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ON COMPLEXITY OF REPRESENTATIONS OF QUIVERS

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ABSTRACT

It is shown that, given a representation of a quiver over a finite field, one can check in polynomial time whether it is absolutely indecomposable.

1. SOME RESULTS ON ABSOLUTELY INDECOMPOSABLE REPRESENTATIONS OF QUIVERS

Let Γ be a finite graph without self-loops (but several edges connecting two vertices are allowed), and let \mathcal{V} denote the set of its vertices. The graph Γ with an orientation Ω of its edges is called a *quiver*. A *representation* of the quiver (Γ, Ω) over a field \mathbb{F} is a collection of finite-dimensional vector spaces $\{U_v\}_{v \in \mathcal{V}}$ over \mathbb{F} and linear maps $\{U_v \rightarrow U_w\}$ for each oriented edge $v \rightarrow w$. Homomorphisms and isomorphisms of two representations are defined in the obvious way. The *direct sum* of two representations $(\{U_v\}, \{U_v \rightarrow U_w\})$ and $(\{U'_v\}, \{U'_v \rightarrow U'_w\})$ is the representation

$$(\{U_v \oplus U'_v\}, \{U_v \oplus U'_v \rightarrow U_w \oplus U'_w\}),$$

where maps are the direct sums of maps. A representation π is called *indecomposable* if it is not isomorphic to a direct sum of two non-zero representations; π is called *absolutely indecomposable* if it is indecomposable over the algebraic closure $\bar{\mathbb{F}}$ of the field \mathbb{F} .

Let $r = \#\mathcal{V}$ and let $Q = \bigoplus_{v \in \mathcal{V}} \mathbb{Z}\alpha_v$ be a free abelian group of rank r with a fixed basis $\{\alpha_v\}_{v \in \mathcal{V}}$. Let $Q_+ = \bigoplus_v \mathbb{Z}_{\geq 0} \alpha_v \subset Q$. The *dimension* of a representation $\pi = \{U_v\}_{v \in \mathcal{V}}$ is the element

$$\dim \pi = \sum_{v \in \mathcal{V}} (\dim U_v) \alpha_v \in Q_+.$$

The *Cartan matrix* of the graph Γ is the symmetric matrix $A = (a_{uv})_{u,v \in \mathcal{V}}$, where $a_{vv} = 2$ and $-a_{uv}$ is the number of edges, connecting u and v if $u \neq v$. Define a $\frac{1}{2}\mathbb{Z}$ -valued symmetric bilinear form on Q , such that $(\alpha|\alpha) \in \mathbb{Z}$, by

$$(\alpha_u|\alpha_v) = \frac{1}{2}a_{uv}, \quad u, v \in \mathcal{V},$$

and the following (involutive) automorphisms r_v , $v \in \mathcal{V}$, of the free abelian group Q

$$r_v(\alpha_u) = \alpha_u - a_{uv}\alpha_v, \quad u \in \mathcal{V}.$$

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The group $W \subset \text{Aut } Q$, generated by all $r_v, v \in \mathcal{V}$, is called the *Weyl group* of the graph Γ . It is immediate to see that the bilinear form $(\cdot | \cdot)$ is invariant with respect to all $r_v, v \in \mathcal{V}$, hence with respect to the Weyl group W .

It is well known that the group W is finite if and only if the Cartan matrix A is positive definite, which happens if and only if all connected components of Γ are Dynkin diagrams of simple finite-dimensional Lie algebra of type A_r, D_r, E_6, E_7, E_8 (see e.g. [K90]). Gabriel's theorem [G72] states that for a quiver (Γ, Ω) the number of indecomposable representations, up to isomorphism, is finite if and only if the group W is finite. Moreover, in this case the map $\pi \mapsto \dim \pi$ establishes a bijective correspondence between isomorphism classes of indecomposable representations of (Γ, Ω) and the set of positive roots $\Delta_+ \subset Q_+$ of the semisimple Lie algebra with Dynkin diagram Γ , where

$$(1) \quad \Delta_+ = \bigcup_{v \in \mathcal{V}} ((W \cdot \alpha_v) \cap Q_+).$$

For an arbitrary graph Γ denote by Δ_+^{re} the RHS of (1); note that $(\alpha | \alpha) = 1$ for all $\alpha \in \Delta_+^{\text{re}}$. Furthermore, let

$$(2) \quad \mathcal{C} = \{\alpha \in Q_+ \setminus \{0\} \mid (\alpha | \alpha_v) \leq 0, v \in \mathcal{V}, \text{ and } \text{supp } \alpha \text{ is connected}\},$$

where for $\alpha = \sum_{v \in \mathcal{V}} n_v \alpha_v$, we let $\text{supp } \alpha = \{v \mid n_v \neq 0\}$. We let

$$\Delta_+^{\text{im}} = W \cdot \mathcal{C}, \quad \Delta_+ = \Delta_+^{\text{re}} \cup \Delta_+^{\text{im}}.$$

It is easy to see that $\Delta_+^{\text{im}} \subset Q_+$ and that $(\alpha | \alpha) \in \mathbb{Z}_{\leq 0}$ for $\alpha \in \Delta_+^{\text{im}}$. The set $\Delta_+ \subset Q_+$ is the set of *positive roots* of the Kac-Moody algebra $\mathfrak{g}(A)$, associated to the Cartan matrix A , and Δ_+^{im} is empty if and only if the matrix A is positive definite [K80], [K90].

Theorem 1. *Let $\mathbb{F} = \mathbb{F}_q$ be a field of q elements.*

- (a) *The number of absolutely indecomposable representations over \mathbb{F}_q of dimension $\alpha \in Q_+$ of a quiver (Γ, Ω) is independent of the orientation Ω . It is zero if $\alpha \notin \Delta_+$, and it is given by a monic polynomial $P_{\Gamma, \alpha}(q)$ of degree $1 - (\alpha | \alpha)$ with integer coefficients. In particular, $P_{\Gamma, \alpha}(q) = 1$ if $\alpha \in \Delta_+^{\text{re}}$.*
- (b) *The constant term $P_{\Gamma, \alpha}(0)$ equals to the multiplicity of the root α in $\mathfrak{g}(A)$.*
- (c) *All coefficients of $P_{\Gamma, \alpha}(q)$ are non-negative.*
- (d) *Consequently, for any quiver (Γ, Ω) and any $\alpha \in \Delta_+$ there exists an absolutely indecomposable representation over \mathbb{F}_q of dimension α .*

Claim (a) was proved in [K80] and [K83]; claims (b) and (c) were conjectured in [K80], [K83], and proved in [H10] and [HLRV13] respectively. For indivisible $\alpha \in \Delta_+$ both claims (b) and (c) were proved earlier in [CBVB04].

2. QUASI-NILPOTENT SUBALGEBRAS OF $\text{End}_{\mathbb{F}} U$.

Consider a finite-dimensional vector space U over a field \mathbb{F} . An endomorphism a of U is called *quasi-nilpotent* if all its eigenvalues are equal; denote these eigenvalues by $\text{eig}(a)$. They are elements of the algebraic closure $\bar{\mathbb{F}}$ of the field \mathbb{F} . An associative subalgebra A of $\text{End}_{\mathbb{F}} U$ is called *quasi-nilpotent* if it consists of quasi-nilpotent elements. For an

associative algebra A we denote by A_- the Lie algebra obtained from A by taking the bracket $[a, b] = ab - ba$. We also let $\overline{A} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} A$, $\overline{U} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} U$.

Lemma 1. *Let A be a subalgebra of the associative algebra $\text{End}_{\mathbb{F}} U$.*

- (a) *If A is a quasi-nilpotent subalgebra, then in some basis of \overline{U} , all endomorphisms $a \in A$ have upper triangular matrices with $\text{eig}(a)$ on the diagonal. In particular, $\text{eig}(a + b) = \text{eig}(a) + \text{eig}(b)$ for $a, b \in A$, and A_- is a nilpotent Lie algebra.*
- (b) *If A_- is a nilpotent Lie algebra and A has a basis, consisting of quasi-nilpotent endomorphisms, then A is a quasi-nilpotent subalgebra.*

Proof. Burnside's theorem says that any subalgebra of the $\overline{\mathbb{F}}$ -algebra $\text{End}_{\overline{\mathbb{F}}} \overline{U}$, where \overline{U} is a finite-dimensional vector space over $\overline{\mathbb{F}}$, which acts irreducibly on \overline{U} , coincides with $\text{End}_{\overline{\mathbb{F}}} \overline{U}$. Hence, in some basis of \overline{U} the algebra \overline{A} consists of upper triangular block matrices with blocks $\text{End}_{\overline{\mathbb{F}}} \overline{\mathbb{F}}^{m_i}$ on the diagonal, where $m_i \geq 1$, $\sum_i m_i = \dim \overline{U}$.

If A is a quasi-nilpotent subalgebra, then so is \overline{A} , and, in particular, $\text{End}_{\overline{\mathbb{F}}} \overline{\mathbb{F}}^{m_i}$ for all i . This implies that all $m_i = 1$. Hence \overline{A} consists of upper triangular quasi-nilpotent matrices. This proves (a).

In order to prove (b), note that if A_- is a nilpotent Lie algebra, then so is \overline{A}_- , and, in particular so are all $(\text{End}_{\overline{\mathbb{F}}} \overline{\mathbb{F}}^{m_i})_-$. It follows that all $m_i = 1$, so that \overline{A}_- consists of upper triangular matrices in some basis of \overline{U} . Since A has a basis, consisting of quasi-nilpotent elements, the subalgebra A is quasi-nilpotent. This proves (b). \square

Corollary 1. *A subalgebra A of the associative algebra $\text{End}_{\mathbb{F}} U$ is quasi-nilpotent if and only if the Lie algebra A_- is nilpotent and A has a basis, consisting of quasi-nilpotent endomorphisms.* \square

3. CRITERION OF ABSOLUTE INDECOMPOSABILITY.

Let $\pi = (\{U_v\}, \{U_v \rightarrow U_w\})$ be a representation of a quiver (Γ, Ω) over a field \mathbb{F} , of dimension $\alpha = \sum_{v \in \mathcal{V}} n_v \alpha_v$. Let $U = \bigoplus_{v \in \mathcal{V}} U_v$. Then the space $\text{Hom}_{\mathbb{F}}(U_v, U_w)$ is naturally identified with a subspace of $\text{End}_{\mathbb{F}} U$, so that the representation π is identified with a collection of endomorphisms for each oriented edge $v \rightarrow w$ of the quiver (Γ, Ω) : $\{\pi_{v,w} : U_v \rightarrow U_w\} \subset \text{End}_{\mathbb{F}} U$. An endomorphism a of π decomposes as $a = \sum_{v \in \mathcal{V}} a_v$, where $a_v \in \text{End}_{\mathbb{F}} U_v \subset \text{End}_{\mathbb{F}} U$, and the condition that $a \in \text{End } \pi$, the algebra of endomorphisms of π , means that

$$(3) \quad a_w \pi_{v,w} = \pi_{v,w} a_v \text{ for all oriented edges } v \rightarrow w.$$

This simply means that the block diagonal endomorphism a commutes with all endomorphisms $\pi_{v,w}$ in the algebra $\text{End}_{\mathbb{F}} U$. Note that (3) has an obvious solution $a_v = c I_{U_v}$, $v \in \mathcal{V}$, where $c \in \mathbb{F}$, hence $\dim \text{End } \pi \geq 1$. In the case of equality, α lies in Δ_+ , and it is called a *Schur vector*; in this and only in this case a generic representation of dimension α is absolutely indecomposable [K82].

Lemma 2. *The representation π is absolutely indecomposable if and only if the algebra of its endomorphisms $\text{End } \pi$ is quasi-nilpotent in $\text{End}_{\mathbb{F}} U$.*

Proof. An endomorphism $a \in \text{End } \pi \subset \text{End}_{\mathbb{F}} U \subset \text{End}_{\overline{\mathbb{F}}} \overline{U}$ decomposes in a sum of commuting endomorphisms $a = a_{(s)} + a_{(n)}$, where the endomorphism $a_{(s)}$ is diagonalizable and the endomorphone $a_{(n)}$ is nilpotent (Jordan decomposition). Condition (3) means that a commutes with $\pi_{v,w}$ for all oriented edges $v \rightarrow w$. By a well-known fact of linear algebra, it follows that the $\pi_{v,w}$ commute with $a_{(s)}$. But then the decomposition of \overline{U} in a direct sum of eigenspaces of $a_{(s)}$ is a decomposition of the representation π in a direct sum of representation of the quiver (Γ, Ω) . Thus, π is absolutely indecomposable if and only if $a_{(s)}$ is a scalar endomorphism of \overline{U} , which is equivalent to say that a is a quasi-nilpotent endomorphism of U . \square

4. MAIN THEOREM

The following is the main result of the paper.

Theorem 2. *Let \mathbb{F}_q be a fixed finite field. Then there exists an algorithm which, given as input a quiver (Γ, Ω) and its representation $\pi = (\{U_v\}, \{U_v \rightarrow U_w\})$ over \mathbb{F}_q of dimension $\sum_{v \in \mathcal{V}} n_v \alpha_v$, can decide in polynomial in $N := \sum_v n_v$ time whether π is absolutely indecomposable or not.*

Proof. By Lemma 2 one has to check whether $\text{End } \pi \subset \text{End}_{\mathbb{F}_q} U$, where $U = \bigoplus_{v \in \mathcal{V}} U_v$, consists of quasi-nilpotent elements. By Corollary 1 one has to check two things :

- (i) $\text{End } \pi$ has a basis, consisting of quasi-nilpotent elements;
- (ii) the Lie algebra $(\text{End } \pi)_-$ is nilpotent.

For this we identify U_v with the vector space $\mathbb{F}_q^{n_v}$, so that U is identified with \mathbb{F}_q^N and $\text{End}_{\mathbb{F}_q} U$ with the algebra of $N \times N$ -matrices over \mathbb{F}_q . $\text{End } \pi$ is a subspace of $\text{End}_{\mathbb{F}_q} U$, given by linear homogeneous equation (3), hence, using Gauss elimination, we can construct in polynomial in N time a basis a_1, \dots, a_m of $\text{End } \pi$, where $m \leq N$.

First, we check that all the a_i are quasi-nilpotent. This simply means that

$$(4) \quad \det_U(\lambda I_N + a_i) = (\lambda + \gamma_i)^N, \text{ where } \gamma_i \in \overline{\mathbb{F}_q}.$$

The left-hand side of (4) can be computed in polynomial in N time by Gauss elimination. By the separability of $\overline{\mathbb{F}_q}$ over \mathbb{F}_q , (4) implies that all γ_i lie in \mathbb{F}_q . Hence we have to check that (4) holds for each i and some element $\gamma_i \in \mathbb{F}_q$, which can be done in polynomial in N time.

Second, we check that $(\text{End } \pi)_-$ is a nilpotent Lie algebra. Recall that a Lie algebra \mathfrak{g} of dimension m is nilpotent if and only if the member \mathfrak{g}^m of the sequence of subspaces, defined inductively by

$$\mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^j = [\mathfrak{g}, \mathfrak{g}^{j-1}] \text{ for } j \geq 2,$$

is zero. Given a basis $\{a_i\}$ of \mathfrak{g} (which we already have), the subspace \mathfrak{g}^2 is the span over \mathbb{F}_q of all commutators $[a_i, a_j]$. Using Gauss elimination, construct a basis $\{b_i\}$ of \mathfrak{g}^2 . Next, \mathfrak{g}^3 is the span of commutators $[a_i, b_j]$, and again, using Gauss elimination, choose a basis $\{c_i\}$ of \mathfrak{g}^3 , etc. The Lie algebra \mathfrak{g} is nilpotent if and only if $\mathfrak{g}^m = 0$. \square

5. A BRIEF DISCUSSION ON P VS NP

In terms of matrices over \mathbb{F}_q , a representation π over \mathbb{F}_q of a quiver (Γ, Ω) of dimension $\alpha = \sum_{v \in \mathcal{V}} n_v \alpha_v \in Q_+$ is a collection of $n_w \times n_v$ matrices $\pi_{v,w}$ over \mathbb{F}_q for each oriented edge $v \rightarrow w$. An endomorphism of π is a collection of $n_v \times n_v$ matrices a_v over \mathbb{F}_q for each vertex $v \in \mathcal{V}$, such that the linear homogeneous equations (3) hold. The representation π is absolutely indecomposable if for each endomorphism of π all matrices $a_v, v \in \mathcal{V}$, are quasi-nilpotent (equivalently, by Corollary 1, $\text{End } \pi$ has a basis of elements with this property).

The following discussion was outlined to me by Mike Sipser. Given a representation π over a fixed finite field \mathbb{F}_q of a quiver (Γ, Ω) of dimension $\alpha \in \Delta_+$, which is a collection of $M_\alpha := \sum_{v \rightarrow w} n_v n_w$ numbers from \mathbb{F}_q , the output is YES if π is absolutely indecomposable and NO otherwise. Call this problem INDEC; it is a P problem, according to Theorem 2. Define a generalization of INDEC, where some of the numbers are replaced by variables $x_i, i = 1, \dots, M$, where M is an integer, such that $1 \leq M \leq M_\alpha$, and call this problem $\text{INDEC}[x_1, \dots, x_M]$. Say YES for the latter problem if there exist $\gamma_1, \dots, \gamma_M \in \mathbb{F}_q$ we can substitute for x_1, \dots, x_M , such that the resulting INDEC problem is YES. Obviously INDEC is in P implies that $\text{INDEC}[x_1, \dots, x_M]$ is in NP.

Now assume that $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is actually in P. We give a polynomial in M_α time procedure to output an absolutely indecomposable representation. Test $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$. The answer is YES by Theorem 1(d). Now reduce M_α by 1, by trying all possible numbers from \mathbb{F}_q in place of x_{M_α} and test $\text{INDEC}[x_1, \dots, x_{M_\alpha-1}]$ for each of these numbers. The answer must be YES for at least one of these numbers. Repeat this procedure until we find all M_α numbers. That is our answer.

6. CONJECTURES AND EXAMPLES

Conjecture 1. $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is not in P.

Conjecture 2. $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P for any quiver (Γ, Ω) if $\alpha \in \Delta_+$ is a Schur vector.

Conjecture 3. $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P for any quiver (Γ, Ω) if $\alpha \in \mathcal{C}$ (defined by (2)).

Example 1. Let Γ be a Dynkin diagram of type A_r, D_r, E_6, E_7, E_8 . In this case for any orientation Ω of Γ all indecomposable representations have been constructed explicitly in [G72], which shows that in this case $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P.

Example 2. Let Γ be the extended (connected) Dynkin diagram, so that $\#\mathcal{V} = r + 1$ and $\det A = 0$. These are the only connected graphs, for which the Cartan matrix is positive semidefinite and singular. In this case all absolutely indecomposable representations for any orientation Ω have been constructed in [N73] and in [DF73]. which shows that in this case $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P as well. Note that in this case [K80] $\Delta_+^{im} = \mathbb{Z}_{\geq 1}\delta$, where $A\delta = 0$ and $(\delta|\delta) = 0$, and one can show that $P_{\Gamma, n\delta}(q) = q + r$ for $n \in \mathbb{Z}_{\geq 1}$.

Example 3. Let Γ_m be the quiver with two vertices v_1 and v_2 , and m arrows from v_1 to v_2 . For $m = 1$ and 2 this is a quiver from Examples 1 and 2 respectively. For $m \geq 3$ the explicit expressions for the polynomials $P_{\Gamma_m, \alpha}(q)$ for an arbitrary $\alpha \in \Delta_+^{im}$ are unknown. Note that in this case $\Delta_+^{re} \text{ (resp. } im) = \{\alpha = n_1\alpha_1 + n_2\alpha_2 \mid n_i \in \mathbb{Z}_{\geq 0} \text{ and } n_1^2 + n_2^2 - mn_1n_2 = 1 \text{ (resp. } < 0)\}$.

Now, let (Γ, Ω) be a quiver, and let v be a vertex, which is a source or a sink. In [BGP73] an explicitly computable reflection functor R_v was constructed, which sends a representation π of dimension $\alpha \neq v$ of (Γ, Ω) to a representation $R_v(\pi)$ of the reflected quiver $(\Gamma, R_v(\Omega))$ of dimension $r_v(\alpha)$, preserving indecomposability, see also [K80]. It follows that if the problem $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P for the quiver (Γ, Ω) and dimension $\alpha \neq v$, and v is a source or a sink of (Γ, Ω) , then it is in P for the quiver $(\Gamma, R_v(\Omega))$ and dimension $r_v(\alpha)$.

Remark 1. If v is a source or a sink of the quiver (Γ, Ω) and $\alpha \in \Delta_+ \setminus \{v\}$ is a Schur vector, then $r_v(\alpha)$ is a Schur vector for $(\Gamma, R_v(\Omega))$. Also, if α is such that $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P, then the same holds for $r_v(\alpha)$.

Remark 2. For an arbitrary quiver (Γ, Ω) the set \mathcal{C} consists of Schur vectors, except for the vectors with $(\alpha|\alpha) = 0$ [K80], in which case, $\text{supp } \alpha$ is a graph from Example 2. Hence Conjecture 2 implies Conjecture 3.

Remark 3. Let Γ_m be a quiver from Example 3. Then, using the reflection functors, we see that for all $\alpha \in \Delta_+^{re}$, $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P. Since for this quiver $(\alpha|\alpha) < 0$ for all $\alpha \in \mathcal{C}$, we see that all $\alpha \in \Delta_+^{im}$ are Schur vectors [K80], and it follows from Remark 1 and Conjecture 2 that for all $\alpha \in \Delta_+^{im}$, $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P as well.

However, in general, $\alpha \in \Delta_+$ is not a Schur vector, so that a generic representation of a quiver (Γ, Ω) of dimension $\alpha \in \Delta_+$ is not absolutely indecomposable. In this case $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ becomes a problem of finding a needle in a haystack, which leads me to (naively) believe in Conjecture 1.

In fact, I believe that for any connected quiver, different from those in Examples 1, 2, and 3, there exists $\alpha \in \Delta_+$, for which $\text{INDEC}[x_1, \dots, x_{M_\alpha}]$ is not in P.

Remark 4. As explained in [K83], claim (a) of Theorem 1 extends to the case of Γ with self-loops. Claim (c) is proved in [HLRV13] in this generality. Theorem 2 holds in this generality as well.

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