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GENERALIZED SPRINGER THEORY AND WEIGHT FUNCTIONS

G. Lusztig

Introduction

0.1. The generalized Springer correspondence [L1] is a bijection beween, on the one hand, the set of pairs consisting of a unipotent class in a connected reductive group G and an irreducible G-equivariant local system on it and, on the other hand, the union of the sets of irreducible representations of a collection of Weyl groups associated to G. (The classical case involves only some irreducible local systems and only one Weyl group.) In this paper we show that each Weyl group appearing in the collection has a natural weight function (see 0.2). We also show how to extend each of these weight functions to an affine Weyl group; in fact, we describe two such extensions, one in terms of G and one in terms of the dual group G^*. The one in terms of G^* has a surprising representation theoretic interpretation, see 3.3.

0.2. Notation. Let G be a connected reductive group over $\mathbf{C}$. We fix a prime number l. By local system we mean a $\bar{\mathbf{Q}}_l$-local system. The centralizer of an element x of a group Γ is denoted by $Z_\Gamma(x)$. The identity component of an algebraic group H is denoted by H^0. For an algebraic group H let $\mathcal{Z}_H$ be the centre of H. For a connected affine algebraic group H let U_H be the unipotent radical of H. If (W,S) is a Coxeter group with length function $\underline{l}$ we say that $\mathcal{L}: W \to \mathbf{N}$ is a weight function if $\mathcal{L}(ww') = \mathcal{L}(w) + \mathcal{L}(w')$ whenever w, w' in W satisfy $\underline{l}(ww') = \underline{l}(w) + \underline{l}(w')$.

1. A weighted Weyl group

1.1. Induction data. An *induction datum* for G is a triple $(L, \mathcal{O}, \mathcal{E})$ where L is a Levi subgroup of a parabolic subgroup of G, $\mathcal{O}$ is a unipotent conjugacy class of L and $\mathcal{E}$ is an irreducible L-equivariant local system on $\mathcal{O}$ (up to isomorphism) which is *cuspidal* (in a sense that will be made precise in 1.3). To an induction datum $(L, \mathcal{O}, \mathcal{E})$ we will associate a complex of sheaves K on G as follows. We choose a parabolic subgroup P for which L is a Levi subgroup; let $pr : \mathcal{Z}_L^0 \mathcal{O} U_P \to \mathcal{O}$ be

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the projection (we identify $\mathcal{Z}_L^0\mathcal{O}U_P$, a subvariety of P, with $\mathcal{Z}_L^0 \times \mathcal{O} \times U_P$). We have a diagram

$$\mathcal{Z}_L^0 \times \mathcal{O} \xleftarrow{a} \tilde{\mathfrak{P}} \xrightarrow{b} \mathfrak{P} \xrightarrow{c} G$$

where
$\tilde{\mathfrak{P}} = \{(h,g) \in G \times G; h^{-1}gh \in \mathcal{Z}_L^0\mathcal{O}U_P\}$,
$\mathfrak{P} = \{(hP,g) \in G/P \times G; h^{-1}gh \in \mathcal{Z}_L^0\mathcal{O}U_P\}$,
$a(h,g) = pr(h^{-1}gh)$, $b(h,g) = (hP,g)$, $c(hP,g) = g$.
We have $a^*(\bar{\mathbf{Q}}_l \boxtimes \mathcal{E}) = b^*\tilde{\mathcal{E}}$ where $\tilde{\mathcal{E}}$ is a well defined local system on $\mathfrak{P}$. Thus, $K = c_!\tilde{\mathcal{E}}$ is well defined. According to [L1], K is an intersection cohomology complex on G whose support is $\cup_{h\in G} h\mathcal{Z}_L^0\bar{\mathcal{O}}U_P h^{-1}$; $\bar{\mathcal{O}}$ is the closure of $\mathcal{O}$).

Let X_G be the (finite) set consisting of all pairs $(\mathcal{C},\mathcal{S})$ where $\mathcal{S}$ is a unipotent conjugacy class in G and $\mathcal{S}$ is an irreducible G-equivariant local system on $\mathcal{C}$ (up to isomorphism). Let $[L,\mathcal{O},\mathcal{E}]$ be the set of all $(\mathcal{C},\mathcal{S}) \in X_G$ such that $\mathcal{S}$ is a direct summand of the local system on $\mathcal{C}$ obtained by restricting some cohomology sheaf of $K|_{\mathcal{C}}$. Note that subset $[L,\mathcal{O},\mathcal{E}]$ depends only on the G-conjugacy class of $(L,\mathcal{O},\mathcal{E})$.

1.2. For example, if L is a maximal torus of G (so that P is a Borel subgroup, $\mathcal{O} = \{1\}$ and $\mathcal{E} = \bar{\mathbf{Q}}_l$), we have $\mathfrak{P} = \{(hP,g) \in G/P \times G; h^{-1}gh \in P\}$ and $c : \mathfrak{P} \to G$ is the Springer resolution; in this case, $K = c_!\bar{\mathbf{Q}}_l$.

1.3. Blocks of X_G. Following [L1] we define a partition of X_G into subsets called *blocks*. If $(\mathcal{C},\mathcal{S}) \in X_G$ we say that $\mathcal{S}$ is cuspidal if $\{(\mathcal{C},\mathcal{S})\}$ is a block by itself said to be a cuspidal block. The definition of blocks is by induction on $\dim G$. If $G = \{1\}$, then X_G has a single element; it forms a block. For general G, the non-cuspidal blocks of X_G are exactly the subsets of X_G of the form $[L,\mathcal{O},\mathcal{E}]$, where $(L,\mathcal{O},\mathcal{E})$ is an induction datum for G with $L \neq G$. (Note that the notion of cuspidality of $\mathcal{E}$ is known from the induction hypothesis since $\dim L < \dim G$.) The cuspidal blocks of X_G are the one element subsets of X_G which are not contained in any non-cuspidal block. The correspondence $(L,\mathcal{O},\mathcal{E}) \mapsto [L,\mathcal{O},\mathcal{E}]$ defines a bijection between the set of induction data of G (up to conjugation) and the set of blocks of X_G, see [L1].

1.4. Let $L,\mathcal{O},\mathcal{E},P,c : \mathfrak{P} \to G$ be as in 1.1 and let $x \in \mathcal{O}$. Let $\mathfrak{P}_x = c^{-1}(x)$. Thus, $\mathfrak{P}_x = \{hP \in G/P; h^{-1}xh \in \mathcal{O}U_P\}$. In [L3, §11], $\mathfrak{P}_x$ is called a *generalized flag manifold*. This is justified by the following result in [L3, 11.2] in which $U = U_{Z_G^0(x)}$.

(a) *The conjugation action of $Z_G^0(x)$ on $\mathfrak{P}_x$ is transitive. If $hP \in \mathfrak{P}_x$ then $\beta_P := (hPh^{-1} \cap Z_G^0(x))U$ is a Borel subgroup of $Z_G^0(x)$. The map $hP \to \beta_P$ from $\mathfrak{P}_x$ to the variety of Borel subgroups of $Z_G^0(x)$ is a fibration. The fibres are exactly the orbits of the conjugation action of U on $\mathfrak{P}_x$ hence are affine spaces.*
We have the following result.

(b) $\dim \mathfrak{P}_x = (\dim Z_G^0(x) - \dim Z_L^0(x))/2$.
From [L1, 2.9] we see that the right hand side of (b) is equal to the dimension of

the $Z_G(x)$-orbit of P in G/P and that this orbit is connected so that, by (a), it equals $\mathfrak{P}_x$. This proves (b).

Let W be the Weyl group of G, a finite Coxeter group, and let S_0 be the set of simple reflections of W. For any $J \subset S_0$ let W_J be the subgroup of W generated by J and let w_0^J be the longest element of W_J.

Now P is a parabolic subgroup of type I for a well defined subset I of S_0. Let $\mathcal{W}$ be the set of all $w \in W$ such that $wW_I = W_I w$ and w has minimal length in $wW_I = W_I w$. This is a subgroup of W. For any $s \in S_0 - I$ we have $w_0^{I \cup s} w_0^I = w_0^I w_0^{I \cup s}$ hence $\sigma_s = w_0^{I \cup s} w_0^I = w_0^I w_0^{I \cup s}$ satisfies $\sigma_s^2 = 1$. Moreover we have $\sigma_s \in \mathcal{W}$.

Let $x \in \mathcal{O}$. Let b be the dimension of the variety of Borel subgroups of P that contain x. For any $s \in S_0 - I$ let P_s be the unique parabolic subgroup of type $I \cup s$ that contains P and let

$$\mathfrak{P}_{s,x} = \{hP \in P_s/P; h^{-1}xh \in \mathcal{O}U_P\}.$$

This is the analogue of $\mathfrak{P}_x$ when G is replaced by P_s/U_{P_s} hence is again a generalized flag manifold. We set

$$\mathcal{L}_0(s) = \dim \mathfrak{P}_{s,x}.$$

One can verify that

(c) $\mathcal{W}$ is a Weyl group with Coxeter generators $\{\sigma_s; s \in S_0 - I\}$
(see [L1]) and

(d) $\sigma_s \mapsto \mathcal{L}_0(s)$ is the restriction to $\{\sigma_s; s \in S_0 - I\}$ of a weight function $\tilde{\mathcal{L}}_0$ on $\mathcal{W}$.

To verify (d), we note that $\mathcal{L}_0(s)$ can be computed explicitly in each case using (b) for P_s/U_{P_s} instead of G. (See the next section.)

1.5. We now assume that G is almost simple, simply connected. We describe in each case where L is not a maximal torus, the assignment $(G, L, \mathcal{O}, \mathcal{E}) \mapsto \mathcal{W}$ and the values of the function $\mathcal{L}_0$; we will write (G, L) instead of $(G, L, \mathcal{O}, \mathcal{E})$ and will specify G, L by the type of $G, L/\mathcal{Z}_L$. The notation for Weyl groups is the usual one, with the convention that a Weyl group of type A_0 is $\{1\}$.

(a) $$(A_{kn-1}, A_{n-1}^k) \mapsto A_{k-1}, \quad n \geq 2, k \geq 1; \mathcal{L}_0 = n, n, \ldots, n;$$

(b) $$(C_{2t^2+t+k}, C_{2t^2+t}) \mapsto C_k, \quad t \geq 1, k \geq 0; \mathcal{L}_0 = 1, 1, \ldots, 1, 2t+1;$$

(c) $$(C_{2t^2+3t+k+1}, C_{2t^2+3t+1}) \mapsto C_k, \quad t \geq 0, k \geq 0; \mathcal{L}_0 = 1, 1, \ldots, 1, 2t+2;$$

(d) $$(B_{2t^2+2t+k}, B_{2t^2+2t}) \mapsto B_k, \quad t \geq 1, k \geq 0; \mathcal{L}_0 = 1, 1, \ldots, 1, 2t+1;$$

(e) $$(B_{4t^2+3t+2k}, B_{4t^2+3t}\times A_1^k)\mapsto C_k,\quad t\ge 1, k\ge 0; \mathcal{L}_0=2,2,\dots,2,4t+2;$$

(f) $$(B_{4t^2+5t+2k+1}, B_{4t^2+5t+1}\times A_1^k)\mapsto C_k,\quad t\ge 0, k\ge 0; \mathcal{L}_0=2,2,\dots,2,4t+1;$$

(g) $$(D_{2t^2+k}, D_{2t^2})\mapsto B_k,\quad t\ge 1, k\ge 0; \mathcal{L}_0=1,1,\dots,1,2t;$$

(h) $$(D_{4t^2+t+2k}, D_{4t^2+t}\times A_1^k)\mapsto C_k,\quad t\ge 1, k\ge 0; \mathcal{L}_0=2,2,\dots,2,4t-1;$$

(i) $$(D_{4t^2-t+2k}, D_{4t^2-t}\times A_1^k)\mapsto C_k,\quad t\ge 1, k\ge 0; \mathcal{L}_0=2,2,\dots,2,4t;$$

(j) $$(E_6, A_2^2)\mapsto G_2; \mathcal{L}_0=1,3;$$

(k) $$(E_7, A_1^3)\mapsto F_4; \mathcal{L}_0=1,1,2,2;$$

(l) $$(E_8, E_8)\mapsto A_0;$$

(m) $$(F_4, F_4)\mapsto A_0;$$

(n) $$(G_2, G_2)\mapsto A_0.$$

(In the case where $\mathcal{W}$ is of type $B_k=C_k$ the name we have chosen is such that it agrees with the type of the affine Weyl group $\hat{\mathcal{W}}$ in 1.5.)

In the case where L is a maximal torus that is, $(L,\mathcal{O},\mathcal{E})$ is as in 1.2, we have $\mathcal{W}=W$; the function $\mathcal{L}_0$ is constant equal to 1.

1.6. Let $L,\mathcal{O},\mathcal{E},P$ be as in 1.1 and let $x\in\mathcal{O}$. Let Ω be the set of P-orbits on G/P (under the action by left translation). For $\omega\in\Omega$ we set $\mathfrak{P}_x^\omega=\mathfrak{P}_x\cap\omega$ so that we have a partition $\mathfrak{P}_x=\sqcup_\omega\mathfrak{P}_x^\omega$ where each $\mathfrak{P}_x^\omega$ is locally closed in $\mathfrak{P}_x$. Let NL be the normalizer of L in G. We can identify NL/L with a subset of Ω by $nL\mapsto P-$ orbit of nP where $n\in NL$. We can also identify $NL/L=\mathcal{W}$ canonically so that we can identify $\mathcal{W}$ with a subset of Ω. One can show that

(a) *If $w\in\mathcal{W}$ then $\mathfrak{P}_x^w$ is an affine space of dimension $\tilde{\mathcal{L}}_0(w)$.*

Let w_0 be the longest element of $\mathcal{W}$. Since $\mathfrak{P}_x^{w_0}$ is open in $\mathfrak{P}_x$ we deduce that

(b) $$\dim\mathfrak{P}_x=\tilde{\mathcal{L}}_0(w_0).$$

2. A weighted affine Weyl group

2.1. In this subsection we describe an affine analogue of the generalized Springer theory. We assume that G is almost simple, simply connected and that $(L,\mathcal{O},\mathcal{E})$ are as in 1.1. Let $\hat{G}=G(\mathbf{C}((\epsilon)))$ where ϵ is an indeterminate. We can find a parahoric subgroup $\hat{P}$ of $\hat{G}$ whose prounipotent radical $U_{\hat{P}}$ satisfies $\hat{P}=U_{\hat{P}}L$, $U_{\hat{P}}\cap L=\{1\}$. Let $\hat{W}$ be the affine Weyl group defined by $\hat{G}$. It is a Coxeter group with set of simple reflections $\hat{S}_0$. We have $S_0\subset\hat{S}_0$ naturally and the subgroup of $\hat{W}$ generated by S_0 can be identified with W. In particular the subset $I\subset S_0$ can be viewed as a subset of $\hat{S}_0$. Let $\hat{S}'_0$ be the set of $s\in\hat{S}_0-I$ such that $I\cup s$ generate a finite subgroup of $\hat{W}$; this set contains S_0-I. Let $\hat{\mathcal{W}}$ be the subgroup of $\hat{W}$ defined in terms of $\hat{W},W,u=1$ as in [L4, 25.1]. This is a Coxeter group (in fact an affine Weyl group) with generators $\{\sigma_s;s\in\hat{S}'_0\}$. It contains $\mathcal{W}$ as the subgroup generated by S_0-I.

For any $g\in\hat{G}$ let $\hat{\mathfrak{P}}_g=\{h\hat{P}\in\hat{G}/\hat{P};h^{-1}gh\in\mathcal{Z}_L^0\mathcal{O}U_{\hat{P}}\}$. If $g\in\hat{G}$ is regular semisimple, then $\hat{\mathfrak{P}}_g$ can be viewed as an increasing union of algebraic varieties of bounded dimension. Moreover, $\mathcal{E}$ gives rise to a local system $\hat{\mathcal{E}}$ on $\hat{\mathfrak{P}}_g$ in the same way as $\mathcal{E}$ gives rise to a a local system $\tilde{\mathcal{E}}$ on $\mathfrak{P}$ in 1.1. Then the homology groups $H_i(\hat{\mathfrak{P}}_g,\hat{\mathcal{E}})$ are defined; they are (possibly infinite dimensional) $\bar{\mathbf{Q}}_l$-vector spaces. Using the method in [L5] (patching together various generalized Springer representations for groups of rank 2 considered in [L1]) we see that $\hat{\mathcal{W}}$ acts naturally on $H_i(\hat{\mathfrak{P}}_g,\hat{\mathcal{E}})$.

We now describe the type of the affine Weyl group $\hat{\mathcal{W}}$.

In 1.5(a), $\hat{\mathcal{W}}$ has type $\tilde{A}_{k-1}$.
In 1.5(b), $\hat{\mathcal{W}}$ has type $\tilde{C}_k$.
In 1.5(c), $\hat{\mathcal{W}}$ has type $\tilde{C}_k$.
In 1.5(d), $\hat{\mathcal{W}}$ has type $\tilde{B}_k$.
In 1.5(e), $\hat{\mathcal{W}}$ has type $\tilde{C}_k$.
In 1.5(f), $\hat{\mathcal{W}}$ has type $\tilde{C}_k$.
In 1.5(g), $\hat{\mathcal{W}}$ has type $\tilde{B}_k$.
In 1.5(h), $\hat{\mathcal{W}}$ has type $\tilde{C}_k$.
In 1.5(i), $\hat{\mathcal{W}}$ has type $\tilde{C}_k$.
In 1.5(j), $\hat{\mathcal{W}}$ has type $\tilde{G}_2$.
In 1.5(k), $\hat{\mathcal{W}}$ has type $\tilde{F}_4$.
In 1.5(l),(m),(n), $\hat{\mathcal{W}}$ has type $\tilde{A}_0$.

In [L2, 2.6] it is shown that the Weyl group $\mathcal{W}$ can be identified with the Weyl group of $Z_G^0(x)/U_{Z_G^0(x)}$ where $x\in\mathcal{O}$. The results above show that $\hat{\mathcal{W}}$ can be identified with the affine Weyl group associated with $Z_G^0(x)/U_{Z_G^0(x)}$.

2.2. For any $s\in\hat{S}'_0$ let $\hat{P}_s$ be a parahoric subgroup of type $I\cup\{s\}$ containing $\hat{P}$ and let $U_{\hat{P}_s}$ the prounipotent radical of $\hat{P}_s$. Then $(L,\mathcal{O},\mathcal{E})$ can be viewed as an induction datum for the connected reductive group $\hat{P}_s/U_{\hat{P}_s}$. Let $\mathcal{L}_0(s)$ be the dimension of the generalized flag manifold associated to the induction datum $(L,\mathcal{O},\mathcal{E})$ of $\hat{P}_s/U_{\hat{P}_s}$. (When $s\in S_0-I$ this agrees with the definition of $\mathcal{L}_0(s)$ in 1.4.) One can verify that

(a) $\sigma_s\mapsto\mathcal{L}_0(s)$ is the restriction to $\{\sigma_s; s\in\hat{S}'_0\}$ of a weight function $\tilde{\mathcal{L}}$ on the Coxeter group $\hat{\mathcal{W}}$.

2.3. Let $x\in\mathcal{O}\subset L\subset\hat{G}$. We say that $\hat{\mathfrak{P}}_x=\{h\hat{P}\in\hat{G}/\hat{P}; h^{-1}xh\in\mathcal{O}U_{\hat{P}}\}$ is a generalized affine flag manifold. Let $\hat{\Omega}$ be the set of $\hat{P}$-orbits on $\hat{G}/\hat{P}$ (under the action by left translation). For $\omega\in\hat{\Omega}$ we set $\hat{\mathfrak{P}}_x^\omega=\hat{\mathfrak{P}}_x\cap\omega$ so that we have a partition $\hat{\mathfrak{P}}_x=\sqcup_\omega\hat{\mathfrak{P}}_x^\omega$ where each $\hat{\mathfrak{P}}_x^\omega$ is an algebraic variety. In analogy with 1.6, we can identify $\hat{\mathcal{W}}$ with a subset of $\hat{\Omega}$. It is likely that the following affine analogue of 1.6(a) holds.

(a) *If $w\in\hat{\mathcal{W}}$ then $\hat{\mathfrak{P}}_x^w$ is an affine space of dimension $\tilde{\mathcal{L}}_0(w)$.*

3. Another weighted affine Weyl group

3.1. We again assume that G is almost simple, simply connected. We denote by G^* a simple adjoint group over $\mathbf{C}$ of type dual to that of G. Let $(L,\mathcal{O},\mathcal{E})$ be an induction datum for G. Let G^* (resp. L^*) be a connected reductive group over $\mathbf{C}$ of type dual to that of G (resp. L); we can regard L^* as the Levi subgroup of a parabolic subgroup of G^*. Let $\underline{\mathcal{E}}=j_!(\bar{\mathbf{Q}}_l\boxtimes\mathcal{E})$ where $j:\mathcal{Z}_L^0\times\mathcal{O}=\mathcal{Z}_L^0\mathcal{O}\to L$ is the obvious imbedding. Then $\underline{\mathcal{E}}[d]$ (where $d=\dim(\mathcal{Z}_L^0\mathcal{O})$) is a character sheaf on L. The classification of character sheaves of L associates to $\underline{\mathcal{E}}[d]$ a triple $(s,C,\mathbf{c})$ where s is a semisimple element of finite order of L^*, C is a connected component of $H=Z_{L^*}(s)$ and $\mathbf{c}$ is a two-sided cell of the Weyl group W' of H^0 which is stable under the conjugation by any element of C. (The triple $(s,C,\mathbf{c})$ is defined up to L^*-conjugacy.) Let W^a be the affine Weyl group associated to $(Z^0_{G^*}(s)/\text{centre})(\mathbf{C}((\epsilon)))$. Then W' can be viewed as a finite (standard) parabolic subgroup of W^a. Note that conjugation by an element of C induces a Coxeter group automorphism $\gamma:W^a\to W^a$ which leaves W' stable.

We describe in each case where L is not a maximal torus, th assignment $(G,L,\mathcal{O},\mathcal{E})\mapsto(W^a,W')$; we will write (G,L) instead of $(G,L,\mathcal{O},\mathcal{E})$ and will specify G,L by the type of $G,L/\mathcal{Z}_L$. The notation for Weyl groups and affine Weyl groups is the usual one, with the convention that a Weyl group or affine Weyl group of type A_0,B_0,C_0,D_0,D_1 is $\{1\}$. The cases (a)-(n) below correspond to the cases (a)-(n) in 1.5.

(a) $$(A_{kn-1},A^k_{n-1})\mapsto(\tilde{A}^n_{k-1},A_0),\quad n\ge 2,k\ge 1;$$

(b) $$(C_{2t^2+t+k},C_{2t^2+t})\mapsto(\tilde{B}_{t^2+t+k}\times\tilde{D}_{t^2},B_{t^2+t}\times D_{t^2}),\quad t\ge 1,k\ge 0;$$

(c) $$(C_{2t^2+3t+k+1},C_{2t^2+3t+1})\mapsto(\tilde{D}_{t^2+2t+k+1}\times\tilde{B}_{t^2+t},D_{t^2+2t+1}\times B_{t^2+t}),\quad t\ge 0,k\ge 0;$$

(d) $$(B_{2t^2+2t+k}, B_{2t^2+2t}) \mapsto (\tilde{C}_{t^2+t} \times \tilde{C}_{t^2+t+k}, C_{t^2+t} \times C_{t^2+t}), \quad t \geq 1, k \geq 0;$$

(e) $$\begin{aligned}&(B_{4t^2+3t+2k}, B_{4t^2+3t} \times A_1^k)\\ &\mapsto (\tilde{C}_{t^2+t+k} \times \tilde{A}_{2t^2+t-1} \times \tilde{C}_{t^2+t+k}, C_{t^2+t} \times A_{2t^2+t-1} \times C_{t^2+t}), \quad t \geq 1, k \geq 0;\end{aligned}$$

(f) $$\begin{aligned}&(B_{4t^2+5t+2k+1}, B_{4t^2+5t+1} \times A_1^k)\\ &\mapsto (\tilde{C}_{t^2+t} \times \tilde{A}_{2t^2+3t+2k} \times \tilde{C}_{t^2+t}, C_{t^2+t} \times A_{2t^2+3t} \times C_{t^2+t}), \quad t \geq 0, k \geq 0;\end{aligned}$$

(g) $$(D_{2t^2+k}, D_{2t^2}) \mapsto (\tilde{D}_{t^2} \times \tilde{D}_{t^2+k}, D_{t^2} \times D_{t^2}), \quad t \geq 1, k \geq 0;$$

(h) $$\begin{aligned}&(D_{4t^2+t+2k}, D_{4t^2+t} \times A_1^k)\\ &\mapsto (\tilde{D}_{t^2} \times \tilde{A}_{2t^2+t+2k-1} \times \tilde{D}_{t^2}, D_{t^2} \times A_{2t^2+t-1} \times D_{t^2}), \quad t \geq 1, k \geq 0;\end{aligned}$$

(i) $$\begin{aligned}&(D_{4t^2-t+2k}, D_{4t^2-t} \times A_1^k)\\ &\mapsto (\tilde{D}_{t^2+k} \times \tilde{A}_{2t^2-t-1} \times \tilde{D}_{t^2+k}, D_{t^2} \times A_{2t^2-t-1} \times D_{t^2}), \quad t \geq 1, k \geq 0;\end{aligned}$$

(j) $$(E_6, A_2^2) \mapsto (\tilde{D}_4, A_0);$$

(k) $$(E_7, A_1^3) \mapsto (\tilde{E}_6, A_0);$$

(l) $$(E_8, E_8) \mapsto (\tilde{E}_8, E_8);$$

(m) $$(F_4, F_4) \mapsto (\tilde{F}_4, F_4);$$

(n) $$(G_2, G_2) \mapsto (\tilde{G}_2, G_2).$$

We set $n_t = 1$ if t is even, $n_t = 2$ if t is odd. In (a) with $k \geq 1$, γ has order n; it permutes cyclically the n copies of A_{k-1}; in (a) with $k = 1$, we have $\gamma = 1$. In (b) with $t \geq 2$, γ has order n_t; it acts only on the $\tilde{D}$-factor; in (b) with $t = 1$, we have $\gamma = 1$. In (c) with $(t, k) \neq (0, 0)$, γ has order n_{t+1}; it acts only on the $\tilde{D}$-factor; in (c) with $(t, k) = (0, 0)$, we have $\gamma = 1$. In (d) we have $\gamma = 1$. In (e), γ has order 2; it interchanges the two $\tilde{C}$-factors and acts nontrivially on the $\tilde{A}$-factor. In (f) with $(t,k) \neq (0,0)$, γ has order 2; it interchanges the two $\tilde{C}$-factors and acts nontrivially on the $\tilde{A}$-factor; in (f) with $(t,k) = (0,0)$, we have $\gamma = 1$. In (g) with $(t,k) \neq (1,0)$, γ has order n_t; it acts on the $\tilde{D}_{t^2+k}$-factor. In (g) with $(t,k) = (1,0)$ we have $\gamma = 1$. In (h) with $(t,k) \neq (1,0)$, γ has order $2n_t$; it interchanges the two $\tilde{D}$ factors and acts nontrivially on the $\tilde{A}$-factor. In (h) with $(t,k) = (1,0)$, γ has order 2. In (i) with $(t,k) \neq (1,0)$, γ has order $2n_t$; it interchanges the two $\tilde{D}$ factors. In (i) with $(t,k) = (1,0)$, we have $\gamma = 1$. In (j), γ has order 3; in (k), γ has order 2. In (l),(m),(n), we have $\gamma = 1$.

We now describe in each case the two-sided cell $\mathbf{c}$ of W'. If $W' = \{1\}$ then $\mathbf{c} = \{1\}$. If $W' \neq \{1\}$, we write $W' = W'_1 \times \ldots \times W'_m$ where W'_i are irreducible Weyl groups and $\mathbf{c} = \mathbf{c}_1 \times \ldots \times \mathbf{c}_m$ where $\mathbf{c}_i$ is a two-sided cell in W'_i. For any i such that W'_i is of type $A_r, r \geq 1$, we have $r+1 = (h^2+h)/2$ for some h and $\mathbf{c}_i$ is the two-sided cell associated to a unipotent cuspidal representation of a nonsplit group of type A_r over $\mathbf{F}_q$. For any i such that W'_i is of type B_r or C_r with $r \geq 2$, we have $r = h^2+h$ for some h and $\mathbf{c}_i$ is the two-sided cell associated to a unipotent cuspidal representation of a group of type B_r or C_r over $\mathbf{F}_q$. For any i such that W'_i is of type D_r with $r \geq 4$, we have $r = h^2$ for some h and $\mathbf{c}_i$ is the two-sided cell associated to a unipotent cuspidal representation of a group of type D_r over $\mathbf{F}_q$ (which is split if h is even, nonsplit if h is odd). If W' is of type E_8, F_4 or G_2, $\mathbf{c}$ is the two-sided cell associated to a unipotent cuspidal representation of a group of type E_8, F_4 or G_2 over $\mathbf{F}_q$.

3.2. We associate to an induction datum $(L, \mathcal{O}, \mathcal{E})$ of G an affine Weyl group $\mathcal{W}^a$. We define $\mathcal{W}^a$ in terms of (W^a, W', γ) as in [L4, 25.1]. In more detail, let S be the set of simple relections of W^a. For any subset J of S let W^a_J be the subgroup of W^a generated by J; when W^a_J is finite let w_0^J be the longest element of W^a_J. Let J' be the set of simple reflections of W'. Let $\tilde{\mathcal{W}}^a$ be the set of all $w \in W^a$ such that $wW^a_{J'} = W^a_{J'}w$ and w has minimal length in $wW^a_{J'} = W^a_{J'}w$ and let $\mathcal{W}^a$ be the fixed point set of $\gamma : \tilde{\mathcal{W}}^a \to \tilde{\mathcal{W}}^a$. Note that $\tilde{\mathcal{W}}^a, \mathcal{W}^a$ are subgroup of W^a.

Let K be the set of all γ-orbits k on $S - J'$ such that $W^a_{J' \cup k}$ is finite. In each case (a)-(n), for any $k \in K$ we have $w_0^{J' \cup k} w_0^{J'} = w_0^{J'} w_0^{J' \cup k}$ hence $\tau_k = w_0^{J' \cup k} w_0^{J'} = w_0^{J'} w_0^{J' \cup k}$ satisfies $\tau_k^2 = 1$. Moreover we have $\tau_k \in \mathcal{W}_a$. Let $\mathbf{a} : W^a \to \mathbf{N}$ be the $\mathbf{a}$-function of the Coxeter group W^a (with standard length function), see [L4, §13]. We define $\mathcal{L} : K \to \mathbf{N}$ by $\mathcal{L}(k) = \mathbf{a}(\mathbf{c}\tau_k) - \mathbf{a}(\mathbf{c})$ where $\mathbf{a}(\mathbf{c}\tau_k)$, $\mathbf{a}(\mathbf{c})$ denotes the (constant) value of the $\mathbf{a}$-function on $\mathbf{c}\tau_k, \mathbf{c}$ (see [L4, 9.13]). One can verify that $\mathcal{W}^a$ is an affine Weyl group with Coxeter generators $\{\tau_k; k \in K\}$ and that $\tau_k \mapsto \mathcal{L}(k)$ is the restriction to $\{\tau_k; k \in K\}$ of a weight function on $\mathcal{W}^a$.

We describe below the type of the affine Weyl group $\mathcal{W}^a$ and the values of the weight function $\mathcal{L}$ on K.

In 3.1(a), $\mathcal{W}^a$ has type $\tilde{A}_{k-1}$, $\mathcal{L} = n, n, \ldots, n$.

In 3.1(b), $\mathcal{W}^a$ has type $\tilde{B}_k$, $\mathcal{L} = 1, 1, \ldots, 1, 2t+1$.

In 3.1(c), $\mathcal{W}^a$ has type $\tilde{B}_k$, $\mathcal{L} = 1, 1, \ldots, 1, 2t+2$.

In 3.1(d), $\mathcal{W}^a$ has type $\tilde{C}_k$, $\mathcal{L} = 1, 1, \dots, 1, 2t + 1$.
In 3.1(e), $\mathcal{W}^a$ has type $\tilde{C}_k$, $\mathcal{L} = 2, 2, \dots, 2, 4t + 2$.
In 3.1(f), $\mathcal{W}^a$ has type $\tilde{C}_k$, $\mathcal{L} = 1, 2, 2, \dots, 2, 4t + 1$.
In 3.1(g), $\mathcal{W}^a$ has type $\tilde{B}_k$, $\mathcal{L} = 1, 1, \dots, 1, 2t$.
In 3.1(h), $\mathcal{W}^a$ has type $\tilde{C}_k$, $\mathcal{L} = 1, 2, 2, \dots, 2, 4t - 1$.
In 3.1(i), $\mathcal{W}^a$ has type $\tilde{B}_k$, $\mathcal{L} = 2, 2, \dots, 2, 4t$.
In 3.1(j), $\mathcal{W}^a$ has type $\tilde{G}_2$, $\mathcal{L} = 1, 1, 3$.
In 3.1(k), $\mathcal{W}^a$ has type $\tilde{F}_4$, $\mathcal{L} = 1, 1, 1, 2, 2$.
In 3.1(l),(m),(n), $\mathcal{W}^a$ has type $\tilde{A}_0$.

In the case where L is a maximal torus that is, $(L, \mathcal{O}, \mathcal{E})$ is as in 1.2, we have $s = 1$, W^a is an affine Weyl group of type dual to that of G, $W' = \{1\}$, $\mathbf{c} = 1$, and $\gamma = 1$; $\mathcal{W}^a = W^a$; the function $\mathcal{L}$ is constant equal to 1.

We see that $\mathcal{W}$ in 1.4 is naturally imbedded (as a Coxeter group) in $\mathcal{W}^a$ so that $\mathcal{W}^a$ is an affine Weyl group associated to $\mathcal{W}$ and that $\mathcal{L}_0$ in 1.4 is the restriction of $\mathcal{L}$.

3.3. Let $\bar{\mathbf{F}}_q$ be an algebraic closure of the finite field $\mathbf{F}_q$. The pair $Z^0_{G^*}(s) \supset Z^0_{L^*}(s)$ has a version $\mathcal{G}' \supset \mathcal{G}'_0$ with $\mathcal{G}', \mathcal{G}'_0$ being connected reductive groups over $\bar{\mathbf{F}}_q$ of the same type as $(Z^0_{G^*}(s), Z^0_{L^*}(s))$. Let $\mathcal{G} \supset \mathcal{G}_0$ be obtained from $\mathcal{G}' \supset \mathcal{G}'_0$ by dividing by the centre of $\mathcal{G}'$. Let $F : \mathcal{G} \to \mathcal{G}$ be the Frobenius map for an $\mathbf{F}_q$-rational structure on $\mathcal{G}$ which induces on the Weyl group of $\mathcal{G}$ the same automorphism as γ in 3.1. We can then form the corresponding group $\mathcal{G}(\mathbf{F}_q((\epsilon)))$ where ϵ is an indeterminate and its subgroup $\mathcal{G}_0(\mathbf{F}_q)$. This subgroup can be regarded as the reductive quotient of a parahoric subgroup $\mathcal{P}$ of $\mathcal{G}(\mathbf{F}_q((\epsilon)))$; moreover this subgroup carries a unipotent cuspidal representation as in the last paragraph of 3.1. We can induce this representation from $\mathcal{P}$ to $\mathcal{G}(\mathbf{F}_q((\epsilon)))$. The endomorphism algebra of this induced representation is known to be an extended affine Hecke algebra with explicitly known (possibly unequal) parameters. An examination of the cases (a)-(n) in 3.2 shows that these parameters are exactly those described by the function $\mathcal{L}$ in 3.2.

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Department of Mathematics, M.I.T., Cambridge, MA 02139