z}) \to H_2(\mathbb{P}^1) \cong \mathbb{Z} \to 0 \,. \tag{8.6}$$

Any section ζ_x gives a noncanonical splitting of the above exact sequence.

In particular, the degrees $\beta \in H_2(X^\sim, \mathbb{Z})$ such that $p(\beta) = [\mathbb{P}^1]$ form a single $H_2(X, \mathbb{Z})$-coset of sections. For β in this coset, we consider

$$M^\sim(\beta) = \mathsf{ev}^{-1}(X_0 \times X_\infty) \subset \overline{M}_{0,2}(X^\sim, \beta) \,,$$

where $X_0, X_\infty \subset X$ are the fibers of p over $0, \infty \in \mathbb{P}^1$.

8.1.6 Example

Take

$$X = T^*\mathbb{P}^1 = \mathcal{M}_{1,0}(1,2)$$

for the quiver Q with one vertex and no arrows. This is the quotient of pairs

$$A = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^1)\,, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^2)$$

such that $AB = 0$ and $A \neq 0$ by the action of $G_{\mathsf{v}} = GL(1)$. Take

$$\sigma(z) = \begin{pmatrix} z & \\ & 1 \end{pmatrix} \in G_{\mathsf{w}}\,.$$

Then $X^\sigma = \{x_0, x_\infty\}$, where

$$x_0 = \{a_2 = 0, B = 0\}\,, \quad x_\infty = \{a_1 = 0, B = 0\}\,.$$

The variety $X^\sim$ is the relative cotangent bundle to the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ given by

$$\mathrm{Bl}_{\mathrm{point}}\,\mathbb{P}^2 \to \mathbb{P}^1\,.$$

We have

$$[\zeta_{x_0}] = \text{line in } \mathbb{P}^2\,, \quad [\zeta_{x_\infty}] = \text{exceptional divisor}\,,$$

and so $[\zeta_{x_0}] - [\zeta_{x_\infty}]$ is the generator $[\mathbb{P}^1]$ of $H_2(X, \mathbb{Z})$.

8.1.7

We use the spaces $X^\sim$ to define shift operators

$$\mathsf{S}(\sigma) : H^{\bullet}_{\mathsf{G}^\sim}(X_\infty) \longrightarrow H^{\bullet}_{\mathsf{G}^\sim}(X_0) \otimes \mathbb{Q}[[\mathrm{Eff}(X^\sim)]]$$

as follows.

Given $\gamma_1 \in H^{\bullet}_{\mathsf{G}^\sim}(X_0)$ and $\gamma_2 \in H^{\bullet}_{\mathsf{G}^\sim}(X_\infty)$, we define the matrix element

$$(\gamma_1, \mathsf{S}(\sigma) \cdot \gamma_2) = \sum_{p(\beta)=[\mathbb{P}^1]} q^\beta \int_{[M^\sim(\beta)]^{vir}} \mathsf{ev}^{-1}(\gamma_1 \times \gamma_2)\,. \tag{8.7}$$

By definition, the matrix coefficients (8.7) take values in formal power series in q^β with coefficients in the localized $\mathsf{G}^\sim$-equivariant cohomology of a point, although the operator $\mathsf{S}(\sigma)$ itself is integral. In particular, (8.7) depends on

$$\varepsilon \in \mathrm{Lie}(\mathbb{C}^\times_\varepsilon) \subset \mathrm{Lie}(\mathsf{G}^\sim)\,.$$

Our eventual goal will be to find $\sigma, \gamma_1, \gamma_2$ such that the integral in (8.7) is proper of correct dimension, thus independent of all equivariant parameters.

8.2 Intertwining property

Given $D \in H^2_{\mathsf{G}^\sim}(X^\sim)$, we set

$$\frac{\partial}{\partial D} q^\beta = \left(\int_\beta D \right) q^\beta \,, \quad \beta \in H_2(X^\sim)\,.$$

Note that this is nonequivariant, that is, depends only on the class of D in nonequivariant cohomology.

If we consider the restriction $D_0 = D\Big|_{X_0} \in H^2_{\mathsf{G}^\sim}(X_0)$, let $\mathsf{M}_{D_0}(q)$ denote the operator of quantum multiplication by D_0, and similarly for X_∞.

Proposition 8.2.1. *For any D as above, the operator* (8.7) *satisfies*

$$\varepsilon \frac{\partial}{\partial D} \mathsf{S}(\sigma) = \mathsf{M}_{D_0}(q) \circ \mathsf{S}(\sigma) - \mathsf{S}(\sigma) \circ \mathsf{M}_{D_\infty}(q)\,. \tag{8.8}$$

Proof. For brevity, set $Y = X^\sigma$.

We compute $\mathsf{S}(\sigma)$ by localization with respect to the $\mathbb{C}^\times_\varepsilon$-factor in (8.7). The domain of an $\mathbb{C}^\times_\varepsilon$-fixed map in $(C, f) \in M^\sim(\beta)$ is a union

$$C = C_0 \cup C_1 \cup C_\infty$$

where $f : C_0 \to X_0$ is a σ-fixed map, $f(C_\infty) \subset X_\infty$, and C_1 is of the form (8.4)

$$C_1 = \zeta_y$$

for some point $y \in Y$.

Standard localization arguments (see e.g. Chapter 27 in [54]) give a factorization

$$\mathsf{S}(\sigma) = \Psi_0 \, \Psi_1 \, \Psi_\infty \,, \tag{8.9}$$

with the following factors. We define

$$(\gamma_1, \Psi_0 \cdot \gamma_2) = \sum_{\beta \in H_2(X_0, \mathbb{Z})} q^\beta \int_{\left[\overline{M}_{0,2}(X_0,\beta)\right]^{vir}} \frac{\mathsf{ev}^*(\gamma_1 \times \gamma_2)}{\varepsilon - \psi_2}\,. \tag{8.10}$$

Here ψ_2 is the cotangent line at the second marked point and the integral is computed in equivariant cohomology. The unstable $\beta = 0$ contributions to (8.10) are defined to give

$$\Psi_0 = 1 + O(q^\beta), \quad \beta > 0\,. \tag{8.11}$$

If we evaluate Ψ_0 via virtual localization, we obtain precisely the localization contribution of the C_0.

In the definition of Ψ_∞, one replaces $\varepsilon - \psi_2$ by $-\varepsilon - \psi_1$ and X_0 by X_∞.

Note the virtual class in (8.10) is the ordinary, nonreduced virtual fundamental class. The reduced virtual class gives

$$\Psi_0 = 1 + O(\hbar)\,, \tag{8.12}$$

if X is holomorphic symplectic.

The middle factor Ψ_1 is of the form

$$\Psi_1 = \iota_{0*}\, q^\zeta\, \Gamma\, \iota_\infty^*$$

where ι_0, ι_∞ denote the inclusion of Y into X_0 and X_∞ respectively, Γ is multiplication by a class in $H^{\cdot}(Y)$ that absorbs the deformation and obstruction contributions of C_1. The class ζ was defined in (8.5); q^ζ is a monomial which varies depending on the connected component of Y. Note that by localization

$$\varepsilon\,(\zeta, D) = \iota_0^*(D) - \iota_\infty^*(D)\,. \tag{8.13}$$

It is standard [20, 54] to abbreviate

$$\tau_k(\gamma_2) = \psi_2^k\, \mathsf{ev}^*(\gamma_2)\,.$$

The convention (8.11) means that

$$\langle \gamma_1\, \tau_k(\gamma_2)\rangle_0 = \delta_{k+1} \int_X \gamma_1 \cup \gamma_2$$

where angle brackets denote equivariant genus 0 GW-invariants of X and subscript refers to invariants of degree $\beta = 0$. With this convention, the string and divisor equations yield

$$\langle \gamma_1, \tau_k(\gamma_2), D\rangle_\beta = \left(\int_\beta D\right) \langle \gamma_1, \tau_k(\gamma_2)\rangle_\beta + \langle \gamma_1, \tau_{k-1}(\gamma_2 \cup D)\rangle_\beta\;, \tag{8.14}$$

for all $k \geqslant 0$ and all degrees β. Similarly, the topological recursion relations (see e.g. Section 26.4 in [54]) read

$$\langle \gamma_1, \tau_k(\gamma_2), D\rangle_\beta = \sum \langle \gamma_1, D, \eta\rangle_{\beta'}\, \langle \eta^\vee\, \tau_{k-1}(\gamma_2)\rangle_{\beta-\beta'} \tag{8.15}$$

for all $k \geqslant 0$ and all degree splittings, where $\sum \eta \otimes \eta^\vee$ is the Poincaré dual of the diagonal in X^2.

Combining (8.14) with (8.15) gives

$$\varepsilon \frac{\partial}{\partial D} \Psi_0 = \mathsf{M}_{D_0}(q) \circ \Psi_0 - \Psi_0 \circ \mathsf{M}_{D_0}(0) \tag{8.16}$$

where $\mathsf{M}_{D_0}(0)$ denotes the operator of classical multiplication by D_0. By the same reasoning

$$\varepsilon \frac{\partial}{\partial D} \Psi_\infty = \mathsf{M}_{D_\infty}(0) \circ \Psi_\infty - \Psi_\infty \circ \mathsf{M}_{D_\infty}(q)\,. \tag{8.17}$$

Finally, (8.13) gives

$$\varepsilon \frac{\partial}{\partial D} \Psi_1 = \mathsf{M}_{D_0}(0) \circ \Psi_1 - \Psi_1 \circ \mathsf{M}_{D_\infty}(0)\,. \tag{8.18}$$

The combination of (8.16), (8.17), and (8.18) completes the proof. □

8.3 Shift operators are quantum operators

In this section, we extend Proposition 8.2.1; as a consequence, we will see that shift operators are quantum operators after passing to the $\varepsilon = 0$ limit.

Let

$$M^{\sim}_{\bullet}(\beta) = \mathsf{ev}_{1,2}^{-1}(X_0 \times X_\infty) \subset \overline{M}_{0,3}(X^{\sim}, \beta)$$

denote the moduli space of twisted maps from last section, equipped with an extra marked point $\bullet$.

Given $\gamma \in H^{\cdot}_{\mathsf{G}^{\sim}}(X_0)$, we define the operator $\mathsf{S}_0(\sigma;\gamma)$ by

$$(\gamma_1, \mathsf{S}_0(\sigma;\gamma)\cdot\gamma_2) = \sum_{p(\beta)=[\mathbb{P}^1]} q^\beta \int_{[M^{\sim}_{\bullet}(\beta)]^{vir}} \mathsf{ev}_{1,2}^{-1}(\gamma_1 \times \gamma_2) \cup \mathsf{ev}_{\bullet}^{*}(\iota_{0,*}\gamma) \tag{8.19}$$

Lemma 8.3.1. *We have the factorization*

$$\mathsf{S}_0(\sigma;\gamma) = \mathsf{M}_\gamma(q) \circ \mathsf{S}(\sigma)$$

where $\mathsf{M}_\gamma(q)$ denotes quantum multiplication operator for γ.

Proof. We follow the approach of Proposition 8.2.1. If we compute $\mathsf{S}_0(\sigma;\gamma)$ by localization with respect to the $\mathbb{C}^\times_\varepsilon$-factor. As before, this gives a factorization

$$\mathsf{S}_0(\sigma;\gamma) = \Psi_0^\gamma\, \Psi_1\, \Psi_\infty\,, \tag{8.20}$$

where the second and third factor are as before and the first factor is defined by

$$(\gamma_1, \Psi_0^\gamma \cdot \gamma_2) = \varepsilon \cdot \sum_{\beta \in H_2(X)} q^\beta \int_{\left[\overline{M}_{0,3}(X,\beta)\right]^{vir}} \frac{\mathsf{ev}^*(\gamma_1 \times \gamma_2 \times \gamma)}{\varepsilon - \psi_2}\,. \tag{8.21}$$

When we expand this expression, the leading term with no ψ_2 is simply quantum multiplication by γ. The terms with positive powers of ψ_2 can be expanded using the topological recursion relation of (8.15) to give quantum multiplication by γ composed with the $\beta > 0$ contribution to Ψ_0.

The result is the factorization

$$\Psi_0^\gamma = \mathsf{M}_\gamma(q) \circ \Psi_0.$$

Combining with (8.9) gives the statement of the lemma. □

Given $\gamma \in H^{\cdot}_{\mathsf{G}^\sim}(X_\infty)$, we can define the operator $\mathsf{S}_\infty(\sigma;\gamma)$ in the analogous manner, and we can derive the formula

$$\mathsf{S}_\infty(\sigma;\gamma) = \mathsf{S}(\sigma) \circ \mathsf{M}_\gamma(q)$$

in the same way.

If we restrict to G^σ-equivariant cohomology by setting $\varepsilon = 0$, then for $\gamma \in H^{\cdot}_{\mathsf{G}^\sim}(X)$, we have $\iota_{0,*}\gamma = \iota_{\infty,*}\gamma$ after this specialization. In particular, we have

$$\mathsf{S}_0(\sigma;\gamma) = \mathsf{S}_\infty(\sigma;\gamma)$$

after setting $\varepsilon = 0$.

As a corollary, we see that the shift operator $\mathsf{S}(\sigma)|_{\varepsilon=0}$ commutes with all quantum multiplication operators. If we fix a splitting of (8.6), it can thus be identified with quantum multiplication by

$$\mathsf{S}(\sigma)\Big|_{\varepsilon=0}(1) \in H^{\cdot}_{\mathsf{G}^\sigma}(X) \otimes \mathbb{Q}[[\mathrm{Eff}(X)]].$$

Chapter 9

Minuscule shifts and R-matrices

9.1 Setup

9.1.1

In this chapter, we consider shift operators $\mathsf{S}(\sigma)$ satisfying the following additional assumptions:

1. X is a symplectic resolution,
2. σ preserves the symplectic form ω,
3. σ is minuscule,

see Section 2.6 for a discussion of the last condition.

9.1.2

We define

$$\mathrm{Stab}_{\pm} : H^*(X^\sigma) \to H^*(X) \,. \tag{9.1}$$

to be the stable envelope maps corresponding to the two chambers $\gtrless 0$ of $\mathrm{Lie}\,\mathbb{C}^\times_\sigma$ and an arbitrary choice of polarization. We will see a close relation between $\mathsf{S}(\sigma)$ and

$$R_\sigma = \mathrm{Stab}_-^{-1}\,\mathrm{Stab}_+ \,. \tag{9.2}$$

9.1.3

It follows from our assumptions and (8.2) that

$$c_1\big(X^{\sim}\big) = 2\,[\mathit{Fiber}]$$

and hence that

$$\text{vir}\dim M^{\sim}(\beta) = \dim X\,. \tag{9.3}$$

for all β in (8.7). This means $\mathsf{S}(\sigma)$ has cohomological degree 0.

9.1.4

Lemma 9.1.1. *With the assumptions of Section 9.1.1, all σ-weights in the normal bundle to X^σ are ± 1.*

Proof. Choose a proper map $X \to V$, where V is a linear representation of σ with weights in $\{0, \pm 1\}$. For any x, the σ-orbit of x is either contracted by the map to V or is mapped isomorphically to a line of weight ± 1.

If there is a weight $k \neq \pm 1$ in the normal bundle to some component Y of X^σ then the corresponding normal directions are mapped to a point in V. Hence, their closure meets another component Y' of X^σ, where same weight k has to occur again. Using induction on $\prec$ and finiteness of the number of component of X^σ, we see that this is impossible. □

9.1.5

Recall from (8.5) that every component of X^σ defines a curve class $H_2(X^{\sim}, \mathbb{Z})$. If we fix a splitting of (8.6), we can project to obtain curve classes in $H_2(X, \mathbb{Z})$.

A more convenient way of making this choice is as follows. Choose a σ-linearization for a basis $\mathcal{L}_1, \mathcal{L}_2, \dots$ of $\operatorname{Pic}(X)$. Given $x \in X^\sigma$, we define $\overline{\zeta_x} \in H_2(X, \mathbb{Z})$ so that

$$\int_{\overline{\zeta_x}} c_1(\mathcal{L}_i) = \deg \widetilde{\mathcal{L}}_i\Big|_{\zeta_x}$$

where ζ_x is the section (8.4) and $\widetilde{\mathcal{L}}_i$ is the lift of $\mathcal{L}_i$ to $X^{\sim}$ that uses the fixed linearization. A change of linearization adds a overall constant to $\overline{\zeta}$.

For Nakajima varieties, the entire group of automorphisms G acts naturally on all tautological bundles and their associated determinant bundles.

In this case, we can arrange for the linearization of $\mathrm{Pic}(X)$ to be compatible with this natural linearization. If we know that these tautological divisor classes span $\mathrm{Pic}(X)$, this completely determines the linearization and thus gives a preferred normalization of the map $\overline{\zeta}$.

9.1.6

In particular, take σ to be the action corresponding to a tensor product of Nakajima varieties, that is to

$$\mathsf{w} = z\,\mathsf{w}' + \mathsf{w}''\,, \quad \mathsf{v} = z\,\mathsf{v}' + \mathsf{v}''\,,$$

as in Section 2.4. Then

$$(\overline{\zeta}, c_1(\mathcal{V}_i)) = \mathsf{v}'_i\,.$$

In other words, connected components of X^σ are distinguished by the value of v' and

$$\begin{array}{ccc} \overline{\zeta} & \longmapsto & H_2(\mathcal{M}_{\theta,\zeta}(\mathsf{v},\mathsf{w}),\mathbb{Z}) \\ \Big\downarrow & & \Big\downarrow \\ \mathsf{v}' & \longmapsto & \mathfrak{h}^* \end{array} \tag{9.4}$$

under the natural map on the right.

For example, for $T^*\mathbb{P}^1$ as in Section 8.1.6, we get

$$q^{\overline{\zeta}} = \begin{pmatrix} q & \\ & 1 \end{pmatrix}$$

in the basis $\{x_0, x_\infty\}$.

9.1.7

Recall that $\boldsymbol{\kappa}$ is defined as $c_1(T^{\mathbf{1/2}})$ modulo 2, where $T^{\mathbf{1/2}}$ is a half of tangent bundle as in (2.7).

Lemma 9.1.2. *For minuscule σ*

$$(\zeta, \boldsymbol{\kappa}) = \tfrac{1}{2}\operatorname{codim} X^\sigma \mod 2\,.$$

Proof. Since σ is minuscule, the σ-weights of $T^{\mathbf{1/2}}$ when restricted to fixed loci lie in the set $\{0, \pm 1\}$. Moreover, the number of nonzero weights equals $\frac{1}{2}\operatorname{codim} X^\sigma$. Therefore, the bundle $\widetilde{T^{\mathbf{1/2}}}$ restricted to ζ_x is a sum of $\mathcal{O}(k)$, $k \in \{0, \pm 1\}$, and the number of nontrivial terms in this sum equals $\frac{1}{2}\operatorname{codim} X^\sigma$. □

9.2 Properness

Proposition 9.2.1. *For X and σ be as above, the convolution*

$$\mathrm{Stab}_-^\tau \circ \mathsf{S}(\sigma) \circ \mathrm{Stab}_- \tag{9.5}$$

is proper and defines a Steinberg correspondence in $X^\sigma \times_{X_0} X^\sigma$.

Proof. Since σ is minuscule, we have proper $\mathbb{C}_\sigma^\times$-equivariant maps

$$X \to X_0 \subset V = \bigoplus_{|i| \leqslant 1} V_i \,,$$

where $\mathbb{C}_\sigma^\times$ acts on V_i with weight i. Applying (8.1), we get a proper map

$$X^\sim \to \widetilde{V} = \bigoplus_{|i| \leqslant 1} V_i \otimes \mathcal{O}(i) \,,$$

to a vector bundle $\widetilde{V}$ over $\mathbb{P}^1$. Moreover, the fiberwise image of Stab_- is contained in the subbundle

$$\widetilde{V}_{\leqslant 0} = V_0 \otimes \mathcal{O} \oplus V_{-1} \otimes \mathcal{O}(-1) \subset \widetilde{V} \,.$$

Now let

$$(x_4, x_3, x_2, x_1) \in X^\sigma \times X \times X \times X^\sigma \tag{9.6}$$

be a quadruple of points in the definition of the convolution (9.5). Since $\pi(x_2), \pi(x_3) \in V_{\leqslant 0}$ and $\mathcal{O}(-1)$ has no sections, we must have

$$\pi(x_2), \pi(x_3) \in V_0 \,.$$

Moreover, by Proposition 3.5.2, we have

$$\pi(x_4) = \pi(x_3) \,, \quad \pi(x_2) = \pi(x_1) \,.$$

Since a section of $\mathcal{O}(i)$ is fixed by evaluation at $i+1$ points, we conclude

$$\pi(x_4) = \pi(x_3) = \pi(x_2) = \pi(x_1)$$

and any section of $X^\sim$ connecting x_3 and x_2 maps to the corresponding constant section of $\widetilde{V}$. Since the fibers of this projection are proper, the proposition follows. □

9.3 Computation of $\mathsf{S}(\sigma)$

9.3.1

Let

$$R_{\sigma,\infty} \in \operatorname{End} H^{\cdot}_{\mathsf{G}^{\sim}}(X^{\sigma}_{\infty}) \otimes \mathbb{Q}(\operatorname{Lie}\mathsf{G}^{\sim})$$

be the $\mathsf{G}^{\sim}$-equivariant R-matrix for the fiber X_{∞}. Since $\mathbb{C}^{\times}_{\varepsilon}$ acts on X_{∞} via the cocharacter σ^{-1}, $R_{\sigma,\infty}$ is obtained from the G^{σ}-equivariant R-matrix by the substitution

$$\xi \mapsto \xi - \varepsilon\sigma(1)\,, \quad \xi \in \operatorname{Lie}\mathsf{G}^{\sigma}\,.$$

Theorem 9.3.1. *For X and σ as in Section 9.1.1, and ζ as in Section 9.1.5,*

$$\operatorname{Stab}_{+}^{-1}\,\mathsf{S}(\sigma)\,\operatorname{Stab}_{+} = (-1)^{(\zeta,\boldsymbol{\kappa}_X)}\,q^{\zeta}\,R_{\sigma,\infty}\,. \tag{9.7}$$

Proof. Using (9.2) and Lemma 9.1.2 we may restate this as commutativity of the following diagram

$$\begin{array}{ccc}
H^{\cdot}_{\mathsf{G}^{\sim}}(X^{\sigma}_{\infty}) & \xrightarrow{(-1)^{\operatorname{codim}/2}\,q^{\zeta}} & H^{\cdot}_{\mathsf{G}^{\sim}}(X^{\sigma}_{0}) \\
\Big\downarrow{\scriptstyle \operatorname{Stab}_{-}} & & \Big\uparrow{\scriptstyle \operatorname{Stab}^{\tau}_{-}} \\
H^{\cdot}_{\mathsf{G}^{\sim}}(X_{\infty}) & \xrightarrow{\mathsf{S}(\sigma)} & H^{\cdot}_{\mathsf{G}^{\sim}}(X_{0})
\end{array}$$

where $\operatorname{codim}/2$ denotes the locally constant function on X^{σ} given by taking half the codimension in X. By Proposition 9.2.1, we may compute the composition

$$\operatorname{Stab}^{\tau}_{-} \circ \mathsf{S}(\sigma) \circ \operatorname{Stab}_{-}$$

with any choice of equivariant parameters.

We choose $\hbar = 0$ and $\varepsilon \to \infty$, where ε is the equivariant parameter for the $\mathbb{C}^{\times}_{\varepsilon}$-action in (8.3). In particular, since $\hbar = 0$, stable envelopes are diagonal and we must have

$$x_4 = x_3, \quad x_2 = x_1\,,$$

in (9.6) above. Also $\hbar = 0$ implies

$$\Psi_0 = \Psi_{\infty} = 1$$

by (8.12) above.

Next consider the operator Ψ_1. It counts constant sections C_1 of $X^\sim$ corresponding to

$$x_4 = x_3 = x_2 = x_1 \in X^\sigma .$$

By Lemma 9.1.1, the normal bundle to C_1 is

$$\mathcal{N} = N_1(1) \oplus N_{-1}(-1) \oplus T_{x_1} X^\sigma$$

where $N_{\pm 1}$ are σ-eigenspaces in the normal bundle N to X^σ in X and twists are by $\mathcal{O}(i)$, $i = \pm 1$.

It follows that $C_1 \in M^\sim(\zeta)$ is unobstructed with tangent space

$$T_{C_1} M^\sim(\zeta) = (N_1)_0 \oplus (N_1)_\infty \oplus T_{x_1} X^\sigma$$

where the subscripts $0, \infty \in \mathbb{P}^1$ denote the fibers of $N_+(1)$ over the respective points. We observe that these correspond precisely to the normal directions to $\mathrm{Stab}_-(x_i)$.

In the end, all contributions to the integral cancel except for the sign in the definition of adjoint Stab_-^τ. This sign gives $(-1)^{\mathrm{codim}/2}$. □

9.3.2

In particular, for σ as in Section 9.1.6, we have

$$X^\sigma = \coprod \mathcal{M}(\mathsf{v}', \mathsf{w}') \times \mathcal{M}(\mathsf{v}'', \mathsf{w}'')$$

and

$$q^{\bar{\zeta}} \mapsto q^{\mathsf{v}'} = q^{\mathsf{v}} \otimes 1 \,,$$

after restricting to functions on $\mathfrak{h}$ as in (9.4). Our computation of $\mathsf{S}(\sigma)$ together with the results of Section 8.3 imply the following

Corollary 9.3.2. *For tensor products of Nakajima varieties, the operator*

$$(q^{\mathsf{v}} \otimes 1)\, R(u)$$

belongs to the algebra of modified operators of quantum multiplication.

Chapter 10

Quantum multiplication by divisors

10.1 Classical multiplication by divisors

10.1.1

A vector $\lambda \in \mathfrak{h} = \mathbb{C}^I$ corresponds to a divisor

$$c_1(\lambda) = \sum \lambda^i \, c_1(\mathcal{V}_i) \tag{10.1}$$

which we identify with the corresponding cup product operator. Using

$$[e_\alpha, e_{-\alpha}] = h_\alpha \tag{10.2}$$

we obtain from (4.35)

$$c_1(\mathsf{w}) = \mathsf{E}\left(|\mathsf{w}\rangle\langle\mathsf{w}| \, u^2\right) + \hbar \sum_{\substack{\alpha>0 \\ \theta\cdot\alpha<0}} \alpha(\mathsf{w}) \, e_\alpha e_{-\alpha} + \dots \tag{10.3}$$

where the sum over roots is with multiplicity,

$$\alpha(\mathsf{w}) = \mathsf{w}(h_\alpha) = \alpha \cdot \mathsf{w} \,,$$

and dots stand for a quadratic polynomial in v, that is, an element of $\mathcal{U}(\mathfrak{h})$ of degree at most two.

10.1.2

Since $\mathsf{E}\,(|\mathsf{w}\rangle\langle\mathsf{w}|\,u^2)$ comes from the $1/u^2$ coefficient of the R-matrix, its coproduct will involve itself and the $1/u$ coefficient, that is, the classical **r**-matrix. From Theorem 5.3.11, we compute

$$\Delta\,\mathsf{E}\left(|\mathsf{w}\rangle\langle\mathsf{w}|\,u^2\right) = \mathsf{E}\left(|\mathsf{w}\rangle\langle\mathsf{w}|\,u^2\right)\otimes 1 + 1\otimes\mathsf{E}\left(|\mathsf{w}\rangle\langle\mathsf{w}|\,u^2\right) + \dots \\ + \hbar\sum_{\alpha,\beta}\langle\mathsf{w}|\,e_{-\beta}e_{\alpha}\,|\mathsf{w}\rangle\; e_{-\alpha}\otimes e_{\beta}\,, \quad (10.4)$$

where dots stand for terms in $\mathcal{U}(\mathfrak{h})^{\otimes 2}$. Using (10.2), we compute

$$\langle\mathsf{w}|\,e_{-\beta}e_{\alpha}\,|\mathsf{w}\rangle = \begin{cases}-\alpha\cdot\mathsf{w}\,, & \beta=\alpha>0\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

We deduce the following

Theorem 10.1.1. *We have*

$$\Delta c_1(\lambda) = c_1(\lambda)\otimes 1 + 1\otimes c_1(\lambda) - \hbar\sum_{\theta\cdot\alpha>0}\alpha(\lambda)\,e_{-\alpha}\otimes e_{\alpha} + \dots$$

where the sum is over roots α of $\mathfrak{g}_Q$ with multiplicities and dots stand for terms in $\mathcal{U}(\mathfrak{h})^{\otimes 2}$.

10.1.3

In particular, we have

$$R(u)\,\Delta c_1(\lambda)\,R(u)^{-1} = \Delta^{\mathrm{op}}c_1(\lambda) \\ = \Delta\,c_1(\lambda) + \hbar\sum_{\theta\cdot\alpha>0}\alpha(\lambda)\,(e_{-\alpha}\otimes e_{\alpha} - e_{\alpha}\otimes e_{-\alpha})\,. \quad (10.5)$$

10.2 Quantum operators

10.2.1

We denote by $\mathsf{Q}(\lambda)$ the operator of modified quantum multiplication by the divisor (10.1). By construction

$$\mathsf{Q}(\lambda) = c_1(\lambda) + \hbar\sum_{\beta}(-1)^{(\beta,\kappa)}\,q^{\beta}\,\lambda(\beta)\,\mathsf{Q}_{2,\mathrm{red}}(\beta)\,,$$

where $\beta \in H_2(\mathcal{M}, \mathbb{Z})$ is an effective curve class and $\mathsf{Q}_{2,\mathrm{red}}(\beta)$ is the image of the corresponding reduced virtual class under the evaluation map. The quantum part of $\mathsf{Q}(\lambda)$ is a linear combination of Steinberg correspondences.

10.2.2

Theorem 10.2.1. *We have*

$$\mathsf{Q}(\lambda) = c_1(\lambda) - \hbar \sum_{\theta\cdot\alpha>0} \alpha(\lambda) \, \frac{q^\alpha}{1-q^\alpha} \, e_\alpha e_{-\alpha} + \dots$$

where the sum is over roots of $\mathfrak{g}_Q$ with multiplicity and dots denote a scalar operator.

The scalar operator is fixed by the requirement that the purely quantum part of $\mathsf{Q}(\lambda)$ annihilates the identity.

Proof. For brevity, we write $\mathsf{Q} = \mathsf{Q}(\lambda)$.

Let $\Delta\mathsf{Q}$ be the pullback of the operator Q under the stable envelope map

$$H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathcal{M}(\mathsf{w}')) \otimes H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathcal{M}(\mathsf{w}'')) \to H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathcal{M}(\mathsf{w}' + \mathsf{w}'')) \,.$$

We can decompose it

$$\Delta\mathsf{Q} = \sum_\alpha \Delta_\alpha \mathsf{Q}$$

according to the weights of $\mathfrak{h} \otimes 1$. Here

$$[h \otimes 1, \Delta_\alpha \mathsf{Q}] = \alpha(h) \, \Delta_\alpha \mathsf{Q} \,.$$

In other words, $\Delta_\alpha \mathsf{Q}$ increases v' by α. By Proposition 7.5.1 and Theorem 10.1.1, we have

$$\Delta_0 \mathsf{Q} = \mathsf{Q} \otimes 1 + 1 \otimes \mathsf{Q} \,.$$

By Corollary 9.3.2,

$$[(q^{\mathsf{v}} \otimes 1) \, R(u), \Delta\mathsf{Q}] = 0 \,,$$

which means

$$R(u) \, \Delta\mathsf{Q} \, R(u)^{-1} = \sum q^{-\alpha} \, \Delta_\alpha \mathsf{Q} \,.$$

The purely quantum part in Q is a Steinberg correspondence, hence commutes with R-matrices. Taking into account the classical part, we get from (10.5)

$$\sum_\alpha (1 - q^{-\alpha}) \, \Delta_\alpha \mathsf{Q} = \hbar \sum_{\theta\cdot\alpha>0} \alpha(\lambda) \left(e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha \right) \,,$$

which uniquely determines all $\Delta_\alpha \mathsf{Q}$ with $\alpha \neq 0$.

Now consider

$$\mathsf{Q}_{\text{remainder}} = \sum_{\beta} (-1)^{(\beta,\kappa)}\, q^\beta\, \lambda(\beta)\, \mathsf{Q}_{2,\text{red}}(\beta) + \sum_{\theta\cdot\alpha>0} \alpha(\lambda)\, \frac{q^\alpha}{1-q^\alpha}\, e_\alpha e_{-\alpha}\,.$$

By Lemma 5.3.12, this is a Steinberg correspondence. Moreover, it commutes with $\mathfrak{h}$ and is primitive in the sense that

$$\Delta \mathsf{Q}_{\text{remainder}} = \mathsf{Q}_{\text{remainder}} \otimes 1 + 1 \otimes \mathsf{Q}_{\text{remainder}}\,.$$

The following Proposition finishes the proof. □

10.2.3

Proposition 10.2.2. *Let Θ be a family of Steinberg correspondences*

$$\Theta_{\mathsf{v},\mathsf{w}} \subset \mathcal{M}_{\theta,0}^{\times 2}$$

defined for all v,w. If it is primitive

$$\Delta\Theta = \Theta \otimes 1 + 1 \otimes \Theta$$

and commutes with $\mathfrak{h}$ then $\Theta \in \overline{\mathfrak{h}}_Q$.

Recall that $\overline{\mathfrak{h}}_Q$ acts by multiplication by linear function of v and w. Again, by Δ in the above proposition, we mean the pullback of Θ under the stable envelope map.

Proof. By hypothesis, Θ preserves the decomposition

$$H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\mathsf{w})) = \bigoplus_{\mathsf{v}} H^{\cdot}_{\mathsf{G}}(\mathcal{M}(\mathsf{w},\mathsf{v}))$$

into $\overline{\mathfrak{h}}_Q$-weight subspaces. In particular,

$$[\Theta, |\mathsf{w}\rangle\langle\mathsf{w}|] = 0 \tag{10.6}$$

where $|\mathsf{w}\rangle\langle\mathsf{w}|$ is the projector onto the $\mathsf{v}=0$ part.

Since Θ is a Steinberg correspondence on each component,

$$[R(u), \Delta\Theta] = 0$$

This and (10.6) implies

$$\left[\operatorname{tr}_1\left((|\mathsf{w}\rangle\langle\mathsf{w}|\otimes 1)\circ R(u)\right),\Theta\right]=0$$

where the trace is over the first tensor factor and Θ acts in the second tensor factor. By the results of Section 4.7, this means that $\Theta_{\mathsf{v},\mathsf{w}}$ commutes with operators of classical multiplication by all characteristic classes of the tautological bundles. Proposition 5.4.1 implies that $\Theta_{\mathsf{v},\mathsf{w}}$ is itself an operator of classical multiplication. Since it has cohomological degree 0, it can only be a multiple of the identity. The primitivity condition forces this multiple to be a linear function of v and w. □

Chapter 11

Cotangent bundles of Grassmannians

In this chapter, we illustrate the general theory for the simplest possible quiver Q — that with one vertex and no arrows. The corresponding Nakajima varieties are cotangent bundles of Grassmann varieties.

Grassmann varieties are among the oldest objects of study in algebraic geometry; in particular, their quantum cohomology has been described by many authors from many different angles, see e.g. [7, 15, 16, 49, 63, 78, 105, 110]. The modest goal of this chapter is to help the reader align his favorite point of view on Grassmannians with the direction of this paper.

11.1 Quantum cohomology

11.1.1 Setup

For the quiver Q with one vertex and no arrows, the Nakajima quiver data is a pair of matrices

$$\mathbb{C}^n \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{C}^k$$

where $\mathbb{C}^n = \mathbb{C}^{\mathsf{w}_1}$ is the framing space and $k = \mathsf{v}_1$. Let X be the corresponding quiver variety

$$X = \mathcal{M}_{\theta,0}(k,n) = \{(A,B), AB = 0\} /\!\!/_{\theta}\, GL(k)\,,$$

where

$$\text{stable points} = \begin{cases} \operatorname{rk} A = k\,, & \theta > 0\,, \\ \operatorname{rk} B = k\,, & \theta < 0\,. \end{cases}$$

In either case, $X = \varnothing$ if $k > n$. The map

$$(A, B) \mapsto L = \begin{cases} \operatorname{Ker} A\,, & \theta > 0\,, \\ \operatorname{Im} B\,, & \theta < 0\,, \end{cases}$$

makes X a vector bundle over the Grassmannian

$$\mathsf{Gr} = \begin{cases} Gr(n-k, n)\,, & \theta > 0\,, \\ Gr(k, n)\,, & \theta < 0 \end{cases}$$

of possible $L \subset \mathbb{C}^n$. The fiber of this vector bundle is $\operatorname{Hom}(\mathbb{C}^n/L, L)$, whence

$$X = T^*\mathsf{Gr}\,.$$

Of course, Grassmann varieties of complementary dimension are isomorphic, but this isomorphism is not canonical, in particular not $GL(n)$-equivariant. Here we are interested in G-equivariant quantum cohomology of X, where

$$\mathsf{G} = GL(n) \times \mathbb{C}^\times_\hbar\,.$$

The second factor in G scales the cotangent directions with weight $-\hbar$.

11.1.2 Divisors

The tautological bundle $\mathcal{V} = \mathcal{V}_1$ is identified as follows

$$\mathcal{V} = \begin{cases} \mathbb{C}^n/L\,, & \theta > 0\,, \\ L\,, & \theta < 0\,, \end{cases}$$

that is, $\mathcal{V}$ is the universal quotient bundle for $\theta > 0$ and the universal subbundle for $\theta < 0$. The line bundle

$$\mathcal{O}(1) = \left(\Lambda^{\mathrm{top}}\mathcal{V}\right)^{\operatorname{sgn}\theta}$$

is the very ample generator of $\operatorname{Pic} X$. The corresponding projective embedding of the Grassmannian is classically known as the Plücker embedding.

It is elementary to see that $c_1(\mathcal{O}(1))$ generates $H^{\cdot}_{\mathsf{G}}(X)$. Therefore quantum multiplication by this class uniquely determines the algebra of quantum multiplication.

11.1.3 The affine quotient

Let

$$\pi : X \to X_0$$

be the affinization of X. Its target X_0 may be described in terms of square-zero matrices D, or differentials. Let

$$\mathfrak{D} = \left\{ D \,\middle|\, D^2 = 0 \right\} \subset \operatorname{End} \mathbb{C}^n \,.$$

denote the set of square-zero matrices. It is stratified by $GL(n)$-orbits

$$\mathfrak{D}_r = \{ \operatorname{rk} D = r \} \,, \quad r = 0, 1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor \,. \tag{11.1}$$

The map

$$(A, B) \mapsto D = BA$$

gives

$$X_0 \cong \mathfrak{D}_{\leqslant r} \,, \quad r = \min(k, n-k) \,.$$

The fibers of π are Grassmann varieties, namely

$$\pi^{-1}(D) \cong \left\{ L \,\middle|\, \operatorname{Im} D \subset L \subset \operatorname{Ker} D \right\} \,.$$

In particular, $\pi^{-1}(0) = \mathsf{Gr}$.

11.1.4 The Steinberg variety

By definition, the Steinberg variety is

$$\mathfrak{S} = X \times_{X_0} X \,.$$

The stratification (11.1) gives a decomposition into irreducible components

$$\mathfrak{S} = \bigcup_d \mathfrak{S}_d \,,$$

where $\mathfrak{S}_d$ is the closure of $X \times_{\mathfrak{D}_{r-d}} X$. In particular,

$$\mathfrak{S}_0 = \text{diagonal} \,,$$
$$\mathfrak{S}_r = \mathsf{Gr} \times \mathsf{Gr} \,.$$

For us, the most important stratum is $\mathfrak{S}_1$.

11.1.5 Lines on X

Let $\ell \in H_2(\mathsf{Gr}, \mathbb{Z})$ be the effective generator. Curves of class ℓ are lines in the Plücker embedding. Two points $L_1 \neq L_2 \in \mathrm{Gr}$ lie on a line ℓ_{L_1,L_2} if

$$\dim L_1 \cap L_2 = \dim L_1 - 1\,,$$

in which case

$$\ell_{L_1,L_2} = \left\{ L \,\middle|\, L_1 \cap L_2 \subset L \subset L_1 + L_2 \right\}\,.$$

Lines on X are the lines in the fibers of π. Therefore $\mathfrak{S}_1$ is formed by pairs of points that lie on a line.

11.1.6 Torus-fixed curves

Let $\mathsf{A} \subset GL(n)$ be the diagonal torus. Since $X_0^{\mathsf{A}} = \{0\}$, we have

$$X^{\mathsf{A}} = \mathsf{Gr}^{\mathsf{A}}\,.$$

This is a finite set formed by coordinate subspaces

$$L_S = \bigoplus_{s \in S} \mathbb{C} e_s$$

where $\{e_1, \dots, e_n\} \in \mathbb{C}^n$ is the coordinate basis and $S \subset \{1, \dots, n\}$ ranges over subsets of cardinality $\dim L$.

The set of reduced irreducible A-invariant curves in X is also finite, formed by lines $\ell_{S,S'}$ joining fixed points L_S and $L_{S'}$ with $|S \triangle S'| = 2$. Their tangent A-weights have the form

$$\pm(a_i - a_j)\,, \quad \{i, j\} = S \triangle S'\,,$$

from which one concludes the following

Lemma 11.1.1. *The only unbroken* A*-fixed chains in* X *are covers of lines branched over fixed points.*

11.1.7 Quantum product by divisor

For $d = 1, 2, \dots$ let

$$\mathsf{Q}_d \subset H_{\mathrm{middle}}(X \times X)$$

be the following Steinberg correspondence

$$\mathsf{Q}_d = d\,(-1)^{nd}\,\mathsf{ev}_*\left[\overline{M}_{0,2}(X,d\ell)\right]_{\text{virtual,reduced}}\,. \tag{11.2}$$

The sign $(-1)^{nd}$ is taken from the definition of modified Gromov-Witten invariants, that is, it comes from pairing $d\ell$ with

$$\kappa_X = c_1(\mathsf{Gr}) = n\,\mathcal{O}(1)\,.$$

The factor of d is introduced in (11.2) so that

$$\mathsf{Q}_{\text{quantum}} = \hbar \sum_{d>0} q^{d\ell}\,\mathsf{Q}_d \tag{11.3}$$

is the modified purely quantum multiplication by $\mathcal{O}(1)$.

Proposition 11.1.2. *For all $d > 0$ we have*

$$\mathsf{Q}_d = \mathsf{Q}_1 = \pm\mathfrak{S}_1\,. \tag{11.4}$$

Proof. As a first step, we compute the push-forward (11.2) modulo terms supported on the diagonal. We do this by A-equivariant localization.

Recall that only unbroken maps contribute to localization of reduced virtual classes. Suppose the marked points of an unbroken map f evaluate to distinct points of X^{A}. Then by Lemma 11.1.1 f has the form

$$f : \mathbb{P}^1 \xrightarrow{z\mapsto z^d} \ell_{S,S'} \subset \mathsf{Gr}\,,$$

ramified over $L_S, L_{S'} \in \mathsf{Gr}^{\mathsf{A}}$. In particular

$$\operatorname{Aut} f = \mathbb{Z}/d\,,$$

and hence f contributes

$$-(-1)^{nd}\,\text{Euler}'\,H^{\cdot}(f^*TX)^{-1} \in \mathbb{Q}(\mathfrak{a})\,,$$

to localization of Q_d. Here Euler$'$ is the product of nonzero A-weights in the virtual A-module $H^{\cdot}(f^*TX)$.

To be precise, there are two zero weights in this module. One occurs in $H^0(f^*T\ell_{S,S'})$ and is taken out by the automorphism of a 2-pointed $\mathbb{P}^1$. The other occurs in $H^1(f^*T^*\ell_{S,S'})$ and is taken out by passing to the reduced invariants. The minus sign appears because $T^*\ell_{S,S'}$ has weight $-\hbar$ under the $\mathbb{C}^\times_\hbar$-action while in (11.3) we take out a factor of $\hbar$.

Since

$$f^*TX = \mathcal{T} \oplus \mathcal{T}^*\,, \quad \mathcal{T} = f^*T\mathsf{Gr}\,,$$

Lemma 11.1.3 below shows

$$\frac{\mathsf{Q}_d\big|_p}{\operatorname{Euler} T_p X \times X} = \frac{(-1)^{\dim \mathsf{Gr}}}{\operatorname{Euler} T_p \mathsf{Gr} \times \mathsf{Gr}} \tag{11.5}$$

for any off-diagonal $p \in \mathfrak{S}_1^{\mathsf{A}}$, that is, for any

$$p = (L_S, L_{S'})\,, \quad |S \triangle S'| = 2\,.$$

This proves (11.4) modulo a class supported on the diagonal.

To show the contribution of the diagonal vanishes, it suffices to note that Q_1 annihilates the identity in cohomology and so does $\mathfrak{S}_1$. Indeed, the fibers of π are positive-dimensional over $\mathfrak{D}_{r-1}$. □

Lemma 11.1.3. *Let* A *be a torus and let* $\mathcal{T}$ *be an* A*-equivariant bundle on* $\mathbb{P}^1$ *without zero weights in the fibers* $\mathcal{T}_0, \mathcal{T}_\infty$ *over fixed points. Then*

$$\operatorname{Euler}' H^{\cdot}(\mathcal{T} \oplus \mathcal{T}^*) = (-1)^{\deg \mathcal{T} + \operatorname{rk} \mathcal{T} + \#z} \operatorname{Euler}(\mathcal{T}_0 \oplus \mathcal{T}_1)\,,$$

where $\#z = \dim H^1(\mathcal{T} \oplus \mathcal{T}^*)^{\mathsf{A}}$.

The sign in (11.4) is easily determined from (11.5), but we will not need it in what follows.

11.2 The stable basis

11.2.1 Tensor product structure

As usual, we define

$$\mathcal{M}(n) = \bigsqcup_k \mathcal{M}_{\theta,0}(k,n)\,.$$

The A-action makes $\mathcal{M}(n)$ a tensor product

$$\mathcal{M}(n) = \mathcal{M}(1)^{\otimes n}\,, \quad \mathcal{M}(1) = 2 \text{ points}\,.$$

We write

$$H^{\cdot}(\mathcal{M}(1),\mathbb{Q}) = \mathbb{Q}\,|0\rangle \oplus \mathbb{Q}\,|1\rangle = \mathbb{Q}^2\,,$$

where

$$\mathsf{v}\,|k\rangle = k\,|k\rangle\,.$$

Similarly,

$$H^{\cdot}\left(\mathcal{M}(n)^{\mathsf{A}}\right) = \left(\mathbb{Q}^2\right)^{\otimes n} = \bigoplus_{S\subset\{1,\dots,n\}} \mathbb{Q}\,|S\rangle$$

where we identify

$$\text{subsets of } \{1,\dots,n\} \leftrightarrow \{0,1\}^n$$

using indicator functions. In G-equivariant cohomology, we replace $\mathbb{Q}$ above with the equivariant cohomology ring of a point.

11.2.2 Polarization

Recall from Example 3.3.3 that we have a canonical choice for polarization of any Nakajima variety. In the case at hand, this gives

$$\begin{aligned}\mathrm{Stab}_{\mathfrak{C}}\,|S\rangle\Big|_{L_S} &= \mathrm{Euler}\,\mathrm{Hom}(\mathcal{V},\mathbb{C}^n\ominus\mathcal{V})\\ &= (\pm 1)^{\dim\mathsf{Gr}}\,\mathrm{Euler}\,T_{L_S}\mathsf{Gr}\,,\end{aligned} \tag{11.6}$$

depending on the sign of θ. Here the Euler class is the product of A-weights.

Note the two possibilities in (11.6) differ by an overall scalar, which means that all geometric operators act canonically in the stable basis.

11.2.3 Classical r-matrix

We claim

$$\mathfrak{g}_Q = \mathfrak{gl}(2)$$

with its natural action on $\mathbb{Q}^2$ and, by tensor product, on $H^{\cdot}\left(\mathcal{M}(n)^{\mathsf{A}}\right)$. Indeed, the classical $\mathbf{r}$-matrix is computed as follows in terms of the matrix units $e_{ij}\in\mathfrak{gl}(2)$.

Proposition 11.2.1.

$$\mathbf{r} = e_{00}\otimes e_{11} + e_{11}\otimes e_{00} - e_{01}\otimes e_{10} - e_{10}\otimes e_{01}\,.$$

Proof. For $\mathcal{M}(1) \otimes \mathcal{M}(1)$ this was computed in Section 4.1.2. In general, it follows by additivity of the classical **r**-matrix. □

Other ways to write the **r**-matrix include

$$\begin{aligned}\mathbf{r} &= \mathsf{w} \otimes \mathsf{w} - \sum_{ij} e_{ij} \otimes e_{ji} \\ &= -e \otimes f - f \otimes e + \dots ,\end{aligned}$$

where

$$e = e_{10}\,, \quad f = e_{01}\,, \quad \mathsf{w} = e_{00} + e_{11}\,,$$

and dots stand for a diagonal operator.

11.2.4 Quantum multiplication in stable basis

Recall the operators Q_d are Steinberg correspondences. Therefore, by Theorem 4.6.1, their action in the stable basis does not depend on the choice of a chamber $\mathfrak{C}$ for A.

The following Proposition gives a direct verification of Theorem 10.2.1 for cotangent bundles of Grassmannians.

Proposition 11.2.2. *We have*

$$\mathsf{Q}_{\text{quantum}} = \hbar \frac{q^\ell}{1-q^\ell}\, ef + \dots$$

where dots stand for a diagonal operator.

Proof. By Proposition 11.1.2, the statement to prove is

$$\mathsf{Q}_1 = ef + \dots .$$

Since $\dim \mathcal{M}^{\mathsf{A}} = 0$, theorem 4.4.1 gives

$$\mathsf{Q}_{1,\mathsf{A}} \left|S\right\rangle = \sum_{S'} (-1)^{\dim \mathsf{Gr}} \big(\operatorname{Stab}_{-\mathfrak{C}} \left|S'\right\rangle \otimes \operatorname{Stab}_{\mathfrak{C}} \left|S\right\rangle, \mathsf{Q}_1 \big) \left|S'\right\rangle \tag{11.7}$$

The coefficient in (11.7) may be computed using (11.6) and (11.5) and recall that we can set $\hbar = 0$ in this computation, which makes stable envelopes diagonal. For either sign of θ, this gives

$$\left\langle S'\right| \mathsf{Q}_{1,\mathsf{A}} \left|S\right\rangle = \begin{cases} 0\,, & |S \triangle S'| > 2\,, \\ 1\,, & |S \triangle S'| = 2\,, \end{cases}$$

proving the proposition. □

11.2.5

It is an interesting combinatorial and geometric question to compute the transition matrix between the stable basis and the fixed-point basis in $H^{\cdot}_{\mathsf{G}}(X)$.

In the quantum integrable system language, the fixed-point basis corresponds to the eigenbasis at $q = 0$, while the stable basis is the coordinate basis, that is, the spin basis of the spin chain for $X = T^*\mathsf{Gr}$. Thus, the question is equivalent to explicit diagonalization of the Hamiltonian at $q = 0$.

For the inhomogeneous XXX spin chain, the answer was known to Nekrasov and Shatashvili. The corresponding symmetric functions are rational analogs of the interpolation Schur functions. Just like Schur functions may be deformed to Macdonald polynomials associated to root systems of type A and, more generally, to nonreduced BC root systems, these rational interpolation Schur functions naturally lie in the family of special functions studied by E. Rains in [104].

In [107], D. Shenfeld shows how this identification is a example of the general abelianization procedure for stable bases.

11.3 Yangian action

11.3.1 The Yangian of $\mathfrak{gl}(2)$

Yangians of finite-dimensional Lie algebras have been studied in great detail, see for example the exposition in [17, 33, 82, 83]. We recall $\mathsf{Y}(\mathfrak{gl}(2))$ is generated by countably many generators, the coefficients $\mathsf{E}_{ij}^{(k)}$ in the generating series

$$\mathsf{E}_{ij}(u) = \delta_{ij} + \sum_{k>0} \frac{\mathsf{E}_{ij}^{(k)}}{u^k}, \quad i, j \in \{1, 2\},$$

subject to the RTT=TTR relations. These relations are written in terms of the matrix

$$\mathsf{E}(u) = \begin{pmatrix} \mathsf{E}_{11}(u) & \mathsf{E}_{12}(u) \\ \mathsf{E}_{21}(u) & \mathsf{E}_{22}(u) \end{pmatrix} \in \operatorname{End} \mathbb{Q}^2 \otimes \mathsf{Y}(\mathfrak{gl}(2))[[u^{-1}]]$$

and have the form

$$R(u-v)\, \mathsf{E}(u)\, \mathsf{E}(v) = \mathsf{E}(v)\, \mathsf{E}(u)\, R(u-v) \,. \tag{11.8}$$

The equality in (11.8) is an equality in

$$(11.8) \in \operatorname{End}(\mathbb{Q}^2 \otimes \mathbb{Q}^2) \otimes \mathsf{Y}(\mathfrak{gl}(2))[[u^{-1}, v^{-1}]] \,.$$

The R-matrix in (11.8) is

$$R(u) = \Big(1 - \frac{\mathbf{s}}{u}\Big) \Big/ \Big(1 - \frac{1}{u}\Big) \in \operatorname{End}(\mathbb{Q}^2 \otimes \mathbb{Q}^2)[[u^{-1}]] \,,$$

where $\mathbf{s}$ is the permutation of tensor factors. The scalar factor, which plays no role in (11.8), is chosen here so that $R(u)$ equals the R-matrix for $\mathcal{M}(1) \otimes \mathcal{M}(1)$ computed in (4.1) for $\hbar = 1$.

11.3.2 Evaluation representation

Consider the map $\mathsf{Y}(\mathfrak{gl}(2)) \to \operatorname{End} \mathbb{Q}^2$ given by

$$\mathsf{E}_{ij}(u) \mapsto \Big(\delta_{ij} - \frac{e_{ji}}{u}\Big) \Big/ \Big(1 - \frac{1}{u}\Big) \,. \tag{11.9}$$

This takes $\mathsf{E}(u)$ to $R(u)$ and is indeed a representation of $\mathsf{Y}(\mathfrak{gl}(2))$ by the Yang-Baxter equation. We denote by $\mathbb{Q}^2(a)$ this representation precomposed with the translation automorphism of the Yangian. It is well-known, and can be seen as in Section 5.5.3, that

$$\bigcap_{n=1}^{\infty} \operatorname{Ker} \mathbb{Q}^2(a_1) \otimes \cdots \otimes \mathbb{Q}^2(a_n) = 0 \,. \tag{11.10}$$

Traditionally, a different representation of the Yangian, namely

$$\mathsf{E}_{ij}(u) \mapsto \delta_{ij} + \frac{e_{ij}}{u} \,.$$

is called the evaluation representation. The two are related by a composition of automorphisms

$$\mathsf{E}(u) \mapsto \mathsf{E}(-u)^T, \quad \mathsf{E}(u) \mapsto f(u)\mathsf{E}(u)$$

of $\mathsf{Y}(\mathfrak{gl}(2))$, where the superscript T denotes transposition and $f(u) = 1 + O(u^{-1}) \in \mathbb{Q}[[u^{-1}]]$ is an arbitrary scalar factor.

11.3.3 Comparison of Yangians

Let Y_Q denote the Yangian constructed in Chapter 5. This is an algebra over $\Bbbk = \mathbb{Q}[\hbar]$. The Yangian Y_Q is graded by cohomological degree and $\hbar$ has cohomological degree 2. Therefore, Y_Q is uniquely reconstructed, via the Rees algebra construction, from its specialization at $\hbar = 1$, with the induced filtration. We set $\hbar = 1$ in what follows.

Proposition 11.3.1.

$$\mathsf{Y}(\mathfrak{gl}(2)) \cong \mathsf{Y}_Q$$

Proof. Since the generators of Y_Q satisfy the RTT=TTR relation (5.10), we have a surjective homomorphism $\mathsf{Y}(\mathfrak{gl}(2)) \to \mathsf{Y}_Q$. Its injectivity follows from (11.10). □

11.3.4 The center of $\mathsf{Y}(\mathfrak{gl}(2))$

For any Lie algebra $\mathfrak{g}$, we have

$$\mathrm{Center}\,\mathcal{U}(\mathfrak{g}[u]) = \mathcal{U}(\mathrm{Center}(\mathfrak{g})[u])\,,$$

see e.g. Section 2.12 in [83]. The center of $\mathcal{U}(\mathfrak{gl}(2)[u])$ deforms to the center Z of $\mathsf{Y}(\mathfrak{gl}(2))$, which is freely generated by the coefficients in the expansion

$$\mathrm{qdet}\,\mathsf{E}(u) = 1 + \sum_{k>0} \mathrm{qdet}_k\; u^{-k}$$

of the quantum determinant

$$\mathrm{qdet}\,\mathsf{E}(u) = \mathsf{E}_{11}(u)\,\mathsf{E}_{22}(u-1) - \mathsf{E}_{21}(u)\,\mathsf{E}_{12}(u-1)\,.$$

The quantum determinant is group-like

$$\Delta\,\mathrm{qdet}\,\mathsf{E}(u) = \mathrm{qdet}\,\mathsf{E}(u) \otimes \mathrm{qdet}\,\mathsf{E}(u)$$

and

$$\mathrm{qdet}\,\mathsf{E}(u)\Big|_{\mathbb{Q}^2(a)} = \frac{u-a}{u-a-1}\,.$$

Whence the equality of ideals

$$\Big(\mathrm{qdet}_k\Big)_{k>0} = \Big(\mathrm{ch}_k\,\mathcal{W}\Big)_{k\geqslant 0} \subset \mathsf{Z}\,. \tag{11.11}$$

11.3.5 The core Yangian

The nonequivariant Cartan matrix for Q is $\mathsf{C} = (2)$, which is invertible. Therefore

$$\mathbb{Y}_Q \subset \mathsf{Y}_Q \,.$$

The classical $\mathbf{r}$ matrix for the core Yangian $\mathbb{Y}_Q$ equals

$$\mathbf{r} - \mathsf{C}^{-1}\, \mathsf{w} \otimes \mathsf{w} = -\tfrac{1}{2}\, h \otimes h - e \otimes f - f \otimes e \,,$$

where

$$h = e_{11} - e_{00} \in \mathfrak{sl}(2) \,.$$

This is the classical $\mathbf{r}$-matrix for $\mathfrak{sl}(2)$. This means the core Yangian $\mathbb{Y}_Q$ is a filtered deformation of $\mathcal{U}(\mathfrak{sl}(2)[u])$.

Proposition 11.3.2.

$$\mathbb{Y}_Q \cong \mathsf{Y}(\mathfrak{sl}(2)) \,.$$

Proof. Let $\mathfrak{z} \cong \mathfrak{gl}(1)$ denote the center of $\mathfrak{gl}(2)$. By deformation from

$$\mathcal{U}(\mathfrak{gl}(2)[u]) \cong \mathcal{U}(\mathfrak{sl}(2)[u]) \otimes \mathcal{U}(\mathfrak{z}[u])$$

we get

$$\mathbb{Y}_Q \otimes \mathsf{Z} \cong \mathsf{Y}_Q \cong \mathsf{Y}(\mathfrak{gl}(2)) \cong \mathsf{Y}(\mathfrak{sl}(2)) \otimes \mathsf{Z} \,.$$

Taking the quotient by the ideal (11.11) gives the desired isomorphisms. □

11.3.6

Baxter subalgebras in $\mathsf{Y}(\mathfrak{gl}(2))$ appeared in mathematical physics as quantum integrals of motion of the XXX spin chain with quasi-periodic boundary conditions.

Proposition 11.2.2 and Section 6.5 shows the operator

$$\mathsf{Q} = c_1(\mathcal{O}(1)) \cup + \mathsf{Q}_{\text{quantum}}$$

of modified quantum multiplication by $c_1(\mathcal{O}(1))$ lies in the Baxter subalgebra corresponding to

$$g = q^{\vee} \in GL(2) \,.$$

Since the operator $c_1(\mathcal{O}(1))\cup$ in $H^{\bullet}_{\mathsf{G}}(X)$ has distinct eigenvalues, the algebra of quantum multiplication equals the Baxter subalgebra in $\mathsf{Y}(\mathfrak{gl}(2))$. This is one of the most basic examples in Nekrasov-Shatashvili theory.

Part II

Instanton moduli

Chapter 12

Classical r-matrix and $\widehat{\mathfrak{gl}(1)}$.

12.1 Setup

12.1.1 Moduli of framed sheaves

We now specialize our general discussion to the quiver Q with one vertex and one loop. We take $\zeta = 0$, $\theta > 0$ and denote

$$(r, n) = (\mathsf{w}_1, \mathsf{v}_1)\,.$$

The corresponding Nakajima variety $\mathcal{M}(r, n)$ is the moduli space of framed torsion-free sheaves $\mathcal{F}$ with

$$\operatorname{rk} \mathcal{F} = r\,, \quad c_2(\mathcal{F}) = n\,,$$

on $\mathbb{P}^2$, see [88]. Framing means a choice of trivialization of $\mathcal{F}$ along $\mathbb{P}^2 \backslash \mathbb{C}^2$. It implies $c_1(\mathcal{F}) = 0$. As usual, we set

$$\mathcal{M}(r) = \bigsqcup_n \mathcal{M}(r, n)\,.$$

In particular,

$$\mathcal{M}(1) = \operatorname{Hilb} = \bigsqcup_n \operatorname{Hilb}_n$$

is the Hilbert scheme of points of $\mathbb{C}^2$.

Our goal in the rest of the paper is to make the general theory explicit in this very important special case.

12.1.2 Uhlenbeck space

The affine variety

$$\mathcal{U}(r,n) = \mathcal{M}_{0,0}(r,n)$$

is the Uhlenbeck compactification of the moduli of framed instantons. The canonical map

$$\mathcal{M}(r,n) \to \mathcal{U}(r,n)$$

takes a torsion free sheaf $\mathcal{F}$ to the vector bundle $\mathcal{F}^{\vee\vee}$ together with the support of $\mathcal{F}^{\vee\vee}/\mathcal{F}$, counting multiplicity.

12.1.3 Group actions

Concretely, $\mathcal{M}(r,n)$ is the $GL(\mathbb{C}^n)$-quotient of the spaces of quadruples

$$X_1, X_2 : \mathbb{C}^n \to \mathbb{C}^n\,, \quad A : \mathbb{C}^r \to \mathbb{C}^n\,, \quad B : \mathbb{C}^n \to \mathbb{C}^r$$

satisfying the equation

$$[X_1, X_2] + AB = 0 \tag{12.1}$$

and stability condition: the image of A must generate $\mathbb{C}^n$ under the action of X_1 and X_2.

The framing group $G_{\mathsf{w}} = GL(r)$ acts by the automorphisms of $\mathbb{C}^r$ or by changing the framing in the sheaf description. The group $G_{\text{edge}} = GL(2)$ acts by

$$g \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} g_{11}\, X_1 + g_{12}\, X_2 \\ g_{21}\, X_1 + g_{22}\, X_2 \end{pmatrix}\,, \quad g \cdot A = A\,, \quad g \cdot B = \det(g)\, B\,.$$

in the quiver description. In the sheaf description, it acts by automorphisms of $\mathbb{P}^2$ preserving $\mathbb{C}^2$.

We fix a maximal torus $\mathsf{A} \subset GL(r)$ with

$$\mathfrak{a} = \operatorname{Lie} \mathsf{A} = \operatorname{diag}(a_1, \dots, a_r)$$

and take $\mathsf{G} = \mathsf{A} \times GL(2)$. Note that A is central in G.

12.1.4 Fixed loci

By Example 3.2.2, we have

$$\mathcal{M}(r)^{\mathsf{A}} = \mathcal{M}(1)^{\times r}\,, \quad \mathcal{M}(1) = \bigsqcup_n \mathrm{Hilb}_n\,.$$

In the sheaf description, $\mathcal{M}(r)^{\mathsf{A}}$ is the locus of direct sums

$$\mathcal{M}(r)^{\mathsf{A}} = \left\{ \bigoplus_{i=1}^{r} I_i \right\}$$

of ideals $I_i \subset \mathbb{C}[x_1, x_2] = \mathcal{O}_{\mathbb{C}^2}$.

12.1.5 Polarization

Our general prescription for polarizations of Nakajima varieties gives the following for instanton moduli.

Following Example 3.3.2, consider the $\mathbb{C}^\times$-action on $\mathbb{C}^2$ that scales one coordinate axis, say the x_2-axis. This scales ω with weight -1. One of the components $X^{\mathbb{C}^\times}$ is the following Quot-scheme

$$Q_n = \left\{ \mathcal{F} \,\middle|\, x_2 \mathcal{O}^{\oplus r} \subset \mathcal{F} \subset \mathcal{O}^{\oplus r} \right\} \subset \mathcal{M}(r,n)\,.$$

It is middle-dimensional. Since ω pairs $\mathbb{C}^\times$-weight spaces of total weight 1, it is Lagrangian. In the quiver description, it is given by

$$X_2 = 0\,, \quad B = 0\,,$$

that is, by representations of one half of the quiver $\overline{Q}$.

As out polarization, we take weights that are normal to Q_n. Those are easily identified, giving

$$\varepsilon = \prod_{i=1}^{r} \prod_{j \neq i} (a_j - a_i)^{n_i} \tag{12.2}$$

for the component

$$\mathrm{Hilb}_{n_1} \times \cdots \times \mathrm{Hilb}_{n_r} \subset \mathcal{M}(r)^{\mathsf{A}}\,.$$

12.1.6 R-matrices

Our general theory produces an R-matrix

$$\mathbf{R}(u) \in \operatorname{End}\left(H^{\cdot}_{\mathsf{G}_\mathsf{A}}(\mathrm{Hilb})^{\otimes 2}\right) \otimes \mathbb{Q}(\mathfrak{g}_\mathsf{A})$$

which solves the Yang-Baxter equation with spectral parameter

$$u = a_1 - a_2 \,.$$

Our goal now is to identify $\mathbf{R}(u)$ and the corresponding Yangian. We use the boldface letter to denote this particular R-matrix. It will be characterized in terms of the Virasoro algebra in Chapter 14.

12.1.7

As a first step, in Section 12.4 we show the corresponding classical **r**-matrix is the **r**-matrix for $\widehat{\mathfrak{gl}(1)}$, modulo zero modes. The action of $\widehat{\mathfrak{gl}(1)}$ on the cohomology of Hilbert schemes was constructed by Nakajima [86] and Grojnowski [51]. Its extension to higher rank is due to Baranovsky [6].

12.1.8

In principle, $\mathbf{R}(u)$ may be computed from the R-matrix of $\mathsf{Y}(\mathfrak{gl}(\infty))$ using the factorization in Theorem 4.3.1, see [114]. In particular, the classical **r**-matrix is very easy to determine in this approach. Here we take a different route to the same result.

12.2 Baranovsky operators

12.2.1

We recall from [6] the definition of Baranovsky operators β_k. For $k > 0$, consider the locus

$$\mathfrak{B} \subset \mathcal{M}(r, n+k) \times \mathbb{C}^2 \times \mathcal{M}(r, n) \tag{12.3}$$

of triples $(\mathcal{F}', x, \mathcal{F})$ such that $\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{F}/\mathcal{F}'$ is a length k sheaf supported at x. We have [6]

$$\dim \mathfrak{B} = 2rn + rk + 1 \,,$$

which is the middle dimension of the product. Note that $\mathfrak{B}$ is G-invariant.

12.2.2

Next, $\mathfrak{B}$ is a Lagrangian Steinberg correspondence between the first factor and the other two, which can be seen as follows. We embed

$$\mathbb{C}^2 \ni (c_1, c_2) \mapsto ((x_1 - c_1)^k, x_2 - c_2) \in \mathrm{Hilb}_k \ .$$

Note this pulls back the translation-invariant symplectic form. Consider the maps

$$\begin{array}{ccccc} \mathrm{Hilb}_k \times \mathcal{M}(r,n) & \hookrightarrow & \mathcal{M}(r+1, n+k) & \hookleftarrow & \mathrm{Hilb}_0 \times \mathcal{M}(r, n+k) \\ & & \downarrow & & \\ & & \mathcal{U}(r+1, n+k)\,. & & \end{array}$$

Here the horizontal arrows are formed by taking direct sums and the vertical is the canonical projection to the Uhlenbeck space. It is clear that points on the correspondence $\mathfrak{B}$ map to the same points of $\mathcal{U}(r+1, n+k)$.

12.2.3

The correspondence $\mathfrak{B}$ defines a map

$$\Theta_{\mathfrak{B}} : H^{\cdot}_{\mathsf{G}}(\mathbb{C}^2) \otimes H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r,n)) \to H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r, n+k))\,.$$

We define the operators β_{-k}, $k > 0$, as the matrix elements of $\Theta_{\mathfrak{B}}$ with respect to the $\mathbb{C}^2$-factor, that is

$$\beta_{-k}(\gamma) \cdot \eta = \Theta_{\mathfrak{B}}(\gamma \otimes \eta)\,, \quad \gamma \in H^{\cdot}_{\mathsf{G}}(\mathbb{C}^2)\,.$$

12.2.4

For $k > 0$, we define $\beta_k(\gamma)$ as the matrix elements of the adjoint operator

$$\Theta^{\tau}_{\mathfrak{B}} : H^{\cdot}_{\mathsf{G}}(\mathbb{C}^2) \otimes H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r, n+k)) \to H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r,n)) \otimes \mathbb{K}\,,$$

see Section 3.1.3. A larger base ring $\mathbb{K}$ is required because the adjoint correspondence is not proper and equivariant localization is needed to define it as an operator. We will see that

$$\mathbb{K} = H^{\cdot}_{\mathsf{G}}(\mathsf{pt}) \left[\frac{1}{\det_{\mathbb{C}^2}} \right]$$

where

$$\det_{\mathbb{C}^2}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = t_1 t_2 \in \mathbb{Q}[\mathfrak{gl}(2)]$$

is the determinant of the defining representation.

Also note that since we permute the source and target of $\Theta_{\mathfrak{B}}$, the operator $\Theta^{\tau}_{\mathfrak{B}}$ gets a sign, namely

$$(-1)^{rk} = (-1)^{\frac{1}{2}\dim \mathcal{M}(r,n+k)+\frac{1}{2}\dim \mathcal{M}(r,n)} .$$

12.2.5

For $r = 1$, Baranovsky operators specialize to the original Nakajima operators, up to normalization. We denote them by $\alpha_k(\gamma)$. It is a theorem of Nakajima that these satisfy

$$[\alpha_k(\gamma_1), \alpha_l(\gamma_2)] = k\, \delta_{k+l}\, \tau(\gamma_1 \cup \gamma_2) \tag{12.4}$$

see [86]. Recall from Section 3.1.3 that τ involves a sign. Since γ_i are cohomology classes on a surface,

$$\tau(\gamma) = -\int_{\mathbb{C}^2} \gamma\, ,$$

where the integral is defined as an equivariant residue. In particular

$$\tau(1) = -\frac{1}{\det_{\mathbb{C}^2}}\, . \tag{12.5}$$

12.2.6

Since $\beta_k(\gamma)$ is a Steinberg correspondence, there exist a Steinberg correspondence $\beta_k(\gamma)_{\mathsf{A}}$ that makes the following diagram commute

$$\begin{array}{ccc} H^{\cdot}_{\mathsf{G}}(\mathcal{M}(1))^{\otimes r} & \xrightarrow{\mathrm{Stab}_{\mathfrak{C}}} & H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r)) \\ \downarrow{\scriptstyle \beta_k(\gamma)_{\mathsf{A}}} & & \downarrow{\scriptstyle \beta_k(\gamma)} \\ H^{\cdot}_{\mathsf{G}}(\mathcal{M}(1))^{\otimes r} & \xrightarrow{\mathrm{Stab}_{\mathfrak{C}}} & H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r)) \end{array}$$

for every chamber $\mathfrak{C}$ and every $k < 0$. Here we use that A does not act on the $\mathbb{C}^2$ factor in (12.3). By taking adjoints, the same holds for $k > 0$ after tensoring with $\mathbb{K}$.

Theorem 12.2.1. *We have*

$$\beta_k(\gamma)_{\mathsf{A}} = \sum_{i=1}^{r} 1 \otimes \cdots \otimes \alpha_k(\gamma) \otimes \cdots \otimes 1\,,$$

where $\alpha_k(\gamma)$ acts in the ith tensor factor.

12.2.7

In particular, Theorem 12.2.1 and commutation relations (12.4) imply

$$[\beta_n(\gamma_1), \beta_m(\gamma_2)] = rn\,\delta_{n+m}\tau(\gamma_1 \cup \gamma_2)\,, \tag{12.6}$$

which is a theorem of Baranovsky, see [6].

12.3 Proof of Theorem 12.2.1

12.3.1

By Theorem 4.6.1, the operator in question is given by a correspondence supported on $\mathfrak{B}^{\mathsf{A}}$. From definitions

$$\mathfrak{B}^{\mathsf{A}} = \Big\{\Big(\bigoplus_{i=1}^{r} I_i, x, \bigoplus_{i=1}^{r} J_i\Big)\Big\} \tag{12.7}$$

where $I_i, J_i \in \mathcal{M}(1)$, $I_i \subset J_i$, and

$$\operatorname{supp} J_i/I_i \subset \{x\}$$

for all i. The connected components of $\mathfrak{B}^{\mathsf{A}}$ are classified by the second Chern classes of I_i, J_i, and their dimensions are computed as follows

$$\dim = 2 + \sum_i \max\left(c_2(I_i) - c_2(J_i) - 1, 0\right).$$

In particular,

$$\mathfrak{B}^{\mathsf{A}} = \bigcup_{i=1}^{r} \mathfrak{B}_1^{(i)} \cup \text{lower dimension}\,,$$

where $\mathfrak{B}_1^{(i)}$ denotes the corresponding correspondence for $r = 1$ acting in the ith factor.

12.3.2

The top dimensional components $\mathfrak{B}_1^{(i)}$ are irreducible. Therefore, to compute the Lagrangian residue of $\mathfrak{B}$, it suffices to find a smooth point $\mathfrak{b}$ of $\mathfrak{B}$ on each of them.

By symmetry, we may assume $i = 1$. In (12.7), we take a point $\mathfrak{b} \in \mathfrak{B}^{\mathsf{A}}$ such that

$$x = 0 \in \mathbb{C}^2\,, \quad I_1 = (x_1^k, x_2)\,, \quad J_1 = \mathcal{O}\,,$$

while $0 \notin \operatorname{supp} I_i, \operatorname{supp} J_i$ for $i > 1$. Lemma 12.3.2 below gives a rational map

$$f : \mathcal{M}(r,k) \times \mathcal{M}(r,n) \dashrightarrow \mathcal{M}(r,n+k)$$

which is an isomorphism in a neighborhood of $\mathfrak{b}$. Denoting the correspondence (12.3) by $\mathfrak{B}_{r,n,k}$, we have

$$f^*(\mathfrak{B}_{r,n,k}) = \mathfrak{B}_{r,0,k} \times \operatorname{diag}_{\mathcal{M}(r,n)}$$

in a neighborhood of $\mathfrak{b}$.

Note that polarizations in Theorem 4.6.1 enter in the combination $\varepsilon_X\, \bar{\varepsilon}_Y$. Therefore, the residue of the diagonal is always the diagonal and the computation is reduced to the case $n = 0$.

12.3.3

The correspondence

$$\mathfrak{B}_{r,0,k} \subset \mathcal{M}(r,k) \times \mathbb{C}^2$$

has the following quiver description:

$$\mathfrak{B}_{r,0,k} = \{B = 0, (X_1 - x_1)^k = 0, (X_2 - x_2)^k = 0\}\,,$$

where $x = (x_1, x_2) \in \mathbb{C}^2$. Our reference point $\mathfrak{b}$ on it is given by

$$X_1 = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad X_2 = 0\,, \quad A = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix}.$$

Lemma 12.3.1. *The variety $\mathfrak{B}_{r,0,k}$ is smooth at $\mathfrak{b}$ and its nonzero tangent A-weights are*

$$(a_1 - a_i)^{\oplus n}\,, \quad i = 2, \dots, r\,.$$

Proof. In a neighborhood, the operator $X_1 - x_1$ will remain a regular nilpotent and the $(1,1)$-entry of A will remain nonzero, hence the triple (X_1, X_2, A) may be brought to the normal form

$$X_1 = \begin{pmatrix} x_1 & & & \\ 1 & x_1 & & \\ 0 & 1 & x_1 & \\ & & \ddots & \ddots \end{pmatrix}, \quad X_2 = P(X_1)\,, \quad A = \begin{pmatrix} 1 & * & * & \dots \\ 0 & * & * & \\ 0 & * & * & \\ \vdots & & & \ddots \end{pmatrix}$$

by a unique element of $GL(n)$. Here P is a polynomial of degree $< n$ and stars stand for arbitrary numbers. Thus a neighborhood of F in $\mathfrak{B}$ is isomorphic to $\mathbb{C}^{rn+1}$. The computation of the tangent weights is straightforward. □

12.3.4

In particular, we see that

$$T_{\mathfrak{b}}\mathfrak{B}/T_{\mathfrak{b}}\mathfrak{B}^{\mathsf{A}} \cong T_{\mathfrak{b}}Q_k/T_{\mathfrak{b}}Q_k^{\mathsf{A}}$$

as A-modules.

Since $\mathfrak{B}$ is smooth at $\mathfrak{b}$, its Lagrangian residue is $\pm\mathfrak{B}^{\mathsf{A}}$. Further, the normal weights to $\mathfrak{B}$ agree with the polarization (12.2). This finishes the proof modulo the following lemma used above.

12.3.5

Lemma 12.3.2. *Suppose the eigenvalues of X_1 may be partitioned*

$$\text{Eigenvalues}(X_1) = \bigsqcup E_i$$

into a nontrivial disjoint union. Then a neighborhood of $(X_1, X_2, A, 0)$ in $\mathcal{M}(r,n)$ is $GL(r)$-equivariantly isomorphic to an open set in $\prod \mathcal{M}(r, |E_i|)$.

We are grateful to the referee for pointing out that this statment, with a different proof, is the *factorization property* of [11].

Proof. For nearby X_1 we can still group the eigenvalues according to the same partition. We denote

$$P_i : \mathbb{C}^n \to \mathbb{C}^n\,, \quad i = 1, \dots, \ell(\mu)$$

the corresponding spectral projectors.

The projectors P_i are canonically defined and, in particular, commute with the centralizer of X_1 in $GL(n)$. We thus may assume they project onto coordinate subspaces and replace the $GL(n)$-quotient by $\prod GL(|E_i|)$. For each i, the quadruple

$$(Z_i, W_i, A_i, B_j) = (P_i\, X_1, P_i\, X_2\, P_i, P_i\, A, B\, P_i)$$

solves (12.1). Because the starting point $(X_1, X_2, A, 0)$ is stable, each of these blocks remains stable in a certain neighborhood. Thus we get a map to $\prod \mathcal{M}(r, |E_i|)$. Clearly, it is $GL(r)$-equivariant.

The original data (X_1, X_2, A, B) may be reconstructed as follows. Since $\sum P_i = 1$, all we need is to recover

$$W_{ij} = P_i X_2 P_j$$

for $i \neq j$. It is a solution of

$$Z_i W_{ij} - W_{ij} Z_j = -A_i B_j\,,$$

which exists and is unique because the spectra of Z_i and Z_j are disjoint.

□

12.4 Classical r-matrix

12.4.1

Denote $\Bbbk = H^{\cdot}_{\mathsf{G}}(\mathsf{pt})$ and let

$$|\rangle \in H^{\cdot}_{\mathsf{G}}(\mathcal{M}(1,0)) \cong \Bbbk$$

be the identity element. We abbreviate

$$\alpha_n = \alpha_n(1)$$

in this section. It is a theorem of Nakajima [86] that the map

$$\Bbbk\big[\alpha_{-1}, \alpha_{-2}, \alpha_{-3}, \dots\big] \to H^{\cdot}_{\mathsf{G}}(\mathcal{M}(1))$$

given by

$$f \mapsto f\,|\rangle\,,$$

is an isomorphism. We will use it to identify its source and target.

12.4.2

We define

$$\mathsf{F} = \mathbb{K}\big[\alpha_{-1}, \alpha_{-2}, \alpha_{-3} \dots . \big]$$

The operators α_n, $n > 0$, act on F satisfying (12.4) and annihilating the vector

$$\mathsf{vac} = 1 = |\rangle \, .$$

Following the tradition in quantum field theory, F is called a Fock space. The operators α_n generate a Heisenberg algebra in $\mathrm{End}(\mathsf{F})$.

12.4.3

Consider

$$\mathbf{R}(u) \in \mathrm{End}\,(\mathsf{F} \otimes \mathsf{F}) \otimes_{\mathbb{K}} \mathbb{Q}(\mathfrak{g}_{\mathsf{A}}) \, .$$

By Theorem 12.2.1, it commutes with the operators

$$\beta_n(1)_{\mathsf{A}} = \alpha_n \otimes 1 + 1 \otimes \alpha_n \, .$$

We define

$$\alpha_n^{\pm} = \alpha_n \otimes 1 \pm 1 \otimes \alpha_n \, .$$

These satisfy

$$[\alpha_k^{\varepsilon}, \alpha_l^{\eta}] = 2k\tau(1)\, \delta_{k+l}\, \delta_{\varepsilon,\eta} \, , \tag{12.8}$$

where $\varepsilon, \eta \in \{\pm\}$ as a consequence of (12.4).

We see the operators $\alpha_n^{\pm}$ generate two new commuting Heisenberg subalgebras of $\mathsf{F} \otimes \mathsf{F}$ and $\mathbf{R}$ commutes with one of them.

12.4.4

Using the operators $\alpha_n^{\pm}$, we can write

$$\mathsf{F} \otimes \mathsf{F} = \mathsf{F}^{+} \otimes \mathsf{F}^{-} \, . \tag{12.9}$$

We denote by End^{-} the image of $\mathrm{End}(\mathsf{F}^{-})$ in $\mathrm{End}(\mathsf{F}^{\otimes 2})$,

Lemma 12.4.1. *The operator* $\mathbf{R}$ *belongs to* End^{-}.

Proof. The operators α_n^{+} act irreducibly on F^{+} and commute with $\mathbf{R}$. □

12.4.5

Lemma 12.4.2. *An operator in* End^- *is uniquely determined by its matrix elements in the subspace*

$$\mathsf{vac} \otimes \mathsf{F} \subset \mathsf{F}^{\otimes 2} .$$

Proof. Let $A \in \operatorname{End}^-$ and suppose that

$$\left(A \cdot \mathsf{vac} \otimes v_1, \mathsf{vac} \otimes v_2\right) = 0 \tag{12.10}$$

for all $v_1, v_2 \in \mathsf{F}$, while

$$\left(A \prod \alpha^-_{-\mu_i} \mathsf{vac} \otimes \mathsf{vac}, \prod \alpha^-_{-\nu_i} \mathsf{vac} \otimes \mathsf{vac}\right) \neq 0 \tag{12.11}$$

for some partitions μ, ν. We may further assume, the partitions μ, ν in (12.11) are chosen minimal with respect to $|\mu|, |\nu|$. Then taking

$$v_1 = \prod \alpha_{-\mu_i} \mathsf{vac} , \quad v_2 = \prod \alpha_{-\nu_i} \mathsf{vac} ,$$

in (12.10) and expanding

$$1 \otimes \alpha_n = \tfrac{1}{2}(\alpha_n^+ - \alpha_n^-)$$

we get a contradiction. □

12.4.6

The subspace

$$\mathsf{vac} \otimes \mathsf{F} \subset H^{\cdot}_{\mathsf{G}}(\mathcal{M}(2)^{\mathsf{A}}) \otimes \mathbb{K}$$

is a vacuum subspace in the sense of Section 4.7. By Theorem 4.7.1, the corresponding matrix element of $\mathbf{R}(u)$ is the operator of classical multiplication by

$$\frac{e(N_-)}{e(N_- \otimes \hbar)} = 1 + \frac{\hbar \operatorname{rk} N_-}{a_1 - a_2} + \dots \tag{12.12}$$

where

$$-u = a_2 - a_1$$

is the A-weight of N_-. By formula (2.15) ,

$$\operatorname{rk} N_-\big|_{\mathcal{M}(1,0)\times\mathcal{M}(1,n)} = n .$$

In the sheaf interpretation, the unstable normal bundle N_- to

$$\mathcal{M}(1) \ni I \mapsto \mathcal{O} \oplus I \in \mathcal{M}(2)$$

is the tautological bundle of the Hilbert scheme

$$N_- \cong \mathrm{Taut} = H^0(\mathcal{O}/I)\,.$$

Hence $\mathrm{rk}\, N_-$ is indeed the number of points.

12.4.7

Consider the operator

$$L_0 = -\sum_{k>0} \alpha_{-k}(1)\, \alpha_k(\mathsf{pt})$$

where

$$\mathsf{pt} = [0] = \det_{\mathbb{C}^2} \in H^2_{\mathsf{G}}(\mathbb{C}^2)$$

is the class of the origin. Note that since pt and 1 are proportional, they may be distributed arbitrarily between the two factors. From

$$[\alpha_k(\mathsf{pt}), \alpha_l(1)] = -k\, \delta_{k+l}\,,$$

one has the following

Lemma 12.4.3. *L_0 acts by multiplication by n in $H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathcal{M}(1,n))$.*

12.4.8

Theorem 12.4.4. *The classical* $\mathbf{r}$*-matrix for* $\mathcal{M}(1) \times \mathcal{M}(1)$ *equals*

$$\mathbf{r} = -\sum_{n>0} \alpha^-_{-n}(1)\, \alpha^-_n(\mathsf{pt})\,. \tag{12.13}$$

Proof. This commutes with α_n^+ and has correct vacuum matrix elements by Lemma 12.4.3. We conclude by Lemma 12.4.2. □

12.4.9

Expanding out (12.13), we get the following formula for the action of **r** on cohomology of $\mathcal{M}(r_1) \times \mathcal{M}(r_2)$

$$\mathbf{r} = \mathsf{v} \otimes \mathsf{w} + \mathsf{w} \otimes \mathsf{v} + \sum_{k \neq 0} \beta_{-k}(1) \otimes \beta_k(\mathsf{pt}) \tag{12.14}$$

where

$$\mathsf{w} = r, \quad \mathsf{v} = \mathsf{c}_2$$

act by multiplication by the rank and instanton charge, respectively, compare with (5.23).

12.4.10

We conclude

$$\mathfrak{g}_{\mathsf{Q}} \cong \widehat{\mathfrak{gl}(1)} \otimes \mathbb{K} \Big/ \text{zero modes}\,,$$

where zero modes (or constant loops) refer to central elements $\beta_0(\gamma)$. The brackets in this Lie algebra

$$\begin{aligned} [\mathsf{v}, \beta_n(\gamma)] &= -n\beta_n(\gamma)\,, \\ [\beta_n(\gamma), \beta_m(\gamma')] &= \tau(\gamma \cup \gamma')\, n\, \delta_{n+m}\, \mathsf{w}\,, \end{aligned} \tag{12.15}$$

are a special case of the relation (5.11).

Chapter 13

Free bosons

13.1 Fock spaces

13.1.1

Anticipating application to algebraic surfaces other than $\mathbb{C}^2$, we will put the commutation relations (12.15) in a more abstract framework, in which the insertions γ take values in a general commutative Frobenius algebra $\mathbb{H}$ over a ring $\mathbb{K}$.

To go back to framed sheaves on $\mathbb{C}^2$, one takes

$$\mathbb{H} = H^{\cdot}_{\mathsf{G}}(\mathbb{C}^2)\left[\frac{1}{\det_{\mathbb{C}^2}}\right], \quad \mathbb{K} = H^{\cdot}_{\mathsf{G}}(\mathsf{pt})\left[\frac{1}{\det_{\mathbb{C}^2}}\right] \tag{13.1}$$

with the trace map

$$\tau : \mathbb{H} \to \mathbb{K}$$

given by $\tau(\gamma) = -\int_{\mathbb{C}^2} \gamma$. We denote this Frobenius algebra $\mathbb{H}(\mathbb{C}^2)$.

Most of the material in this section is completely standard and is recalled mainly for setting up the notation.

13.1.2 Heisenberg algebras

Let $\mathbb{H}$ be a free $\mathbb{K}$-module with a nondegenerate symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$. Consider the space

$$\mathbb{H}[z^{\pm 1}] = \mathbb{H} \otimes \mathbb{K}[z^{\pm 1}]$$

of polynomial loops, that is, Laurent polynomials $f(z)$ with values in $\mathbb{H}$. This has a natural skew-symmetric form

$$\{f,g\} = \int (df, g)\,, \qquad \int = \oint \frac{dz}{2\pi i z}\,. \tag{13.2}$$

For example

$$\{\gamma z^n, \eta z^{-n}\} = n\,(\gamma,\eta)\,, \quad \gamma,\eta \in \mathbb{H}\,.$$

The form (13.2) makes $\mathbb{H}[z^{\pm 1}] \oplus \mathbb{K}$ a Heisenberg Lie algebra. We denote by $\mathfrak{Heis} = \mathfrak{Heis}(\mathbb{H})$ its universal enveloping algebra and denote by $\alpha_n(\gamma) \in \mathfrak{Heis}$ the image of γz^n.

Note that $\mathfrak{Heis}$ has a center, generated by the identity and the zero modes $\alpha_0(\gamma)$, $\gamma \in \mathbb{H}$.

13.1.3 Translation automorphisms

The additive group of $\mathbb{H}$ acts on $\mathfrak{Heis}(\mathbb{H})$ by automorphisms

$$\varsigma_\gamma\,(\alpha_n(\eta)) = \alpha_n(\eta) - \delta_{n,0}\,(\gamma,\eta)\,, \quad \gamma,\eta \in \mathbb{H}\,.$$

We denote

$$\mathfrak{Heis}^\sim = \mathbb{K}[\mathbb{H}_{\text{add}}] \ltimes \mathfrak{Heis}\,,$$

where $\mathbb{K}[\mathbb{H}_{\text{add}}]$ denotes the group algebra of the additive group of $\mathbb{H}$. By definition, it is spanned by linear combinations of ς_γ, $\gamma \in \mathbb{H}$.

Introduce the corresponding Lie algebra elements

$$\alpha_{\log}(\gamma) = \log \varsigma_\gamma$$

which satisfy the relations

$$[\alpha_n(\gamma), \alpha_{\log}(\eta)] = \delta_{n,0}\,(\gamma,\eta)\,.$$

13.1.4 Fields

The commutation relations in $\mathfrak{Heis}^\sim$ are best summarized using fields, or generating functions. Consider

$$\boldsymbol{\phi}(\gamma; z) = \alpha_{\log}(\gamma) + \alpha_0(\gamma)\,\log z - \sum_{n\neq 0} \frac{\alpha_n(\gamma)}{n}\, z^{-n} \tag{13.3}$$

where $z \in \mathbb{C}^\times$ is a variable. Then

$$[\boldsymbol{\phi}(\gamma; z), \boldsymbol{\phi}(\eta; w)] = (\gamma, \eta) \, \left(\log(z-w)_{|z|>|w|} - \log(w-z)_{|w|>|z|}\right) \, .$$

Here

$$\log(z-w)_{|z|>|w|} = \log z - \sum_{n>0} \frac{(w/z)^n}{n} \, ,$$

is the series expansion in the region $|z| > |w|$. We will also consider

$$\boldsymbol{\alpha}(\gamma; z) = \partial \, \boldsymbol{\phi}(\gamma; z) = \sum_n \alpha_n(\gamma) \, z^{-n} \, , \tag{13.4}$$

where

$$\partial = z \frac{\partial}{\partial z} \, .$$

The coefficients of the fields (13.4) generate $\mathfrak{Heis}(\mathbb{H})$.

13.1.5 Fock spaces

The Fock representation of $\mathfrak{Heis}^\sim$ is generated by the vacuum vector $|0\rangle$ such that

$$\alpha_n(\gamma) \, |0\rangle = 0 \, , \quad n \geqslant 0 \, .$$

We denote

$$|\eta\rangle = \varsigma_{-\eta} \, |0\rangle \, .$$

These satisfy

$$\alpha_0(\gamma) \, |\eta\rangle = -(\gamma, \eta) \, |\eta\rangle$$

and generate an irreducible $\mathfrak{Heis}$-module that we denote $\mathsf{F}(\eta)$. We have

$$\mathsf{F}(\eta) \cong \mathsf{F}(0) \cong S^{\cdot}\left(z^{-1}\mathbb{H}[z^{-1}]\right) \quad \text{as vector spaces} \, ,$$

the first isomorphism being the action of ς_η. The module structure of $\mathsf{F}(\eta)$ varies with η, but only in how the center of $\mathfrak{Heis}$ acts.

13.1.6 Adjoints

There is an anti-involution on $\mathfrak{Heis}_\zeta$ defined by

$$(\alpha_n(\gamma))^* = \alpha_{-n}(\gamma)\,, \quad \boldsymbol{\varsigma}_\eta^* = \boldsymbol{\varsigma}_{-\eta}\,,$$

that is,

$$\boldsymbol{\phi}(\gamma, z)^* = -\boldsymbol{\phi}(\gamma, z^{-1})\,.$$

The Fock representation has a unique inner product for which $|\eta\rangle$ are orthonormal and the anti-involution $*$ coincides with taking the adjoint operator. We will use this inner product to define matrix elements of operators.

13.1.7 Normally ordered products

Consider a product $\boldsymbol{\alpha}(\gamma, z)\,\boldsymbol{\alpha}(\eta, w)$ of two fields. Its matrix elements are given by convergent series in the region $|z| > |w|$. At $z = w$ they have a singularity. This is regularized by commuting all annihilation operators to the right. In other words, one defines the normally ordered product by

$$\boldsymbol{\alpha}(\gamma, z)\,\boldsymbol{\alpha}(\eta, w) = (\gamma, \eta)\,\frac{zw}{(z-w)^2} + :\boldsymbol{\alpha}(\gamma, z)\,\boldsymbol{\alpha}(\eta, w):, \tag{13.5}$$

where the first, singular, term is to be expanded in the region $|z| > |w|$. The normally ordered term in (13.5) is regular at $z = w$, in fact

$$\big(:\boldsymbol{\alpha}(\gamma, z)\,\boldsymbol{\alpha}(\eta, w): f_1, f_2\big) \in \mathbb{K}[z^{\pm 1}, w^{\pm 1}] \tag{13.6}$$

for all f_1, f_2 in the Fock space.

By linearity, we can say that the normally ordered product $:\boldsymbol{\alpha}(\gamma, z)\,\boldsymbol{\alpha}(\eta, z):$ takes an element $\gamma \otimes \eta \in \mathbb{H}^{\otimes 2}$ as an argument.

A generalization of (13.5), known as Wick's theorem, explains how to normally order any product of normally ordered monomials in $\boldsymbol{\alpha}(\gamma_i, z_i)$. See for example [45, 25, 58].

13.1.8 Grading

Recall we assume $(\,\cdot\,,\,\cdot\,)$ to be nondegenerate and let $g^{-1} \in \mathbb{H}^{\otimes 2}$ be the inverse quadratic form. Then $:\boldsymbol{\alpha}^2:(g^{-1}, z)$ is a well-defined operator-valued Laurent series, from which we can extract the constant term $\int :\boldsymbol{\alpha}^2: (g^{-1}, z)$. The following computation is standard

Lemma 13.1.1. *Let the Fock space be graded by*

$$\deg|\eta\rangle = (\eta,\eta)/2\,, \quad \deg\alpha_{-n} = n\,.$$

Then $\frac{1}{2}\int :\boldsymbol{\alpha}^2:(g^{-1},z)$ *is the the grading operator.*

This is a generalization of Lemma 12.4.3.

13.2 Insertions and coproducts

13.2.1

Note that in (13.6) we evaluate both operators at the same point $z = w \in \mathbb{C}^\times$, but they still take two distinct cohomology insertions γ and η, or, equivalently, a element of $\gamma \otimes \eta \in \mathbb{H}^{\otimes 2}$ as an argument.

To write an operator with a single cohomology insertion, we need a coassociative coproduct

$$\Delta : \mathbb{H} \to \mathbb{H}^{\otimes 2}\,,$$

and its iterates

$$\mathbb{H} \ni \gamma \mapsto \gamma^{\Delta n} \in \mathbb{H}^{\otimes n}\,.$$

We can then construct an operator

$$:\boldsymbol{\alpha}^n:(\gamma, z) \stackrel{\text{def}}{=} :\boldsymbol{\alpha}^n:(\gamma^{\Delta n}, z)$$

which depends on a single point $z \in \mathbb{C}^\times$ and also depends linearly on a single cohomology insertion γ.

13.2.2

For example, for $\mathbb{H} = H^{\bullet}_{\mathsf{G}}(\mathbb{C}^2)\left[\frac{1}{\det_{\mathbb{C}^2}}\right]$ we have

$$1^\Delta = -1 \otimes \mathsf{pt} = -\mathsf{pt} \otimes 1\,.$$

This is because the comultiplication, as adjoint to multiplication, gets the sign $-1 = (-1)^{\frac{1}{2}\dim\mathbb{C}^2}$. Therefore, the formula (12.13) can be recast in the following form

$$\mathbf{r} = \frac{1}{2}\int :(\boldsymbol{\alpha}^-)^2:(1)\,, \tag{13.7}$$

modulo zero modes $\alpha_0(\gamma)$.

13.2.3

Because of the Frobenius algebra structure on $\mathbb{H}$, Wick's formula for the operators $:\boldsymbol{\alpha}^n:(\gamma)$ takes the following particularly nice form.

For any symmetric Frobenius algebra, there is a canonical central element $\mathsf{e} \in \mathbb{H}$ such that

$$m(\gamma^{\Delta}) = \mathsf{e}\gamma$$

for all $\gamma \in \mathbb{H}$. Here $m : \mathbb{H}^{\otimes 2} \to \mathbb{H}$ is the multiplication map. This is associated with gluing a handle in the context of 2-dimensional topological quantum field theories, see for example [65]. One has

$$\tau(\mathsf{e}) = \operatorname{rk}_{\mathbb{K}} \mathbb{H} \,.$$

In particular, if $\mathbb{H} = H^*(S)$ then this is the Euler characteristic of S (recall we assume $\mathbb{H}$ is commutative for simplicity).

Lemma 13.2.1.

$$:\boldsymbol{\alpha}^n:(\gamma_1)(z_1) :\boldsymbol{\alpha}^m:(\gamma_2)(z_2) = \sum_{k=0}^{\min(n,m)} c_k(z_1,z_2) :\boldsymbol{\alpha}^{n-k}(z_1)\boldsymbol{\alpha}^{m-k}(z_2):\left(\gamma_1\gamma_2\mathsf{e}^{k-1}\right) \,, \quad (13.8)$$

where

$$c_k(z_1,z_2) = \frac{(-n)_k(-m)_k}{k!} \left(\frac{z_1 z_2}{(z_1-z_2)^2}\right)^k \,. \quad (13.9)$$

Here $(n)_k = n(n+1)\cdots(n+k-1)$ and the combinatorial factor in (13.9) is the number of ways to form k pairs of elements from $\{1,\dots,n\}$ and $\{1,\dots,m\}$, respectively.

Two terms in (13.8) require a special discussion. For $k=0$, the insertion is defined to be

$$\gamma_1^{\Delta n} \otimes \gamma_2^{\Delta m} \in \mathbb{H}^{\otimes(n+m)} \,.$$

For $n=m=k$, the whole term is defined to be

$$c_k(z_1,z_2)\, \tau(\gamma_1\, \gamma_2\, \mathsf{e}^{k-1}) \,.$$

Proof. This is an exercise in matching the Wick's formula with the graphical calculus for Frobenius algebras, as explained, for example, in [65]. The tensor operations

$$\mathbb{H}^{\otimes 2} \to \mathbb{H}^{\otimes(n+m-2k)}$$

that arise from Wick's formula, are interpreted graphically as surface of genus $k-1$ with two incoming and $n+m-2k$ outgoing holes, hence equal to

$$\gamma_1 \otimes \gamma_2 \to \left(\gamma_1 \gamma_2 \mathsf{e}^{k-1}\right)^{\Delta(n+m-2k)} .$$

Note for $k=0$, the surface is disconnected, whence the need to consider this case separately. The other special case $n=m=k$ is the case of no outgoing holes. In this case, there is only the scalar operator left in Wick's formula. □

It is straightforward to generalize this Lemma to more than two normally ordered monomials.

13.3 Virasoro algebra

13.3.1

For an arbitrary $\kappa \in \mathbb{H}$, define

$$\boldsymbol{T}(\gamma,\kappa) = \tfrac{1}{2} : \boldsymbol{\alpha}^2 : (\gamma) + \partial \boldsymbol{\alpha}(\gamma\kappa) - \tfrac{1}{2}\tau(\gamma\,\kappa^2) . \tag{13.10}$$

This field generates a Virasoro-like subalgebra of the Heisenberg algebra, known as the Feigin-Fuchs or background charge Virasoro algebra. The statement for an arbitrary $\mathbb{H}$ should also be considered known, see for example the discussion in Section 5 of [67].

13.3.2

We denote by $\mathsf{L}_n(\gamma,\kappa)$ the coefficients of $\boldsymbol{T}(\gamma,\kappa)$, that is,

$$\boldsymbol{T}(\gamma,\kappa) = \sum_{n\in\mathbb{Z}} \mathsf{L}_n(\gamma,\kappa)\, z^{-n} .$$

Theorem 13.3.1. *The operators* $\mathsf{L}_n(\gamma,\kappa)$ *satisfy*

$$[\mathsf{L}_n(\gamma_1), \mathsf{L}_m(\gamma_2)] = (n-m)\,\mathsf{L}_{n+m}(\gamma_1\,\gamma_2) + \tau(\gamma_1\gamma_2(\mathsf{e}-12\kappa^2))\,\delta_{n+m}\frac{n^3-n}{12} . \tag{13.11}$$

These are the familiar Virasoro relations adorned with cohomology labels. The element

$$\mathbf{c} = \mathsf{e} - 12\kappa^2 \in \mathbb{H} \tag{13.12}$$

plays the role of the central charge.

13.3.3 OPEs

The most efficient way to encode the commutation relations for the operators $\boldsymbol{T}$ is via the operator product expansion. This goes as follows. Let the fields

$$A(z) = \sum_{n\in\mathbb{Z}} a_n\, z^{-n}\,, \quad B(z) = \sum_{n\in\mathbb{Z}} b_n\, z^{-n}\,,$$

satisfy a commutation relation of the form

$$[A(z), B(w)] = \sum_{k\geqslant 0} C_k(w) \left(w\,\frac{\partial}{\partial w}\right)^k \delta(z,w)\,,$$

where $\delta(z,w) = \sum_{n\in\mathbb{Z}}(z/w)^n$ and $C_k(w)$ are some fields like A and B. Then

$$\begin{aligned} A(z)B(w) &= [A_-(z), B(w)] + {:}A(z)B(w){:} \\ &\sim \sum_k C_k(w) \left(w\frac{\partial}{\partial w}\right)^k \frac{w}{z-w} \end{aligned} \tag{13.13}$$

where $A_-(z) = \sum_{n>0} a_n\, z^{-n}$ and $\sim$ means equality modulo terms that remain regular as $z \to w$. In particular, in (13.13) we dropped the normally ordered term.

13.3.4 Proof of Theorem 13.3.1

Let

$$G = \frac{\sqrt{zw}}{z-w} = \frac{1}{e^{x/2} - e^{-x/2}}\,, \quad x = \ln(z/w)\,,$$

denote one of the Green's functions of the $\bar\partial$ operator on the cylinder. Since we will only deal with expansions as $z \to w$, we may ignore the monodromy of G.

Proposition 13.3.2. *The field $\boldsymbol{T}$ satisfies the following OPE*

$$\begin{aligned} &\boldsymbol{T}(\gamma_1)(z)\,\boldsymbol{T}(\gamma_2)(w) \sim \\ &\qquad \tfrac{1}{2}G^4\,\tau(\gamma_1\,\gamma_2(\mathsf{e} - 12\kappa^2)) + 2G^2\,\boldsymbol{T}(\gamma_1\gamma_2)(w) + G\,\partial\boldsymbol{T}(\gamma_1\gamma_2)(w)\,, \end{aligned} \tag{13.14}$$

where $\mathsf{e} \in H$ is the handle-gluing element.

Proof. Direct computation using Lemma 13.2.1. □

This proposition finishes the proof of Theorem 13.3.1.

13.3.5 Lowest weight

From definitions, we compute

$$\mathsf{L}_n(\gamma,\kappa)\,|\eta\rangle = 0\,, \quad n>0\,,$$

while

$$\mathsf{L}_0(\gamma,\kappa)\,|\eta\rangle = \tfrac{1}{2}\,\tau(\gamma(\eta^2-\kappa^2))\;|\eta\rangle\;.$$

For $\gamma = 1$ and $\kappa = 0$ this specializes to Lemma 13.1.1. The element

$$\mathbf{d} = \tfrac{1}{2}\,(\eta^2-\kappa^2) \in \mathbb{H} \tag{13.15}$$

should thus be viewed as the conformal dimension of $|\eta\rangle$, that is, the lowest weight of the Virasoro module $\mathsf{F}(\eta)$.

13.3.6 Irreducibility

We have the following standard

Lemma 13.3.3. *The Virasoro module $\mathsf{F}(\eta)$ is irreducible for generic η.*

Proof. For $\eta\to\infty$, Virasoro algebra degenerates to Heisenberg algebra which acts irreducibly. □

13.4 Reflection operator

13.4.1

Lemma 13.3.3 implies for generic η, $\mathsf{F}(\eta)$ is a Verma module for Virasoro algebra with central charge (13.12) and lowest weight (13.15). Note, however, that the map

$$(\eta,\kappa)\mapsto(\mathbf{d},\mathbf{c})$$

is many-to-one, in particular, the 4 points $(\pm\eta,\pm\kappa)$ give isomorphic Virasoro modules for generic parameters. This implies the following

Proposition 13.4.1. *For generic η and any choice of signs, there exists a unique, up to multiple, operator $R_{\pm,\pm}$ that makes the following diagram*

$$\begin{array}{ccc}
\mathsf{F}(\eta) & \xrightarrow{\ \boldsymbol{T}(\kappa)\ } & \mathsf{F}(\eta) \\
{\scriptstyle R_{\pm,\pm}}\downarrow & & \downarrow{\scriptstyle R_{\pm,\pm}} \\
\mathsf{F}(\pm\eta) & \xrightarrow{\ \boldsymbol{T}(\pm\kappa)\ } & \mathsf{F}(\pm\eta)
\end{array}$$

commute. It depends rationally on $\eta, \kappa \in \mathbb{H}$.

The first $\pm$ in $R_{\pm,\pm}$ is for η, the second — for κ. The intertwiner $R_{\pm,\pm}$ is a rational function of $\eta, \kappa \in \mathbb{H}$ because it solves linear equations in which η and κ enter polynomially. We normalize it so that

$$R_{\varepsilon_1,\varepsilon_2} |\eta\rangle = |\varepsilon_1 \eta\rangle \,.$$

13.4.2

In down-to-earth terms,

$$R_{\pm\pm} \prod \mathsf{L}_{-\mu_i}(\gamma_i, \kappa) \; |\eta\rangle = \prod \mathsf{L}_{-\mu_i}(\gamma_i, \pm\kappa) \; |\pm\eta\rangle$$

for all partitions μ. For generic η, these vectors form a basis of $\mathsf{F}(\pm\eta)$.

In particular, $R_{\pm\pm}$ preserves the grading by $|\mu|$, hence is a direct sum of finite-dimensional operators.

13.4.3

It is easy to see that

$$R_{--}\, \alpha_n(\gamma)\, R_{--}^{-1} = -\alpha_n(\gamma)\,. \tag{13.16}$$

Thus of the four operators $R_{\pm\pm}$ only one is really nontrivial. Also, we note

$$\eta = 0 \Rightarrow R_{+-} = R_{--}\,. \tag{13.17}$$

13.4.4

The operator R_{-+} is known as the reflection operator in Liouville CFT, see [118], while we will identify $\mathbf{R}(u)$ with the operator R_{+-} for

$$\mathbb{H} = H^{\bullet}_{\mathsf{G}}(\mathbb{C}^2) \left[\tfrac{1}{\det_{\mathbb{C}^2}} \right]$$

in Chapter 14. Thus, the Liouville reflection operator will be identified with

$$\mathbf{R}^{\vee} = (12)\, \mathbf{R}\,.$$

The Yang-Baxter equation satisfied by $\mathbf{R}(u)$ is a new and unexpected aspect of the theory.

13.4.5

In addition to the inner product discussed in Section 13.1.6, the action of the Virasoro algebra equips the Fock space with the Shapovalov inner product, such that

$$\mathsf{L}_n^\dagger = \mathsf{L}_{-n}\,,$$

where dagger denotes the adjoint operator with respect to the Shapovalov product.

We have

$$R_{-+}\,\mathsf{L}_n^\dagger = \mathsf{L}_n^*\,R_{-+}\,,$$

therefore R_{-+} is precisely the operator that relates the two inner products. In particular, the determinant of the graded pieces of $\mathbf{R}$ is very closely related to Kac determinant for Virasoro algebra, see [56, 40]. We will see the classical results of Feigin and Fuchs on it from a new perspective in Chapter 14.

Chapter 14

The full R-matrix

14.1 Zero modes

In Section 12.4.10, we identified the Lie algebra $\mathfrak{g}_Q$ for instanton moduli $\mathcal{M}(r)$ with the algebra $\widehat{\mathfrak{gl}(1)}$ modulo the zero modes. On the other hand, we saw in Chapter 13 the convenience and importance of including the zero modes in the considerations.

Later, a different normalization of $\mathbf{R}(u)$ will be introduced which will reconcile these two points of view. For now, until the Section 14.3.1, we set zero modes to zero.

14.2 Cup product by divisor

14.2.1

Generalizing the formula for $\mathbf{r}$, we define

$$\mathbf{\Phi}_n = \frac{1}{n!} \int :\boldsymbol{\alpha}^n: (1)\,.$$

These are examples of Fourier coefficients of vertex operators, see e.g. [45, 58]. The following operator $\mathbf{\Omega}$, while not a Fourier coefficient of a vertex operator, plays an important role in the theory.

14.2.2

Define the operator $|\partial|$ by

$$|\partial| \cdot z^n = |n| \, z^n \,.$$

This is a composition of $\partial = z\frac{d}{dz}$ and the Hilbert transform. We define

$$\mathbf{\Omega} = \frac{1}{2} \int :\boldsymbol{\alpha} |\partial| \boldsymbol{\alpha}: (1) = \sum_{n>0} n \, \alpha_{-n} \alpha_n (1^\Delta) \,.$$

14.2.3

The operator $\mathbf{\Omega}$ appears in the following formula due to M. Lehn [66]. Recall that

$$\mathcal{O}(1) = \Lambda^{\mathrm{top}} \, \mathrm{Taut} \,, \quad \mathrm{Taut} = \mathcal{V}_1 = H^1_{\mathbb{P}^2}(\mathcal{F}(-1))$$

is the ample generator of the Picard group of $\mathcal{M}(r)$.

Theorem 14.2.1 ([66])**.** *The operator of cup product by $c_1(\mathcal{O}(1))$ in $H^{\bullet}_{\mathsf{T}}(\mathrm{Hilb})$ is given by*

$$c_1(\mathcal{O}(1)) \, \cup \, \cdots = -\mathbf{\Phi}_3 + (a - \tfrac{1}{2}\,\hbar)\mathbf{\Phi}_2 + \tfrac{1}{2}\,\hbar\,\mathbf{\Omega} \,. \qquad (14.1)$$

Here a is the weight of the framing torus $\mathsf{A} \cong \mathbb{C}^\times$ that acts trivially on $\mathcal{M}(1)$ itself, but nontrivially, namely with weight a, on the tautological bundle. Such an insignificant additional parameter is usually suppressed and, in particular, it is not present in Lehn's formulation.

Lehn's theorem may be also deduced from the factorization of $\mathbf{R}(u)$ into R-matrices for $\mathsf{Y}(\mathfrak{gl}(\infty))$ given in Theorem 4.3.1, see [114].

14.2.4

Lehn's theorem identifies the operator of cup product by $c_1(\mathcal{O}(1))$ with the second quantized trigonometric Calegero-Sutherland Hamiltonian, see for example [19] for a comprehensive discussion.

The explicit form of the Calogero-Sutherland operator in the basis of power-sum symmetric functions (that is, in the natural basis of the bosonic Fock space) was computed by Richard Stanley [117] and rediscovered many times since. The equivalence between Lehn's and Stanley's formulas was noticed, apparently, by many people, [76] being one of the early references, see the discussion in [19].

Note that classes of torus-fixed points in $H^{\cdot}_{\mathsf{T}}(\text{Hilb})$ are trivially eigenfunctions of cup product operators and their identification with Jack polynomials, that is, CS eigenfunctions, was noted earlier, see in particular [85]. At about the same time, it was recognized by Mark Haiman that the more general Macdonald polynomials correspond to the classes of fixed points in the equivariant K-theory of Hilbert schemes, see for example [52].

14.2.5

We will see the analogous integrable system for $\mathcal{M}(r,n)$ is a coupled r-tuple of Calogero-Sutherland systems. The coupling is triangular, so the spectrum is additive, which is obvious from the geometric description of torus-fixed points. Independently of our work, the same quantum integrable system appeared in [28].

The algebra of operators of quantum multiplication gives a one-parameter deformation of cup product operators and thus a deformation of the Calogero-Sutherland quantum integrable system. It has been identified with the quantum Intermediate Long Wave equation [94]. In particular, this allows to determine the spectrum of the latter as well as to give an explicit construction of integrals of motion.

14.2.6

Taking the expansion (12.12) one step further, we get

$$\frac{e(N_-)}{e(N_- \otimes \hbar)} = 1 + \frac{\hbar\, \mathrm{rk}}{u} + \frac{\hbar\, c_1(\mathcal{O}(1)) + \frac{1}{2}\,\hbar^2\, \mathrm{rk}(\mathrm{rk}+1)}{u^2} + \dots \tag{14.2}$$

where $u = a_1 - a_2$ and $c_1(\mathcal{O}(1))$ is the operator from Theorem 14.2.1 with $a = 0$. This is because we already accounted for the fact that N_- has weight $-u$ with respect to the rank 2 framing torus.

14.2.7

Proposition 14.2.2. *We have*

$$\mathbf{R}(u) = 1 + \frac{\hbar}{u}\,\mathbf{\Phi}_2^- + \frac{\hbar}{u^2}\,\mathbf{\Phi}_3^- + \frac{\hbar^2}{2u^2}\left(\mathbf{\Phi}_2^-\right)^2 + O(u^{-3})\,. \tag{14.3}$$

Here and in what follows, $\mathbf{\Phi}_n^-$ denotes the result of substituting $\boldsymbol{\alpha}^-$ for $\boldsymbol{\alpha}$ in the definition of $\mathbf{\Phi}_n$.

Proof. Denote by P the orthogonal projection onto $\mathsf{vac} \in \mathsf{F}$. We compute

$$\begin{aligned} \mathsf{P}^{(1)}\, \mathbf{\Phi}_3^-\, \mathsf{P}^{(1)} &= -\mathbf{\Phi}_3^{(2)}\,, \\ \mathsf{P}^{(1)} \left(\mathbf{\Phi}_2^-\right)^2 \mathsf{P}^{(1)} &= \left(\mathbf{\Phi}_2^{(2)}\right)^2 + \mathbf{\Omega}^{(2)}\,, \end{aligned} \tag{14.4}$$

where upper indices like the one in $\mathsf{P}^{(1)}$ denote an operator acting in the corresponding tensor factor of $\mathsf{F} \otimes \mathsf{F}$. It is very instructive to see how Fourier coefficients of vertex operator produce something which isn't one upon taking vacuum matrix elements.

Now the result follows from comparing (14.1) with (14.2). □

Note, for example, that

$$\mathbf{R}(-u)_{12} = 1 - \frac{\hbar}{u}\, \mathbf{\Phi}_2^- - \frac{\hbar}{u^2}\, \mathbf{\Phi}_3^- + \frac{\hbar^2}{2u^2} \left(\mathbf{\Phi}_2^-\right)^2 + O(u^{-3})\,,$$

because the permutation of tensor factors flips the sign of α^-. This illustrates general results on unitarity of R-matrices, see Section 4.5.

14.2.8

We now consider an $(r+1)$-fold tensor power of F and denote by

$$\boldsymbol{\alpha}^{(i)}\,, \quad i = 0, \dots, r\,,$$

the Heisenberg operators in the corresponding tensor factors. We denote

$$\mathbf{\Phi}_n^{(ij)} = \frac{1}{n!} \int :(\boldsymbol{\alpha}^{(i)} - \boldsymbol{\alpha}^{(j)})^n: (1)$$

and

$$\mathbf{\Omega}^{(ij)} = \sum_{n>0} n\, \alpha_{-n}^{(i)} \alpha_n^{(j)} (1^\Delta)\,.$$

In particular, $\mathbf{\Phi}_n^{(12)} = \mathbf{\Phi}_n^-$ and $\mathbf{\Omega}^{(ii)}$ is the operator $\mathbf{\Omega}$ acting in the ith tensor factor. Generalizing (14.4), we compute

$$\mathsf{P}^{(0)}\, \mathbf{\Phi}_2^{(0j)} \mathbf{\Phi}_2^{(0i)}\, \mathsf{P}^{(0)} = \mathbf{\Phi}_2^{(j)} \mathbf{\Phi}_2^{(i)} + \mathbf{\Omega}^{(ji)}\,. \tag{14.5}$$

14.2.9

We consider $X = \mathcal{M}(r)$ and the action of the maximal torus A of $GL(r)$ on it. Fix a chamber $\mathfrak{C} \subset \mathfrak{a}$ and denote by Q_{cl} the operator that makes the following diagram commute

$$\begin{array}{ccc} \mathsf{F}^{\otimes r} & \xrightarrow{\mathrm{Stab}_{\mathfrak{C}}} & H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r)) \otimes \mathbb{K} \\ \Big\downarrow \mathsf{Q}_{cl} & & \Big\downarrow \cup c_1(\mathcal{O}(1)) \\ \mathsf{F}^{\otimes r} & \xrightarrow{\mathrm{Stab}_{\mathfrak{C}}} & H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r)) \otimes \mathbb{K}\,. \end{array}$$

Consider the following modified step function

$$\varrho(x) = \begin{cases} 1\,, & x > 0\,, \\ 1/2\,, & x = 0\,, \\ 0\,, & x < 0\,, \end{cases}$$

and define

$$\varrho_{\mathfrak{C}}(i,j) = \varrho\left((a_i - a_j)|_{\mathfrak{C}}\right)\,.$$

Theorem 14.2.3. *The operator Q_{cl} is given by*

$$\mathsf{Q}_{cl} = \sum_{i=1}^{r} \left(-\mathbf{\Phi}_3^{(i)} + (a_i - \tfrac{1}{2}\hbar)\mathbf{\Phi}_2^{(i)}\right) + \hbar \sum_{i,j=1}^{r} \varrho_{\mathfrak{C}}(i,j)\, \mathbf{\Omega}^{(j,i)}\,. \tag{14.6}$$

This is a special case of Theorem 10.1.1. We recall the proof.

Proof. Using Theorem 4.7.1 and equation (14.2), in particular, the operator Q_{cl} may be computed from the $1/u^2$ coefficient of the R-matrix from Example 4.2.4. We substitute the formula from Proposition 14.2.2 and expand using (14.5). This gives the result. □

14.2.10

Note for the standard chamber $\mathfrak{C}$, we have

$$\sum_{i,j=1}^{r} \varrho_{\mathfrak{C}}(i,j)\, \mathbf{\Omega}^{(j,i)} = \frac{1}{4}\int :\boldsymbol{\beta}\,|\partial|\,\boldsymbol{\beta}:(1) + \frac{1}{2}\sum_{i<j} \int \boldsymbol{\alpha}^{(i)}\,\partial\,\boldsymbol{\alpha}^{(j)}(1) \tag{14.7}$$

where

$$\boldsymbol{\beta} = \boldsymbol{\alpha}^{(1)} + \cdots + \boldsymbol{\alpha}^{(r)}\,.$$

For general $\mathfrak{C}$, the final sum in (14.7) is over all i,j such that $a_i - a_j$ is positive on $\mathfrak{C}$.

14.3 R-matrix as a Virasoro intertwiner

14.3.1

Theorem 14.3.1. *The operator* $\mathbf{R}(u)$ *is obtained by substitution*

$$\boldsymbol{\alpha} = \frac{1}{\sqrt{2}}\,\boldsymbol{\alpha}^- \,, \quad \eta = \frac{u}{\sqrt{2}} \,, \quad \kappa = \frac{1}{\sqrt{2}}\,\hbar \,. \tag{14.8}$$

into the Virasoro intertwiner R_{+-} *for* $\mathbb{H} = \mathbb{H}(\mathbb{C}^2)$.

Here $u = a_1 - a_2$ and $\mathbb{H}(\mathbb{C}^2)$ is the Frobenius algebra (13.1). The square roots in (14.8) are needed because of the factor 2 in (12.8). In other words, they are there because the vector $(1, -1)$ has length $\sqrt{2}$.

14.3.2 Proof of Theorem 14.3.1

From Lemma 12.4.1, we know that $\mathbf{R}(u)$ acts only in the F^- factor in (12.9). To find out how it acts in F^-, we will use the intertwining relation with the operators Q_{cl} for the two chambers

$$a_1 \gtrless a_2 \,.$$

We express Q_{cl} in terms of $\boldsymbol{\alpha}^\pm$ and note that $\boldsymbol{\alpha}^+$ commutes with $\mathbf{R}$. In particular, the first term in the right-hand side of (14.7) commutes with $\mathbf{R}$. Therefore we have, for $\mathfrak{C}_\pm = \{a_1 \gtrless a_2\}$

$$-2\mathsf{Q}_{cl} = \cdots + \frac{1}{4}\int \boldsymbol{\alpha}^+ :(\boldsymbol{\alpha}^-)^2:(1) + \int \boldsymbol{\alpha}^+ \left(\frac{a_2 - a_1}{2}\,\boldsymbol{\alpha}^- \pm \frac{\hbar}{2}\,\partial\boldsymbol{\alpha}^-\right)(1) \,, \tag{14.9}$$

where dots stand for terms that commute with $\mathbf{R}$.

Since $\mathbf{R}$ commutes with $\boldsymbol{\alpha}^+$, it has to intertwine the coefficients of its modes in (14.9), therefore it has to intertwine the operators

$$\boldsymbol{T}_\pm(\gamma) = \frac{1}{4}:(\boldsymbol{\alpha}^-)^2:(\gamma) + \left(\frac{a_2 - a_1}{2}\,\boldsymbol{\alpha}^- \pm \frac{\hbar}{2}\,\partial\boldsymbol{\alpha}^-\right)(\gamma) + \dots \tag{14.10}$$

for all $\gamma \in H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathbb{C}^2)$. Here dots stand for a scalar operator that will be fixed in a minute.

Strictly speaking, since $\boldsymbol{\alpha}^+$ does not include zero modes, the above argument shows $\mathbf{R}$ intertwines all coefficients of $\boldsymbol{T}_\pm$ except the constant term $\int \boldsymbol{T}_\pm(\gamma)$. However, this constant term can be obtained as commutator of other coefficients of $\boldsymbol{T}_\pm$, by Virasoro commutation relations.

We now compare (14.10) with (13.10). The two operators become identical if we substitute

$$\boldsymbol{\alpha} = \frac{1}{\sqrt{2}}\,\boldsymbol{\alpha}^-\,, \quad \kappa = \frac{1}{\sqrt{2}}\,\hbar\,,$$

and make the zero mode present in (13.10) act via the identification

$$H^{\cdot}_{\mathsf{G}}\left(\mathcal{M}(2)^{\mathsf{A}}\right) \otimes \mathbb{K} \cong \mathsf{F}(a_1) \otimes \mathsf{F}(a_2)\,. \tag{14.11}$$

This identification fixes the constant term left as dots in (14.10). Thus $\mathbf{R}(u)$ is identified with R_{+-} by the uniqueness of the latter.

14.3.3 The determinant of $\mathbf{R}(u)$

By construction, $\mathbf{R}(u)$ is a product of two triangular operators, namely of the composition

$$H^{\cdot}_{\mathsf{G}}(X^{\mathsf{A}}) \xrightarrow{\text{Stab}_{\mathfrak{C}}} H^{\cdot}_{\mathsf{G}}(X) \xrightarrow{\text{Restriction}} H^{\cdot}_{\mathsf{G}}(X^{\mathsf{A}})\,,$$

and the inverse of the analogous composition for the other chamber. Each of these operators has simple diagonal parts, yielding a factorization for the determinant of the graded pieces of $\mathbf{R}$. This gives an alternative derivation of the product formula for the determinant of the Shapovalov form [56, 40].

14.3.4

From (13.16) and (13.17), we conclude

$$\mathbf{R}(0) = (12)$$

where (12) is the permutation of the two factors. This is because

$$(12)\,\boldsymbol{\alpha}^-(12) = -\boldsymbol{\alpha}^-\,.$$

14.4 The $1/u$ expansion of R

14.4.1

In this section we derive an expansion of $\log \mathbf{R}(u)$ in inverse powers of the spectral parameter u. We write

$$\boldsymbol{T}(\gamma)_\pm = -\frac{u}{2}\,\boldsymbol{\alpha}^-(\gamma) + \boldsymbol{T}'(\gamma)_\pm\,, \quad \boldsymbol{T}'_\pm = \tfrac{1}{4}\,{:}(\boldsymbol{\alpha}^-)^2{:}(\gamma) \pm \tfrac{\hbar}{2}\,\partial\boldsymbol{\alpha}^-(\gamma) + \dots\,,$$

where dots stand for a constant term that cancels out of the equation

$$\mathbf{R}\,\boldsymbol{T}_+(\gamma)\,\mathbf{R}^{-1} = \boldsymbol{T}_-(\gamma)\,. \tag{14.12}$$

We look for solutions in the form

$$\mathbf{R} = \exp\left(\sum_{n>0} \frac{\mathbf{r}^{(n)}}{u^n}\right)$$

where, in particular,

$$\mathbf{r}^{(1)} = \frac{1}{2}\int {:}(\boldsymbol{\alpha}^-)^2{:}(\hbar)$$

is, up to normalization, the familiar classical R-matrix. We denote by $\mathbf{R}^{(m)} = \exp\left(\sum_{0<n\leqslant m} \frac{\mathbf{r}^{(n)}}{u^n}\right)$ the successive approximations. The recurrence relations for $n > 1$ take the form

$$\left[\mathbf{r}^{(n)}, \boldsymbol{\alpha}^-(\gamma)\right] = 2[u^{-n+1}]\ \exp(\mathrm{ad}(\log \mathbf{R}^{(n-1)})) \cdot \boldsymbol{T}'_+(\gamma)\,. \tag{14.13}$$

where $[u^{-n+1}]$ denotes the coefficient of u^{-n+1}. These fix $\mathbf{r}^{(n)}$ uniquely up to an additive constant. The constant is determined by the requirement that $\mathbf{r}^{(n)}$ annihilates the vacuum vector.

14.4.2

Solving equations (14.13), we obtain

$$\begin{aligned}
\mathbf{r}^{(2)} &= \frac{1}{6}\int {:}(\boldsymbol{\alpha}^-)^3(\hbar){:} \\
\mathbf{r}^{(3)} &= \frac{1}{12}\int {:}(\boldsymbol{\alpha}^-)^4{:}(\hbar) - \frac{1}{12}\int {:}(\boldsymbol{\alpha}^-)^2{:}(\hbar\mathsf{e}) \\
&\quad - \frac{1}{12}\int {:}\left(\partial\boldsymbol{\alpha}^-\right)^2{:}(2\hbar^3 + \hbar\mathsf{e})\ ,
\end{aligned} \tag{14.14}$$

where

$$\mathsf{e} = -\det_{\mathbb{C}^2} \in H^{\bullet}_{\mathsf{G}}(\mathsf{pt})$$

is the handle-gluing element. Of course, since our Frobenius algebra is 1-dimensional, all cohomology insertions may be converted to coefficients in the formula.

14.4.3

Further structures in this expansion will be discussed elsewhere. Here we only note the following. The normally ordered polynomials in the field $\boldsymbol{\alpha}^-$ and its derivatives are, from definitions, *vertex operators* in the Heisenberg vertex algebra. Integrals of such operators are known as *residues* of vertex operators. They act as infinitesimal automorphisms of the Heisenberg vertex algebra.

Theorem 14.4.1. *The logarithm of* $\mathbf{R}$ *is a residue of a vertex operator, that is*

$$\mathbf{r}^{(n)} = \int : P_n(\boldsymbol{\alpha}^-, \partial\boldsymbol{\alpha}^-, \partial^2\boldsymbol{\alpha}^-, \dots; \hbar, \mathsf{e}) : (1)\,,$$

for some polynomials P_n.

Proof. The commutator of a vertex operator with a residue of a vertex operator is again a vertex operator. Therefore, by induction, the equation for $\mathbf{r}^{(n)}$ has the form

$$\left[\mathbf{r}^{(n)}, \boldsymbol{\alpha}^-(\gamma)\right] = \text{vertex operator}\,.$$

One can see explicitly that this equation is solved by a residue of a vertex operator. □

14.4.4

Also note that in the grading such that

$$\deg \boldsymbol{\alpha} = \deg \hbar = 1\,, \quad \deg e = 2$$

the polynomial P_n is homogeneous of degree

$$\deg P_n = n + 2\,.$$

Chapter 15

Quantum multiplication for $\mathcal{M}(r, n)$

We can now return to the formulas for quantum multiplication for $\mathcal{M}(r, n)$ using the computations of the last chapters.

15.1 Explicit formulas

Let us first state explicitly the operator for modified quantum multiplication by $c_1(\mathcal{O}(1))$. We will express them in terms of the Heisenberg operators $\alpha_k^{(i)}(\mathsf{pt})$ and $\alpha_{-k}^{(i)}(1)$ for $k > 0$ and $1 \leqslant i \leqslant r$. These satisfy the commutation relations

$$[\alpha_k^{(i)}(\mathsf{pt}), \alpha_{-k}^{(j)}(1)] = -\delta_{i,j} k = \delta_{i,j} \cdot k \cdot \tau(\mathsf{pt}).$$

Up to a scalar operator, we have

$$\mathsf{Q} = \text{Cubic} + \text{Quadratic} + \text{Purely Quantum}$$

where we have decompose the contribution of classical multiplication into cubic and quadratic expressions in the Heisenberg generators. The formula for the cubic term is

$$\text{Cubic} = \sum_{i=1}^{r} -\frac{1}{2} \sum_{n,m>0} \left(t_1 t_2 \alpha_{-n}^{(i)}(1) \alpha_{-m}^{(i)}(1) \alpha_{n+m}^{(i)}(\mathsf{pt}) + \alpha_{-n-m}^{(i)}(1) \alpha_n^{(i)}(\mathsf{pt}) \alpha_m^{(i)}(\mathsf{pt}) \right) .$$

The classical quadratic term is

$$\text{Quadratic} = -\sum_{i=1}^{r}\sum_{n>0}(t_1+t_2)\cdot(a_i+\frac{1-n}{2})\cdot\alpha_{-n}^{(i)}(1)\alpha_n^{(i)}(\mathsf{pt})$$
$$+\sum_{i<j}\sum_{n>0}(t_1+t_2)\cdot n\cdot\alpha_{-n}^{(j)}(1)\alpha_n^{(i)}(\mathsf{pt})\,.$$

The purely quantum term is

$$\text{Purely quantum} = (t_1+t_2)\sum_{n>0}\frac{nq^n}{1-q^n}\cdot\beta_{-n}(1)\beta_n(\mathsf{pt})\,,$$

where

$$\beta_{-n}(1)=\sum_{i=1}^{r}\alpha_{-n}^{(i)}(1) \text{ and } \beta_n(\mathsf{pt})=\sum_{i=1}^{r}\alpha_n^{(i)}(\mathsf{pt})$$

are the Baranovsky operators.

We can determine the scalar discrepancy as follows. For $r>1$, there is no correction required. For $r=1$, we need to add the scalar term

$$-(t_1+t_2)\frac{q}{1-q}\sum_{n>0}\alpha_{-n}(1)\alpha_n(\mathsf{pt}).$$

This follows from the evaluation of $\mathsf{Q}\cdot 1$ which comes via the following lemma.

Lemma 15.1.1. *We have the following vanishing statement:*

$$\beta_k(\mathsf{pt})\cdot 1 = 0\,,\ \text{if } k\geqslant 2,\ \text{or } k=1\,, r\geqslant 2. \tag{15.1}$$

Proof. The dimension of the fiber of the punctual Baranovsky correspondence in (12.3) over a generic point of $\mathcal{M}(r,n)$ is

$$r\cdot k-1$$

which is positive under the hypotheses of the Lemma. Therefore, the push-forward of the fundamental class under this projection vanishes. □

15.2 Generation statement

As a corollary, we can deduce the following:

Theorem 15.2.1. *The divisor $c_1(\mathcal{O}(1))$ generates the quantum cohomology ring of $\mathcal{M}(r,n)$.*

Proof. It suffices to show that $\mathsf{Q}(q,t_1,t_2,a_1,\ldots,a_r)$ has distinct eigenvalues for generic values of the parameters.

First, notice that by taking the substitution

$$t_1 = t, t_2 = t^{-1}, a_i = ta_i$$

and studying the limit

$$\mathsf{Q}_0 = \lim_{t\to\infty} \frac{1}{t}\mathsf{Q}$$

as $t \to \infty$, we can ignore the cubic term, and show the remaining operator has distinct eigenvalues.

For $n \geqslant 1$, let

$$V_n = \bigoplus_{i=1}^{r} \mathbb{Q}e_n^{(i)} .$$

We have an identification

$$\mathsf{F}^{\otimes r} = \mathrm{Sym}^*(\bigoplus_n V_n)$$

characterized by sending $\mathsf{vac}^{\otimes r}$ to 1 and requiring $\alpha_{-n}^{(i)}(1)$ to act by multiplication by $e_n^{(i)}$ on the right-hand side. In other words, if we think of the left-hand side as r-tuples of partitions, then the right-hand side is the decomposition into parts of size k.

We can decompose Q_0 in terms of V_n as follows. Let

$$A_n(q,a_1,\ldots,a_r) = -n\sum_{i=1}^{r}\left(a_i + \frac{1-n}{2}\right)E_{ii} + n^2\sum_{i<j}E_{ji} + \frac{n^2q^n}{1-q^n}\sum_{i,j}E_{ji}$$

be a matrix valued function acting on V_n, where E_{ji} is the matrix with 1 in position (j,i) and 0 elsewhere. We extend A_n by zero to an operator on

$\bigoplus V_n$ and, by the Leibniz rule, to a derivation $D(A_n)$ on $\mathrm{Sym}^*(\bigoplus V_n)$. Then it follows from our formulas that

$$\mathsf{Q}_0 = \sum_n D(A_n).$$

In particular, the eigenvalues of Q_0 are non-negative linear combinations of the eigenvalues of A_n. The nondegeneracy of the spectrum of Q is a consequence of the following lemma. □

Lemma 15.2.2. *For very general values of $a_1, \ldots, a_r$ and q, there is no nontrivial finite linear relation*

$$\sum_{n,i} c_{n,i}\gamma_n^{(i)} = 0 \tag{15.2}$$

between the eigenvalues $\{\gamma_n^{(i)}\}$ of $A_n(q, a_1, \ldots, a_r)$, with $c_{n,i} \in \mathbb{Q}$.

Proof. Suppose otherwise. Then there exists such a relation that is valid for all values of parameters for which the operators A_n are well-defined. Let n be the largest index appearing in the relation with some nonzero coefficient $c_{n,i}$.

Fix a base point $p = [q = 0, a_1, \ldots, a_r] \in \mathbb{C} \times \mathbb{C}^{\times r}$ so that the a_i are distinct. The eigenvalues of $A_n(p)$ are

$$\gamma_n^{(i)}(p) = -n\left(a_i + \frac{1-n}{2}\right), \quad i = 1, \ldots, r.$$

Let $U \subset \mathbb{C} \times \mathbb{C}^{\times r}$ be the complement of the discriminant loci for $A_j(q, a_i)$ with $j \leqslant n$; each A_j has nondegenerate spectrum over U. Since $p \in U$, we know that U is nonempty.

Let $\zeta = e^{2\pi i/n}$ be a primitive n-th root of unity. Choose an analytic path $\Gamma : [0, 1) \to U$ such that $\Gamma(0) = p$,

$$\lim_{s\to 1} \Gamma(s) = (\zeta, a_1', \ldots, a_r'),$$

and that Γ meets the hypersurface $q = \zeta$ transversely at this limit point. As $q \to \zeta$, the last term in the formula for A_n dominates the others. Since the matrix $\sum_{i,j} E_{ij}$ has eigenvalues $\{1, 0, \ldots, 0\}$, it follows from perturbation theory of linear operators that one of the eigenvalues of A_n goes to infinity on the order of $\frac{1}{|q-\zeta|}$ as $s \to 1$, while the others grow at a slower rate (or remain bounded). Without loss of generality, we can assume that it is $\gamma_n^{(1)}$. Furthermore, for $j < k$, the operator A_j has a well-defined limit as $q \to \zeta$, so its eigenvalues remain bounded.

Therefore, if we take the relation (15.2) along the path Γ, $\gamma_n^{(1)}$ dominates the other terms, so this forces its coefficient to vanish:

$$c_{n,1} = 0.$$

For $1 < i \leqslant r$, if we choose a permutation σ of $1, \ldots, r$ that sends 1 to i, then we can choose a path from p to $\sigma(p)$, contained in the hyperplane $q = 0$, and concatenate with the path $\sigma(\Gamma)$ starting from $\sigma(p)$. Under this concatenation, the eigenvalue $\gamma_n^{(i)}$ is now the dominant term, so this forces

$$c_{n,i} = 0$$

for all i. This is a contradiction, so no nontrivial relation exists. □

Chapter 16

Gamma functions

16.1 The bundle $\widehat{\mathcal{V}}$

16.1.1

Recall that the main ingredient in the construction of the core Yangian $\mathbb{Y}$ is the Chern character of

$$\widehat{\mathcal{V}} = \mathcal{V} - \hbar^{-1} \otimes \mathsf{C}^{-1} \mathcal{W} .$$

We begin by identifying this K-theory class for the moduli spaces of framed sheaves.

Let t_1, t_2 denote the weights of the $G_{\text{edge}} = GL(2)$ action on $\mathbb{C}^2$. Then $\hbar = t_1^{-1} t_2^{-1}$, written multiplicatively, and the equivariant Cartan matrix equals

$$\mathsf{C} = (1 - t_1)(1 - t_2) ,$$

as already discussed in Section 2.5.9. If

$$\mathsf{w} = \sum a_i$$

is the character of the framing space then

$$\hbar^{-1} \otimes \mathsf{C}^{-1} \mathcal{W} = \frac{\sum a_i}{(1 - t_1^{-1})(1 - t_2^{-1})} = \text{character } H^0(\mathbb{C}^2, \mathcal{O}^{\otimes r}) \tag{16.1}$$

where $GL(2)$ acts on $\mathbb{C}^2$ and $G_{\mathsf{w}} = GL(r)$ acts by automorphisms of the trivial bundle $\mathcal{O}^{\otimes r}$. In gauge theory, G_{w} is known as the group of constant gauge transformations.

This gives us the following interpretation of $\widehat{\mathcal{V}}$.

16.1.2

In the sheaf language, the tautological bundle $\mathcal{V}$ is interpreted as the bundle with fiber $H^1(\mathbb{P}^2, \mathcal{F}(-1))$ over $\mathcal{F} \in \mathcal{M}(r)$, where (-1) denotes twisting down by the line at infinity. We claim

$$\widehat{\mathcal{V}} = -H^0(\mathbb{C}^2, \mathcal{F})$$

in K-theory of $\mathcal{M}(r)$. Indeed, consider the following exact sequence of sheaves on $\mathbb{P}^2$

$$0 \to \mathcal{F}(-1) \to \mathcal{F}(+\infty) \to \bigoplus_{d \geqslant 0} \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus r} \to 0\,,$$

where $\mathbb{P}^1 = \mathbb{P}^2 \backslash \mathbb{C}^2$ is the line at infinity. From the corresponding long exact sequence and its special case $\mathcal{F} = \mathcal{O}$, we obtain

$$0 \to H^0(\mathbb{C}^2, \mathcal{F}) \to H^0(\mathbb{C}^2, \mathcal{O}^{\oplus r}) \to \mathcal{V} \to 0\,,$$

as desired.

16.2 Barnes' Γ-function

16.2.1

Moduli spaces of framed sheaves provide a nice example of the Γ-function regularization from Section 6.1.10. In particular, the bundle (6.10) for $\mathsf{w} = \mathsf{w}' = 1$ specializes to the negative of (16.1) with $r = 1$ and $a_1 = 1$.

We have

$$\text{character } H^0(\mathbb{C}^2, \mathcal{O})^\vee = \sum_{i,j \geqslant 0} a^{-1} t_1^i t_2^j$$

thus, symbolically,

$$c(H^0(\mathbb{C}^2, \mathcal{O})^\vee, u) = \text{“}\prod_{i,j \geqslant 0} (u - a + t_1 i + t_2 j)\text{”}\,.$$

This is regularized using Barnes' multiple Γ-function (specifically, double Γ-function), see [106] for a modern reference, with the result that

$$c(H^0(\mathbb{C}^2, \mathcal{O})^\vee, u) = \Gamma(u - a |\, t_1, t_2)^{-1}\,. \tag{16.2}$$

Note that the same regularization (and, essentially, for the same reason) appears as the perturbative part of Nekrasov partition functions, see [93].

16.2.2

By definition,

$$\log\Gamma(u|\,t_1,t_2) = \frac{\partial}{\partial s}\zeta(s,u|\,t_1,t_2)\Big|_{s=0}\,,$$

where

$$\zeta(s,u\,|\,t_1,t_2) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{dz}{z}\,z^s\,\frac{e^{-uz}}{(1-e^{-t_1 z})(1-e^{-t_2 z})}\,,\quad \Re s > 2\,.$$

An asymptotic expansion of $\zeta(s,u\,|\,t_1,t_2)$ as $u\to+\infty$ may be obtained by expanding

$$\frac{1}{(1-e^{-t_1 z})(1-e^{-t_2 z})} = \sum_{k\geqslant -2} z^k\,\mathrm{ch}_k\,H^0(\mathbb{C}^2,\mathcal{O})$$

and integrating term-wise to get

$$\zeta(s,u\,|\,t_1,t_2) = \sum_{k\geqslant -2}\frac{\Gamma(s+k)}{\Gamma(s)u^{s+k}}\,\mathrm{ch}_k\,H^0(\mathbb{C}^2,\mathcal{O})\,. \tag{16.3}$$

Since

$$\frac{\partial}{\partial s}\,\frac{\Gamma(s+k)}{\Gamma(s)\,u^{s+k}}\bigg|_{s=0} = (-1)^{k+1}\ln^{(k)} u\,,$$

this verifies the agreement between (6.9) and (16.2).

16.3 The matrix $\widehat{\mathbf{R}}$

16.3.1

For $\mathsf{w} = a_1$ and $\mathsf{w}' = a_2$ the Γ-factor from (6.11) specializes to

$$\Gamma(u\,|\,\mathsf{w},\mathsf{w}') = \frac{c(H^0(\mathbb{C}^2,\mathcal{O})^\vee, u-\hbar)}{c(H^0(\mathbb{C}^2,\mathcal{O})^\vee, u)} = \\ = \frac{\Gamma(u|\,t_1,t_2)}{\Gamma(u+t_1+t_2\,|\,t_1,t_2)} = u\,\Gamma(u|\,t_1)\Gamma(u|\,t_2)\,, \tag{16.4}$$

where $u = a_1 - a_2$ and $\Gamma(u\,|\,t_1)$ is the single Barnes's Γ-function, defined similarly[1]. We define $\widehat{\mathbf{R}} = \Gamma(u\,|\,\mathsf{w},\mathsf{w}')\,\mathbf{R}$.

[1] It is related to Euler's Γ-function by

$$\Gamma(u\,|\,t_1) = \frac{\exp((u/t_1 - 1/2)\ln t_1)}{\sqrt{2\pi}}\,\Gamma(u/t_1)\,.$$

16.3.2 Zero modes and the singular part of $\widehat{\mathrm{R}}$

From (16.3), or the Stirling formula, we compute

$$\frac{1}{\hbar}\ln\Gamma(u-a\,|\,\mathsf{w},\mathsf{w}') = \tau(1)\,\ln^{(-1)}u - \tau(a)\,\ln u \\ + \left(\frac{1}{2}\tau(a^2) - \frac{1}{12}\right)\frac{1}{u} + O\left(\frac{1}{u^2}\right) \quad (16.5)$$

as $u \to \infty$. This gives the following identification of the central operators c_{-2} and c_{-1} from Section 6.1.11. Write $M_{\varnothing,\varnothing}$ for the $\langle \mathsf{w}|\,\cdot\,|\mathsf{w}\rangle$ vacuum matrix element of an operator M corresponding to $\mathsf{w}=1$. Then

$$(\mathsf{c}_{-2})_{\varnothing,\varnothing} = \tau(1)\,r\,, \quad (\mathsf{c}_{-1})_{\varnothing,\varnothing} = \beta_0(1)\,,$$

where $r = \mathsf{v}$ is the rank and

$$\beta_0 = \sum_{i=1}^{r} 1 \otimes \cdots \otimes \alpha_0 \otimes \cdots \otimes 1$$

is the 0th Baranovsky operator. Here and in what follows we identify

$$H^{\cdot}_{\mathsf{G}}\left(\mathcal{M}(r)^{\mathsf{A}}\right) \otimes \mathbb{K} \cong \mathsf{F}(a_1) \otimes \cdots \otimes \mathsf{F}(a_r)\,, \quad (16.6)$$

generalizing (14.11) to arbitrary rank. Thus the zero modes appear in the Yangian.

Note by construction the operators $(\mathsf{c}_{-i})_{\varnothing,\varnothing}$ have the same span as the operators $\mathrm{ch}_i\,\widehat{\mathcal{V}}$ for $i \in \{-2,-1\}$.

16.3.3

We stress that in what follows we adopt the identification (16.6) and that, for now on, all formulas involving $\boldsymbol{\alpha}$ include the zero modes.

16.3.4

Similarly, consider the vacuum-vacuum matrix element of the regular part $\widehat{\mathbf{R}}_{\mathrm{reg}}$ of $\widehat{\mathbf{R}}$, as in Section 6.1.11. The new terms coming from (16.5) give

$$\frac{1}{\hbar}\left[\frac{1}{u}\right]\left(\widehat{\mathbf{R}}_{\mathrm{reg}}\right)_{\varnothing,\varnothing} = \sum_{i=1}^{r} 1 \otimes \cdots \otimes \widehat{L}_0 \otimes \cdots \otimes 1$$

where

$$\widehat{L}_0 = \tfrac{1}{2} \int :\boldsymbol{\alpha}^2:(1) - \tfrac{1}{12}\,,$$

where we keep the zero modes, compare with (13.7). Note the familiar $\zeta(-1) = -\frac{1}{12}$ term.

16.3.5

Recall the classical **r**-matrix (12.14) and note its matrix elements gave

$$\beta_n(1), \beta_{-n}(\mathsf{pt}) \in \mathsf{Y}(\widehat{\mathfrak{gl}(1)})\,, \quad n > 0\,.$$

Since the core Yangian is an algebra over $\Bbbk[\boldsymbol{\delta}^{-1}]$ where

$$\boldsymbol{\delta} = t_1 t_2\,,$$

we have

$$\beta_{-n}(1) = \boldsymbol{\delta}^{-1}\,\beta_{-n}(\mathsf{pt}) \in \mathbb{Y}(\widehat{\mathfrak{gl}(1)})\,.$$

Chapter 17

Core Yangian modulo $\hbar$

17.1 Semiclassical R-matrix

17.1.1

Since $\hbar = -t_1 - t_2$ does not divide $\boldsymbol{\delta} = t_1 t_2$ we may study $\mathbb{Y}$ modulo $\hbar$, which leads to great simplifications.

Define the semiclassical R-matrix $\mathfrak{R}_{\mathrm{sc}}$ by

$$\widehat{\mathbf{R}}(u) = 1 + \hbar\, \mathfrak{R}_{\mathrm{sc}}(u) + O(\hbar^2)\,.$$

Modulo $\hbar$, the generators of $\mathbb{Y}$ are primitive and act by matrix coefficients of $\mathfrak{R}_{\mathrm{sc}}$.

The Yang-Baxter equation becomes the classical Yang-Baxter equation for $\mathfrak{R}_{\mathrm{sc}}$. It implies the generators of $\mathbb{Y}/\hbar\mathbb{Y}$ form a Lie algebra $\mathfrak{g}_{\mathrm{sc}}$ and

$$\mathbb{Y}/\hbar\mathbb{Y} \cong \mathcal{U}(\mathfrak{g}_{\mathrm{sc}})\,.$$

17.1.2

The Lie algebra $\mathfrak{g}_{\mathrm{sc}}$ may be described explicitly by its action in the basis of stable envelopes of $\mathcal{M}(r)^{\mathsf{A}}$, where

$$\mathsf{A} \subset SL(2) \times GL(r)$$

is a maximal torus. Since $\mathcal{M}(r)^{\mathsf{A}}$ is finite, the classes of A-fixed points form an eigenbasis for operators of classical multiplication.

In A-equivariant cohomology, stable envelopes are proportional to fixed points, and thus diagonalize operators of classical multiplication. Steinberg correspondences act nicely in this basis by the general principles explained in Section 4.6.

17.1.3

The fixed points of the maximal torus of $SL(2)$ on the Hilbert schemes are Nakajima varieties of type A_∞, see in particular Section 4.3.6. We will see a close connection between $\mathfrak{g}_{\mathrm{sc}}$ and the corresponding Lie algebra $\mathfrak{gl}(\infty)$.

17.2 Stable basis for Hilb_n

17.2.1

The stable basis for $\mathcal{M}(1) = \mathrm{Hilb}$ is identified as follows. Let

$$\left\{ \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \right\} \subset SL(2)$$

be the standard maximal torus. To match standard symmetric functions conventions, we choose the $z \to \infty$ chamber, that is,

$$\mathfrak{C} = \{u < 0\} \,,$$

where $u = \log z$. The other choice may be obtained by a permutation of coordinates.

A subscheme of $\mathbb{C}^2$ has a $z \to \infty$ limit if and only if it is set-theoretically supported on the x_2-axis

$$\ell_2 = \{x_1 = 0\} \,.$$

In particular, the stable basis must be a $\mathbb{Q}$-linear combination of the Nakajima descendents of the x_2-axis

$$p_\mu = \prod \alpha_{-\mu_i}(\ell_2) \; |\rangle \,.$$

The notation is chosen to agree with the traditional map of the equivariant cohomology of the Hilbert scheme to symmetric function that takes

$$\alpha_{-k}(\ell_2) \mapsto \text{multiplication by } p_k \,. \tag{17.1}$$

17.2.2

Recall the sign-twisted inner product on cohomology from Section 3.1.3 and transport it to symmetric functions using (17.1). This gives the Jack inner product on symmetric functions

$$[p_k^\tau, p_l] = \delta_{kl}\, k\,(-t_1/t_2)$$

with parameter $-t_1/t_2$. In [70], this parameter is denoted α.

Gram-Schmidt orthogonalization of monomial symmetric function m_λ with respect to this inner product gives, by definition, the basis of Jack symmetric functions. We define

$$\mathsf{J}_\lambda = t_2^{|\lambda|} \cdot \text{integral Jack polynomial as in [70]}\,.$$

This is normalized so that

$$\mathsf{J}_\lambda = \prod_{\square\in\lambda} (t_2(l(\square)+1) - t_1\, a(\square))\, m_\lambda + \dots \tag{17.2}$$

and is a polynomial in t_1, t_2 of degree $|\lambda|$. Here

$$a(\square) = \lambda_i - j\,, \quad l(\square) = \lambda'_j - i$$

denote the arm- and leg-length of a square $\square = (i,j)$ in the diagram λ. Note that the product in (17.2) is the Euler class of N_+ at the monomial ideal

$$\mathsf{I}_\lambda = \left(x_1^{\lambda_i} x_2^{i-1}\right)_{i=1,2,\dots} \in \mathrm{Hilb}\,. \tag{17.3}$$

17.2.3

The following is well-known and is a a consequence of the orthogonality of classes of fixed points $[\,\mathsf{I}_\lambda]$ in cohomology

Proposition 17.2.1 ([85, 124, 69])**.** *The map* (17.1) *sends* $[\,\mathsf{I}_\lambda]$ *to* J_λ.

17.2.4

Let us polarize $\mathrm{Hilb}^{\mathsf{A}}$ by the Euler class of N_-. We then have the following

Proposition 17.2.2. *The map* (17.1) *sends the stable envelope of* I_λ *to the Schur function* s_λ.

Proof. Schur functions are triangular with respect to J_λ and proportional to them modulo $\hbar$. This shows stable envelopes are proportional to Schur functions. By (17.2) we have

$$\mathsf{J}_\lambda = e(N_+)\, s_\lambda + \dots \,,$$

which fixes the normalization. □

17.3 Differential operators on $\mathbb{C}^\times$ and $\mathfrak{gl}(\infty)$

17.3.1

Let e_a denote the function

$$e_a(x) = e^{ax} \,. \tag{17.4}$$

Let $\varepsilon \in \mathbb{C}^\times$ be a parameter and consider

$$\mathcal{D}_{\text{assoc}} = \mathbb{C}\left\langle D, e_{\pm\varepsilon}\right\rangle \,, \quad D = \frac{d}{dx} \,.$$

It may be identified with differential operators on $\mathbb{C}^\times$ via the map $z = e_\varepsilon$. The parameter ε may be scaled away but it will be convenient to keep it. We denote by

$$\mathcal{D} = (\mathcal{D}_{\text{assoc}})_{\text{Lie}}$$

the same algebra viewed as a Lie algebra.

The center of $\mathcal{D}$ is spanned by $1 \in \mathcal{D}_{\text{assoc}}$ which we denote by D^0 to avoid confusion.

17.3.2

The natural action of $\mathcal{D}$ on $e_s\,\mathbb{C}[e_{\pm\varepsilon}]$, $s \in \mathbb{C}$, gives a family of embeddings

$$\rho_s : \mathcal{D} \hookrightarrow \mathfrak{gl}(\infty)$$

into the Lie algebra $\mathfrak{gl}(\infty)$ of all infinite matrices with finitely many nonzero diagonals. Its image is the unipotent Jordan block of the automorphism of $\mathfrak{gl}(\infty)$ that corresponds to the shift of the Dynkin diagram.

The diagram shift automorphism is the deck transformation of the universal cover of the quiver with one vertex and one loop. From this point of view, the description of $\mathcal{D}$ as automorphism-finite vectors in $\mathfrak{gl}(\infty)$ is intrinsic, while its identification with differential operators is less so.

The Lie algebra $\mathfrak{gl}(\infty)$ has a central extension $\widehat{\mathfrak{gl}(\infty)}$ which may be pulled back to a central extension

$$0 \to \mathbb{C}\mathbf{c} \to \widehat{\mathcal{D}} \to \mathcal{D} \to 0\,. \tag{17.5}$$

This extension does not depend on s.

Representation theory of $\widehat{\mathcal{D}}$ was studied by Kac and Radul [59] and many others. Here we will see the simplest representations: those obtained from the half-infinite wedge representations of $\mathfrak{gl}(\infty)$.

17.3.3

By construction, the representation $\pi_s = \bigwedge^{\infty/2} \rho_s$ is the $\widehat{\mathcal{D}}$ module with basis

$$|\lambda; s\rangle = \bigwedge_{i=1}^{\infty} e_{(\lambda_i - i)\,\varepsilon + s}\,, \tag{17.6}$$

where

$$\lambda = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$$

is a partition. Usual rules of linear algebra give a well-defined answer for the action of the off-diagonal elements of $\mathcal{D}$ in this basis. For the diagonal elements, it is convenient to use the ζ-regularization

$$\text{“}\sum_{i=1}^{\infty} ((\lambda_i - i)\,\varepsilon + s)^k\text{”} = k!\,[x^k]\,e_s \sum_{i=1}^{\infty} e_{(\lambda_i - i)\,\varepsilon}\,,$$

where $e_a = e_a(x)$ as in (17.4). Note

$$\sum_{i=1}^{\infty} e_{(\lambda_i - i)\varepsilon} = \frac{1}{e_\varepsilon - 1} + \sum_{i=1}^{\infty} \left[e_{(\lambda_i - i)\,\varepsilon} - e_{-i\varepsilon}\right]$$

where the second term is a Laurent polynomial in e_ε. In particular,

$$\pi_s(D^0) = \frac{s}{\varepsilon} - \frac{1}{2}\,.$$

The central extension (17.5) is normalized so that

$$\pi_s(\mathbf{c}) = 1\,.$$

17.3.4

For I_λ as in (17.3) we have

$$\operatorname{ch} \widehat{\mathcal{V}}\Big|_{\mathsf{I}_\lambda} = -\frac{e^a}{1-e^{-t_1}} \sum_{i=1}^{\infty} e^{-\lambda_i t_1 - (i-1)\, t_2}$$

where a is the framing weight and t_1, t_2 are the tangent weights of the two coordinate axes. We see that if

$$t_1 = -t_2 = -\varepsilon$$

then the map

$$F(a) \ni \operatorname{Stab}[\mathsf{I}_\lambda] \mapsto |\lambda; a + \varepsilon/2\rangle \tag{17.7}$$

identifies

$$\operatorname{ch} \widehat{\mathcal{V}} = \frac{\mathbf{c}}{\varepsilon\big(e^{\varepsilon/2} - e^{-\varepsilon/2}\big)} + \frac{1}{e^{\varepsilon/2} - e^{-\varepsilon/2}} \exp D\,. \tag{17.8}$$

Here $\exp(D)$ is a generating function for the operators $D^k \in \widehat{\mathcal{D}}$, in other words

$$\pi_s\left(\exp D\right) = \sum_{k \geqslant 0} \frac{1}{k!}\, \pi_s\big(D^k\big) \neq \exp(\pi_s(D))\,.$$

17.3.5

Generalizing (17.8), we have

Proposition 17.3.1. *The identification* (17.7) *gives*

$$\mathfrak{g}_{\mathrm{sc}} \cong \widehat{\mathcal{D}}\,.$$

Proof. It remains to check that it takes

$$\alpha_{-k}(\ell_2) \mapsto e_{\epsilon k} \in \widehat{\mathcal{D}}\,,$$

which is easy. For example, mapping both sides of (17.7) to the Schur function s_λ, this becomes the classical rule for multiplication of Schur functions by power-sum functions. □

17.4 Plücker relations

17.4.1

Let ψ_a be the operator of wedge product by e_a

$$\psi_a v = e_a \wedge v$$

and let ψ_a^* be the adjoint operator with respect to inner product in which the vectors (17.6) are orthonormal. More canonically, the operators ψ_a^* are associated to bases of representations dual to ρ_s.

17.4.2

Consider the operator

$$\Omega = \sum_{a \in s+\mathbb{Z}\varepsilon} \psi_a \otimes \psi_a^*$$

which depends only on the $\mathbb{Z}\varepsilon$-coset of s. It defines a map

$$\Omega : \pi_s \otimes \pi_{s'} \mapsto \pi_{s+\varepsilon} \otimes \pi_{s'-\varepsilon}$$

provided

$$s' \equiv s \mod \mathbb{Z}\varepsilon \,.$$

This map commutes with $\mathfrak{gl}(\infty)$ and, hence, with $\widehat{\mathcal{D}}$.

17.4.3

Classically, Ω is used to describe the image of the natural embedding

$$GL(V) \hookrightarrow GL(\Lambda^{\cdot} V)\,,$$

where V a vector space, which for simplicity can be assumed to be finite-dimensional, see [57, 81]. Matrix elements of $g \in GL(V)$ acting on $\Lambda^{\cdot} V$ are the minors of g.

Commutation with Ω gives quadratic relations for minors of g, analogous to the better known Plücker relations among maximal minors of a rectangular matrix (that is, among the Plücker coordinates on the Grassmann variety). Here we use the term *Plücker relations* in the broader sense.

17.4.4

We denote by

$$\begin{aligned}\mathsf{E}(\lambda,\mu;s,u) &= \langle \mu; s|\, \mathfrak{R}_{\mathrm{sc}}(u)\, |\lambda; s\rangle \\ &= \sum_{k\geqslant -1} \mathsf{E}(\lambda,\mu;s)_k \, \ln^{(k)}(u)\,.\end{aligned}$$

matrix elements of $\mathfrak{R}_{\mathrm{sc}}$ in the first (by convention) tensor factor. Here $\mathsf{E}(\lambda,\mu;s)_k \in \mathfrak{g}_{\mathrm{sc}}$ and the singular central terms

$$\mathrm{ch}_{-2}\,\widehat{\mathcal{V}} = \frac{\mathbf{c}}{\varepsilon^2}\,, \quad \mathrm{ch}_{-1}\,\widehat{\mathcal{V}} = \frac{D^0}{\varepsilon}$$

are only present if $\lambda = \mu$. By construction, $\mathsf{E}(\lambda,\mu;s,u)$ only depend on $u+s$ in the sense that

$$\forall t \quad \mathsf{E}(\lambda,\mu;s+t,u-t) = \mathsf{E}(\lambda,\mu;s,u)\,. \tag{17.9}$$

17.4.5

By construction, $\mathsf{E}(\lambda,\mu;s)_k$ generate $\mathbb{Y}/\hbar\mathbb{Y}$ and all relations between these generators are linear. Among them are the Plücker relations, which become linear

$$[\xi\otimes 1 + 1\otimes \xi, \Omega] = 0\,, \quad \xi \in \mathfrak{g}_{\mathrm{sc}}\,, \tag{17.10}$$

at the Lie algebra level.

Proposition 17.4.1. *Plücker relations and* (17.9) *span all linear relations among matrix elements of* $\mathfrak{R}_{\mathrm{sc}}$.

This statement is a variation on the classical theme. For convenience, we give a proof.

17.4.6

We divide the proof of Proposition 17.4.1 into a sequence of lemmas.

Lemma 17.4.2. *Suppose* $\psi_a^* \,|\lambda\rangle \neq 0$ *and* $|\mu\rangle \neq \psi_b\, \psi_a^*\, |\lambda\rangle$ *for all* b*. Then*

$$\langle \mu|\, \xi\, |\lambda\rangle = \langle \mu|\, \psi_a\, \xi\, \psi_a^*\, |\lambda\rangle$$

for all $\xi \in \mathfrak{g}_{\mathrm{sc}}$.

Note the hypothesis of the Lemma implies $\mu \neq \lambda$.

Proof. Expand

$$0 = \langle \mu, \lambda | \, 1 \otimes \psi_a \left[\xi \otimes 1 + 1 \otimes \xi, \Omega \right] \psi_a^* \otimes 1 \, | \lambda, \lambda \rangle \tag{17.11}$$

where $|\mu, \lambda\rangle = |\mu\rangle \otimes |\lambda\rangle$. □

Corollary 17.4.3. *Plücker relations imply*

$$\mathsf{E}(\lambda, \mu; s, u) \neq 0 \Longrightarrow |\mu\rangle = \psi_b \, \psi_a^* \, |\lambda\rangle$$

for some $a, b \in s + \mathbb{Z}\varepsilon$.

In the language of Chapter 11, this means the corresponding points of the half-infinite Grassmannian must lie on a line.

Proof. Otherwise, we can find a in Lemma 17.4.2 such that $\langle \mu | \, \psi_a \; = 0$. □

17.4.7

Let $\lambda \neq \mu$ lie on a line, which means that there exists $k, l \in \mathbb{Z}$ such that

$$\{k\} = \mathfrak{S}(\lambda) \backslash \mathfrak{S}(\mu) \,, \quad \{l\} = \mathfrak{S}(\mu) \backslash \mathfrak{S}(\lambda) \,,$$

where $\mathfrak{S}(\lambda) = \{\lambda_i - i\} \subset \mathbb{Z}$. Using Lemma 17.4.2, we can add or remove elements in $\mathfrak{S}(\lambda) \cap \mathfrak{S}(\mu)$, which means there exists $\mathsf{E}_{kl}(s, u)$ such that

$$\mathsf{E}(\lambda, \mu; s, u) = \pm \mathsf{E}_{kl}(s, u)$$

with the sign determined from the action of the operators ψ_a^* in the basis $|\lambda\rangle$. Lemma 17.4.2 further implies

$$\begin{aligned} \mathsf{E}_{k+1,l+1}(s, u) &= \mathsf{E}_{kl}(s + \varepsilon, u) \\ &= \mathsf{E}_{kl}(s, u + \varepsilon) \,, \end{aligned} \tag{17.12}$$

where the second step is based on (17.9).

17.4.8

Now consider diagonal matrix elements of $\mathfrak{R}_{\mathrm{sc}}$. Here we have the following

Lemma 17.4.4. *Suppose $\psi_a^* |\lambda\rangle \neq 0$ and $\psi_a^* |\mu\rangle \neq 0$. Then*

$$\langle\lambda|\, \xi \,|\lambda\rangle - \langle\lambda|\, \psi_a \, \xi \, \psi_a^* \,|\lambda\rangle = \langle\mu|\, \xi \,|\mu\rangle - \langle\mu|\, \psi_a \, \xi \, \psi_a^* \,|\mu\rangle \tag{17.13}$$

for all $\xi \in \mathfrak{g}_{\mathrm{sc}}$.

Proof. Expand $\langle\lambda, \mu|\, 1 \otimes \psi_a \left[\xi \otimes 1 + 1 \otimes \xi, \Omega\right] \psi_a^* \otimes 1 \;|\lambda, \mu\rangle$. □

We denote the difference of the matrix elements in (17.13) by $\mathsf{E}_{kk}(s, u)$, where $a = k\varepsilon + s$. For example,

$$\mathsf{E}_{0,0}(s, u) = \mathsf{E}(\varnothing, \varnothing; s + \varepsilon, u) - \mathsf{E}(\varnothing, \varnothing; s, u)\,.$$

One can choose a different parameter $s' \in s + \mathbb{Z}\varepsilon$ for μ in (17.13) which shows the relation (17.12) is valid for $k = l$.

17.4.9

Symbolically, Lemma 17.4.4 and (17.12) shows

$$\mathsf{E}(\lambda, \lambda; s, u) = \text{``} \sum_{k \in \mathfrak{S}(\lambda)} \mathsf{E}_{00}(s, u + k\varepsilon) \text{''}\,.$$

A better way to write this relation is the following.

For each partition λ, define

$$\mathrm{corners}_\lambda : \mathbb{Z} \to \{\pm 1, 0\}$$

as the difference of the following indicator functions

$$\mathrm{corners}_\lambda = \sum_{\text{inner corners } \square} \delta_{c(\square)} - \sum_{\text{outer corners } \square} \delta_{c(\square)}\,.$$

Here $c(\square) = j - i$ is the content of the square $\square = (i, j)$. This may also be defined using the identity

$$\sum_k \mathrm{corners}_\lambda(k)\, t^k = (t - 1) \sum t^{\lambda_i - i}\,.$$

Lemma 17.4.5.

$$\mathsf{E}(\lambda, \lambda; s, u) = \sum_k \mathrm{corners}_\lambda(k)\; \mathsf{E}(\varnothing, \varnothing; s, u + k\varepsilon)\,.$$

Proof. Follows from Lemma 17.4.4 and (17.12). □

17.4.10

Proof of Proposition 17.4.1. Previous lemmas reduce the matrix elements of $\mathfrak{R}_{\mathrm{sc}}$ to shifts in u of the operators $\mathsf{E}(\varnothing, \varnothing; s, u)$ and $\mathsf{E}_{k0}(s, u)$, $k \neq 0$.

The algebra $\mathfrak{g}_{\mathrm{sc}}$ is graded by eigenvalues of the adjoint action of D, this is the grading by the difference $k - l$ of E_{kl}. Each graded piece is further filtered by the degree in u, with 1-dimensional factors. This shows there are no further linear relations among the coefficients of $\mathsf{E}(\varnothing, \varnothing; s, u)$ and $\mathsf{E}_{k0}(s, u)$, $k \neq 0$. □

17.4.11

The factorization of Section 4.3.6 gives the following formula for the semi-classical R-matrix

$$\mathfrak{R}_{\mathrm{sc}} = \sum_{i,j,k \in \mathbb{Z}} \frac{E_{ij} \otimes E_{j+k,i+k}}{u - k\varepsilon} \tag{17.14}$$

in terms of the classical R-matrix

$$\mathbf{r}_{\mathfrak{gl}(\infty)} = \sum_{i,j \in \mathbb{Z}} E_{ij} \otimes E_{ji}$$

for $\mathfrak{gl}(\infty)$. The operator (17.14) acts in half-infinite wedge representations of $\mathfrak{gl}(\infty)$ via the ζ-regularization discussed in Section 16.2.2.

It is instructive to retrace the steps of the above proof with this explicit formula.

Chapter 18

The Yangian of $\widehat{\mathfrak{gl}(1)}$

18.1 Generators of the core Yangian

18.1.1

By Theorem 6.1.4, $\mathbb{Y}$ is generated by the Baranovsky operators β_n and $\mathrm{ch}_k \widehat{\mathcal{V}}$ for $k = -2, -1, \dots$. Here, for brevity, we write $\beta_n = \beta_n(1)$.

The following theorem shows it suffices to add a single operator $\mathrm{ch}_1 \widehat{\mathcal{V}}$ to the Baranovsky operators to generate the Yangian.

Theorem 18.1.1. *The core Yangian $\mathbb{Y}$ is generated by the Baranovsky operators $\beta_{\pm 1}$, and the operator of cup product by*

$$\widehat{\mathsf{Q}}_{\mathrm{cl}} = \mathrm{ch}_1 \widehat{\mathcal{V}}$$

of cup product by ch_1 of the bundle $\widehat{\mathcal{V}} = -H^0(\mathbb{C}^2, \mathcal{F})$.

Proof. Follows from the corresponding statement modulo $\hbar$. □

18.1.2

The generation statement can be made more effective using the the following geometric fact. Parallel results were proven by M. Lehn for the cohomology of Hilbert schemes and by O. Schiffmann and E. Vasserot for the K-theory of $\mathcal{M}(r)$.

Proposition 18.1.2. *For any k and l,*

$$\Big[\mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^k \beta_1, \mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^l \beta_{-1}\Big] \tag{18.1}$$

is an operator of classical multiplication.

Proof. Recall that the Baranovsky operators $\beta_{\pm 1}$ are defined using the correspondence

$$\mathfrak{B}_1 = \{(\mathcal{G}, x, \mathcal{F})\} \subset \mathcal{M}(r, n+1) \times \mathbb{C}^2 \times \mathcal{M}(n)$$

formed by exact sequences

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{O}_x \to 0\,. \tag{18.2}$$

On this correspondence, we have a tautological line bundle $\mathcal{F}/\mathcal{G}$ and the action of $\mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})$ introduces a factor of

$$c_1(\mathcal{F}/\mathcal{G}) = -\operatorname{ch}_1(H^0(\mathbb{C}^2, \mathcal{G})) + \operatorname{ch}_1(H^0(\mathbb{C}^2, \mathcal{F})) \in H^2_{\mathsf{G}}(\mathfrak{B}_1)\,.$$

Therefore

$$\Big(\mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^k \beta_1(\gamma)\Big) \circ \Big(\mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^l \beta_{-1}(\gamma')\Big) = \\ (-1)^r (\pi_{13})_* \left((-c_1(\mathcal{F}_1/\mathcal{G}))^k \, c_1(\mathcal{F}_2/\mathcal{G})^l \, \pi_{45}^*(\gamma \times \gamma')\right)$$

where π_{ij} are the projections to respective factors in the correspondence

$$\{(\mathcal{F}_1, \mathcal{G}, \mathcal{F}_2, x_1, x_2)\} \subset \mathcal{M}(r,n) \times \mathcal{M}(r, n+1) \times \mathcal{M}(r,n) \times \mathbb{C}^2 \times \mathbb{C}^2$$

in which $\mathcal{F}_i/\mathcal{G} \cong \mathcal{O}_{x_i}$. The $(-1)^r$ factors comes from our sign conventions, see Section 12.2.4.

The product $\mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^l \beta_{-1} \, \mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^k \beta_1$ in the opposite order is, similarly, computed by pushing forward

$$(-1)^r \, c_1(\mathcal{G}'/\mathcal{F}_1)^l (-c_1(\mathcal{G}'/\mathcal{F}_2))^k$$

along the $\mathcal{G}'$-factor in the correspondence defined by

$$\mathcal{G}'/\mathcal{F}_i = \mathcal{O}_{x_{3-i}}\,, \quad i = 1, 2\,.$$

We now note that outside of the diagonal $\mathcal{F}_1 \cong \mathcal{F}_2$ the two correspondences are canonically isomorphic, because necessarily

$$\mathcal{G}' = \mathcal{F}_1 + \mathcal{F}_2 \,, \quad \mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2 \,,$$

as subsheaves of the common double dual $\mathcal{F}_1^{\vee\vee} = \mathcal{F}_2^{\vee\vee}$. Clearly,

$$\mathcal{F}_i/\mathcal{G} \cong \mathcal{G}'/\mathcal{F}_{3-i}$$

which identifies the integrands and shows the commutator (18.1) is supported on the diagonal $\mathcal{F}_1 \cong \mathcal{F}_2$. This means it is an operator of classical multiplication. □

From the proof above we have the following

Corollary 18.1.3.

$$\left[\mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^k \beta_1, \mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^l \beta_{-1}\right] = (-1)^k \left[\beta_1, \mathrm{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})^{k+l} \beta_{-1}\right]$$

18.1.3

The commutator in Proposition 18.1.2 can be explicitly identified. We do it using equivariant localization following [108].

To set up equivariant localization, we need to identify the the normal bundle to $\mathfrak{B}_1$. We have the following

Proposition 18.1.4. *The tangent bundle to $\mathfrak{B}_1$ fits into an exact sequence of the form*

$$0 \to T\mathfrak{B}_1 \to T\mathcal{M}(r, n+1) \oplus T\mathcal{M}(r, n) \to \\ \to \mathrm{Ext}^1(\mathcal{G}, \mathcal{F}(-1)) \to \mathbb{C}(-\hbar) \to 0 \,, \quad (18.3)$$

where $\mathbb{C}(-\hbar)$ is the trivial bundle with equivariant weight $-\hbar$.

In particular, $\mathfrak{B}_1$ is smooth, which is a special case of Theorem 5.7 in [87]. The sequence (18.3) may also be found there for more general Hecke correspondences among Nakajima varieties.

Proof. Let

$$\xi = (\xi_{\mathcal{G}}, \xi_{\mathcal{F}}) \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}(-1)) \oplus \operatorname{Ext}^1(\mathcal{G}, \mathcal{G}(-1))$$

be a tangent vector to $\mathcal{M}(r, n+1) \times \mathcal{M}(r, n)$. A sheaf homomorphism (in our case, inclusion)

$$\phi : \mathcal{G} \to \mathcal{F}$$

deforms with ξ to first order when the commutator

$$[\xi, \phi] = \xi_{\mathcal{F}}\, \phi - \phi\, \xi_{\mathcal{G}} \in \operatorname{Ext}^1(\mathcal{G}, \mathcal{F}(-1)) \tag{18.4}$$

vanishes. Here

$$\operatorname{Ext}^i(\mathcal{A}, \mathcal{B}) \otimes \operatorname{Ext}^j(\mathcal{B}, \mathcal{C}) \to \operatorname{Ext}^{i+j}(\mathcal{A}, \mathcal{C})$$

is the usual composition of Ext groups. Note that

$$\operatorname{rk} \operatorname{Ext}^1(\mathcal{G}, \mathcal{F}(-1)) = 2rn + r\,,$$

while

$$\dim \mathcal{M}(r, n+1) \times \mathcal{M}(r, n) - \dim \mathfrak{B}_1 = 2rn + r - 1\,.$$

In fact, the obstruction $[\xi, \phi]$ to deforming ϕ lies in the following corank 1 subbundle of $\operatorname{Ext}^1(\mathcal{G}, \mathcal{F}(-1))$.

For every deformation of $\mathcal{F}$ there is some deformation of $\mathcal{G} \subset \mathcal{F}$. This means the image of $\xi_{\mathcal{F}} \mapsto \xi_{\mathcal{F}}\, \phi$ lies in the image of $\xi_{\mathcal{G}} \mapsto \phi\, \xi_{\mathcal{G}}$ and hence the obstruction $[\xi, \phi]$ lies in the image of the first arrow in the following piece of the long exact sequence

$$\begin{aligned} &\operatorname{Ext}^1(\mathcal{G}, \mathcal{G}(-1)) \to \operatorname{Ext}^1(\mathcal{G}, \mathcal{F}(-1)) \to \operatorname{Ext}^1(\mathcal{G}, \mathcal{O}_x) \to \\ \to &\operatorname{Ext}^2(\mathcal{G}, \mathcal{G}(-1))\,. \end{aligned}$$

By Serre duality,

$$\operatorname{Ext}^2(\mathcal{G}, \mathcal{G}(-1)) = 0,$$

while

$$\operatorname{Ext}^1(\mathcal{G}, \mathcal{O}_x)^\vee \otimes \mathcal{O}(-\hbar) = \operatorname{Ext}^1(\mathcal{O}_x, \mathcal{G})\,.$$

We have

$$\operatorname{Ext}^1(\mathcal{O}_x, \mathcal{G})\big|_{\mathfrak{B}_1} \cong \mathbb{C}\,,$$

canonically trivialized by the class of the extension (18.2). This gives the exact sequence stated. □

18.1.4

Suppose we are at a fixed point $(\mathcal{G}, 0, \mathcal{F}) \in \mathfrak{B}_1$ of the torus action. Consider a free resolution of $\mathcal{F}$ and its restriction to $\mathbb{C}^2$

$$0 \to \bigoplus \mathcal{O}_{\mathbb{C}^2}(w_i) \to \bigoplus \mathcal{O}_{\mathbb{C}^2}(v_i) \to \mathcal{F}\Big|_{\mathbb{C}^2} \to 0 \tag{18.5}$$

where $v_i, w_j \in (\operatorname{Lie} \mathsf{G})^*$ are the equivariant weight of the generators and relations. (Note that these include the framing weights.) We have

$$\operatorname{ch} \mathcal{F} = \sum e^{v_i} - \sum e^{w_i} ,$$

and

$$\operatorname{ch} \mathcal{G} = \operatorname{ch} \mathcal{F} - e^{v_k}(1 - e^{-t_1})(1 - e^{-t_2})$$

if the generator with weight v_k surjects onto $\mathcal{F}/\mathcal{G}$. The characters of the Ext-groups in (18.3) are computed as follows

$$\operatorname{ch} \operatorname{Ext}^1(\mathcal{G}, \mathcal{F}(-1)) = \frac{(1 - \overline{\operatorname{ch} \mathcal{G}} \operatorname{ch} \mathcal{F})}{(1 - e^{-t_1})(1 - e^{-t_2})} ,$$

where bar denotes the dual representation, that is, $\overline{e^v} = e^{-v}$.

Let $N_{(\mathcal{G},x,\mathcal{F})}\mathfrak{B}_1$ denote the normal bundle to the Baranovsky correspondence at the at the point $(\mathcal{G}, x, \mathcal{F})$

Lemma 18.1.5. *We have*

$$\begin{aligned} \operatorname{ch} N_{(\mathcal{G},x,\mathcal{F})}\mathfrak{B}_1 - \operatorname{ch} T_{(\mathcal{G},x,\mathcal{F})}\mathfrak{B}_1 &= e^{-\hbar - v_k} \operatorname{ch} \mathcal{G} - e^{v_k} \overline{\operatorname{ch} \mathcal{G}} - e^{-\hbar} + 1 \\ &= e^{-\hbar - v_k} \operatorname{ch} \mathcal{F} - e^{v_k} \overline{\operatorname{ch} \mathcal{F}} - e^{-\hbar} + 1 , \end{aligned}$$

where v_k is the weight of $\mathcal{G}/\mathcal{F}$.

Note that the trivial weight 1 here cancels with the trivial weight that comes from the expansion of $e^{v_k} \overline{\operatorname{ch} \mathcal{G}}$, and similarly for the weights $-e^{-\hbar}$.

Proof. Direct computation from (18.3) . □

Proposition 18.1.6. *We have*

$$\left[\beta_1(\gamma_1), \frac{1}{u - \operatorname{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})} \beta_{-1}(\gamma_2) \right] = \frac{1}{\hbar} \int_{\mathbb{C}^2} \left(1 - \frac{c(\mathcal{F}^\vee \otimes \hbar, u)}{c(\mathcal{F}^\vee, u)} \right) \gamma_1 \gamma_2 , \tag{18.6}$$

where $\mathcal{F}$ is the universal sheaf on $\mathcal{M}(r) \times \mathbb{C}^2$, the right-hand side is viewed as operator of cup-product by this cohomology class in $H^{\cdot}_{\mathsf{G}}(\mathcal{M}(r))$.

Note, for example, that the $1/u$ term here gives the familiar result

$$[\beta_1(\gamma_1), \beta_{-1}(\gamma_2)] = -\int_{\mathbb{C}^2} \gamma_1\gamma_2 \operatorname{rk}\mathcal{F} = r\,\tau(\gamma_1\gamma_2)\,.$$

It is clear from Grothedieck-Riemann-Roch that the right-hand side of (18.6) generates the same algebra as $\operatorname{ch}_k \widehat{\mathcal{V}}$.

Proof. We use equivariant localization. Let $\mathcal{F}$ be a torus-fixed sheaf as in (18.5) and let

$$[\mathcal{F}] \in H^{2\dim}_{\mathsf{G}}(\mathcal{M}(r,n))\,,$$

denote the class of this fixed point. The computation of

$$\left(\beta_1(\gamma_1) \circ \frac{1}{u - \operatorname{ad}(\widehat{\mathsf{Q}}_{\mathrm{cl}})}\, \beta_{-1}(\gamma_2)\cdot[\mathcal{F}], [\mathcal{F}]\right) \Big/ ([\mathcal{F}],[\mathcal{F}])$$

is given by summing $1/(u - v_i)$ over all generators of $\mathcal{F}$ with a certain equivariant weight that accounts for the normal bundle to $\mathfrak{B}_1$ and for the tangent bundle to $\mathcal{M}(r,n+1)\times\mathcal{M}(r)\times(\mathbb{C}^2)^2$. This equivariant weight is determined from Lemma 18.1.5.

The product in the opposite order involves summation over all relations w_i in the resolution (18.5), because they correspond to torus-fixed sheaves that contain $\mathcal{F}$. The new generator has weight $w_i - \hbar$, therefore we sum $1/(u - w_i + \hbar)$ with a weight which is again computed from Lemma 18.1.5.

The resulting sum simplifies using the elementary identity

$$\sum_k \frac{\hbar}{u - v_k} \prod_{i\neq k} \frac{v_k - v_i + \hbar}{v_k - v_i} \prod_i \frac{v_k - w_i}{v_k - w_i + \hbar} - \sum_k \frac{\hbar}{u - w_k + \hbar} \prod_{i\neq k} \frac{w_k - w_i - \hbar}{w_k - w_i} \prod_i \frac{w_k - v_i}{w_k - v_i - \hbar} = \prod_i \frac{u - v_i + \hbar}{u - v_i} \prod_i \frac{u - w_i}{u - w_i + \hbar} - 1\,, \quad (18.7)$$

which is proven by observing that it is a partial fraction expansion in the variable u. (This identity also appears in [108].) Since

$$c(\mathcal{F}^\vee, u) = \prod_i (u - v_i) \Big/ \prod_i (u - w_i)$$

the result follows. □

Another proof of Theorem 18.1.1. Follows from the above proposition and Theorem 6.1.4. □

18.2 Slices and screening operators

18.2.1

In Section 6.2 we constructed geometrically core Yangian intertwiners from slices. In this section, we identify algebraically the intertwiner corresponding to the slices from Section 2.5.9. They turn out to be the well-known screening operators for Virasoro modules.

By the boson-fermion correspondence, screening operators specialize to Plücker relations modulo $\hbar$. Thus, by Proposition 17.4.1, they generate the relations in the core Yangian of $\widehat{\mathfrak{gl}(1)}$. Hence, for $\mathbb{Y}(\widehat{\mathfrak{gl}(1)})$, the answer to the question from Section 6.4.3 is affirmative.

18.2.2

We recall some basic notion, in the generality of Chapter 13.

A field $Y(\eta, z) = \sum_n Y_n(\eta)\, z^{-n}$ is called *primary* of dimension $\lambda \in \mathbb{H}$ if it satisfies the OPE

$$\boldsymbol{T}(\gamma, z)\, Y(\eta, w) \sim \frac{zw}{(z-w)^2}\, Y(\lambda\gamma\eta, w) + \frac{zw}{z-w}\, \frac{\partial}{\partial w} Y(\gamma\eta, w)\,.$$

Equivalently,

$$[\mathsf{L}_n(\gamma), Y_m(\eta)] = Y_{m+n}\big((n\lambda - n - m))\, \gamma\eta\big)\,.$$

In particular, if $\lambda = 1$ then the operator

$$Y_0(\eta) = \int Y(\eta, z)$$

commutes with all operators $\mathsf{L}_n(\gamma)$.

18.2.3

Define normally ordered exponential of a field $Y(\gamma, z)$ by

$$:\exp Y(z):(\gamma) = \tau(\gamma) + :Y(z):(\gamma) + \frac{1}{2} :Y(z)^2:(\gamma) + \dots\,,$$

where terms of the form $:Y(z)^n:(\gamma)$ are defined using the n-fold coproduct $\mathbb{H} \to \mathbb{H}^{\otimes n}$ as in Section 13.2.

These satisfy the usual rules like

$$\frac{\partial}{\partial z} :\exp Y(z):(\gamma) =: \left(\frac{\partial}{\partial z} Y(z)\right) \exp Y(z):(\gamma)\,.$$

18.2.4

Let η be an eigenvector of multiplication operators in $\mathbb{H}$. We define $\eta^\vee$ by

$$\gamma\eta = (\gamma, \eta^\vee)\,\eta\,, \tag{18.8}$$

for all $\gamma \in \mathbb{H}$. Define

$$\mathsf{V}_\mu(z) =: \exp \mu \boldsymbol{\phi}^-(z):(\eta)$$

where

$$\boldsymbol{\phi}^- = \boldsymbol{\phi}^{(1)} - \boldsymbol{\phi}^{(2)}$$

is the antiderivative of the field $\boldsymbol{\alpha}^-$, see Section 13.1.4. In particular, we have

$$\boldsymbol{\alpha}^-(\gamma, z)\,\boldsymbol{\phi}^-(\eta, w) \sim \frac{2z}{z-w}\,(\gamma, \eta) + \dots \tag{18.9}$$

18.2.5

Since the operator V_μ involves $\alpha_{\log}$, it has nontrivial commutation relations with α_0^-, namely

$$\left[\alpha_0^-(\gamma), \mathsf{V}_\mu(z)\right] = 2\mu(\gamma, \eta^\vee)\,\mathsf{V}_\mu(z)\,.$$

This means

$$V_\mu : \mathsf{F}(a_1) \otimes \mathsf{F}(a_2) \to \mathsf{F}(a_1 - \mu\,\eta^\vee) \otimes \mathsf{F}(a_2 + \mu\,\eta^\vee)\,.$$

18.2.6

Proposition 18.2.1. *If η is an eigenvector of multiplication as in* (18.8) *then the operator*

$$z^{\mu^2(\mathsf{e}, \eta^\vee)} V_\mu(z)$$

is primary for $\boldsymbol{T}(z, K)$ of dimension

$$\lambda = \mu^2 \mathsf{e} - \mu K\,.$$

Here $\mathsf{e} \in \mathbb{H}$ is the handle-gluing element.

Proof. This is a standard computation that uses (18.9) and Lemma 13.2.1. □

18.2.7

In particular, primary of dimension 1 can give rise to Virasoro intertwiners. In the case

$$\mathbb{H} = H^{\bullet}_{\mathsf{G}}(\mathbb{C}^2)\left[\frac{1}{\det^2_{\mathbb{C}}}\right], \quad K = \hbar = -t_1 - t_2\,, \quad \eta^\vee = \mathsf{e} = -t_1 t_2\,, \qquad (18.10)$$

we have

$$\mu^2 \mathsf{e} - \mu K = 1 \;\Rightarrow\; \mu = \frac{1}{t_1}, \frac{1}{t_2}\,.$$

For the integral $\int z^{\mu^2(\mathsf{e},\eta^\vee)} V_\mu(z)$ to be well-defined, the integrand has to have integral powers of z. The nonintegral powers of z come from the $\log z$ term in $\boldsymbol{\phi}^-$, namely

$$e^{\mu \log z\, \alpha_0^-}(\eta)\Big|_{\mathsf{F}(a_1)\otimes\mathsf{F}(a_2)} = z^{-\mu(a_1-a_2,\eta^\vee)}\,\tau(\eta)\,.$$

For the case (18.10), this integrality constrain becomes

$$(\mu^2\mathsf{e} - \mu(a_1 - a_2), \eta^\vee) = -\frac{t_2}{t_1} - \frac{a_1 - a_2}{t_1} = -n \in \mathbb{Z}\,, \quad \mu = \frac{1}{t_1}\,,$$

and similarly for $\mu = 1/t_2$.

18.2.8

Theorem 18.2.2. *For every $n \in \mathbb{Z}$ the screening operator*

$$\int z^{-\frac{t_2}{t_1}}\, \mathsf{V}_{\frac{1}{t_1}}(z) : \mathsf{F}(a_2 + nt_1 - t_2) \otimes \mathsf{F}(a_2) \to \mathsf{F}(a_2 + nt_1) \otimes \mathsf{F}(a_2 - t_2)$$

is a map of Y*-modules.*

Proof. The operator clearly commutes with the Baranovsky operators and intertwines the Virasoro operators $\boldsymbol{T}_+(z)$ by Proposition 18.2.1. Formula (14.9) expresses the operator of classical multiplication by divisor in terms of the Baranovsky operators and $\boldsymbol{T}_+(z)$, therefore the screening operator intertwines it as well. Now Theorem 18.1.1 finishes the proof. □

18.2.9

Note, in particular, the screening operators annihilates the vacuum vector for $n < 0$. This is reflected in the poles of the $\mathbf{R}(u)$ at

$$u = \hbar, \hbar - t_1, \hbar - 2t_1\,, \dots\,.$$

Chapter 19

Yangian and vertex algebras

19.1 The operator $\widehat{\mathsf{Q}}_{\mathrm{cl}}$

19.1.1

Since the operator $\widehat{\mathsf{Q}}_{\mathrm{cl}}$ plays an important role in Theorem 18.1.1, we give a formula for it that modifies the formula in Theorem 14.2.3.

More compact formulas are obtained for Chern character of $\widehat{\mathcal{V}} \otimes \hbar^{1/2}$, where $\operatorname{ch} \hbar^{1/2} = e^{\hbar/2}$. This is the familiar twist by the square root of the canonical bundle (of $\mathbb{C}^2$, in this case). However, only the overall shape of the formula will be used below, not the details.

We define

$$\widehat{\mathbf{\Omega}} = \tfrac{1}{2} \int :\boldsymbol{\beta}\, |\partial|\, \boldsymbol{\beta}:(1)\,,$$

as in Section 14.2.2 and denote by

$$\mathfrak{C}_{>} = \{a_1 > \cdots > a_r\}$$

the standard chamber for A. The analog of Theorem 14.2.3 is the following

Proposition 19.1.1. *Under the identification*

$$\mathsf{F}(a_1) \otimes \cdots \otimes \mathsf{F}(a_r) \xrightarrow{\mathrm{Stab}_{\mathfrak{C}_>}} H^{\bullet}_{\mathsf{G}}(\mathcal{M}(r)) \otimes \mathbb{K}\,,$$

as in (16.6), *we have*

$$\mathrm{ch}_1 \left(\widehat{\mathcal{V}} \otimes \hbar^{1/2}\right) = -\sum_i \tfrac{1}{6} \int :\left(\boldsymbol{\alpha}^{(i)}\right)^3 :(1) + \sum_i \tfrac{1}{24} \int :\boldsymbol{\alpha}^{(i)} :(\hbar^2 + 2\mathsf{e}) \\ + \tfrac{1}{2}\hbar \sum_{i<j} \int \boldsymbol{\alpha}^{(i)}\, \partial\, \boldsymbol{\alpha}^{(j)}(1) + \tfrac{1}{2}\hbar\, \widehat{\boldsymbol{\Omega}}\,. \quad (19.1)$$

For other chambers, one rearranges the $\sum_{i<j}$ term accordingly.

Proof. The inclusion of zero modes and the $\otimes\hbar^{1/2}$ twist removes the $\boldsymbol{\Phi}_2$-term from formula (14.6). Therefore, the two sides of (19.1) differ by a scalar operator that we can determine by evaluating on the vacuum vector. This is straightforward, using

$$-\,\mathrm{ch}_1 \frac{e^{a-\hbar/2}}{(1-e^{-t_1})(1-e^{-t_2})} = \tfrac{1}{6}\tau(a^3) - \tfrac{1}{24}\tau((\hbar^2+2\mathsf{e})a)\,,$$

where $\hbar = -t_1 - t_2$, $\mathsf{e} = -t_1 t_2$. □

19.1.2

Let

$$\mathfrak{V}_r \subset \mathrm{End}\, \mathsf{F}(a_1) \otimes \cdots \otimes \mathsf{F}(a_r)$$

denote the algebra generated by all Fourier coefficients of vertex operators, that is,

$$\int z^n :P(\boldsymbol{\alpha}^{(i)}, \partial\boldsymbol{\alpha}^{(i)}, \partial^2\boldsymbol{\alpha}^{(i)}, \dots):\ \in \mathfrak{V}_r$$

for any $n \in \mathbb{Z}$ and any normally ordered polynomial P in the fields $\boldsymbol{\alpha}^{(i)}$, $i = 1, \dots, r$, and their derivatives.

Proposition 19.1.2. *The action of* $\mathbb{Y}$ *on* $\mathsf{F}(a_1)\otimes\cdots\otimes\mathsf{F}(a_r)$ *factors through a map*

$$\mathbb{Y} \to \mathfrak{V}_r\big[\widehat{\boldsymbol{\Omega}}\big]\,. \quad (19.2)$$

The map (19.2) *is equivariant with respect to the translation automorphism.*

Note the translation automorphism

$$\varsigma_c\left(\boldsymbol{\alpha}^{(i)}\right) = \boldsymbol{\alpha}^{(i)} - \tau(c)$$

of the Heisenberg vertex algebra has a natural extension to $\mathfrak{V}_r\big[\widehat{\boldsymbol{\Omega}}\big]$. This extension leaves $\widehat{\boldsymbol{\Omega}}$ invariant.

Proof. This follows at once from formula (19.1) and Theorem 18.1.1. □

19.1.3

It may be curious to notice that the $\widehat{\boldsymbol{\Omega}}$ term disappears from the corresponding *quantum* operator $\widehat{\mathsf{Q}}$ upon averaging over all $|q| = 1$ in the principal value sense.

19.2 Yangian and $\mathcal{W}$-algebras

19.2.1

Our goal in this section is to describe the image of the map (19.2) in terms of the so-called $\mathcal{W}$ vertex operator algebras. This provides a link to the ideas of Alday, Gaiotto, and Tachikawa [2], the existence of which was suggested to us by Nakajima and Tachikawa.

The $\mathcal{W}$-algebras first appeared in mathematical physics as extended symmetry algebras of conformal field theories, see for example [10] for a survey. Following Feigin and Frenkel [38, 39], they may be described as explicit subalgebras of the Heisenberg vertex algebra. This is the description that we use here.

19.2.2

Let $\mathbb{K}$ be a commutative ring and let

$$\mathbb{H} \cong \mathbb{K}^r$$

be a free $\mathbb{K}$-module of rank r with a nondegenerate quadratic form. The setup is like in Chapter 13, except neither product nor coproduct on $\mathbb{H}$ is required. One should view $\mathbb{H}$ as a Cartan subalgebra of a reductive Lie algebra, with the restriction of a an invariant bilinear form. Here we need the form

$$(x,x) = \sum_1^r x_i^2\,, \tag{19.3}$$

that corresponds to the Lie algebra $\mathfrak{gl}(r)$.

One defines the Heisenberg algebra $\mathfrak{Heis}(\mathbb{H})$ as in Chapter 13 and the algebra of Fourier coefficients of vertex operators

$$\mathfrak{V}(\mathbb{H}) \supset \mathfrak{Heis}(\mathbb{H})$$

as in Section 19.1.2. For any orthogonal decomposition $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$, we have

$$\mathfrak{V}(\mathbb{H}) = \mathfrak{V}(\mathbb{H}_1) \,\widehat{\otimes}\, \mathfrak{V}(\mathbb{H}_2) \,,$$

where the completion is the usual completion required to collect terms in a product of two series.

19.2.3

Let

$$\eta = (0, \dots, 1, -1, \dots, 0) \,,$$

range over the simple positive roots of $\mathfrak{gl}(r)$. For each η, consider the corresponding Heisenberg field

$$\alpha_\eta(x) = \sum_n \left(\alpha_n^{(i)} - \alpha_n^{(i+1)} \right) \, x^{-n-1} \,.$$

Here we denote the argument by x to emphasize a small discrepancy between the conventions of Chapter 13 and standard CFT conventions. In Chapter 13, the arguments of the fields were coordinates on $\mathbb{C}^\times$. Here x is a coordinate on $\mathbb{C}$ and the exponents of x are shifted by 1, that is, by the conformal dimension of the field.

Since $(\eta, \eta) = 2$, we have

$$\alpha_\eta(x)\, \alpha_\eta(y) \sim \frac{2}{(x-y)^2} \,,$$

and the field

$$T_\eta = \frac{1}{4} :\alpha_\eta^2: + \frac{\kappa}{2} \frac{\partial \alpha_\eta}{\partial x}$$

generates a Virasoro vertex algebra which we denote by

$$\mathfrak{Vir}_\eta \subset \mathfrak{V}(\mathbb{K}\eta) \,.$$

Here κ is a parameter that enters the definition of the $\mathcal{W}$-algebra. To match it to conventions in the literature, we note that

$$\kappa = \frac{\beta}{\sqrt{2}} - \frac{\sqrt{2}}{\beta}$$

in the book [45] and that the central charge of $\mathfrak{Vir}_\eta$ equals $1 - 6\kappa^2$.

19.2.4

By definition, see for example [45], a vertex operator algebra is a collection of operator-valued distributions, called *vertex operators*, satisfying certain axioms. In CFT, these correspond to local chiral operators and, as in any mathematical formulation of QFT, the locality of these operators is really the key axiom. A specific feature of 2-dimensional conformal field theories is the presence of the Virasoro algebra among its chiral operators.

While the language of vertex operators is very rich and concise, for our current purposes it will be sufficient to work with the following classical algebraic structures associated to a vertex algebra:

— the *associative* algebra generated by the Fourier coefficients of vertex operators, such as $\mathfrak{V}(\mathbb{H})$, $\mathcal{W}(\mathfrak{gl}(r))$, or $\mathfrak{Vir}_\eta$,

— the *Lie algebra* generated by the Fourier coefficients of vertex operators with respect to the commutator, which will be indicated by a subscript like $\mathfrak{V}_{\mathrm{Lie}}(\mathbb{H})$, $\mathcal{W}_{\mathrm{Lie}}(\mathfrak{gl}(r))$, or $\mathfrak{Vir}_{\eta,\mathrm{Lie}}$

see Chapter 4 in [45]. Clearly, the latter generates the former.

To describe the $\mathcal{W}(\mathfrak{gl}(r))$ as a subalgebra of $\mathfrak{V}(\mathbb{H})$, we will use the following characterization due to Feigin and Frenkel. Recall that for each η we have

$$\mathfrak{V}(\mathbb{H}) = \mathfrak{V}(\mathbb{K}\eta) \,\widehat{\otimes}\, \mathfrak{V}(\eta^\perp)\,.$$

Theorem 19.2.1 ([38, 39])**.** *The algebra $\mathcal{W}_{\mathrm{Lie}}(\mathfrak{gl}(r))$ is the intersection*

$$\mathcal{W}_{\mathrm{Lie}}(\mathfrak{gl}(r)) = \bigcap_\eta \mathfrak{Vir}_{\eta,\mathrm{Lie}} \,\widehat{\otimes}\, \mathfrak{V}_{\mathrm{Lie}}(\eta^\perp)\,, \tag{19.4}$$

where η ranges over the simple positive roots of $\mathfrak{gl}(r)$.

The following outline of the argument was kindly provided by E. Frenkel.

Proof. The proof proceeds in 4 steps.

First, for generic values of the parameter, the vertex $\mathcal{W}$-algebra is equal to the intersection of the kernels of the screening operators. This is the most non-trivial step, proved in two ways: first, in Proposition 3 of [38] (this proof is reproduced in Theorem 15.4.12 of [45]) and second, in Theorem 4.6.9 of [39]).

Second, in the case of $\mathfrak{sl}(2)$, the kernel of the screening operator is the Virasoro vertex algebra for generic values of the parameter. This is proved in Proposition 4 of [38] (this proof is reproduced in 15.4.14 of [45]) and in Proposition 4.4.4 of [39].

Next, the kernel of the i-th screening operator is equal to the the tensor product of the Virasoro vertex algebra along the i-th simple root and the Heisenberg vertex algebra orthogonal to the i-th root. The proof is given in the proof of Proposition 5 in [38] and in 15.4.15 of [45].

Finally, the same results hold for the algebras of Fourier coefficients of vertex operators. This is proved in Proposition 2 of [38] and Theorem 4.6.11 of [39]. □

We define

$$\mathcal{W}(\mathfrak{sl}(r)) = \mathcal{W}(\mathfrak{gl}(r)) \cap \mathfrak{V}(\mathfrak{Z}^\perp)$$

where $\mathfrak{Z} \subset \mathbb{H} \subset \mathfrak{gl}(r)$ is the center. This implies

$$\mathcal{W}(\mathfrak{gl}(r)) = \mathfrak{V}(\mathfrak{Z}) \, \widehat{\otimes} \, \mathcal{W}(\mathfrak{sl}(r)) \,.$$

19.2.5

To compare this with our formulas, we take

$$\gamma = 1 \in H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathbb{C}^2)$$

in the formula (14.10). Since $(1,1)_{H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathbb{C}^2)} = \tau(1)$, we have

$$\text{our } \alpha_n(1) = \sqrt{\tau(1)} \text{ standard } \alpha_n$$

where Heisenberg operators associated to the quadratic form (19.3) are considered standard. Further, since

$$\Delta 1 = \frac{1 \otimes 1}{\tau(1)}$$

we have in (14.10)

$$\left[z^{-n}\right] \boldsymbol{T}_+(1) = \left[x^{-n-2}\right] T_\eta \Big|_{\alpha_0 \mapsto \alpha_0 + \frac{1}{2}\kappa\eta} \tag{19.5}$$

where $\eta = (1, -1)$ is the root of $\mathfrak{gl}(2)$ and

$$\kappa = \hbar\sqrt{\tau(1)} = -\frac{t_1 + t_2}{\sqrt{-t_1 t_2}}\,. \tag{19.6}$$

The shift of zero modes

$$\left(\alpha_0^{(1)}, \alpha_0^{(2)}\right) \mapsto \left(\alpha_0^{(1)} + \tfrac{1}{2}\,\hbar, \alpha_0^{(2)} - \tfrac{1}{2}\,\hbar\right) \tag{19.7}$$

compensates for the difference between $\partial = z\frac{\partial}{\partial z}$ in $\boldsymbol{T}_+$ and $\frac{\partial}{\partial x}$ in T_η.

19.2.6

Generalizing (19.7), we incorporate the shift of the zero modes by $\kappa\rho$, where ρ is the half-sum of positive roots, in the definition of $\mathcal{W}(\mathfrak{gl}(r))$. This is an automorphism of the ambient Heisenberg vertex algebra.

19.2.7

Nakajima varieties produce lowest weight Yangian modules. Any action of a graded algebra $A = \bigoplus A_n$ on a lowest weight module canonically extends to a certain completion $\overline{A} \supset A$. Neighborhoods of zero in this completions are left ideals generated by $\bigoplus_{n<-N} A_n$.

Proposition 19.2.2. *The action of* $\mathbb{Y}$ *on* $\mathsf{F}(a_1)\otimes\cdots\otimes\mathsf{F}(a_r)$ *factors through a map*

$$\mathbb{Y} \to \overline{\mathfrak{V}(\mathfrak{Z})}\,\widehat{\otimes}\,\mathcal{W}(\mathfrak{sl}(r))\,, \tag{19.8}$$

where $\overline{\mathfrak{V}(\mathfrak{Z})} \supset \mathfrak{V}(\mathfrak{Z})$ *is a completion as above.*

Proof. Extract the η-component from the operator (19.1) as in Section 14.3.2. Using (19.1) and (19.5), we conclude

$$\widehat{\mathsf{Q}}_{\mathrm{cl}} \in \mathcal{W}_{\mathrm{Lie}}(\mathfrak{gl}(r)) + \mathbb{K}\,\widehat{\boldsymbol{\Omega}} \otimes 1\,,$$

where the second term is written with respect to the decomposition $\mathbb{H} = \mathfrak{Z} \oplus \mathfrak{Z}^\perp$. Therefore

$$\widehat{\mathsf{Q}}_{\mathrm{cl}}, \beta_{\pm 1} \in \overline{\mathfrak{V}(\mathfrak{Z})}\,\widehat{\otimes}\,\mathcal{W}(\mathfrak{sl}(r))$$

and Theorem 18.1.1 completes the proof. □

19.2.8

The proof of (19.4) by Feigin and Frenkel uses a screening operators characterization of $\mathfrak{Vir}_\eta$. Those can be matched to the screening operators of Section 18.2.

19.2.9

Proposition 19.2.3. *The map*

$$\overline{\mathbb{Y}} \to \overline{\mathcal{W}(\mathfrak{gl}(r))} \tag{19.9}$$

induced by (19.8) *is surjective.*

Proof. Follows from the corresponding statement for $\hbar = \kappa = 0$ proven by Frenkel, Kac, Radul, and Wang in [46]. When $\hbar = 0$, the nonlocal term $\mathbf{\Omega}$ drops out and the surjectivity

$$\mathbb{Y}/\hbar\mathbb{Y} \cong \mathcal{U}(\widehat{\mathcal{D}}) \to \mathcal{W}(\mathfrak{gl}(r))\Big|_{\kappa=0} \to 0$$

is true without completion, see [46]. Clearly, it implies the surjectivity after completion. □

19.2.10

One of the goals of [2] is a characterization of interesting cohomology classes in terms of the $\mathcal{W}$-action. For example, one can consider the vector of identities

$$\mathbf{1} \in H^{\boldsymbol{\cdot}}_{\mathsf{G}}(\mathcal{M}(r))$$

in the cohomology of each $\mathcal{M}(r,n)$. In this direction, there is the following simple result. Define

$$\beta_n^{[k]} = \left(\mathrm{ad}\, \widehat{\mathsf{Q}}_{\mathrm{cl}}\right)^k \cdot \beta_n \,.$$

Proposition 19.2.4. *The vector of identities* $\mathbf{1}$ *satisfies*

$$\beta_n^{[k]}(\mathsf{pt}) \cdot \mathbf{1} = \begin{cases} 0\,, & k < rn - 1 \\ -\mathbf{1}\,, & n = 1, k = r-1\,. \end{cases}$$

Proof. The operator $\beta_n(\mathsf{pt})$ is defined by a proper push-forward with fibers of generic dimension $rn-1$, therefore it annihilates any cohomology class of degree less than $rn-1$. This proves the first claim.

If $n = 1$ then generic fibers are projective spaces $\mathbb{P}^{r-1}$ on which the generator $\mathrm{ch}_1 \widehat{\mathcal{V}}$ restricts to the hyperplane class $c_1(\mathcal{O}(1))$, up-to equivariant corrections. Therefore

$$\beta_1^{[r-1]}(\mathsf{pt}) = (-1)^{r-1}\beta_1 \, \widehat{\mathsf{Q}}_{\mathrm{cl}}^{r-1} \cdot \mathbf{1} = \\ -\left(\int_{\mathbb{P}^{r-1}} c_1(\mathcal{O}(1))^{r-1}\right) \cdot \mathbf{1} = -\mathbf{1}\,, \quad (19.10)$$

where an extra $(-1)^r$ comes from the definition of $\beta_1 = \beta^\tau_{-1}$, see Section 12.2.4.

□

Bibliography

[1] M. Aganagic and A. Okounkov, *Elliptic stable envelopes*, `arXiv:1604.00423`.

[2] L. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010), no. 2, 167–197.

[3] P. Aluffi, L. Mihalcea, J. Schürmann and C. Su, *Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells*, `arXiv:1709.08697`.

[4] R. Anno, R. Bezrukavnikov, and I. Mirkovic, *A thin stringy moduli space for Slodowy slices*, `arXiv:1108.1563`.

[5] N. Arbesfeld, in preparation.

[6] V. Baranovsky, *Moduli of sheaves on surfaces and action of the oscillator algebra*, J. Differential Geom. **55** (2000), no. 2, 193–227.

[7] A. Bertram, *Quantum Schubert calculus*, Adv. Math. 128 (1997), no. 2, 289–305.

[8] R. Bezrukavnikov, A. Braverman, P. Etingof, A. Okounkov, V. Toledano-Laredo, et al, *Quantum Cohomology, Quantized Algebraic Varieties, and Representation Theory*, FRG proposal, 2008.

[9] R. Bezrukavnikov and A. Okounkov, in preparation.

[10] P. Bouwknegt and K. Schoutens, *$\mathcal{W}$ symmetry in conformal field theory*, Phys. Rep. **223** (1993), no. 4, 183–276.

[11] A. Braverman, M. Finkelberg, and D. Gaitsgory, *Uhlenbeck spaces via affine Lie algebras* The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006.

[12] A. Braverman, D. Maulik, A. Okounkov, *Quantum cohomology of the Springer resolution*, Adv. Math. 227 (2011), no. 1, 421–458.

[13] T. Bridgeland, *Stability conditions on a non-compact Calabi-Yau threefold*, Comm. Math. Phys. **266** (2006), no. 3, 715–733.

[14] T. Bridgeland, *Spaces of stability conditions*, Algebraic geometry–Seattle 2005, Part 1, 1–21, Proc. Sympos. Pure Math., **80**, Part 1, AMS Providence, RI, 2009.

[15] A. Buch, *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), no. 2, 227–235.

[16] A. Buch, A. Kresch, and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. 16 (2003), no. 4, 901–915.

[17] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995.

[18] I. Ciocan-Fontanine, B. Kim, and D. Maulik, *Stable quasimaps to GIT quotients*, J. Geom. Phys. **75** (2014), 17-47.

[19] K. Costello and I. Grojnowski, *Hilbert schemes, Hecke algebras and the Calogero-Sutherland system*, `arXiv:math/0310189`.

[20] D. Cox, S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, 68, American Mathematical Society, Providence, RI, 1999.

[21] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. 126 (2001), no. 3, 257–293.

[22] W. Crawley-Boevey, *Normality of Marsden-Weinstein reductions for representations of quivers*, Math. Ann. 325 (2003), no. 1, 55–79.

[23] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 2010.

[24] D.-E. Diaconescu, *Moduli of ADHM sheaves and the local Donaldson-Thomas theory*, J. Geom. Phys. **62** (2012), no. 4, 763–799.

[25] Ph. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[26] S. Donaldson and P. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990.

[27] V. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR 283 (1985), no. 5, 1060–1064.

[28] B. Estienne, V. Pasquier, R. Santachiara, D. Serban, *Conformal blocks in Virasoro and W theories: duality and the Calogero-Sutherland model*, Nuclear Phys. B **860** (2012), no. 3, 377–420.

[29] P. Etingof, *Representation theory in complex rank*, unfinished manuscript.

[30] P. Etingof and V. Ginzburg, *Noncommutative complete intersections and matrix integrals*, Pure Appl. Math. Q. **3** (2007), no. 1, 107–151.

[31] P. Etingof, V. Ginzburg, E. Rains, *Toroidal Yangians*, in preparation.

[32] P. Etingof, T. Schedler, O. Schiffmann, *Explicit quantization of dynamical r-matrices for finite dimensional semisimple Lie algebras*, Journal of AMS, 13(2000), no. 3, 595–609.

[33] P. Etingof and O. Schiffmann, *Lectures on quantum groups*, Lectures in Mathematical Physics, International Press, Somerville, MA, 2002.

[34] L. Faddeev, *How the algebraic Bethe ansatz works for integrable models*, Symétries quantiques (Les Houches, 1995), 149–219, North-Holland, 1998.

[35] L. Faddeev, N. Reshetikhin, L. Takhtadzhyan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990), no. 1, 193–225.

[36] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Quantum continuous $\mathfrak{gl}_\infty$: Semi-infinite construction of representations*, Kyoto J. Math. **51** (2011), no. 2, 337-364.

[37] B. Feigin, M. Finkelberg, I. Frenkel, L. Rybnikov, *Gelfand-Tsetlin algebras and cohomology rings of Laumon spaces*, Selecta Math. (N.S.) 17 (2011), no. 2, 337–361.

[38] B. Feigin and E. Frenkel, *Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras*, Int. Jour. Mod. Phys. A7, Supplement 1A (1992) 197–215

[39] B. Feigin and E. Frenkel, *Integrals of motion and quantum groups*, Integrable systems and quantum groups (Montecatini Terme, 1993), 349–418, Lecture Notes in Math., **1620**, Springer, Berlin, 1996.

[40] B. Feigin and D. Fuks, *Invariant skew-symmetric differential operators on the line and Verma modules over the Virasoro algebra*, Functional Analysis and Its Applications **16** (1982), no. 2, 114–126

[41] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, S. Yanagida *A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials*, J. Math. Phys. **50** (2009), no. 9.

[42] B. Feigin and A. Tsymbaliuk, *Equivariant K-theory of Hilbert schemes via shuffle algebra*, Kyoto J. Math. **51** (2011), no. 4, 831–854

[43] B. Feigin and A. Zelevinsky, *Representations of contragredient Lie algebras and the Kac-Macdonald identities*, Representations of Lie groups and Lie algebras (Budapest, 1971), 2577, Akad. Kiadó, Budapest, 1985.

[44] G. Felder, Y. Markov, V. Tarasov, and A. Varchenko, *Differential equations compatible with KZ equations*, Math. Phys. Anal. Geom. **3** (2000), no. 2, 139–177.

[45] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, **88**, American Mathematical Society, Providence, RI, 2004.

[46] E. Frenkel, V. Kac, A. Radul, W. Wang, *$\mathcal{W}_{1+\infty}$ and $\mathcal{W}(\mathfrak{gl}_N)$ with central charge N*, Comm. Math. Phys. **170** (1995), no. 2, 337–357.

[47] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60.

[48] S. Gautam, V. Toledano-Laredo, *Monodromy of the trigonometric Casimir connection for sl_2*, Noncommutative birational geometry, representations and combinatorics, 137-176, Contemp. Math., 592, AMS, 2013.

[49] V. Gorbounov, R. Rimanyi, V. Tarasov, and A. Varchenko, *Quantum cohomology of the cotangent bundle of a flag variety as a Yangian Bethe algebra*, J. Geom. Phys. **74** (2013), 56-86.

[50] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83.

[51] I. Grojnowski, *Instantons and affine algebras, I. The Hilbert scheme and vertex operators*, Math. Res. Lett. **3** (1996), no. 2, 275–291.

[52] M. Haiman, *Macdonald polynomials and geometry*, New Perspectives in Geometric Combinatorics, MSRI Publications **37** (1999), 207–254.

[53] D. Halpern-Leistner, D. Maulik and A. Okounkov, in preparation.

[54] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow, *Mirror symmetry*, Clay Mathematics Monographs, 1, American Mathematical Society, Providence, RI, 2003.

[55] M. Jimbo, T. Miwa, *Algebraic analysis of solvable lattice models*, American Mathematical Society, 1995.

[56] V. Kac, *Contravariant form for infinite-dimensional Lie algebras and superalgebras*, Group theoretical methods in physics, Lecture Notes in Physics, **94**, Springer-Verlag, Berlin-New York, 1979, pp. 441–445

[57] V. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, Cambridge, 1990.

[58] V. Kac, *Vertex algebras for beginners*, Second edition. University Lecture Series, **10**, American Mathematical Society, Providence, RI, 1998.

[59] V. Kac and A. Radul, *Quasifinite highest weight modules over the Lie algebra of differential operators on the circle*, Comm. Math. Phys. **157** (1993), no. 3, 429–457.

[60] D. Kaledin, *Geometry and topology of symplectic resolutions*, Algebraic geometry—Seattle 2005. Part 2, 595–628, Proc. Sympos. Pure Math., 80, Part 2, Amer. Math. Soc., 2009.

[61] D. Kaledin, *Symplectic singularities from the Poisson point of view*, J. Reine Angew. Math. **600** (2006), 135–156.

[62] S. Khoroshkin and V. Tolstoy, *Yangian double*, Lett. Math. Phys. **36** (1996), no. 4, 373–402.

[63] B. Kim, *On equivariant quantum cohomology*, Internat. Math. Res. Notices 1996, no. 17, 841–851.

[64] V. Korepin, N. Bogoliubov, A. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, 1993.

[65] J. Kock, *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts, **59**, Cambridge University Press, Cambridge, 2004.

[66] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. **136** (1999), no. 1, 157–207.

[67] M. Lehn, *Lectures on Hilbert schemes*, Algebraic structures and moduli spaces, 1–30, CRM Proc. Lecture Notes, 38, Amer. Math. Soc., Providence, RI, 2004.

[68] S. Levendorskij and A. Sudbery, *Yangian construction of the Virasoro algebra*, Lett. Math. Phys. **37** (1996), no. 3, 243–247.

[69] W.-P. Li, Z. Qin, W. Wang, *The cohomology rings of Hilbert schemes via Jack polynomials*, Algebraic structures and moduli spaces, 249–258, CRM Proc. Lecture Notes, 38, Amer. Math. Soc., Providence, RI, 2004.

[70] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995.

[71] A. Malkin, *Tensor product varieties and crystals: the ADE case*, Duke Math. J. 116 (2003), no. 3, 477–524.

[72] A. Malkin, V. Ostrik, and M. Vybornov, *Quiver varieties and Lusztig's algebra*, Adv. Math. **203** (2006), no. 2, 514–536.

[73] D. Maulik and A. Oblomkov, *Quantum cohomology of the Hilbert scheme of points on A_n-resolutions*, J. Amer. Math. Soc. **22** (2009), no. 4, 1055–1091.

[74] D. Maulik, A. Oblomkov, A. Okounkov, R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Invent. Math. 186 (2011), no. 2, 435–479.

[75] D. Maulik and A. Okounkov, in preparation.

[76] Y. Matsuo, *Matrix theory, Hilbert scheme and integrable system*, Modern Phys. Lett. A **13** (1998), no. 34, 2731–2742.

[77] K. McGerty and T. Nevins, *Kirwan surjectivity for quiver varieties*, `arXiv:1610.08121`.

[78] L. Mihalcea, *Equivariant quantum Schubert calculus*, Adv. Math. **203** (2006), no. 1, 1–33.

[79] K. Miki, *A (q, γ) analog of the $W_{1+\infty}$ algebra*, J. Math. Phys. **48** (2007).

[80] J. Millson and V. Toledano Laredo, *Casimir operators and monodromy representations of generalised braid groups*, Transform. Groups **10** (2005), no. 2, 217–254.

[81] T. Miwa, M. Jimbo, E. Date, *Solitons*, Cambridge Tracts in Mathematics, **135**, Cambridge University Press, Cambridge, 2000.

[82] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, **143** American Mathematical Society, Providence, RI, 2007.

[83] A. Molev, M. Nazarov, and G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Surveys **51** (1996), no. 2, 205–282.

[84] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. 76 (1994), no. 2, 365–416.

[85] H. Nakajima, *Jack polynomials and Hilbert schemes of points on surfaces*, `alg-geom/9610021`.

[86] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), no. 2, 379–388.

[87] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. 91 (1998), no. 3, 515–560.

[88] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, **18**, American Mathematical Society, Providence, RI, 1999.

[89] H. Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238.

[90] H. Nakajima, *Quiver varieties and tensor products*, Invent. Math. 146 (2001), no. 2, 399–449.

[91] H. Nakajima, *Quiver varieties and tensor products, II*, Symmetries, integrable systems and representations, 403-428, Springer Proc. Math. Stat., 40, Springer, 2013.

[92] Y. Namikawa, *Extension of 2-forms and symplectic varieties*, J. Reine Angew. Math. **539** (2001), 123–147.

[93] N. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2003), no. 5, 831–864.

[94] N. Nekrasov and A. Okounkov, *Quantum cohomology and quantum intermediate long wave equation*, in preparation.

[95] N. Nekrasov and A. Okounkov, *Membranes and Sheaves*, Algebraic Geometry, **3** (2016), no. 3, 320–369.

[96] N. Nekrasov and S. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, Nuclear Phys. B Proc. Suppl. 192/193 (2009), 91–112.

[97] N. Nekrasov and S. Shatashvili, *Quantum integrability and supersymmetric vacua*, Progr. Theor. Phys. **177** (2009) 105–119.

[98] N. Nekrasov and S. Shatashvili, *Quantization of integrable systems and four dimensional gauge theories*, XVIth International Congress on Mathematical Physics, 265–289, World Sci. Publ., 2010

[99] A. Okounkov, *Lectures on K-theoretic computations in enumerative geometry*, arXiv:1512.07363.

[100] A. Okounkov, *Enumerative geometry and geometric representation theory*, Proceedings of the 2015 AMS Algebraic Geomtery Summer Institute, arXiv:1701.00713.

[101] A. Okounkov and A. Smirnov, in preparation.

[102] A. Okounkov and A. Smirnov, *Quantum difference equations for Nakajima varieties*, arXiv:1602.09007.

[103] A. Okounkov and R. Pandharipande, *Quantum cohomology of the Hilbert scheme of points in the plane*, Invent. Math. 179 (2010), no. 3, 523–557.

[104] E. Rains, *BC_n-symmetric Abelian functions* Duke Math. J. **135** (2006), no. 1, 99–180.

[105] K. Rietsch, *Quantum cohomology rings of Grassmannians and total positivity*, Duke Math. J. **110** (2001), no. 3, 523–553.

[106] S. N. M. Ruijsenaars, *On Barnes' multiple zeta and gamma functions*, Adv. Math. **156** (2000), no. 1, 107–132.

[107] D. Shenfeld, *Abelianization of Stable Envelopes in Symplectic Resolutions*, PhD thesis, Princeton, 2013.

[108] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of* $\mathbb{A}^2$, Duke Math. J. **162** (2013), no. 2, 279-366.

[109] O. Schiffmann and E. Vasserot, *Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on* $\mathbb{A}^2$, Publ. Math. Inst. Hautes Études Sci. **118** (2013), 213-342.

[110] B. Siebert and G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. **1** (1997), no. 4, 679–695.

[111] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. 134 (1991), 375–422.

[112] N. Slavnov, *The algebraic Bethe ansatz and quantum integrable systems*, Russian Math. Surveys 62 (2007), no. 4, 727–766.

[113] B. Shou, J.-F. Wu, M. Yu, *AGT conjecture and AFLT states: a complete construction*, `arXiv:1107.4784`.

[114] A. Smirnov, *On the instanton R-matrix*, Comm. Math. Phys. **345** (2016), no. 3, 703-740.

[115] Y. Soibelman, *Meromorphic tensor categories, quantum affine and chiral algebras*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 437–451, Contemp. Math., 248, Amer. Math. Soc., Providence, RI, 1999.

[116] C. Soulé, *Lectures on Arakelov geometry*, With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer, Cambridge Studies in Advanced Mathematics, **33**, Cambridge University Press, Cambridge, 1992.

[117] R. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), no. 1, 76–115.

[118] J. Teschner, *Liouville theory revisited*, Classical Quantum Gravity **18** (2001), no. 23, R153–R222.

[119] V. Toledano Laredo, *A Kohno-Drinfeld theorem for quantum Weyl groups*, Duke Math. J. **112** (2002), no. 3, 421–451.

[120] V. Toledano Laredo, *The trigonometric Casimir connection of a simple Lie algebra*, J. Algebra 329 (2011), 286–327.

[121] M. Varagnolo, *Quiver varieties and Yangians*, Lett. Math. Phys. 53 (2000), no. 4, 273–283.

[122] M. Varagnolo and E. Vasserot, *Standard modules of quantum affine algebras*, Duke Math. J. 111 (2002), no. 3, 509–533.

[123] M. Varagnolo and E. Vasserot, *Perverse sheaves and quantum Grothendieck rings*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 345-365.

[124] E. Vasserot, *Sur l'anneau de cohomologie du schéma de Hilbert de* C^2, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 1, 7–12.

[125] E. Vinberg and V. Popov, *Invariant theory*, Encyclopaedia of Mathematical Sciences, 55, Springer-Verlag, 1994.