PROJECTIVE AND INJECTIVE BANACH SPACES

by

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ABSTRACT

The author defines projective and injective Banach spaces and the classes $Pr(\lambda)$ and $In(\lambda)$, $1 \leq \lambda < \infty$, of such spaces respectively. Necessary and sufficient conditions for a space to be projective (injective) are established and it is shown that every projective (injective) space is a member of $Pr(\lambda)$ ($In(\lambda)$) for some λ. It is shown that a necessary condition that a space be projective is that weak and strong convergence of sequences coincide. Various alternative definitions and reductions in the original definitions of projectivity (injectivity) are shown to be equivalent to the original definitions. A necessary geometric condition for a real Banach space to be injective is established and this condition is used to prove that if X is a real Banach space which is a dual space and which is a member of $In(1+\epsilon)$ for every $\epsilon > 0$, then X is a member of $In(1)$. It is shown that the class $Pr(1)$ consists of only the zero space. It is shown that the dual space of a projective space is injective and that the dual space of every injective space is projective if the dual spaces of a certain class of injective spaces are projective. The notion of a $*$-projective Banach space is defined and it is shown that the dual space of a $*$-projective space is injective. Finally the author proves that a non-zero Banach space is separable and projective if and only if it is equivalent to $\ell_1(S)$ for some at most countably infinite set S. Open questions are discussed.

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Notation and Terminology

In this section we define explicitly the various notations and terminologies which we shall be using. Additional notation and terminology will be introduced in particular chapters as needed.

If A and B are sets, $A \subset B$ will mean that A is a subset of B, with the possibility that $A = B$ not excluded. The empty set will be denoted by ϕ. If A and B are non-empty sets, AxB will denote the cartesian product of A and B, i.e. $AxB = \{(a,b) \mid a \in A, b \in B\}$.

If A and B are Banach spaces over the same field of scalars, $A \oplus B$ denotes AxB equipped with the following Banach space structure: Addition of two elements (a_1, b_1) and (a_2, b_2) of AxB is defined by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$, multiplication of $(a, b) \in AxB$ by a scalar α is defined by $\alpha(a, b) = (\alpha a, \alpha b)$, and the norm of (a, b) is defined to be $\|(a, b)\|_{A \oplus B} = \|a\|_A + \|b\|_B$ where $\| \ \|_A$ and $\| \ \|_B$ denote the norms in A and B respectively.

If A and B are non-empty subsets of the same vector space, $A + B$ denotes the set $\{a + b \mid a \in A, b \in B\}$. If B consists of only one vector, say x, we write $A + B = A + x = \{a + x \mid a \in A\}$. If α is a scalar, αA denotes the set $\{\alpha a \mid a \in A\}$.

If X and Y are normed linear spaces, we say that X is equivalent to Y if there exists a one-one bounded linear transformation T from X onto Y with bounded inverse. We say that the normed linear spaces X and Y are congruent if there exists a linear transformation T from X onto Y such that $\|Tx\| = \|x\|$ for all $x \in X$. A linear transformation T from a normed linear space X into a normed linear space Y with the property that $\|Tx\| = \|x\|$ for all $x \in X$ will be called an isometry. We shall also refer to congruent normed linear spaces as being isometric spaces and isometrically isomorphic spaces.

If X is a vector space, a linear transformation T from X into X is called a projection if $T^2 = T$, i.e. $T(T(x)) = T(x)$ for each $x \in X$.

When we say that a subset Y of a normed linear space X is closed, we mean that Y is closed with respect to the topology induced by the metric ρ on $X \times X$ defined by $\rho(x_1, x_2) = \|x_1 - x_2\|$, $x_1, x_2 \in X$. Occasionally we shall refer to such a closed set as being strongly closed.

When we say that a subset Y of a normed linear space is a subspace, we mean that Y is a linear subspace and if we speak of the norm of an element $y \in Y$, we mean unless we specify otherwise the norm of y when considered as an element of X.

If X is a Banach space and Y is a closed subspace of X, we say that Y has a <u>closed complement</u> in X if there exists a closed subspace W of X such that $X = Y + W$ and $Y \cap W = \{0\}$. If Y has a closed complement in X, we say that Y is <u>complemented</u> in X.

If A, B, and C are non-empty sets and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions, we shall denote the composition mapping $h : A \longrightarrow C$ defined by $h(a) = g(f(a))$, $a \in A$, by gf. If $f : A \longrightarrow B$ is a function and D is a non-empty subset of A, we denote the restriction of f to D by $f|D$.

If X is a normed linear space, $x_o \in X$, and r is a non-negative real number, the <u>closed sphere of radius r with center x_o</u> is the set $\{x \in X \mid \|x - x_o\| \leq r\}$. The <u>open sphere of radius r with center x_o</u> is the set $\{x \in X \mid \|x - x_o\| < r\}$.

If X is a normed linear space, the dual space of X, i.e. the space of continuous linear functionals on X, is denoted by X^*.

We say that a family of vectors $\{x_s\}_{s \in S}$, indexed by a non-empty set S, in a normed linear space X is <u>summable</u> and use the notation $\sum_{s \in S} x_s < \infty$ if there exists a vector $x \in X$ such that if $\epsilon > 0$, there exists a finite subset S_ϵ of S such that if S_F is any finite non-empty subset of S containing S_ϵ, we have $\|x - \sum_{s \in S_F} x_s\| < \epsilon$. If such a vector x exists, it is unique and we may write $\sum_{s \in S} x_s = x$ and call x the sum of the family $\{x_s\}_{s \in S}$. For the basic properties of this type of summability, the reader is referred to Kelley [20, pages 77-79, exercise G and page 214, exercise S][1], Halmos [16, pages 17-19], or Day [7, Chapter IV]. We assume the reader is familiar with these properties. For example, if X is complete, then $\sum_{s \in S} x_s < \infty$ if and only if for each $\epsilon > 0$, there

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[1] The numbers in brackets refer to the Bibliography at the end.

exists a finite subset S_ϵ of S such that for every non-empty finite subset S_F of S such that $S_\epsilon \cap S_F = \phi$, we have $\left\|\sum_{s \in S_F} x_s\right\| < \epsilon$.

Let $\{x_n\}_{n=1,2,\ldots}$ be a sequence of vectors in a normed linear space X and let $y_n = \sum_{i=1}^{n} x_i$, $n = 1,2,\ldots$ If the sequence $\{y_n\}_{n=1,2,\ldots}$ converges to $x \in X$, we write $\sum_{n=1}^{\infty} x_n = x$ and we say that <u>the infinite series $\sum_{n=1}^{\infty} x_n$ converges to x</u>. Clearly if $\{x_n\}_{n=1,2,\ldots}$ is summable with sum x, the series $\sum_{n=1}^{\infty} x_n$ converges to x (but not conversely). The notation $\sum_{n=1}^{\infty} x_n < \infty$ will mean that $\{x_n\}_{n=1,2,\ldots}$ is summable. The notation $\sum_{n=1}^{\infty} x_n = x$ without any comment should be understood by the reader to mean that the series $\sum_{n=1}^{\infty} x_n$ converges to x, without any implication of the summability of $\{x_n\}_{n=1,2,...}$ (although it will often by the case that $\{x_n\}_{n=1,2,...}$ is indeed **summable**).

If S is a non-empty set and $1 \leq p < \infty$, we denote by $\ell_p(S)$ the Banach space of all scalar valued functions α defined on S such that $\sum_{s \in S} |\alpha(s)|^p < \infty$. $\|\alpha\|_{\ell_p(S)}$ is defined to be $(\sum_{s \in S} |\alpha(s)|^p)^{\frac{1}{p}}$. We denote by $\ell_\infty(S)$ the Banach space of all scalar valued functions α defined on S such that $\sup_{s \in S} \{|\alpha(s)|\} < \infty$. $\|\alpha\|_{\ell_\infty(S)}$ is defined to be $\sup_{s \in S} \{|\alpha(s)|\}$. Addition and multiplication by scalars of elements of $\ell_p(S)$ $(\ell_\infty(S))$ are defined pointwise. If $S = \{s_1, s_2, ...\}$ is a countably infinite set, we shall often for conveninece, denote the elements α of $\ell_p(S)$ $(\ell_\infty(S))$ by $\alpha = (\alpha_1, \alpha_2, ...)$ where $\alpha_i = \alpha(s_i)$, $i = 1,2,...$ We denote by $c_o(S)$ the closed subspace of $\ell_\infty(S)$ consisting of those functions α such that for each $\epsilon > 0$, the set $\{s \in S \mid |\alpha(s)| > \epsilon\}$ is finite. Although we may neglect to say so in each specific instance, whenever we refer to a set S in connection with $\ell_p(S)$, $\ell_\infty(S)$, etc., S will always be a non-empty set.

If S is a non-empty topological space, we denote by $C(S)$ the closed subspace of continuous functions in $\ell_\infty(S)$. Usually S will be compact and Hausdorff.

Let S be a non-empty set and let $\{A_s\}_{s \in S}$ be a family of Banach spaces indexed by S. We denote by $\sum_{s \in S} \oplus_1 A_s$ the <u>ℓ_1 direct sum of the spaces A_s</u>, i.e. $\sum_{s \in S} \oplus_1 A_s$ is the Banach space of all functions α from S into $\bigcup_{s \in S} A_s$ such that $\alpha(s) \in A_s$ for each $s \in S$ and such that $\sum_{s \in S} \|\alpha(s)\|_{A_s} < \infty$ where $\|\alpha(s)\|_{A_s}$ denotes the norm of the element $\alpha(s) \in A_s$. Addition and multiplication by scalars of elements of $\sum_{s \in S} \oplus_1 A_s$ are defined pointwise and the norm of $\alpha \in \sum_{s \in S} \oplus_1 A_s$ is defined to be $\sum_{s \in S} \|\alpha(s)\|_{A_s}$.

Let $-\infty < a < b < \infty$. $L_1[a, b]$ denotes the Banach space of all equivalence classes of Lebesgue integrable scalar valued functions defined on the closed interval $[a, b]$, two such functions being equivalent if their difference is zero except on a set of Lebesgue measure zero. As is ordinarily the case, we shall refer to the members of $L_1[a, b]$ as if they were functions, rather than equivalence classes of functions. Addition and multiplication by scalars of "functions" in $L_1[a, b]$ are defined pointwise and the norm of $f \in L_1[a, b]$ is $\int_a^b |f(x)|dx$ where the integral is of course the Lebesgue integral.

INTRODUCTION

Although the adjective "injective" is borrowed from homological algebra and is consequently relatively recent in origin, the study of injective spaces can be said to have originated in the late 1920's and early 1930's with the Hahn-Banach theorem. This theorem asserts, in one of its several forms, that a continuous linear functional defined on a subspace of a normed linear space can be extended to a continuous linear functional defined on the whole space with the same norm as the original functional. One is immediately led to consider the following problem: Given a normed linear space I of dimension greater than one, and a bounded linear transformation T defined on a subspace Y of a normed linear space X and having values in I, does there exist a bounded linear transformation $\tilde{T} : X \longrightarrow I$ such that $\|\tilde{T}\| = \|T\|$ and such that $\tilde{T}(y) = T(y)$ for all $y \in Y$? The answer to this question is in general no even if we relax the requirement that $\|\tilde{T}\| = \|T\|$. An early example to show that the answer is no can be found in Fichtenholz and Kantorovitch [9]. Since the answer is no in general, one is led to consider those spaces I for which the answer is affirmative and to formulate the notion of an injective space and in particular the notion of an

In(λ) space, $1 \leq \lambda < \infty$.[1] One of the earliest examples of such a space was given by Phillips [33] which we present in Chapter I (Theorem 1.9). Our proof is similar to that of Phillips although he obtained the theorem as a corollary to his theorem giving the general form of a bounded linear transformation from a Banach space to $\ell_\infty(S)$.

Sobczyk announced in an abstract (Sobczyk [37]) a result which is equivalent to the statement "If a Banach space is injective, then it is a member of the class In(λ) for some finite λ." He proved this result in Sobczyk [38], but by rather complicated methods. We present in Chapter III an elementary proof of Sobczyk's theorem, based on a necessary (and sufficient) condition for injectivity which we establish in Chapter II.

Nachbin [30] suggested for investigation the study of "injective spaces" if we restrict some of the spaces in the definition of injective space to certain categories of spaces. In Chapter V we present some results of our investigation of this topic. Our results are of the

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[1] Precise definitions of these terms will of course be given in Chapter I.

type where it appears that we define a "weaker" sort of injective space, but in reality the "weaker" type of injective turns out to be injective in our original sense. Some of the results in Chapter V were announced without proof by the author in Metas [26].

For some time it has been known that if a Banach space X has the property (which we shall call property $\mathcal{P}$) that for all Banach spaces Y which contain X as a normed linear subspace, there exists a bounded projection from Y onto X, then X is injective and conversely.[1] We have avoided using this definition of injective as long as possible (Chapter VI) for the simple reason that it is cumbersome to show that property $\mathcal{P}$ is preserved under congruence. Since we need this definition for our later work, we present a complete treatment of the equivalence of the two definitions of injectivity as well as their equivalence to two other definitions. In order to establish the equivalence of these various definitions of injectivity we require the lemma that property $\mathcal{P}$ is preserved under congruence and we present

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[1] Indeed most of the examples of non-injective spaces were established by showing that there does not exist a bounded projection from some superspace onto the space.

a complete proof of this fact (Lemma 6.4). It may be pointed out that most writers in this field have used this fact without explicitly stating it. Goodner [11] states it as a lemma, but overlooks a logical difficulty in his proof.[1] We trust that we have not overlooked anything in our proof of it. The equivalence between some of the definitions of injectivity considered in Chapter VI were observed by Akilov [2] and Phillips [33].

Nachbin [29] proved that a real Banach space X is a member of the class In(1) if and only if the set of all closed spheres in X has the binary intersection property. We generalize the binary intersection property to the λ-intersection property, $1 \leq \lambda < \infty$, (so that our 1-intersection property is the binary intersection property) and we generalize the necessity part of Nachbin's theorem so that it reads "If a real Banach space X is a member of the class In(λ), then the set of all closed spheres in X has the λ-intersection property." We present two proofs of this result, the

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[1] In his Lemma 2.3 Goodner [11, page 90] constructs a set $Z = X \cup W'$ and proceeds to define a one-one mapping U from Z onto a space W. Implicit in his definition of U is the condition that X and W' are disjoint and it just is not necessarily the case that they are disjoint.

first of which imitates Nachbin's proof for the case $\lambda = 1$, and the second of which assumes Nachbin's result and deduces the result for $\lambda > 1$ with the aid of our necessary and sufficient condition for injectivity of Chapter II. We use our generalization of Nachbin's result to prove a special case of a theorem on injective spaces announced without proof by Lindenstrauss [21].

We introduce the notion of a projective Banach space by going directly to the definition of "projective" as it appears in homological algebra (see for example Northcott [31]) with Banach spaces as our objects and bounded linear transformations as our maps. This definition is obtained by merely reversing the arrows in the diagram which describes the definition of "injective". We then proceed to prove (Proposition 1.4) that projectivity is equivalent to the existence of a bounded linear transformation which lifts (with respect to the canonical quotient map) a given bounded linear transformation from a Banach space into a quotient space. This alternate definition of projectivity enables us to define the classes $Pr(\lambda)$, $1 \leq \lambda < \infty$ of projective spaces.

In Chapter II, we establish a necessary and sufficient condition for a Banach space to be projective and in Chapter III we use this result to prove the analogue for projective spaces of the theorem of Sobczyk, namely that every projective Banach space is a member of $Pr(\lambda)$ for some λ. This result, in slightly different notation, was announced by the author in Metas [25]. In Chapter IV we establish a necessary condition for projectivity which allows us to construct examples of non-projective spaces. In Chapter V we introduce some apparently weaker definitions of projectivity and proceed to show that these definitions are actually equivalent to projectivity. Some of these results were announced by the author in Metas [26]. In Chapter VIII we show that the analogue for projective spaces of the theorem of Lindenstrauss on injective spaces (which we referred to earlier) is false by showing that the class $Pr(1)$ consists of only the zero space. In Chapter IX we consider dual spaces of projective and injective spaces.

In Chapter X we give a complete proof of a difficult theorem of Pelczynski which states that if S is a countably infinite set and X is an infinite dimensional closed subspace of $\ell_1(S)$ with a closed complement, then X is equivalent to $\ell_1(S)$. The proof we give differs in several respects from the one given by Pelczynski [32]. We avoid Pelczynski's use of previous results of Bessaga and Pelczynski [6] and of Nikolskii and replace them by direct arguments, thereby making the proof completely self-contained. We have also corrected a number of incorrect statements that appear in Pelczynski's proof.

Finally using the theorem of Pelczynski together with our necessary and sufficient condition for projectivity of Chapter II and the well known result that a separable Banach space is the image under a continuous linear transformation of $\ell_1(S)$ for some countably infinite set S (of which we give a proof in Chapter II), we obtain a characterization of all separable projective Banach spaces.

CHAPTER I

Definitions and Examples

In this chapter we shall define the notion of an injective Banach space and the notion of a projective Banach space. The classes $In(\lambda)$ and $Pr(\lambda)$, $1 \leq \lambda < \infty$, of such spaces respectively will be introduced and examples of injective and projective spaces will be given.

1.1 Definition. A Banach space B is said to be injective if for all Banach spaces X and Y and all bounded linear transformations i and g where i maps Y onto a closed subspace of X in a one-one manner and g maps Y into B, there exists a bounded linear map $\tilde{g} : X \longrightarrow B$ such that $g = \tilde{g}i$.

Using the diagrammatic notation of exact sequences we can express the situation of the preceding definition as follows

(1.1)
$$\begin{array}{ccccc} 0 \longrightarrow & Y & \longrightarrow & X & \text{(exact)} \\ & \downarrow & \swarrow & & \\ & B & & & \end{array}$$

and we can say roughly that every bounded linear map from Y into B "extends" to a map from X into B such that the diagram is commutative. If we reverse all the arrows in (1.1), we obtain the following diagram:

$$
\begin{array}{ccccc}
0 \leftarrow & Y & \leftarrow & X & \text{(exact)} \\
 & \uparrow & \nearrow & & \\
 & B & & &
\end{array} \tag{1.2}
$$

We are thus led to the following definition which is in a sense dual to Definition 1.1.

1.2 Definition. A Banach space B is said to be projective if for all Banach spaces X and Y and all bounded linear transformations g and f where g maps X onto Y and f maps B into Y there exists a bounded linear transformation $\tilde{f} : B \to X$ such that $f = g\tilde{f}$. Roughly we can say that every bounded linear map f from B into Y "lifts" to a map $\tilde{f}$ into X and we call $\tilde{f}$ a "lift" for f or we may say that "$\tilde{f}$ lifts f".

1.3 Remark. The preceding two definitions are meaningful in the case where all our Banach spaces are complex as well as in the case where all our Banach spaces are real. From this point on, when no mention is made of the scalar field associated with the Banach space (or spaces) under discussion, it is to be understood that the statements made are valid both for the real and the complex cases.

The purpose of our first proposition is to show that we can narrow down somewhat the classes of Banach spaces and bounded linear maps that one must examine in order to establish that a given Banach space is projective or injective.

1.4 Proposition.

(a) Let P be a Banach space. Then P is projective if and only if for every Banach space X, every closed subspace X_o of X and every bounded linear transformation $T : P \to X/X_o$, there exists a bounded linear transformation $\tilde{T} : P \to X$ such that $T = Q\tilde{T}$ where Q is the canonical quotient map from X onto X/X_o.

(b) Let I be a Banach space. Then I is injective if and only if for every Banach space X, every closed subspace Y of X, and every bounded linear transformation $g : Y \to I$, there exists a bounded linear transformation $\tilde{g} : X \to I$ such that the restriction of $\tilde{g}$ to Y is g.

Proof. (=>) If P is projective, it is clear that the map $\tilde{T}$ with the properties asserted in (a) exists. If I is injective we can let $i : Y \to X$ be the identity mapping and then the map $\tilde{g} : X \to I$ with the property that $\tilde{g}i = g$ has g as its restriction to Y.

(<=) (a) Assume that every bounded linear transformation from P into a quotient space lifts and suppose that g is a bounded linear transformation from X onto Y, X and Y arbitrary Banach spaces, and f is a bounded linear transformation from P into Y. Let $X_o = g^{-1}(\{0\})$. Then X_o is closed in X and so X/X_o is a Banach space.

Let $Q : X \longrightarrow X/X_o$ be the canonical quotient map. Then there exists a one-one linear transformation ψ from X/X_o onto Y such that $\psi Q = g$. Indeed ψ is continuous. For let Y_1 be an open set in Y. Then $\psi^{-1}(Y_1) = Q(g^{-1}(Y_1))$ and since g is continuous, $g^{-1}(Y_1)$ is open in X. Since Q is an open map, $Q(g^{-1}(Y_1))$ is open in X/X_o. So ψ is continuous. So $\psi^{-1} : Y \longrightarrow X/X_o$ is continuous. We have the following situation:

$$\begin{array}{ccc} & & P \\ & & \downarrow f \\ (1.3) & & Y \\ & & \downarrow \psi^{-1} \\ X & \xrightarrow[Q]{} & X/X_o \end{array}$$

By hypothesis, there exists a bounded linear transformation $\tilde{f} : P \longrightarrow X$ such that $Q\tilde{f} = \psi^{-1}f$. Hence $\psi Q\tilde{f} = f$, that is, $g\tilde{f} = f$. So P is projective.

(b) Assume that every bounded linear transformation from a closed subspace extends. Suppose that i is a one-one bounded linear transformation from a Banach space Y onto a closed subspace $i[Y]$ of a Banach space X and let $g : Y \longrightarrow I$ be a bounded linear transformation. Let $j : i[Y] \longrightarrow Y$ be defined by $j(i(y)) = y$. j is well defined, linear, and bounded.

So $gj : i[Y] \longrightarrow I$ maps a closed subspace of X into I. By hypothesis, there exists a bounded linear transformation $\tilde{g} : X \longrightarrow I$ such that the restriction of $\tilde{g}$ to $i[Y]$ is gj. So for $y \in Y$, $\tilde{g}(i(y)) = g(j(i(y))) = g(y)$. So $\tilde{g}i = g$. Hence I is injective. Q.E.D.

Thus for the projective case we need merely consider spaces $Y = X/X_o$ with g (in Definition 1.2) the canonical quotient map, and for the injective case we need merely consider closed subspaces Y of X. Indeed we shall use Proposition 1.4 as our definition of projective and injective Banach spaces almost exclusively from now on. Later we shall show how the class of spaces X and Y in our original definition can be narrowed down even further.

1.5 Definition. Let $1 \leq \lambda < \infty$. An injective Banach space I is said to be a member of the class $In(\lambda)$ if the map $\tilde{g}$ in Proposition 1.4(b) can be chosen such that $\|\tilde{g}\| \leq \lambda\|g\|$. A projective Banach space P is said to be a member of the class $Pr(\lambda)$ if the map $\tilde{T}$ of Proposition 1.4(a) can be chosen such that $\|\tilde{T}\| \leq \lambda\|T\|$.

The familiar Hahn-Banach theorem states that the real field and the complex field are both in $In(1)$. Indeed one can view the study of injective Banach spaces as a study of those Banach spaces for which a generalized Hahn-Banach theorem holds.

We shall now proceed to construct examples of projective and injective Banach spaces. We require a lemma first.

1.6 Lemma. Let S be a non-empty set and let $\{P_s\}_{s\in S}$ be a family of projective Banach spaces indexed by the set S. Assume that each $P_s \in Pr(\lambda_s)$ and that $\sup_{s\in S} \{\lambda_s\} < \infty$. Let $L = \sum_{s\in S} \oplus_1 P_s$ be the ℓ_1 direct sum of the spaces P_s. Then L is projective and is a member of $Pr(\lambda)$ where $\lambda = \sup_{s\in S} \{\lambda_s\}$.

Proof. For each $s \in S$ let i_s be the natural injection of P_s into L. In other words, if $y \in P_s$, then $i_s(y)$ is that function in L whose value is $0_{s'}$ for $s' \neq s$ (where $0_{s'}$ denotes the zero element in $P_{s'}$) and whose value at s is y. It is easy to see that i_s is linear and that $\|i_s(y)\|_L = \|y\|_{P_s}$.[1] Let X be a Banach space and X_o a closed subspace

[1] When there is no danger of confusion, we shall in the future usually omit the name of the space as a subscript and merely write $\| \; \|$.

of X. Let $f : L \longrightarrow X/X_o$ be a bounded linear transformation. Let $f_s = fi_s$. Then f_s is a bounded linear transformation from P_s into X/X_o and since $P_s \in Pr(\lambda_s)$, there exists a bounded linear transformation $\tilde{f}_s : P_s \longrightarrow X$ such that $Q\tilde{f}_s = f_s$ (where Q is the quotient map from X onto X/X_o) and $\|\tilde{f}_s\| \leq \lambda_s\|f_s\|$.

We want to define a map $\tilde{f} : L \longrightarrow X$. It seems reasonable to define $\tilde{f}$ as follows:

$$(1.4) \qquad \tilde{f}(\alpha) = \sum_{s \in S} \tilde{f}_s(\alpha(s)), \quad \alpha \in L$$

However we must first show that the right hand side of (1.4) does indeed define an element of X, i.e. that $\sum_{s \in S} \tilde{f}_s(\alpha(s)) < \infty$. Now for every non-empty finite subset S_F of S we have

$$\sum_{s \in S_F} \|\tilde{f}_s(\alpha(s))\| \leq \sum_{s \in S_F} \|\tilde{f}_s\| \; \|\alpha(s)\| \leq \sum_{s \in S_F} \lambda_s \|f_s\| \; \|\alpha(s)\|$$

$$(1.5) \qquad \leq \sum_{s \in S_F} \lambda \|fi_s\| \; \|\alpha(s)\| \leq \sum_{s \in S_F} \lambda \|f\| \; \|\alpha(s)\|$$

$$\leq \lambda \|f\| \sum_{s \in S} \|\alpha(s)\| = \lambda \|f\| \; \|\alpha\| < \infty.$$

So the family $\{\|\tilde{f}_s(\alpha(s))\|\}_{s \in S}$ is summable which implies the summability of the family $\{\tilde{f}_s(\alpha(s))\}_{s\in S}$. So the right hand side of (1.4) does indeed define an element of X and so if we define $\tilde{f}(\alpha)$ by (1.4), we have a mapping from L into X. It is easy to see that $\tilde{f}$ is linear.

Now to prove that $\tilde{f}$ lifts f, i.e. that $Q\tilde{f} = f$. Let $\alpha \in L$. First we show that the family of vectors $\{i_s(\alpha(s))\}_{s\in S}$ in L is summable to α. Let $\epsilon > 0$. Now since $\|\alpha\|_L = \sum_{s\in S} \|\alpha(s)\|_{P_s}$, there exists a finite subset S_ϵ of S such that if $S_\epsilon \subset S_F \subset S$, S_F finite and non-empty, then $\left| \|\alpha\|_L - \sum_{s\in S_F} \|\alpha(s)\|_{P_s} \right| < \epsilon$.

Now $\alpha - \sum_{s\in S_F} i_s(\alpha(s))$ is that function in L whose value is $\alpha(s)$ if $s \notin S_F$ and whose value is $0_s \in P_s$ if $s \in S_F$. So

$$\|\alpha - \sum_{s\in S_F} i_s(\alpha(s))\|_L = \sum_{s\in S-S_F} \|\alpha(s)\|_{P_s}$$

$$= \|\alpha\|_L - \sum_{s\in S_F} \|\alpha(s)\|_{P_s} \le \left| \|\alpha\|_L - \sum_{s\in S_F} \|\alpha(s)\|_{P_s} \right| < \epsilon.$$

(We have used here the fact that if $\{x_s\}_{s\in S}$ is a summable family of vectors and if

$S = A \cup B$, $A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$, then the families $\{x_s\}_{s\in A}$ and $\{x_s\}_{s\in B}$ are both summable and indeed $\sum_{s\in S} x_s = \sum_{s\in A} x_s + \sum_{s\in B} x_s$.) So we have shown that $\alpha = \sum_{s\in S} i_s(\alpha(s))$. Now

$$Q\tilde{f}(\alpha) = Q\Big(\sum_{s\in S} \tilde{f}_s(\alpha(s))\Big) = \sum_{s\in S} Q\tilde{f}_s(\alpha(s))^{1}$$

(1.6)

$$= \sum_{s\in S} f_s(\alpha(s)) = \sum_{s\in S} f(i_s(\alpha(s))) = f\Big(\sum_{s\in S} i_s(\alpha(s))\Big)^{1} = f(\alpha).$$

So $Q\tilde{f} = f$. Finally we have to show that $\|\tilde{f}\| \leq \lambda\|f\|$.

Now $\|\tilde{f}(\alpha)\| = \|\sum_{s\in S} \tilde{f}_s(\alpha(s))\|$ which equals (since the norm on a normed linear space is a continuous function)

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[1] The pulling of Q inside the summation sign and the pulling of f outside the summation sign is justified by the fact that Q and f are continuous and that our definition of the sum x of a summable family of vectors $\{x_s\}_{s\in S}$ in a normed linear space X coincides precisely with the definition of limit of a generalized sequence if we take as our generalized sequence the set of all finite sums $\sum_{s\in d} x_s$, d a finite non-empty subset of S. More precisely, let D denote the set of all finite non-empty subsets of S and let $\varphi : D \longrightarrow X$ be defined by $\varphi(d) = \sum_{s\in d} x_s$, $d \in D$. If we make D into a directed

$\lim_D \|\sum_{s\in d} \tilde{f}_s(\alpha(s))\|$ where D is the directed set of all non-empty finite subsets of S (see Footnote 1, page 28). But by (1.5)

$$\|\sum_{s\in d} \tilde{f}_s(\alpha(s))\| \leq \sum_{s\in d} \|\tilde{f}_s(\alpha(s))\| \leq \lambda\|f\| \ \|\alpha\|$$

which implies that $\lim_D \|\sum_{s\in d} \tilde{f}_s(\alpha(s))\| \leq \lambda\|f\| \ \|\alpha\|$.

So $\|\tilde{f}(\alpha)\| \leq \lambda\|f\| \ \|\alpha\|$. So $\|\tilde{f}\| \leq \lambda\|f\|$. Q.E.D.

Using Lemma 1.6 we can now prove a theorem which provides us with examples of projective Banach spaces.

1.7 Theorem. Let S be a non-empty set. Then $\ell_1(S)$ is projective and belongs to $Pr(1+\epsilon)$ for every $\epsilon > 0$.

Proof. Let K denote the scalar field. For each $s \in S$, let $K_s = K$. Then $\ell_1(S) = \sum_{s\in S} \oplus_1 K_s$ and so by Lemma 1.6 it suffices to show that $K \in Pr(1+\epsilon)$ for every $\epsilon > 0$. Let X be a Banach space, X_0 a

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[1] set by taking as our relation ordinary set inclusion for the finite subsets of S, then "the generalized sequence $\varphi : D \longrightarrow X$ converges to x (symbolically $\lim_D \varphi(d) = x$)" means precisely that "$\{x_s\}_{s\in S}$ is summable to x." (See Dunford and Schwartz [8 , page 26, Definition 1 and page 27, Lemma 4]).

closed subspace of X, Q the canonical quotient map from X onto X/X_o and f a bounded linear transformation from K to X/X_o. Let $\epsilon > 0$ be given. Now since

$$\inf_{x \in f(1)} \{\|x\|_X\} = \|f(1)\|_{X/X_o} \leq \|f\| |1| = \|f\|,$$

there exists an $x_\epsilon \in f(1)$ such that $\|x_\epsilon\| \leq (1 + \epsilon)\|f\|$. Define $\tilde{f} : K \longrightarrow X$ by $\tilde{f}(\alpha) = \alpha x_\epsilon$. Then $\tilde{f}$ is linear and $\|\tilde{f}\| \leq (1 + \epsilon)\|f\|$. Also $Q\tilde{f}(\alpha) = \alpha Q(x_\epsilon) = \alpha f(1)$ $= f(\alpha 1) = f(\alpha)$. So $Q\tilde{f} = f$. Q.E.D.

1.8 Remark. A natural question to ask at this point is whether $\ell_1(S) \in Pr(1)$. It will be shown in a later chapter when a more general question is answered that the answer is no.

Our next theorem provides us with examples of injective Banach spaces.

1.9 Theorem. Let S be a non-empty set. Then $\ell_\infty(S) \in In(1)$.

Proof. For each $s \in S$ define a linear functional f_s on $\ell_\infty(S)$ by $f_s(\alpha) = \alpha(s)$. Then since $|f_s(\alpha)|$ $= |\alpha(s)| \leq \sup_{t \in S} \{|\alpha(t)|\} = \|\alpha\|_{\ell_\infty(S)}$, we have $\|f_s\| \leq 1$.

Let X be a Banach space, Y a closed subspace of X, and g a bounded linear transformation from Y to $\ell_\infty(S)$.

We want to extend g to $\tilde{g} : X \longrightarrow \ell_\infty(S)$ with $\|\tilde{g}\| \leq \|g\|$. Let $g_s = f_s g$. Then g_s is a linear functional on Y and indeed $\|g_s\| = \|f_s g\| \leq \|f_s\| \, \|g\| \leq \|g\|$. By the Hahn-Banach theorem each g_s extends to a linear functional $\tilde{g}_s$ on X with $\|\tilde{g}_s\| = \|g_s\|$.

Define $\tilde{g} : X \longrightarrow \ell_\infty(S)$ as follows. For $x \in X$, define $(\tilde{g}(x))(s) = \tilde{g}_s(x)$. Now $\sup_{s \in S} \{|(\tilde{g}(x))(x)|\}$ $= \sup_{s \in S} \{|\tilde{g}_s(x)|\} \leq \sup_{s \in S} \{\|\tilde{g}_s\| \, \|x\|\} \leq \|g\| \, \|x\|$ (since $\|\tilde{g}_s\| = \|g_s\| \leq \|g\|$ for all $s \in S$) $< \infty$ and so $\tilde{g}(x)$ $\in \ell_\infty(X)$. That $\tilde{g}$ is linear is clear and since $\|\tilde{g}(x)\|_{\ell_\infty(S)} \leq \|g\| \, \|x\|$, it follows that $\|\tilde{g}\| \leq \|g\|$. All that remains to be shown is that $\tilde{g}$ restricted to Y is g. Let $y \in Y$. Then for each $s \in S$, we have

$$(\tilde{g}(y))(s) = \tilde{g}_s(y) = g_s(y) = f_s(g(y)) = (g(y))(s).$$

So $\tilde{g}$ extends g. Q.E.D.

In a later chapter we shall give a geometrical proof of Theorem 1.9 for the case where our scalar system is the real field.

1.10 <u>Remark</u>. The reader will have noticed that our examples of projective and injective Banach spaces were in fact members of the classes $Pr(\lambda)$ and $In(\lambda)$ respectively. That this was not accidental will be shown in a later chapter when the following striking results will be established: If a Banach space is injective, then it is a member of In(λ) for some finite λ. If a Banach space is projective, then it is a member of Pr(λ) for some finite λ.

1.11 Remark. In the next chapter we shall show that if a Banach space X is injective (projective) and X is equivalent to a Banach space Y, then Y is injective (projective). If we assume this fact, it is easy to see that a finite dimensional Banach space B is injective and projective. For let B have dimension $n > 0$. (If the dimension of B is zero, the result is immediate.) Then B is equivalent to the space $\ell_1(S_n)$ and to the space $\ell_\infty(S_n)$ where $S_n = \{1, 2, \ldots, n\}$.

In later chapters we shall give examples of non-projective Banach spaces as well as examples of non-injective Banach spaces.

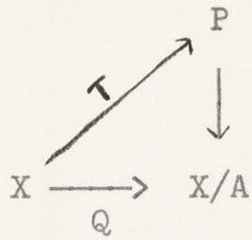
CHAPTER II

Necessary and Sufficient Conditions for Injectivity and Projectivity

In this chapter we shall prove a theorem (2.4) which establishes necessary and sufficient conditions for a Banach space to be projective and an analogous theorem (2.8) which establishes necessary and sufficient conditions for a Banach space to be injective. Unfortunately these conditions are not very helpful for constructing concrete examples of such spaces. They will be used however in subsequent chapters for establishing various theoretical results.

2.1 Lemma. Let T be a bounded linear transformation from a Banach space X onto a projective Banach space P. Then there exist closed subspaces A and Y of X such that $X = Y + A$, $A \cap Y = \{0\}$ and Y is equivalent to P.

Proof. Put $A = T^{-1}(\{0\})$. Then A is a closed subspace of X. Let Q be the quotient map from X onto X/A. As in the proof of Proposition 1.4, part (a), it follows that there exists a one-one bicontinuous linear transformation $\bar{T}$ from P onto X/A such that $\bar{T}T = Q$. So we have the following situation:



Since P is projective, there exists a bounded linear transformation $S : P \longrightarrow X$ such that $QS = \overline{T}$. Since $\overline{T}$ is invertible and $\overline{T} = QS = \overline{T}TS$, it follows that $TS = 1_P$ = the identity map on P. Let $Y = S(P)$. First we shall establish that $X = Y + A$, $A \cap Y = \{0\}$. Let $x \in X$ and let $y = S(T(x))$. Then $y \in Y$ and $T(x-y) = Tx - Ty = Tx - T(ST(x)) = Tx - Tx$ (since $TS = 1_P$) $= 0$. So $x - y \in A$ and so $X = Y + A$ since $x = y + x - y$. Now suppose $z \in A \cap Y$. Then $z = S(p)$ for some $p \in P$ and $Tz = 0$. So $p = 1_P(p) = TS(p) = Tz = 0$. So $z = S(0) = 0$. So $A \cap Y = \{0\}$.

To show Y is closed in X, let $y_n \in Y$, $n = 1,2,3,\ldots$, and let $\lim_{n \to \infty} y_n = x \in X$. We want to show that $x \in Y$. Let $y_n = S(p_n)$, $p_n \in P$. Then $S(p_n) \longrightarrow x$ and so $QS(p_n) = \overline{T}(p_n) \to Q(x)$. But $\overline{T}^{-1}$ is continuous and hence $p_n = \overline{T}^{-1}(\overline{T}(p_n)) \longrightarrow \overline{T}^{-1}(Q(x))$. Since S is continuous and $p_n \to \overline{T}^{-1}(Q(x))$, it follows that $Sp_n \longrightarrow S\overline{T}^{-1}Q(x)$. But $Sp_n \longrightarrow x$ and so $x = S(\overline{T}^{-1}Q(x)) \in S(P) = Y$. So Y is closed and in particular Y is a Banach space.

Finally to show that Y is equivalent to P, let $S(p) = 0$. Then $p = 1_P(p) = TS(p) = 0$. So S is a one-one continuous linear transformation from the Banach space P onto the Banach space Y. By the closed graph theorem, $S^{-1} : Y \longrightarrow P$ is continuous. So Y is equivalent to P. Q.E.D.

2.2 Lemma. Let B be a Banach space. Then

(1) there exist a set S and a bounded linear transformation T with $\|T\| \leq 1$ from $\ell_1(S)$ onto B; and

(2) each $y \in B$ has at least one pre-image in $\ell_1(S)$ with the same norm as y.

If B is separable, there exists a countably infinite set S satisfying (1) above.

Proof. Let $U = \{x \in B \mid \|x\| \leq 1\}$ and let $S = U$. Let $\alpha \in \ell_1(S)$ and let S_F denote any finite non-empty subset of S. Then

$$\sum_{x\in S_F} \|\alpha(x)x\| = \sum_{x\in S_F} |\alpha(x)|\,\|x\| \leq \sum_{x\in S_F} |\alpha(x)| \leq \sum_{x\in S} |\alpha(x)| = \|\alpha\| < \infty.$$

So the family $\{\|\alpha(x)x\|\}_{x\in S}$ is summable which implies the summability of the family $\{\alpha(x)x\}_{x\in S}$. So $\sum_{x\in S} \alpha(x)x$

defines a vector in B and indeed $\|\sum_{x \in S} \alpha(x)x\| \leq \|\alpha\|$.

Define $T : \ell_1(S) \longrightarrow B$ by $T(\alpha) = \sum_{x \in S} \alpha(x)x$. Then

T is bounded, linear, and $\|T\| \leq 1$. Now for each $x \in S$, define ϵ_x on S by $\epsilon_x(s) = 1$ if $s = x$ and $\epsilon_x(s) = 0$ if $s \neq x$. It is clear that $\epsilon_x \in \ell_1(S)$. If $y \in B$ and $y \neq 0$, then $\frac{y}{\|y\|} \in S$ and $T(\|y\|\epsilon_{\frac{y}{\|y\|}}) = \|y\|T(\epsilon_{\frac{y}{\|y\|}})$

$= \|y\| \frac{y}{\|y\|} = y$. So T maps $\ell_1(S)$ onto B and an element $y \neq 0$ in B is the image under T of the element $\|y\|\epsilon_{\frac{y}{\|y\|}}$ and $\|\,\|y\|\epsilon_{\frac{y}{\|y\|}}\|_{\ell_1(S)} = \|y\|\|\epsilon_{\frac{y}{\|y\|}}\|_{\ell_1(S)} = \|y\|1 = \|y\|$.

So (1) and (2) are proved.

If B is separable, then so is the closed unit sphere $U = \{x \in B \mid \|x\| \leq 1\}$ of B. Let $S = \{y_1, y_2, \ldots\}$ be a countably infinite subset of U which is dense in U.[1] We construct our map T from $\ell_1(S)$ to B as before. Indeed in this case we can for convenience denote our elements $\alpha \in \ell_1(S)$ by

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[1] If B consists of only one element, namely 0, then there does not exist such an infinite set, but in this case the lemma is trivial since we can take S to be any countably infinite set and T to be the zero map from $\ell_1(S)$ onto B.

$\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_i = \alpha(y_i)$, $i = 1, 2, \ldots,$

and write $T(\alpha) = \sum_{i=1}^{\infty} \alpha_i y_i$. T is as before bounded and linear and $\|T\| \leq 1$. We want to show that T maps $\ell_1(S)$ onto B. It suffices to show that $U \subset T(\ell_1(S))$, for if $z \in B$, $z \neq 0$, then $w = \frac{z}{\|z\|} \in U$ and hence $w = T(\alpha)$ for some $\alpha \in \ell_1(S)$ and therefore $z = \|z\|w = T(\|z\|\alpha)$.

Let $u \in U$, $u \neq 0$, be fixed. Let $u_n = \frac{1}{2^n} u$, $n = 1,2,\ldots,$ and let $c_n = \frac{3\|u\|}{2^n}$, $n = 1,2,\ldots$. Let G_1 denote the open sphere in B of radius $\frac{1}{6}$ $(= \frac{1}{c_1} \frac{\|u\|}{2^{1+1}})$ and center at $\frac{1}{c_1} u_1$. It is easy to see that G_1 consists of all vectors in B of the form $\frac{1}{c_1}(u_1 + a)$ where $\|a\| < \frac{\|u\|}{4}$. $G_1 \subset U$ since

$$\left\|\frac{1}{c_1}(u_1 + a)\right\| \leq \frac{2}{3\|u\|}(\|u_1\| + \|a\|) < \frac{2}{3\|u\|}\left(\frac{\|u\|}{2} + \frac{\|u\|}{4}\right)$$

$$< \frac{2}{3\|u\|} \frac{3\|u\|}{2} = 1.$$

Since S is dense in U, there exists a point (indeed infinitely many points) of S in G_1. Let n_1 be the smallest positive integer such that $y_{n_1} \in G_1$. Then

$y_{n_1} = \frac{1}{c_1}(u_1 + a_1)$ for some $a_1 (a_1 = c_1 y_{n_1} - u_1)$ with $\|a_1\| < \frac{\|u\|}{4}$. Let $b_1 = u_1 + a_1$. Then $y_{n_1} = \frac{b_1}{c_1}$. Let G_2 denote the open sphere in B of radius $\frac{1}{c_2} \frac{\|u\|}{2^{2+1}}$ and center at $\frac{1}{c_2}(u_2 - a_1)$. G_2 consists of all vectors in B of the form $\frac{1}{c_2}(u_2 - a_1 + a)$ where $\|a\| < \frac{\|u\|}{2^3}$.

$G_2 \subset U$ since $\|\frac{1}{c_2}(u_2 - a_1 + a)\| \leq \frac{4}{3\|u\|}(\|u_2\| + \|a_1\| + \|a\|)$ $< \frac{4}{3\|u\|}(\frac{\|u\|}{4} + \frac{\|u\|}{4} + \frac{\|u\|}{8}) < \frac{4}{3\|u\|} \frac{3\|u\|}{4} = 1$. Since S is dense in U, there exist infinitely many points of S in G_2. Let n_2 be the smallest positive integer greater than n_1 such that $y_{n_2} \in G_2$. Then $y_{n_2} = \frac{1}{c_2}(u_2 - a_1 + a_2)$ for some $a_2 (a_2 = c_2 y_{n_2} - u_2 + a_1)$ with $\|a_2\| < \frac{\|u\|}{2^3}$. Let $b_2 = u_2 - a_1 + a_2$. Then $y_{n_2} = \frac{b_2}{c_2}$. If we let $a_o = 0$, then by induction there exist sequences $\{a_n\}_{n=o,1,2,...}$ and $\{b_n\}_{n=1,2,...}$ in B such that $\|a_n\| < \frac{\|u\|}{2^{n+1}}$, $n = 0,1,2,...,$ $b_n = u_n - a_{n-1} + a_n$, $n = 1,2,...,$ and an increasing sequence of positive integers $\{n_i\}_{i=1,2,...}$ and a sequence $\{y_{n_i}\}_{i=1,2,\ldots}$ of distinct vectors in S such that $y_{n_i} = \frac{b_i}{c_i}$, $i = 1,2,\ldots$ For suppose m is a positive integer and $a_o, a_1, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ and $n_1, n_2, \ldots, n_m$ and $y_{n_1}, y_{n_2}, \ldots, y_{n_m}$ have been chosen and satisfy the aforementioned conditions. We want to choose a_{m+1}, b_{m+1}, n_{m+1}, and $y_{n_{m+1}}$. Let G_{m+1} denote the open sphere in B of radius $\frac{1}{c_{m+1}} \frac{\|u\|}{2^{m+2}}$ and center at $\frac{1}{c_{m+1}}(u_{m+1} - a_m)$. G_{m+1} consists of all vectors in B of the form $\frac{1}{c_{m+1}}(u_{m+1} - a_m + a)$ where $\|a\| < \frac{\|u\|}{2^{m+2}}$. $G_{m+1} \subset U$ since

$$\left\|\frac{1}{c_{m+1}}(u_{m+1} - a_m + a)\right\| \leq \frac{2^{m+1}}{3\|u\|}(\|u_{m+1}\| + \|a_m\| + \|a\|)$$

$$< \frac{2^{m+1}}{3\|u\|}\left(\frac{\|u\|}{2^{m+1}} + \frac{\|u\|}{2^{m+1}} + \frac{\|u\|}{2^{m+2}}\right)$$

$$< \frac{2^{m+1}}{3\|u\|} \frac{3\|u\|}{2^{m+1}} = 1.$$

There exist infinitely many points of S in G_{m+1}. Let n_{m+1} be the smallest positive integer greater than n_m such that $y_{n_{m+1}} \in G_{m+1}$. Then $y_{n_{m+1}} = \frac{1}{c_{m+1}}(u_{m+1} - a_m + a_{m+1})$

for some a_{m+1} with $\|a_{m+1}\| < \frac{\|u\|}{2^{m+2}}$. We let

$b_{m+1} = u_{m+1} - a_m + a_{m+1}$ and hence $y_{n_{m+1}} = \frac{b_{m+1}}{c_{m+1}}$ and

our induction is completed.

Now it is clear that $\sum_{n=1}^{\infty} u_n = u$ and that

$\lim_{n \to \infty} a_n = 0$. Also since

$$b_1 + b_2 + \dots + b_n = u_1 + a_1 + u_2 - a_1 + a_2 + \dots + u_n - a_{n-1} + a_n$$

$$= u_1 + u_2 + \dots + u_n + a_n$$

$$= \sum_{i=1}^{n} u_i + a_n$$

$$\to u \text{ as } n \to \infty,$$

it follows that $\sum_{n=1}^{\infty} b_n = u$. Let $S_1 = \{y_{n_1}, y_{n_2}, y_{n_3}, \dots\}$.

Define a scalar valued function φ on S by

$$\varphi(s) = 0 \text{ if } s \notin S_1,$$

$$\varphi(y_{n_i}) = c_i, \quad i = 1, 2, \dots$$

Since $c_i = \frac{3}{2^i}\|u\|$, $i = 1, 2, \dots$, it is clear that

$\varphi \in \ell_1(S)$. Also $T(\varphi) = \sum_{i=1}^{\infty} c_i y_{n_i} = \sum_{i=1}^{\infty} c_i \frac{b_i}{c_i} =$

$\sum_{i=1}^{\infty} b_i = u$. So $U \subset T(\ell_1(S))$ and hence T maps $\ell_1(S)$ onto B. Q.E.D.

2.3 Lemma. Let P be a projective Banach space and let X be a Banach space equivalent to P under the mapping $\varphi : P \longrightarrow X$. Then X is projective. If $P \in Pr(\lambda)$, then $X \in Pr(\lambda')$ where $\lambda' = \lambda\|\varphi\| \, \|\varphi^{-1}\|$ (and in particular if X is congruent to P, $X \in Pr(\lambda)$).

Proof. Let Y be a Banach space, Y_o a closed subspace of Y, Q the quotient map from Y onto Y/Y_o, and f a bounded linear transformation from X to Y/Y_o. Then $f\varphi : P \longrightarrow Y/Y_o$ lifts to $\tilde{g} : P \longrightarrow Y$. But then $\tilde{g}\varphi^{-1} : X \longrightarrow Y$ is a lift for f since $Q(\tilde{g}\varphi^{-1}) = f\varphi\varphi^{-1}$ (since $\tilde{g}$ lifts $f\varphi$) $= f$. So X is projective. Finally if $P \in Pr(\lambda)$, the map $\tilde{g}$ can be chosen so that $\|\tilde{g}\| \leq \lambda\|f\varphi\|$ and hence

$\|\tilde{g}\varphi^{-1}\| \leq \|\tilde{g}\| \, \|\varphi^{-1}\| \leq \lambda\|f\varphi\| \, \|\varphi^{-1}\| \leq \lambda\|\varphi\| \, \|\varphi^{-1}\| \, \|f\|$.

So $X \in Pr(\lambda')$. Q.E.D.

2.4 Theorem. A Banach space P is projective if and only if P is equivalent to a closed subspace with a closed complement of some $\ell_1(S)$.

Proof. (a) Assume first that P is projective. By Lemma 2.2 there exists a set S and a bounded linear transformation T from $\ell_1(S)$ onto P. By Lemma 2.1 there exist closed subspaces A and Y of $\ell_1(S)$ such that $\ell_1(S) = Y + A$, $A \cap Y = \{0\}$, and Y is equivalent to P.

(b) Now assume that there exists a set S, closed subspaces Y and A of $\ell_1(S)$ with $Y \cap A = \{0\}$ and $\ell_1(S) = Y + A$ and such that Y is equivalent to P. We want to show that P is projective. By Lemma 2.3 it suffices to prove that Y is projective. Now because $\ell_1(S) = Y + A$, and A and Y are closed subspaces with $A \cap Y = \{0\}$, there exists a bounded projection T from $\ell_1(S)$ onto Y (see Dunford and Schwartz [8 , page 480]). Let X be a Banach space, X_o a closed subspace of X, f a bounded linear transformation from Y to X/X_o and Q the quotient map from X onto X/X_o. Let $i : Y \to \ell_1(S)$ be the identity map and let $\varphi = fT$. We have the following situation:

$$\begin{array}{ccc} & \ell_1(S) & \\ & i \uparrow \downarrow T & \\ & Y & \\ & \downarrow f & \\ X & \xrightarrow[Q]{} & X/X_o \end{array}$$

Since $\ell_1(S)$ is projective (Theorem 1.7) there exists a bounded linear transformation $\tilde{\varphi} : \ell_1(S) \longrightarrow X$ such that $Q\tilde{\varphi} = \varphi = fT$. Define $\tilde{f} : Y \longrightarrow X$ by $\tilde{f} = \tilde{\varphi}i$. Then $\tilde{f}$ lifts f since for $y \in Y$ we have $Q\tilde{f}(y) = Q\tilde{\varphi}i(y) = Q\tilde{\varphi}(y) = \varphi(y) = fT(y) = f(y)$ since the restriction of a projection to its image (in our case T restricted to Y) is the identity map on that image. So Y is projective and hence so is P. Q.E.D.

2.5 <u>Lemma</u>. Let T be a one-one bounded linear transformation from an injective Banach space I onto a closed subspace of a Banach space X. Then there exist closed subspaces A and Y of X such that $X = Y + A$, $A \cap Y = \{0\}$ and Y is equivalent to I. If T is an isometry, Y is congruent to I.

<u>Proof</u>. Let $Y = T(I)$. Then Y is closed in X and hence is a Banach space. By the closed graph theorem, $T^{-1} : Y \longrightarrow I$ is bounded. Since I is

injective, T^{-1} extends to a bounded linear transformation $S : X \longrightarrow I$. Let $A = S^{-1}(\{0\})$. A is closed in X and if $z \in A \cap Y$, then $T^{-1}(z) = S(z) = 0$ and so $z = 0$ (since T^{-1} is one-one). So $A \cap Y = \{0\}$. If $x \in X$, then $x = y + x - y$ where $y = TS(x)$. Certainly $y \in Y$ while $S(y) = S(TS(x)) = T^{-1}(TS(x)) = S(x)$ so that $x - y \in S^{-1}(\{0\}) = A$. Hence $X = Y + A$. The equivalence between I and Y is clear and so is the congruence between them if T is an isometry. Q.E.D.

2.6 <u>Lemma</u>. Let B be a Banach space. Then there exists an isometry T from B onto a closed subspace of $\ell_\infty(S)$ for some set S.

<u>Proof</u>. Let K denote the scalar field and let $S = \{f \in B^* \mid \|f\| \leq 1\}$. For $x \in B$, define $Tx : S \longrightarrow K$ by $(Tx)(f) = f(x)$. Then $|(Tx)(f)| = |f(x)| \leq \|f\|\,\|x\| \leq \|x\|$ and so $Tx \in \ell_\infty(S)$. The mapping T from B into $\ell_\infty(S)$ defined by $x \longrightarrow Tx$ is clearly linear and bounded since

$$\|Tx\|_{\ell_\infty(S)} = \sup_{f \in S} \{|Tx(f)|\} = \sup_{f \in S} \{|f(x)|\}$$

$$\leq \sup_{f \in S} \{\|f\|\,\|x\|\} \leq \|x\|.$$ Indeed T is an isometry since

$$\|x\| = \sup_{f \in S} \{|f(x)|\} = \sup_{f \in S} \{|Tx(f)|\} = \|Tx\|_{\ell_\infty(S)} .$$

It remains for us to show that $T(B)$ is closed in $\ell_\infty(S)$.

Let $y_n \in T(B)$, $n = 1, 2, \ldots$, and assume

$\lim_{n \to \infty} y_n = y \in \ell_\infty(S)$. We want to show that $y \in T(B)$.

Now since the sequence $\{y_n\}_{n=1,2,\ldots}$ converges, it is Cauchy. Let $y_n = Tx_n$, $x_n \in B$, $n = 1, 2, \ldots$. Then it follows that the sequence $\{x_n\}_{n=1,2,\ldots}$ is also Cauchy and so there exists an element $x \in B$ such that

$\lim_{n \to \infty} x_n = x$. So $T(x_n) \longrightarrow T(x)$. But $T(x_n) = y_n \longrightarrow y$.

So $y = T(x) \in T(B)$. So $T(B)$ is closed in $\ell_\infty(S)$. Q.E.D.

2.7 <u>Lemma</u>. Let I be an injective Banach space and let X be a Banach space equivalent to I under the mapping $\varphi : I \longrightarrow X$. Then X is injective. If $I \in \mathrm{In}(\lambda)$, then $X \in \mathrm{In}(\lambda')$ where $\lambda' = \lambda\|\varphi\|\,\|\varphi^{-1}\|$ (and in particular if X is congruent to I, $X \in \mathrm{In}(\lambda)$).

<u>Proof</u>. Let A be a Banach space, Y a closed subspace of A, and f a bounded linear transformation from Y into X. Let $g = \varphi^{-1}f$. Then g is a bounded linear transformation from Y into I. Since I is injective, g extends to a bounded linear transformation $\tilde{g}$ from A into I. Define $\tilde{f} : A \longrightarrow X$ by $\tilde{f} = \varphi\tilde{g}$. Then $\tilde{f}$ extends f since for $y \in Y$,

$\tilde{f}(y) = \varphi\tilde{g}(y) = \varphi g(y) = \varphi\varphi^{-1}f(y) = f(y)$. So X is injective. Finally if $I \in In(\lambda)$, the map $\tilde{g}$ can be chosen so that $\|\tilde{g}\| \leq \lambda\|g\| = \lambda\|\varphi^{-1}f\|$ and so

$\|\tilde{f}\| = \|\varphi\, \tilde{g}\| \leq \|\varphi\|\, \|\tilde{g}\| \leq \lambda\|\varphi^{-1}f\|\, \|\varphi\| \leq \lambda\|\varphi^{-1}\|\, \|\varphi\|\, \|f\|$.

So $X \in In(\lambda')$. Q.E.D.

2.8 <u>Theorem</u>. A Banach space I is injective if and only if I is congruent to a closed subspace with a closed complement of some $\ell_\infty(S)$.

<u>Proof</u>. (a) Assume first that I is injective. By Lemma 2.6 there exists an isometry T from I onto a closed subspace of $\ell_\infty(S)$ for some set S. By Lemma 2.5 there exist closed subspaces A and Y of $\ell_\infty(S)$ such that $\ell_\infty(S) = Y + A$, $A \cap Y = \{0\}$ and Y is congruent to I.

(b) Now assume that there exists a set S, closed subspaces Y and A of $\ell_\infty(S)$ with $Y \cap A = \{0\}$ and $\ell_\infty(S) = Y + A$ and such that Y is congruent to I. We want to show that I is injective. By Lemma 2.7 it suffices to show that Y is injective. Now because $\ell_\infty(S) = Y + A$ and Y and A are closed subspaces with $Y \cap A = \{0\}$, there exists a bounded projection T from $\ell_\infty(S)$ onto Y. Let X be a Banach space, B a closed subspace of X, and f a bounded linear transformation from B into Y. Let $i : Y \longrightarrow \ell_\infty(S)$ be the identity map and let $g = if$. Since $\ell_\infty(S)$ is injective (Theorem 1.9), there exists a bounded linear transformation $\tilde{g} : X \longrightarrow \ell_\infty(S)$ which extends g. Define $\tilde{f} : X \longrightarrow Y$ by $\tilde{f} = T\tilde{g}$. Then $\tilde{f}$ extends f since for $b \in B$, we have $\tilde{f}(b) = T\tilde{g}(b) = Tg(b) = Tif(b) = Tf(b) = f(b)$. So Y is injective and hence so is I. Q.E.D.

The two theorems in this chapter reduce the study of projective Banach spaces to the study of closed subspaces with closed complements of the spaces $\ell_1(S)$ and the study of injective Banach spaces to the study of closed subspaces with closed complements of the spaces $\ell_\infty(S)$. We shall use this reduction in some subsequent chapters to deduce various theoretical results.

CHAPTER III

<u>Projectivity Implies</u> $Pr(\lambda)$; <u>Injectivity Implies</u> $In(\lambda)$

In this chapter we shall establish the results mentioned in Remark 1.10.

3.1 <u>Theorem</u>. If a Banach space P is projective, then P is a member of the class $Pr(\lambda)$ for some finite λ.

<u>Proof</u>. Since P is projective, there exist by Theorem 2.4 a set S and closed subspaces Y and A of $\ell_1(S)$ with $Y \cap A = \{0\}$ and $\ell_1(S) = Y + A$ and such that P is equivalent to Y. We shall show that Y is a member of $Pr(\lambda_1)$ for some λ_1 from which it will follow by Lemma 2.3 that $P \in Pr(\lambda)$. Because $\ell_1(S) = Y + A$, Y and A closed subspaces with $Y \cap A = \{0\}$, there exists a bounded projection T from $\ell_1(S)$ onto Y. Let X be any Banach space, X_o a closed subspace of X, Q the quotient map from X onto X/X_o and f a bounded linear transformation from Y into X/X_o. Let $g : \ell_1(S) \longrightarrow X/X_o$ be defined by $g = fT$. Because $\ell_1(S) \in Pr(1 + \epsilon)$ for every $\epsilon > 0$ (Theorem 1.7), there exists (for a fixed $\epsilon > 0$) a bounded linear transformation $\tilde{g} : \ell_1(S) \longrightarrow X$ such that $Q\tilde{g} = g = fT$ and such that

$\|\tilde{g}\| \leq (1 + \epsilon)\|g\| = (1 + \epsilon)\|fT\|$. Let $i : Y \longrightarrow \ell_1(S)$ be the identity map. Then $\tilde{g}i$ lifts f since for $y \in Y$, we have $Q\tilde{g}i(y) = Q\tilde{g}(y) = fT(y) = f(y)$. Also $\|\tilde{g}i\| \leq \|\tilde{g}\| \ \|i\| = \|\tilde{g}\| \leq (1 + \epsilon)\|g\| = (1 + \epsilon)\|fT\|$ $\leq (1 + \epsilon)\|T\| \ \|f\|$. So $Y \in Pr(\lambda_1)$ with $\lambda_1 = (1 + \epsilon)\|T\|$ and so $P \in Pr(\lambda)$ for some finite λ. Q.E.D.

The next theorem is the analogue of Theorem 3.1 for injective Banach spaces.

3.2 Theorem. If a Banach space I is injective, then I is a member of the class $In(\lambda)$ for some finite λ.

Proof. Since I is injective, there exist by Theorem 2.8 a set S and closed subspaces Y and A of $\ell_\infty(S)$ with $Y \cap A = \{0\}$ and $\ell_\infty(S) = Y + A$ and such that I is congruent to Y. We shall show that $Y \in In(\lambda)$ from which it will follow by Lemma 2.7 that $I \in In(\lambda)$ also. Because $\ell_\infty(S) = Y + A$, Y and A closed subspaces with $Y \cap A = \{0\}$, there exists a bounded projection T from $\ell_\infty(S)$ onto Y. Let X be any Banach space, B a closed subspace of X and $f : B \longrightarrow Y$ a bounded linear transformation. Let $i : Y \longrightarrow \ell_\infty(S)$ be the identity map and let $g : B \longrightarrow \ell_\infty(S)$ be defined by $g = if$. Because $\ell_\infty(S)$

ϵ In(1) (Theorem 1.9), there exists a bounded linear transformation $\tilde{g} : X \longrightarrow \ell_\infty(S)$ which extends g and such that $\|\tilde{g}\| = \|g\|$. Define $\tilde{f} : X \longrightarrow Y$ by $\tilde{f} = T\tilde{g}$. Then $\tilde{f}$ extends f since for $b \in B$, we have $\tilde{f}(b) = T\tilde{g}(b) = Tg(b) = Tif(b) = Tf(b) = f(b)$. Also $\|\tilde{f}\| = \|T\tilde{g}\| \leq \|T\| \|\tilde{g}\| = \|T\| \|g\| = \|T\| \|if\| \leq \|T\| \|f\|$. So $Y \in In(\lambda)$ with $\lambda = \|T\|$ and hence $I \in In(\lambda)$. Q.E.D.

Thus we see that although in our concepts of projective and injective as expressed by Proposition 1.4, we did not require any restricting relation between the norm of the extension map and the norm of the map being extended, or between the norm of the lifting map and the norm of the map being lifted, we do in fact have a pleasant and surprising relation between the two.

CHAPTER IV

Some Non-projective Banach Spaces

The purpose of this chapter is to give some examples of Banach spaces which are not projective. Our method for accomplishing this is to prove that a necessary condition for projectivity is that weak and strong convergence of sequences coincide. Thus any example of a Banach space in which weak and strong convergence of sequences do not coincide is automatically an example of a non-projective Banach space.

4.1 Definition. Let X be a normed linear space. We say that a sequence of elements $\{x_n\}_{n=1,2,\ldots}$ in X converges weakly to an element $x \in X$ if $\lim_{n \to \infty} f(x_n) = f(x)$ for every $f \in X^*$. We say that $\{x_n\}_{n=1,2,\ldots}$ converges strongly to x if $\lim_{n \to \infty} \|x_n - x\| = 0$. (Thus strong convergence is ordinary convergence with respect to the norm which we have already had occasion to use although we have not called it by any special name.) We say that weak and strong convergence of sequences in X coincide if "$\{x_n\}$ converges to x weakly" implies "$\{x_n\}$ converges to x strongly."

4.2 Remark. It is easy to see that if $\{x_n\}$ converges weakly to x and to y, then $x = y$. Also if $\{x_n\}$ converges strongly to x, then $\{x_n\}$ converges weakly to x. Thus the term "coincide" in the preceding definition is justified.

4.3 Lemma. Let X be a Banach space with the property that weak and strong convergence of sequences coincide. Let Y be a Banach space equivalent to X. Then weak and strong convergence of sequences in Y coincide.

Proof. Let the sequence $\{y_n\}_{n=1,2,\ldots}$ in Y converge weakly to $y \in Y$. We want to show that $\|y_n - y\|_Y \longrightarrow 0$ as $n \longrightarrow \infty$. Let $T : X \longrightarrow Y$ be the mapping defining the equivalence between X and Y, let $y_n = Tx_n$, $n = 1,2,\ldots$, and let $y = Tx$. So $T^{-1}y_n = x_n$ and $T^{-1}y = x$. Let f_X be any continuous linear functional on X. Define a functional f_Y on Y by $f_Y(z) = f_X(T^{-1}(z))$, $z \in Y$. f_Y is linear and since $|f_Y(z)| = |f_X(T^{-1}z)| \leq \|f_X\| \|T^{-1}z\|$ $\leq \|f_X\| \|T^{-1}\| \|z\|$, f_Y is continuous. Since $y_n \longrightarrow y$ weakly, $f_Y(y_n) \longrightarrow f_Y(y)$. In other words $f_X(T^{-1}y_n) \longrightarrow f_X(T^{-1}y)$, that is $f_X(x_n) \longrightarrow f_X(x)$. So $x_n \longrightarrow x$ weakly and hence $\|x_n - x\|_X \longrightarrow 0$. So

$\|y_n - y\|_Y = \|Tx_n - Tx\|_Y \leq \|T\| \, \|x_n - x\|_X \longrightarrow 0$. So $y_n \longrightarrow y$ strongly. Q.E.D.

4.4 Lemma. Let Y be a subspace of a normed linear space X. Let $\{y_n\}_{n=1,2,\ldots}$ and y be elements of Y. Then $\{y_n\}$ converges weakly to y when considered as elements of the normed linear space Y if and only if $\{y_n\}$ converges weakly to y when considered as elements of the normed linear space X.

Proof. (a) Assume $\{y_n\}$ converges weakly to y when considered as elements of Y. Let $f \in X^*$. We want to show that $f(y_n) \longrightarrow f(y)$ as $n \longrightarrow \infty$. Let g be the restriction of f to Y. Then $g \in Y^*$ and so $g(y_n) \longrightarrow g(y)$. But $g(y_n) = f(y_n)$ and $g(y) = f(y)$. So $f(y_n) \longrightarrow f(y)$.

(b) Assume $\{y_n\}$ converges weakly to y when considered as elements of X. Let $f \in Y^*$. We must show that $f(y_n) \longrightarrow f(y)$. By the Hahn-Banach theorem, there exists an $\tilde{f} \in X^*$ such that the restriction of $\tilde{f}$ to Y is f. $\tilde{f}(y_n) \longrightarrow \tilde{f}(y)$. But $\tilde{f}(y_n) = f(y_n)$ and $\tilde{f}(y) = f(y)$. So $f(y_n) \longrightarrow f(y)$. Q.E.D.

4.5 Theorem. If P is a projective Banach space, then weak and strong convergence of sequences in P coincide.

Proof. Because P is projective, P is equivalent to a closed subspace, say Y, of some $\ell_1(S)$. Now weak and strong convergence of sequences in $\ell_1(S)$ coincide (see Day [7 , page 33, Corollary 2]). By Lemma 4.4 weak and strong convergence of sequences in Y (when considered as elements of the Banach space Y) coincide. For if $y_n \longrightarrow y$ weakly $(y_n\ y \in Y,\ n = 1,2,\ldots)$ with respect to Y, then $y_n \longrightarrow y$ weakly with respect to $\ell_1(S)$ which implies that $\|y_n - y\|_{\ell_1(S)} \longrightarrow 0$ which is of course the same as saying $\|y_n - y\|_Y \longrightarrow 0$. By Lemma 4.3 weak and strong convergence of sequences in P coincide. Q.E.D.

4.6 Corollary. If $1 < p < \infty$ and $S = \{s_1, s_2, \ldots\}$ is a countably infinite set, the Banach space $\ell_p(S)$ is not projective.

Proof. Since S is countably infinite, we shall use the convenient standard notation. For $n = 1,2,\ldots,$ let $f_n = (\delta_{n1}, \delta_{n2}, \delta_{n3}, \ldots)$ where $\delta_{ni} = 0$ for $i \neq n$ and $\delta_{nn} = 1$. Each $f_n \in \ell_p(S)$. The sequence $\{f_n\}$ converges weakly to 0. For let φ be any continuous linear functional on $\ell_p(S)$. Then there exists $g = (g_1, g_2, \ldots) \in \ell_q(S)$ where $\frac{1}{p} + \frac{1}{q} = 1$

such that for all $y = (y_1, y_2, \ldots) \in \ell_p(S), \varphi(y)$ is given by $\sum_{i=1}^{\infty} g_i y_i$. In particular $\varphi(f_n) = g_1\delta_{n1} + g_2\delta_{n2} + \ldots = g_n$. But $\lim_{n \to \infty} |g_n|^q = 0$ since $(g_1, g_2, \ldots) \in \ell_q(S)$ and so $\lim_{n \to \infty} g_n = 0$. So $\lim_{n \to \infty} \varphi(f_n) = 0$ which means that the sequence $\{f_n\}$ converges weakly to 0. But $\{f_n\}$ does not converge strongly to 0 since $\|f_n\|_{\ell_p(S)} = 1$ for each n. So $\ell_p(S)$ for $1 < p < \infty$ is not projective. Q.E.D.

4.7 <u>Corollary</u>. The complex Banach space $L_1[0,1]$ is not projective.

<u>Proof</u>. For $n = 1, 2, \ldots,$ and for $x \in [0,1]$ define $f_n(x) = e^{inx}$. $f_n \in L_1[0,1]$. Let φ be a continuous linear functional on $L_1[0,1]$. Then there exists a bounded measurable (and hence integrable) function g on [0,1] such that for all $f \in L_1[0,1]$, $\varphi(f) = \int_0^1 f(x)g(x)dx$. In particular,

$\varphi(f_n) = \int_0^1 e^{inx}g(x)dx \longrightarrow 0$ (by the Riemann-Lebesgue lemma[1]) $= \int_0^1 0g(x)dx = \varphi(0)$. So the sequence $\{f_n\}$

[1] See Wiener [42, page 14] or McShane [23, pages 231-232].

converges weakly to 0. However $\{f_n\}$ does not converge strongly to 0 since $\|f_n\|_{L_1[0,1]} = \int_0^1 |e^{inx}|dx = \int_0^1 1dx = 1$ for each n. So $L_1[0,1]$ is not projective. Q.E.D.

4.8 Remark. There is nothing special about complex $L_1[0,1]$ that it fails to be projective, nor about the finite interval [0,1]. For example, the real Banach space $L_1[0,\pi]$ is not projective. For let $f_n(x) = \sin nx$, $x \in [0,\pi]$, $n = 1, 2, 3, \ldots$. Then $\{f_n\}$ converges weakly to 0, again by the Riemann-Lebesgue lemma, but $\{f_n\}$ does not converge strongly to 0 since $\|f_n\|_{L_1[0,\pi]} = \int_0^\pi |\sin nx|\, dx = 2$ for each n.

CHAPTER V

Some Further Reductions in the Definitions of Projective and Injective

In this chapter we shall show that we can narrow down even further the class of Banach spaces X one must consider in Proposition 1.4 in order to establish that a particular Banach space is injective or projective.

5.1 Definition. A Banach space I is said to be dually injective if for every Banach space X which is congruent to the dual space of some Banach space, every closed subspace Y of X, and every bounded linear transformation T from Y into I, there exists a bounded linear transformation $\tilde{T}$ from X into I which extends T. Clearly every injective Banach space is dually injective.

5.2 Lemma. Let X be a Banach space which is congruent to the dual space of some Banach space and let I be a dually injective Banach space. Let T be a one-one bounded linear transformation from I onto a closed subspace of X. Then there exist closed subspaces A and Y of X such that $X = Y + A$, $A \cap Y = \{0\}$ and Y is equivalent to I. If T is an isometry, Y is congruent to I.

Proof. Let $Y = T(I)$. Then Y is closed in X and hence is a Banach space. By the closed graph theorem, $T^{-1} : Y \longrightarrow I$ is bounded. Since I is dually injective, T^{-1} extends to a bounded linear transformation $S : X \longrightarrow I$. Let $A = S^{-1}(\{0\})$. The rest of the proof proceeds exactly as the proof of Lemma 2.5 starting at the point where A is defined. Q.E.D.

5.3 Lemma. If I is a dually injective Banach space, then there exist a set S and closed subspaces Y and A of $\ell_\infty(S)$ with $\ell_\infty(S) = Y + A$, $Y \cap A = \{0\}$ and such that I is congruent to Y.

Proof. By Lemma 2.6 there exists an isometry T from I onto a closed subspace of $\ell_\infty(S)$ for some set S. Now $\ell_\infty(S)$ is congruent to the dual space of $\ell_1(S)$ (see Day [7, pages 29-30]). By Lemma 5.2 there exist closed subspaces Y and A of $\ell_\infty(S)$ such that $\ell_\infty(S) = Y + A$, $Y \cap A = \{0\}$ and such that Y is congruent to I. Q.E.D.

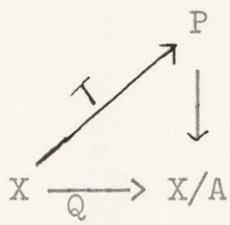
So we have shown that a dually injective Banach space is congruent to a closed subspace with a closed complement of some $\ell_\infty(S)$. But by Theorem 2.8 we know that if a Banach space I is congruent to a closed subspace with a closed complement of some $\ell_\infty(S)$, then I is injective. Hence we conclude

5.4 Theorem. A Banach space I is injective if and only if I is dually injective.

5.5 Definition. A Banach space P is said to be dually projective if for every Banach space X which is congruent to the dual space of some Banach space, every closed subspace X_o of X, and every bounded linear transformation T from P to X/X_o, there exists a bounded linear transformation $\tilde{T}$ from P to X such that $Q\tilde{T} = T$ where Q is the quotient map from X onto X/X_o. Clearly every projective Banach space is dually projective.

5.6 Lemma. Let X be a Banach space which is congruent to the dual space of some Banach space and let P be a dually projective Banach space. Let T be a bounded linear transformation from X onto P. Then there exist closed subspaces A and Y of X such that $X = Y + A$, $Y \cap A = \{0\}$ and Y is equivalent to P.

Proof. Let $A = T^{-1}(\{0\})$. A is a closed subspace of X. Let Q be the quotient map from X onto X/A. There exists a one-one bicontinuous linear transformation $\overline{T}$ from P onto X/A such that $\overline{T}T = Q$. So we have the following situation:



Since P is dually projective, there exists a bounded linear transformation $S : P \longrightarrow X$ such that $QS = \overline{T}$. Since $\overline{T}$ is invertible and $\overline{T} = QS = (\overline{T}T)S$, it follows that $TS = 1_P$ = the identity map on P. Let $Y = S(P)$. The rest of the proof proceeds exactly as the proof of Lemma 2.1 starting at the point where Y is defined. Q.E.D.

5.7 Lemma. If P is a dually projective Banach space, then there exist a set S and closed subspaces Y and A of $\ell_1(S)$ with $\ell_1(S) = Y + A$, $Y \cap A = \{0\}$ and such that P is equivalent to Y.

Proof. By Lemma 2.2 there exist a set S and a bounded linear transformation T from $\ell_1(S)$ onto P. Now $\ell_1(S)$ is congruent to the dual space of $c_o(S)$ (see Day [7 , pages 29-30]). By Lemma 5.6 there exist closed subspaces A and Y of $\ell_1(S)$ such that $\ell_1(S) = Y + A$, $Y \cap A = \{0\}$ and Y is equivalent to P. Q.E.D.

So we have shown that a dually projective Banach space is equivalent to a closed subspace with a closed complement of some $\ell_1(S)$. But by Theorem 2.4 we know that if a Banach space P is equivalent to a closed subspace with a closed complement of some $\ell_1(S)$, then P is projective. Hence we conclude

5.8 Theorem. A Banach space P is projective if and only if P is dually projective.

5.9 Remark. We can define a Banach space P to be ℓ_1-projective if for every $\ell_1(S)$, every closed subspace X_o of $\ell_1(S)$ and every bounded linear transformation T from P to $\ell_1(S)/X_o$, there exists a bounded linear transformation $\tilde{T}$ from P to $\ell_1(S)$ such that $Q\tilde{T} = T$ where Q is the quotient map from $\ell_1(S)$ onto $\ell_1(S)/X_o$. It is clear that the argument that was used to establish Theorem 5.8 also proves that P is ℓ_1-projective if and only if P is projective. Similarly we can define a Banach space I to be ℓ_∞-injective if for every $\ell_\infty(S)$, every closed subspace Y of $\ell_\infty(S)$, and every bounded linear transformation T from Y into I, there exists a bounded linear transformation $\tilde{T}$ from $\ell_\infty(S)$ into I which extends T. It is clear again that the argument that was used to establish Theorem 5.4 also proves that I is injective if and only if I is ℓ_∞-injective. Finally we can define a Banach space I to be C(S)-injective if for every Banach space X which is congruent to a space C(S) of all continuous scalar valued functions defined on a non-empty compact Hausdorff space S, every closed subspace Y of X, and every bounded linear transformation T from Y into I, there exists a bounded linear transformation $\tilde{T}$ from X into I which extends T. Again the argument that we used to establish Theorem 5.4 can be used to prove that I is injective if and only if I is C(S)-injective although in this case the following details ought to be mentioned. Lemma 5.2 goes through as before with the hypothesis that X is a Banach space congruent to a C(S) space and I is a C(S)-injective Banach space. Now if we put the discrete topology on an arbitrary set S, then S becomes a completely regular topological space and $\ell_\infty(S)$ is nothing but the set of all bounded continuous scalar valued functions on S. So $\ell_\infty(S)$ is congruent to C(βS) where βS denotes the Stone-Čech compactification of S.[1] Then Lemma 5.3 goes through with the hypothesis that I is a C(S)-injective Banach space and so we obtain the conclusion that a C(S)-injective

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[1] See Day [7], Kelley [20], or Simmons [35].

Banach space is congruent to a closed subspace with a closed complement of some $\ell_\infty(S)$ and is therefore injective.

5.10 <u>Definition</u>. A Banach space P is said to be injectively projective if given any injective Banach space I, any closed subspace I_o of I, and any bounded linear transformation T from P to I/I_o, there exists a bounded linear transformation $\tilde{T}$ from P to I such that $Q\tilde{T} = T$ where Q denotes the quotient map from I onto I/I_o.

It is clear that a projective Banach space is injectively projective. The following theorem establishes the converse.

5.11 <u>Theorem</u>. If a Banach space P is injectively projective, then it is projective.

<u>Proof</u>. Let X be any Banach space, X_o any closed subspace of X, T a bounded linear transformation from P to X/X_o, and Q the quotient map from X onto X/X_o. By Lemma 2.6 there exists an isometry φ from X onto a closed subspace, say X', of some $\ell_\infty(S)$. In particular, X is congruent to a closed subspace of an injective Banach space. Let $X_o' = \varphi(X_o)$. X_o' is also a closed subspace of $\ell_\infty(S)$. Indeed X_o' is closed in X' since $X_o' = X' \cap X_o'$.

Let $X_o + x \in X/X_o$ $(x \in X)$. Define $\psi : X/X_o \longrightarrow X'/X_o'$ by $\psi(X_o + x) = X_o' + \varphi(x)$. To show that ψ is well defined, suppose $X_o + x = X_o + x_1$. Then $x - x_1 \in X_o$. So $\varphi(x - x_1) \in X_o'$, i.e. $\varphi(x) - \varphi(x_1) \in X_o'$ which means that $X_o' + \varphi(x) = X_o' + \varphi(x_1)$. So ψ is well defined. ψ is linear. For consider two elements $X_o + x_1$ and $X_o + x_2$ of X/X_o. Then $\psi(X_o + x_1 + X_o + x_2)$ $= \psi(X_o + x_1 + x_2) = X_o' + \varphi(x_1 + x_2) = X_o' + \varphi(x_1) + \varphi(x_2)$ $= X_o' + \varphi(x_1) + X_o' + \varphi(x_2) = \psi(X_o + x_1) + \psi(X_o + x_2)$. Also if α is a scalar, $\psi(\alpha(X_o + x_1)) = \psi(X_o + \alpha x_1)$ $= X_o' + \varphi(\alpha x_1) = X_o' + \alpha\varphi(x_1) = \alpha(X_o' + \varphi(x_1))$ $= \alpha\psi(X_o + x_1)$. So ψ is linear. ψ is bounded. For

$$\|\psi(X_o + x)\|_{X'/X_o'} = \|X_o' + \varphi(x)\|_{X'/X_o'}$$

$$= \inf_{x_o' \in X_o'} \{\|x_o' + \varphi(x)\|_{X'}\}$$

$$= \inf_{x_o \in X_o} \{\|\varphi(x_o) + \varphi(x)\|_{X'}\} \text{ (because } \varphi : X_o \longrightarrow X_o' \text{ is onto)}$$

$$= \inf_{x_o \in X_o} \{\|\varphi(x_o + x)\|_{X'}\}$$

$$= \inf_{x_o \in X_o} \{\|x_o + x\|_X\} \text{ (because } \varphi \text{ is an isometry)} = \|X_o + x\|_{X/X_o}.$$

So ψ is an isometry (and hence bounded).[1]

Now $X'/X_o' \subset \ell_\infty(S)/X_o'$. Let $i : X'/X_o' \longrightarrow \ell_\infty(S)/X_o'$ be the identity mapping. We have the following situation.

(5.1)
$$\begin{array}{ccc} & & P \\ & & \big\downarrow T \\ & & X/X_o \\ & & \big\downarrow \psi \\ & & X'/X_o' \\ & & \big\downarrow i \\ \ell_\infty(S) & \xrightarrow[Q_I]{} & \ell_\infty(S)/X_o' \end{array}$$

where Q_I denotes the quotient map from $\ell_\infty(S)$ onto $\ell_\infty(S)/X_o'$.

Since P is injectively projective, there exists a bounded linear transformation $T_1 : P \longrightarrow \ell_\infty(S)$ which lifts $i\psi T$, that is, $Q_I T_1 = i\psi T$. We claim that $T_1(P) \subset X'$. For consider $T_1(p)$, $p \in P$. Now $i\psi T(p) \in X'/X_o'$,

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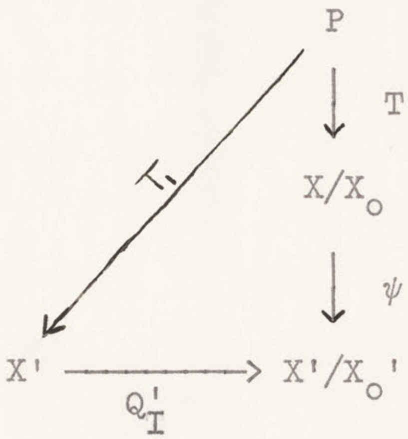
[1] Actually X/X_o and X'/X_o' are congruent under the mapping ψ, but we won't use this fact.

say $i\psi T(p) = X_o' + x'$. But $Q_I(T_1(p)) = i\psi T(p)$. So $X_o' + T_1(p) = X_o' + x'$. So $T_1(p) - x' \in X_o'$ and hence $T_1(p) = x' + x_o' \in X'$ where x_o' denotes some element in X_o'. So $T_1(P) \subset X'$. Let Q_I' denote the restriction of Q_I to X'. So we have the following commutative diagram, i.e. $Q_I'T_1 = \psi T$.

(5.2)



Let $\tilde{T} = \varphi^{-1}T_1 : P \longrightarrow X$. We claim that $\tilde{T}$ lifts T. For let $p \in P$. We want to show that $Q\tilde{T}(p) = T(p)$. Let $T(p) = X_o + x_p$, $x_p \in X$. We must show that $X_o + \tilde{T}(p) = X_o + x_p$ or in other words that $\tilde{T}(p) - x_p \in X_o$. But $\tilde{T}(p) - x_p \in X_o$ if and only if $\varphi(\tilde{T}(p) - x_p) \in X_o'$. Now $\varphi\tilde{T}(p) - \varphi(x_p) = \varphi\varphi^{-1}T_1(p) - \varphi(x_p) = T_1(p) - \varphi(x_p)$. So it suffices to show that $T_1(p) - \varphi(x_p) \in X_o'$ or equivalently that $X_o' + T_1(p) = X_o' + \varphi(x_p)$. But by diagram (5.2), $X_o' + T_1(p) = \psi T(p) = \psi(X_o + x_p)$ $= X_o' + \varphi(x_p)$. So $\tilde{T}$ lifts T. Q.E.D.

5.12 Remark. In the proof of the preceding theorem, we did not make use of the fact that $\ell_\infty(S)$ is injective, that is to say, we did not have to extend any bounded linear transformations. All we used $\ell_\infty(S)$ for was to arrange matters so that we could arrive at a diagram (5.1) which enabled us to invoke the hypothesis that P was injectively projective. We can define a Banach space P to be ℓ_∞-projective if for every $\ell_\infty(S)$, every closed subspace X_o of $\ell_\infty(S)$, and every bounded linear transformation T from P into $\ell_\infty(S)/X_o$, there exists a bounded linear transformation $\tilde{T}$ from P to $\ell_\infty(S)$ such that $Q\tilde{T} = T$ where Q denotes the quotient map from $\ell_\infty(S)$ onto $\ell_\infty(S)/X_o$. It is clear from the proof of Theorem 5.11 that P is ℓ_∞-projective if and only if P is projective. Similarly we can define a Banach space P to be C(S)-projective if for every Banach space X which is congruent to a space C(S) of all continuous scalar valued functions defined on a non-empty compact Hausdorff space S, every closed subspace X_o of X, and every bounded linear transformation T from P into X/X_o, there exists a bounded linear transformation $\tilde{T}$ from P to X such that $Q\tilde{T} = T$

where Q denotes the quotient map from X onto X/X_o. If we observe as in Remark 5.9 that $\ell_\infty(S)$ is congruent to $C(\beta S)$ where βS denotes the Stone-Čech compactification of S with the discrete topology, the proof of Theorem 5.11 yields the result that P is projective if and only if P is $C(S)$-projective.

5.13 Definition. A Banach space I is said to be projectively injective if given any projective Banach space P, any closed subspace Y of P, and any bounded linear transformation T from Y into I, there exists a bounded linear transformation $\tilde{T}$ from P into I which extends T.

It is clear that every injective Banach space is projectively injective. The following theorem establishes the converse.

5.14 Theorem. If a Banach space I is projectively injective, then it is injective.

Proof. Let X be any Banach space, Y any closed subspace of X, and T a bounded linear transformation from Y into I. By Lemma 2.2 there exist a set S and a bounded linear transformation g from $\ell_1(S)$ onto X. Let M be the kernel of g and let $Y_1 = g^{-1}(Y)$. M and Y_1 are both closed subspaces of $\ell_1(S)$ and $M \subset Y_1$ since $g(M) = \{0\} \subset Y$. Let g_1 be the restriction of g to Y_1 and define a map φ from Y_1 to I by $\varphi = Tg_1$. φ is bounded and linear. Since I is projectively injective, there exists a bounded linear transformation $\tilde{\varphi}$ from $\ell_1(S)$ to I which extends φ. We now want to define a map $\tilde{T}$ from X into I which extends T and we proceed as follows. Let $x \in X$. Choose any $p \in \ell_1(S)$ such that $g(p) = x$. Define $\tilde{T}(x) = \tilde{\varphi}(p)$. $\tilde{T}$ is well defined. For suppose $g(p_1) = g(p_2) = x$. We must show that $\tilde{\varphi}(p_1) = \tilde{\varphi}(p_2)$. Now $g(p_1) = g(p_2)$ implies $g(p_1 - p_2) = 0$ which implies that $p_1 - p_2 \in M$. But M is a subset of the kernel of $\tilde{\varphi}$ since for $m \in M$ we have $\tilde{\varphi}(m) = \varphi(m) = Tg(m)$ $= T(0) = 0$. So $\tilde{\varphi}(p_1 - p_2) = 0$ or $\tilde{\varphi}(p_1) = \tilde{\varphi}(p_2)$. So $\tilde{T}$ is well defined. $\tilde{T}$ is linear. For let x_1 and x_2 be in X. Let $p_1, p_2 \in \ell_1(S)$ be such that $g(p_1) = x_1$ and $g(p_2) = x_2$. Then $g(p_1 + p_2) = x_1 + x_2$ and so $\tilde{T}(x_1 + x_2) = \tilde{\varphi}(p_1 + p_2) = \tilde{\varphi}(p_1) + \tilde{\varphi}(p_2)$ $= \tilde{T}(x_1) + \tilde{T}(x_2)$. Similarly if α is a scalar, $g(\alpha p_1) = \alpha g(p_1) = \alpha x_1$ and so $\tilde{T}(\alpha x_1) = \tilde{\varphi}(\alpha p_1)$ $= \alpha\tilde{\varphi}(p_1) = \alpha\tilde{T}(x_1)$. So $\tilde{T}$ is linear. To show that $\tilde{T}$ is bounded, we observe the following:

By Lemma 2.2, each x in X has at least one pre-image p in $\ell_1(S)$ with $\|x\|_X = \|p\|_{\ell_1(S)}$. We have also shown that $\tilde{T}(x)$ does not depend on which pre-image of x we choose. Now we want to show that there exists a constant K such that for all $x \in X$, we have $\|\tilde{T}x\| \leq K\|x\|$. So given an $x \in X$, choose $p \in \ell_1(S)$ such that $g(p) = x$ and such that $\|p\| = \|x\|$. Then $\|\tilde{T}(x)\| = \|\varphi(p)\| \leq \|\tilde{\varphi}\| \|p\| = \|\tilde{\varphi}\| \|x\|$. So we may take K to be $\|\tilde{\varphi}\|$. So $\tilde{T}$ is bounded. Finally it remains to be shown that $\tilde{T}$ extends T. Let $y \in Y$. Then $\tilde{T}(y) = \tilde{\varphi}(p)$ where $g(p) = y$. $y \in Y$ implies that $p \in Y_1$ and so $\tilde{\varphi}(p) = \varphi(p) = Tg_1(p) = Tg(p) = T(y)$. So $\tilde{T}$ extends T. Q.E.D.

5.15 <u>Remark</u>. We can define a Banach space I to be ℓ_1-injective if for every $\ell_1(S)$, every closed subspace Y of $\ell_1(S)$, and every bounded linear transformation T from Y to I, there exists a bounded linear transformation $\tilde{T}$ from $\ell_1(S)$ to I which extends T. It is clear from the proof of Theorem 5.14 that I is injective if and only if it is ℓ_1-injective.

5.16 <u>Remark</u>. We can obtain the analogues of Theorems 5.11 and 5.14 very easily. More explicitly, we define a Banach space P to be projectively projective if given any projective Banach space X, any closed subspace X_0 of X, any bounded linear transformation T from P into X/X_0, there exists a bounded linear transformation $\tilde{T}$ from P to X such that $Q\tilde{T} = T$ where Q denotes the quotient map from X onto X/X_0. Then clearly a projectively projective Banach space is ℓ_1-projective and hence by Remark 5.9 is also projective. Similarly we can define a Banach space I to be injectively injective if given any injective Banach space X, any closed subspace Y of X, and any bounded linear transformation T from Y to I, there exists a bounded linear transformation $\tilde{T}$ from X to I which extends T. Then clearly an injectively injective Banach space is ℓ_∞-injectiveand hence is injective by Remark 5.9.

To summarize, in this chapter we have shown the following:

(a) A Banach space is injective if and only if it is dually injective if and only if it is ℓ_∞-injective if and only if it is $C(S)$-injective if and only if it is projectively injective if and only if it is injectively injective if and only if it is ℓ_1-injective.

(b) A Banach space is projective if and only if it is dually projective if and only if it ℓ_1-projective if and only if it is injectively projective if and only if it is projectively projective if and only if it is ℓ_∞-projective if and only if it is $C(S)$-projective.

CHAPTER VI

Some Alternative Definitions of Injective and Projective Banach Spaces

In this chapter we shall show that projectivity and injectivity are each equivalent to certain alternative conditions that one might impose on a Banach space. Unlike the conditions in the preceding chapter, these alternative conditions are not a weakening of the conditions of Proposition 1.4. Using one of the alternative conditions for projectivity, we shall obtain a result on the **lifting of** linear functionals.

6.1 Proposition. The following three conditions on a Banach space P are equivalent:

(1) P is projective

(2) For every Banach space X and every bounded linear transformation T from X onto P, there exists a bounded linear transformation T_1 from P to X such that TT_1 is the identity map on P.

(3) For every Banach space X and every bounded linear transformation T_1 from X onto P and for every Banach space Y and every bounded linear transformation T from Y to P, there exists a bounded linear transformation $\tilde{T}$ from Y to X such that $T_1\tilde{T} = T$.

<u>Proof</u>. Let 1_P denote the identity map on P.

(1) => (2) We have the following situation:

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow 1_P & & \\ X & \xrightarrow[T]{} & P & \longrightarrow & 0 \quad \text{(exact)} \end{array}$$

Since P is projective, there exists a bounded linear transformation $\tilde{1}_P : P \longrightarrow X$ such that $T\tilde{1}_P = 1_P$. We can take T_1 to be $\tilde{1}_P$.

(2) => (3) We have the following situation:

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow T & & \\ X & \xrightarrow[T_1]{} & P & \longrightarrow & 0 \quad \text{(exact)} \end{array}$$

By (2) there exists a bounded linear transformation $T_2 : P \longrightarrow X$ such that $T_1T_2 = 1_P$.

Define $\tilde{T} : Y \longrightarrow X$ by $\tilde{T} = T_2T$. $\tilde{T}$ is bounded and linear, and for $y \in Y$, $T_1\tilde{T}(y) = T_1T_2T(y) = 1_PT(y) = T(y)$.

(3) => (2) We have the following situation:

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow 1_P & & \\ X & \xrightarrow[T]{} & P & \longrightarrow & 0 \quad \text{(exact)} \end{array}$$

By (3) there exists a bounded linear transformation $\tilde{1}_P$ from P to X such that $T\tilde{1}_P = 1_P$. We can take T_1 to be $\tilde{1}_P$.

(2) => (1) Let X and Y be arbitrary Banach spaces, T_1 a bounded linear transformation from X onto Y, and T a bounded linear transformation from P into Y. We want to lift T. By Lemma 2.2, there exists a non-empty set S and a bounded linear transformation T_2 from $\ell_1(S)$ onto P. Since $\ell_1(S)$ is projective, there exists a bounded linear transformation $T_3 : \ell_1(S) \longrightarrow X$ such that $T_1T_3 = TT_2$. By (2) there exists a bounded linear transformation $T_4 : P \longrightarrow \ell_1(S)$ such that $T_2T_4 = 1_P$. Define $\tilde{T} : P \longrightarrow X$ by $\tilde{T} = T_3T_4$. $\tilde{T}$ is bounded and linear and lifts T since for $p \in P$ we have $T_1\tilde{T}(p) = T_1T_3T_4(p) = TT_2T_4(p) = T1_P(p) = T(p)$. Q.E.D.

6.2 <u>Corollary</u>. If X and Y are arbitrary Banach spaces, f a continuous linear functional on Y, and g a non-zero continuous linear functional on X, then there exists a bounded linear transformation $\tilde{f} : Y \longrightarrow X$ such that $g\tilde{f} = f$.

<u>Proof</u>. Since g is non-zero, it is onto the

scalar field. The field of scalars is projective (see the proof of Theorem 1.7) and so by condition (3) of the proposition, there exists a bounded linear transformation $\tilde{f} : Y \longrightarrow X$ such that $g\tilde{f} = f$. Q.E.D.

So in addition to knowing that we can always extend continuous linear functionals (Hahn-Banach theorem), the preceding corollary tells us that we can also lift them.

6.3 Definition. A Banach space X is said to have property $\mathcal{P}$ if for every Banach space Y which contains X as a closed subspace, there exists a bounded projection T from Y onto X.

6.4 Lemma. If a Banach space X has property $\mathcal{P}$, and X is congruent to a Banach space Y, then Y has property $\mathcal{P}$.

Proof. Let $\varphi : X \longrightarrow Y$ be a mapping establishing the congruence between X and Y. Let W be a Banach space containing Y as a closed subspace. If $W = Y$, the identity mapping on W is a bounded projection from W onto Y. So assume that $W \neq Y$. Let Y' denote the set theoretic complement of Y with respect to W. Let Y_1 be a set with the same cardinality as Y' and disjoint from X.[1] Let ψ be a one-one mapping from Y_1 onto Y'. Let $Z = X \cup Y_1$. Define $\tilde{\varphi}: Z \longrightarrow W$ by $\tilde{\varphi}(x) = \varphi(x)$ if $x \in X$ and $\tilde{\varphi}(y_1) = \psi(y_1)$ if $y_1 \in Y_1$. $\tilde{\varphi}$ is a one-one mapping from Z onto W. We make Z into a vector space by defining addition and scalar multiplication as follows. If z_1 and z_2 are elements of Z, let $w = \tilde{\varphi}(z_1) + \tilde{\varphi}(z_2) \in W$. There exists a unique $z \in Z$ such that $\tilde{\varphi}(z) = w$. Define $z_1 + z_2$ to be that z. If α is a scalar and $z \in Z$, we define αz to be that unique element $z' \in Z$ such that $\tilde{\varphi}(z') = \alpha\tilde{\varphi}(z)$. We observe that if x_1 and x_2 are in X, their sum when they are considered as elements of Z is the same as their sum when they are considered as elements of our given Banach space X. For $x_1 + x_2$ (in Z) is that element $z \in Z$ such that

$\tilde{\varphi}(z) = \tilde{\varphi}(x_1) + \tilde{\varphi}(x_2) = \varphi(x_1) + \varphi(x_2) = \varphi(x_1 + x_2)$

since φ is linear. (The last plus sign refers to

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[1] If X and Y' are disjoint, we can take Y' to be Y_1. If X and Y' are not disjoint, we construct a set Y_1 as follows: If no element of X is an ordered pair, we let $Y_1 = \{(y,1) \mid y \in Y'\}$. If some elements of X are ordered pairs, let α denote an element which is not the second member of any of the ordered pairs in X and let $Y_1 = \{(y,\alpha) \mid y \in Y'\}$.

our original addition in X.) So $x_1 + x_2$ (in Z) = $x_1 + x_2$ (in X) since $x_1 + x_2$ (in X) has the property that $\tilde{\varphi}$ maps it into $\varphi(x_1) + \varphi(x_2)$. Similarly for $x \in X$ and α a scalar, our definition of αx in Z agrees with scalar multiplication in X.

We proceed to verify that Z with addition and scalar multiplication so defined is a vector space. Addition in Z is commutative. For let $z_1, z_2 \in Z$. Now $z_1 + z_2$ is that element $z_3 \in Z$ such that $\tilde{\varphi}(z_3) = \tilde{\varphi}(z_1) + \tilde{\varphi}(z_2)$ and $z_2 + z_1$ is that element $z_4 \in Z$ such that $\tilde{\varphi}(z_4) = \tilde{\varphi}(z_2) + \tilde{\varphi}(z_1)$. But $\tilde{\varphi}(z_1) + \tilde{\varphi}(z_2) = \tilde{\varphi}(z_2) + \tilde{\varphi}(z_1)$ and so $z_3 = z_4$. Addition in Z is associative. For let z_1, z_2, and z_3 be elements of Z and consider $(z_1 + z_2) + z_3$ and $z_1 + (z_2 + z_3)$. Now $(z_1 + z_2) + z_3$ is that element z_4 of Z such that $\tilde{\varphi}(z_4) = \tilde{\varphi}(z_1 + z_2) + \tilde{\varphi}(z_3) = (\tilde{\varphi}(z_1) + \tilde{\varphi}(z_2)) + \tilde{\varphi}(z_3)$ (by definition of addition in Z). $z_1 + (z_2 + z_3)$ is that element z_5 of Z such that $\tilde{\varphi}(z_5) = \tilde{\varphi}(z_1) + \tilde{\varphi}(z_2 + z_3) = \tilde{\varphi}(z_1) + (\tilde{\varphi}(z_2) + \tilde{\varphi}(z_3))$. Addition in W is associative and so

$\tilde{\varphi}(z_1) + (\tilde{\varphi}(z_2) + \tilde{\varphi}(z_3)) = (\tilde{\varphi}(z_1) + \tilde{\varphi}(z_2)) + \tilde{\varphi}(z_3)$ from which we conclude the associativity of addition in Z. The zero element of X, 0_X, has the property that $0_X + z = z$ for every $z \in Z$. For $0_X + z$ is that element z_1 in Z such that $\tilde{\varphi}(z_1) = \tilde{\varphi}(0_X) + \tilde{\varphi}(z) = \varphi(0_X) + \tilde{\varphi}(z) =$ (since φ is linear) $0_Y + \tilde{\varphi}(z)$ (where 0_Y denotes the zero element of Y) = (since Y is a subspace of W) $0_W + \tilde{\varphi}(z)$ (where 0_W denotes the zero element of W) = $\tilde{\varphi}(z)$. So $z_1 = z$ or in other words, $0_X + z = z$. So Z has a zero element $0_Z (= 0_X)$. For each $z \in Z$, $z + (-1z) = 0_X$. For $z + (-1z)$ is that element $z' \in Z$ such that $\tilde{\varphi}(z') = \tilde{\varphi}(z) + \tilde{\varphi}(-1z)$. But $-1z$ is that element $z'' \in Z$ such that $\tilde{\varphi}(z'') = -\tilde{\varphi}(z)$. So $\tilde{\varphi}(z') = \tilde{\varphi}(z) - \tilde{\varphi}(z) = 0_W = 0_Y$. So $z' = 0_X$. For each $z \in Z$, $1z = z$ since $1z$ is that element z_1 in Z such that $\tilde{\varphi}(z_1) = 1\tilde{\varphi}(z) = \tilde{\varphi}(z)$. For each $z \in Z$ and each pair of scalars α and β, we have $(\alpha + \beta)z = \alpha z + \beta z$. For $(\alpha + \beta)z$ is that element z_1 in Z such that $\tilde{\varphi}(z_1) = (\alpha + \beta)\tilde{\varphi}(z) = \alpha\tilde{\varphi}(z) + \beta\tilde{\varphi}(z) = \tilde{\varphi}(\alpha z) + \tilde{\varphi}(\beta z)$ while $\alpha z + \beta z$ is that element z_2 in Z such that $\tilde{\varphi}(z_2) = \tilde{\varphi}(\alpha z) + \tilde{\varphi}(\beta z)$. If z_1 and z_2 are elements of Z and α is a scalar, $\alpha(z_1 + z_2) = \alpha z_1 + \alpha z_2$. For $\alpha(z_1 + z_2)$ is that element z_3 of Z such that $\tilde{\varphi}(z_3) = \alpha\tilde{\varphi}(z_1 + z_2) = \alpha(\tilde{\varphi}(z_1) + \tilde{\varphi}(z_2))$ while $\alpha z_1 + \alpha z_2$ is that element z_4 of Z such that $\tilde{\varphi}(z_4) = \tilde{\varphi}(\alpha z_1) + \tilde{\varphi}(\alpha z_2) = \alpha\tilde{\varphi}(z_1) + \alpha\tilde{\varphi}(z_2) = \alpha(\tilde{\varphi}(z_1) + \tilde{\varphi}(z_2))$. Finally if α and β are scalars and $z \in Z$, $\alpha(\beta z) = (\alpha\beta)(z)$. For $\alpha(\beta z)$ is that element z_1 in Z such that $\tilde{\varphi}(z_1) = \alpha(\tilde{\varphi}(\beta z)) = \alpha(\beta\tilde{\varphi}(z)) = (\alpha\beta)(\tilde{\varphi}(z))$ while $(\alpha\beta)(z)$ is that element z_2 in Z such that $\tilde{\varphi}(z_2) = (\alpha\beta)(\tilde{\varphi}(z))$. So Z is a vector space. It is easily seen that X is a linear subspace of Z and that $\tilde{\varphi} : Z \longrightarrow W$ is a linear transformation by the very definition of addition and scalar multiplication in Z.

We define a norm on Z as follows: $\|z\|_Z = \|\tilde{\varphi}(z)\|_W$. We observe that if $x \in X$, $\|x\|_Z = \|x\|_X$ since $\|x\|_Z = \|\tilde{\varphi}(x)\|_W = \|\varphi(x)\|_W = \|\varphi(x)\|_Y = \|x\|_X$ since φ is an isometry. This "norm" on Z does indeed satisfy the reqirements of a norm. For each $z \in Z$, $\|z\|_Z \geq 0$ and

if $\|z\|_Z = 0$, then $\|\tilde{\varphi}(z)\|_W = 0$ which implies that $\tilde{\varphi}(z) = 0_W$ which implies that $z = 0_Z$ since $\tilde{\varphi}$ is linear and one-one. Clearly $\|0_Z\|_Z = 0$.
$\|\alpha z\|_Z = \|\tilde{\varphi}(\alpha z)\|_W = \|\alpha\tilde{\varphi}(z)\|_W = |\alpha|\ \|\tilde{\varphi}(z)\|_W =$ $|\alpha|\ \|z\|_Z$. If $z_1, z_2 \in Z$, $\|z_1 + z_2\|_Z = \|\tilde{\varphi}(z_1 + z_2)\|_W =$ $\|\tilde{\varphi}(z_1) + \tilde{\varphi}(z_2)\|_W \leq \|\tilde{\varphi}(z_1)\|_W + \|\tilde{\varphi}(z_2)\|_W = \|z_1\|_Z + \|z_2\|_Z$. So Z is a normed linear space and $\tilde{\varphi}$ is an isometry from Z onto W. Z is a Banach space. For suppose $\{z_n\}_{n=1,2,\ldots}$ is a Cauchy sequence in Z. Then the sequence $\{\tilde{\varphi}(z_n)\}_{n=1,2,\ldots}$ is Cauchy and since W is complete, there exists a $w \in W$ such that $\lim_{n \to \infty} \tilde{\varphi}(z_n) = w$. Let $z \in Z$ be such that $\tilde{\varphi}(z) = w$. Then $\lim_{n \to \infty} z_n = z$ since $\|z - z_n\| = \|\tilde{\varphi}(z - z_n)\| = \|\tilde{\varphi}(z) - \tilde{\varphi}(z_n)\| =$ $\|w - \tilde{\varphi}(z_n)\| \longrightarrow 0$. So Z is complete. Finally X is a closed subspace of Z. For let $x_n \in X$, $n = 1, 2, \ldots$ and suppose $\lim_{n \to \infty} x_n = z \in Z$. We want to show that $z \in X$. Now the sequence $\{x_n\}_{n=1,2,\ldots}$ is Cauchy since it converges. Because X is complete, there exists an element $x \in X$ such that $\lim_{n \to \infty} x_n = x$. But $\lim_{n \to \infty} x_n = z$. So $z = x \in X$. So X is closed in Z.

So we have managed to arrange matters so that our hypothesis that X has property $\mathcal{P}$ is applicable. There exists a bounded projection T from Z onto X. Define a mapping $T' : W \longrightarrow Y$ by $T' = \tilde{\varphi} T \tilde{\varphi}^{-1}$. T' is bounded and linear and $T'T' = T'$ since

$$T'T' = \tilde{\varphi} T \tilde{\varphi}^{-1} \tilde{\varphi} T \tilde{\varphi}^{-1} = \tilde{\varphi} TT \tilde{\varphi}^{-1} = \tilde{\varphi} T \tilde{\varphi}^{-1} = T'.$$

Finally T' maps W onto Y since $T'(y) = y$ for each $y \in Y$ since $T'(y) = \tilde{\varphi}(T(\tilde{\varphi}^{-1}(y))) = \tilde{\varphi}(\tilde{\varphi}^{-1}(y))$ (since $\tilde{\varphi}^{-1}(y) \in X$ and T restricted to X is the identity mapping) $= y$. So we have succeeded in showing that there exists a bounded projection from W onto Y for any Banach space W which contains Y as a closed subspace. So Y has property $\mathcal{P}$. Q.E.D.

6.5 Remark. If the projection T from Z onto X in the proof of the preceding lemma is such that $\|T\| \leq \lambda$ where $1 \leq \lambda < \infty$, then it is seen easily from the definition of the projection T' from W onto Y that $\|T'\| \leq \lambda$ also.

6.6 Proposition. The following four conditions on a Banach space I are equivalent:

(1) I is injective

(2) For every Banach space X which contains I as a closed subspace, for every Banach space Y and for every bounded linear transformation T from I into Y, there exists a bounded linear transformation $\tilde{T}$

from X to Y which extends T.

(3) I has property $\mathcal{P}$.

(4) For every Banach space X and every isometry T from I onto a closed subspace of X, there exists a bounded linear transformation T_1 from X onto I such that T_1T is the identity mapping on I.

<u>Proof</u>. Let i denote the identity mapping on I.

(2) => (3). Let X be any Banach space which contains I as a closed subspace. We have the following situation:

$$\begin{array}{ccc} X & & \\ \cup & & \\ I & \xrightarrow[i]{} & I \end{array}$$

If we let our space I be the space Y of (2) and i be the map T of (2), there exists by (2) a bounded linear transformation $\tilde{i}$ from X to I which extends i. Indeed $\tilde{i}$ maps X onto I since i does and hence is a bounded projection from X onto I.

(3) => (2) By (3) there exists a bounded projection, say P, from X onto I. Define a map $\tilde{T}$ from X to Y by $\tilde{T} = TP$. $\tilde{T}$ is bounded and linear and extends T.

(1) => (3) Let X be any Banach space which contains I as a closed subspace. We have the following situation:

$$\begin{array}{ccc} X & & \\ \cup & & \\ I & \xrightarrow[i]{} & I \end{array}$$

Because I is injective, there exists a bounded linear transformation $\tilde{i}$ from X to I which extends i. Indeed $\tilde{i}$ maps X onto I since i does and hence is a bounded projection from X onto I.

(3) => (1) Let X be a closed subspace of a Banach space Y and let $T : X \longrightarrow I$ be a bounded linear transformation. We want to extend T to Y. By Lemma 2.6 I is congruent to a closed subspace, say I_1, of some $\ell_\infty(S)$. Let $T_1 : X \longrightarrow \ell_\infty(S)$ be defined by $T_1 = \varphi T$ where $\varphi : I \longrightarrow \ell_\infty(S)$ is the map establishing the congruence between I and I_1. Since $\ell_\infty(S)$ is injective, T_1 extends to $\tilde{T}_1 : Y \longrightarrow \ell_\infty(S)$. Now since I has property $\mathcal{P}$, so does I_1 by Lemma 6.4 and so there exists a bounded projection ψ from $\ell_\infty(S)$ onto I_1. Define $\tilde{T} : Y \longrightarrow I$ by $\tilde{T} = \varphi^{-1}\psi\tilde{T}_1$. $\tilde{T}$ is bounded and linear and extends T.

(1) => (4). Let $X_1 = T(I)$. X_1 is closed in X and hence is a Banach space. So $T^{-1} : X_1 \longrightarrow I$ is bounded. So there exists a bounded linear transformation

$T_1 : X \longrightarrow I$ which extends T^{-1}. T_1 is onto I since T^{-1} is and $T_1 T$ is the identity mapping on I since for $y \in I$, $T_1(T(y)) = T^{-1}T(y) = y$.

(4) => (3). Let X be any Banach space containing I as a closed subspace. If we take the map T of (4) to be i, there exists a bounded linear transformation T_1 from X onto I such that $T_1 i = i$. Take $\psi : X \longrightarrow I$ to be T_1. Then ψ is a bounded projection from X onto I since for
$x \in X$, $\psi(\psi(x)) = T_1(T_1(x)) = T_1(i(T_1(x))) = i(T_1(x)) =$
$T_1(x) = \psi(x)$. So I has property $\mathcal{P}$. Q.E.D.

6.7 Remark. If a Banach space I is a member of the class $In(\lambda)$ (and hence injective), it is easy to see from the proof of Proposition 6.6 that the bounded projection of (3) has norm less than or equal to λ. If we define the class $\mathcal{P}_\lambda$ as consisting of those Banach spaces X that have property $\mathcal{P}$ together with the requirement that the bounded projections from the superspaces Y containing X can always be chosen with norms less than or equal to λ, then the proof of Proposition 6.6, together with the fact that $\ell_\infty(S) \in In(1)$ and Remark 6.5, shows that if $X \in \mathcal{P}_\lambda$, then $X \in In(\lambda)$. So the classes $In(\lambda)$ and $\mathcal{P}_\lambda$ are identical and in particular Theorem 3.2 implies that if a Banach space X has property $\mathcal{P}$, then X is a member of the class $\mathcal{P}_\lambda$ for some finite λ.

Similarly $I \in In(\lambda)$ implies that $\tilde{T}$ in (2) can be chosen so that $\|\tilde{T}\| \leq \lambda\|T\|$. Conversely if we require that $\tilde{T}$ in (2) can be chosen so that $\|\tilde{T}\| \leq \lambda\|T\|$, it follows that $I \in In(\lambda)$. Finally, $I \in In(\lambda)$ implies that the map T_1 in (4) can always be chosen so that $\|T_1\| \leq \lambda$, and conversely if we require that the map T_1 in (4) can be chosen such that $\|T_1\| \leq \lambda$, it follows that $I \in In(\lambda)$.

CHAPTER VII

Geometric Properties of Injective Banach Spaces

In this chapter we shall prove a necessary geometric condition for a real Banach space to be injective. A special case of this condition will be shown to be sufficient. We then give a geometric proof that real $\ell_\infty(S)$ is injective and finally as an application of our necessary condition, we prove a theorem about real Banach spaces which are dual spaces and which belong to a certain class of injectives.

We first note the following fact. In Proposition 6.6 we showed that injectivity for a Banach space is equivalent to the Banach space's having property $\mathcal{P}$. In our formulation of property $\mathcal{P}$, we required that the superspaces containing our given Banach space be Banach spaces also. Actually the superspaces need not be complete. More precisely, let a Banach space X have property $\mathcal{P}$. Then X also has the property that for each normed linear space Y containing X as a closed subspace, there exists a bounded projection T from Y onto X. For let $\hat{Y} \supset Y$ be the completion of Y, (If Y is already complete, there is nothing to prove.) and let T be a bounded projection from $\hat{Y}$ onto X. Let T_1 be the restriction of T to Y. Then T_1 is of course bounded, linear, and $T_1^2 = T_1$. T_1 maps Y onto X since $T_1(X) = X$. We shall need this fact later in this chapter when we show that a certain Banach space X is not injective by constructing a normed linear space Y containing X as a closed subspace onto which there exists no bounded projection from Y.

If x_o is a point in a normed linear space X and r is a non-negative real number, $S(x_o, r)$ will denote the closed sphere in X with center x_o and radius r, i.e. $S(X_o, r) = \{x \in X \mid \|x - x_o\| \leq r\}$. Occasionally we may use the notation $S_X(x_o, r)$ to emphasize that the sphere we are dealing with is in X. Whenever we use the word "sphere" in this chapter, we shall mean <u>closed</u> sphere (possibly with radius zero) unless we specify otherwise.

Our first lemma will be used throughout this section.

7.1 <u>Lemma</u>. Let X be a normed linear space and let $S(x_1, r_1)$ and $S(x_2, r_2)$ be two spheres in X. Then $S(x_1, r_1) \cap S(x_2, r_2) \neq \phi$ if and only if $\|x_1 - x_2\| \leq r_1 + r_2$, i.e. the distance between their centers does not exceed the sum of their radii.

Proof. (=>). Assume that $S(x_1, r_1) \cap S(x_2, r_2) \neq \phi$. Let $y \in S(x_1, r_1) \cap S(x_2, r_2)$. Then $\|x_1 - x_2\| \leq \|x_1 - y\| + \|y - x_2\| \leq r_1 + r_2$.

(<=). Now assume that $\|x_1 - x_2\| \leq r_1 + r_2$. If $\|x_1 - x_2\| = 0$, then $x_1 = x_2 \in S(x_1, r_1) \cap S(x_2, r_2)$. So we may assume that $\|x_1 - x_2\| > 0$. Now our hypothesis that $\|x_1 - x_2\| \leq r_1 + r_2$ implies that $\|x_1 - x_2\| - r_1 \leq r_2$ which implies that

$$\frac{\|x_1 - x_2\| - r_1}{\|x_1 - x_2\|} \leq \frac{r_2}{\|x_1 - x_2\|} \tag{7.1}$$

Choose any real number λ such that

$$\frac{\|x_1 - x_2\| - r_1}{\|x_1 - x_2\|} \leq \lambda \leq \frac{r_2}{\|x_1 - x_2\|} \tag{7.2}$$

and such that $0 \leq \lambda \leq 1$. It is clear that we can always choose a λ satisfying (7.2) but that we can also choose it so that $0 \leq \lambda \leq 1$ perhaps requires some discussion. First of all, if $r_2 = 0$, the right hand side of (7.2) is zero and so we can choose $\lambda = 0$. If $r_2 > 0$, then the right hand side of (7.2) is positive. We note that the left hand side of (7.2) is ≤ 1, for if $\dfrac{\|x_1 - x_2\| - r_1}{\|x_1 - x_2\|} > 1$, then

$\|x_1 - x_2\| - r_1 > \|x_1 - x_2\|$ which implies that $r_1 < 0$

which is impossible. So we have in this case

$\dfrac{\|x_1 - x_2\| - r_1}{\|x_1 - x_2\|} \leq 1$ and $\dfrac{r_2}{\|x_1 - x_2\|} > 0$ in addition to

(7.1) and clearly we can choose a λ satisfying (7.2)
and such that $0 \leq \lambda \leq 1$.

So we have $\dfrac{\|x_1 - x_2\| - r_1}{\|x_1 - x_2\|} \leq \lambda$ which implies

that $(1 - \lambda)\|x_1 - x_2\| \leq r_1$ and $\lambda \leq \dfrac{r_2}{\|x_1 - x_2\|}$ which

implies that $\lambda\|x_1 - x_2\| \leq r_2$. Let $z = \lambda x_1 + (1 - \lambda)x_2$.
Then $\|x_1 - z\| = \|x_1 - \lambda x_1 - (1 - \lambda)x_2\| =$
$\|(1 - \lambda)x_1 - (1 - \lambda)x_2\| = (1 - \lambda)\|x_1 - x_2\| \leq r_1$ which
means that $z \in S(x_1, r_1)$. Also we have $\|x_2 - z\| =$
$\|x_2 - \lambda x_1 - (1 - \lambda)x_2\| = \|- \lambda x_1 + \lambda x_2\| = \lambda\|x_2 - x_1\| \leq r_2$
which means that $z \in S(x_2, r_2)$. So
$z \in S(x_1, r_1) \cap S(x_2, r_2)$. Q.E.D.

7.2 <u>Definition</u>. Let X be a normed linear space. Let Y be a normed linear space containing X as a normed linear subspace and having the property that if Z is a linear subspace of Y containing X and such that $Z \neq X$, then $Z = Y$. Then Y is called an immediate extension of X.

Our next two lemmas will be needed for the first theorem in this section.

7.3 <u>Lemma</u>. Let ρ be a real valued function on a normed linear space X and assume that ρ satisfies the following four conditions:

(I) $\rho(x) + \rho(y) \geq \|x - y\|$ for all $x, y \in X$

(II) $\rho(x) - \rho(y) \leq \|x - y\|$ for all $x, y \in X$

(III) $\rho(\lambda x + (1 - \lambda)y) \leq \lambda\rho(x) + (1 - \lambda)\rho(y)$ for all $x, y \in X$ and $0 \leq \lambda \leq 1$.

(IV) $\rho(x) > 0$ for all $x \in X$.

Then there exists an immediate extension Y of X and a point ζ in Y but not in X such that $\rho(x) = \|x - \zeta\|_Y$ for all $x \in X$.

<u>Proof</u>. Let $Y_1 = \{(x, \alpha) \mid x \in X, \alpha \in \text{scalars}\}$. If we define addition of two elements (x_1, α_1) and (x_2, α_2) in Y_1 by $(x_1, \alpha_1) + (x_2, \alpha_2) = (x_1 + x_2, \alpha_1 + \alpha_2)$ and multiplication of an element

$(x, \alpha) \in Y_1$ by a scalar β by $\beta(x, \alpha) = (\beta x, \beta\alpha)$, it is easy to see that Y_1 becomes a vector space. The set $X_1 = \{(x_1, \alpha) \in Y_1 \mid \alpha = 0\}$ is easily seen to be a linear subspace of Y_1. Suppose $X_1 \subset Z_1 \subset Y_1$, where Z_1 is a linear subspace of Y_1 and $X_1 \neq Z_1$. Then there exists an element $(x_o, \alpha_o) \in Z_1$ such that $\alpha_o \neq 0$. If (x, α) is an arbitrary element in Y_1, then $(x, \alpha) \in Z_1$. For if $\alpha = 0$, then $(x, \alpha) \in X_1 \subset Z_1$ and if $\alpha \neq 0$, then

$$(x, \alpha) = \frac{\alpha}{\alpha_o}\left(\left(\frac{\alpha_o}{\alpha} x - x_o, 0\right) + (x_o, \alpha_o)\right) \in Z_1.$$

Let $T : X \longrightarrow X_1$ be defined by $T(x) = (x, 0)$. It is clear that T is a one-one linear transformation from X onto X_1. Let X_1' denote the set theoretic complement of X_1 with respect to Y_1, i.e. $X_1' = \{(x, \alpha) \in Y_1 \mid \alpha \neq 0\}$. If X and X_1' are disjoint, let $Y = X \cup X_1'$. If $X \cap X_1' \neq \phi$, let A be a set with the same cardinality as X_1' and disjoint from X and let $Y = X \cup A$. As in the proof of Lemma 6.4, we define the operations of addition and multiplication by scalars on Y in such a way that X with its original addition and scalar multiplication becomes a linear subspace of Y and there a one-one linear transformation $\tilde{T}$ from Y onto Y_1 such that the restriction of $\tilde{T}$ to X is T. Suppose $X \subset Z \subset Y$, where Z is a linear subspace of Y and $Z \neq X$. Then we claim that $Z = Y$. For choose an element $z \in Z$, but not in X. Let $\tilde{T}(Z) = Z_1$. Then $X_1 \subset Z_1 \subset Y_1$ and $X_1 \neq Z_1$ since $\tilde{T}(z) \in Z_1$ and $\tilde{T}(z) \notin X_1$ for if $\tilde{T}(z) = (x, 0) \in X_1$, we also have $(x, 0) = T(x) = \tilde{T}(x)$ and so $z = x \in X$ (since $\tilde{T}$ is one-one) which is impossible since $z \notin X$. $X_1 \subset Z_1 \subset Y_1$ and $X_1 \neq Z_1$ imply $Z_1 = Y_1$ which implies that $Z = Y$. For let $y \in Y$ and consider $\tilde{T}(y) = z_1 \in Y_1$. But $z_1 = \tilde{T}(z')$ for some $z' \in Z$. So $y = z' \in Z$ since $\tilde{T}$ is one-one. So $Z = Y$.

So we have succeeded in showing that there exists a linear space Y (not as yet normed) containing X as a linear subspace with the property that if $X \subset Z \subset Y$, where Z is a linear subspace of Y and $X \neq Z$, then $Z = Y$. Choose an element $\zeta \in Y$ but not in X. Every element $y \in Y$ can be represented as $y = x + \lambda\zeta$ for some $x \in X$ and some scalar λ. For the elements of the form $x + \lambda\zeta$ constitute a linear subspace Z of

Y and $Z \neq X$. Also the x and λ are unique for if $x_1 + \lambda_1\zeta = x_2 + \lambda_2\zeta$, then $x_1 - x_2 = (\lambda_2 - \lambda_1)\zeta$. If $\lambda_1 \neq \lambda_2$, then $\frac{1}{\lambda_2 - \lambda_1}(x_1 - x_2) = \zeta$. But $\frac{1}{\lambda_2 - \lambda_1}(x_1 - x_2) \in X$ and $\zeta \notin X$. So $\lambda_1 = \lambda_2$ from which $x_1 = x_2$ follows immediately. We define a function μ on Y by

$$\mu(y) = \|x\|_X \quad \text{if} \quad \lambda = 0$$

$$\mu(y) = |\lambda|\ \rho(\tfrac{-x}{\lambda}) \quad \text{if} \quad \lambda \neq 0.$$

We shall show that μ is a norm on Y. Certainly $\mu(y) \geq 0$ for all $y \in Y$ and $\mu(0) = 0$. If $y = x + \lambda\zeta \neq 0$, then either $\lambda \neq 0$ in which case $\mu(y) = |\lambda|\rho(\frac{-x}{\lambda}) > 0$ by property (IV), or else $\lambda = 0$ and $x \neq 0$ in which case $\mu(y) = \|x\|_X > 0$. So we have shown for all $y \in Y$ that $\mu(y) \geq 0$ and $\mu(y) = 0$ if and only if $y = 0$. To show that $\mu(\alpha y) = |\alpha|\mu(y)$ for all $y \in Y$ and all scalars α, we first note that if $\alpha = 0$, then $\mu(\alpha y) = \mu(0) = 0 = 0\mu(y) = |\alpha|\mu(y)$. If $\alpha \neq 0$ and $\lambda = 0$, then $\mu(\alpha y) = \mu(\alpha x) = \|\alpha x\|_X = |\alpha|\ \|x\|_X = |\alpha|\mu(y)$. If $\alpha \neq 0$ and $\lambda \neq 0$, then $\mu(\alpha y) = \mu(\alpha x + \alpha\lambda\zeta) = |\alpha\lambda|\rho(\frac{-\alpha x}{\alpha\lambda}) = |\alpha|\ |\lambda|\rho(\frac{-x}{\lambda}) = |\alpha|\mu(y)$.

So for all scalars α and for all $y \in Y$, we have $\mu(\alpha y) = |\alpha|\mu(y)$. There remains for us to prove that μ satisfies the triangle inequality. Let $y, w \in Y$. We break this part of the proof into various cases.

Case (a). y and w both belong to X. Then $\mu(y + w) = \|y + w\|_X \leq \|y\|_X + \|w\|_X = \mu(y) = \mu(w)$.

Case (b). One and only one of the vectors y and w belongs to X, say $w \in X$ and $y \notin X$. Let $y = x + \lambda\zeta$ where $\lambda \neq 0$, $x \in X$. We want to show that

$$(7.3) \qquad |\lambda|\rho(-\frac{x + w}{\lambda}) \leq |\lambda|\rho(\frac{-x}{\lambda}) + \|w\|_X .$$

If in property (II) we replace x and y by $-\frac{x + w}{\lambda}$ and $-\frac{x}{\lambda}$ respectively, we obtain

$$\rho(-\frac{x + w}{\lambda}) - \rho(-\frac{x}{\lambda}) \leq \|-\frac{x + w}{\lambda} + \frac{x}{\lambda}\|_X = \|-\frac{w}{\lambda}\|_X$$

from which (7.3) follows immediately.

Case (c). $y \notin X$ and $w \notin X$. Let $y = x + \lambda\zeta$ and $w = u + \tau\zeta$ where $x, u \in X$ and $\lambda \neq 0$, $\tau \neq 0$. We break case (c) into the following subcases.

Case (c_1). λ and τ have the same sign. If $\lambda > 0$ and $\tau > 0$, we have to prove that

$$(7.4) \qquad (\lambda + \tau)\rho(-\frac{x + u}{\lambda + \tau}) \leq \lambda\rho(\frac{-x}{\lambda}) + \tau\rho(\frac{-u}{\tau})$$

If in property (III) we replace x, y, and λ by $\frac{-x}{\lambda}$, $\frac{-u}{\tau}$, and $\frac{\lambda}{\lambda + \tau}$ respectively, we obtain

$$\rho(\frac{\lambda}{\lambda + \tau}(\frac{-x}{\lambda}) + (1 - \frac{\lambda}{\lambda + \tau})(\frac{-u}{\tau}))$$

$$\leq \frac{\lambda}{\lambda + \tau} \rho(\frac{-x}{\lambda}) + (1 - \frac{\lambda}{\lambda + \tau})\rho(\frac{-u}{\tau}),$$

i.e., $\rho(- \frac{x + u}{\lambda + \tau}) \leq \frac{\lambda}{\lambda + \tau} \rho(\frac{-x}{\lambda}) + \frac{\tau}{\lambda + \tau} \rho(\frac{-u}{\tau})$

from which (7.4) follows immediately. If $\lambda < 0$ and $\tau < 0$, we have to prove

$$(7.5) \qquad - (\lambda + \tau)\rho(- \frac{x + u}{\lambda + \tau}) \leq - \lambda\rho(\frac{-x}{\lambda}) - \tau\rho(\frac{-u}{t}).$$

Again in property (III), we replace x, y, and λ by $\frac{-x}{\lambda}$, $\frac{-u}{\tau}$, and $\frac{\lambda}{\lambda + \tau}$ respectively, and we obtain

$$(7.6) \qquad \rho(- \frac{x + u}{\lambda + \tau}) \leq \frac{\lambda}{\lambda + \tau} \rho(\frac{-x}{\lambda}) + \frac{\tau}{\lambda + \tau} \rho(\frac{-u}{\tau}),$$

and multiplying both sides of (7.6) by $- (\lambda + \tau) > 0$, we obtain (7.5).

Case (c_2) λ and τ have different signs. Without loss of generality we may assume $\lambda > 0$ and $\tau < 0$. If $\lambda + \tau \neq 0$, then either $\lambda + \tau > 0$ or $\lambda + \tau < 0$. If $\lambda + \tau > 0$, we must prove that

$$(7.7) \qquad (\lambda + \tau)\rho(- \frac{x + u}{\lambda + \tau}) \leq \lambda\rho(\frac{-x}{\lambda}) - \tau\rho(\frac{-u}{\tau}),$$

which is equivalent to

$$(7.8) \quad (\lambda + \tau)\rho(- \frac{x + u}{\lambda + \tau}) - (\lambda + \tau)\rho(\frac{-x}{\lambda}) \leq - \tau\rho(\frac{-x}{\lambda}) - \tau\rho(\frac{-u}{\tau})$$

In property (II) we replace x and y by $- \frac{x + u}{\lambda + \tau}$ and $\frac{-x}{\lambda}$ respectively, thus obtaining

$$\rho(- \frac{x + u}{\lambda + \tau}) - \rho(\frac{-x}{\lambda}) \leq \|- \frac{x + u}{\lambda + \tau} + \frac{x}{\lambda}\|_X = \|\frac{- \lambda u + \tau x}{(\lambda + \tau)\lambda}\|_X .$$

So $(\lambda + \tau)\rho(- \frac{x + u}{\lambda + \tau}) - (\lambda + \tau)\rho(\frac{-x}{\lambda}) \leq \|- u + \frac{\tau}{\lambda} x\|_X$, i.e. the left hand side of (7.8) is less than or equal to $\|- u + \frac{\tau}{\lambda} x\|_X$. On the other hand, if in property (I) we replace x and y by $\frac{-x}{\lambda}$ and $\frac{-u}{\tau}$ respectively, we obtain

$$(7.9) \qquad \rho(\frac{-x}{\lambda}) + \rho(\frac{-u}{\tau}) \geq \|\frac{-x}{\lambda} + \frac{u}{\tau}\|_X = \|\frac{- \tau x + \lambda u}{\lambda\tau}\|_X$$

and multiplying both sides of (7.9) by $- \tau > 0$, we obtain

$$- \tau\rho(\frac{-x}{\lambda}) - \tau\rho(\frac{-u}{\tau}) \geq \frac{-\tau}{-\tau} \|\frac{-\tau x}{\lambda} + u\|_X = \|- u + \frac{\tau x}{\lambda}\|_X ,$$

i.e. the right hand side of (7.8) is greater than or equal to $\|- u + \frac{\tau x}{\lambda}\|_X$. Hence (7.8) is proved.

If $\lambda + \tau < 0$ ($\lambda > 0$, $\tau < 0$ still), we must prove that

$$(7.10) \qquad - (\lambda + \tau)\rho(- \frac{x + u}{\lambda + \tau}) \leq \lambda\rho(\frac{-x}{\lambda}) - \tau\rho(\frac{-u}{\tau}),$$

which is equivalent to

$$(7.11) \qquad - (\lambda + \tau)\rho(- \frac{x + u}{\lambda + \tau}) + (\lambda + \tau)\rho(\frac{-u}{\tau})$$

$$\leq \lambda\rho(\frac{-u}{\tau}) + \lambda\rho(\frac{-x}{\lambda}).$$

If in property (II) we replace x and y by $- \frac{x + u}{\lambda + \tau}$ and $\frac{-u}{\tau}$ respectively, we obtain

$$\rho(- \frac{x + u}{\lambda + \tau}) - \rho(\frac{-u}{\tau}) \leq \|- \frac{x + u}{\lambda + \tau} + \frac{u}{\tau}\|_X = \| \frac{- \tau x + \lambda u}{(\lambda + \tau)\tau}\|_X .$$

So $- (\lambda + \tau)\rho(- \frac{x + u}{\lambda + \tau}) + (\lambda + \tau)\rho(\frac{-u}{\tau}) \leq \|- x + \frac{\lambda}{\tau} u\|_X$,

i.e. the left hand side of (7.11) is less than or equal to $\|- x + \frac{\lambda}{\tau} u\|_X$. On the other hand, if in property (I) we replace x and y by $\frac{-x}{\lambda}$ and $\frac{-u}{\tau}$ respectively, we obtain

$$(7.12) \qquad \rho(\frac{-x}{\lambda}) + \rho(\frac{-u}{\tau}) \geq \|\frac{-x}{\lambda} + \frac{u}{\tau}\|_X = \|\frac{- \tau x + \lambda u}{\lambda\tau}\|_X$$

and multiplying both sides of (7.12) by λ, we obtain

$$\lambda\rho(\frac{-x}{\lambda}) + \lambda\rho(\frac{-u}{\tau}) \geq \|- x + \frac{\lambda u}{\tau}\|_X, \quad \text{i.e.}$$

the right hand side of (7.11) is greater than or equal to $\|- x + \frac{\lambda u}{\tau}\|_X$. Hence (7.11) is established.

Finally there remains under case (c_2) the case where $\lambda + \tau = 0$, i.e. $\lambda = - \tau$ ($\lambda > 0$, $\tau < 0$ still). We have to prove that $\|x + u\|_X \leq \lambda\rho(\frac{-x}{\lambda}) + \lambda\rho(\frac{u}{\lambda})$. If in property (I) we replace x and y by $\frac{-x}{\lambda}$ and $\frac{u}{\lambda}$ respectively, we obtain

$$\rho(\frac{-x}{\lambda}) + \rho(\frac{u}{\lambda}) \geq \|- \frac{x}{\lambda} - \frac{u}{\lambda}\| = \frac{1}{\lambda}\|x + u\|_X$$

from which the desired inequality follows immediately.

So we have shown that μ is a norm on Y. Since $\mu(x) = \|x\|_X$ if $x \in X$, X is a <u>normed</u> linear subspace of Y and by what we have already shown, it follows that Y is an immediate extension of X. Finally if $x \in X$, $\mu(x - \zeta)(= \|x - \zeta\|_Y) = |- 1|\rho(\frac{-x}{-1}) = \rho(x)$. Q.E.D.

7.4 <u>Lemma</u>. Let r be a real valued function defined on a normed linear space X such that $r(x) + r(y) \geq \|x - y\|$ for all $x, y \in X$. Then there exists a real valued function ρ on X satisfying conditions (I), (II), and (III) of Lemma 7.3 and such that $r(x) \geq \rho(x)$ for all $x \in X$.

Proof. Let Δ be the set of all real valued functions f defined on X, satisfying the inequality $f(x) + f(y) \geq \|x - y\|$ for all $x, y \in X$, and such that $r(x) \geq f(x)$ for all $x \in X$. Δ is not empty since $r \in \Delta$. We define an order relation $\prec$ on Δ by $f \prec g$ if and only if $f(x) \geq g(x)$ for all $x \in X$. It is clear that under the relation $\prec$, Δ is a partially ordered set. Let $\Gamma = \{f_i\}_{i \in I}$ be a non-empty totally ordered subset of Δ. We shall show that Γ has an upper bound φ in Δ. First we note that if $f \in \Delta$, then $f(x) \geq 0$ for all $x \in X$ since $2f(x) = f(x) + f(x) \geq \|x - x\| = 0$. Now for each $x \in X$, define $\varphi(x) = \inf_{i \in I} \{f_i(x)\}$. We note that $\varphi(x)$ is finite since $f_i(x) \geq 0$ for each $i \in I$. To show that $\varphi \in \Delta$, let $x, y \in X$ and let $\epsilon > 0$. There exists a function $f_1 \in \Gamma$ such that $f_1(x) \leq \varphi(x) + \epsilon$ and also a function $f_2 \in \Gamma$ such that $f_2(y) \leq \varphi(y) + \epsilon$. Since Γ is totally ordered, we have either $f_1 \prec f_2$ or $f_2 \prec f_1$, that is $f_1(z) \geq f_2(z)$ or $f_2(z) \geq f_1(z)$ respectively for all $z \in X$. If $f_2 \prec f_1$, then we have

$$\|x - y\| \leq f_1(x) + f_1(y) \leq \varphi(x) + \epsilon + f_2(y)$$

$$\leq \varphi(x) + \epsilon + \varphi(y) + \epsilon = \varphi(x) + \varphi(y) + 2\epsilon.$$

If $f_1 \prec f_2$, then we have $\|x - y\| \leq f_2(x) + f_2(y) \leq f_1(x) + \varphi(y) + \epsilon \leq \varphi(x) + \varphi(y) + 2\epsilon$. In either case since ϵ was arbitrary, we conclude that $\|x - y\| \leq \varphi(x) + \varphi(y)$. Also for each $z \in X$, $\varphi(z) \leq f_i(z) \leq r(z)$ for all $i \in I$. So $\varphi \in \Delta$ and clearly $f_i \prec \varphi$ for each $f_i \in \Gamma$. So Γ has an upper bound in Δ and hence by Zorn's lemma, Δ has a maximal element ρ.

Now ρ of course satisfies condition (I) since $\rho \in \Delta$. To show that ρ satisfies condition (II), let x and y be fixed (but arbitrary) elements of X. Now for any $t \in X$, we have

$$\rho(t) + \rho(y) \geq \|t - y\| = \|(t - x) - (- x + y)\|$$

$$\geq \|t - x\| - \|x - y\|.$$

So $\|x - y\| + \rho(y) \geq \|t - x\| - \rho(t)$ from which we conclude that

$$\|x - y\| + \rho(y) \geq \sup_{t \in X} \{\|t - x\| - \rho(t)\}. \tag{7.13}$$

In particular, $\sup_{t \in X} \{\|t - x\| - \rho(t)\} < \infty$. Let h be a non-negative real number such that $h \geq \sup_{t \in X} \{\|t - x\| - \rho(t)\}$. Define a function ψ on X by

$$\psi(t) = \begin{cases} h & \text{if } t = x \\ \rho(t) & \text{if } t \neq x. \end{cases}$$

We claim that $\psi(t) + \psi(z) \geq \|t - z\|$ for all t, $z \in X$, or equivalently $\psi(t) \geq \|t - z\| - \psi(z)$. We break the proof of this claim into various cases.

Case 1. $t = x$ and $z = x$. We must show that $h \geq \|x - x\| - h$, i.e. that $h \geq - h$. But clearly $h \geq - h$ since $h \geq 0$.

Case 2. $t = x$ and $z \neq x$. We must show that $h \geq \|x - z\| - \rho(z)$. But this inequality follows immediately from the fact that $h \geq \sup_{t \in X} \{\|t - x\| - \rho(t)\}$.

Case 3. $t \neq x$ and $z = x$. We must prove that $\rho(t) \geq \|t - x\| - h$ or equivalently that $h \geq \|t - x\| - \rho(t)$. But this last inequality follows again from the fact that $h \geq \sup_{t \in X} \{\|t - x\| - \rho(t)\}$.

Case 4. $t \neq x$ and $z \neq x$. We must show that $\rho(t) \geq \|t - z\| - \rho(z)$ or equivalently that

$\rho(t) + \rho(z) \geq \|t - z\|$. But this last inequality is true because $\rho \in \Delta$.

So $\psi(t) + \psi(z) \geq \|t - z\|$ for all t, $z \in X$.

Moreover, $h \geq \rho(x)$. For suppose $h < \rho(x)$. Then for all $z \in X$, we have $\psi(z) \leq \rho(z)$ since for $z \neq x$, $\psi(z) = \rho(z) \leq \rho(z)$ and for $z = x$ we have $\psi(x) = h < \rho(x)$. Since $\rho(z) \leq r(z)$ for all $z \in X$, we have then that $\psi \in \Delta$, $\rho \prec \psi$ and $\rho \neq \psi$. But ρ is a maximal element in Δ and hence we cannot have $\rho \prec \psi$. $\rho \neq \psi$. So we must conclude that $h \geq \rho(x)$.

So we have shown that a non-negative real number h satisfying $h \geq \sup_{t \in X} \{\|t - x\| - \rho(t)\}$ also satisfies the inequality $h \geq \rho(x)$. In particular if we take h to be $\|x - y\| + \rho(y)$, then (7.13) tells us that $h \geq \sup_{t \in X} \{\|t - x\| - \rho(t)\}$ and since $\rho(y) \geq 0$ (since $\rho \in \Delta$), we also have $h \geq 0$. So $h \geq \rho(x)$, i.e. $\|x - y\| + \rho(y) \geq \rho(x)$, i.e. $\|x - y\| \geq \rho(x) - \rho(y)$. So ρ satisfies condition (II).

Finally to show that ρ satisfies condition (III), let $0 \leq \lambda \leq 1$. Let $z = \lambda x + (1 - \lambda)y$. Now for any $t \in X$, we have

$$\begin{aligned} \|t - z\| &= \|\lambda(t - x) + (1 - \lambda)(t - y)\| \\ &\leq \lambda\|t - x\| + (1 - \lambda)\|t - y\| \\ &\leq \lambda(\rho(t) + \rho(x)) + (1 - \lambda)(\rho(t) + \rho(y)). \end{aligned}$$

So $\|t - z\| - \rho(t) \leq \lambda\rho(x) + (1 - \lambda)\rho(y)$ and so

$$\sup_{t \in X} \{\|t - z\| - \rho(t)\} \leq \lambda\rho(x) + (1 - \lambda)\rho(y). \tag{7.14}$$

If we take $h = \lambda\rho(x) + (1 - \lambda)\rho(y)$, then (7.14) together with the fact that $h \geq 0$ (since $0 \leq \lambda \leq 1$ and $\rho(w) \geq 0$ for all $w \in X$) allows us to conclude that $h \geq \rho(z)$, i.e. $\lambda\rho(x) + (1 - \lambda)\rho(y) \geq \rho(\lambda x + (1 - \lambda)y)$. So ρ satisfies (III). Q.E.D.

7.5 Definition. Let X be a normed linear space and let $1 \leq \lambda < \infty$. A non-empty collection $\mathcal{C}$ of closed spheres in X is said to have the λ-intersection property if for every non-empty subcollection $\mathcal{C}_o = \{S(x_i, r_i)\}_{i \in I}$ of $\mathcal{C}$ with the property that every two spheres of $\mathcal{C}_o$ have a non-empty intersection, $\bigcap_{i \in I} S(x_i, \lambda r_i) \neq \phi$.

7.6 Theorem. Let the real Banach space X be a member of $\text{In}(\lambda)$. Then the collection of all closed spheres in X has the λ-intersection property.

Proof. Suppose our theorem is false. Then there exists a non-empty collection $\mathcal{C}_o = \{S(x_i, r_i)\}_{i \in I}$ of closed spheres in X with the property that any two spheres in $\mathcal{C}_o$ have a non-empty intersection, and such that $\bigcap_{i \in I} S(x_i, \lambda r_i) = \phi$. Note that $x_i = x_j$, $r_i \neq r_j$ is a possibility for $i, j \in I$, $i \neq j$. In other words two distinct spheres in $\mathcal{C}_o$ may have the same center. Now let $A = \{x \in X \mid x = x_i$ for at least one $i \in I\}$. For each $x \in A$, let $r(x)$ denote the greatest lower bound of the radii of all the spheres in $\mathcal{C}_o$ that have x as their center. Then for all $x, y \in A$, we have $r(x) + r(y) \geq \|x - y\|$. For suppose not. Then there exist points x and y in A such that $r(x) + r(y) < \|x - y\|$. Let $\epsilon = \|x - y\| - r(x) - r(y) > 0$. There exist a sphere $S(x_i, r_i)$, $i \in I$ with $x_i = x$ and radius $r_i < r(x) + \frac{\epsilon}{4}$ and a sphere $S(x_j, r_j)$, $j \in I$ with $x_j = y$ and radius $r_j < r(y) + \frac{\epsilon}{4}$. Since $\mathcal{C}_o$ has the property that any two spheres in $\mathcal{C}_o$ have a non-empty intersection, we have $S(x_i, r_i) \cap S(x_j, r_j) \neq \phi$. So the distance between the centers of $S(x, r_i)$ and $S(y, r_j)$ must be less than or equal to the sum of their radii. So we have $\|x - y\| \leq r_i + r_j < r(x) + r(y) + \frac{\epsilon}{2}$. So $\epsilon = \|x - y\| - r(x) - r(y) < \frac{\epsilon}{2}$ which is impossible. So for all $x, y \in A$ we have the inequality $r(x) + r(y) \geq \|x - y\|$.

Let $\mathcal{C}_1 = \{S(x, r(x)) \mid x \in A\}$. By the inequality just established, every two spheres in $\mathcal{C}_1$ have a non-empty intersection. We claim that $\bigcap_{x \in A} S(x, \lambda r(x)) = \phi$. For suppose not. Let $z \in \bigcap_{x \in A} S(x, \lambda r(x))$. Then $\|z - x\| \leq \lambda r(x)$ for each $x \in A$. But this implies that $z \in \bigcap_{i \in I} S(x_i, \lambda r_i)$ since for an arbitrary sphere $S(x_i, r_i)$, $i \in I$ $(x_i \in A)$, we have $r(x_i) \leq r_i$ (by definition of $r(x_i)$) and hence $\|z - x_i\| \leq \lambda r(x_i) \leq \lambda r_i$. Since $\bigcap_{i \in I} S(x_i, \lambda r_i) = \phi$, we must conclude that $\bigcap_{x \in A} S(x, \lambda r(x)) = \phi$.

Now distinct members of $\mathcal{C}_1$ have distinct centers. If A is a proper subset of X, choose a point $\xi \in A$ and define a set $\mathcal{C}_2$ of closed spheres in X by $\mathcal{C}_2 = \{S(x, \|x - \xi\| + r(\xi)) \mid x \in A'\}$, where A' denotes the complement of A with respect to X. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Since for each $x \in A'$, the sphere with center x and radius $\|x - \xi\| + r(\xi)$ contains the sphere with center ξ and radius $r(\xi)$, it follows that the collection $\mathcal{C}$ has the property that any two spheres in $\mathcal{C}$ have a non-empty intersection, but the intersection

$$E = (\bigcap_{x \in A} S(x, \lambda r(x))) \cap (\bigcap_{x \in A'} S(x, \lambda(\|x - \xi\| + r(\xi))))$$

is empty since $\bigcap_{x \in A} S(x, \lambda r(x)) = \phi$. Every point $x \in X$ is the center of one and only one member of $\mathcal{C}$. We extend the domain of definition of the function r to all of X by defining $r(x)$ for $x \in A'$ to be the radius of the sphere in $\mathcal{C}$ with center at x, i.e. $r(x) = \|x - \xi\| + r(\xi)$. Since every two spheres in $\mathcal{C}$ have a non-empty intersection the following inequality holds for all points $x, y \in X$: $r(x) + r(y) \geq \|x - y\|$. Moreover for any given $w \in X$, the inequality $\lambda r(x) \geq \|x - w\|$ cannot hold for all $x \in X$, for otherwise we would have $w \in E$.

By Lemma 7.4 there exists a real valued function ρ defined on X satisfying conditions (I), (II), and (III) of Lemma 7.3 and such that $r(x) \geq \rho(x)$ for all $x \in X$. It is then clear that for any given $w \in X$, the inequality $\rho(x) \geq \|x - w\|$ cannot hold for all $x \in X$. For if it did, then we would have $\lambda r(x) \geq r(x) \geq \rho(x) \geq \|x - w\|$ for all $x \in X$ which we have just shown is impossible. It follows that $\rho(x) > 0$ for all $x \in X$. For suppose $\rho(w) \leq 0$ for some $w \in X$.

Then for all $x \in X$ we have $\rho(x) \geq \rho(x) + \rho(w) \geq \|x - w\|$ (since ρ satisfies condition(I)) and so $\rho(x) \geq \|x - w\|$ which we know is impossible. So $\rho(x) > 0$ for all $x \in X$, i.e. ρ satisfies condition (IV) of Lemma 7.3 also. By Lemma 7.3 there exists an immediate extension Y of X and a point ζ in Y but not in X such that $\rho(x) = \|x - \zeta\|$ for all $x \in X$.

We claim that there exists no projection P from Y onto X such that $\|P\| \leq \lambda$. For suppose there did exist such a projection P. Let $w = P(\zeta) \in X$. Then $\|P(y) - w\| = \|P(P(y) - \zeta)\| \leq \|P\| \, \|P(y) - \zeta\| \leq \lambda\|P(y) - \zeta\|$ for all $y \in Y$. Since every $x \in X$ is equal to $P(y)$ for some $y \in Y$, we would then have $\|x - w\| \leq \lambda\|x - \zeta\| = \lambda\rho(x)$ for all $x \in X$ and hence $\|x - w\| \leq \lambda\rho(x) \leq \lambda r(x)$ which we know cannot hold for all $x \in X$.

So there does not exist any projection P from Y onto X such that $\|P\| \leq \lambda$. So X is not a member of $In(\lambda)$ (see Proposition 6.6 and Remark 6.7) and thus we have arrived at a contradiction. Hence we must conclude that the collection of all closed spheres in X has the λ-intersection property. Q.E.D.

7.7 Remark. The preceding theorem is false if X is a complex injective Banach space as the following example illustrates. Let S be a set consisting of only one point, say a, and let X be complex $\ell_\infty(S) \in In(1)$. Then the set of all closed spheres in X does not have the 1-intersection property. To see this we observe that X consists of all functions f from $\{a\}$ to the complexes such that

$\|f\|_\infty = \sup_{s \in S} \{|f(s)|\} = |f(a)| < \infty$. In other words X

can be regarded as the set of all complex numbers with the understanding that a complex number z represents the complex valued function defined on $S = \{a\}$ whose value (at a) is z and the norm of this function is $|z|$, the ordinary absolute value of z. So the closed spheres in X are "round" and it is easy to construct a family of closed spheres, any two of which have a non-empty intersection, but such that the intersection of all the spheres in the family is empty. For example, consider the family consisting of the following three spheres: $S_1 = S(0, 1)$, $S_2 = S(2, 1)$, and $S_3 = S(1 - i\sqrt{3}, 1)$. Then $1 \in S_1 \cap S_2$, $\frac{1}{2} - i\,\frac{\sqrt{3}}{2} \in S_1 \cap S_3$, and $\frac{3}{2} - i\,\frac{\sqrt{3}}{2} \in S_2 \cap S_3$. Indeed $S_1 \cap S_2 = \{1\}$. For suppose a function f in X is in $S_1 \cap S_2$. Letting $f(a) = x + iy$, we have $|x + iy| \leq 1$ and $|x + iy - 2| \leq 1$. If $|x| > 1$, then $|x + iy| > 1$

which is impossible. If $|x| < 1$, then either $0 \leq x < 1$, or $-1 < x < 0$. If $0 \leq x < 1$, then $x - 2 < -1$ which means that $|x - 2| > 1$ which means that $|x + iy - 2| > 1$ which is impossible. If $-1 < x < 0$, then $-1 < -x$ and so $1 < 2 - x$ which again implies that $|x + iy - 2| > 1$. So $x = 1$. If $y \neq 0$, then $|1 + iy| > 1$ which is impossible. So f must be the function whose value at a is 1. It is clear now that $S_1 \cap S_2 \cap S_3 = \phi$ since 1 is not a member of S_3.

For $\lambda = 1$, we can prove the converse of Theorem 7.6, namely,

7.8 <u>Theorem</u>. Let I be a Banach space such that the collection of all closed spheres in I has the 1-intersection property. Then $I \in In(1)$.

<u>Proof</u>. Let X be a Banach space, Y a closed subspace of X, and T a bounded linear transformation from Y to I. Let Ω denote the set of all ordered pairs (W, T_W) where W is a linear subspace (not necessarily closed) of X which contains Y, T_W is a bounded linear transformation from W to I such that $T_W(y) = T(y)$ for all $y \in Y$ and such that $\|T_W\| = \|T\|$. Ω is not empty since it contains the pair (Y, T). We define an order relation $\prec$ on Ω by

$(W_1, T_{W_1}) \prec (W_2, T_{W_2})$ if and only if $W_1 \subset W_2$ and

$T_{W_2}(w) = T_{W_1}(w)$ for all $w \in W_1$. It is easy to see that Ω is a partially ordered set under the relation $\prec$. Let $\Gamma = \{(W_j, T_{W_j})\}_{j \in J}$ be a non-empty totally ordered subset of Ω. We shall show that Ω contains an upper bound for Γ. Let $Z = \bigcup_{j \in J} W_j$. If z_1 and z_2 are in Z, then $z_1 \in W_{j_1}$ for some $j_1 \in J$ and $z_2 \in W_{j_2}$ for some $j_2 \in J$. Since Γ is totally ordered, either $W_{j_1} \subset W_{j_2}$ or $W_{j_2} \subset W_{j_1}$. So z_1 and z_2 are both in W_{j_2} or both in W_{j_1} and so $z_1 + z_2 \in W_{j_2}$ or $z_1 + z_2 \in W_{j_1}$. Hence $z_1 + z_2 \in Z$. Similarly if $z \in Z$ and α is a scalar, $z \in W_j$ for some $j \in J$ and so $\alpha z \in W_j$ and hence $\alpha z \in Z$. So Z is a linear subspace of X and clearly $W_j \subset Z$ for each $j \in J$. Define a mapping $T_Z : Z \longrightarrow I$ as follows. If $z \in Z$, then $z \in W_j$ for some $j \in J$ and we define $T_Z(z)$ to be $T_{W_j}(z)$. T_Z is well defined for if z is also in $W_{j'}$, $j' \in J$, then either $(W_j, T_{W_j}) \prec (W_{j'}, T_{W_{j'}})$ or $(W_{j'}, T_{W_{j'}}) \prec (W_j, T_{W_j})$. In either case we have

$T_{W_j}(z) = T_{W_{j'}}(z)$ and so T_Z is well defined. T_Z is linear for if $z_1, z_2 \in Z$, z_1 and z_2 are both in W_j for some $j \in J$ and $T_Z(z_1 + z_2) = T_{W_j}(z_1 + z_2) = T_{W_j}(z_1) + T_{W_j}(z_2) = T_Z(z_1) + T_Z(z_2)$. Similarly if $z \in Z$ and α is a scalar, $z \in W_j$ for some $j \in J$ and so $\alpha z \in W_j$ and $T_Z(\alpha z) = T_{W_j}(\alpha z) = \alpha T_{W_j}(z) = \alpha T_Z(z)$. T_Z extends T for if $y \in Y$, $T_Z(y) = T_{W_j}(y)$ for every $j \in J$ and since each T_{W_j} extends T, we have $T_Z(y) = T(y)$. T_Z is bounded for if $z \in Z$, $z \in W_j$ for some $j \in J$ and $\|T_Z(z)\| = \|T_{W_j}(z)\| \leq \|T_{W_j}\| \, \|z\| = \|T\| \, \|z\|$ and so $\|T_Z\| \leq \|T\|$. Since T_Z extends T, we also have $\|T\| \leq \|T_Z\|$. So $\|T\| = \|T_Z\|$. So $(Z, T_Z) \in \Omega$ and clearly $(W_j, T_{W_j}) \prec (Z, T_Z)$ for each $j \in J$. So Γ has an upper bound in Ω. By Zorn's lemma, Ω contains a maximal element, say (W_M, T_{W_M}).

We claim that $W_M = X$. In order to establish this, we shall show that an arbitrary $(W, T_W) \in \Omega$ such that W is a proper subset of X cannot be a maximal member of Ω. Choose a $\zeta \in X$ such that $\zeta \notin W$. Let $U = T_W(W) \subset I$. For each $u \in U$ define

$\rho(u) = \|T\| \inf_{x \in T_W^{-1}(\{u\})} \{\|x - \zeta\|\}$. If $u_1, u_2 \in U$ and

$x_1 \in T_W^{-1}(\{u_1\})$, $x_2 \in T_W^{-1}(\{u_2\})$, then

$$(7.15) \quad \|u_1 - u_2\| = \|T_W(x_1) - T_W(x_2)\| = \|T_W(x_1 - x_2)\|$$

$$\leq \|T_W\| \, \|x_1 - x_2\| = \|T\| \, \|x_1 - x_2\|.$$

Also by the triangle inequality we have

$$(7.16) \quad \|x_1 - x_2\| \leq \|x_1 - \zeta\| + \|x_2 - \zeta\|$$

and so

$$(7.17) \quad \|u_1 - u_2\| \leq \|T\| \, \|x_1 - \zeta\| + \|T\| \, \|x_2 - \zeta\|$$

from which we conclude that

$$(7.18) \qquad \|u_1 - u_2\| \leq \rho(u_1) + \rho(u_2).$$

For each $u \in U$, let $S_u = S_I(u, \rho(u)) =$

$\{t \in I \mid \|t - u\| \leq \rho(u)\}$ and let $\mathcal{S} = \{S_u \mid u \in U\}$.

Then (7.18) says that any two spheres in $\mathcal{S}$ have a non-empty intersection. Since by hypothesis the set of all closed spheres in I has the 1-intersection property, we conclude that $\bigcap_{u \in U} S_u \neq \phi$. Choose a point

$\xi \in \bigcap_{u \in U} S_u$. Then $\|\xi - u\| \leq \rho(u)$ for each $u \in U$ and so if $x \in T_W^{-1}(\{u\})$ we have

$$(7.19) \qquad \|\xi - T_W(x)\| \leq \rho(u) \leq \|T\| \, \|x - \zeta\| .$$

Since $U = T_W(W)$, (7.19) is true for each $x \in W$. Let W_1 be the linear subspace of X generated by the set $W \cup \{\zeta\}$. W_1 consists of all elements in X of the form $w + \alpha\zeta$, where $w \in W$ and α is a scalar and it is easy to see that each element $w_1 \in W_1$ has a unique representation $w + \alpha\zeta$. Define a mapping $T_{W_1} : W_1 \longrightarrow I$ by $T_{W_1}(w + \alpha\zeta) = T_W(w) + \alpha\xi$. T_{W_1} is linear since

$T_{W_1}(w + \alpha\zeta + w' + \alpha'\zeta) = T_{W_1}(w + w' + (\alpha + \alpha')\zeta)$

$= T_W(w + w') + (\alpha + \alpha')\xi = T_W(w) + T_W(w') + \alpha\xi + \alpha'\xi$

$= T_{W_1}(w + \alpha\zeta) + T_{W_1}(w' + \alpha'\zeta)$ and $T_{W_1}(\beta(w + \alpha\zeta))$

$= T_{W_1}(\beta w + \beta\alpha\zeta) = T_W(\beta w) + (\beta\alpha)\xi = \beta(T_W(w) + \alpha\xi)$

$= \beta T_{W_1}(w + \alpha\zeta)$. Also T_{W_1} extends T_W since for $w \in W$, $T_{W_1}(w) = T_{W_1}(w + 0\zeta) = T_W(w) + 0\xi = T_W(w)$.

Finally T_{W_1} is bounded and indeed

$\|T_{W_1}\| \leq \|T\|$, i.e. $\|T_W(w) + \alpha\xi\| \leq \|T\| \, \|w + \alpha\zeta\|$ for all $w \in W$ and all scalars α. To see this, we note first that if $\alpha = 0$, we have $\|T_W(w)\| \leq \|T_W\| \, \|w\| = \|T\| \, \|w\|$. If $\alpha \neq 0$, the inequality that we must establish is equivalent to the inequality

$$(7.20) \qquad \|T_W(-\tfrac{1}{\alpha} w) - \xi\| \leq \|T\| \, \|-\tfrac{1}{\alpha} w - \zeta\|$$

which follows immediately from (7.19). Since T_{W_1} extends T_W, we also have $\|T\| = \|T_W\| \leq \|T_{W_1}\|$ and so $\|T\| = \|T_{W_1}\|$. Since T_{W_1} extends T_W and T_W extends T (since $(W, T_W) \in \Omega$), T_{W_1} extends T. So $(W_1, T_{W_1}) \in \Omega$ and we have $(W, T_W) \prec (W_1, T_{W_1})$. But $(W, T_W) \neq (W_1, T_{W_1})$ since $W \neq W_1$. Hence (W, T_W) cannot be a maximal element in Ω. So for our maximal element (W_M, T_{W_M}) in Ω we must have $X = W_M$. In other words there exists a bounded linear extension T_{W_M} of T to all of X with $\|T_{W_M}\| = \|T\|$. So $I \in In(1)$. Q.E.D.

7.9 Remark. It may not be immediately clear to the reader where the hypothesis that $\lambda = 1$ was used in the proof of the preceding theorem. Suppose instead that $\lambda > 1$. Then (7.19) would be

$$(7.19)' \qquad \|\xi - T_W(x)\| \leq \lambda\rho(u) \leq \lambda\|T\| \, \|x - \zeta\|$$

for each $x \in W$ and we would be unable to conclude that $\|T_{W_1}\| = \|T\|$. Nevertheless we may ask whether the converse of Theorem 7.6 is true for $\lambda > 1$, i.e. if the set of all closed spheres in a Banach space has the λ-intersection property, is the space a member of $In(\lambda)$? The following proposition shows that the converse is false for $\lambda > 1$.

7.10 Proposition. Let X be a Banach space. Then the set of all closed spheres in X has the $(2 + \epsilon)$-intersection property for every $\epsilon > 0$.

Proof. Let $\mathcal{S} = \{S(x_j, r_j)\}_{j \in J}$ be a non-empty family of closed spheres in X such that any two spheres in $\mathcal{S}$ have a non-empty intersection. We want to show that for any $\epsilon > 0$, we have $\bigcap_{j \in J} S(x_j, (2 + \epsilon)r_j) \neq \phi$.

We have two cases to consider, the case where $\inf_{j \in J} \{r_j\} = 0$ and the case where $\inf_{j \in J} \{r_j\} = \rho_0 > 0$.

If $\inf_{j \in J} \{r_j\} = \rho_0 > 0$, then given any $\epsilon > 0$, $\rho_0 + \epsilon\rho_0 > \rho_0$ and so there exists $j(\epsilon) \in J$ such that

$r_{j(\epsilon)} \leq \rho_o + \epsilon\rho_o$. Then for every $j \in J$, since $S(x_j, r_j)$ and $S(x_{j(\epsilon)}, r_{j(\epsilon)})$ have a non-empty intersection, the distance between their centers cannot exceed the sum of their radii, i.e. $\|x_j - x_{j(\epsilon)}\| \leq r_j + r_{j(\epsilon)}$. But $r_j + r_{j(\epsilon)} \leq r_j + \rho_o + \epsilon\rho_o \leq r_j + r_j + \epsilon r_j = (2 + \epsilon)r_j$. So $\|x_j - x_{j(\epsilon)}\| \leq (2 + \epsilon)r_j$ which means that $x_{j(\epsilon)} \in \bigcap_{j \in J} S(x_j, (2 + \epsilon)r_j)$. If $\inf_{j \in J} \{r_j\} = 0$, let $j_1, j_2, j_3, \ldots$ denote a sequence of elements in J such that $\lim_{n \to \infty} r_{j_n} = 0$. The sequence $\{x_{j_n}\}_{n=1,2,\ldots}$ is then a Cauchy sequence. For if $\delta > 0$ is given, choose a positive integer N such that for all integers $n > N$, we have $r_{j_n} < \frac{\delta}{2}$. Then we have for all integers $p, q > N$, $\|x_{j_p} - x_{j_q}\| \leq r_{j_p} + r_{j_q}$ (since every two spheres in $\mathcal{S}$ have a non-empty intersection) $< \frac{\delta}{2} + \frac{\delta}{2} = \delta$. So the sequence $\{x_{j_n}\}_{n=1,2,\ldots}$ is a Cauchy sequence. Since X is complete, $\{x_{j_n}\}_{n=1,2,\ldots}$ converges to an element $x_o \in X$. We shall show that $x_o \in \bigcap_{j \in J} S(x_j, r_j)$ and hence $x_o \in \bigcap_{j \in J} S(x_j, (2 + \epsilon)r_j)$.

Indeed given any $\epsilon_1 > 0$, let n be such that $r_{j_n} < \frac{1}{2}\epsilon_1$ and $\|x_{j_n} - x_o\| < \frac{1}{2}\epsilon_1$. Then for any $j \in J$ we have $\|x_j - x_o\| \leq \|x_j - x_{j_n}\| + \|x_{j_n} - x_o\|$ $\leq r_j + r_{j_n} + \frac{1}{2}\epsilon_1 < r_j + \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_1 = r_j + \epsilon_1$. Since ϵ_1 was arbitrary, we conclude that $x_o \in S(x_j, r_j)$ for all $j \in J$, i.e. $x_o \in \bigcap_{j \in J} S(x_j, r_j)$. Q.E.D.

If we accept the fact that there exist non-injective Banach spaces, the preceding proposition shows us that for an arbitrary $\lambda > 1$, we cannot conclude that a Banach space $X \in \text{In}(\lambda)$ if the set of all closed spheres in X has the λ-intersection property. Of course if $1 < \lambda \leq 2$, the preceding proposition does not provide us with a counterexample and we may again ask whether the converse of Theorem 7.6 is true for $1 < \lambda \leq 2$. We consider briefly some possible modifications of the proof of Theorem 7.8 to see what difficulties occur if $1 < \lambda \leq 2$. If we define Ω as in the proof of Theorem 7.8, then as already pointed out in Remark 7.9 we cannot conclude that $\|T_{W_1}\| = \|T\|$. We can conclude however that $\|T_{W_1}\| \leq \lambda\|T\|$, but this inequality does not imply that $(W_1, T_{W_1}) \in \Omega$ and hence we cannot conclude that (W, T_W) is not maximal in Ω. If we define Ω as in the proof of Theorem 7.8 with the exception that we require the norms of the extensions T_W of T to be such that $\|T_W\| \leq \lambda\|T\|$, we are still able to deduce that this new Ω has a maximal element (W_M, T_{W_M}). However we run into difficulty when we try to show that $W_M = X$. First of all we are forced to define $\rho(u)$ to be $\lambda\|T\| \inf_{x\in T_W^{-1}(\{u\})} \{\|x - \zeta\|\}$ (if we hope to make use of our hypothesis at all, i.e. if we hope to construct a class of mutually intersecting spheres) and we still deduce (7.18) i.e. $\|u_1 - u_2\| \leq \rho(u_1) + \rho(u_2)$. However (7.19) becomes

$$(7.19)'' \quad \|\xi - T_W(x)\| \leq \lambda\rho(u) \leq \lambda\cdot\lambda\|T\| \, \|x - \zeta\|$$

and when we try to show that an element $(W, T_W) \in \Omega$ such that $W \neq X$ cannot be a maximal element of Ω, we are unable to show that the pair $(W_1, T_{W_1}) \in \Omega$, i.e. all we are able to conclude is that $\|T_{W_1}\| \leq \lambda^2\|T\|$. If we try to define our transformation T_{W_1} in a manner other than that which we used in the proof of Theorem 7.8, we again run into difficulties.

Finally if we define Ω as in the proof of Theorem 7.8 with the exception that all we require of the extensions T_W is that they be bounded and linear, we run into difficulty when we try to show that Ω has a maximal element. More specifically we have difficulty when we try to show that a non-empty totally ordered subset Γ of Ω has an upper bound in Ω, i.e. we are unable to show that the linear transformation T_Z is bounded.

Perhaps it is asking too much to expect that a Banach space X be a member of $In(\lambda)$ if the set of all closed spheres in X has the λ-intersection property. Perhaps a more realistic "converse" to aim for is the following: If the set of all closed spheres in a Banach space X has the λ-intersection property, then X is a member of the class $In(f(\lambda))$ where f is some well-behaved function (and $f(1) = 1$).

We proved Theorem 7.6 for any $\lambda \geq 1$. It is interesting to note that if we accept the truth of Theorem 7.6 for the case $\lambda = 1$, the proof of the theorem for the case $\lambda > 1$ follows quite readily as we now demonstrate.

7.11 Theorem. Let the real Banach space X be a member of $In(\lambda)$, $\lambda > 1$. Then the set of all closed spheres in X has the λ-intersection property.

Proof. $X \in In(\lambda)$ implies that there exists a set S and closed subspaces Y and A of $\ell_\infty(S)$ such that $\ell_\infty(S) = Y + A$, $Y \cap A = \{0\}$ and X is congruent to Y. Let $T : X \longrightarrow Y$ be the isometry defining the congruence between X and Y. Let $\mathcal{S} = \{S(x_i, r_i)\}_{i \in I}$ be a non-empty family of closed spheres in X such that every two spheres in $\mathcal{S}$ have a non-empty intersection. We want to show that $\bigcap_{i \in I} S(x_i, \lambda r_i) \neq \phi$. Let $S_Y(Tx_i, r_i) = \{y \in Y \mid \|y - Tx_i\|_Y \leq r_i\}$. Now since T is an isometry, the sphere $S(x_i, r_i)$ in X maps onto the sphere $S_Y(Tx_i, r_i)$. For let $x \in X$ be a point in $S(x_i, r_i)$, i.e. $\|x - x_i\|_X \leq r_i$. Then $\|Tx - Tx_i\|_Y = \|T(x - x_i)\|_Y = \|x - x_i\|_X \leq r_i$ and so $Tx \in S_Y(Tx_i, r_i)$. Similarly if $y \in Y$ is such that $\|y - Tx_i\|_Y \leq r_i$, let $x \in X$ be such that $y = Tx$. Then $\|x - x_i\|_X = \|T(x - x_i)\|_Y = \|Tx - Tx_i\|_Y = \|y - Tx_i\|_Y \leq r_i$ and so $x \in S(x_i, r_i)$, i.e. every point of the sphere

$S_Y(Tx_i, r_i)$ in Y is the image of a point in the sphere $S(x_i, r_i)$ in X. So consider the family of spheres $\mathcal{S}_Y = \{S_Y(Tx_i, r_i)\}_{i \in I}$ in Y. Every two spheres in $\mathcal{S}_Y$ have a non-empty intersection. For consider any two spheres, say $S_Y(Tx_1, r_1)$ and $S_Y(Tx_2, r_2)$ in $\mathcal{S}_Y$. Choose a point $x_o \in X$ such that $x_o \in S(x_1, r_1) \cap S(x_2, r_2)$. Then $Tx_o \in S_Y(Tx_1, r_1) \cap S_Y(Tx_2, r_2)$ since $\|Tx_o - Tx_1\|_Y = \|x_o - x_1\|_X \leq r_1$ and $\|Tx_o - Tx_2\|_Y = \|x_o - x_2\|_X \leq r_2$. So the family of spheres $\mathcal{S}_Y$ in Y has the property that every two spheres in $\mathcal{S}_Y$ have a non-empty intersection.

Now clearly $\{y \in Y \mid \|y - Tx_i\|_Y \leq r_i\} \subset \{z \in \ell_\infty(S) \mid \|z - Tx_i\|_{\ell_\infty(S)} \leq r_i\}$. For each Tx_i, $i \in I$, let $S_{\ell_\infty(S)}(Tx_i, r_i)$ denote the closed sphere in $\ell_\infty(S)$ with center Tx_i and radius r_i, i.e. $S_{\ell_\infty(S)}(Tx_i, r_i) = \{z \in \ell_\infty(S) \mid \|z - Tx_i\|_{\ell_\infty(S)} \leq r_i\}$

and let $\mathcal{S}_{\ell_\infty(S)} = \{S_{\ell_\infty(S)}(Tx_i, r_i)\}_{i\in I}$. Then every two spheres in $\mathcal{S}_{\ell_\infty(S)}$ have a non-empty intersection since $S_Y(Tx_i, r_i) \subset S_{\ell_\infty(S)}(Tx_i, r_i)$ and every two spheres in $\mathcal{S}_Y$ have a non-empty intersection. Since $\ell_\infty(S) \in In(1)$, we have $\bigcap_{i\in I} S_{\ell_\infty(S)}(Tx_i, r_i) \neq \phi$ by Theorem 7.6 for the case $\lambda = 1$. Now $Y \in In(\lambda)$ by Lemma 2.7 and so there exists a bounded projection P from $\ell_\infty(S)$ onto Y with $\|P\| \leq \lambda$ (see Remark 6.7). Let $z \in \bigcap_{i\in I} S_{\ell_\infty(S)}(Tx_i, r_i)$. Then $Pz \in \bigcap_{i\in I} S_Y(Tx_i, \lambda r_i)$. For $\|Pz - Tx_i\|_Y = \|Pz - PTx_i\|_Y = \|P(z - Tx_i)\|_Y \leq \|P\| \, \|z - Tx_i\|_{\ell_\infty(S)} \leq \lambda\|z - Tx_i\|_{\ell_\infty(S)} \leq \lambda r_i$ for each $i \in I$. Finally $T^{-1}(Pz) \in \bigcap_{i\in I} S(x_i, \lambda r_i)$ since $\|T^{-1}Pz - x_i\|_X = \|T^{-1}Pz - T^{-1}Tx_i\|_X = \|T^{-1}(Pz - Tx_i)\|_X = \|Pz - Tx_i\|_Y \leq \lambda r_i$ for each $i \in I$. So $\bigcap_{i\in I} S(x_i, \lambda r_i) \neq \phi$. Q.E.D.

In chapter I we said that we were going to give in a later chapter a geometric proof of the theorem that real $\ell_\infty(S) \in In(1)$. We are now in a position to do this.

(7.12) Proof. We shall prove our theorem by showing that the collection of all closed spheres in $\ell_\infty(S)$ has the 1-intersection property. Then by Theorem 7.8 we can conclude that $\ell_\infty(S) \in In(1)$. So let $\mathcal{S} = \{S(f_i, r_i)\}_{i\in I}$ be any non-empty collection of closed spheres in real $\ell_\infty(S)$ such that every two spheres in $\mathcal{S}$ have a non-empty intersection. We want to show that $\bigcap_{i\in I} S(f_i, r_i) \neq \phi$, i.e. we want to define a bounded real valued function f on S such that $\|f - f_i\| \leq r_i$ for all $i \in I$. Let $s \in S$ be fixed and consider the set $\{f_i(s) + r_i\}_{i\in I}$ and the set $\{f_i(s) - r_i\}_{i\in I}$. We claim that $\sup_{i\in I} \{f_i(s) - r_i\} \leq \inf_{i\in I} \{f_i(s) + r_i\}$. Suppose not. Then $\inf_{i\in I} \{f_i(s) + r_i\} < \sup_{i\in I} \{f_i(s) - r_i\}$. So there exists an $i_1 \in I$ such that $\inf_{i\in I} \{f_i(s) + r_i\} \leq f_{i_1}(s) + r_{i_1} < \sup_{i\in I} \{f_i(s) - r_i\}$. Also there exists an $i_2 \in I$ such that $\inf_{i \in I} \{f_i(s) + r_i\} \leq f_{i_1}(s) + r_{i_1} <$

$f_{i_2}(s) - r_{i_2} \leq \sup_{i \in I} \{f_i(s) - r_i\}$. So

$r_{i_1} + r_{i_2} < f_{i_2}(s) - f_{i_1}(s) \leq |f_{i_2}(s) - f_{i_1}(s)| \leq$

$\|f_{i_2} - f_{i_1}\|_{\ell_\infty(S)}$. So the sum of the radii of the

spheres $S(f_{i_1}, r_{i_1})$ and $S(f_{i_2}, r_{i_2})$ is less than

the distance between their centers. Hence

$S(f_{i_1}, r_{i_1}) \cap S(f_{i_2}, r_{i_2}) = \phi$ which contradicts a

property of $\mathcal{S}$. So we must have

$\sup_{i \in I} \{f_i(s) - r_i\} \leq \inf_{i \in I} \{f_i(s) + r_i\}$. Notice that both

sides of this last inequality are finite since for any

$i_3 \in I$ we have $\inf_{i \in I} \{f_i(s) + r_i\} \leq f_{i_3}(s) + r_{i_3} < \infty$

and similarly for any $i_4 \in I$ we have $-\infty < f_{i_4}(s) - r_{i_4} \leq$

$\sup_{i \in I} \{f_i(s) - r_i\}$. Let α_s be any real number such that

$\sup_{i \in I} \{f_i(s) - r_i\} \leq \alpha_s \leq \inf_{i \in I} \{f_i(s) + r_i\}$ and define

$f(s) = \alpha_s$. Since $s \in S$ was arbitrary, we have thus

defined a real valued function f on S. To show

that $f \in \ell_\infty(S)$ we proceed as follows. Let $s \in S$.

Then $f_i(s) - r_i \leq \sup_{i\in I} \{f_i(s) - r_i\} \leq f(s) \leq$

$\inf_{i\in I} \{f_i(s) + r_i\} \leq f_i(s) + r_i$ for any $i \in I$. So

$f_i(s) - r_i \leq f(s) \leq f_i(s) + r_i$ and hence

$f_i(s) - f(s) \leq r_i$ and $f(s) - f_i(s) \leq r_i$. So

$|f(s) - f_i(s)| \leq r_i$ for any $i \in I$. So

$|f(s)| - |f_i(s)| \leq r_i$ and hence $|f(s)| \leq r_i + |f_i(s)|$

$\leq r_i + \|f_i\|_{\ell_\infty(S)}$. Hence $\sup_{s\in S} |f(s)| \leq r_i + \|f_i\|_{\ell_\infty(S)}$

(for any $i \in I$). So $f \in \ell_\infty(S)$. Finally we show that

$f \in \bigcap_{i\in I} S(f_i, r_i)$. Let $i \in I$. Then for any $s \in S$

we have $|f(s) - f_i(s)| \leq r_i$, i.e. $|(f - f_i)(s)| \leq r_i$.

So $\sup_{s\in S} |(f - f_i)(s)| \leq r_i$, i.e. $\|f - f_i\|_{\ell_\infty(S)} \leq r_i$.

So $f \in \bigcap_{i\in I} S(f_i, r_i)$ and hence the collection of all closed spheres in real $\ell_\infty(S)$ has the 1-intersection property. So real $\ell_\infty(S) \in In(1)$. Q.E.D.

As an application of Theorem 7.6, we prove the following theorem.

7.13 <u>Theorem</u>. Let X be a real Banach space which is a dual space and which is in $In(1 + \epsilon)$ for every $\epsilon > 0$. Then $X \in \mathrm{In}(1)$.

Proof. Since $X \in \mathrm{In}(1 + \epsilon)$, the set of all closed spheres in X has the $(1 + \epsilon)$-intersection property for every $\epsilon > 0$. Let $\mathcal{S} = \{S(x_i, r_i)\}_{i \in I}$ be a non-empty collection of closed spheres in X such that every two spheres in $\mathcal{S}$ have a non-empty intersection. We want to show that $\bigcap_{i \in I} S(x_i, r_i) \neq \phi$ for this will imply by Theorem 7.8 that $X \in \mathrm{In}(1)$. Now we know that $\bigcap_{i \in I} S(x_i, (1 + \epsilon)r_i) \neq \phi$ for every $\epsilon > 0$. Let $\epsilon_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$ If $m \geq n$ we have $\bigcap_{i \in I} S(x_i, (1 + \epsilon_m)r_i) \subset \bigcap_{i \in I} S(x_i, (1 + \epsilon_n)r_i)$. For if z is a member of the left hand side, we have

$\|z - x_i\| \leq (1 + \epsilon_m)r_i = (1 + \frac{1}{m})r_i \leq (1 + \frac{1}{n})r_i = (1 + \epsilon_n)r_i$

for all $i \in I$ which means that z is a member of the right hand side. Let $I_n = \bigcap_{i \in I} S(x_i, (1 + \epsilon_n)r_i)$, $n = 1, 2, 3, \ldots$ Then we have

$$I_1 \supset I_2 \supset I_3 \supset \ldots$$

and each I_n is non-empty.

Let us now consider our space X endowed with the weak-* topology. The (strongly) closed unit sphere

$S(0, 1)$ in X, i.e. $\{x \in X \mid \|x\| \leq 1\}$ is closed and compact in the weak-* topology. (X with the weak-* topology is a Hausdorff topological linear space. $S(0, 1)$ is compact in the weak-* topology by Alaoglu's theorem and so $S(0, 1)$ is closed in the weak-* topology since it is a compact subset of a Hausdorff space.) Since X with the weak-* topology is a topological linear space, for an arbitrary fixed vector $x \in X$ and an arbitrary non-empty subset A of X, the mapping $T_x : A \longrightarrow A + x$ defined by $T_x(a) = a + x, a \in A$ is a homeomorphism between A and $A + x$. Similarly if α is a fixed non-zero scalar, the mapping $T_\alpha : A \longrightarrow \alpha A$ defined by $T_\alpha(a) = \alpha a, a \in A$ is a homeomorphism between A and αA. Now given any strongly closed sphere, say $S(x_o, r_o)$ of X with positive radius r_o, we can obtain $S(x_o, r_o)$ from the strongly closed unit sphere $S(0, 1)$ of X by a composition of mappings of the type just discussed. More explicitly the mapping $T_{x_o} T_{r_o}$ maps the unit sphere $S(0, 1)$ onto the sphere $S(x_o, r_o)$. For let $x \in S(0, 1)$. Then $\|T_{x_o} T_{r_o}(x) - x_o\| = \|r_o x + x_o - x_o\| = \|r_o x\| = r_o\|x\| \leq r_o$ and so $T_{x_o} T_{r_o}(S(0,1)) \subset S(x_o, r_o)$.

Also if $y \in S(x_o, r_o)$ then the vector

$\frac{1}{r_o}(y - x_o) \in S(0, 1)$ since $\|\frac{1}{r_o}(y - x_o)\| =$

$\frac{1}{r_o}\|y - x_o\| \leq \frac{r_o}{r_o} = 1$ and $T_{x_o}T_{r_o}(\frac{1}{r_o}(y - x_o)) =$

$T_{x_o}(y - x_o) = y - x_o + x_o = y$ and so $T_{x_o}T_{r_o}$ maps $S(0, 1)$ onto $S(x_o, r_o)$. So an arbitrary strongly closed sphere $S(x_o, r_o)$ in X with positive radius is compact and closed in the weak-* topology since $T_{r_o}(S(0, 1))$ is compact and closed in the weak-* topology (since T_{r_o} is a homeomorphism) and hence so is $T_{x_o}(T_{r_o}(S(0, 1)))$. Indeed even a strongly closed sphere in X with radius equal to zero is compact and closed in the weak-* topology since the sphere consists of only one point and a finite set is compact in any topology and hence closed since the weak-* topology on X is Hausdorff.

Now for any sphere $S(x_i, r_i) \in \mathcal{S}$, we have

$$S(x_i, (1 + \epsilon_i)r_i) \supset I_1 \supset I_2 \supset I_3 \supset \dots$$

Each I_n is the intersection of weak-* closed sets and hence is weak-* closed. So we have a descending sequence of non-empty weak-* closed sets in a weak-* compact space, namely $S(x_i, (1 + \epsilon_1)r_i)$. So $\bigcap_{n=1}^{\infty} I_n \neq \phi$.[1]

Let $y_o \in \bigcap_{n=1}^{\infty} I_n$. We claim that $y_o \in \bigcap_{i \in I} S(x_i, r_i)$.

For suppose $y_o \notin \bigcap_{i \in I} S(x_i, r_i)$. Then

$y_o \notin S(x_{i_o}, r_{i_o})$ for some $i_o \in I$. So

$\|y_o - x_{i_o}\| = \delta > r_{i_o}$. Choose a positive integer n_o

sufficiently large so that $(1 + \frac{1}{n_o})r_{i_o} < \delta$. Then

$y_o \notin S(x_{i_o}, (1 + \epsilon_{n_o})r_{i_o})$ since $\|y_o - x_{i_o}\| = \delta > (1 + \epsilon_{n_o})r_{i_o}$.

So $y_o \notin I_{n_o} = \bigcap_{i \in I} S(x_i, (1 + \epsilon_{n_o})r_i)$. So $y_o \notin \bigcap_{n=1}^{\infty} I_n$

and this contradicts the fact that y_o was chosen to

be a member of $\bigcap_{n=1}^{\infty} I_n$. So we must conclude that

$y_o \in \bigcap_{i \in I} S(x_i, r_i)$. So the set of all closed spheres in

X has the 1-intersection property and hence

$X \in In(1)$. Q.E.D.

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[1] See for example Kelley [20, page 163, exercise H].

CHAPTER VIII

The Class Pr(1)

In this chapter we shall answer the question raised in Remark 1.8 by showing that the class Pr(1) contains only the trivial space, i.e. the space consisting of only the zero vector. This result shows incidentally that the analogue of Theorem 7.13 is false for projective Banach spaces of positive dimension.

Before we present the proof that Pr(1) contains only the trivial space, we must take care of a preliminary matter. In the course of the proof we shall need the fact that there exists a Banach space Y with a closed linear subspace X and a point $y_0 \in Y$ but not in X such that the distance from y_0 to X is not attained, that is to say, there exists no point $x_0 \in X$ such that $\|y_0 - x_0\| = \inf_{x \in X} \{\|y_0 - x\|\}$. So we first construct such an example.

8.1 Example. Let $Y = L_1[0, 1]$. Define a linear functional T on $L_1[0, 1]$ by $Tf = \int_0^1 tf(t)dt$, $f \in L_1[0, 1]$. Now for $0 < \epsilon < 1$ we have

$$
\begin{aligned}
|Tf| &\leq \int_0^{1-\epsilon} t|f(t)|dt + \int_{1-\epsilon}^1 t|f(t)|dt \\
(8.1) \qquad & \\
&\leq (1 - \epsilon) \int_0^{1-\epsilon} |f(t)|dt + \int_{1-\epsilon}^1 |f(t)|dt.
\end{aligned}
$$

Now if f differs from zero on a set of positive measure, we have $\int_0^1 |f(t)|dt > 0$ and indeed $\int_0^{1-\epsilon} |f(t)|dt > 0$ for some $0 < \epsilon < 1$. Choose such an ϵ. Then from (8.1) we have

$$
\begin{aligned}
|Tf| &\leq \int_0^{1-\epsilon} |f(t)dt - \epsilon \int_0^{1-\epsilon} |f(t)|dt + \\
(8.2) \quad & \int_{1-\epsilon}^1 |f(t)|dt = \int_0^1 |f(t)|dt - \epsilon \int_0^{1-\epsilon} |f(t)|dt < \\
& \int_0^1 |f(t)|dt = \|f\|_{L_1[0,1]}.
\end{aligned}
$$

Denoting $\|f\|_{L_1[0,1]}$ by $\|f\|$ for simplicity, we have shown that if $f \in L_1[0, 1]$ differs from 0 on a set of positive measure, then $|Tf| < \|f\|$. Clearly if f is 0 almost everywhere, $Tf = 0 = \|f\|$ and so $|Tf| \leq \|f\|$ for all $f \in L_1[0, 1]$. So $\|T\| \leq 1$. Indeed $\|T\| = 1$. For consider the functions

f_n, $n = 1, 2, \ldots$ defined by $f_n(t) = t^n$, $t \in [0, 1]$.

Then each $f_n \in L_1[0, 1]$ and we have

$$\frac{|Tf_n|}{\|f_n\|} = \frac{\int_0^1 t^{n+1}dt}{\int_0^1 t^n dt} = \frac{\frac{1}{n+2}}{\frac{1}{n+1}} = \frac{n+1}{n+2} .$$

So $|Tf_n| = \frac{n+1}{n+2} \|f_n\|$. But $\lim_{n \to \infty} \frac{n+1}{n+2} = 1$ which implies that $\|T\| = 1$.

Let $X = T^{-1}(\{0\})$. X is a closed subspace of Y and clearly $X \neq Y$. Choose a function $y_o \in L_1[0, 1]$ such that $Ty_o = 1$ (for example $y_o(t) = 3t$, $t \in [0, 1]$). We claim that there does not exist any function $f_o \in X$ such that $\|y_o - f_o\| = \inf_{f \in X} \{\|y_o - f\|\}$. Let

$\delta = \inf_{f \in T^{-1}(\{1\})} \{\|f\|\}$. Now $f \in T^{-1}(\{1\})$ implies f differs from zero on a set of positive measure (for otherwise Tf would be zero) and so

$1 = Tf = |Tf| < \|f\|$. So $\delta \geq 1$. On the other hand let $\epsilon_1 > 0$. Then $0 < \frac{1}{1 + \epsilon_1} < 1 = \|T\|$ and since

$\|T\| = \sup_{\|f\|=1} \{|Tf|\}$, there exists a function f_1 such that $\|f_1\| = 1$ and $|Tf_1| > \frac{1}{1+\epsilon_1}$. Let $f_2 = \frac{f_1}{Tf_1}$.

Then $Tf_2 = 1$ and

$$\|f_2\| = \int_0^1 \frac{|f_1(t)|}{|Tf_1|}\,dt = \frac{\|f_1\|}{|Tf_1|} = \frac{1}{|Tf_1|} < 1 + \epsilon_1.$$

So $\delta \leq 1 + \epsilon_1$. Since $\epsilon_1 > 0$ is arbitrary, we conclude that $\delta \leq 1$. Hence $\delta = 1$.

Now it is easy to see that $y_o - X = T^{-1}(\{1\})$. So $\inf_{f \epsilon X} \{\|y_o - f\|\} = \inf_{f \epsilon T^{-1}(\{1\})} \{\|f\|\} = \delta = 1$. So in order to show that there does not exist any function $f_o \in X$ such that $\|y_o - f_o\| = \inf_{f \epsilon X} \{\|y_o - f\|\}$, it suffices to show that $\|y_o - f\| > 1$ for all $f \in X$. Now for any $f \in X$, $T(y_o - f) = 1$, and so $y_o - f$ differs from zero on a set of positive measure. Hence $T(y_o - f) < \|y_o - f\|$, i.e. $1 < \|y_o - f\|$.

8.2 <u>Theorem</u>. The class Pr(1) contains only the Banach space consisting of the zero vector alone.

<u>Proof</u>. First of all it is clear that $\{0\} \in Pr(1)$. So now assume that a Banach space $P \in Pr(1)$ where the

dimension of P is positive. Choose a vector $x \in P$ such that $\|x\| = 1$ and let V be the subspace (one dimensional, hence closed) of P spanned by x. Take an $f \in P^*$ such that $\|f\| = 1$ and $f(x) = 1$ and define $U : P \longrightarrow V$ by $U(p) = f(p)x$, $p \in P$. Then U is linear and bounded since $\|U(p)\| = \|f(p)x\| = |f(p)|\|x\| \leq \|f\| \|p\| \|x\| \leq \|p\|$. $\|U\| \leq 1$ and since $\|U(x)\| = \|f(x)x\| = \|1x\| = \|x\|$, we conclude that $\|U\| = 1$. Also U maps P onto V since an arbitrary element of V can be written as αx for some scalar α and $\alpha = f(p)$ for some $p \in P$ (since f is onto the scalar field because f is not the identically zero linear functional). Finally $U^2 = U$ since

$U(U(p)) = U(f(p)x) = f(f(p)x)x = f(p)f(x)x = f(p)x = U(p)$.

So U is a bounded projection from P onto V. Using the same type of argument used in the proof of Theorem 3.1 (with our V as the subspace Y and $\ell_1(S)$ replaced by our $P \in Pr(1)$), we see that $V \in Pr(1)$. Now our scalar field K is congruent to V via the map $\alpha \longrightarrow \alpha x$, $\alpha \in K$. Hence by Lemma 2.3, K is also a member of $Pr(1)$. We shall now deduce a contradiction.

Let Y be a Banach space, Z a closed subspace of Y and y_o an element of Y but not of Z such that $\inf_{z \in Z} \{\|y_o - z\|\} = \|y_o - z_o\|$ for no vector $z_o \in Z$.

Let Y_o be the one dimensional subspace of Y spanned by y_o and let $Z_1 = Z + Y_o$. Z_1 is a linear subspace of Y and indeed Z_1 is closed.[1] So Z_1 is a Banach space. Also Z is closed in Z_1 since Z is closed in Y. Let $Q : Z_1 \longrightarrow Z_1/Z$ be the canonical quotient map. Define f from K to Z_1/Z by $f(\alpha) = Q(\alpha y_o)$, $\alpha \in K$. It is easy to see that f is bounded and linear. We note that

(8.3) $$f(1) = Q(y_o).$$

We have the following situation:

$$\begin{array}{ccc} & & K \\ & & \big\downarrow f \\ Z_1 & \xrightarrow[Q]{} & Z_1/Z \end{array}$$

$K \in Pr(1)$ implies that there exists a bounded linear transformation $g : K \longrightarrow Z_1$ such that $Qg = f$ and

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[1] If Z is a closed linear subspace and Y_o a finite dimensional subspace of a topological linear space (in particular of our Banach space Y), then $Z + Y_o$ is closed.

(8.4) $\|g\| \leq \|f\|$.

Also we have

(8.5) $\|f\| = \|Q(y_o)\|$

since $\|f\| = \sup_{|\alpha|=1} \{\|f(\alpha)\|\} = \sup_{|\alpha|=1} \{\|f(\alpha 1)\|\}$

$= \sup_{|\alpha|=1} \{\|\alpha f(1)\|\} = \sup_{|\alpha|=1} \{|\alpha|\|f(1)\|\}$

$= \|f(1)\| = \|Q(y_o)\|$ by (8.3).

So $\|g(1)\| \leq \|g\||1| = \|g\| \leq \|f\|$ (by (8.4)) $= \|Q(y_o)\|$.

Now $Q(g(1)) = f(1) = Q(y_o)$ by (8.3) and so $Q(g(1) - y_o) = 0$.

So $g(1) - y_o \in Z$ and so $g(1) \in y_o + Z = Q(y_o) = y_o - Z$, say $g(1) = y_o - z_o$, $z_o \in Z$.

So $\|Q(y_o)\| = \inf_{w \in Q(y_o)} \{\|w\|\} \leq \|g(1)\|$ and since we have already shown that $\|g(1)\| \leq \|Q(y_o)\|$, we conclude that

$\|Q(y_o)\| = \|g(1)\| = \|y_o - z_o\|$. But

$\|Q(y_o)\| = \inf_{z \in Z} \{\|y_o - z\|\} \neq \|y_o - z_o\|$ for any $z_o \in Z$

by the way we chose Y, Z, and y_o. So K cannot be a member of Pr(1) and hence $P \notin Pr(1)$. Q.E.D.

CHAPTER IX

Dual Spaces of Injective and Projective Spaces

In this chapter we shall prove that the dual space of a projective Banach space is injective. We present two proofs of this theorem, the first of which is a direct proof in the sense that it does not make use of our previous results and the second of which does make use of previous results and is simpler. In the first proof we actually prove somewhat more than is claimed in the statement of the theorem and we shall discuss these implications at the end of this chapter. Also we show that the corresponding question of whether the dual space of an injective Banach space is projective can be reduced to the question of whether the dual spaces of a certain class of injective Banach spaces are projective.

9.1 Theorem. If P is a projective Banach space, then P^* is injective.

Proof. Let X be a Banach space, Y a closed subspace of X and $g : Y \longrightarrow P^*$ a bounded linear transformation. We want to construct a bounded linear transformation $G : X \longrightarrow P^*$ which extends g. Let $J : P \longrightarrow P^{**}$ and $j : X \longrightarrow X^{**}$ be the canonical injections (i.e. if $\varphi \in P^*$, $p \in P$, then $J(p)(\varphi) = \varphi(p)$ and if $\psi \in X^*$, $x \in X$, then $j(x)(\psi) = \psi(x)$). Let $r : X^* \longrightarrow Y^*$ be the restriction mapping, i.e. if $\psi \in X^*$, $r(\psi) = \psi|Y$. It is clear that r is linear and $\|r\| \leq 1$. Since every continuous linear functional on Y can be extended to a continuous linear functional on X (Hahn-Banach theorem), we have $r(X^*) = Y^*$. Let $Y^{\perp} = \{\psi \in X^* \mid \psi(y) = 0 \text{ for all } y \in Y\} = r^{-1}(\{0\})$. Since r is bounded, $Y^{\perp}$ is a closed subspace of X^*. Let $Q : X^* \longrightarrow X^*/Y^{\perp}$ be the canonical quotient mapping. We have the situation

$$\begin{array}{ccccc} X^* & \xrightarrow{\ r\ } & Y^* & \longrightarrow & 0 \quad \text{(exact)} \\ \downarrow{\scriptstyle Q} & & & & \\ X^*/r^{-1}(\{0\}) & & & & \end{array}$$

and so there exists a one-one linear transformation T which maps $X^*/Y^{\perp}$ onto Y^* in a bicontinuous manner and such that $TQ = r$.

Let $g^* : P^{**} \longrightarrow Y^*$ be the adjoint mapping of g, i.e. if $p^{**} \in P^{**}$ and $y \in Y$, $g^*(p^{**})(y) = p^{**}(g(y))$. Define $f : P \longrightarrow X^*/Y^{\perp}$ by

$$f(p) = T^{-1}(g^*(J(p))), \quad p \in P. \tag{9.1}$$

We have the situation

$$\begin{array}{ccc} & & P \\ & & \Big\downarrow f \\ X^* & \xrightarrow{\ Q\ } & X^*/Y^{\perp} \end{array} .$$

Since P is projective, f lifts to a bounded linear transformation $F : P \longrightarrow X^*$ such that

$$QF(p) = f(p), \quad p \in P. \tag{9.2}$$

Now $TQ = r$, so $Q = T^{-1}r$ and hence from (9.2) we have

$$f(p) = T^{-1}(r(F(p))), \quad p \in P.$$

Since $r(\psi) = \psi|Y$, $\psi \in X^*$, we have for $y \in Y$, $p \in P$, $r(F(p))(y) = F(p)(y)$ while $r(F(p)) = TQ(F(p)) = T(f(p))$ by (9.2). So

$$F(p)(y) = T(f(p))(y), \quad p \in P, \quad y \in Y.$$

Now

$$\begin{aligned} T(f(p))(y) &= T(T^{-1}g^*J(p))(y) \quad \text{by (9.1)} \\ &= (g^*J(p))(y) \\ &= J(p)(g(y)) \\ &= g(y)(p). \end{aligned}$$

So

(9.3) $$F(p)(y) = g(y)(p), \quad p \in P, \quad y \in Y.$$

Let $F^* : X^{**} \longrightarrow P^*$ be the adjoint mapping of F and let $G = F^*j : X \longrightarrow P^*$. G is bounded and linear and we claim that G extends g. For let $y \in Y$. Then we have for any $p \in P$

$$\begin{aligned} G(y)(p) &= F^*(j(y))(p) \\ &= j(y)(F(p)) \\ &= F(p)(y) \\ &= g(y)(p) \quad \text{by } (9.3). \end{aligned}$$

So G extends g and hence P^* is injective. Q.E.D.

Now every projective Banach space is a member of $Pr(\lambda)$ for some $1 \leq \lambda < \infty$ and every injective Banach space is a member of $In(\lambda')$ for some $1 \leq \lambda' < \infty$. So for a given projective space $P \in Pr(\lambda)$, the preceding theorem tells us that $P^* \in In(\lambda')$ for some $1 \leq \lambda' < \infty$. Indeed we can take λ' to be λ as the following corollary shows.

9.2 <u>Corollary</u>. If a Banach space P is a member of $Pr(\lambda)$, then P^* is a member of $In(\lambda)$.

<u>Proof</u>. We shall use the same notations as in the proof of Theorem 9.1. We establish first that $\|T^{-1}\| \leq 1$.

Let $\alpha \in Y^*$ and let $\tilde{\alpha} \in X^*$ be such that $r(\tilde{\alpha}) = \alpha$ and $\|\tilde{\alpha}\| = \|\alpha\|$. (Such an $\tilde{\alpha}$ exists by the Hahn-Banach theorem.) Since $TQ = r$, we have $\alpha = r(\tilde{\alpha}) = TQ(\tilde{\alpha}) = T(\tilde{\alpha} + Y^{\perp})$. So $T^{-1}(\alpha) = \tilde{\alpha} + Y^{\perp}$ and so

$$\|T^{-1}(\alpha)\| = \|\tilde{\alpha} + Y^{\perp}\| = \inf_{\psi \in Y^{\perp}} \{\|\tilde{\alpha} + \psi\|\} \leq \|\tilde{\alpha} + 0\| = \|\tilde{\alpha}\| = \|\alpha\|.$$

So $\|T^{-1}\| \leq 1$. Now because $P \in Pr(\lambda)$ we can assume that $\|F\| \leq \lambda\|f\|$. If $x \in X$, we have $G(x) = F^*j(x) = j(x)(F)$ and so $\|G(x)\| \leq \|j(x)\| \|F\| = \|x\| \|F\| \leq \lambda\|x\| \|f\| = \lambda\|x\| \|T^{-1}g^*J\| \leq \lambda\|x\| \|T^{-1}\| \|g^*\| \|J\| \leq \lambda\|x\| \|g\|$. So $\|G\| \leq \lambda\|g\|$ and so $P^* \in In(\lambda)$. Q.E.D.

Before we present the alternate proof of Theorem 9.1, we require some lemmas.

9.3 <u>Lemma</u>. Let X be a Banach space with closed subspaces V and W such that $X = V + W$ and $V \cap W = \{0\}$. Then X^* is equivalent to $V^* \oplus W^*$.

<u>Proof</u>. For $\varphi \in X^*$, let $\varphi_V = \varphi|V$ and $\varphi_W = \varphi|W$. Define $T : X^* \longrightarrow V^* \oplus W^*$ by $T(\varphi) = (\varphi_V, \varphi_W)$, $\varphi \in X^*$. It is clear that T is linear. Also $\|T(\varphi)\| = \|(\varphi_V, \varphi_W)\| = \|\varphi_V\| + \|\varphi_W\| = \sup_{\substack{v \in V \\ \|v\|=1}} \{|\varphi(v)|\} + \sup_{\substack{w \in W \\ \|w\|=1}} \{|\varphi(w)|\} \leq 2\|\varphi\|$ and so T is

bounded. Since for each $x \in X$ we have $x = v + w$, $v \in V$, $w \in W$, v and w unique, it follows that T is one-one. For suppose $0 = T(\varphi) = (\varphi_V, \varphi_W) = (0, 0)$. Then for $x \in X$ we have $\varphi(x) = \varphi(v + w) = \varphi(v) + \varphi(w) = \varphi_V(v) + \varphi_W(w) = 0 + 0 = 0$ and hence $\varphi = 0$ on X. Finally T is onto $V^* \oplus W^*$. For let $(\psi, \chi) \in V^* \oplus W^*$. Then for $x = v + w \in X$, define $\varphi(x) = \psi(v) + \chi(w)$. φ is linear and $\varphi|V = \psi$, $\varphi|W = \chi$ and so all that remains to be established to show that T is onto is the continuity of φ on X. Now $|\varphi(x)| = |\psi(v) + \chi(w)| \leq \|\psi\|_{V^*} \|v\| + \|\chi\|_{W^*} \|w\| \leq K(\|v\| + \|w\|)$ where $K = \max\{\|\psi\|_{V^*}, \|\chi\|_{W^*}\}$. Hence if we can prove that there exists a constant K_1 such that $\|v\| + \|w\| \leq K_1 \|x\|$ for all $x = v + w \in X$, the continuity of φ will have been established. Define a new norm $\| \ \|_1$ on X by $\|x\|_1 = \|v\| + \|w\|$. Then since $\|x\| = \|v + w\| \leq \|v\| + \|w\| = \|x\|_1$, the identity mapping from X with the norm $\| \ \|_1$ to X with its original norm $\| \ \|$ is continuous. If X with the norm $\| \ \|_1$ is complete, then the closed graph theorem tells us that the identity mapping from X with the norm $\| \ \|$ to X with the norm $\| \ \|_1$ is continuous, that is, there exists a constant K_1 such that for all $x = v + w \in X$, $\|x\|_1 = \|v\| + \|w\| \leq K_1\|x\|$. So we proceed to show that X with the norm $\| \ \|_1$ is complete. Let $\{x_n\}_{n=1,2,...}$ be a Cauchy sequence in X with the norm $\| \ \|_1$ and let $x_n = v_n + w_n$, $n = 1,2,...$ Then each of the sequences $\{v_n\}_{n=1,2,...}$ and $\{w_n\}_{n=1,2,...}$ is Cauchy in V and W respectively with respect to the norm $\| \ \|$. For given $\epsilon > 0$, there exists a positive integer N such that $\|x_n - x_m\|_1 < \epsilon$ if $n, m > N$, that is, $\|v_n - v_m\| + \|w_n - w_m\| < \epsilon$ if $n, m > N$ from which we conclude that $\{v_n\}_{n=1,2,...}$ and $\{w_n\}_{n=1,2,...}$ are Cauchy. Since V and W are closed subspaces of X, they are complete and so there exist $v \in V$ and $w \in W$ such that $\lim_{n \to \infty} \|v - v_n\| = 0$ and $\lim_{n \to \infty} \|w - w_n\| = 0$. But then $\lim_{n \to \infty} \|v + w - x_n\|_1 = 0$ since $\|v + w - x_n\|_1 = \|v - v_n\| + \|w - w_n\|$. So X with the norm $\| \ \|_1$ is complete, and hence φ is continuous. So T is a one-one continuous linear transformation from the Banach space X^* onto the Banach space $V^* \oplus W^*$ and hence by the closed graph theorem, T^{-1} is continuous. So X^* and $V^* \oplus W^*$ are equivalent. Q.E.D.

9.4 <u>Lemma</u>. Let A and B be Banach spaces and suppose $A \oplus B$ is injective. Then A is injective.

<u>Proof</u>. Let X be a Banach space, Y a closed subspace of X, and g a bounded linear transformation from Y into A. Define $f : A \longrightarrow A \oplus B$ by $f(a) = (a, 0)$. Clearly f is linear and since $\|f(a)\| = \|(a,0)\| = \|a\|$, f is bounded. Let $f_1 = fg : Y \longrightarrow A \oplus B$. f_1 is bounded and linear and since $A \oplus B$ is injective, there exists a bounded linear transformation $\tilde{f}_1 : X \longrightarrow A \oplus B$ which extends f_1. Define $h : A \oplus B \longrightarrow A$ by $h(a, b) = a$. Clearly h is linear and since $\|h(a, b)\| = \|a\| \leq \|a\| + \|b\| = \|(a, b)\|$, h is bounded. Define $\tilde{g} : X \longrightarrow A$ by $\tilde{g} = h\tilde{f}_1$. $\tilde{g}$ is bounded and linear and $\tilde{g}$ extends g. For if $y \in Y$, $\tilde{g}(y) = h\tilde{f}_1(y) = hf_1(y) = hfg(y) = h(g(y), 0) = g(y)$. So A is injective. Q.E.D.

9.5 <u>Alternate Proof of Theorem 9.1</u>. Since P is projective, there exist a set S and closed subspaces A and B of $\ell_1(S)$ such that $A \cap B = \{0\}$, $\ell_1(S) = A + B$, and A is equivalent to P. By Lemma 9.3, $(\ell_1(S))^*$ is equivalent to $A^* \oplus B^*$. But $(\ell_1(S))^*$ is congruent to $\ell_\infty(S)$ and so $\ell_\infty(S)$ is equivalent to $A^* \oplus B^*$. Since $\ell_\infty(S)$ is injective, $A^* \oplus B^*$ is injective. By Lemma 9.4, A^* is injective. But A^* is equivalent to P^* since A is equivalent to P and so P^* is injective. Q.E.D.

Before we prove our next theorem which was motivated by considering the question answered by Theorem 9.1 with "projective" and "injective" interchanged, we require the analogue for projective spaces of Lemma 9.4.

9.6 Lemma. Let A and B be Banach spaces and suppose $A \oplus B$ is projective. Then A is projective.

Proof. Let X be a Banach space, X_o a closed subspace of X, Q the canonical quotient map from X onto X/X_o, and f a bounded linear transformation from A to X/X_o. Define $f_1 : A \oplus B \longrightarrow X/X_o$ by $f_1(a, b) = f(a)$. Clearly f_1 is linear and since $\|f_1(a, b)\| = \|f(a)\| \leq \|f\| \|a\| \leq \|f\|(\|a\| + \|b\|) = \|f\| \|(a, b)\|$, f_1 is bounded. Since $A \oplus B$ is projective, there exists a bounded linear transformation $\tilde{f}_1 : A \oplus B \longrightarrow X$ such that $Q\tilde{f}_1 = f_1$. Define a map $\tilde{f} : A \longrightarrow X$ by $\tilde{f}(a) = \tilde{f}_1(a, 0)$. $\tilde{f}$ is linear and since $\|\tilde{f}(a)\| = \|\tilde{f}_1(a, 0)\| \leq \|\tilde{f}_1\| \|(a, 0)\| = \|\tilde{f}_1\| \|a\|$, $\tilde{f}$ is bounded. Finally $\tilde{f}$ lifts f since $Q\tilde{f}(a) = Q\tilde{f}_1(a, 0) = f_1(a, 0) = f(a)$. So A is projective. Q.E.D.

9.7 Theorem. The dual space of every injective Banach space is projective if and only if $(\ell_\infty(S))^*$ is projective for every non-empty set S.

Proof. (=>) If injective Banach spaces have projective dual spaces, then clearly $(\ell_\infty(S))^*$ is projective for every S since $\ell_\infty(S)$ is injective.

(<=) Now assume that $(\ell_\infty(S))^*$ is projective for every S. Let I be any injective Banach space. Then there exist a non-empty set S and closed subspaces A and B of $\ell_\infty(S)$ such that $\ell_\infty(S) = A + B$, $A \cap B = \{0\}$ and A is congruent to I. By Lemma 9.3 $(\ell_\infty(S))^*$ is equivalent to $A^* \oplus B^*$. Also by our assumption $(\ell_\infty(S))^*$ is projective and hence so is $A^* \oplus B^*$. By Lemma 9.6 A^* is projective and hence so is I^* since A^* is equivalent to I^*. Q.E.D.

9.8 Remark. In the first proof of Theorem 9.1 we did not make full use of the hypothesis that P is projective. More precisely the map f that we lifted was not just a bounded linear transformation from P into a quotient space A/B, A an arbitrary Banach space and B an arbitrary closed subspace of A. Our quotient space A/B was of a very special type, namely A was a dual space (X^*) and B was $Y^{\perp}$, the space of all continuous linear functionals defined on X which vanish on the closed subspace Y of X. Now such a subspace $Y^{\perp}$ of X^* is closed in the weak-* topology on X^*. For let $x \in Y$ and let $F_x = \{ \varphi \in X^* \mid \varphi(x) = 0\}$. It is easy to see that $Y^{\perp} = \bigcap_{x \in Y} F_x$. Now each F_x is weak-* closed. For consider the linear functional J_x on X^* defined by $J_x(\varphi) = \varphi(x)$, $\varphi \in X^*$. J_x is continuous with respect to the weak-* topology on X^* (by definition of the weak-* topology) and $F_x = J_x^{-1}(\{0\})$. So F_x is weak-* closed and so $Y^{\perp}$, being the intersection of a family of weak-* closed sets, is weak-* closed. Hence if the only hypothesis imposed on our Banach space P were that for any dual space A, any weak-* closed subspace B of A, and any bounded linear trasnformation f from P to A/B, there exists a bounded linear transformation $\tilde{f}$ from P to A such that $Q\tilde{f} = f$ (where Q is the quotient map from A onto A/B), we would still be able to conclude that P* is injective. A Banach space satisfying the criteria imposed on P will be said to be *-projective and Theorem 9.1 can be reworded to read "The dual space of a *-projective Banach space is injective". We can define the class $Pr^*(\lambda)$ $1 \leq \lambda < \infty$, as consisting of those *-projective Banach spaces for which the map $\tilde{f}$ can be chosen so that $\|\tilde{f}\| \leq \lambda\|f\|$ and Corollary 9.2 becomes "The dual space of a member of $Pr^*(\lambda)$ is a member of $In(\lambda)$." Indeed the dual space of any *-projective Banach space (whether a member of $Pr^*(\lambda_1)$ or not for some $1 \leq \lambda_1 < \infty$) is a member of $In(\lambda)$ for some $1 \leq \lambda < \infty$ since the dual space is injective and hence is a member of $In(\lambda)$ for some $1 \leq \lambda < \infty$.

CHAPTER X

Separable Projective Banach Spaces

In this chapter we shall prove that if P is a separable projective Banach space, then either P is finite dimensional or else P is equivalent to $\ell_1(S)$ where S is a countably infinite set. We shall accomplish this by noting the following. If P is separable and projective, there exist a countably infinite set S and closed subspaces X and Y of $\ell_1(S)$ such that $\ell_1(S) = X + Y$, $X \cap Y = \{0\}$, and X is equivalent to P. Hence it suffices to establish that an infinite dimensional closed subspace of $\ell_1(S)$, S countably infinite, with a closed complement is equivalent to $\ell_1(S)$. On the other hand if S is a non-empty at most countably infinite set, $\ell_1(S)$ is a separable projective Banach space. Thus a non-zero Banach space P is separable and projective if and only if P is equivalent to $\ell_1(S)$ for some non-empty at most countably infinite set S and hence we obtain a characterization of separable projective Banach spaces.

We require several lemmas, some of which are rather trivial, but we include them for completeness.

10.1 Notation. Let A and B be Banach spaces. We write "$A \sim B$" to denote that A is equivalent to B.

10.2 Lemma. Let A, B, and C be Banach spaces and let $A \sim B$. Then $A \oplus C \sim B \oplus C$.

Proof. Let $T : A \longrightarrow B$ be a map defining the equivalence between A and B. Define $\varphi : A \oplus C \longrightarrow B \oplus C$ by $\varphi(a, c) = (Ta, c)$, $(a, c) \in A \oplus C$. φ is linear and if $\varphi(a, c) = \varphi(a_1, c_1)$, then $c = c_1$ and $Ta = Ta_1$ which implies that $a = a_1$ since T is one-one. So φ is one-one. If $(b, c) \in B \oplus C$, let $a \in A$ be such that $Ta = b$. Then $\varphi(a, c) = (b, c)$ and so φ is onto. φ is bounded for $\|\varphi(a, c)\| = \|(Ta, c)\| = \|Ta\| + \|c\| \leq \|T\| \; \|a\| + \|c\| \leq K(\|a\| + \|c\|) = K\|(a, c)\|$ where $K = \max\{\|T\|, 1\}$. So φ is a one-one bounded linear transformation from the Banach space $A \oplus C$ onto the Banach space $B \oplus C$. By the closed graph theorem, φ^{-1} is continuous. So $A \oplus C \sim B \oplus C$. Q.E.D.

10.3 Lemma. Let A, B, and C be Banach spaces. Then $(A \oplus B) \oplus C \sim A \oplus (B \oplus C)$.

Proof. Define $T : (A \oplus B) \oplus C \longrightarrow A \oplus (B \oplus C)$ by $T((a, b), c) = (a, (b, c))$, $((a, b), c) \in (A \oplus B) \oplus C$.

It is easy to see that T is linear. T is one-one for if $T(\alpha_1) = T(\alpha_2)$, then $(a_1, (b_1, c_1)) = (a_2, (b_2, c_2))$ which implies that $a_1 = a_2$ and $(b_1, c_1) = (b_2, c_2)$ which implies that $b_1 = b_2$ (and so $(a_1, b_1) = (a_2, b_2)$) and $c_1 = c_2$ and hence $\alpha_1 = \alpha_2$. T is onto for if $(a, (b, c)) \in A \oplus (B \oplus C)$, $(a, (b, c)) = T((a, b), c)$. T is bounded for

$$\begin{aligned} \|T((a, b), c)\| &= \|(a, (b, c))\| = \|a\| + \|(b, c)\| \\ &= \|a\| + \|b\| + \|c\| = \|(a, b)\| + \|c\| \\ &= \|((a, b), c)\|. \end{aligned}$$

So T is a one-one bounded linear transformation from the Banach space $(A \oplus B) \oplus C$ onto the Banach space $A \oplus (B \oplus C)$. By the closed graph theorem, T^{-1} is continuous. So $(A \oplus B) \oplus C \sim A \oplus (B \oplus C)$. Q.E.D.

10.4 Lemma. Let A and B be Banach spaces and suppose $A \sim B$. Let S be a non-empty set. For each $s \in S$, let $A_s = A$ and $B_s = B$. Let

$X = \sum_{s \in S} \oplus_1 A_s$ and $Y = \sum_{s \in S} \oplus_1 B_s$. Then $X \sim Y$.

Proof. Let $T : A \longrightarrow B$ be a map defining the equivalence between A and B. We want to define a one-one bounded linear transformation $\tilde{T}$ from X

onto Y. Let $f \in X$. Define $\tilde{T}f : S \longrightarrow B$ by $(\tilde{T}f)(s) = T(f(s))$, $s \in S$. $\tilde{T}f \in Y$. For we have $\|(\tilde{T}f)(s)\| = \|Tf(s)\| \leq \|T\| \, \|f(s)\|$ for $s \in S$ and since $\sum_{s \in S} \|f(s)\| < \infty$ (since $f \in X$), it follows that

$$\sum_{s \in S} \|T\| \, \|f(s)\| < \infty \text{ and hence } \sum_{s \in S} \|(\tilde{T}f)(s)\| < \infty.$$

So we have a mapping $\tilde{T} : X \longrightarrow Y$ given by $f \longrightarrow \tilde{T}f$. $\tilde{T}$ is linear. For let $f_1, f_2 \in X$ and $s \in S$. Then

$$\tilde{T}(f_1 + f_2)(s) = T((f_1 + f_2)(s)) = T(f_1(s) + f_2(s))$$
$$= T(f_1(s)) = T(f_2(s)) = (\tilde{T}f_1)(s) + (\tilde{T}f_2)(s)$$
$$= (\tilde{T}f_1 + \tilde{T}f_2)(s).$$

So $\tilde{T}(f_1 + f_2) = \tilde{T}f_1 + \tilde{T}f_2$. Also if α is a scalar,

$$(\tilde{T}(\alpha f_1))(s) = T((\alpha f_1)(s)) = T(\alpha(f_1(s))) = \alpha(T(f_1(s)))$$
$$= \alpha((\tilde{T}f_1)(s)) = (\alpha(\tilde{T}f_1))(s).$$

So $\tilde{T}(\alpha f_1) = \alpha(\tilde{T}f_1)$. So $\tilde{T}$ is linear. $\tilde{T}$ is one-one. For suppose $\tilde{T}f_1 = \tilde{T}f_2$, $f_1, f_2 \in X$. Then for all $s \in S$, we have $T(f_1(s)) = T(f_2(s))$ which implies $f_1(s) = f_2(s)$ (since T is one-one) and so $f_1 = f_2$.

$\tilde{T}$ is onto Y. For let $g \in Y$. Define $f : S \longrightarrow A$ by $f(s) = T^{-1}(g(s))$, $s \in S$. $f \in X$ for if $s \in S$ we have $\|f(s)\| = \|T^{-1}(g(s))\| \leq \|T^{-1}\| \, \|g(s)\|$ and since $g \in Y$, $\sum_{s \in S} \|g(s)\| < \infty$ which implies that $\sum_{s \in S} \|T^{-1}\| \, \|g(s)\| < \infty$ and hence $\sum_{s \in S} \|f(s)\| < \infty$.

$\tilde{T}f = g$ for if $s \in S$ we have $(\tilde{T}f)(s) = T(f(s)) = TT^{-1}g(s) = g(s)$. So $\tilde{T}$ is onto Y. Finally $\tilde{T}$ is bounded. For if $f \in X$, we have $\|(\tilde{T}f)(s)\| = \|T(f(s))\| \leq \|T\| \, \|f(s)\|$ for each $s \in S$ and since $\sum_{s \in S} \|T\| \, \|f(s)\| < \infty$ (since $f \in X$), it follows that $\sum_{s \in S} \|(\tilde{T}f)(s)\| < \infty$. Indeed

$$\sum_{s \in S} \|(\tilde{T}f)(s)\| \leq \sum_{s \in S} \|T\| \, \|f(s)\| = \|T\| \sum_{s \in S} \|f(s)\| = \|T\| \, \|f\|.$$

Since $\|\tilde{T}f\| = \sum_{s \in S} \|(\tilde{T}f)(s)\|$, we conclude that $\|\tilde{T}f\| \leq \|T\| \, \|f\|$, i.e. that $\tilde{T}$ is bounded. So $\tilde{T}$ is a one-one bounded linear transformation from the Banach space X onto the Banach space Y. By the closed graph theorem, $\tilde{T}^{-1}$ is continuous. Hence $X \sim Y$. Q.E.D.

10.5 Lemma. Let A and B be Banach spaces and let S be a non-empty set. For each $s \in S$, let $A_s = A$, $B_s = B$, and $D_s = A \oplus B$. Let

$$X = \sum_{s \in S} \oplus_1 A_s, \quad Y = \sum_{s \in S} \oplus_1 B_s, \text{ and } \quad W = \sum_{s \in S} \oplus_1 D_s.$$

Then $W \sim X \oplus Y$.

Proof. We want to define a one-one bounded linear transformation T from W onto $X \oplus Y$. Define a map $g_1 : A \oplus B \longrightarrow A$ by $g_1(a, b) = a$ and a map $g_2 : A \oplus B \longrightarrow B$ by $g_2(a, b) = b$. g_1 and g_2 are clearly linear. Now if $f \in W$ and $s \in S$, $f(s) \in A \oplus B$ and $f(s) = (g_1 f(s), g_2 f(s))$. Define a map $f_A : S \longrightarrow A$ by $f_A(s) = g_1 f(s)$ and a map $f_B : S \longrightarrow B$ by $f_B(s) = g_2 f(s)$. Now $\|f_A(s)\|_A = \|g_1 f(s)\|_A \leq \|g_1 f(s)\|_A + \|g_2 f(s)\|_B = \|f(s)\|_{A \oplus B}$. Since $f \in W$, we have $\sum_{s \in S} \|f(s)\|_{A \oplus B} < \infty$ from which it follows that $\sum_{s \in S} \|f_A(s)\|_A < \infty$. So $f_A \in X$.

Similarly

$$\|f_B(s)\|_B = \|g_2 f(s)\|_B \leq \|g_1 f(s)\|_A + \|g_2 f(s)\|_B = \|f(s)\|_{A \oplus B}$$

and so $\sum_{s \in S} \|f_B(s)\|_B < \infty$. So $f_B \in Y$. So

$(f_A, f_B) \in X \oplus Y$ and we define a map $T : W \longrightarrow X \oplus Y$ by $Tf = (f_A, f_B)$, $f \in W$.

T is linear. For let f and g be in W and $s \in S$. Then $T(f + g) = ((f + g)_A, (f + g)_B)$ and $Tf + Tg = (f_A, f_B) + (g_A, g_B) = (f_A + g_A, f_B + g_B)$.

$$\begin{aligned}(f + g)_A(s) &= g_1((f + g)(s)) = g_1(f(s) + g(s)) \\ &= g_1 f(s) + g_1 g(s) = f_A(s) + g_A(s) \\ &= (f_A + g_A)(s).\end{aligned}$$

So $(f + g)_A = f_A + g_A$. Similarly $(f + g)_B = f_B + g_B$ and hence $T(f + g) = Tf + Tg$. Also if α is a scalar, $T(\alpha f) = ((\alpha f)_A, (\alpha f)_B)$ and $\alpha(Tf) = \alpha(f_A, f_B) = (\alpha(f_A), \alpha(f_B))$. Now $(\alpha f)_A(s) = g_1(\alpha f(s)) = \alpha(g_1 f(s)) = \alpha(f_A(s))$. So $(\alpha f)_A = \alpha(f_A)$ and similarly

$(\alpha f)_B = \alpha(f_B)$ and so $T(\alpha f) = \alpha(Tf)$. So T is linear.

T is one-one. For suppose $Tf = Tg$, $f, g \in W$. Then $(f_A, f_B) = (g_A, g_B)$ and so $f_A = g_A$ and $f_B = g_B$. So for $s \in S$, $f(s) = (f_A(s), f_B(s)) = (g_A(s), g_B(s)) = g(s)$. Hence $f = g$. T is onto $X \oplus Y$. For let $(\varphi, \psi) \in X \oplus Y$. Define a map $f : S \longrightarrow A \oplus B$ by $f(s) = (\varphi(s), \psi(s))$. Now $\|f(s)\|_{A \oplus B} = \|(\varphi(s), \psi(s))\|_{A \oplus B} = \|\varphi(s)\|_A + \|\psi(s)\|_B$. Since $\varphi \in X$, $\sum_{s \in S} \|\varphi(s)\|_A < \infty$ and similarly $\psi \in Y$ implies $\sum_{s \in S} \|\psi(s)\|_B < \infty$.

So $\sum_{s \in S} (\|\varphi(s)\|_A + \|\psi(s)\|_B) < \infty$ and hence $\sum_{s \in S} \|f(s)\|_{A \oplus B} < \infty$. So $f \in W$. We claim that $Tf = (\varphi, \psi)$. For $Tf = (f_A, f_B)$ and if $s \in S$, $f(s) = (f_A(s), f_B(s)) = (\varphi(s), \psi(s))$ (by definition of f). So $f_A(s) = \varphi(s)$ and $f_B(s) = \psi(s)$ for all $s \in S$, i.e. $f_A = \varphi$, $f_B = \psi$. So T is onto $X \oplus Y$.

Finally T is bounded. For if $f \in W$,

$$\|Tf\|_{X \oplus Y} = \|(f_A, f_B)\| = \|f_A\|_X + \|f_B\|_Y$$

$$= \sum_{s \in S} \|f_A(s)\|_A + \sum_{s \in S} \|f_B(s)\|_B$$

$$= \sum_{s \in S} (\|f_A(s)\|_A + \|f_B(s)\|_B)$$

$$= \sum_{s \in S} \|(f_A(s), f_B(s))\|_{A \oplus B}$$

$$= \sum_{s \in S} \|f(s)\|_{A \oplus B}$$

$$= \|f\|_W.$$

So $\|Tf\|_{X \oplus Y} = \|f\|_W$ and hence T is bounded. So T is a one-one bounded linear transformation from the Banach space W onto the Banach space $X \oplus Y$. By the closed graph theorem, T^{-1} is continuous. Hence $W \sim X \oplus Y$. Q.E.D.

10.6 <u>Lemma</u>. Let A and B be Banach spaces. Then $A \oplus B \sim B \oplus A$.

Proof. Define $T : A \oplus B \longrightarrow B \oplus A$ by $T(a, b) = (b, a)$, $a \in A$, $b \in B$. T is onto $B \oplus A$ since an arbitrary $(b, a) \in B \oplus A$ is the image under T of $(a, b) \in A \oplus B$. It is clear that T is one-one and linear. T is bounded since

$$\|T(a, b)\| = \|(b, a)\| = \|b\| + \|a\| = \|a\| + \|b\| = \|(a, b)\|.$$

So T is a one-one bounded linear transformation from the Banach space $A \oplus B$ onto the Banach space $B \oplus A$ and hence by the closed graph theorem, T^{-1} is continuous. So $A \oplus B \sim B \oplus A$. Q.E.D.

10.7 Lemma. Let A, B, and C be Banach spaces and suppose $A \sim B$. Then $C \oplus A \sim C \oplus B$.

Proof. By Lemma 10.2 we have $A \oplus C \sim B \oplus C$. By Lemma 10.6, $C \oplus A \sim A \oplus C$ and $B \oplus C \sim C \oplus B$. So $C \oplus A \sim C \oplus B$ since $\sim$ is an equivalence relation. Q.E.D.

10.8 Lemma. Let S and S' be two non-empty sets with the same cardinality. Let A be a Banach space. For each $s \in S$, let $A_s = A$ and for each $s' \in S'$, let $A_{s'} = A$. Let $X = \sum_{s \in S} \oplus_1 A_s$ and $Y = \sum_{s' \in S'} \oplus_1 A_{s'}$. Then $X \sim Y$.

<u>Proof</u>. Since S and S' have the same cardinality, there exists a one-one mapping φ from S onto S'. Let $f \in X$. Define a function $Tf : S' \longrightarrow A$ by $(Tf)(s') = f(\varphi^{-1}(s'))$, $s' \in S'$. $Tf \in Y$ since

$$\sum_{s' \in S'} \|Tf(s')\| = \sum_{s' \in S'} \|f(\varphi^{-1}(s'))\| = \sum_{s \in S} \|f(s)\| < \infty.$$

So we have a mapping $T : X \longrightarrow Y$ given by $f \longrightarrow Tf$.
T is linear. For let $f, g \in X$. Then

$$(T(f+g))(s') = (f+g)(\varphi^{-1}(s')) = f(\varphi^{-1}(s')) + g(\varphi^{-1}(s'))$$
$$= (Tf)(s') + (Tg)(s') = (Tf + Tg)(s').$$

So $T(f+g) = Tf + Tg$. Similarly if α is a scalar, $(T(\alpha f))(s') = (\alpha f)(\varphi^{-1}(s')) = \alpha(f(\varphi^{-1}(s'))) = \alpha(Tf(s')) = (\alpha(Tf))(s')$. So $T(\alpha f) = \alpha(Tf)$ and hence T is linear.

T is one-one. For suppose $Tf_1 = Tf_2$, $f_1, f_2 \in X$. Then for all $s' \in S$, we have $Tf_1(s') = Tf_2(s')$, i.e. $f_1(\varphi^{-1})(s')) = f_2(\varphi^{-1}(s'))$. But every $s \in S$ is $\varphi^{-1}(s')$ for some $s' \in S'$ and so $f_1(s) = f_2(s)$ for all $s \in S$. So $f_1 = f_2$. T is onto Y. For let $h \in Y$. Define $w : S \longrightarrow A$ by $w(s) = h(\varphi(s))$, $s \in S$.

$w \in X$ since $\sum_{s \in S} \|w(s)\| = \sum_{s \in S} \|h(\varphi(s))\| = \sum_{s' \in S'} \|h(s')\| < \infty.$

Also $Tw = h$ since $(Tw)(s') = w(\varphi^{-1}(s')) = h(\varphi(\varphi^{-1}(s')))$ $= h(s')$ for all $s' \in S'$. So T is onto Y. Finally

$$\|Tf\| = \sum_{s' \in S'} \|Tf(s')\| = \sum_{s' \in S'} \|f(\varphi^{-1}(s'))\| = \sum_{s \in S} \|f(s)\| =$$

$\|f\|$. So T is bounded. So T is a one-one bounded linear transformation from the Banach space X onto the Banach space Y. By the closed graph theorem, T^{-1} is continuous. Hence $X \sim Y$. Q.E.D.

10.9 <u>Lemma</u>. Let S be an infinite set and let A be a Banach space. For each $s \in S$, let $A_s = A$.

Let $X = \sum_{s \in S} \oplus_1 A_s$. Then $X \oplus A \sim X$.

<u>Proof</u>. Let s_o be a point which is not in S. Then since S is infinite, the set $S' = S \cup \{s_o\}$ has the same cardinality as S. For each $s' \in S'$, let $A_{s'} = A$ and let $Y = \sum_{s' \in S'} \oplus_1 A_{s'}$. Let $z \in X \oplus A$. Then $z = (f, a)$, $f \in X$, $a \in A$. Define a map $Tz : S' \longrightarrow A$ as follows:

$$(Tz)(s) = f(s) \quad \text{if} \quad s \in S$$

$$(Tz)(s_o) = a.$$

We claim that $Tz \in Y$. We must show that

$\sum_{s' \in S'} \|(Tz)(s')\| < \infty$. Let $\epsilon > 0$. Since $f \in X$,

$\sum_{s \in S} \|f(s)\| < \infty$. So there exists a finite subset S_ϵ of

S such that if S_F is any finite non-empty subset of

S such that $S_F \cap S_\epsilon = \phi$, we have $\sum_{s \in S_F} \|f(s)\| < \epsilon$.

Let $S'_\epsilon = S_\epsilon \cup \{s_o\}$. If S'_F is any finite non-empty

subset of S' such that $S'_F \cap S'_\epsilon = \phi$, then $S'_F \subset S$

and $S'_F \cap S_\epsilon = \phi$ and so $\sum_{s' \in S'_F} \|(Tz)(s')\| = \sum_{s' \in S'_F} \|f(s')\| < \epsilon$.

So $\sum_{s' \in S'} \|(Tz)(s')\| < \infty$. So we have a map T from

$X \oplus A$ to Y given by $z \longrightarrow Tz$. T is linear. For

let $z_1 = (f_1, a_1)$ and $z_2 = (f_2, a_2)$ be members of

$X \oplus A$. Then $z_1 + z_2 = (f_1 + f_2, a_1 + a_2)$ and so for

$s \in S$, $T(z_1 + z_2)(s) = (f_1 + f_2)(s) = f_1(s) + f_2(s)$

$$= Tz_1(s) + Tz_2(s) = (Tz_1 + Tz_2)(s),$$

while $T(z_1 + z_2)(s_o) = a_1 + a_2 = Tz_1(s_o) + Tz_2(s_o) =$ $(Tz_1 + Tz_2)(s_o)$. So $T(z_1 + z_2) = Tz_1 + Tz_2$. If α is a scalar, $\alpha z_1 = (\alpha f_1, \alpha a_1)$ and so for $s \in S$, $(T(\alpha z_1))(s) = (\alpha f_1)(s) = \alpha(f_1(s)) = \alpha(Tz_1(s)) = (\alpha(Tz_1))(s)$, while $(T(\alpha z_1))(s_o) = \alpha a_1 = \alpha((Tz_1)(s_o)) = (\alpha(Tz_1))(s_o)$. So $T(\alpha z_1) = \alpha(Tz_1)$. So T is linear. T is one-one. For suppose $Tz_1 = Tz_2$. Then for all $s \in S$, $f_1(s) = Tz_1(s) = Tz_2(s) = f_2(s)$. So $f_1 = f_2$. Also $a_1 = Tz_1(s_o) = Tz_2(s_o) = a_2$. So $z_1 = z_2$ and hence T is one-one. T is onto Y. For let $g \in Y$. Define a map $f : S \longrightarrow A$ by $f(s) = g(s)$ and let $a = g(s_o)$. Then $f \in X$ since $\sum_{s \in S} \|f(s)\| = \sum_{s \in S} \|g(s)\| \leq \sum_{s' \in S'} \|g(s')\| < \infty$. So $(f, a) \in X \oplus A$ and $T(f, a) = g$. So T is onto Y. Finally T is bounded. For if $z = (f, a) \in X \oplus A$, $\|Tz\| = \sum_{s' \in S'} \|Tz(s')\| = \sum_{s \in S} \|f(s)\| + \|a\| = \|f\| + \|a\| = \|(f, a)\| = \|z\|$. So T is a one-one bounded linear transformation from the Banach space $X \oplus A$ onto the Banach space Y. By the closed graph theorem, T^{-1} is continuous. Hence $X \oplus A \sim Y$. Now by Lemma 10.8, $Y \sim X$. So $X \oplus A \sim X$. Q.E.D.

10.10 Lemma. Let S be an infinite set. For each $s \in S$, let $A_s = \ell_1(S)$ and let $X = \sum_{s \in S} \oplus_1 A_s$. Then $X \sim \ell_1(S)$.

Proof. Since S is infinite, the cardinality of $S \times S$ equals the cardinality of S. So there exists a one-one mapping φ from $S \times S$ onto S. We want to define a one-one bounded linear transformation T from X onto $\ell_1(S)$. Let $f \in X$. We define a scalar valued function Tf on S as follows. For each $s \in S$, let $(x, y) = \varphi^{-1}(s)$ and define $(Tf)(s) = (f(y))(x)$. We claim that $Tf \in \ell_1(S)$. Now for each $y \in S$, we have

$\sum_{x \in S} |f(y)(x)| < \infty$ since $f(y) \in \ell_1(S)$ and indeed

$\sum_{x \in S} |f(y)(x)| = \|f(y)\|$. Also $\sum_{y \in S} \|f(y)\| < \infty$ since

$f \in X$. For each $(x, y) \in S \times S$, $|f(y)(x)| \geq 0$. Hence

it follows that $\sum_{(x,y) \in S \times S} |f(y)(x)| < \infty$.[1] The summability

of the family $\{|f(y)(x)|\}_{(x,y) \in S \times S}$ implies that

[1] See for example Kelley [20, page 78, exercise G(h)(ii)].

$\sum_{s \in S} |Tf(s)| < \infty$. For let $\alpha = \sum_{(x,y) \in SxS} |f(y)(x)|$ and let $\epsilon > 0$. Then there exists a non-empty finite subset $(SxS)_\epsilon$ of SxS such that if $(SxS)_F$ is any finite subset of SxS containing $(SxS)_\epsilon$, we have

$$(10.1) \qquad \left| \alpha - \sum_{(x,y) \in (SxS)_F} |f(y)(x)| \right| < \epsilon .$$

Let $S_\epsilon = \varphi((SxS)_\epsilon)$. S_ϵ is a finite non-empty subset of S. Let S_F be any finite subset of S containing S_ϵ. Let $(SxS)_F = \varphi^{-1}(S_F)$. $(SxS)_F$ is finite and contains $(SxS)_\epsilon$ and so (10.1) holds. But

$\sum_{s \in S_F} |Tf(s)| = \sum_{(x,y) \in (SxS)_F} |f(y)(x)|$ and so

$\left| \alpha - \sum_{S \in S_F} |Tf(s)| \right| < \epsilon$. So the family $\{|Tf(s)|\}_{s \in S}$ is summable and indeed

$$(10.2) \qquad \sum_{s \in S} |Tf(s)| = \alpha = \sum_{(x,y) \in SxS} |f(y)(x)|$$

So $Tf \in \ell_1(S)$ and so we have a mapping $T : X \longrightarrow \ell_1(S)$

given by $f \longrightarrow Tf$. T is linear. For suppose $f, g \in X$ and $s \in S$. Let $s = \varphi(x, y)$. Then

$$T(f + g)(s) = ((f + g)(y))(x) = (f(y) + g(y))(x)$$
$$= f(y)(x) + g(y)(x) = (Tf)(s) + (Tg)(s)$$
$$= (Tf + Tg)(s).$$

So $T(f + g) = Tf + Tg$. Similarly if α is a scalar, $T(\alpha f) = \alpha(Tf)$ and so T is linear. T is bounded. For if $f \in X$, $\|Tf\| = \sum_{s \in S} |Tf(s)| = \sum_{(x,y) \in S \times S} |f(y)(x)|$

$$\text{(by (10.2))} = \sum_{y \in S} \Big(\sum_{x \in S} |f(y)(x)| \Big)^{1} = \sum_{y \in S} \|f(y)\| = \|f\|.$$

So T is bounded and indeed an isometry, hence one-one. Finally T is onto $\ell_1(S)$. For let $g \in \ell_1(S)$. For each $y \in S$, define a scalar valued function $f(y)$ on S by $(f(y))(x) = g(\varphi(x, y))$, $x \in S$. We claim that the family $\{|f(y)(x)|\}_{(x,y) \in S \times S}$ is summable (with sum $\|g\|$). For let $\epsilon_1 > 0$. Then there exists a finite non-empty subset S_{ϵ_1} of S such that if S_{F_1} is any finite subset of S containing S_{ϵ_1}, we have

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[1] See for example Kelley [20, page 78, exercise G(h)(i)].

(10.3) $$\left| \|g\| - \sum_{s \in S_{F_1}} |g(s)| \right| < \epsilon_1 .$$

Let $(SxS)_{\epsilon_1} = \varphi^{-1}(S_{\epsilon_1})$. $(SxS)_{\epsilon_1}$ is a finite non-empty subset of SxS. Let $(SxS)_{F_1}$ be any finite subset of SxS containing $(SxS)_{\epsilon_1}$ and let $S_{F_1} = \varphi((SxS)_{F_1})$. S_{F_1} is finite and contains S_{ϵ_1}, so (10.3) holds. But $\sum_{s \in S_{F_1}} |g(s)| = \sum_{(x,y)\in(SxS)_{F_1}} |f(y)(x)|$

and so $\left| \|g\| - \sum_{(x,y)\in(SxS)_{F_1}} |f(y)(x)| \right| < \epsilon_1$. So the family $\{|f(y)(x)|\}_{(x,y)\in SxS}$ is summable and hence for each fixed $y \in S$, the family $\{|f(y)(x)|\}_{x\in S}$ is summable, i.e. $f(y) \in \ell_1(S)$ for each $y \in S$. Now $\|f(y)\| = \sum_{x\in S} |f(y)(x)|$ and the summability of $\{|f(y)(x)|\}_{(x,y)\in SxS}$ implies the summability of $\{\|f(y)\|\}_{y\in S}$, i.e. $\sum_{y\in S} \|f(y)\| < \infty$ which means that the mapping f from S to $\ell_1(S)$ given by $y \longrightarrow f(y)$, $y \in S$, is in X. Finally $Tf = g$. For let $s \in S$ and let $(x, y) = \varphi^{-1}(s)$. Then $(Tf)(s) = f(y)(x) = g(\varphi(x, y)) = g(s)$. So T is a one-one bounded linear transformation from the Banach space X onto the Banach space $\ell_1(S)$. By the closed graph theorem T^{-1} is continuous. So $X \sim \ell_1(S)$. Q.E.D.

10.11 <u>Lemma</u>. Let X be a Banach space and let Y and W be closed subspaces of X such that $X = Y + W$, $Y \cap W = \{0\}$. Then $X \sim Y \oplus W$.

<u>Proof</u>. Define $T : Y \oplus W \longrightarrow X$ by $T(y, w) = y + w$. T is clearly linear. Also if $T(y_1, w_1) = T(y_2, w_2)$, then $y_1 + w_1 = y_2 + w_2$ which implies that $y_1 - y_2 = w_2 - w_1$. Since $Y \cap W = \{0\}$, we must conclude that $y_1 - y_2 = 0$ and $w_2 - w_1 = 0$. So T is one-one. T is onto X for by hypothesis each $x \in X$ can be written as $x = y + w$, $y \in W$, $w \in W$ and so $T(y, w) = x$. T is bounded since $\|T(y, w)\| = \|y + w\| \leq \|y\| + \|w\| = \|(y, w)\|$. So T is a one-one bounded linear transformation from the Banach space $Y \oplus W$ onto the Banach space X. By the closed graph theorem, T^{-1} is continuous. So $X \sim Y \oplus W$. Q.E.D.

10.12 Lemma. Let S be an infinite set. Let X and W be closed subspaces of $\ell_1(S)$ such that $\ell_1(S) = X + W$ and $X \cap W = \{0\}$. Let Y and Y_1 be closed subspaces of X such that $X = Y + Y_1$, $Y \cap Y_1 = \{0\}$, and suppose $Y \sim \ell_1(S)$. Then $X \sim \ell_1(S)$.

Proof. For each $s \in S$, let $A_s = \ell_1(S)$, let $B_s = X \oplus W$, let $X_s = X$, and let $W_s = W$. Let

$$A = \sum_{s \in S} \oplus_1 A_s, \quad \text{let} \quad B = \sum_{s \in S} \oplus_1 B_s, \quad \text{let} \quad Z = \sum_{s \in S} \oplus_1 X_s,$$

and let $R = \sum_{s \in S} \oplus_1 W_s$. Then we have

$$\begin{aligned}
\ell_1(S) = X + W &\sim X \oplus W && \text{by Lemma 10.11} \\
&\sim (Y \oplus Y_1) \oplus W && \text{by Lemmas 10.11 and 10.2} \\
&\sim (\ell_1(S) \oplus Y_1) \oplus W && \text{by hypothesis and Lemma 10.2} \\
&\sim \ell_1(S) \oplus (Y_1 \oplus W) && \text{by Lemma 10.3} \\
&\sim A \oplus (Y_1 \oplus W) && \text{by Lemmas 10.10 and 10.2} \\
&\sim B \oplus (Y_1 \oplus W) && \text{by Lemmas 10.11, 10.4, and 10.2} \\
&\sim (Z \oplus R) \oplus (Y_1 \oplus W) && \text{by Lemmas 10.5 and 10.2}
\end{aligned}$$

$\sim (Z \oplus R) \oplus (W \oplus Y_1)$ by Lemmas 10.6 and 10.7

$\sim Z \oplus (R \oplus (W \oplus Y_1))$ by Lemma 10.3

$\sim Z \oplus ((R \oplus W) \oplus Y_1)$ by Lemmas 10.3 and 10.7

$\sim Z \oplus (R \oplus Y_1)$ by Lemmas 10.9, 10.2 and 10.7

$\sim (Z \oplus R) \oplus Y_1$ by Lemma 10.3

$\sim B \oplus Y_1$ by Lemmas 10.5 and 10.2

$\sim A \oplus Y_1$ by Lemmas 10.11, 10.4, and 10.2

$\sim \ell_1(S) \oplus Y_1$ by Lemmas 10.10 and 10.2

$\sim Y \oplus Y_1$ by hypothesis and Lemma 10.2

$\sim X$ by Lemma 10.11.

So $\ell_1(S) \sim X$. Q.E.D.

10.13 Definition. Let X be a Banach space and let $\{x_i\}_{i=1,2,3,\ldots}$ be a sequence of vectors in X. We say that the sequence $\{x_i\}_{i=1,2,\ldots}$ is a Schauder basis for X if for each $x \in X$, there exists a unique sequence $\{\alpha_i\}_{i=1,2,\ldots}$ of scalars such that the sequence $\{x - \sum_{i=1}^{n} \alpha_i x_i\}_{n=1,2,\ldots}$ converges to 0 as $n \longrightarrow \infty$, i.e. $\sum_{i=1}^{\infty} \alpha_i x_i$ converges to x.

We assume the reader is familiar with the basic properties of Schauder bases as contained for example in Banach [5 , chapter VII] or Day [7 , chapter IV]. In particular we assume the reader is familiar with the following:

(a) No vector in a Schauder basis is zero.

(b) If X is a Banach space with a Schauder basis $\{x_i\}_{i=1,2,\ldots}$, let $X_{\{x_i\}_{i=1,2,\ldots}}$ denote the set of all sequences of scalars $\{\alpha_i\}_{i=1,2,\ldots}$ such that the series $\sum_{i=1}^{\infty} \alpha_i x_i$ converges. Under the usual definition of addition of two sequences and multiplication of a sequence by a scalar, $X_{\{x_i\}_{i=1,2,\ldots}}$ is a vector space and if we define $\|\alpha\|$ for $\alpha = \{\alpha_i\}_{i=1,2,\ldots} \in X_{\{x_i\}_{i=1,2,\ldots}}$ by $\|\alpha\| = \sup_{1 \leq n < \infty} \{\|\sum_{i=1}^{n} \alpha_i x_i\|\}$, then $X_{\{x_i\}_{i=1,2,\ldots}}$ is a Banach space equivalent to X under the mapping $T : X_{\{x_i\}_{i=1,2,\ldots}} \longrightarrow X$ defined by $T(\alpha) = \sum_{i=1}^{\infty} \alpha_i x_i$.

For each $i = 1,2,\ldots$ we define a scalar valued function

x_i^* on X by $x_i^*(x) = x_i^*(\sum_{n=1}^{\infty} \alpha_n x_n) = \alpha_i$. Then x_i^* is

a continuous linear functional on X and

$\|x_i^*\| \leq \frac{2\|T^{-1}\|}{\|x_i\|}$. The sequence $\{x_i^*\}_{i=1,2,\ldots}$ in X^*

is called the sequence orthonormal to $\{x_i\}_{i=1,2,\ldots}$.

(c) If $S = \{s_1, s_2, s_3, \ldots\}$ is a countably infinite set, let e_i, $i = 1,2,\ldots$, be that element in $\ell_1(S)$ defined by $e_i(s_i) = 1$ and $e_i(s) = 0$, $s \neq s_i$. Then the sequence $\{e_i\}_{i=1,2,\ldots}$ is a Schauder basis for $\ell_1(S)$. If $f \in \ell_1(S)$, then $f = \sum_{i=1}^{\infty} \alpha_i e_i$ where $\alpha_i = f(s_i)$, $i = 1,2,\ldots$, and

$$\|f\| = \sum_{i=1}^{\infty} |\alpha_i|.$$

10.14 <u>Notation</u>. Let X be a Banach space and let $\{x_n\}_{n=1,2,\ldots}$ be a sequence of elements in X. We shall denote by $[x_n]_{n=1,2,\ldots}$ the smallest (in the sense of set inclusion) closed subspace of X containing each of the vectors x_n, $n = 1,2,\ldots$

10.15 Lemma. Let X be a Banach space and let $\{z_n\}_{n=1,2,\ldots}$ be a sequence of elements in X. Suppose there exists a constant $M \geq 1$ such that

$$\|\sum_{i=1}^{n} \alpha_i z_i\| \leq M \|\sum_{i=1}^{m} \alpha_i z_i\|$$

for all positive integers m and n with $m \geq n$ and all scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$. Assume that no z_n equals 0. Then the sequence $\{z_n\}_{n=1,2,\ldots}$ is a Schauder basis for $[z_n]_{n=1,2,\ldots}$.

Proof. Consider the set Z of all vectors $z \in X$ for which there exists a sequence (not necessarily unique) of scalars $\{\alpha_n\}_{n=1,2,\ldots}$ such that the sequence

$$\{z - \sum_{i=1}^{n} \alpha_i z_i\}_{n=1,2,\ldots} \text{ converges to } 0 \text{ as } n \longrightarrow \infty.$$

In other words Z consists of all those vectors $z \in X$ which can be expressed as a convergent infinite series of the form $\sum_{i=1}^{\infty} \alpha_i z_i$. It is clear that each z_n is in

Z and that Z is a linear subspace of X. Also $Z \subset [z_n]_{n=1,2,\ldots}$. For if we let Z_V denote the linear subspace (not necessarily closed) of X generated by the vectors in the sequence $\{z_n\}_{n=1,2,\ldots}$, every element z of Z is the limit of a sequence of elements in Z_V, namely the sequence of partial sums of an infinite series converging to z, and hence z is in the closure of Z_V, i.e. in $[z_n]_{n=1,2,\ldots}$.

What we want to show is that $Z = [z_n]_{n=1,2,\ldots}$ and that the infinite series which converges to an element $z \in Z$ is unique. We shall show uniqueness first. Suppose $z = \sum_{i=1}^{\infty} \alpha_i z_i = \sum_{i=1}^{\infty} \alpha_i' z_i$. Then

$0 = \sum_{i=1}^{\infty} (\alpha_i - \alpha_i') z_i$ and so it suffices to show that

the expansion of 0 into an infinite series (which is of course always possible by taking all the coefficients to be zero) is unique. So let $0 = \sum_{i=1}^{\infty} \gamma_i z_i$, where the γ_i's are scalars, and suppose some $\gamma_i \neq 0$. Let i_o be a positive integer such that $\gamma_{i_o} \neq 0$, but

$\gamma_i = 0$ for all $0 \leq i < i_o$. Then for all integers $n \geq i_o$, we have, by hypothesis,

$$(10.4) \qquad \left\|\sum_{i=1}^{i_o} \gamma_i z_i\right\| \leq M \left\|\sum_{i=1}^{n} \gamma_i z_i\right\|.$$

Now since $0 = \sum_{i=1}^{\infty} \gamma_i z_i = \lim_{n \to \infty} \sum_{i=1}^{n} \gamma_i z_i$ and since the norm is a continuous function it follows that

$$\lim_{n \to \infty} \left\|\sum_{i=1}^{n} \gamma_i z_i\right\| = \left\|\lim_{n \to \infty} \sum_{i=1}^{n} \gamma_i z_i\right\| = \|0\| = 0.$$ Hence given $\epsilon_1 > 0$, there exists a positive integer N such that for all $n \geq N$, we have

$$\left| 0 - \left\|\sum_{i=1}^{n} \gamma_i z_i\right\| \right| < \frac{\epsilon_1}{M}, \quad \text{that is,} \quad \left\|\sum_{i=1}^{n} \gamma_i z_i\right\| < \frac{\epsilon_1}{M}.$$

If in (10.4) we choose our $n \geq N$ (as well as $n \geq i_o$), we have $\left\|\sum_{i=1}^{i_o} \gamma_i z_i\right\| < \epsilon_1$. But $\sum_{i=1}^{i_o} \gamma_i z_i = \gamma_{i_o} z_{i_o}$ and so $\|\gamma_{i_o} z_{i_o}\| < \epsilon_1$. Since $\epsilon_1 > 0$ was arbitrary, we conclude that $\gamma_{i_o} z_{i_o} = 0$. But we assumed that

$\gamma_{i_o} \neq 0$ and so we must conclude that $z_{i_o} = 0$. But $z_{i_o} = 0$ contradicts our hypothesis that no $z_n = 0$. So we must conclude that if $0 = \sum_{i=1}^{\infty} \gamma_i z_i$, then $\gamma_i = 0$ for all i. So uniqueness is established.

There remains for us to prove that $Z = [z_n]_{n=1,2,\ldots}$. It suffices to show that Z is closed since Z is a linear subspace of X containing each z_n, $Z \subset [z_n]_{n=1,2,\ldots}$, and $[z_n]_{n=1,2,\ldots}$ is the smallest (with respect to set inclusion) closed linear subspace of X containing each z_n. Let $\{x_n\}_{n=1,2,\ldots}$ be a sequence of elements in Z and suppose $\lim_{n \to \infty} x_n = x \in [z_n]_{n=1,2,\ldots}$. We shall prove that $x \in Z$, thus establishing that Z is closed. Now since $\{x_n\}_{n=1,2,\ldots}$ is a convergent sequence, it is Cauchy.

Let $x_n = \sum_{i=1}^{\infty} \beta_i(x_n) z_i$, $n = 1,2,\ldots$, where $\beta_i(x_n)$ denotes the unique coefficient of z_i in the infinite series expansion of x_n. Let k be a non-negative integer. Define a map U_k from Z to Z as follows:

If $k \geq 1$ and $x = \sum_{i=1}^{\infty} \alpha_i z_i \in Z$, define $U_k(x) = \sum_{i=1}^{k} \alpha_i z_i$;

if $k = 0$ define $U_o(x) = 0$. It is clear that each map U_k is linear. Also $\|U_k\| \leq M^2$ for all k. To see this last inequality we may assume that $k \geq 1$, since $\|U_o\| = 0 < M^2$. We first note that for each

$x = \sum_{i=1}^{\infty} \alpha_i z_i \in Z$, we have

$$(10.5) \qquad \left\|\sum_{i=1}^{n} \alpha_i z_i\right\| \leq M^2\|x\| \quad \text{for all positive integers } n.$$

For suppose that for some positive integer n_o we have $\left\|\sum_{i=1}^{n_o} \alpha_i z_i\right\| > M^2\|x\|$. Then for all $n \geq n_o$, we have

$M\left\|\sum_{i=1}^{n} \alpha_i z_i\right\| \geq \left\|\sum_{i=1}^{n_o} \alpha_i z_i\right\| > M^2\|x\|$ and hence

$\|x\| = \lim_{n \to \infty} \left\|\sum_{i=1}^{n} \alpha_i z_i\right\| \geq \frac{1}{M}\left\|\sum_{i=1}^{n_o} \alpha_i z_i\right\| > M\|x\| \geq \|x\|$ which

is impossible. So (10.5) is established and hence

$\|U_k(x)\| = \|\sum_{i=1}^{k} \alpha_i z_i\| \leq M^2\|x\|$. So $\|U_k\| \leq M^2$.

Let $0 \leq k < j$, j,k integers and let $U_{kj} = U_j - U_k$. Then we have $\|U_{kj}(x_m - x_n)\| = \|(U_j - U_k)(x_m - x_n)\| = \|U_j(x_m - x_n) - U_k(x_m - x_n)\| \leq \|U_j(x_m - x_n)\| + \|U_k(x_m - x_n)\| \leq 2M^2\|x_m - x_n\|$. In particular $\|\beta_j(x_m)z_j - \beta_j(x_n)z_j\| = \|\beta_j(x_m - x_n)z_j\| = \|U_{j-1,j}(x_m - x_n)\| \leq 2M^2\|x_m - x_n\|$, and so the sequence $\{\beta_j(x_n)z_j\}_{n=1,2,...}$ is Cauchy for each fixed positive integer j. Hence the sequence $\{\beta_j(x_n)z_j\}_{n=1,2,...}$ converges to an element $a_jz_j \in [z_n]_{n=1,2,...}$ for each such j.[1] We shall now show that the sequence

$\{\sum_{i=1}^{n} a_i z_i\}_{n=1,2,...}$ is Cauchy. Let $\epsilon > 0$ and let M_ϵ be a positive integer such that $n > m > M_\epsilon$ implies $\|x_m - x_n\| < \frac{\epsilon}{M^2}$. We have

---

[1] It is trivial to show that if a sequence of vectors $\{\alpha_n x\}_{n=1,2,...}$ (α_n scalars, x a fixed vector) in a normed linear space converges to y, then $y = \alpha x$ for some scalar α.

$$\|U_{kj}x_m - \sum_{k<i\leq j} a_i z_i\| = \|U_{kj}x_m - U_{kj}x_n + U_{kj}x_n - \sum_{k<i\leq j} a_i z_i\|$$

$$= \|U_{kj}(x_m - x_n) + U_{kj}x_n - \sum_{k<i\leq j} a_i z_i\|$$

$$\leq 2M^2\|x_m - x_n\| + \|U_{kj}x_n - \sum_{k<i\leq j} a_i z_i\| = 2M^2\|x_m - x_n\|$$

$$+ \|\beta_{k+1}(x_n)z_{k+1} + \beta_{k+2}(x_n)z_{k+2} + \ldots + \beta_j(x_n)z_j - \sum_{k<i\leq j} a_i z_i\|$$

$$\leq 2M^2\|x_m - x_n\| + \|\beta_{k+1}(x_n)z_{k+1} - a_{k+1}z_{k+1}\| + \ldots + \|\beta_j(x_n)z_j - a_j z_j\|$$

In particular if we choose our integers m and n such that $n > m > M_\epsilon$, we have

(10.6) $$\|U_{kj}x_m - \sum_{k<i\leq j} a_i z_i\| < 2\epsilon + \|\beta_{k+1}(x_n)z_{k+1} - a_{k+1}z_{k+1}\| + \ldots + \|\beta_j(x_n)z_j - a_j z_j\|.$$

Since for each positive integer j the sequence $\{\beta_j(x_n)z_j\}_{n=1,2,\ldots}$ converges to $a_j z_j$, we can by choosing n sufficiently large make the right hand side of (10.6) less than $2\epsilon + \delta$ for any given $\delta > 0$ from which we conclude that

(10.7) $$\|U_{kj}x_m - \sum_{k<i\leq j} a_i z_i\| \leq 2\epsilon \quad \text{for all} \quad 0 \leq k < j$$

and all $m > M_\epsilon$.

Now let $m > M_\epsilon$ be fixed. Then since

$$\lim_{k \to \infty} \sum_{i=1}^{k} \beta_i(x_m)z_i = x_m,$$ we have

$$\lim_{k \to \infty} \|\sum_{i=1}^{k} \beta_i(x_m)z_i - x_m\| = 0.$$ But for $k \geq 1$,

$$\sum_{i=1}^{k} \beta_i(x_m)z_i = U_k x_m$$ and so we have

$$\lim_{k \to \infty} \|U_k(x_m) - x_m\| = 0.$$ So there exists a positive

integer K_ϵ such that $k > K_\epsilon$ implies

$\|U_k(x_m) - x_m\| < \frac{\epsilon}{2}$. Hence for all integers j and k

such that $j > k > K_\epsilon$, we have

$$\|U_{kj}x_m\| = \|(U_j - U_k)x_m\| = \|U_j x_m - U_k x_m\|$$

(10.8) $$= \|U_j x_m - x_m + x_m - U_k x_m\| \leq \|U_j x_m - x_m\|$$

$$+ \|U_k x_m - x_m\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now $\|U_{kj}x_m - \sum_{k<i\leq j} a_i z_i\| \geq \|\sum_{k<i\leq j} a_i z_i\| - \|U_{kj}x_m\|$ and so

$$\left\| \sum_{k<i\leq j} a_i z_i \right\| \leq \left\| U_{kj} x_m - \sum_{k<i\leq j} a_i z_i \right\| + \| U_{kj} x_m \|$$

(10.9)

$$< 2\epsilon + \epsilon = 3\epsilon \quad \text{for all} \quad j > k > K_\epsilon$$

by (10.7) and (10.8). (10.9) shows that the sequence $\{ \sum_{i=1}^{n} a_i z_i \}_{n=1,2,\ldots}$ is Cauchy.

Since the terms of the sequence $\{ \sum_{i=1}^{n} a_i z_i \}_{n=1,2,\ldots}$ are all in $[z_n]_{n=1,2,\ldots}$ and since $[z_n]_{n=1,2,\ldots}$ is closed and hence complete, the sequence $\{ \sum_{i=1}^{n} a_i z_i \}_{n=1,2,\ldots}$ converges to some element $y \in [z_n]_{n=1,2,\ldots}$, i.e.

$$y = \lim_{n \to \infty} \sum_{i=1}^{n} a_i z_i = \sum_{i=1}^{\infty} a_i z_i .$$ So $y \in Z$ by the definition of Z. We shall prove that $x = y$ and this will complete the proof of our lemma. We shall establish that $x = y$ by showing that our original sequence $\{x_n\}_{n=1,2,\ldots}$

converges to y. Let $\epsilon' > 0$. Let $M_{\epsilon'}$ be a positive integer such that $n > m > M_{\epsilon'}$ implies $\|x_m - x_n\| < \frac{\epsilon'}{M^2}$. Let $m > M_{\epsilon'}$ be fixed. Then by (10.7)[1] we have

$$\|U_{kj}x_m - \sum_{k<i\leq j} a_i z_i\| \leq 2\epsilon' \quad \text{for all} \quad 0 \leq k < j.$$

In particular for $k = 0$ and j an arbitrary positive integer, we have

$$\|U_{oj}x_m - \sum_{i=1}^{j} a_i z_i\| \leq 2\epsilon'. \tag{10.10}$$

But $U_{oj} = U_j - U_o = U_j$ and so (10.10) becomes

$$\|U_j x_m - \sum_{i=1}^{j} a_i z_i\| \leq 2\epsilon'. \tag{10.11}$$

Now $x_m - y = \sum_{i=1}^{\infty} \beta_i(x_m) z_i - \sum_{i=1}^{\infty} a_i z_i$

$$= \sum_{i=1}^{\infty} (\beta_i(x_m) - a_i) z_i$$

$$= \lim_{j \to \infty} \sum_{i=1}^{j} (\beta_i(x_m) - a_i) z_i.$$

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[1] We are using "ϵ'" instead of "ϵ" in this part of the proof because we already used "ϵ". However ϵ was arbitrary and so the various inequalities that we deduced are valid with the appropriate changes in notation.

So $\|x_m - y\| = \|\lim_{j \to \infty} \sum_{i=1}^{j} (\beta_i(x_m) - a_i)z_i\|$

$$= \lim_{j \to \infty} \|\sum_{i=1}^{j} (\beta_i(x_m) - a_i)z_i\|$$

$$= \lim_{j \to \infty} \|U_j x_m - \sum_{i=1}^{j} a_i z_i\|.$$

But by (10.11) $\|U_j x_m - \sum_{i=1}^{j} a_i z_i\| \leq 2\epsilon'$ for all positive integers j and so $\lim_{j \to \infty} \|U_j x_m - \sum_{i=1}^{j} a_i z_i\| \leq 2\epsilon'$.

So $\|x_m - y\| \leq 2\epsilon'$ for $m > M_{\epsilon'}$ which means that $\lim_{n \to \infty} x_n = y$. Since by our assumption $\lim_{n \to \infty} x_n = x$, we must conclude that $x = y \in Z$. Q.E.D.

10.16 Lemma. Let $S = \{s_1, s_2, \ldots\}$ be a countably infinite set. Let $\{N_m\}_{m=0,1,2,\ldots}$ be a sequence of integers such that $N_o = 0$ and $N_m < N_{m+1}$, $m = 0,1,2,\ldots$. Let $\{z_m\}_{m=1,2,\ldots}$ be a sequence of vectors in $\ell_1(S)$ such that $z_m \neq 0$, $m = 1,2,\ldots$, and such that

$$z_m = \sum_{i=N_{m-1}+1}^{N_m} t_i^m e_i, \quad m = 1,2,\ldots$$

Then the sequence $\{z_m\}_{m=1,2,\ldots}$ is a Schauder basis for $[z_m]_{m=1,2,\ldots}$, $[z_m]_{m=1,2,\ldots}$ is congruent to $\ell_1(S)$, and $[z_m]_{m=1,2,\ldots}$ is the image of a continuous projection P with $\|P\| = 1$ from $\ell_1(S)$. (In particular $[z_m]_{m=1,2,\ldots}$ is complemented in $\ell_1(S)$.)

<u>Proof</u>. Let k be a positive integer and $\lambda_1, \lambda_2, \ldots, \lambda_k$ arbitrary scalars. Then

$$(10.12) \qquad \Big\|\sum_{m=1}^{k} \lambda_m z_m\Big\| = \sum_{m=1}^{k} |\lambda_m| \; \|z_m\|$$

since $\|\sum_{m=1}^{k} \lambda_m z_m\| = \|\sum_{m=1}^{k} \sum_{i=N_{m-1}+1}^{N_m} \lambda_m t_i^m e_i\| =$

$$\|\lambda_1 t_1^1 e_1 + \lambda_1 t_2^1 e_2 + \ldots + \lambda_1 t_{N_1}^1 e_{N_1} + \lambda_2 t_{N_1+1}^2 e_{N_1+1} + \ldots + \lambda_2 t_{N_2}^2 e_{N_2} + \ldots + \lambda_k t_{N_k}^k e_{N_k}\|$$

$$= \sum_{m=1}^{k} \sum_{i=N_{m-1}+1}^{N_m} |\lambda_m t_i^m| = \sum_{m=1}^{k} |\lambda_m| \Big(\sum_{i=N_{m-1}+1}^{N_m} |t_i^m| \Big) = \sum_{m=1}^{k} |\lambda_m| \; \|z_m\|.$$

Now clearly if p and q are positive integers with $p \leq q$, and $\alpha_1, \alpha_2, \ldots, \alpha_p, \ldots, \alpha_q$ are arbitrary scalars, we have $\sum_{m=1}^{p} |\alpha_m| \, \|z_m\| \leq \sum_{m=1}^{q} |\alpha_m| \, \|z_m\|$

and hence by (10.12)

$$(10.13) \qquad \left\| \sum_{m=1}^{p} \alpha_m z_m \right\| \leq \left\| \sum_{m=1}^{q} \alpha_m z_m \right\|.$$

By Lemma 10.15 it follows that the sequence $\{z_m\}_{m=1,2,\ldots}$ is a Schauder basis for $[z_m]_{m=1,2,\ldots}$.

We want now to define an isometry T from $\ell_1(S)$ onto $[z_m]_{m=1,2,\ldots}$. Let $x = \sum_{i=1}^{\infty} t_i e_i \in \ell_1(S)$ and consider the infinite series $\sum_{i=1}^{\infty} \left\| t_i \frac{z_i}{\|z_i\|} \right\|$. This infinite series converges since $\left\| t_i \frac{z_i}{\|z_i\|} \right\| = |t_i|$, $i=1,2,\ldots$, and $\sum_{i=1}^{\infty} |t_i|$ converges. The convergence of the series $\sum_{i=1}^{\infty} \left\| t_i \frac{z_i}{\|z_i\|} \right\|$ implies the convergence of the series

$\sum_{i=1}^{\infty} t_i \frac{z_i}{\|z_i\|}$. We define

$$T(x) = \sum_{i=1}^{\infty} t_i \frac{z_i}{\|z_i\|}, \quad x = \sum_{i=1}^{\infty} t_i e_i \in \ell_1(S).$$

Since $[z_m]_{m=1,2,\ldots}$ is closed, $T(x) \in [z_m]_{m=1,2,\ldots}$.

It is clear that T is linear and since

$$\|Tx\| = \|\sum_{i=1}^{\infty} t_i \frac{z_i}{\|z_i\|}\| = \|\lim_{k \to \infty} \sum_{i=1}^{k} t_i \frac{z_i}{\|z_i\|}\|$$

$$= \lim_{k \to \infty} \|\sum_{i=1}^{k} t_i \frac{z_i}{\|z_i\|}\| = \lim_{k \to \infty} \sum_{i=1}^{k} |t_i| \text{ (by (11.12))}$$

$$= \sum_{i=1}^{\infty} |t_i| = \|x\|,$$

T is an isometry. Finally T is onto $[z_m]_{m=1,2,\ldots}$. For let $y \in [z_m]_{m=1,2,\ldots}$. Then $y = \sum_{i=1}^{\infty} \gamma_i z_i$ since $\{z_m\}_{m=1,2,\ldots}$ is a Schauder basis for $[z_m]_{m=1,2,\ldots}$.

The series $\sum_{i=1}^{\infty} \gamma_i \|z_i\| e_i$ defines an element in $\ell_1(S)$, i.e. converges, since the series $\sum_{i=1}^{\infty} \|\gamma_i \|z_i\| e_i\|$ converges

since $\|\gamma_i \|z_i\| e_i\| = |\gamma_i| \|z_i\|$ and

$$\sum_{i=1}^{\infty} |\gamma_i| \|z_i\| = \lim_{k \to \infty} \sum_{i=1}^{k} |\gamma_i| \|z_i\|$$

$$= \lim_{k \to \infty} \|\sum_{i=1}^{k} \gamma_i z_i\| \quad \text{(by (10.12))}$$

$$= \|\lim_{k \to \infty} \sum_{i=1}^{k} \gamma_i z_i\| = \|y\|, \text{ and}$$

$$T(\sum_{i=1}^{\infty} \gamma_i \|z_i\| e_i) = \sum_{i=1}^{\infty} \gamma_i \|z_i\| \frac{z_i}{\|z_i\|} = y. \text{ So}$$

T is onto $[z_m]_{m=1,2,\ldots}$ and hence $\ell_1(S)$ and $[z_m]_{m=1,2,\ldots}$ are congruent..

Finally we want to define a projection of norm one from $\ell_1(S)$ onto $[z_m]_{m=1,2,\ldots}$. Let E_m, $m=1,2,\ldots,$ denote the normed linear subspace of $\ell_1(S)$ spanned by the vectors $e_{N_{m-1}+1}, e_{N_{m-1}+2}, \ldots, e_{N_m}$. Each z_m, $m=1,2,\ldots,$ is by its very form a member of E_m.

Hence there exists a continuous linear functional f_m defined on E_m such that $f_m(z_m) = 1$ and $\|f_m\| = \frac{1}{\|z_m\|}$, $m=1,2,\ldots$. Again let $x = \sum_{i=1}^{\infty} t_i e_i \in \ell_1(S)$ and consider the series

$$\sum_{m=1}^{\infty} (f_m(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i)) z_m. \tag{10.14}$$

Now for $m = 1,2,\ldots,$ we have

$$\begin{aligned} |f_m(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i)| &\leq \|f_m\| \, \|\sum_{i=N_{m-1}+1}^{N_m} t_i e_i\| \\ &= \frac{1}{\|z_m\|} \sum_{i=N_{m-1}+1}^{N_m} |t_i|. \end{aligned} \tag{10.15}$$

If we let $A_m = \sum_{i=N_{m-1}+1}^{N_m} |t_i|$, $m = 1,2,\ldots,$ then the norm of the general term (i.e. the $m^{\underline{th}}$ term) of the series (10.14) is less than or equal to A_m. Since the series $\sum_{m=1}^{\infty} A_m$ converges $(\sum_{m=1}^{\infty} A_m = \sum_{i=1}^{\infty} |t_i| = \|x\|)$, it follows that the series (10.14) converges and indeed to an element of $[z_m]_{m=1,2,\ldots}$ since $[z_m]_{m=1,2,\ldots}$ is closed. We define a map $P : \ell_1(S) \longrightarrow [z_m]_{m=1,2,\ldots}$ by

$$P(x) = \sum_{m=1}^{\infty} (f_m(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i))z_m$$

where $x = \sum_{i=1}^{\infty} t_i e_i \in \ell_1(S)$. It is easy to see that P is linear. P is bounded. For we have

$$\|Px\| = \|\sum_{m=1}^{\infty} (f_m(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i))z_m\|$$

$$= \|\lim_{k \to \infty} \sum_{m=1}^{k} (f_m(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i))z_m\|$$

$$= \lim_{k \to \infty} \|\sum_{m=1}^{k} (f_m(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i))z_m\|$$

$$= \lim_{k \to \infty} \sum_{m=1}^{k} |f_m(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i)| \, \|z_m\| \quad \text{by (10.12).}$$

Now $|f_m(\sum_{i=N_{m-1}+1}^{N_m} t_ie_i)| \leq \frac{1}{\|z_m\|} \sum_{i=N_{m-1}+1}^{N_m} |t_i|$ by (10.15)

and hence for all positive integers k

$$\sum_{m=1}^{k} |f_m(\sum_{i=N_{m-1}+1}^{N_m} t_ie_i)| \|z_m\| \leq \sum_{m=1}^{k} (\frac{1}{\|z_m\|} \sum_{i=N_{m-1}+1}^{N_m} |t_i|) \|z_m\|$$

$$= \sum_{m=1}^{k} (\sum_{i=N_{m-1}+1}^{N_m} |t_i|)$$

$$\leq \sum_{m=1}^{\infty} (\sum_{i=N_{m-1}+1}^{N_m} |t_i|)$$

$$= \sum_{i=1}^{\infty} |t_i| = \|x\|.$$

So $\|Px\| = \lim_{k \to \infty} \sum_{m=1}^{k} |f_m(\sum_{i=N_{m-1}+1}^{N_m} t_ie_i)| \|z_m\| \leq \|x\|$,

i.e. P is bounded and $\|P\| \leq 1$. Now $z_m = \sum_{i=N_{m-1}+1}^{N_m} t_i^m e_i$

and so

(10.16) $$P(z_m) = f_m(z_m)z_m = z_m, \quad m = 1,2,\ldots$$

Hence $\|P\| = 1$. Also $P^2 = P$. For if

$$x = \sum_{i=1}^{\infty} t_i e_i \in \ell_1(S), \ P(P(x))$$

$$= P\left(\sum_{m=1}^{\infty} \left(f_m\left(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i\right)\right) z_m\right)$$

$$= P\left(\lim_{k \to \infty} \sum_{m=1}^{k} \left(f_m\left(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i\right)\right) z_m\right)$$

$$= \lim_{k \to \infty} P\left(\sum_{m=1}^{k} \left(f_m\left(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i\right)\right) z_m\right)$$

$$= \lim_{k \to \infty} \sum_{m=1}^{k} \left(f_m\left(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i\right)\right) z_m \quad \text{by (10.16)}$$

$$= \sum_{m=1}^{\infty} \left(f_m\left(\sum_{i=N_{m-1}+1}^{N_m} t_i e_i\right)\right) z_m = P(x).$$

Finally P maps $\ell_1(S)$ onto $[z_m]_{m=1,2,\ldots}$. For if

$y \in [z_m]_{m=1,2,\ldots}$, $y = \sum_{i=1}^{\infty} \gamma_i z_i$ and

$$P(y) = P(\sum_{i=1}^{\infty} \gamma_i z_i) = \sum_{i=1}^{\infty} \gamma_i P(z_i) = \sum_{i=1}^{\infty} \gamma_i z_i = y.$$ So P

is a projection of norm one from $\ell_1(S)$ onto

$[z_m]_{m=1,2,\ldots}$. Q.E.D.

10.17 Lemma. Let X be a Banach space and let $\{x_n\}_{n=1,2,\ldots}$ be a sequence of non-zero elements in X which satisfy the inequality $\|\sum_{i=1}^{n} \alpha_i x_i\| \leq \|\sum_{i=1}^{m} \alpha_i x_i\|$ for all positive integers m and n with $m \geq n$ and all scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$. (In particular by Lemma 10.15, $\{x_n\}_{n=1,2,\ldots}$ is a Schauder basis for $[x_n]_{n=1,2,\ldots}$.) Let $\{y_n\}_{n=1,2,\ldots}$ be a sequence of non-zero elements in X and suppose

$$\sum_{n=1}^{\infty} \|x_n^*\| \|y_n - x_n\| = \delta < 1$$

where $\{x_n^*\}_{n=1,2,\ldots}$ is the sequence in $[x_n^*]_{n=1,2,\ldots}$ orthonormal to $\{x_n\}_{n=1,2,\ldots}$. Then $\{y_n\}_{n=1,2,\ldots}$ is a Schauder basis for $[y_n]_{n=1,2,\ldots}$.

<u>Proof</u>. By Lemma 10.15 it suffices to show that there exists a constant $M \geq 1$ such that for all positive integers p and q with $p \leq q$ and all scalars $t_1, t_2, \ldots, t_q$, we have

$$\|t_1y_1 + \ldots + t_py_p\| \leq M\|t_1y_1 + \ldots + t_py_p + \ldots + t_qy_q\|.$$

Now

$$t_1y_1 + \ldots + t_py_p = t_1x_1 + \ldots + t_px_p + t_1y_1 + \ldots + t_py_p - t_1x_1 - t_2x_2 - \ldots - t_px_p$$

and so

$$\|\sum_{i=1}^{p} t_iy_i\| \leq \|\sum_{i=1}^{p} t_ix_i\| + \|\sum_{i=1}^{p} t_i(y_i - x_i)\|$$

(10.17)

$$\leq \|\sum_{i=1}^{p} t_ix_i\| + \sum_{i=1}^{p} |t_i|\ \|y_i - x_i\|.$$

Now for $1 \leq j \leq p$, we have

$$|t_j| = |x_j^*(t_1x_1 + \ldots + t_px_p)| \leq \|x_j^*\|\ \|t_1x_1 + \ldots + t_px_p\|$$

and so from (10.17)

$$\left\|\sum_{i=1}^{p} t_i y_i\right\| \leq \left\|\sum_{i=1}^{p} t_i x_i\right\|$$

$$+ \sum_{i=1}^{p} \|x_i^*\| \, \|t_1 x_1 + \ldots + t_p x_p\| \, \|y_i - x_i\|$$

$$= \left\|\sum_{i=1}^{p} t_i x_i\right\|$$

$$+ \|t_1 x_1 + \ldots + t_p x_p\| \sum_{i=1}^{p} \|x_i^*\| \, \|y_i - x_i\|$$

$$\leq \left\|\sum_{i=1}^{p} t_i x_i\right\| + \|t_1 x_1 + \ldots + t_p x_p\| \sum_{i=1}^{\infty} \|x_i^*\| \, \|y_i - x_i\|$$

$$= (1 + \delta) \left\|\sum_{i=1}^{p} t_i x_i\right\|.$$

We have thus established

$$\left\|\sum_{i=1}^{p} t_i y_i\right\| \leq (1 + \delta) \left\|\sum_{i=1}^{p} t_i x_i\right\|. \tag{10.18}$$

We shall now proceed to establish

$$\left\|\sum_{i=1}^{q} t_i y_i\right\| \geq (1 - \delta) \left\|\sum_{i=1}^{q} t_i x_i\right\|. \tag{10.19}$$

Now

$$\left\| \sum_{i=1}^{q} t_i y_i \right\| = \left\| \sum_{i=1}^{q} t_i x_i - \sum_{i=1}^{q} t_i (x_i - y_i) \right\|$$

(10.20)

$$\geq \left\| \sum_{i=1}^{q} t_i x_i \right\| - \left\| \sum_{i=1}^{q} t_i (x_i - y_i) \right\|.$$

For $1 \leq j \leq q$ we have

$$|t_j| = |x_j^*(t_1 x_1 + \dots + t_q x_q)| \leq \|x_j^*\| \left\| \sum_{i=1}^{q} t_i x_i \right\|$$

and so

$$\left\| \sum_{i=1}^{q} t_i (x_i - y_i) \right\| \leq \sum_{i=1}^{q} |t_i| \, \|x_i - y_i\|$$

$$\leq \|t_1 x_1 + \dots + t_q x_q\| \sum_{i=1}^{q} \|x_i^*\| \, \|x_i - y_i\|$$

$$\leq \|t_1 x_1 + \dots + t_q x_q\| \sum_{i=1}^{\infty} \|x_i^*\| \, \|x_i - y_i\|$$

$$= \delta \left\| \sum_{i=1}^{q} t_i x_i \right\| .$$

So $- \left\| \sum_{i=1}^{q} t_i (x_i - y_i) \right\| \geq - \delta \left\| \sum_{i=1}^{q} t_i x_i \right\|$ and hence

$$\left\|\sum_{i=1}^{q} t_i x_i\right\| - \left\|\sum_{i=1}^{q} t_i(x_i - y_i)\right\|$$

(10.21) $$\geq \left\|\sum_{i=1}^{q} t_i x_i\right\| - \delta \left\|\sum_{i=1}^{q} t_i x_i\right\|$$

$$= (1 - \delta) \left\|\sum_{i=1}^{q} t_i x_i\right\|.$$

(10.19) now follows from (10.20) and (10.21). Now

$$\left\|\sum_{i=1}^{p} t_i y_i\right\| \leq (1 + \delta) \left\|\sum_{i=1}^{p} t_i x_i\right\| \quad \text{by (10.18)}$$

$$\leq (1 + \delta) \left\|\sum_{i=1}^{q} t_i x_i\right\| \quad \text{by hypothesis}$$

$$\leq \left(\frac{1 + \delta}{1 - \delta}\right) \left\|\sum_{i=1}^{q} t_i y_i\right\| \quad \text{by (10.19).}$$

So we may take M to be $\frac{1 + \delta}{1 - \delta}$. Q.E.D.

10.18 Lemma. Let X be a Banach space and let $\{x_n\}_{n=1,2,...}$ be a sequence of elements in X such that $\{x_n\}_{n=1,2,...}$ is a Schauder basis for $[x_n]_{n=1,2,...}$.

Let $\{y_n\}_{n=1,2,\ldots}$ be a sequence of elements in X such that $\sum_{n=1}^{\infty} \|x_n{*}\|\,\|x_n - y_n\| = \delta < 1$ where $\{x_n{*}\}_{n=1,2,\ldots}$ is the sequence in $[x_n]^*_{n=1,2,\ldots}$ orthonormal to $\{x_n\}_{n=1,2,\ldots}$. Let A be the set of all sequences of scalars $\{t_n\}_{n=1,2,\ldots}$ such that $\sum_{n=1}^{\infty} t_n x_n$ converges and let B be the set of all sequences of scalars $\{t_n'\}_{n=1,2,\ldots}$ such that $\sum_{n=1}^{\infty} t_n' y_n$ converges. Then $A = B$.

<u>Proof</u>. Let $\{t_n\}_{n=1,2,\ldots} \in A$. We will show that $\{t_n\}_{n=1,2,\ldots} \in B$. Now $\{t_n\}_{n=1,2,\ldots} \in A$ means that the sequence $\{s_n\}_{n=1,2,\ldots}$ converges, where $s_n = \sum_{i=1}^{n} t_i x_i$, $n = 1,2,\ldots$. Let $s_n' = \sum_{i=1}^{n} t_i y_i$, $n = 1,2,\ldots$. We want to show that the sequence $\{s_n'\}_{n=1,2,\ldots}$ converges. Since X is a Banach space, it suffices to show that

$\{s_n'\}_{n=1,2,\ldots}$ is Cauchy. Let $\epsilon > 0$. Since $\{s_n\}_{n=1,2,\ldots}$ converges, it is Cauchy. Hence there exists a positive integer N such that for $p, q > N$, p and q integers, we have $\|s_p - s_q\| < \frac{\epsilon}{1 + \delta}$. Let p, q be integers such that $p, q > N$ and assume $p > q$. Then

$$s_p' - s_q' = \sum_{i=q+1}^{p} t_i y_i = \sum_{i=q+1}^{p} t_i x_i + \sum_{i=q+1}^{p} t_i(y_i - x_i)$$

and so

$$\|s_p' - s_q'\| \leq \Big\|\sum_{i=q+1}^{p} t_i x_i\Big\| + \Big\|\sum_{i=q+1}^{p} t_i(y_i - x_i)\Big\|$$

$$= \|s_p - s_q\| + \Big\|\sum_{i=q+1}^{p} t_i(y_i - x_i)\Big\|$$

$$\leq \|s_p - s_q\| + \sum_{i=q+1}^{p} |t_i| \; \|y_i - x_i\| .$$

Now for $q + 1 \leq j \leq p$ we have

$$|t_j| = |x_j^*(t_{q+1}x_{q+1} + \ldots + t_p x_p)| \leq \|x_j^*\| \; \|s_p - s_q\|$$

and so

$$\|s_p' - s_q'\| \leq \|s_p - s_q\| + \sum_{i=q+1}^{p} \|x_i^*\| \, \|s_p - s_q\| \, \|y_i - x_i\|$$

$$\leq \|s_p - s_q\| + \|s_p - s_q\| \sum_{i=1}^{\infty} \|x_i^*\| \, \|y_i - x_i\|$$

$$= (1 + \delta) \|s_p - s_q\| < (1 + \delta) \frac{\epsilon}{1 + \delta} = \epsilon$$

since $p, q > N$.

So $\{s_n'\}_{n=1,2,\ldots}$ is Cauchy and hence $\sum_{i=1}^{\infty} t_i y_i$ converges.

So $A \subset B$.

Now let $\{t_n'\}_{n=1,2,\ldots} \in B$. We shall show that $\{t_n'\}_{n=1,2,\ldots} \in A$. Let $w_n = \sum_{i=1}^{n} t_i' x_i$ and $w_n' = \sum_{i=1}^{n} t_i' y_i$, $n = 1,2,\ldots$. $\{t_n'\}_{n=1,2,\ldots} \in B$ implies that $\{w_n'\}_{n=1,2,\ldots}$ converges and is therefore Cauchy. We want to show that $\{w_n\}_{n=1,2,\ldots}$ converges. Let $\epsilon_1 > 0$. Since $\{w_n'\}_{n=1,2,\ldots}$ is Cauchy, there exists a positive integer M such that for $k, m > M$, k and m

integers, we have $\|w_k' - w_m'\| < \epsilon_1(1 - \delta)$. Let k and m be integers such that $k, m > M$ and assume $k > m$. Now

$$w_k - w_m = t_{m+1}'x_{m+1} + \dots + t_k'x_k$$

$$= 0x_1 + 0x_2 + \dots + 0x_m + t_{m+1}'x_{m+1} + \dots + t_k'x_k$$

and so by (10.19) of Lemma 10.17 we have

$$\|w_k - w_m\| \leq \frac{1}{1 - \delta} \|0y_1 + 0y_2 + \dots + 0y_m + t_{m+1}'y_{m+1} + \dots + t_k'y_k\|$$

$$= \frac{1}{1 - \delta} \|w_k' - w_m'\| < \frac{\epsilon_1(1 - \delta)}{1 - \delta} = \epsilon_1.$$

So $\{w_n\}_{n=1,2,\dots}$ is a Cauchy sequence and hence it converges, i.e. $\sum_{n=1}^{\infty} t_n'x_n$ converges. So $B \subset A$ and hence $A = B$. Q.E.D.

10.19 <u>Lemma</u>. Let $\{x_n\}_{n=1,2,\dots}$ and $\{x_n'\}_{n=1,2,\dots}$ be sequences of non-zero elements in a Banach space X such that $\{x_n\}_{n=1,2,\dots}$ and $\{x_n'\}_{n=1,2,\dots}$ are Schauder bases for $[x_n]_{n=1,2,\dots}$ and $[x_n']_{n=1,2,\dots}$ respectively. Let U be a bounded projection from X onto $[x_n]_{n=1,2,\dots}$ such that

$\|U\| \sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n - x_n'\| = \delta < 1$, where $\{x_n^*\}_{n=1,2,\dots}$ is the sequence in $[x_n]^*_{n=1,2,\ldots}$ orthonormal to $\{x_n\}_{n=1,2,\ldots}$. Then $[x'_n]_{n=1,2,\ldots}$ is complemented in X.

Proof. Since $\{x_n\}_{n=1,2,\ldots}$ is a sequence of non-zero vectors, U is not the zero projection and hence $\|U\| \geq 1$. So $\sum_{n=1}^{\infty} \|x_n{}^*\| \, \|x_n - x'_n\| < 1$ and it follows by Lemma 10.18 that if $\{t_n\}_{n=1,2,\ldots}$ is a sequence of scalars such that $\sum_{n=1}^{\infty} t_n x_n$ converges, then $\sum_{n=1}^{\infty} t_n x'_n$ also converges. Let $x \in X$. Then $U(x) \in [x_n]_{n=1,2,\ldots}$ and hence there exists a unique sequence of scalars, namely $\{x_n{}^*(U(x))\}_{n=1,2,\ldots}$, such that $U(x) = \sum_{n=1}^{\infty} x_n{}^*(U(x))x_n$. So $\sum_{n=1}^{\infty} x_n{}^*(U(x))x'_n$ converges. We define a mapping $A : X \longrightarrow X$ by

$$A(x) = x - U(x) + \sum_{n=1}^{\infty} x_n{}^*(U(x))x'_n, \quad x \in X.$$

It is easy to see that A is linear. We want to show that A is bounded. In order to establish that A is bounded, it suffices to show that the linear mapping $I - A$ is bounded where I is the identity mapping on X. For if $B = I - A$ is bounded, then $A = I - B$ is the sum of two bounded linear transformations and is therefore bounded. Now

$$\begin{aligned}
\|I - A\| &= \sup_{\|x\|\leq 1} \{\|(I - A)x\|\} = \sup_{\|x\|\leq 1} \{\|x - Ax\|\} \\
&= \sup_{\|x\|\leq 1} \{\|x - (x - U(x) + \sum_{n=1}^{\infty} x_n^*(U(x))x_n')\|\} \\
&= \sup_{\|x\|\leq 1} \{\|U(x) - \sum_{n=1}^{\infty} x_n^*(U(x))x_n'\|\} \\
&= \sup_{\|x\|\leq 1} \{\|\sum_{n=1}^{\infty} x_n^*(U(x))x_n - \sum_{n=1}^{\infty} x_n^*(U(x))x_n'\|\} \\
&= \sup_{\|x\|\leq 1} \{\|\sum_{n=1}^{\infty} x_n^*(U(x))(x_n - x_n')\|\}.
\end{aligned}$$

Now for any $x \in X$ such that $\|x\| \leq 1$ and for any positive integer k we have

$$\left\|\sum_{n=1}^{k} x_n^*(U(x))(x_n - x_n')\right\| \leq \sum_{n=1}^{k} |x_n^*(U(x))| \, \|x_n - x_n'\|$$

$$\leq \sum_{n=1}^{k} \|x_n^*\| \, \|U\| \, \|x\| \, \|x_n - x_n'\|$$

(10.22) $$\leq \|U\| \sum_{n=1}^{k} \|x_n^*\| \, \|x_n - x_n'\|$$

$$\leq \|U\| \sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n - x_n'\|$$

$$= \delta < 1.$$

Since $\left\|\sum_{n=1}^{\infty} x_n^*(U(x))(x_n - x_n')\right\| = \left\|\lim_{k \to \infty} \sum_{n=1}^{k} x_n^*(U(x))(x_n - x_n')\right\|$

$$= \lim_{k \to \infty} \left\|\sum_{n=1}^{k} x_n^*(U(x))(x_n - x_n')\right\|,$$

it follows from (10.22) that

(10.23) $\left\|\sum_{n=1}^{\infty} x_n^*(U(x))(x_n - x_n')\right\| \leq \delta < 1$, if $\|x\| \leq 1$

and hence

$$\|I - A\| = \sup_{\|x\| \leq 1} \{ \| \sum_{n=1}^{\infty} x_n^*(U(x))(x_n - x_n') \| \} \leq \delta < 1.$$

So $I - A$ is bounded and hence so is A. Indeed from the inequality $\|I - A\| < 1$ it follows that A is one-one, maps X onto X, and $A^{-1} : X \longrightarrow X$ is continuous.[1]

Now $A([x_n]_{n=1,2,\ldots}) \subset [x_n']_{n=1,2,\ldots}$. For let $y \in [x_n]_{n=1,2,\ldots}$. Then $y = U(y)$ and

$$A(y) = y - U(y) + \sum_{n=1}^{\infty} x_n^*(U(y))x_n' = \sum_{n=1}^{\infty} x_n^*(U(y))x_n' \in [x_n']_{n=1,2,\ldots}$$

since $[x_n']_{n=1,2,\ldots}$ is closed. Indeed $A([x_n]_{n=1,2,\ldots}) = [x_n']_{n=1,2,\ldots}$.

For let $x' = \sum_{n=1}^{\infty} \alpha_n' x_n' \in [x_n']_{n=1,2,\ldots}$. By Lemma 10.18,

$\sum_{n=1}^{\infty} \alpha_n' x_n$ converges to an element, call it y, and

$y \in [x_n]_{n=1,2,\ldots}$ since $[x_n]_{n=1,2,\ldots}$ is closed, and

we have

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[1] See for example Taylor [40, page 164, Theorem 4.1-D].

$$A(y) = A(\lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i' x_i) = \lim_{n \to \infty} A(\sum_{i=1}^{n} \alpha_i' x_i)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i' A(x_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i' x_i' = x'.$$

So $A([x_n]_{n=1,2,\ldots}) = [x_n']_{n=1,2,\ldots}$. Consider the mapping $P : X \to X$ defined by $P = A\, U\, A^{-1}$. P is bounded and linear and $P^2 = A\, U\, A^{-1}\, A\, U\, A^{-1} = A\, U^2 A^{-1} = A\, U\, A^{-1} = P$. It is clear that $P(X) \subset [x_n']_{n=1,2,\ldots}$ and indeed $P(X) = [x_n']_{n=1,2,\ldots}$. For if $x' \in [x_n']_{n=1,2,\ldots}$, let $y \in [x_n]_{n=1,2,\ldots}$ be such that $A(y) = x'$. Then $P(x') = A\, U\, A^{-1}(x') = A\, U\,(y) = A(y) = x'$. So P is a bounded projection from X onto $[x_n']_{n=1,2,\ldots}$, i.e. $[x_n']_{n=1,2,\ldots}$ is complemented in X. Q.E.D.

10.20 Lemma. Let $S = \{s_1, s_2, \ldots\}$ be a countably infinite set and let X be an infinite dimensional closed subspace of $\ell_1(S)$. Then X contains a subspace Y such that Y is closed in $\ell_1(S)$, Y is equivalent to $\ell_1(S)$, and Y has a closed complement in $\ell_1(S)$.

Proof. If $f \in \ell_1(S)$ and $f(s_i) = \alpha_i$, $i = 1,2,\ldots,$ we shall occasionally, for simplicity of notation, write $f = (\alpha_1, \alpha_2, \ldots)$. Let N be a positive integer. Because X is infinite dimensional, there exists a linearly independent set $\{x_1, x_2, \ldots, x_{N+1}\}$ of $N + 1$ vectors in X. Let $x_i = (\beta_1^i, \beta_2^i, \ldots)$, $i = 1,2,\ldots, N + 1$, and consider the system of N linear homogeneous equations in the $N + 1$ unknowns $\gamma_1, \gamma_2, \ldots, \gamma_{N+1}$

$$\sum_{i=1}^{N+1} \gamma_i \beta_n^i = 0, \quad n = 1,2,\ldots, N.$$

Since the number of unknowns exceeds the number of equations, there exists a non-trivial solution $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_2$, $\ldots$, $\gamma_{N+1} = \alpha_{N+1}$ of this system, i.e. some $\alpha_i \neq 0$. Let $x = \sum_{i=1}^{N+1} \alpha_i x_i \in X$. $x \neq 0$ since the set $\{x_1, x_2, \ldots, x_{N+1}\}$ is linearly independent. Let $x = (\beta_1, \beta_2, \ldots)$. Then for $1 \leq i \leq N$ we have

$$\beta_i = \sum_{j=1}^{N+1} \alpha_j \beta_i^j = 0$$ by the way the α_i's were chosen. So

we have established for any given positive integer N, the existence of a non-zero element $x \in X$ whose first N entries (i.e. whose values at the points $s_1, s_2, \ldots, s_N$) are 0 and indeed we can choose x such that $\|x\| = 1$.

We shall now define by induction a sequence of vectors $\{y_i\}_{i=1,2,\ldots} = \{(y_1^i, y_2^i, y_3^i, \ldots)\}_{i=1,2,\ldots}$ in X. For y_1 pick any element in X whose norm is 1. Now $y_1 \in \ell_1(S)$ implies that $\sum_{n=1}^{\infty} |y_n^1| < \infty$ which

implies that there exists a positive integer N such that $\sum_{n=N}^{\infty} |y_n^1| < \frac{1}{16} = \frac{1}{2^4}$. Let N_1 be the smallest such N. We note that $N_1 > 1$ since $\|y_1\| = 1$. For y_2 pick an element in X whose first N_1 entries are 0 and such that $\|y_2\| = 1$. Let N_2 be the smallest positive integer such that $\sum_{n=N_2}^{\infty} |y_n^2| < \frac{1}{2^5}$. For y_3 pick an element in X whose first N_2 entries are 0

and such that $\|y_3\| = 1$ and in general for y_i, $i = 2,3,4,\ldots,$ pick an element in X with norm equal to one and whose first N_{i-1} entries are 0 where N_{i-1} is the smallest positive integer such that

$$(10.24) \qquad \sum_{n=N_{i-1}}^{\infty} |y_n^{i-1}| < \frac{1}{2^{i-1+3}} \, .$$

We claim that $N_i < N_{i+1}$ for $i = 1,2,\ldots.$ For suppose $N_{i+1} \leq N_i$ for some such i. Then

$$\|y_{i+1}\| = \sum_{n=1}^{\infty} |y_n^{i+1}| = \sum_{n=1}^{N_{i+1}} |y_n^{i+1}| + \sum_{n=N_{i+1}+1}^{\infty} |y_n^{i+1}|$$

$$= 0 + \sum_{n=N_{i+1}+1}^{\infty} |y_n^{i+1}| \quad \text{(since the first } N_i \text{ entries of } y_{i+1} \text{ are } 0 \text{ and we are assuming that } N_{i+1} \leq N_i)$$

$$\leq \sum_{n=N_{i+1}}^{\infty} |y_n^{i+1}| < \frac{1}{2^{i+1+3}} < 1,$$

which is impossible since $\|y_{i+1}\| = 1$. So we have a strictly increasing sequence of positive integers

$$1 < N_1 < N_2 < N_3 < \ldots$$

If we let $N_o = 0$, we can write

$$y_1 = \sum_{i=N_o+1}^{\infty} y_i^1 e_i, \quad y_2 = \sum_{i=N_1+1}^{\infty} y_i^2 e_i, \quad y_3 = \sum_{i=N_2+1}^{\infty} y_i^3 e_i, \quad \text{and}$$

in general,

$$(10.25) \qquad y_m = \sum_{i=N_{m-1}+1}^{\infty} y_i^m e_i, \quad \|y_m\| = 1, \quad m = 1,2,\ldots.$$

Also for each $m = 1,2,\ldots,$ and for each positive integer

$k \geq N_m + 1$, we have $\left\| \sum_{i=N_m+1}^{k} y_i^m e_i \right\| \leq \sum_{i=N_m+1}^{k} \|y_i^m e_i\|$

$$= \sum_{i=N_m+1}^{k} |y_i^m| \leq \sum_{i=N_m+1}^{\infty} |y_i^m| \leq \sum_{i=N_m}^{\infty} |y_i^m| < \frac{1}{2^{m+3}} \text{ by (10.24).}$$

Since

$$\left\| \sum_{i=N_m+1}^{\infty} y_i^m e_i \right\| = \left\| \lim_{k \to \infty} \sum_{i=N_m+1}^{k} y_i^m e_i \right\|$$

$$= \lim_{k \to \infty} \left\| \sum_{i=N_m+1}^{k} y_i^m e_i \right\|,$$

we conclude

(10.26) $$\left\| \sum_{i=N_m+1}^{\infty} y_i^m e_i \right\| < \frac{1}{2^{m+3}}, \quad m = 1,2,\ldots.$$

Let

$$z_m = \sum_{i=N_{m-1}+1}^{N_m} y_i^m e_i, \quad m = 1,2,\ldots.$$

Then $y_m - z_m = \sum_{i=N_{m-1}+1}^{\infty} y_i^m e_i - \sum_{i=N_{m-1}+1}^{N_m} y_i^m e_i = \sum_{i=N_m+1}^{\infty} y_i^m e_i$

and so by (10.26)

(10.27) $$\|y_m - z_m\| < \frac{1}{2^{m+3}}, \quad m = 1,2,\ldots$$

So $z_m \neq 0$, $m = 1,2,\ldots$. For if $z_m = 0$, then $\|y_m - z_m\| = \|y_m\| = 1$ by (10.25) and $1 > \frac{1}{2^{m+3}}$.

So the sequence $\{z_m\}_{m=1,2,\ldots}$ satisfies the hypothesis of Lemma 10.16 and hence there exists a projection P with $\|P\| = 1$ from $\ell_1(S)$ onto $[z_m]_{m=1,2,\ldots}$, and the sequence $\{z_m\}_{m=1,2,\ldots}$ is a Schauder basis for $[z_m]_{m=1,2,\ldots}$. Let $\{z_m^*\}_{m=1,2,\ldots}$ be the sequence in $[z_m]^*_{m=1,2,\ldots}$ orthonormal to $\{z_m\}_{m=1,2,\ldots}$. We claim that

(10.28) $$\|z_m^*\| \leq \frac{2}{\|z_m\|}, \quad m = 1,2,\ldots$$

In order to establish (10.28) we first note that the sequence $\{\frac{z_m}{\|z_m\|}\}_{m=1,2,\ldots}$ is also a Schauder basis for $[z_m]_{m=1,2,\ldots}$. For if $z \in [z_m]_{m=1,2,\ldots}$, then $z = \sum_{i=1}^{\infty} t_i z_i = \sum_{i=1}^{\infty} t_i \|z_i\| \frac{z_i}{\|z_i\|}$ and if we also have $z = \sum_{i=1}^{\infty} a_i \frac{z_i}{\|z_i\|}$, then because $\{z_m\}_{m=1,2,\ldots}$ is a Schauder basis for $[z_m]_{m=1,2,\ldots}$, we must conclude that $\frac{a_i}{\|z_i\|} = t_i$, $i = 1,2,\ldots$, that is, $a_i = t_i \|z_i\|$. So $\{\frac{z_m}{\|z_m\|}\}_{m=1,2,\ldots}$ satisfies the definition of a Schauder basis for $[z_m]_{m=1,2,\ldots}$. Let W denote the Banach space $A_{\{\frac{z_m}{\|z_m\|}\}_{m=1,2,\ldots}}$ where A denotes the Banach space $[z_m]_{m=1,2,\ldots}$ and let T be the canonical mapping from W onto A (see (b) after Definition 10.13). We claim that T is an isometry. For if $w = \{w_i\}_{i=1,2,\ldots} \in W$, we have $\|T(w)\| = \|\sum_{i=1}^{\infty} w_i \frac{z_i}{\|z_i\|}\|$ and

$$\|\sum_{i=1}^{\infty} w_i \frac{z_i}{\|z_i\|}\| = \|\lim_{n \to \infty} \sum_{i=1}^{n} w_i \frac{z_i}{\|z_i\|}\| = \lim_{n \to \infty} \|\sum_{i=1}^{n} w_i \frac{z_i}{\|z_i\|}\|$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left|\frac{w_i}{\|z_i\|}\right| \|z_i\| \quad \text{(by (10.12))}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} |w_i| = \sum_{i=1}^{\infty} |w_i|$$

while $\|w\|_W = \sup_{1 \leq n < \infty} \{\|\sum_{i=1}^{n} w_i \frac{z_i}{\|z_i\|}\|\}$

$$= \sup_{1 \leq n < \infty} \{\sum_{i=1}^{n} |w_i|\} \quad \text{(by (10.12))} = \sum_{i=1}^{\infty} |w_i|.$$

So $T : W \to A$ is an isometry and hence so is $T^{-1} : A \to W$. If we let $\{(\frac{z_m}{\|z_m\|})^*\}_{m=1,2,\ldots}$ be the sequence in A^* orthonormal to $\{\frac{z_m}{\|z_m\|}\}_{m=1,2,\ldots}$, then we have (see (b) after Definition 10.13)

$$(10.29) \qquad \left\|\left(\frac{z_m}{\|z_m\|}\right)^*\right\| \leq \frac{2\|T^{-1}\|}{\left\|\frac{z_m}{\|z_m\|}\right\|} = 2, \quad m = 1,2,\ldots.$$

From (10.29) we can easily deduce (10.28). We first note that $\left(\frac{z_m}{\|z_m\|}\right)^* = \|z_m\|z_m{}^*$, $m = 1,2,\ldots$, for if

$z = \sum_{i=1}^{\infty} t_i z_i \in [z_m]_{m=1,2,\ldots}$, then $z_m{}^*(z) = t_m$ while

$$\left(\frac{z_m}{\|z_m\|}\right)^*(z) = \left(\frac{z_m}{\|z_m\|}\right)^*\left(\sum_{i=1}^{\infty} t_i\|z_i\| \frac{z_i}{\|z_i\|}\right) = t_m\|z_m\|.$$

So $\left\|\left(\frac{z_m}{\|z_m\|}\right)^*\right\| = \|\, \|z_m\|z_m{}^* \,\| = \|z_m\| \, \|z_m{}^*\| \leq 2$ by (10.29),

and so $\|z_m{}^*\| \leq \frac{2}{\|z_m\|}$, i.e. (10.28).

Now $\|z_m\| = \|y_m - (y_m - z_m)\| \geq \|y_m\| - \|y_m - z_m\| > 0$ (since $\|y_m\| = 1$ and $\|y_m - z_m\| < \frac{1}{2^{m+3}}$ by (10.25) and (10.27)) and so

$$(10.30) \qquad \frac{2}{\|z_m\|} \leq \frac{2}{\|y_m\| - \|y_m - z_m\|}, \quad m = 1,2,\ldots$$

Now $\|y_m\| - \|y_m - z_m\| \geq \|y_m\| - \frac{1}{2^{m+3}} = 1 - \frac{1}{2^{m+3}} > 0$

and so

(10.31) $$\frac{2}{\|y_m\| - \|y_m - z_m\|} \leq \frac{2}{1 - \frac{1}{2^{m+3}}}, \quad m = 1,2,\ldots$$

So

(10.32) $$\|z_m^*\| \, \|y_m - z_m\| \leq \frac{2}{2^{m+3} - 1}, \quad m = 1,2,\ldots,$$

by (10.28), (10.30), (10.31), and (10.27).

Now it is easily established by induction that

(10.33) $$\frac{1}{2^{m+3} - 1} < \frac{1}{2^{m+2}}, \quad m = 1,2,\ldots$$

(For $m = 1$, (10.33) is clear. Assume (10.33) is true for $m = k$, i.e. assume $2^{k+3} - 1 > 2^{k+2}$. Hence $2^{k+4} - 2 > 2^{k+3}$. But $2^{k+4} - 1 > 2^{k+4} - 2$ and so $2^{k+4} - 1 > 2^{k+3}$ which is equivalent to (10.33) for $m = k + 1$. Hence (10.33) is true for $m = 1,2,\ldots$) So by (10.32) and (10.33) we have

(10.34) $$\|z_m^*\| \, \|y_m - z_m\| < \frac{2}{2^{m+2}}, \quad m = 1,2,\ldots$$

Since $\sum_{m=1}^{\infty} \frac{2}{2^{m+2}} = 2\left(\frac{\frac{1}{8}}{1 - \frac{1}{2}}\right) = \frac{1}{2}$, the series $\sum_{m=1}^{\infty} \|z_m^*\| \, \|y_m - z_m\|$

converges and indeed

$$(10.35) \qquad \sum_{m=1}^{\infty} \|z_m^*\| \, \|y_m - z_m\| = \delta < 1.$$

Now as we already showed, the sequence $\{z_m\}_{m=1,2,\ldots}$ satisfies the hypothesis of Lemma 10.16 and hence the inequality (10.13). Hence by Lemma 10.17, the sequence $\{y_m\}_{m=1,2,\ldots}$ is a Schauder basis for $[y_m]_{m=1,2,\ldots}$.

Since $\|P\| = 1$, we have from (10.35)

$$(10.36) \qquad \|P\| \sum_{m=1}^{\infty} \|z_m^*\| \, \|y_m - z_m\| = \delta < 1.$$

Hence by Lemma 10.19, $[y_m]_{m=1,2,\ldots}$ is complemented in $\ell_1(S)$. We take Y to be $[y_m]_{m=1,2,\ldots}$. Clearly $Y \subset X$ since each $y_m \in X$ and X is closed. The only thing that remains to be shown is that Y is equivalent to $\ell_1(S)$. Now by Lemma 10.16, $[z_m]_{m=1,2,\ldots}$ is congruent to $\ell_1(S)$. Hence it suffices to show that Y is equivalent to $[z_m]_{m=1,2,\ldots}$. Now Y is equivalent to $Y_{\{y_m\}_{m=1,2,\ldots}}$ and $[z_m]_{m=1,2,\ldots} = A$ is equivalent to $A_{\{z_m\}_{m=1,2,\ldots}}$ (see (b) after Definition 10.13).

Let $\{\xi_i\}_{i=1,2,\ldots}$ be a sequence of scalars. By Lemma 10.18, the series $\sum_{i=1}^{\infty} \xi_i z_i$ converges if and only if the series $\sum_{i=1}^{\infty} \xi_i y_i$ converges. Hence the underlying sets in the Banach spaces $Y_{\{y_m\}_{m=1,2,\ldots}}$ and $A_{\{z_m\}_{m=1,2,\ldots}}$ are identical. Let $\varphi : A_{\{z_m\}_{m=1,2,\ldots}} \longrightarrow Y_{\{y_m\}_{m=1,2,\ldots}}$ be the identity map. φ is one-one, linear, and onto. If $\xi = \{\xi_n\}_{n=1,2,\ldots} \in A_{\{z_m\}_{m=1,2,\ldots}}$, then

$$\|\varphi(\xi)\|_{Y_{\{y_m\}_{m=1,2,\ldots}}} = \sup_{1\leq n<\infty} \{\|\sum_{i=1}^{n} \xi_i y_i\|\}.$$

Now by (10.18) of Lemma 10.17, we have

$$\|\sum_{i=1}^{n} \xi_i y_i\| \leq (1 + \delta) \|\sum_{i=1}^{n} \xi_i z_i\|$$ for all positive integers n and hence

$$\sup_{1\leq n<\infty} \{\|\sum_{i=1}^{n} \xi_i y_i\|\} \leq (1 + \delta) \sup_{1\leq n<\infty} \{\|\sum_{i=1}^{n} \xi_i z_i\|\}.$$

In other words

$$\|\varphi(\xi)\|_{Y_{\{y_m\}_{m=1,2,\ldots}}} \leq (1 + \delta)\|\xi\|_{A_{\{z_m\}_{m=1,2,\ldots}}} .$$

So φ is bounded and hence $A_{\{z_m\}_{m=1,2,\ldots}}$ and $Y_{\{y_m\}_{m=1,2,\ldots}}$ are equivalent. So we have

$$\ell_1(S) \sim [z_m]_{m=1,2,\ldots} \sim A_{\{z_m\}_{m=1,2,\ldots}}$$

$$\sim Y_{\{y_m\}_{m=1,2,\ldots}} \sim [y_m]_{m=1,2,\ldots} = Y.$$

So $Y \sim \ell_1(S)$. Q.E.D.

10.21 Lemma. Let S be a countably infinite set. Let X and W be closed subspaces of $\ell_1(S)$ such that $\ell_1(S) = X + W$, $X \cap W = \{0\}$, and let X be infinite dimensional. Then X is equivalent to $\ell_1(S)$.

Proof. By Lemma 10.20 there exist a subspace Y of X such that Y is closed in $\ell_1(S)$ (and hence closed in X) and a closed subspace Z of $\ell_1(S)$ such that $\ell_1(S) = Y + Z$, $Y \cap Z = \{0\}$, and Y is equivalent to $\ell_1(S)$. Let $Y_1 = Z \cap X$. Y_1 is a closed subspace of X and clearly $Y \cap Y_1 = \{0\}$. $X = Y + Y_1$

for if $x \in X$, then $x = y + z$, $y \in Y$, $z \in Z$, (since $\ell_1(S) = Y + Z$) and since $Y \subset X$, $z = x - y \in X$ and hence $z \in Z \cap X = Y_1$. Hence by Lemma 10.12, X is equivalent to $\ell_1(S)$. Q.E.D.

We are now ready to establish formally as a theorem the result we announced at the beginning of this chapter.

10.22 Theorem. A non-zero Banach space P is separable and projective if and only if P is equivalent to $\ell_1(S)$ for some at most countably infinite set S.

Proof. ($\Leftarrow$) If P is equivalent to $\ell_1(S)$ where S is at most countably infinite, then P is separable since $\ell_1(S)$ is separable for such S and of course P is projective since $\ell_1(S)$ is.

($\Rightarrow$) If P is projective, then P is equivalent to a closed subspace X with a closed complement of some $\ell_1(S)$ (Theorem 2.4). If P is infinite dimensional, then so is X and clearly S must be an infinite set. If, in addition, P is separable, then by examining the proof (part (a)) of Theorem 2.4, we see that we can assume that S is countably infinite. (For by Lemma 2.2, a separable Banach space is the image under a bounded linear transformation of $\ell_1(S)$ for some countably infinite set S.) By Lemma 10.21, X is equivalent to $\ell_1(S)$. So if P is separable and projective and infinite dimensional, P is equivalent to $\ell_1(S)$ where S is countably infinite. If P is of finite dimension $n > 0$, then P is equivalent to $\ell_1(S_n)$ where $S_n = \{1, 2, \ldots, n\}$. Q.E.D.

10.23 Remark. Thus we have determined all the separable projective Banach spaces. What can we say about the non-separable ones? Now if S is an uncountably infinite set, $\ell_1(S)$ is an example of a non-separable projective Banach space. The problem of determining all non-separable projective Banach spaces reduces to the problem of determining what the non-separable closed subspaces with closed complements of $\ell_1(S)$, S uncountably infinite, look like. In general if S is an infinite set and X is an infinite dimensional closed subspace with closed complement of $\ell_1(S)$, we cannot conclude that X is equivalent to $\ell_1(S)$ for the same S. The countability of S in Lemma 10.21 is crucial. For example, let S be an uncountable set and let S_1 be a countably infinite subset of S. Let X be the set of all functions in $\ell_1(S)$ which vanish on $S - S_1$, the complement of S_1

with respect to S. It is easy to see that X is a subspace of $\ell_1(S)$. X is closed. For if $f_n \in X$, $n = 1,2,\ldots$, and $\lim_{n \to \infty} f_n = f \in \ell_1(S)$, it follows that $f \in X$, i.e. that f vanishes off S_1. To see this, let $\epsilon > 0$. Then there exists a positive integer $N(\epsilon)$ such that $\|f_n - f\| < \epsilon$ for $n > N(\epsilon)$, i.e.

$$\sum_{s \in S} |f_n(s) - f(s)| = \sum_{s \in S_1} |f_n(s) - f(s)| + \sum_{s \in S - S_1} |f_n(s) - f(s)|$$

$$= \sum_{s \in S_1} |f_n(s) - f(s)| + \sum_{s \in S - S_1} |f(s)| < \epsilon$$

if $n > N(\epsilon)$. But this implies that $\sum_{s \in S - S_1} |f(s)| = 0$, for if $\sum_{s \in S - S_1} |f(s)| = \alpha > 0$, then for $\epsilon < \alpha$, we could not have $\|f_n - f\| < \epsilon$ no matter how large we choose N.

So $\sum_{s \in S-S_1} |f(s)| = 0$ and this of course implies that $f(s) = 0$ if $s \in S - S_1$. So $f \in X$, i.e. X is closed. By the same argument, the subset Y of $\ell_1(S)$ consisting of those functions which vanish on S_1 is a closed linear subspace of $\ell_1(S)$ and it is easy to see that $\ell_1(S) = X + Y$, $X \cap Y = \{0\}$. Consider now the space $\ell_1(S_1)$. If $g \in \ell_1(S_1)$, define $Tg \in \ell_1(S)$ to be that function which vanishes on $S - S_1$ and which agrees with g on S_1. It is easy to see that the map $T : \ell_1(S_1) \longrightarrow \ell_1(S)$ is linear, maps $\ell_1(S_1)$ onto X, and is an isometry, i.e. X and $\ell_1(S_1)$ are congruent. But X is not equivalent to $\ell_1(S)$ because X is separable while $\ell_1(S)$ is not. What this example shows is that the most we can aim for is to try to show that a closed subspace with closed complement of an $\ell_1(S)$ for an arbitrary set S is equivalent to $\ell_1(S_1)$ for some set S_1. Such a result would of course show us that the only non-zero projective Banach spaces are those which are equivalent to $\ell_1(S)$ for some non-empty set S.

CHAPTER XI

Open Questions and Concluding Remarks

In this chapter we shall discuss briefly some open questions in the area of projective and injective spaces together with some known results from the literature and direct the reader to further items in the literature.

The class In(1) has been completely characterized as a result of the work of Goodner, Nachbin, Kelley, and Hasumi. Goodner [11] and Nachbin [29] proved the following theorem:

Theorem Let X be a real Banach space such that the closed sphere in X with center at 0 and radius equal to one has an extreme point. Then $X \in In(1)$ if and only if X is congruent to a space $C(S)$ where S is a compact Hausdorff topological space with the property that the closure of every open set in S is open.[1]

Kelley [19] removed the hypothesis that the closed sphere of radius one and with center at 0 contains an extreme point and Hasumi [17] extended the result of

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[1] A topological space with the property that the closure of every open set is open is often called an extremally disconnected space.

Kelley to complex Banach spaces. The analogous problem of characterizing the classes $In(\lambda)$ for $\lambda > 1$ is unsolved. Indeed no example of a Banach space which is in $In(\lambda)$ for $\lambda > 1$ and which is not equivalent to a space in $In(1)$ is known. Some partial results on the classes $In(\lambda)$, $\lambda > 1$ have been obtained by Amir [3] who proved that (1) if $C(S)$ (S compact Hausdorff) is a member of the class $In(\lambda)$, then every convergent sequence in S is eventually constant; (2) a $C(S)$ space is a member of the class $In(\lambda)$, with $1 \leq \lambda < 2$, if and only if it is a member of the class $In(1)$; and (3) if $C(S) \in In(\lambda)$, then S contains a maximal open and dense extremally disconnected subset. (See also Isbell and Semadeni [18].)

We mentioned in the Introduction that an early example of a non-injective space was provided by Fichtenholz and Kantorovitch [9]. They proved that there does not exist a bounded projection from $\ell_\infty(S)$ onto $C(S)$ where S is the closed interval $[0, 1]$, i.e. $C([0, 1])$ is not injective.[1] Other examples of non-injective spaces were provided by Murray [28] who established the existence

[1] If we accept the result of Amir that if a $C(S)$ space is a member of $In(\lambda)$, then every convergent sequence in S is eventually constant, then we obtain a quick proof that $C([0, 1])$ is not injective.

of closed subspaces of $\ell_p(S)$, S countably infinite, $1 < p \neq 2$, without closed complements. Another example was provided by Sobczyk [36] who proved that there does not exist a bounded projection from $\ell_\infty(S)$ onto $c_o(S)$, S countably infinite.[1] Some recent results on the non-existence of bounded projections can be found in Thorp [41] and Arterburn and Whitley [4].

Definitions of the type found in Chapter V can probably be formed indefinitely although whether one can show that they are equivalent (if indeed they are) to the original definitions of injectivity and projectivity is another matter. Indeed all sorts of variations are possible. For example we can consider those Banach spaces which have in addition a lattice structure (see Dunford and Schwartz [8, page 394]) and consider bounded linear transformations which preserve one or both of the lattice operations $\vee$ and $\wedge$ or the partial order relation (or various combinations of these) and define for example the notion of a projective Banach lattice

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[1] $c_o(S)$, S countably infinite, has the property (proved by Sobczyk [36, 38]) that if X is a separable Banach space containing $c_o(S)$ as a closed subspace, there exists a bounded projection T from X onto $c_o(S)$ with $\|T\| \leq 2$. (See also McWilliams [24].)

and try to characterize the various such spaces that we define. Along the same lines and perhaps more interesting is the following type of problem: Pick a particular category of Banach spaces (for example the C(S) spaces or the $\ell_p(S)$ spaces) and decide what category of Banach spaces (and maps) we must restrict the remaining spaces in the definition of injective (or projective) in order that the members of our chosen category will turn out to be injective (projective). Even better still is the problem of what categories to choose so that our originally chosen category turns out to consist of all the injectives (projectives).

Theorem 7.13 was announced without proof and without the hypothesis that X is a dual space in Lindenstrauss [21]. A proof of Lindenstrauss' theorem appears in Lindenstrauss [22] and is quite involved. It seems plausible that there should exist an elementary proof of Lindenstrauss' theorem, elementary in the sense that it involves purely geometric arguments about the set of all closed spheres in X. It seems intuitively very clear that the $(1+\epsilon)$-intersection property (for every $\epsilon > 0$) should imply the 1-intersection property. Of course we cannot trust our intuition when it comes to infinite dimensional spaces (or even spaces of dimension greater than 3), but we do know that the $(1+\epsilon)$-intersection property (for every $\epsilon > 0$) does indeed imply the 1-intersection property (by the theorem of Lindenstrauss) and so it seems worth trying to seek an elementary proof of Lindenstrauss' theorem.

Chapter VII certainly made clear that the requirement of injectivity on a Banach space has a strong influence on the geometry of the space. It seems worthwhile to investigate whether we can say anything about the geometrical properties of a projective Banach space and whether there exists a geometrical characterization of certain classes of projective Banach spaces.

In Chapter IX we defined the notion of a $*$-projective Banach space, but didn't make any statement as to whether there exist any Banach spaces (other than the projective ones) which are $*$-projective. That there do exist $*$-projective spaces which are not projective follows from Grothendieck [13, Proposition 1]. More precisely, Grothendieck is concerned with what we have called the class $Pr^*(1)$ and he proves (among other things) that a real Banach space $X \in Pr^*(1)$ if and only if the dual space of X is a member of $In(1)$. Now we know from our work in Chapter IV, that real $L_1[0, \pi]$ is not projective. Now the dual space of $L_1[0, \pi]$ is congruent to $L_\infty[0, \pi]$, the space of real valued bounded measurable functions[1] on

[1] More precisely, equivalence classes of such functions.

$[0, \pi]$, and $L_\infty[0, \pi] \in In(1)$ (see Nachbin [29]) and hence so is the dual space of $L_1[0, \pi]$. So $L_1[0, \pi] \in Pr^*(1)$. That real $L_1[0, \pi] \in Pr^*(1)$ also follows from Theorem 1 of Grothendieck [13]. We can similarly define the notion of a $*$-injective space and ask about the nature of its dual space and whether there exist $*$-injective spaces which are not injective. It is known that there exist injective Banach spaces which are not congruent to any dual space (see Isbell and Semadeni [18] and the references there). The question arises: Of those injective spaces which are duals, which are duals of a projective ($*$-projective) space? Do analogues of the theorems in Chapters II and III hold for $*$-projective spaces and $*$-injective spaces?

The problem of determining all the projective Banach spaces is open as we pointed out in the discussion at the end of Chapter X. We might try to abstract as much as possible from the lemmas leading up to the theorem of Pelczynski and perhaps try to define a generalized type (uncountable) of Schauder basis and try to show that closed subspaces (with closed complements) of $\ell_1(S)$, S uncountably infinite, possessing such a generalized type of Schauder basis are equivalent to $\ell_1(S_1)$ for some set S_1. This type of approach might be a first step in attacking the problem of obtaining the non-separable projective Banach spaces. Of course the validity of the converse of Theorem 4.5 is worth investigating, i.e. whether a Banach space with the property that weak and strong convergence of sequences coincide must be projective. It would be very helpful if we had an example of a Banach space which is not equivalent to a space $\ell_1(S)$ and which has the property that weak and strong convergence of sequences coincide. If there are no such spaces, then we've determined all projective Banach spaces.

There seems to be no end to the questions that one can raise in this field which are worth investigating. For example, it is clear that if a closed subspace X of an injective Banach space Y is injective, then X is complemented in Y. Can we replace the word "injective" by the word "projective" and draw the same conclusion? We trust that the references in the Bibliography will raise even more questions.

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