Cost Averaging Techniques for Robust Control of Parametrically Uncertain Systems

by

Nesbitt W. Hagood IV

S.B. Massachusetts Institute of Technology (1985)
S.M. Massachusetts Institute of Technology (1988)

SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy
in
Aeronautics and Astronautics
at the
Massachusetts Institute of Technology
June, 1991

Signature of Author ______________________
Department of Aeronautics and Astronautics
May 17, 1991

Certified by ______________________
Professor Edward F. Crawley
Thesis Committee Chairman, Department of Aeronautics and Astronautics

Certified by ______________________
Professor Andreas von Flotow
Department of Aeronautics and Astronautics

Certified by ______________________
Professor Steven R. Hall
Department of Aeronautics and Astronautics

Certified by ______________________
Professor Michael Athans
Department of Electrical Engineering

Certified by ______________________
Dr. Dennis S. Bernstein
Harris Corporation

Accepted by ______________________
Professor Harold Y. Wachman
Chairman, Department Graduate Committee

Cost Averaging Techniques for Robust Control of Parametrically Uncertain Systems

Nesbitt W. Hagood IV

SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS ON MAY 17, 1991 IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Abstract

A method is presented for analysis and robust control synthesis for linear time-invariant systems with parametric uncertainty structures. The method is based on analysis of the quadratic ($\mathcal{H}_2$) cost averaged over a set of systems whose system matrices are continuous functions of bounded real parameters. A method of describing such a set of systems is presented and related to existing uncertainty modeling paradigms.

Bounded average cost is shown to imply stability over the set of systems. Sufficient conditions for the existence of the average cost and set stability are developed using time domain operator decomposition techniques. Since the average cost cannot be exactly calculated for systems with many uncertainties, the operator decomposition techniques were used to develop computable approximations and bounds for the exact average cost. These computable expressions take the form of coupled systems of modified Lyapunov equations. Their solution is discussed along with conditions for existence of positive definite solutions.

The exact average cost and its approximations and bounds are proposed as cost functionals used to incorporate the effects of the model uncertainties in the controller design process. These cost functionals are used to derive expressions for component and uncertain parameter costs which are applied to the problem of model order reduction and uncertain parameter truncation. Explicit formulae for modal component and parameter costs are derived and compared for the various cost functionals.

The synthesis of robust fixed-order static and dynamic output feedback control is addressed. Necessary conditions are derived for minimization of the proposed cost functionals. Techniques are presented for incorporating these conditions into a numerical solution technique based on homotopic continuation methods. This technique is used to determine the relative robustness properties of the average-based designs. These properties are demonstrated on two simple structural examples; the fourth-order Robust Control Benchmark Problem, and the eight-order Cannon-Rosenthal Problem. The concept of controller efficiency is used to rank the designs based on the trade between nominal performance and achieved robustness.

Thesis Committee Chairman: Dr. Edward F. Crawley
Professor
Department of Aeronautics and Astronautics

Acknowledgments

I dedicate this thesis to my wife, Meredith, for her unwavering love and support, and daughter, Kaileigh, for providing the inspiration for its completion. The months of empty labor were made full by their presence in my life.

This work was developed in the rich environment of the MIT Space Engineering Research Center. I thank Professor Edward Crawley for the freedom to wander unsuspecting into the lair of the controls theoreticians and Professor Andy von Flotow for periodically pulling me out. This thesis would not be the present document if not for the sage criticism of Professors Michael Athans and Steve Hall, and Doctor Dennis Bernstein who invested the time to figure out what was going on. I thank the graduate students in SERC who endured my computational indulgences and in particular Bob Jacques, Doug MacMartin, Jon How, and Mathieu Mercadal who tolerated my numerous questions.

This work was sponsored by the MIT Space Engineering Research Center under NASA grant NAGW-1335.

Contents

List of Figures

Chapter 1

Introduction and Review of Previous Work

1.1 Motivation

One difficulty in feedback control algorithm development is that design and analysis are based on models that do not necessarily accurately reflect the actual system to be analyzed. The difference between a model and the actual system is typically referred to as *model error.* Model error can arise from errors in the values of system parameters used in the model, unmodeled dynamics, and neglected nonlinearities or disturbances. They can be expressed in the time [1, 2] or frequency domain [3] and the form of their expression influences the techniques used to accommodate them.

Once it is recognized that a particular model may not accurately represent an actual system or plant, a natural goal would be to increase the amount of error between the model and the actual plant that a particular control design can tolerate. The property of a controller to tolerate model errors is known as *robustness.* The ability to maintain closed-loop stability in the presence of model errors is known as *stability robustness* [4], while the ability to maintain closed-loop performance in the presence of model errors is known as *performance robustness* [5].

Robust control system design and robust model reduction techniques are useful tools throughout the range of control system applications from chemical process control to control of flight vehicles. This report will concentrate on applications in the field of control of flexible structures. References [6–11] provide a good overview of the flexible structure control problem. Structural plants are characterized by a high density of lightly damped resonant modes.

Future space structures pose a particularly difficult problem for the control designer. Besides the problems of controlling large order systems, the dynamics of the structure on orbit can rarely be tested in full on the ground. This results in sometimes critical uncertainty in the model. Even if components are thoroughly identified, the complete system model will reflect the effects neglected in the assembly procedure such as nonlinearities. References [12–14] concentrate on error modeling for structures.

1.2 Objective

This report will focus on the problem of designing robust controllers for active structural control applications. In particular, it will deal with the problem of developing control algorithms that can simultaneously stabilize a set of linear time invariant plants described in terms of variable real parameters; for instance, a structural system with uncertain natural frequency or damping. This is the robust stability problem which will be stated more explicitly in a later section. The approach taken in this thesis is to examine the quadratic ($\mathcal{H}_2$) cost averaged over a parameterized set of possible systems. The real parameter control problem is one of intense current interest. Some of the efforts in the field will be highlighted in the next section.

This work focusses on stability robustness in the face of parametric model errors and does not attempt to address stability robustness issues relating to unstructured modelling errors in high frequency dynamics. The work presented is intended to be complimentaty to work in the area of unstructured uncertainty. Both types of uncertainty should be considered in practical control design and care must be taken in parameteric uncertainty robust control design to ensure that the system is not made more sensitive to unmodelled dynamics.

1.3 Background

Numerous techniques have been developed for performing robust controller synthesis. The following sections will focus on the dominant paradigms that have emerged for robust control. The various paradigms are distinguished on the basis of how they represent and measure system uncertainty (real parametric or norm bounded), and how they represent, measure, and test for system performance ($\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms) [15]. First, the main frameworks for representing and measuring systems and uncertainties will be presented in the following analysis sections. Then the application of these frameworks to the dominant methods for robust control synthesis will be reviewed.

1.3.1 Frequency Domain Modeling and Control Synthesis

In recent years much attention has been focussed on frequency domain models of systems and uncertainty [16]. In the frequency domain, two possible system norms are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, denoted $\|\cdot\|_2$ and $\|\cdot\|_\infty$ respectively. These can be defined in the frequency domain for stable transfer functions as

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left\{G(j\omega)^H G(j\omega)\right\} d\omega\right)^{\frac{1}{2}} \tag{1.1}$$

$$\|G\|_\infty = \sup_\omega \bar{\sigma}\left[G(j\omega)\right] \quad (\bar{\sigma} = \text{maximum singular value}) \tag{1.2}$$

These are function space norms and must be considered distinct from the matrix and vector norms that have the same symbols. To avoid confusion in the rest of the work, the symbol, $\|\cdot\|_2$, will always represent the $\mathcal{H}_2$-norm of a system while the symbol, $\|\cdot\|$, will denote the induced matrix norm. For more detail on frequency domain modeling consult Refs. [16, 17].

Unstructured Uncertainty

Within the frequency domain framework, plant uncertainty is typically modeled by an unknown but bounded transfer function which modifies the plant. There are several forms of uncertainty for MIMO systems depending on where the uncertainty is reflected relative to the model. These are called additive uncertainties or multiplicative uncertainties reflected at the plant input or output. In the following discussions only output multiplicative uncertainty will be considered. In this case the set of models is described

$$G(j\omega) = \Delta(j\omega)G_0(j\omega) = [I + L(j\omega)]\, G_0(j\omega) \tag{1.3}$$

where $\Delta(j\omega)$ is the multiplicative error matrix and $L(j\omega)$ is the equivalent additive error. The additive model error is bounded for all frequencies

$$\bar{\sigma}\,[L(j\omega)] < l_{max}(\omega) \;\; \forall\, \omega > 0 \tag{1.4}$$

These uncertainties are labeled unstructured because the perturbation is characterized only by a constraint on the maximum singular value of the multiplicative or additive error matrix and no constraints are put on its internal structure.

The dominant analysis tool for determining closed-loop stability of a stable multivariable open-loop plant specified in the frequency domain is the small gain theorem [17] which can be interpreted in terms of the multivariable Nyquist criteria. Consider the system represented in Figure 1.1 with negative feeedback around the system transfer function.

A condition which guarantees stability of the closed-loop system can be obtained by limiting the gain of the loop.

Theorem 1.3.1 (Small Gain Theorem) *Assume that $G(j\omega)$ is stable. Then the closed-loop system described in Figure 1.1 is stable if $\|G(j\omega)\|_\infty < 1, \forall\omega$.*

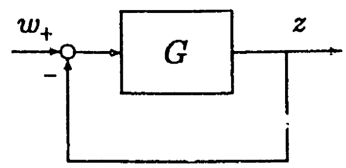

Figure 1.1: System Feedback Configuration

This theorem is important for determining conditions of robust stability for uncertain systems described by Equation (1.3).

Theorem 1.3.2 (Robust Frequency Domain Stability) *Assume that $G_0(j\omega)$ is stable and that the perturbation matrix Δ in Equation (1.3) is a stable transfer function. Then the closed-loop system is stable for all perturbations $\|\Delta\|_\infty < 1$ if and only if $\|G_0\|_\infty < 1$.*

Theorem 1.3.2 is often used to derive conditions for robust stability of plants with additive or multiplicative frequency domain uncertainty.

The theory behind the design of compensators such that the closed-loop transfer function has bounded $\mathcal{H}_\infty$ norm has received attention recently because of state space techniques for deriving such compensation using Riccati Equations [18–21]. This theory can be used for design of robust compensators for parameterized systems by virtue of the small gain theorem and the representation of parameterized uncertainties as a block structured uncertainty matrix using a linear fractional transformation on the original system, Ref. [130, 22]. As will be explained in detail in Chapter 2, when real parameter errors are considered the perturbation matrix, Δ, has the form

$$\Delta = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_r \end{bmatrix} \quad \delta_i \in \mathbb{R} \;\; i = 1, \cdots, r \tag{1.5}$$

$\mathcal{H}_\infty$ theory can thus be used to design compensators that are robust to uncertainty matrices of this form as long as $\|\Delta\|_\infty$ is bounded. In the case of real parameter variations, however, the uncertainty matrix is highly structured. The set of arbitrary error matrices with the property $\|\Delta\|_\infty < \gamma$ is much larger than the set of structured error matrices with this property. Since $\mathcal{H}_\infty$ robust control design only specifies the uncertainty matrix in terms of the norm bound, its design accommodates a much larger set of possible uncertainty matrices than the actual parameterized set. By accommodating more general uncertainty structure, the $\mathcal{H}_\infty$ design stabilizes the system in the face of uncertainties which do not exist. This property of dealing with a larger set of uncertainties than actually exist is known as *conservatism.* For a given performance level, a conservative robust control design procedure can thus lead to higher control cost and actuator gains than a less conservative one.

Structured Uncertainty

The additive or multiplicative frequency domain perturbations can be very conservative if the structure of the perturbations is known. This structure can sometimes be represented in the frequency domain as a matrix possessing block structure [15,23–25]. This type of uncertainty representation will be called structured frequency domain uncertainty, and is much different from the time domain uncertainty structures presented in the next section. The structured frequency domain uncertainty can be viewed as a logical restriction of the unstructured norm bounded uncertainty matrix to the set of matrices satisfying

$$\boldsymbol{\Delta} = \left\{\Delta \ : \Delta = \mathrm{diag}\left(\Delta_1, \Delta_2, \cdots, \Delta_r\right) : \Delta_i \in \mathbf{C}^{k_j \times k_j}\right\} \qquad (1.6)$$

$\boldsymbol{\Delta}$ is thus the set of complex matrices with diagonal block structure. It is also useful to define its bounded subset

$$\mathbf{B}\boldsymbol{\Delta} = \{\Delta \in \boldsymbol{\Delta} : \bar{\sigma}[\Delta] < 1\} \qquad (1.7)$$

The goal of considering a more structured uncertainty matrix is to allow less conservative guarantees for stability robustness. The block structure can also be used to guarantee both robust stability and performance when the performance is measured by singular value bounds [17]. The performance specification can be reinterpreted as a stability robustness problem, and simultaneous robust stability and robust performance can be interpreted as a block structured stability robustness problem. In order to make use of the structure of the uncertainty, it is necessary to define a function that "measures" the distance from a structurally perturbed system to instability at a given frequency. This measure is called μ in [23] and is defined

$$\frac{1}{\mu(G)} = \min_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) : \det(I - G\Delta) = 0\} \tag{1.8}$$

With this definition it is possible to generalize Theorem 1.3.2 to account for the structured perturbations.

Theorem 1.3.3 (Robust Structured Frequency Domain Stability) *Assume the G_0 is stable and that the perturbation $\Delta \in \mathbf{B\Delta}$. Then the actual closed system is stable for all $\Delta \in \mathbf{B\Delta}$ if and only if $\|G_0\|_\mu < 1$. where*

$$\|G\| \equiv \sup_{\omega} \mu\left[G(j\omega)\right]$$

The formalism of μ analysis and the resulting μ synthesis come part of the way to dealing with the nonconservative stability of a set of systems described by real parameter variations. It falls short for two reasons. The first is that since it cannot be calculated exactly for more than 3 blocks, a calculable bound must be employed. The second is that it deals with complex blocks adn is a necessary and sufficient condition for stability *if* the uncertainties are of that form. When the errors are real parameters, Theorem 1.3.3 amounts to a conservative sufficient condition for stability. To understand the difference between real and complex valued uncertainties, consider the input and output signals of the error transfer function. If the uncertainties are real the inputs and outputs either have the same phase or are 180 degrees out of phase. If the uncertainties are complex, the phase relationship between the input and output signals can be arbitrary. The complex block structure is thus a broader uncertainty description than a real block structure. The calculation of the μ -norm entails the use of an bound. When the uncertainties are real scalars this bound can be shown to be very conservative [24]. The problem of restricting the uncertainties to be real parameters is sometimes called the real mu problem.

Control system design using the analysis tools designed for plants with structured frequency domain uncertainty is known as μ–synthesis, Refs. [15, 24],. First a compensator is designed which is stable in the face of $\mathcal{H}_\infty$ norm bounded perturbations (the K loop) and then a frequency-dependent scaling is found which minimizes the conservatism in this design (the D loop). This process is repeated in what is known as D–K iteration. It has been shown that the D–K iteration is a nonconvex minimization process which may not converge to a global minimum. The $\mathcal{H}_\infty$ norm bound and its less conservative scaled version have nevertheless been shown to be very conservative for constant real parameter variations. This conservatism will take the form of requiring higher control gains and costs to achieve system stability and performance.

1.3.2 Time Domain Modeling and Control Synthesis

In this section the method of representing system uncertainties in terms of real parameter variations of elements of the system will be briefly discussed. Much work has been done in the field so the overview will be cursory. A good overview of parametric systems stability analysis and control design is given by Siljak, [26]. When dealing with parameterized systems the methods for analysis and design fall loosely into the two categories of *algebraic methods* and *bounding methods.* The bounding methods have tended to contribute more to the synthesis literature. Real parameter error models are normally (but by no means always) associated with time domain robustness specifications and will be dealt with in this manner in the following sections. The stability analysis of parametrically perturbed systems is linked with the issues of the robust or simultaneous stabilization problem [27–41]. Some stabilizability conditions for parameterized plants have however been discussed in the frequency domain context, [27–31], and considered in parallel with unstructured perturbations [32, 33, 42].

Restricting attention now to the time domain representation of parameter uncertainty, the system matrices are allowed to be nonlinearly dependent on a finite number of perhaps time-varying parameters, α. These parameters define a set of systems which can be simply expressed in the state space as

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + B(\alpha)\tilde{u}(t) \\ \tilde{y}(t) &= C(\alpha)x(t) + D(\alpha)\tilde{u}(t) \end{aligned} \tag{1.9}$$

or in the notation presented in Chapter 2 and used for state space systems in the remainder of this report

$$G(\alpha) = C(\alpha)sI - A(\alpha))^{-1}B(\alpha) + D(\alpha) \triangleq \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C(\alpha) & D(\alpha) \end{array}\right] \tag{1.10}$$

Note that parameter dependent disturbances and measurements can also be included into the model. The parameters themselves are usually considered to be members of some bounded set in $\mathbb{R}^r$. They have also been modeled as random processes such as jump processes or white noise, or random variables with specified continuous or discrete distributions. The random process description of parameterized systems will be discussed in a later section. First the methods which treat the parameters deterministically will be reviewed.

Algebraic and Lyapunov Based Parameter Space Methods

As defined in [26], algebraic approaches to parameterized systems stability analysis establish stability by examining the roots of the characteristic equation and in some sense bounding them. For completeness the well known Routh–Hurwitz stability test [43] will also be placed in this group. The work on interval polynomials and plants [44], most notably the results of Kharitonov's theorem [45] and its generalizations to the stability of matrices through the Edge Theorem [46] and the work on polytope analysis [47], are of particular interest. In general these stability analysis procedures are used for control synthesis by developing conditions on the characteristic polynomial and maximizing the size of the stability region by choice of control parameters.

The other approach which has received much attention both in analysis and synthesis is based upon Lyapunov stability theory [48–74] and its refinements through Vector [75, 76] and majorant [77, 78] Lyapunov analysis. The method of Lyapunov is appealing because it is general enough for non–linear systems and doesn't require calculation of the characteristic polynomial. The general approach is to find a function which bounds the energy of all the possible plants and show that this function is always decreasing. Thus for any of the plants, the energy is decreasing and the system must be stable. A good overview of bounds for uncertain state space systems derived from Lyapunov's method can be found in Bernstein [73, 74] or Yedavalli [61].

Lyapunov stability theory can be used to design controllers which guarantee stability and bound worst case performance over the set of plants. This is done by bounding the portion of the Lyapunov equation that is parameter dependent with a parameter independent bounding function. The solution of the modified Lyapunov equation bounds the possible solutions of the parameter dependent Lyapunov equation. Existence of a positive definite solution to the modified Lyapunov equation will guarantee stability and performance robustness. One such bounding function is called the quadratic bound or Petersen-Hollot bounds. When this bound is used, the resulting modified Lyapunov equation takes the form of a Riccati equation. This bound has been applied to the case of full state feedback design [63], static output feedback [69], and dynamic output feedback [69, 71, 72] control design.

Another Lyapunov bound takes the form of a linear function of the solution to the Lyapunov equation [74]. It is closely related to the Maximum Entropy design approach since the resulting equation for the closed-loop cost can be equated to the solution to a multiplicative white noise model with exponential weighting [69]. The linear bound has been extensively studied and the conditions for existence and uniqueness are known. Necessary conditions for static and dynamic feedback have been derived [69] and solution procedures for the coupled modified Lyapunov and Riccati equations have been formulated, Ref. [95]. This linear bound will be of interest in a later section of this report.

The algebraic and bounding techniques for determining stability robustness of parametrically uncertain plants tend to produce control design techniques which guarantee controlled system stability *a priori*. While this guarantee is sometimes useful, it is sometimes commensurate with loss of performance, that is, higher state and control cost. In the following sections, robust design techniques will be presented which do not necessarily guarantee stability but do increase the system robustness.

Random Process Techniques

Much work has been done on designing controllers when the uncertain parameters are interpreted as random processes, either jump processes [79–81] or multiplicative white noise [82–97]. The use of multiplicative white noise models, called maximum entropy design [93–97], have been applied successfully to the flexible structure control problem [91] and coupled to the optimal projection [97, 98] reduced-order controller design philosophy to provide fixed order controllers which are robust to the unmodeled uncertainty. The only drawbacks of this design technique are the lack of a known correlation between the amount of robustness achieved by the design and the amount of uncertainty modeled and the restrictive form of the uncertainty considered. Only frequency uncertainties have been modeled with this technique. Nevertheless the method has been successfully applied to the structural control problem with good results. It has also been further developed into a structured covariance design approach using the linear bound discussed previously [99]. With slight modification, the maximum entropy design equations, systems of modified Riccati and Lyapunov equations, yield solutions which guarantee *a priori* stability and performance robustness.

The philosophy behing multiplicative white noise uuncertainty modelling and the "robustifying" techniques which will be presented in the following sections is to dispense with methods which bound the system performance over the parameter space (and thereby guarantee *a priori* stability) in favor of a more hueristic approach to increasing the design stability robustness. By abandoning guarantees, it is sometimes possible to find lower cost controllers which achieve the same level of stability robustness as a bound-based design. This idea of moving away from bound-based designs toward a more hueristic "robustifying" methodology is similar in spirit with the techniques presented in this work. Another class of controller design techniques which increase robustness but do not guarantee *a priori* stability are those which minimize the sensitivity of the cost to parameter variations.

Sensitivity Minimization

Another approach to robust control is that of cost or trajectory sensitivity minimization. Many methods have been used to desensitize the cost or trajectory to parameter variation. One such method is to recover the robustness properties of the LQR design in a dynamic compensator using Loop Transfer Recovery Techniques [100]. Another is to modify the conventional quadratic cost to include terms which penalize sensitivity. This section will focus on the work done using the latter technique [101–103]. The sensitized cost is computed using an augmented system called the sensitivity system. The sensitivity system is computed by augmenting the nominal trajectory states by the trajectory sensitivity states and thereby forming a large order system which reflects the parameter sensitivities of the trajectories and the cost. Several cost and trajectory sensitivity cost functions as well as the properties of the sensitivity system are discussed in Ref. [101]. Ref. [101] also presents a full state controller design methodology.

Controller design can be difficult because the sensitivity system order is so large. This issue was addressed in Ref. [102] and [103]. In the former a controller reduction scheme was employed using Q-cover theory and in the later a fixed-form compensator was used. The robustness properties of the fixed-form compensator are well illustrated and compared to Ref. [97] in Ref. [103]. Controllers based on sensitivity system methods have the advantage of dealing with uncertainties in the disturbance and design weighting penalty matrices explicitly. Because only first order sensitivities are used, however, they are unable to reflect the fact that the cost can be a complicated function of the uncertainties. As a result they guarantee no *a priori* bounds and provide only an ad hoc approach to robust control.

Multi–Model Techniques

Another technique used for robust control synthesis is the multi–model design concept. In this concept the controller is designed based on multiple models which represent possible plants. A possibly finite set of design models is used to add robustness over a continuous set of possible plants, represented perhaps by parametric uncertainty or change in operating regimes [104–107]. The cost is averaged in some sense over the design plants in order to detect the contribution of the parameter variation to the cost. Such schemes have been shown to provide robustness to parameter variations in Ref. [104].

Multi–model techniques have been applied to full state feedback of a continuous distribution of plants as given in Ref. [108]. In this case the cost is averaged over an infinite distribution of plants. This methodology of using a continuous distribution of plants is most similar to the one adopted in this work. This will be discussed in more detail in the following section.

1.4 Contributions of this Thesis

The fundamental idea of cost averaging techniques for robust control design is to consider as a performance metric the average of the standard quadratic cost or $\mathcal{H}_2$ system norm over a continuously parameterized model set. The cost averaging techniques are intimately related to the multi-model design methods and the simultaneous stabilization problem discussed above. It will be shown that if the average cost of a parameterized set of systems is bounded, the system must be stable for all values of the parameters. This important property of the average motivates its use as a cost functional for control design.

The difficulty in using the averaged cost for controller design comes in its computation. It is impractical to calculate exactly when there is a large number of uncertain parameters. To solve this problem two techniques were developed. The first is to use computable approximations to the exact average as the performance metric. Two methods for computing the approximate average cost are presented. These are called the *perturbation expansion approximation* (PEA) and the *Bourret approximation*. Since they are only approximations, they will not share the properties of the exact average cost minimization but will provide a method of adding robustness to controllers for parametrically uncertain systems.

The Bourret approximation is widely used as an approximation to the average solution in random wave propagation [118–121] and turbulence modeling [122–124] where its properties make it a better approximation than first or second order perturbation methods. These properties will be discussed in Chapter 3. It is hoped that the quality of the Bourret approximation and the fact that it can represent the destabilizing effect of uncertainties on the cost will make the robustness achieved by this design technique more closely match the desired robustness properties, i.e., the set of stable closed-loop systems is near to the total design set.

The second technique is to bound the average cost and minimize this to locate simultaneously stabilizing controllers. Most current robust design techniques based on Lyapunov bounding (quadratic bound, linear bound, etc.) bound the worst case performance over the set rather than the average performance. They thus guarantee performance as well as stability robustness. This performance guarantee may not be desirable when weighed against its commensurate increase in control cost. As an alternative, bounding the average can guarantee stability without necessarily guaranteeing performance and thus lead to lower cost designs. The bounding however leads to conservatism and thus less efficient designs than those using the exact average. The two techniques of approximation and bounding the exact average are seen as complementary to each other since any real control design will necessarily be iterative.

1.5 Thesis Outline

In the following chapters the tools needed for the analysis of parameterized model set average cost will be developed and applied to the problem of synthesizing robust. Chapter 2 will deal with the issues of modeling systems with parametric uncertainty. Chapter 3 will deal with the analysis of parametrically uncertain systems using the average $\mathcal{H}_2$-norm and its approximations and bounds. Chapter 4 applies these analysis techniques to the problem of static and dynamic output feedback fixed-form compensation. The necessary conditions for minimization of the exact average cost or its approximations and bounds are presented along with the numerical computation of the control. Two structural control examples are presented in Chapter 5 to investigate the properties of the controller derived using the average-related costs. Chapter 6 will address the problem of reducing the order and number of uncertainties of a parametrically uncertain system using average cost decomposition techniques.

Chapter 2

Modeling of Uncertain Systems

2.1 System Notation

This chapter deals with the issues of modeling systems with parametric uncertainties for later analysis and control synthesis. In this section a compact notation for representing a dynamic system will be presented and related to other methods of representing systems. A system is a collection of components which interact over time. This report will deal exclusively with linear systems with time-invariant coefficients. Using standard state space notation, such a system can be represented as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\tilde{u}(t) \\ \tilde{y}(t) &= Cx(t) + D\tilde{u}(t) \end{aligned} \tag{2.1}$$

where the vector $x(t) \in \mathbb{R}^n$, is the system state, $\tilde{u} \in \mathbb{R}^m$ is the system input vector, and $\tilde{y} \in \mathbb{R}^l$ is the system output vector. The transfer function from input to output is found by taking the Laplace transform of the system in (2.1):

$$G(s) = C(sI - A)^{-1}B + D \tag{2.2}$$

A given transfer function can have any number of state space realizations. A simplified notation for a transfer function which incorporates state space realization information in the description can be justified by placing (2.1) into vector notation:

$$\begin{bmatrix} \dot{x} \\ \hline \tilde{y} \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ \hline \tilde{u} \end{bmatrix} \tag{2.3}$$

The information inherent in a given state space model is given by its coefficients so we can simply say without loss of information that

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.4}$$

This is the notation which will be used to specify the state space realization of a given system.

This report will deal with quadratic performance metrics for the above systems such as (2.4). When $\tilde{u}(t)$ is a unity intensity Gaussian white noise process this quadratic cost takes the form:

$$J = E\left\{\lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \tilde{y}(t)\tilde{y}^T(t)\ dt\right)\right\} \tag{2.5}$$

The quadratic cost is numerically equal to the square of the $\mathcal{H}_2$-norm of the system transfer matrix. In this case:

$$J = \|G(s)\|_2^2 \tag{2.6}$$

where the $\mathcal{H}_2$-norm is defined as in Eq. (1.1).

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left\{G(j\omega)^H G(j\omega)\right\} d\omega \tag{2.7}$$

The above form is inconvenient for calculation of the cost associated with a given system. The cost can be more conveniently calculated as

$$J = \operatorname{tr}\left\{QC^TC\right\} \tag{2.8}$$

where Q is the solution of a Lyapunov equation

$$AQ + QA^T + BB^T = 0 \tag{2.9}$$

In this report the matrix $\mathcal{H}_2$-norm will be frequently used as a basis for defining other cost functionals, in particular the average cost function which will be described in the next chapter. Because of the equivalence shown above, the terms "quadratic cost" or "system $\mathcal{H}_2$-norm" will be used interchangeably.

In the next section, the modeling conventions presented above will be expanded to represent systems with inputs and outputs associated with a control system.

2.1.1 Modeling of Controlled Systems

A more specialized form of system notation is sometimes used to specify a given control design problem. This specialized form is derived by dividing the system's inputs and outputs into two groups, respectively

$$\tilde{y} = \begin{bmatrix} z \\ y \end{bmatrix} \quad \tilde{u} = \begin{bmatrix} w \\ u \end{bmatrix} \tag{2.10}$$

where z is the vector of system outputs which contribute to some performance metric, and y is the vector of system outputs which are sensed and are available to the feedback controller. The z and y vectors can be identical, have individual elements in common, or be completely distinct depending on the problem. The inputs w are due to uncontrolled disturbances while the inputs u are due to control inputs. This partitioning of the input and output vectors can be used to partition the coefficient matrices into:

$$\begin{bmatrix} \dot{x} \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix} \tag{2.11}$$

Using the compact notation, the system can therefore be described:

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2.12}$$

Typically some of these matrices are assumed to be zero for a given problem. For structural vibration control problems the D_{11} and D_{22} matrices can typic:ly be set to zero since the disturbances are rarely explicitly included in the performance metric ($D_{11} = 0$) and the transfer function from u to y about which the loop is closed is typically strictly proper ($D_{22} = 0$) since real systems invariably roll off. In command following appications the D_{11} term becomes inportant but can be incorporated into the model by adding states corresponding to the command input. If there is a D_{22} term due to some modeling assumption such as truncated stiffness matrices, it can be incorporated into the system A matrix by adding high frequency rolloff. The systems dealt with in this report will therefor ignore the D_{11} and D_{22} matrices.

The two inputs and two outputs can be used as a basis for partitioning the system transfer function as

$$G(s) = \left[\begin{array}{c|c} G_{11}(s) & G_{12}(s) \\ \hline G_{21}(s) & G_{22}(s) \end{array}\right] \tag{2.13}$$

where, for instance, $G_{11}(s)$ is the transfer function from w to z.

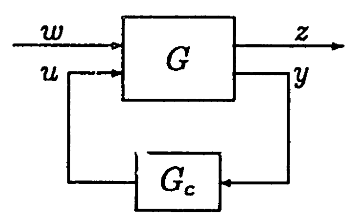

Figure 2.1: The Standard Control Problem

In the standard control problem shown in Figure 2.1, a compensator with arbitrary transfer function, $G_c(s)$, closes the loop between the system outputs, y, and the system inputs, u. Closing the loop between y and u changes the transfer function from w to z. This new transfer function is called the closed-loop transfer function, denoted G_{zw}. It is obtained from a function called the lower linear fractional transformation, which is defined by

$$\mathcal{F}(G, G_c) = G_{11} + G_{12}G_c(I - G_{22}G_c)^{-1}G_{21} \tag{2.14}$$

The closed-loop transfer function is then given by

$$G_{zw} = \mathcal{F}(G, G_c) \tag{2.15}$$

To determine the state space representation of G_{zw}, the state space representation of the compensator, G_c, must be specified. Two control types will be considered in the coming chapters. These are static and dynamic output feedback. For static output feedback, the compensator is a simple constant gain between y and u which can be written as

$$G_c = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_c \end{array}\right] \tag{2.16}$$

Using the system notation presented previously in (2.12) and neglecting the D_{11}, D_{21}, and D_{22} terms, the closed-loop system can be written as

$$G_{zw} = \left[\begin{array}{c|c} A + B_2 D_c C_2 & B_1 \\ \hline C_1 + D_{12} D_c C_2 & 0 \end{array}\right] \tag{2.17}$$

The D_{21} matrix being zero is essentially the statement that there is no noise on the measurements used for control and is a standard assumption for static output feedback.

For dynamic output feedback, the compensator is a strictly proper dynamic system of order n_c which can be expressed by

$$G_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{2.18}$$

The closed-loop system can then be written as

$$G_{zw} = \left[\begin{array}{cc|c} A & B_2 C_c & B_1 \\ B_c C_2 & A_c & B_c D_{21} \\ \hline C_1 & D_{12} C_c & 0 \end{array}\right] \tag{2.19}$$

The order of the system is increased by augmenting the open-loop system dynamics with the controller dynamics. The analysis of parametrically uncertain systems presented in the next chapter is independent of whether the control loop is closed or open. These closed-loop models are presented to provide a needed reference for use when performing control synthesis and to specify the structure of the uncertainty in the closed-loop.

2.1.2 Example: LQG Problem Statement

It is helpful to place the standard LQG problem statement into the matrix norm formalism to help solidify the modeling conventions used in this report. Before doing this, however, the distinction between the *evaluation plant* and *design plant* must be made clear. The evaluation plant is the system upon which a given control design is evaluated for performance and robustness. It represents the actual physical system as closely as possible and is sometimes called the truth model. The design plant is the system which is used in the design of the control system. It is typically the evaluation plant modified by weighting the inputs and outputs to achieve a certain design goal. For instance, a certain output can be heavily weighted in the design process if it is desirable for the control to minimize this output. The weightings on the inputs and outputs do not have to be static gains but can themselves be stable transfer functions which have frequency dependence. When this occurs, the dynamics of the weighting transfer functions is absorbed into the design plant A matrix. The design plant can thus have open-loop dynamics which bear little resemblance to the evaluation plant.

To begin the comparison between the LQG problem statement and the system norm formalism, the system dynamics can be defined by

$$\dot{x}(t) = Ax(t) + Bu(t) + L\xi(t) \tag{2.20}$$

$$y(t) = Cx(t) + \theta(t) \tag{2.21}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$. The two noise input vectors, $\xi(t) \in \mathbb{R}^q$, the process noise, and $\theta(t) \in \mathbb{R}^p$, the sensor noise, are independent, zero mean, Gaussian white noise processes with constant intensity matrices, Ξ and Θ respectively. In addition the cost functional which is to be minimized is defined by

$$J_{LQG} = E\left\{\lim_{T\to\infty}\left(\frac{1}{T}\int_0^T x^T(t)Qx(t) + u^T(t)Ru(t)\ dt\right)\right\} \tag{2.22}$$

which involves a positive semi-definite state weighting, $Q \in \mathbb{R}^{n\times n}$, and a positive definite control weighting, $R \in \mathbb{R}^{m\times m}$.

The evaluation model can be expressed in the standard system notation by first defining the output vector, z and the disturbance vector, w used in (2.12). Let

$$w(t) = \begin{bmatrix} \xi(t) \\ \theta(t) \end{bmatrix} \quad z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{2.23}$$

The evaluation system can now be written

$$G_{eval} = \left[\begin{array}{c|c|c} A & \begin{bmatrix} L & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right] \tag{2.24}$$

To derive the design model, the relative magnitudes of the input disturbances and output variables are explicitly weighted using the noise intensities, Ξ and Θ, and the output weights, Q and R used in the quadratic cost, (2.22). The design plant has the form:

$$G_{des} = \left[\begin{array}{c|c|c} A & \begin{bmatrix} L\Xi^{1/2} & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \\ C & \begin{bmatrix} 0 & \Theta^{1/2} \end{bmatrix} & 0 \end{array}\right] \tag{2.25}$$

Given this definition of the design plant, the $\mathcal{H}_2$-norm of the design system is equivalent to the quadratic cost, that is

$$||G_{des}||_2^2 = J_{LQG} \tag{2.26}$$

and thus the problems of finding the compensator, G_c, to minimize either the $\mathcal{H}_2$-norm of the design plant or the quadratic cost defined in (2.22) are equivalent.

Some of the system notations and concepts which will be used in the rest of the report have now been presented. The next section will deal with modeling systems with parameter uncertainties. The parameterization of the systems describes a set of systems, called the model set, over which the system $\mathcal{H}_2$-norm can be averaged.

2.2 The Model Set

In this section the modeling of parametrically uncertain systems will be approached using the concept of a set of systems called the model set. The model set is a set of plants whose elements are characterized in terms of variable real parameters. The standard system notation set forth in Ref. [19] and in the previous section will be used for the elements of the model set. Two types of sets will be introduced, one with general parametric dependence and one with a more structured parameter dependence which simplifies cost computation.

2.2.1 The General Set of Systems

The behavior of a system is determined by the coefficients of its system matrices. The values of these coefficients can depend on the values of certain parameters of the system such as the stiffness of a structural element or the mass at a particular location. The uncertain systems dealt with in this work arise when the exact values of the model parameters are not known. Instead, only the range over which the parameters can vary is known. If there are r uncertain parameters, each with a specified upper and lower bound then the set of possible parameter vectors can be defined

Definition 2.2.1 (General Parameter Set) *The set, Ω_g, is a compact connected subset of $\mathbb{R}^r$.*

Each vector of parameter values generates a different system. The set of all of the possible systems is called the model set. The *general set of systems* is generated if the functional form of the parameter dependence of the coefficients of the system matrices is unspecified. It will be assumed that the coefficients of the matrices are continuous functions of α, however.

Definition 2.2.2 (General Set of Systems) *The set $\mathcal{G}_g$ of systems is parameterized as follows:*

$$\mathcal{G}_g = \{G_g(\alpha)\; \forall\, \alpha\, \in\, \Omega\} \tag{2.27}$$

where $\Omega_g \subset \mathrm{I\!R}^r$ is defined in Def. 2.2.1 and each element of the set is described in the state space as

$$G_g(\alpha) = \left[\begin{array}{c|cc} A(\alpha) & B_1(\alpha) & B_2(\alpha) \\ \hline C_1(\alpha) & 0 & D_{12}(\alpha) \\ C_2(\alpha) & D_{21}(\alpha) & 0 \end{array}\right] \tag{2.28}$$

where $A(\alpha) \in \mathrm{I\!R}^{n\times n}$, $B_2(\alpha) \in \mathrm{I\!R}^{n\times m}$, $C_2(\alpha) \in \mathrm{I\!R}^{l\times n}$, $B_1(\alpha) \in \mathrm{I\!R}^{n\times p}$, $C_1(\alpha) \in \mathrm{I\!R}^{q\times n}$, $\forall \alpha \in \Omega_g$ and teh elements of the matrices are continuous functions of the parameters over Ω_g.

In addition to the assumptions implicit in the set definition the following assumptions will be made:

(i) For each $\alpha \in \Omega, (A(\alpha), B_1(\alpha))$ is stabilizable, $(C_1(\alpha), A(\alpha))$ is detectable.

(ii) For each $\alpha \in \Omega, (A(\alpha), B_2(\alpha))$ is stabilizable, $(C_2(\alpha), A(\alpha))$ is detectable.

(iii) $D_{12}^T(\alpha)\begin{bmatrix} C_1(\alpha) & D_{12}(\alpha) \end{bmatrix} = \begin{bmatrix} 0 & R(\alpha) \end{bmatrix}, R(\alpha) > 0\; \forall\, \alpha\, \in\, \Omega$

(iv) $D_{21}(\alpha)\begin{bmatrix} B_1^T(\alpha) & D_{21}^T(\alpha) \end{bmatrix} = \begin{bmatrix} 0 & V(\alpha) \end{bmatrix}, V(\alpha) > 0\; \forall\, \alpha\, \in\, \Omega$

(v) The set of systems, $\mathcal{G}_g$, must be simultaneously stabilizable. The conditions for simultaneously stabilizable sets of systems has been considered in Ref. [27, 28, 38–41].

Assumptions (i) and (ii) are made to ensure the observability and controllability of unstable modes from the controller and the disturbability and measureability of the unstable modes in the performance. Assumption (iii) means that C_1x and $D_{12}u$ are orthogonal so that there is no cross weighting between the output and control. This simplifies the form of the necessaary conditions for future use in control design. R is positive definite so that the penalty on z includes a nonsingular penalty on the control. Assumption (iv) is dual to (iii) and ensures the noncorrelation of the process and sensor noise. It is equivalent to the standard conditions assumed for the Kalman filter and is again instated for clarity. Assumption (v) is made to guarantee existence of the controllers derived in the coming chapters.

In association with the set of parameter vectors and the model set, it is convenient to define a distribution function, $\mu(\alpha)$ which is always positive and which is normalized such that

$$\int_{\Omega_g} \mu(\alpha) \ d\alpha = 1 \tag{2.29}$$

This distribution function can be considered a probability distribution function used to establish the relative importance of certain elements of the model set. If, for example, the nominal is highly likely, the distribution function can have a maximum at the value of the parameter vector which corresponds to the nominal system. The distribution function can also be very useful in weighting the average of the costs associated with the elements of the model set. Unless otherwise specified, the distribution function will be assumed uniform in the rest of the report.

It is useful at this point to consider the set of closed-loop systems. For clarity of presentation, only the dynamic output feedback case will be considered here. The closed-loop model set associated with static output feedback will be presented in Chapter 4. Given the set $\mathcal{G}_g$ of open loop systems and the fixed form dynamic compensator of order n_c

$$G_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{2.30}$$

with input y and output u, the set of closed-loop transfer functions from w to z, $\mathcal{G}_{zw}$, can be defined. Each element of $\mathcal{G}_{zw}$ can be expressed in state space form for dynamic output feedback as:

$$\begin{aligned} G_{zw}(\alpha) &= \left[\begin{array}{cc|c} A(\alpha) & B_2(\alpha)C_c & B_1(\alpha) \\ B_cC_2(\alpha) & A_c & B_cD_{21}(\alpha) \\ \hline C_1(\alpha) & D_{12}(\alpha)C_c & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \tilde{A}(\alpha) & \tilde{B}(\alpha) \\ \hline \tilde{C}(\alpha) & 0 \end{array}\right] \end{aligned} \tag{2.31}$$

The B and C matrices of the closed-loop systems are parameter dependent. It can be instructive to consider a more restrictive set of open-loop systems such that the resulting closed loop set only has uncertainty in the A matrix since only uncertainty in the closed-loop A matrix affects the stability of the system.

2.2.2 The Structured Set of Systems

It will prove useful to define a different set of systems with more restrictive assumptions on the functional form of the parameter dependence of the system matrices. The first assumption is that only parameter uncertainties entering into the closed-loop $\tilde{A}$ matrix will be considered. This amounts to restricting the $\tilde{B}$ and $\tilde{C}$ matrices to being parameter independent. This assumption is not overly restrictive for stability robustness considerations since only uncertainties in the closed-loop A matrix affect stability. The uncertainties in the $\tilde{B}$ and $\tilde{C}$ matrices would however efect average performance. This uncertainty restriction is made primarily to enable application of the operator decomposition techniques which will be used to derive approximations and bounds to the average cost. The general uncertain set of systems in (2.28) can be specialized to a more structured set which allows less general parameter dependence. First the structure of the parameter set will be defined.

Definition 2.2.3 (Structured Parameter Set) *The set, Ω_s, of parameter vectors, α, is defined*

$$\Omega = \left\{\alpha \ : \alpha \in \mathbb{R}^r, \delta_i^L \leq \alpha_i \leq \delta_i^U \ \ i = 1, \cdots, r\right\} \tag{2.32}$$

where δ_i^L and δ_i^U are the lower and upper bounds for the i^{th} uncertain parameter.

In addition, the parameter dependence of the elements of the remaining matrices will be assumed to be linear functions of the parameters. This is a very restrictive assumption but necessary if computable approximations for the average are to be derived. If they are in fact not linear functions, then the matrices can be linearized about the nominal values of the parameters. It is also sometimes possible to define a new uncertain parameter which is a nonlinear function of the first but which makes the parameter dependence of the elements of the system matrices linear. This would be the case if, for instance, one used $z = 1/m$ as a variable parameter rather than m itself in a term of the form $1 + 1/m$. Once the parameter dependence has been made linear a more structured set of systems can be defined.

Definition 2.2.4 (Structured Set of Systems) *The set $\mathcal{G}_s$ of systems is parameterized as follows*

$$\mathcal{G}_s = \{G_s(\alpha) \ : \alpha \in \Omega_s\} \tag{2.33}$$

where Ω_s is the structured set of parameter vectors defined in Def. 2.2.3 and each element of $\mathcal{G}_s$ is described in the state space as

$$G_s(\alpha) = \left[\begin{array}{c|cc} A_0 + \sum_{i=1}^{r}\alpha_i A_i & B_1 & B_{2_0} + \sum_{i=1}^{r}\alpha_i B_{2_i} \\ \hline C_1 & 0 & D_{12} \\ C_{2_0} + \sum_{i=1}^{r}\alpha_i C_{2_i} & D_{21} & 0 \end{array}\right] \tag{2.34}$$

where for $i = 0, \ldots, r$; $A_i \in \mathbb{R}^{n \times n}, B_{2_i} \in \mathbb{R}^{n \times m}, C_{2_i} \in \mathbb{R}^{l \times n}$, and $B_1 \in \mathbb{R}^{n \times p} C_1 \in \mathbb{R}^{q \times n}$.

Just as for the general set of systems, a set of closed-loop transfer functions, denoted $\mathcal{G}_{zw}$, can be generated using the structured set of systems. This closed-loop set can be expressed is state space form for dynamic output feedback as

$$\begin{aligned} G_{zw}(\alpha) &= \left[\begin{array}{cc|c} A_0 + \sum_{i=1}^{r} \alpha_i A_i & B_{2_0} C_c + \sum_{i=1}^{r} \alpha_i B_{2_i} C_c & B_1 \\ B_c C_{2_0} + \sum_{i=1}^{r} \alpha_i B_c C_{2_i} & A_c & B_c D_{21} \\ \hline C_1 & D_{12} C_c & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \tilde{A}_0 + \sum_{i=1}^{r} \alpha_i \tilde{A}_i & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] \end{aligned} \tag{2.35}$$

Because of the form assumed for the uncertainty, only the resulting closed-loop A matrix, $\tilde{A}(\alpha) \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, is parameter dependent and the closed-loop system is strictly proper.

2.2.3 Connections to Internal Feedback Loop Modeling

Another way to represent parametric dependency in a state space model is through the use of internal feedback loops. This representation of parametric uncertainty has been used for $\mathcal{H}_\infty$ design for structural systems in Refs. [100, 12, 130, 131]. In this section, the structured set of systems presented in Def. 2.2.4 will be transformed into the internal feedback loop representation to highlight the connection between these modeling techniques.

The internal feedback loop modeling of uncertain parameters involves factoring the parameters out of the model into a fictitious feedback path by augmenting the model with fictitious inputs and outputs. The parameters then represent uncertain feedback gains which the system must tolerate for stability. To see how this is done, we start with the structured set of systems.

$$G_s(\alpha) = \left[\begin{array}{c|cc} A_0 + \sum_{i=1}^{r} \alpha_i A_i & B_1 & B_{2_0} + \sum_{i=1}^{r} \alpha_i B_{2_i} \\ \hline C_1 & 0 & D_{12} \\ C_{2_0} + \sum_{i=1}^{r} \alpha_i C_{2_i} & D_{21} & 0 \end{array}\right] \tag{2.36}$$

The critical assumption of internal feedback loop modeling is that that the uncertainty term associated with α_i

$$\alpha_i T_i = \alpha_i \begin{bmatrix} A_i & 0 & B_{2_i} \\ 0 & 0 & 0 \\ C_{2_i} & 0 & 0 \end{bmatrix} \tag{2.37}$$

can be decomposed as the outer product of two vectors of the form

$$\alpha_i T_i = \begin{bmatrix} M_{x_i} \\ 0 \\ M_{y_i} \end{bmatrix} \alpha_i \begin{bmatrix} N_{x_i} & 0 & N_{u_i} \end{bmatrix} \tag{2.38}$$

This assumption is equivalent to the statement that T_i is of rank one. If it is not of rank one but say of rank two, then the uncertainty term associated with α_i takes the form

$$\alpha_i T_i = \begin{bmatrix} M_{x_i} \\ 0 \\ M_{y_i} \end{bmatrix} \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_i \end{bmatrix} \begin{bmatrix} N_{x_i} & 0 & N_{u_i} \end{bmatrix} \tag{2.39}$$

The single α_i is split into two dependent uncertain parameters. In the internal feedback loop model the parameters must be assumed independent and thus the conservatism of the design is increased. Assuming for now that each T_i is indeed of rank one, then the column and row vectors for the respective parameters can be stacked to give

$$\sum_{i=1}^{r} \alpha_i \begin{bmatrix} A_i & 0 & B_{2_i} \\ 0 & 0 & 0 \\ C_{2_i} & 0 & 0 \end{bmatrix} = \begin{bmatrix} M_x \\ 0 \\ M_y \end{bmatrix} \Delta \begin{bmatrix} N_x & 0 & N_u \end{bmatrix} \tag{2.40}$$

where Δ has the form

$$\Delta = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_r \end{bmatrix} \tag{2.41}$$

To factor the Δ matrix into an internal feedback loop, fictitious inputs and outputs are defined

$$z_\alpha = \begin{bmatrix} N_x & 0 & 0 & N_u \end{bmatrix} \begin{bmatrix} x \\ w_\alpha \\ w \\ u \end{bmatrix} \tag{2.42}$$

$$w_\alpha = \Delta z_\alpha \tag{2.43}$$

and the system inputs and outputs are augmented by the fictitious channels to give a modified system description

$$\tilde{G}_s \longrightarrow \begin{bmatrix} \dot{(x)} \\ \hline z_\alpha \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|ccc} A_0 & M_x & B_1 & B_{2_0} \\ \hline N_x & 0 & 0 & N_u \\ C_1 & 0 & 0 & D_{12} \\ C_{2_0} & M_y & D_{21} & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w_\alpha \\ w \\ u \end{bmatrix} \tag{2.44}$$

The feedback structure of this system is shown in Fig. 2.2. This is also the form of the uncertainty structure when the uncertainty is represented in the frequency domain by norm-bounded uncertain matrices. In this case $\Delta \in \mathbb{R}^{r \times r}$ is an arbitrary complex matrix such that $\|\Delta\|_2 \leq \gamma$. The use of a diagonal matrix of real parameters increases the structure of the uncertainty representation and can thus decrease the conservatism of control design for systems with real parameter uncertainties.

It should be noted that unstructured frequency domain norm-bounded uncertainty is probably more appropriate than parametric error models for representing high frequency errors in the model dynamics when the structure of the high frequency dynamic system errors is unknown. The parametric error modeling can thus be con-

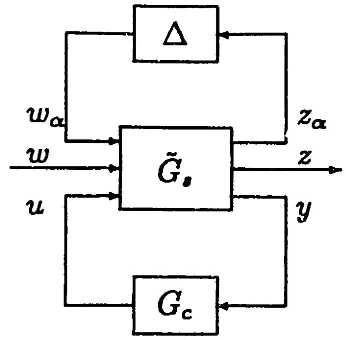

Figure 2.2: Internal Feedback Loop Modeling of Parametric Uncertainties

sidered complementary to the well developed frequency domain uncertainty modeling and should be used when structural information is available about the uncertainty.

The internal feedback loop representation of uncertain parameters is convenient for use with input-output analysis tools, but clouds the effects that the uncertain parameters have on the system dynamics. The parameter dependence of the A matrix is no longer explicitly given but must be inferred from the dynamics of the system with the Δ loop closed. The notation chosen for this report is more suited to calculating and approximating the average over the structured set of systems because it explicitly reflects the parametric dependence of the system matrices.

2.3 Formulation of Parameterized Models

In the previous section, the notation and forms for expressing the parameter dependence of linear time invariant systems have been presented. In addition, the structure of sets of systems described by the parameterization has been defined. In this section, some comments will be made on the process of formulating models of uncertain systems. In essence, this process is one of finding the values and parametric dependence of the system matrices used in the uncertain model. This section will touch on two general areas concerned with model formulation. These are parameterized model generation and model uncertainty representation. The treatments of these topics will be brief since the primary topic of this work is robust control.

2.3.1 Parameterized Model Generation

State space models can be generally categorized according to the source of the information contained in them. Models can be classified as being analysis or measurement based. The *analysis based* model is developed using assumptions on the system properties and nature of the component interactions. The *measurement based* model is generated based on an identification of measured data. There can of course be hybrid models of which certain components are analysis based and others are measurement based. Such models will exhibit characteristics in common with both types of models. These two model types are in some sense complementary since measurement based models provide the accuracy which analysis based models lack while analysis based models provide more insight into the structure of the system. The generation of a parameterized model depends on whether the model is measurement or analysis based.

It is appropriate at this point to discuss the nature of parametric uncertainty. In general, uncertainty can arise from four sources: errors in the values of system parameters used in the model, unmodeled dynamics, neglected nonlinearities, or neglected disturbances [1]. The boundaries between these different types of uncertainties are vague. For instance, neglected dynamics can be modeled as system dynamics with very uncertain coefficients. Take for example a SISO unstructured uncertainty which is described as a complex parameter with bounded magnitude but arbitrary phase. Although this uncertainty can represent a SISO plant with arbitrary order dynamics, the net effect is simply a complex parameter with uncertain magnitude and phase. This complex parameter can be represented by a second order dynamic system with highly uncertain natural frequency and damping. Neglected nonlinearities such as nonlinear joint stiffness in a structure can be accommodated by modeling the nonlinear joint as having uncertain stiffness and damping by using describing functions. The important characteristic of parametric error is that the structure of the interactions is prespecified and only the properties of the interaction are variable. In other words, *parametric errors are specified within a given interaction structure.* This characteristic has strong implications for the utility of parametric error modeling. It is only really useful when the structure of the uncertain model is known. This is more typically true for analysis based models rather than measurement based models.

When a model is measurement based, model errors can be caused by either identification errors or test inaccuracies. Identification errors involve mistakes made in generating the state space system from the given noisy data, be it measured transfer functions or time histories. Such mistakes can entail misrepresenting the system dynamics by not giving the correct pole or zero locations, missing some poles and zeros altogether, or including too many. Parametric error models are better suited to modeling uncertain pole and zero locations than changing the number of poles or zeros. For this case, the uncertain parameters such as modal natural frequency and damping can be identified by comparing the actual data to the identified model data.

Test inaccuracies arise when the conditions under which the system measurement is taken are different from the actual system operating conditions. This is a particularly important class of errors for space structures which can rarely be tested in full scale and which must be tested in the presence of gravity. The differences between the test and the operating environments can arise from several sources. They can be caused by measurements taken on a mis-scaled model, interaction with support structures and their dynamics, or (if the structure is nonlinear) change in vibration amplitude. These types of errors are particularly difficult to model with uncertain parameters since the structure of the error is not known. Parametric error modeling will be of little use unless an analysis model can be compared to the measurement model to test hypotheses on the causes and structures of the errors.

For analysis based models, the errors arise in the assumptions made on either the properties of or the structure of the interactions between the components. The errors in interaction properties are natural candidates for uncertain parameters. If, for instance, the stiffness of a given component is not known exactly, that stiffness can be used as an uncertain parameter in the model. Things are rarely that simple since the dependence of the system matrices on a given parameter is not known *a priori*. For example, a given element of the system matrices, say the second natural frequency of an uncertain structure, can be a complex function of the stiffness assumptions made in a particular finite element of the structure. Since the analytical model is available, however, the stiffness assumption can be varied and the changing values of the natural frequency can be ascertained. This type of modeling will be called perturbation based uncertainty modeling. The ability to test hypothesis about the influence of a given parameter is one of the biggest advantages of analysis based modeling.

Modeling errors in the structure of the interactions among the components is more subtle than modeling errors in the properties of already modeled interactions. Included in the category of uncertain interaction structure are neglected dynamics and incorrect boundary conditions and connectivity. The way to approach modeling this type of error with uncertain parameters is to first assume a general form for the structure and then parameterize the properties of the interactions. As an example, boundary condition assumptions can be replaced by uncertain boundary elements which, for instance, reflect the fact that a fixed boundary condition may in fact have some compliance. The difficulty is to recognize the need for modeling additional interactions rather than modify the coefficients of existing ones. This is an open problem in error model formulation.

2.3.2 Model Uncertainty Representation

In this section, the representation of parametric errors will be discussed. A given transfer function can be represented by any number of state space realizations depending on the coordinate system or choice of states used in the model. Likewise, a given uncertainty can be reflected in the model in many ways depending on the choice of coordinates. For instance, an uncertain stiffness could also be represented as uncertain natural frequencies if the system is transformed into modal coordinates. It will be assumed in the coming section that the system contains some inherent source of uncertainty which must be in some manner represented and accounted for in the model interactions.

A key concept for evaluating the merits of a given uncertainty representation is the idea of *conservatism*. Put loosely, a given uncertainty realization is more conservative than another if the set of represented systems is larger. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are the model sets described by two different representations of the same uncertainty, then $\mathcal{G}_2$ is said to be more conservative than $\mathcal{G}_1$ if $\mathcal{G}_1 \subset \mathcal{G}_2$. To see how this concept can be applied, consider a system with a single uncertain coefficient in its A matrix, for instance, an uncertain stiffness. This single uncertain coefficient can add uncertainty into many different modes of the system. The uncertainty in these modes is correllated because all modal deferences stem from the single uncertain coefficient. If that correlation is not retained in the model and instead each uncertain mode is modeled as an independent uncertainty, the number of uncertain parameters used in the model is increased as is the size of the set of possible systems. The representation of the uncertainty through uncorrelated modes is thus more conservative than representing it through the single uncertain coefficient.

The conservatism of a representation can be reduced by tracking the correlations of the uncertainties in the model rather than modeling correlated uncertainties as independent. This is difficult to do when the core uncertainty is unknown as is the case for measurement based models. With measurement based models, the structure of the system is not well known and thus the core uncertainties are hard to identify. This is another advantage to analysis based models where the correlations of the uncertainties can be determined. For instance an uncertain stiffness can be varied to determine the resulting uncertainty in the modes. Since the source of the modal uncertainties is known, these can be represented as correlated in the model set and the conservatism will not be increased.

In light of the above discussion, the general rule of thumb for uncertainty representation is to *reflect the uncertainties in the most basic model parameters* and determine the correlations of the higher level parameters through perturbation analysis.

2.4 Summary

In this chapter, the basic notation for representing parametrically uncertain systems has been presented. The central concept presented was the idea of the model set to describe a set of systems. Two sets where presented, one in which the parametric dependence of the coefficients of the system matrices were allowed to be arbitrary functions, and one in which the coefficients were assumed to be linear functions of the uncertain parameters. The more structured set was compared to the internal feedback loop model of parameter uncertainties typically used in $\mathcal{H}_\infty$ design. Some thoughts on generating parameterized models and the parametric representation of uncertainties were also presented.

Chapter 3

Analysis of Parameterized Systems

In this chapter the average cost of a parametrically uncertain system will be examined. This is motivated by some useful properties that the average cost has as a cost functional. Averaging the cost over the model set allows it to reflect the effects of parametric uncertainty on the system performance. As will be shown in the next section, the average cost has the property of being finite only if the set of systems is everywhere stable. therefore if the average cost is minimized by a certain controller, the closed-loop system will be guaranteed stable. This method of guaranteeing stability can be compared to the bounding techniques discussed in the introduction. These techniques guarantee stability by minimizing a function which bounds any of the possible costs over the set. This bounding function is called a worst-case bound. The main point made in this chapter is that *stability can be guaranteed by bounding the average cost rather than the worst case cost.* This fact motivates a new class of controllers which minimize quantities related to the average cost in the hopes of achieving design stability with lower nominal system cost than can be achieved using bounding techniques.

In the coming sections, the tools used for the average cost analysis of parameterized systems will be developed and applied to the problem of computing the average $\mathcal{H}_2$-norm of a model set. These tools are to a large extent based on operator decomposition techniques which have been widely used in wave propagation and turbulence modeling and are here applied for the first time, to the best of the author's knowledge, to analysis of parameterized systems. The exact average cost, while possessing useful properties, is difficult to compute. These operator techniques will be used to develop some of the properties of the average cost as well as some computable approximations and bounds to it. The properties of these approximations and bounds will be investigated.

The following sections deal with analysis of parameterized sets of systems. While the analysis techniques are applicable to open-loop sets, they are largely motivated by the control design problem. For development purposes, the sets considered will be closed-loop model sets which are generated when some fixed-form compensator is assumed to close the control loop. The first section introduces the average cost of such a system.

3.1 The Average Cost

In this section the average cost will be defined and discussed as a possible performance metric for parametrically uncertain systems. It is defined as the quadratic ($\mathcal{H}_2$) performance averaged over a parameterized set of linear time-invariant systems. The properties which make the average cost useful for later use in control design will be presented. This section is divided into three parts. First, the definitions, and properties of the average cost are presented. Difficulty in the computation of the average cost motivates the introduction of operator decomposition techniques in the second part. In the third part, these techniques are applied to the analysis of the average cost. The general set definitions and assumptions presented in Section 2.2.1 will be used throughout the development. We will start by considering the definition and properties of the exact average cost.

Definition 3.1.1 (Exact Average Cost) *The exact average cost is defined as the closed-loop $\mathcal{H}_2$-norm averaged over the model set $\mathcal{G}_{zw}$.*

$$J(G_c) = \int_{\Omega} \|G_{zw}(\alpha)\|_2^2 \,\mathrm{d}\mu(\alpha) \tag{3.1}$$

where $G_{zw}(\alpha) \in \mathcal{G}_{zw}$ for each $\forall\, \alpha \in \Omega$ and $\mu(\alpha)$ is the distribution function.

The average cost is thus the average of the $\mathcal{H}_2$-norm of the systems in the model set. As stated in Section 2.2.1, the distribution integrates to unity over the set. As long as it is normalized in this manner it can be an arbitrary function. In later sections, it will be assumed to be a uniform probability distribution. The distribution function can be thought of as a relative weighting of the systems in the model set or the relative probability of a given system. From the point of view of control design, it gives the designer the flexibility to make certain systems more prominent in the average.

The average cost possesses several properties which make it useful as a performance metric for control design. The first property of interest is the relationship between simultaneous stability and bounded average $\mathcal{H}_2$-norm. Before this can be presented, however, it is important to define a subset of Ω called a *set of zero measure.* A set $\Phi \subset \Omega$ has zero measure if

$$\int_{\Phi} \mathrm{d}\mu(\alpha) = 0 \tag{3.2}$$

Which can also be denoted, $\mu(\Phi) = 0$. Since $\mu(\alpha)$ is a positive function on $\Omega \subset \mathbb{R}^r$, sets of zero measure consist of a finite number of isolated single points in Ω. With this concept in hand, the properties associated with bounded average cost can be presented.

Theorem 3.1.1 (Bounded Average Cost) *If the exact averaged cost, Eq. (3.1), of $\mathcal{G}_{zw}$ is bounded*

$$J(G_c) = \int_{\Omega} \|G_{zw}(\alpha)\|_2^2 \,\mathrm{d}\mu(\alpha) < \infty \tag{3.3}$$

then the parameterized closed-loop systems, $G_{zw}(\alpha)$, are stable $\forall\, \alpha \in \Omega$ except possibly on a set of zero measure. Furthermore, no system in $\mathcal{G}_{zw}$ can have eigenvalues with positive real parts.

Proof: First we will show that if the exact averaged cost, Eq. (3.1) is bounded then the closed-loop system is stable $\forall\, \alpha \in \Omega$ except possibly on a set of zero measure. To do this assume that $\exists B \subset \Omega,\ \mu(B) > 0$ such that $\alpha \in B$ implies G_{zw} unstable. Since the norm of an unstable system is infinite, in this case:

$$\int_B \|G_{zw}(\alpha)\|_2^2 \mathrm{d}\mu(\alpha) = \infty \tag{3.4}$$

and thus $J(G_c) = \infty$. Finite average cost therefore implies that there can be no measurable subsets of $\mathcal{G}_{zw}$ with unstable elements.

Next assume that there exists a system, $G_{zw}(\alpha_1)$, with an eigenvalue with positive real part. Denote the open right half plane by $\mathcal{C}^+$. Because $\mathcal{C}^+$ is open, there exists a ball, B_1, about the unstable pole within which poles are also unstable. Now since the coefficients of G_{zw} are continuous functions of α and the eigenvalues are continuous functions of the coefficients, there is a continuous mapping, called $\phi(\alpha)$, from Ω to the unstable eigenvalue in $\mathcal{C}^+$. Because $\phi(\alpha)$ is continuous at α_1, a ball about $\alpha_1 \in \Omega$ can be found whose image is within B_1. If B_2 is this ball in Ω, and $\phi(B_2)$ is its image, then $\phi(B_2) \subset B_1$. Since B_2 has finite measure, the subset of elements of $\mathcal{G}_{zw}$ which have unstable poles has finite measure, and thus the average cost is infinite. The proof is shown in schematic in Fig. 3.1. □

Theorem 3.1.1 states that if the average cost of a set of systems is finite, then almost all of the elements of the set are asymptotically stable. Furthermore, the select few systems that are not asymptotically stable cannot have poles that lie in the right half plane. They can at worst have poles on the imaginary axis. This key theorem provides the motivation for examining the average cost since controllers designed by minimizing the average cost will be guaranteed stable over the model set.

The proof relies heavily on continuity of the map form Ω to the s-plane. Continuity of this map is similar to but not identical to continuity in Vidyasagar's graph topology

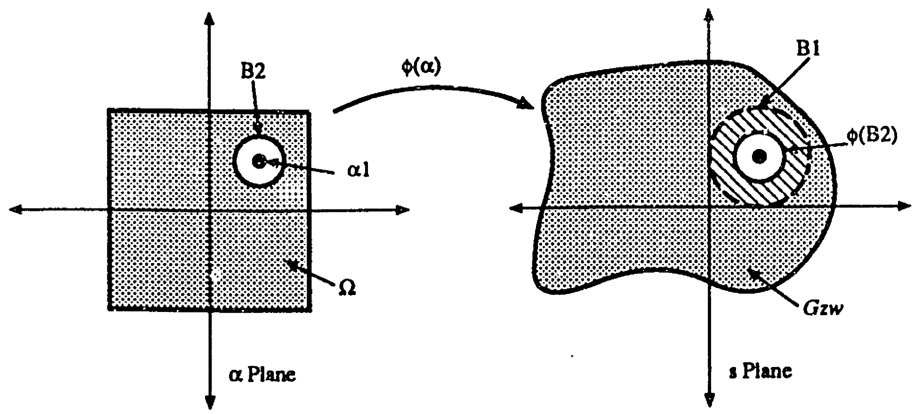

Figure 3.1: Schematic of Mappings from $\alpha_1 \in \Omega$ to the Closed-Loop Eigenvalue in the S-plane

[5]. The difference is that Vidyasagar relates robustness to continuity in the map from the parameter domain to the space of systems, while the present development uses a less restrictive map to the space of eigenvalues.

Since at each value of α the cost is given by the solution of a Lyapunov equation, the next step in the development is to relate the averaged $\mathcal{H}_2$-norm to the averaged solution of a parameterized Lyapunov equation. This gives a possible method of calculating the average cost by calculating the average solution to a linear Lyapunov equation. This is an important step since the approximations and bounds for the average cost will relay on this characterization of the average cost.

Proposition 3.1.1 (Averaged Lyapunov Solution) *Given a specified compensator, G_c, if the exact average cost exists, then it is given by*

$$J(G_c) = \mathrm{tr}\left\{\left\langle \tilde{Q}(\alpha)\tilde{C}^T(\alpha)\tilde{C}(\alpha)\right\rangle\right\} \tag{3.5}$$

where the notation

$$\langle[\cdot]\rangle = \int_\Omega [\cdot]\, \mathrm{d}\mu(\alpha) \tag{3.6}$$

has been adopted, and for each $\alpha \in \Omega$, $\tilde{Q}(\alpha)$ is the unique positive definite solution to

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha) \tag{3.7}$$

Proof: The proof is straightforward since finite average cost guarantees stability for almost all $\alpha \in \Omega$ and for stable systems, the cost is given by the solution of the Lyapunov equation, Ref. [19]. □

There is a problem with calculating the exact averaged cost because of the difficulty of averaging the solution to the parameterized Lyapunov equation, Eq. (3.7). In some instances for low order systems, the solution to Eq. (3.7) can be obtained explicitly as a function of α, and then averaged either numerically of symbolically. There are also numerous numerical techniques for approximating the average solution such as Monte-Carlo or direct numerical integration. These techniques tend to become computationally intensive and impractical as the dimension of Ω is increased.

In an effort to push the problem further analytically before resorting to numerical methods, operator decomposition techniques will be introduced in the next sections and applied to the problem of calculating the exact average cost. They will also be used in later sections to derive explicit bounds and approximations to the averaged solution of the parameterized Lyapunov equation. By turning to the operator decomposition methods of analysis, bounds and approximations to the exact average cost can be derived that are computable even if the number of parameters is large. The next section applies these operator decomposition techniques to the problem of computing the solution of parameterized linear operator equations.

3.1.1 Parameterized Linear Operator

In this section, general results for parameterized linear operators will be derived for analysis of the average cost of a parameterized set of systems. The operator which will be of primary interest is the parameterized Lyapunov equation in (3.7). Using the mathematics presented in this and the following section, the tools needed in the analysis of the properties of this equation and its average solution will be presented.

The following analysis is based in part on the work of Bharucha-Reid, Ref. [114], on the theory of random equations. While the work presented in the following pages is not stochastic in nature it draws heavily on work in the field of linear stochastic operators. First, let's consider a parameter vector, α, taking values on a closed and bounded set, $\Omega \in \mathbb{R}^r$ with distribution function $\mu(\alpha)$. Some useful functions of this parameter vector are presented next.

Definition 3.1.2 (General Parameterized Variable) *A general parameterized variable, $y(\alpha)$, is defined as a mapping from Ω to a Banach space, $\mathcal{H}$, $y(\alpha) : \Omega \rightarrow \mathcal{H}$. The space of general parameterized variables will be denoted $\tilde{\mathcal{H}}$.*

Definition 3.1.3 (Parameterized Linear Operator) *A mapping, $\mathbf{L}(\alpha)$, from the cartesian product space, $\Omega \times \mathcal{H}$, to $\mathcal{H}$ which is linear in $\mathcal{H}$ $\forall\, \alpha \in \Omega$ is called a parameterized linear operator.*

The parameterized variable which will be examined most closely is the solution of (3.7), $\tilde{Q}(\alpha)$. This parameterized variable takes values in $\mathbb{R}^{n\times n}$, the space of $n \times n$ matrices. The parameterized linear operator is the Lyapunov equation (3.7). We are interested in characterizing the solution of the parameterized operator equation,

$$\mathbf{L}(\alpha)\,[y(\alpha)] = x \tag{3.8}$$

where x is a parameter independent element of $\mathcal{H}$,$\mathbf{L}(\alpha)$ is the parameterized linear operator, and $y(\alpha)$ is a general parameterized variable taking values in $\mathcal{H}$ and defined for those α where $\mathbf{L}^{-1}(\alpha)$ exists. To find a solution for $y(\alpha)$ we will introduce the *operator decomposition technique*. This technique involves decomposing $\mathbf{L}(\alpha)$ into the sum of two linear operators, $\mathbf{L}_0$ and $\mathbf{L}_1(\alpha)$, such that $\mathbf{L}_0$ is invertible and parameter independent and $\mathbf{L}_1(\alpha)$ is a parameterized linear operator.

$$\mathbf{L}(\alpha) = \mathbf{L}_0 + \mathbf{L}_1(\alpha) \tag{3.9}$$

then the solution for $y(\alpha)$ can be expressed in terms of the nominal solution, i.e. the solution of (3.8) using only the nominal operator, $\mathbf{L}_0$. This technique of operator decomposition has been used extensively in the computation of solutions to linear stochastic equations, Ref. [116]. Another excellent paper on solution techniques for stochastic equations including operator decomposition is Frisch, Ref. [118].

The general results shown above can be applied to the parameterized Lyapunov equation used to compute the exact average cost in Thm. 3.1.1. Eq. (3.7) will be shown to be a parameterized linear operator which can be decomposed into a nominal and parameter dependent part.

Proposition 3.1.2 (Parameterized Lyapunov Equation) *The parameterized Lyapunov equation presented in Eq. (3.7) and reprinted here for clarity*

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha) \tag{3.10}$$

is a parameterized linear operator equation in the sense of Def. 3.1.3 from $\Omega \times \mathbb{R}^{\tilde{n}\times\tilde{n}} \rightarrow \mathbb{R}^{\tilde{n}\times\tilde{n}}$. *If* $\tilde{A}(\alpha)$ *can be decomposed* $\tilde{A}(\alpha) = \tilde{A}_0 + \tilde{A}_1(\alpha)$ *with* $\tilde{A}_0$ *parameter independent, then Eq. (3.10) has the following decomposition,* $\mathbf{L}(\alpha) = \mathbf{L}_0 + \mathbf{L}_1(\alpha)$ *and ,*

$$\mathbf{L}_0\left[\tilde{Q}\right] + \mathbf{L}_1(\alpha)\left[\tilde{Q}\right] = -\tilde{B}(\alpha)\tilde{B}^T(\alpha) \tag{3.11}$$

$$\mathbf{L}_0\left[\tilde{Q}\right] \quad : \quad \tilde{Q} \rightarrow \tilde{A}_0\tilde{Q} + \tilde{Q}\tilde{A}_0^T \tag{3.12}$$

$$\mathbf{L}_1(\alpha)\left[\tilde{Q}\right] \quad : \quad \tilde{Q} \rightarrow \tilde{A}_1(\alpha)\tilde{Q} + \tilde{Q}\tilde{A}_1^T(\alpha) \tag{3.13}$$

This is a handy operator notation for dealing with cumbersome Lyapunov equations. For example, consider the nominal Lyapunov equation and its nominal solution, $\tilde{Q}^0$.

$$\tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T = 0 \tag{3.14}$$

This equation can be written as

$$\mathbf{L}_0\left[\tilde{Q}^0\right] = -\tilde{B}\tilde{B}^T \tag{3.15}$$

which has the solution:

$$\tilde{Q}^0 = \mathbf{L}_0^{-1}\left[-\tilde{B}\tilde{B}^T\right] \tag{3.16}$$

When the operator is decompcsed into two parts, the operator equation (3.8) can be written as

$$\mathbf{L}_0\,[y(\alpha)] + \mathbf{L}_1(\alpha)\,[y(\alpha)] = x \tag{3.17}$$

which can be manipulated to give

$$y(\alpha) = \mathbf{L}_0^{-1}\,[x] - \mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\,[y(\alpha)] \tag{3.18}$$

Successive substitution for $y(\alpha)$ gives an infinite series for $y(\alpha)$ called the perturbation expansion. This is expressed precisely in the following proposition.

Proposition 3.1.3 (Perturbation Expansion) *Consider the parameterized linear operator,* $\mathbf{L}(\alpha) = \mathbf{L}_0 + \mathbf{L}_1(\alpha)$, *with* $\mathbf{L}_0$ *invertible and* $\left\|\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right\| < 1 \quad \forall\, \alpha \in \Omega$. *Then* $\mathbf{L}(\alpha)$ *is invertible in the sense that the mapping from* x *to* $y(\alpha)$ *is one to one and has continuous inverse* $\forall\, \alpha \in \Omega$ *the inverse is given by*

$$\mathbf{L}^{-1}(\alpha) = \sum_{i=0}^{\infty} \left(-\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^i \mathbf{L}_0^{-1} \tag{3.19}$$

Corollary 3.1.1 *The solution to (3.8),* $y(\alpha) \in \tilde{\mathcal{H}}$, *is a general parameterized variable and can be written as*

$$y(\alpha) = y_0 - \mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)y_0 + \mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)y_0 - \cdots = \sum_{i=0}^{\infty} \left(-\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^i y_0 \tag{3.20}$$

where the nominal solution $y_0 = \mathbf{L}_0^{-1}x$.

Proof: The result is a direct consequence of the von Neumann Lemma, Proposition 22.10 in Ref. [112]. □

Theorem 3.1.3 is the fundamental method used to compute solutions of parameterized linear operator equations. It is not a simple expansion in terms of powers of the parameters as would be obtained from a Taylor expansion on α. Instead it is an expansion in terms of powers of the $\mathbf{L}_1(\alpha)$ operator which could be a complicated function of the parameters.

Two conditions are necessary for the application of the perturbation expansion solution. They are that (i) $\mathbf{L}_0$ must be invertible and that (ii) $\left\|\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right\| < 1 \;\; \forall\, \alpha \in \Omega$. If these conditions are satisfied, then the solution of the parameterized operator equation exists for all α. In the case of the parameterized Lyapunov equation, existence of a solution at every α guarantees system stability at every α. therefore the two necessary conditions for the perturbation expansion solution amount to sufficient conditions for simultaneous stability over the set. In the next proposition, these two conditions will be interpreted in terms of conditions on the system coefficient matrices.

Proposition 3.1.4 (Sufficient Condition for Simultaneous Stability) *Let $\tilde{A}(\alpha)$ be decomposed into $\tilde{A}(\alpha) = \tilde{A}_0 + \tilde{A}_1(\alpha)$, with $\tilde{A}_0$ asymptotically stable and the norm constraint*

$$\left\|\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_1(\alpha) \oplus \tilde{A}_1(\alpha)\right)\right\| < 1 \quad \forall\, \alpha \in \Omega \tag{3.21}$$

then $\tilde{A}(\alpha)$ is asymptotically stable $\forall\, \alpha \in \Omega$.

Proof: The parameterized Lyapunov equation, Eq. (3.7), is a parameterized linear operator with decomposition as given in Proposition 3.1.2. The two conditions for the applicability of Prop. 3.1.3 and therefore existence of a solution $\forall\, \alpha$ are that $\mathbf{L}_0$ is invertible and

$$\left\|\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right\| < 1 \;\forall\, \alpha \in \Omega \tag{3.22}$$

The first condition is satisfied by the condition that $\tilde{A}_0$ is asymptotically stable since stability of $\tilde{A}_0$ guarantees existence of a positive definite solution of the nominal Lyapunov equation by Lyapunov stability theory. Existence of a solution to the nominal Lyapunov equation implies that $\mathbf{L}_0$ is invertible.

The second condition is equivalent to the statement that the solution, $Y \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, to the equation

$$\tilde{A}_0 Y + Y\tilde{A}_0^T = \tilde{A}_1(\alpha)X + X\tilde{A}_1^T(\alpha) \tag{3.23}$$

has norm less than 1 ($\|Y\| < 1 \; \forall \alpha \in \Omega$) when $\|X\| < 1$. This condition can also be expressed using Kronecher notation, Ref. [113]. Equation (3.23) can be rewritten as

$$\text{vec}\{Y\} = \left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1} \left(\tilde{A}_1(\alpha) \oplus \tilde{A}_1(\alpha)\right) \text{vec}\{X\} \tag{3.24}$$

with $\|X\| < 1$, then $\|Y\| < 1 \; \forall \alpha \in \Omega$. For this to be true the induced matrix norm must be less then one.

$$\left\| \left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1} \left(\tilde{A}_1(\alpha) \oplus \tilde{A}_1(\alpha)\right) \right\| < 1 \quad \forall \alpha \in \Omega \tag{3.25}$$

Thus Eq. (3.25) is a sufficient condition for existence of a solution to Eq. (3.7) $\forall \alpha$. From Lyapunov stability theory, existence of the solution to Eq. (3.7) at each α guarantees stability at each α. □

This condition for simultaneous stability really represents a condition on the size of $\tilde{A}_i(\alpha)$ which can be tolerated for stability. The development is a natural result of the application of the operator decompositions techniques on the parameterized Lyapunov equation. The derivation doesn't rely on the use of the average but it will be shown in the next section that the two conditions for simultaneous stability guarantee existence of the average of the parameterized Lyapunov equation. The average solution will be examined in more detail in the next section.

3.1.2 Expressions for the Average

Now that an expression for the solution of a linear operator equation and conditions for its existence have been presented, the problem of determining the average solution can be addressed. In this section, two series expansions for the average are presented. The first to be presented is based on the perturbation expansion derived in Prop. 3.1.3. The series for the average will be derived by averaging the perturbation expansion term by term. The second expression for the average solution of a parameterized Lyapunov equation is called the Dyson equation which has its roots in stochastic operator theory. Both expressions for the average involve infinite series and so their use for computation of the average is limited. Instead, in a later section these two methods will both be used to derive approximations for the average by truncating their respective series. In this section the complete series are presented for later truncation. The first step is to examine the averaging process as an operator.

Definition 3.1.4 (Averaging Operator) *The averaging operation,* $\mathbf{A}: \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ *is given by the Bochner integral*

$$\mathbf{A}[\cdot] = \langle[\cdot]\rangle = \int_{\Omega} [\cdot]\, \mathrm{d}\mu(\alpha) \tag{3.26}$$

provided the integral exists. The average, $\bar{y}$, *of a general parameterized variable,* $y(\alpha) \in \tilde{\mathcal{H}}$, *is a parameter ¿independent element of* $\tilde{\mathcal{H}}$, *defined by*

$$\bar{y} = \mathbf{A}[y(\alpha)] = \int_{\Omega} y(\alpha)\mathrm{d}\mu(\alpha) \tag{3.27}$$

At this point it is useful to introduce some properties and assumptions related to the averaging operator as given in Ref. [118]. These properties and assumptions will used in the remainder of the section.

(i) $\mathbf{A}$ and $\mathbf{I} - \mathbf{A}$ are projectors

$$\mathbf{A}^2 = \mathbf{A} \tag{3.28}$$

(ii) $\mathbf{A}$ commutes with $\mathbf{L}_0^{-1}$ since $\mathbf{L}_0^{-1}$ is parameter independent.

$$\mathbf{A}\mathbf{L}_0^{-1} = \mathbf{L}_0^{-1}\mathbf{A} \tag{3.29}$$

(iii) It is assumed that $\mathbf{L}_1(\alpha)$ is centered (has zero average)

$$\mathbf{A}\mathbf{L}_1(\alpha)\mathbf{A} = 0 \tag{3.30}$$

(iv) Assume that x is parameter independent.

$$\mathbf{A}x = x \tag{3.31}$$

The centering assumption is not limiting since $\mathbf{L}_0$ can be chosen arbitrarily as long as it is invertible. In the case of the parameterized Lyapunov equation, one can choose $\tilde{A}(\alpha) = \tilde{A}_0 + \tilde{A}_1(\alpha)$, where $\tilde{A}_0 = \left\langle \tilde{A}(\alpha) \right\rangle$ and $\tilde{A}_1(\alpha) = \tilde{A}(\alpha) - \tilde{A}_0$. The assumption that x is parameter independent is equivalent to saying the $\tilde{B}\tilde{B}^T$ has no parameter dependence. To enforce this assumption we will restrict our investigation to this class of systems. Note that the structured set of systems, $\mathcal{G}_s$, has this property that the closed-loop $\tilde{B}$ matrix is parameter independent. The restriction to parameter independent closed-loop $\tilde{B}$ matrices doesn't preclude uncertainties in the input and output matrices used for control. The uncertainties in the B_2 and C_2 matrices contribute to the closed-loop $\tilde{A}$ matrix and therefore are allowed.

Using the assumptions stated above, $\mathbf{A}$ can be applied to the series in Eq. (3.20) to obtain the first general expression for the average solution to a parameterized operator equation, called the perturbation expansion for the average.

Proposition 3.1.5 (Perturbation Expansion for Average) *Consider the parameterized operator equation,* $\mathbf{L}(\alpha)y = x$, $\mathbf{L}(\alpha) = \mathbf{L}_0 + \mathbf{L}_1(\alpha)$, *with* $\mathbf{L}_0$ *invertible,* $\mathbf{L}_1(\alpha)$ *centered and uniformly continuous in* $\alpha \in \Omega$, *and* $\left\|\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right\| < 1 \ \forall \, \alpha \in \Omega$. *Then the average of* $y(\alpha)$ *exists and is given by the series*

$$\bar{y} = y_0 + \mathbf{L}_0^{-1}\mathbf{A}\mathbf{L}_1(\alpha)\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)y_0 + \cdots = \sum_{i=0}^{\infty}\mathbf{A}\left(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^{2i} y_0 \tag{3.32}$$

Proof: The crux of the proof is to show that $y(\alpha)$ is a continuous function over a compact set, Ω, and therefore integrable. To show that $y(\alpha)$ is continuous we note that each term in Eq. (3.20) is a uniformly continuous function over Ω and bounded $\forall \, \alpha \in \Omega$ by

$$\left\|\left(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^i y_0\right\| \le \mathcal{B}^i \|y_0\| \quad \forall \, \alpha \in \Omega$$
$$\mathcal{B} = \max_{\alpha}\left\{\left\|\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right\|\right\} < 1 \tag{3.33}$$

Since $\sum_{i=0}^{\infty}\mathcal{B}^i\|y_0\|$ converges, the sequence of partial sums of Eq. (3.20) converges uniformly to $y(\alpha)$ by the Weierstrass M-test. Since each term of $y(\alpha)$ is uniformly continuous and the series converges uniformly to $y(\alpha)$, $y(\alpha)$ is uniformly continuous on a compact set Ω and therefore integrable. □

The convergence properties of Eq. (3.32) are not very good because of the norm constraint on $\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)$. If $\left\|\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right\|$ is close to one (some elements of the set are close to instability), it will take many terms for the series to converge. Also note that since higher order terms involve the average of $\left(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^i$ the number of terms at each power of i can increase geometrically and so therefore can the computational complexity. For example, consider the case of the computation of the 5^{th} term in the series when there are 5 uncertainties. Each uncertainty has associated with it an uncertainty template matrix, $\tilde{A}_i$. Since matrix multiplications do not commute there are 5^5 terms which must be individually accounted for. These two problems, the slow convergence and the large number of terms, make calculation of the exact average cost using the perturbation expansion for the average computationally impractical.

These problems with the perturbation expansion for the average have been encountered in other fields that need to calculate the average solutions of parameterized or stochastic linear equations. To get around these problems, a different sort of equation for the average solution known as the *Dyson Equation*, Ref. [118] which has been widely used in the fields of wave propagation in random media, Refs. [118,119,121], and turbulence modeling, Refs. [122,123]. First the Dyson equation will be presented and then it will be discussed in the context of the problems mentioned above.

Proposition 3.1.6 (Dyson Equation for the Average) *Given the assumptions of Proposition 3.1.5, then $\bar{y} \in \mathcal{H}$ exists and is the solution of the linear equation*

$$\bar{y} = y_0 + \mathbf{L}_0^{-1}\mathbf{M}\bar{y} \tag{3.34}$$

where $\mathbf{M}$ *is a parameter independent operator defined*

$$\mathbf{M} = -\sum_{i=1}^{\infty} \mathbf{A}\mathbf{L}_1(\alpha)\left[-\mathbf{L}_0^{-1}(\mathbf{I}-\mathbf{A})\mathbf{L}_1(\alpha)\right]^i \mathbf{A} \tag{3.35}$$

Proof: By applying $\mathbf{A}$ and $(\mathbf{I}-\mathbf{A})$ to Eq. (3.18) respectively, two equations are obtained

$$\bar{y} = y_0 - \mathbf{L}_0^{-1}\mathbf{A}\mathbf{L}_1(\alpha)\tilde{y}(\alpha) \tag{3.36}$$

$$\tilde{y}(\alpha) = -\mathbf{L}_0^{-1}(\mathbf{I}-\mathbf{A})\mathbf{L}_1(\alpha)(\bar{y}+\tilde{y}(\alpha)) \tag{3.37}$$

where $\tilde{y}(\alpha) = (\mathbf{I}-\mathbf{A})\,y(\alpha)$ is called the reverberant solution and $\bar{y}$ is called the mean solution. Now solving for $\tilde{y}(\alpha)$ in terms of $\bar{y}$ one obtains

$$\tilde{y}(\alpha) = \sum_{i=1}^{\infty}\left[-\mathbf{L}_0^{-1}(\mathbf{I}-\mathbf{A})\mathbf{L}_1(\alpha)\right]^i \bar{y} \tag{3.38}$$

The solution of Eq. (3.38) exists because $y(\alpha)$ is a bounded function on a compact set Ω and $\bar{y}$ exists by Proposition 3.1.5. therefore $\tilde{y}(\alpha) = y(\alpha) - \bar{y}$ exists. Equations (3.34) and (3.35) follow by substitution of Eq. (3.38) into Eq. (3.36). □

Remark 3.1.1 *The Dyson Equation is linear and its solution is given by the parameter independent equation*

$$\bar{y} = \left(\mathbf{I} - \mathbf{L}_0^{-1}\mathbf{M}\right)^{-1} y_0 \tag{3.39}$$

which is equivalent to the infinite series

$$\bar{y} = \sum_{i=0}^{\infty}\left(\mathbf{L}_0^{-1}\mathbf{M}\right)^i y_0 \tag{3.40}$$

Some comparison of the two equations for the average is certainly in order. The perturbation expansion for the average is basically an expansion about the nominal solution, y_0, in powers of the parameter dependent part of the solution $\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)$. The solution takes long to converge when the mean solution is far removed from the nominal. The fundamental difference between the perturbation expansion and the Dyson equation is that the Dyson equation uses an expansion about the mean solution to obtain the mean solution.

To see how this property manifests itself in the mathematics consider Eqs. (3.36) and (3.38). Eq. (3.36) expresses the average solution in terms of an average of the reverberant solution, $\tilde{y}(\alpha)$. This equation can be thought of as a coordinate transformation from averaging over $y(\alpha)$ to averaging over $\tilde{y}(\alpha)$. Equation (3.38) amounts to an expansion for $\tilde{y}(\alpha)$ about the mean in powers of a transformed version of $\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)$ which is written $-\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)+\mathbf{L}_0^{-1}\mathbf{A}\mathbf{L}_1(\alpha)$. So in essence, instead of representing the parameter dependence of $y(\alpha)$ relative to the nominal solution and then averaging as is done in the perturbation expansion, the Dyson equation represents the parameter dependence of $\tilde{y}(\alpha)$ about the mean and then transforms back and averages.

The Dyson equation really incorporates two infinite series. These are the infinite series for the deterministic $\mathbf{M}$ operator given in Eq. 3.35 and the infinite series for the mean solution, Eq. (3.40). The series for $\mathbf{M}$ is the result of expanding about the mean solution, and it contains all of the parameter dependent terms and averages. The series for the mean solution essentially represents the process of untangling the mean solution from the $\mathbf{M}$ operator since this operator is derived using an expansion about the mean which is not yet known. By expanding about the mean solution rather than the nominal the series solution for the Dyson equation tends to converge much more quickly than the perturbation expansion for the average. This will manifest itself in the accuracy of the approximations derived by truncating the two expressions for the average. As will be discussed in the following section, truncation of the Dyson equation entails truncating the number of terms in the infinite series for $\mathbf{M}$ and not the infinite series for the solution.

Both expressions for the exact average involve infinite series of terms which involve $\mathbf{L}_0^{-1}$. Each appearance of $\mathbf{L}_0^{-1}$ signifies an additional Lyapunov equation. Calculating the exact average solution for the parameterized Lyapunov equation will therefore entail an infinite number of Lyapunov equations. In addition, if there are large number of uncertainties (the dimension of Ω is large), then each higher term of the series involves the average over a geometrically increasing number of terms. It is therefore important to derive approximate expressions for the average since the exact average is rarely calculable. These approximations will be discussed in the next section.

3.2 Approximations to the Average Cost

In this section, approximate solutions for the exact average cost will be presented. Explicit equations for the calculation of approximate average costs will be derived. Two types of approximations will be discussed. The first is derived from a truncation of the perturbation expansion for the average solution of the parameterized Lyapunov equation, Eq. (3.7) presented in Prop. 3.1.5. The second is based on a truncation of the Dyson equation, Prop. 3.1.6. As explained in the previous section, the Dyson equation has been widely used in the fields of wave propagation in random media Refs. [118,119,121], and turbulence modeling, Refs. [122,123] as an expression for the average solution of parameterized linear operator equations.

The two approximations for the average solution of the parameterized operator equations will be applied to the problem of approximating the average solution of the parameterized Lyapunov equations presented in Eq. (3.7). These computable approximations will be developed for use as performance metrics for robust control design for systems with a large number of uncertain parameters. The approximations will be used in place of the uncomputable exact average cost in hopes of recovering some of the robustness properties of average cost design.

In the derivations that follow, the structured set of systems, $\mathcal{G}_s$, will be used to model the parameter dependence of the coefficient matrices for the derivation of computable approximations for the average. The set of systems is restricted to the structured set for two reasons. First, the operator equations for the average presented in the previous section only address parameter dependence of the linear operator part of the equation. This translates to a restriction that only the closed-loop $\tilde{A}$ matrix can be parameter dependent. This is satisfied by the structured set of system. Secondly, in order to derive deterministic equations which explicitly deal with the averages of the parameters, the form of the functional dependence of the coefficients of the system matrices must be specified. For simplicity the parameter dependence of the coefficients of the system matrices in the structured set are assumed to be linear. Restricting our attention to the structured set of systems, the parameterized Lyapunov equation to be approximated becomes

$$0 = \left(\tilde{A}_0 + \sum_{i=1}^{r}\alpha_i\tilde{A}_i\right)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\left(\tilde{A}_0 + \sum_{i=1}^{r}\alpha_i\tilde{A}_i\right)^T + \tilde{B}\tilde{B}^T \tag{3.41}$$

The first approximation to be derived is based on the perturbation expansion for the average.

3.2.1 Perturbation Expansion Approximation

The first method of deriving a computable approximation for the average solution of the parameterized Lyapunov equation is to truncate the infinite series given by the perturbation expansion for the average. The truncation retains only the first two terms of the series which are basically the nominal solution and a solution which depends on the square of the uncertain terms. This truncation can be thought of as an approximation to the exact average.

Definition 3.2.1 (Perturbation Series Truncation) *The perturbation series truncation is derived by retaining only the first two terms of the perturbation expansion for the average given in Prop. 3.1.5.*

$$\bar{y}^P = y_0 + \mathbf{A}\left(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^2\mathbf{A}y_0 \tag{3.42}$$

where $y_0 = \mathbf{L}_0^{-1}x$ is the nominal solution.

The perturbation series truncation can be used to derive an approximate average solution to the parameterized Lyapunov equation, Eq. (3.41). This is done by substituting the operator definitions given in Prop. 3.1.2 into Eq. (3.42). This manipulation gives the perturbation expansion approximation.

Proposition 3.2.1 (Perturbation Expansion Approximation) *Given a specified compensator, G_c, if the parameterized closed-loop systems, $G_{zw}(\alpha)$, are stable for almost all $\alpha \in \Omega$ then:*

$$J(G_c) \cong \operatorname{tr}\left\{\tilde{Q}^P\tilde{C}^T\tilde{C}\right\} \tag{3.43}$$

where $\tilde{Q}^P$ is the unique positive definite solution to the following system of Lyapunov equations:

$$0 = \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{3.44}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right) \quad i = 1,\ldots,r \tag{3.45}$$

and $\tilde{Q}^0$ is the solution of the nominal Lyapunov equation

$$0 = \tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T \tag{3.46}$$

and σ_i is defined from the relation:

$$\left\langle\alpha_i^2\right\rangle = \sigma_i^2 \tag{3.47}$$

Proof: The result is a direct consequence of the application of the decomposition of the parameterized Lyapunov equation presented in Prop. 3.1.2 (actually applied to (3.41)) and the definition of the perturbation expansion truncation defined in Def. 3.2.1. □

Remark 3.2.1 (Solution) *The system of Lyapunov equations presented in Eqs. (3.44) and (3.45) are coupled hierarchically. The nominal solution, $\tilde{Q}^0$, can first be solved using Eq. (3.46) and the solution substituted into each of the i equations represented by Eq. (3.45). The solutions for these equations, $\tilde{Q}^i$, can then be used to solve for $\tilde{Q}^P$ using Eq. (3.44).*

Remark 3.2.2 *The system of equations presented in Eqs. (3.44) and (3.45) are similar to those inherent in the sensitivity system design methodology presented in Appendix A of Ref. [109]. This can be seen clearly by putting the equations for the component cost analysis in the notation used here:*

$$0 = \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^{i^T}\tilde{A}_i^T\right) \tag{3.48}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{Q}^0\tilde{A}_i^T\right) \quad i = 1,\ldots,r \tag{3.49}$$

There is only a single term omitted from (3.49) which is in (3.45). The omission of this term essentially cuts in half the contribution of the parameter dependent part of the solution. The nominal solution will be the same but the perturbation expansion approximation will be twice as sensitive to parameter variations as the equivalent sensitivity system model of component cost analysis.

Although the hierarchical nature of the system of Lyapunov equations represented by Eqs. (3.44) and (3.45) makes them simple to solve (just $1+r$ Lyapunov equations). The structure of the equations can be revealed by representing them by a single linear equation for the approximate average solution using Kronecher notation.

$$\text{vec}\left\{\tilde{Q}\right\} = \left(I + \sum_{i=1}^{r}\left(\sigma_i\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)^2\right)\text{vec}\left\{\tilde{Q}^0\right\} \tag{3.50}$$

Equation (3.50) reveals that no finite amount of uncertainty will lead to infinite solutions of the perturbation expansion approximation. It never "blows up." therefore the cost remains finite even when the uncertain set is very large and contains unstable members. From the control design viewpoint this property is a severe drawback. It indicates that not only will minimizing the perturbation approximate average not guarantee stability for the design set but in addition it will not guarantee stability for any subset of the design set. The designer cannot just increase the amount of uncertainty used in the design to obtain a stability guarantee on a less uncertain system. The next approximation for the average solution based on truncation of the Dyson equation will partially address these weaknesses.

3.2.2 Bourret Approximation

In this section, an approximation for the averaged solution of the parameterized Lyapunov equation (3.41) that is based on the Dyson equation for the average will be presented. This approximation is developed for comparison to the perturbation expansion approximation. As explained in Section 3.1.2, the Dyson equation has some interesting characteristics that should make the approximation based on it better suited for control design. In this section, these characteristics will be examined in the context of a computable approximation to the average known as the Bourret approximation. This approximation will be derived by truncating the infinite series associated with the M operator in the Dyson equation.

Definition 3.2.2 (Bourret Equation) *The equation formed by truncating the* M *operator in the Dyson equation, Eqs. (3.34) and (3.35), to include only its first term is called the Bourret equation, given by*

$$\bar{y}^B = y_0 + \mathbf{L}_0^{-1}\mathbf{A}\mathbf{L}_1(\alpha)\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\bar{y}^B \tag{3.51}$$

Remark 3.2.3 *The Bourret equation has the solution, $\bar{y}^B$, appearing on both the right and left sides. To remove the right side dependency, it can be rewritten*

$$\bar{y}^B = \left(\mathbf{I} - \mathbf{A}\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^{-1} y_0 \tag{3.52}$$

which is a shorthand for the series expansion

$$\bar{y}^B = \sum_{i=0}^{\infty}\left(\mathbf{A}\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^i y_0 \tag{3.53}$$

The Bourret equation thus represents an infinite series expansion for the approximate average solution even though it is a truncation of the Dyson equation. This series expansion solution to the Bourret equation contains terms similar to those in the infinite series for the perturbation expansion for the average. Comparing equivalent terms in these two infinite series

$$\begin{aligned}
\left(\mathbf{A}(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha))^2\mathbf{A}\right)^i &\rightarrow \text{Bourret Approximation} \\
\mathbf{A}\left(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^{2i}\mathbf{A} &\rightarrow \text{Perturbation Expansion for the Average}
\end{aligned}$$

One can see that the Bourret terms depend only on the averages of $(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha))^2$. If $\mathbf{L}_1(\alpha)$ is a linear function of the parameters this is equivalent to saying that the Bourret equation only depends on the second moments of the parameters. The perturbation expansion for the exact average depends on higher moments since the averaging operator is outside the product of the terms. In addition, retaining only the first two terms of the Bourret equation is equivalent to retaining the first two terms of the perturbation expansion for the exact average. These two terms correspond to those used in the perturbation expansion approximation. Since the Bourret equation includes additional terms, one would expect it to be a better approximation of the exact average than the perturbation expansion approximation.

At this point, we can apply the Bourret equation to the problem of approximating the average solution of the parameterized Lyapunov equation, (3.41).

Proposition 3.2.2 (Bourret Approximation) *Given a specified compensator, G_c, if the parameterized closed-loop systems, $G_{zw}(\alpha)$ are stable for almost all $\alpha \in \Omega$ then*

$$J(G_c) \cong \operatorname{tr}\left\{\tilde{Q}^B \tilde{C}^T \tilde{C}\right\} \tag{3.54}$$

where $\tilde{Q}^B$ is the unique positive definite solutions to the following system of Lyapunov equations:

$$0 = \tilde{A}_0 \tilde{Q}^B + \tilde{Q}^B \tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r} \sigma_i \left(\tilde{A}_i \tilde{Q}^i + \tilde{Q}^i \tilde{A}_i^T\right) \tag{3.55}$$

$$0 = \tilde{A}_0 \tilde{Q}^i + \tilde{Q}^i \tilde{A}_0^T + \sigma_i \left(\tilde{A}_i \tilde{Q}^B + \tilde{Q}^B \tilde{A}_i^T\right) \quad i = 1, \ldots, r \tag{3.56}$$

where σ_i is defined from Eq. (3.47).

Proof: The result is a direct consequence of the application of the decomposition of the parameterized Lyapunov equation presented in Prop. 3.1.2 and the definition of Bourret approximation defined in Def. 3.2.2. An intermediate variable, $\tilde{Q}^i$, has been introduced as the solution to $\mathrm{L}_0^{-1}\mathrm{L}_1(\alpha)\bar{y}^B$ in (3.51). □

The system of Lyapunov equations presented in Eqs. (3.55-3.56) is very similar to the system generated in Prop. 3.2.1. There is additional coupling occurring in Eq. (3.56). Instead of depending only on the nominal solution, $\tilde{Q}^0$, these equations depend on the total Bourret approximate average solution, $\tilde{Q}^B$. This coupling complicates the solution procedure but leads to a more accurate approximation. The solution is discussed in the next remark.

Remark 3.2.4 (Solution) *The system of Lyapunov equations represented by Eqs. (3.55-3.56) can be solved in two different ways. The first is to represent them by a single linear equation for the approximate average solution using Kronecker notation, as below:*

$$\mathrm{vec}\left\{\tilde{Q}^B\right\} = \left(I - \sum_{i=1}^{r}\left(\sigma_i\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)^2\right)^{-1} \mathrm{vec}\left\{\tilde{Q}^0\right\} \tag{3.57}$$

The second method to solve Eqs. (3.55-3.56) is by iterative solution for the Bourret approximate average, $\tilde{Q}^B$, using the nominal solution, $\tilde{Q}^0$, as the initial guess. In this method, a solution is assumed for $\tilde{Q}^B$ in (3.56) which is then solved for $\tilde{Q}^i$. $\tilde{Q}^i$ is then used to solve for a new $\tilde{Q}^B$ using (3.55) and the process is repeated until the solution for $\tilde{Q}^B$ converges.

For large order systems, the iterative solution procedure is usually superior to the Kronecher notation method because of the size of the matrices in (3.57). If the system is n^{th} order, the matrices have dimension $n^2 \times n^2$. The Kronecher notation form of the Bourret approximation does provide some insight into the conditions under which the Bourret approximation has positive definite solution. This condition is given below.

Proposition 3.2.3 *A sufficient condition for the existence of the inverse in Eq. (3.57) and resulting uniqueness and positive definiteness of the solution is that*

$$\left\|\sum_{i=1}^{r}\left(\sigma_i\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)^2\right\| < 1 \tag{3.58}$$

Proof: This is a direct consequence of the application of the von Neumann Lemma, Prop. (22.10) in Ref. [112] to Eq. (3.57). □

Proposition 3.2.3 represents the condition under which the Bourret equation gives meaningful answers and also gives the amount of uncertainty that the system can tolerate before the Bourret approximation "blows up." This can be thought of as defining the stability region from the Bourret approximations point of view. Notice that Eq. (3.58) doesn't imply that Eq. (3.21), the sufficient condition for simultaneous stability given by:

$$\left\| \sum_{i=1}^{r} \left(\alpha_i \left(\tilde{A}_0 \oplus \tilde{A}_0 \right)^{-1} \left(\tilde{A}_i \oplus \tilde{A}_i \right) \right) \right\| < 1 \quad \forall \alpha \in \Omega \tag{3.59}$$

is satisfied. A finite solution to the Bourret approximation will therefore not necessarily guarantee stability over the model set. It is useful to investigate a set of systems that is guaranteed stable if the Bourret Approximation is finite. This set has the following characterization.

Corollary 3.2.1 *If the sufficient condition for existence of a positive definite solution to the Bourret equation is satisfied, Eq. (3.58) then stability is guaranteed on the set,* $\hat{\Omega} \subset \Omega$*, defined*

$$\hat{\Omega} = \left\{ \alpha \ : \sum_{i=1}^{r} |\alpha_i| \|\mathcal{A}_i\| \leq \left\| \sum_{i=1}^{r} \sigma_i^2 \mathcal{A}_i^2 \right\| \right\} \tag{3.60}$$

where

$$\mathcal{A}_i = \left(\tilde{A}_0 \oplus \tilde{A}_0 \right)^{-1} \left(\tilde{A}_i \oplus \tilde{A}_i \right) \quad i = 1, \cdots, r \tag{3.61}$$

Proof: Using the sufficient condition for positive definiteness of the Bourret equation and the set definition, we have

$$\left\| \sum_{i=1}^{r} \alpha_i \mathcal{A}_i \right\| \leq \sum_{i=1}^{r} |\alpha_i| \|\mathcal{A}_i\| \leq \left\| \sum_{i=1}^{r} \sigma_i^2 \mathcal{A}_i^2 \right\| < 1 \tag{3.62}$$

By Proposition 3.1.4 a positive definite solution of the parameterized Lyapunov equation, Eq. (3.41), exists and hence stability is guarantees for any α which satisfies this condition. □

The general shape of this set of systems in the α plane can be drawn for two uncertainties and is shown in Figure 3.2. The fact that the existence of a solution to the Bourret equation guarantees stability over a definable set is very useful in control design. The values for the σ_i can be set large enough in the controller design process to guarantee stability over the actual set of interest. The designer thus has the flexibility of achieving guaranteed robustness if it is desired by increasing the σ_i. In addition a first cut at the controller can be obtained by by using the σ_i which

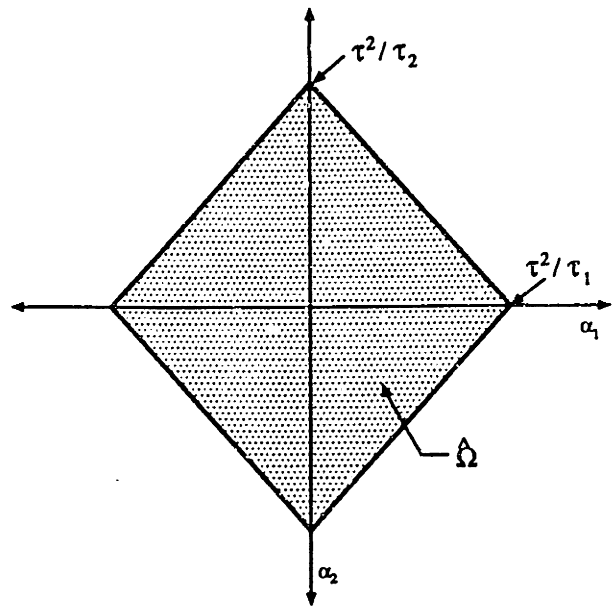

Figure 3.2: The shape of $\hat{\Omega}$ for two uncertain parameters where $\tau_i = \|\mathcal{A}_i\|$ and $\tau^2 = \|\sigma_1^2\mathcal{A}_1^2 + \sigma_2^2\mathcal{A}_2^2\|$

result from the actual set of systems. Because in this case, the Bourret equation is only an approximation to the average over the set, these first cut controllers will not necessarily guarantee stability over the design set. The σ_i can then be increased in the design process if the stability margins are insufficient.

The structure of the Bourret equation suggests possible solution through the use of higher-order Lyapunov equations. In fact, in certain cases the Bourret equation can be solved using a $2\tilde{n}$ order Lyapunov equation and it can always be represented by a $\tilde{n}(1+r)$ order modified Lyapunov equation. These situation will be discussed in the following remarks.

Remark 3.2.5 *If the orthogonality condition,*

$$\mathcal{A}_i\mathcal{A}_j = 0 \ \ \forall i \neq j \tag{3.63}$$

with $\mathcal{A}_i$ defined in (3.61) is satisfied, then the solution of the Bourret equation is given by the solution to

$$\hat{A}\hat{Q} + \hat{Q}\hat{A}^T + \hat{B} = 0 \tag{3.64}$$

where

$$\hat{A} = \begin{bmatrix} \tilde{A}_0 & -\sum_{i=1}^{r}\sigma_i\tilde{A}_i \\ -\sum_{i=1}^{r}\sigma_i\tilde{A}_i & \tilde{A}_0 \end{bmatrix} \quad \hat{Q} = \begin{bmatrix} \tilde{Q}^B & \tilde{Q}^1 \\ \tilde{Q}^1 & \tilde{Q}^B \end{bmatrix} \quad \hat{B} = \begin{bmatrix} \tilde{B}\tilde{B}^T & 0 \\ 0 & \tilde{B}\tilde{B}^T \end{bmatrix} \tag{3.65}$$

The orthogonality condition essentially imposes the constraint that the uncertainties enter into dynamically decoupled subsystems. This constraint is very restrictive. A more general form can be developed using a modified Lyapunov equation.

Remark 3.2.6 *The solution of the Bourret equation can always be given by the solution to the modified $\tilde{n}(1+r)$ order Lyapunov equation*

$$\hat{A}_0\hat{Q} + \hat{Q}\hat{A}_0^T + \hat{A}_i\hat{Q}\hat{I} + \hat{I}\hat{Q}\hat{A}_i^T + \hat{B} = 0 \tag{3.66}$$

where

$$\hat{A}_0 = \begin{bmatrix} \tilde{A}_0 & & & \\ & \tilde{A}_0 & & \\ & & \ddots & \\ & & & \tilde{A}_0 \end{bmatrix} \quad \hat{A}_i = \begin{bmatrix} 0 & \tilde{A}_1 & \cdots & \tilde{A}_r \\ \tilde{A}_1 & 0 & & \\ \vdots & & \ddots & \\ \tilde{A}_r & & & 0 \end{bmatrix} \tag{3.67}$$

$$\hat{Q} = \begin{bmatrix} \tilde{Q}^B & & & \\ & \tilde{Q}_1 & & \\ & & \ddots & \\ & & & \tilde{Q}_r \end{bmatrix} \quad \hat{B} = \begin{bmatrix} \tilde{B}\tilde{B}^T & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \quad \hat{I} = \begin{bmatrix} 0 & I & \cdots & I \\ I & 0 & & \\ \vdots & & \ddots & \\ I & & & 0 \end{bmatrix} \tag{3.68}$$

The above result illustrates the structure of the Bourret equation and may present opportunities for more efficient calculation of the solution.

This concludes the exposition on approximations to the exact average cost. Two bounds on the exact average will be investigated in the next section.

3.3 Bounds on the Average Cost

In this section functions which bound the solution to the parameterized Lyapunov equation, (3.41), will be investigated. The motivation for looking at functions that bound the exact average is the desire for guaranteed stability. If a function which bounds the average has a finite solution, then the average is finite, and thus the set of systems is stable. As discussed in the Introduction, most functions which bound the solution to the parameterized Lyapunov equation do so by bounding all of the solutions to the equation. In Section 3.1, it was shown that it is sufficient to only bound the average solution to guarantee stability. This insight has motivated the search for functions which bound the average and not all of the solutions. It is hypothesized that a function that bounds only the average solution can be used in control design to generate stabilizing controllers which require less control effort and lower control gains than those designed based on functions which bound all of the solutions.

Two bounds to the exact average cost are developed in this section which will be used for control system design in a later section. The first, called the "worst-case" bound, bounds all of the solutions of the parameterized Lyapunov equation. The second, called the average bound, bounds the average solution but not necessarily all of the possible solutions. The preliminary mathematics will first be presented and applied to computing the bounds for average costs associated with the structured set of systems.

To begin it is assumed that a partial ordering of the elements of $\mathcal{H}$ can be defined. That is, the expression $x_1 \leq x_2$, $x \in \mathcal{H}$ has meaning. Such an ordering can for instance be the ordering of positive definite matrices in $\mathbb{R}^{\tilde{n}\times\tilde{n}}$. Several properties of functions can be defined based on this ordering. The first is a new type of function called a bounding function.

Definition 3.3.1 (Bounding Functions) *A function* $\mathbf{N} : \mathcal{H} \rightarrow \mathcal{H}$ *is said to be an parameter independent bound on a parameterized linear operator,* $\mathbf{L}(\alpha)$, *if*

$$\mathbf{L}(\alpha)y < \mathbf{N}y \quad \forall\, \alpha \in \Omega \;\; \forall y \in \mathcal{H} \tag{3.69}$$

The bounding function will prove very useful for deriving bounds on the average.

It is also necessary in the following derivations to assume three new properties for the nominal operator, **L**, and the averaging operator, **A**.

(i) $\mathbf{L}_0^{-1}$ must be *negative semidefinite.*

$$\mathbf{L}_0^{-1} y \leq 0 \quad \forall y \geq 0 \tag{3.70}$$

(ii) $\mathbf{L}_0^{-1}$ must *preserve orderings.*

$$x \leq y \Longrightarrow \mathbf{L}_0^{-1} x \leq \mathbf{L}_0^{-1} y \tag{3.71}$$

(iii) **A** must must be *regular* with respect to the ordering.

$$y(\alpha) \leq x \quad \forall\, \alpha \,\in\, \Omega \Longrightarrow \mathbf{A} y(\alpha) \leq x \tag{3.72}$$

These three assumptions are not restrictive for our problem since they are satisfied when **L** is a Lyapunov equation with $\tilde{A}_0$ asymptotically stable and $\mathcal{H}$ is the space of $\mathbb{R}^{\tilde{n} \times \tilde{n}}$ matrices with parameterized elements.

Having made these assumptions, we can address the problems of finding bounds for the average solution of Eq. (3.41). The worst-case bound will be investigated first for later comparison to the average bound.

3.3.1 Worst Case Bound

The first bound discussed is simply an bound on all of the solutions of the parameterized operator equation, called the "worst-case" bound. It has typically been used to derive bounds on the solutions of parameterized Lyapunov equations discussed in depth in Ref. [73]. The bound presented in this section has been widely investigated and applied to control design problems. It is developed here for later comparison with the average bound. The development will be used to present some of the concepts associated with bounding functions when viewed from the perspective of parameterized operators. The first step is to present the bounding operator equation from which the worst-case bound will be derived.

Proposition 3.3.1 (Worst-Case Equation) *Given a parameter independent bound* N *of* $\mathbf{L}_1(\alpha)$ *and a nominal function,* $\mathbf{L}_0^{-1}$*, that is negative definite and preserves orderings, then the solution to*

$$\bar{y}^W = y_0 - \mathbf{L}_0^{-1}\mathbf{N}\bar{y}^W \tag{3.73}$$

bounds all solutions to the parameterized linear operator equation, Eq. (3.8), decomposed as in Eq. (3.9),

$$y(\alpha) \le \bar{y}^W \quad \forall\, \alpha \in \Omega \tag{3.74}$$

and as a consequence bounds the exact average solution, $\bar{y}$*.*

$$\bar{y} \le \bar{y}^W \tag{3.75}$$

Proof: The proof can be found in Ref. [74]. The crux of it is that the series for $\bar{y}^W$ implicit in Eq. (3.73)

$$\bar{y}^W = y_0 - \mathbf{L}_0^{-1}\mathbf{N}y_0 + \mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-1}\mathbf{N}y_0 - \cdots = \sum_{i=0}^{\infty}\left(-\mathbf{L}_0^{-1}\mathbf{N}\right)^i y_0 \tag{3.76}$$

term by term bounds the parameterized solution, $y(\alpha)$, of (3.20). Comparing the i^{th} terms in the series we have

$$\left(-\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^i y_0 \le \left(-\mathbf{L}_0^{-1}\mathbf{N}\right)^i y_0 \tag{3.77}$$

since $-\mathbf{L}_0^{-1}\mathbf{N}$ is a bound to $-\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)$ because $-\mathbf{L}_0^{-1}$ is positive definite and preserves orderings. □

Prop. 3.3.1 will be applied to the problem of bounding the exact average cost by using the decomposition of the parameterized Lyapunov equation presented in Prop. 3.1.2. The first step in accomplishing this is to determine a bounding function, N, for $\mathbf{L}_1(\alpha)$ as described in Definition 3.3.1. For this bounding function, we will use the linear bound presented as an bounding function for control analysis and design in Refs. [69,71,73,74].

Proposition 3.3.2 (Linear Bound) *Let δ be a arbitrary scalar, and choose $\delta_i \in$ $\mathbb{R} > 0, i = 1 \ldots r$ such that*

$$\sum_{i=1}^{r} \frac{\alpha_i^2}{\delta_i^2} \leq 1 \quad \forall \{\alpha_1, \ldots, \alpha_r\} \in \Omega \tag{3.78}$$

and define the functions

$$\mathbf{L}_1(\alpha)\left[\tilde{Q}\right] \quad : \quad \tilde{Q} \rightarrow \left(\sum_{i=1}^{r} \alpha_i \tilde{A}_i\right) \tilde{Q} + \tilde{Q} \left(\sum_{i=1}^{r} \alpha_i \tilde{A}_i\right)^T \tag{3.79}$$

$$\mathbf{N}\left[\tilde{Q}\right] \quad : \quad \tilde{Q} \rightarrow \delta^2 \tilde{Q} + \sum_{i=1}^{r} \frac{\delta_i^2}{\delta^2} \tilde{A}_i \tilde{Q} \tilde{A}_i^T \tag{3.80}$$

Then N is a linear operator which satisfies

$$\mathbf{L}_1(\alpha)\left[\tilde{Q}\right] \leq \mathbf{N}\left[\tilde{Q}\right] \quad \forall \tilde{Q} \geq 0 \in \mathbb{R}^{\tilde{n} \times \tilde{n}},\ \alpha \in \Omega \tag{3.81}$$

Proof: The proof is identical to that done in Remark 5.4 Ref. [74] and [69]. Note that

$$0 \leq \sum_{i=1}^{r} \left[\left(\frac{\delta \alpha_i}{\delta_i}\right) I_{\tilde{n}} - \left(\frac{\delta_i}{\delta}\right) \tilde{A}_i\right] \tilde{Q} \left[\left(\frac{\delta \alpha_i}{\delta_i}\right) I_{\tilde{n}} - \left(\frac{\delta_i}{\delta}\right) \tilde{A}_i\right] \tag{3.82}$$

$$\leq \delta^2 \sum_{i=1}^{r} \left(\frac{\alpha_i^2}{\delta_i^2}\right) \tilde{Q} + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) \tilde{A}_i \tilde{Q} \tilde{A}_i^T - \sum_{i=1}^{r} \alpha_i \left(\tilde{A}_i \tilde{Q} + \tilde{Q} \tilde{A}_i^T\right) \tag{3.83}$$

$$\leq \delta^2 \tilde{Q} + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) \tilde{A}_i \tilde{Q} \tilde{A}_i^T - \sum_{i=1}^{r} \alpha_i \left(\tilde{A}_i \tilde{Q} + \tilde{Q} \tilde{A}_i^T\right) \tag{3.84}$$

□

The conditions on the δ_i decribe an ellipse in the α-plane. For the linear bound to guarantee that $\mathbf{L}_1(\alpha) \leq \mathbf{N}$ for all the values of $\alpha \in \Omega$, this elipse must circumscribe Ω. The new set of α given by the elipse will be defined as $\acute{\Omega}$. When dealing with the linear bounding function, this set will be used to describe the possible parameter values instead of Ω_s. Another bounding functions which could have been used is the Petersen-Hollot or quadratic bound found in [63, 69, 71, 72]. The linear bound was chosen primarily for its linearity and the extensive analysis associated with its use in robust control design.

Using the above linear bound and the worst-case operator equation presenied in Prop. 3.3.1, the equations which constitute an bound to the exact average cost for the structured set of systems can be written.

Theorem 3.3.1 (Worst Case Bound) *Given a specified compensator, G_c and set of systems, $\mathcal{G}_s$, the exact average cost, $J(G_c)$ is bounded by the worst case cost*

$$J(G_c) \leq J^W(G_c) = \operatorname{tr}\left\{\tilde{Q}^W \tilde{C}^T \tilde{C}\right\} \tag{3.85}$$

where $\tilde{Q}^W$ is the unique positive definite solution to the following Lyapunov equation.

$$0 = \tilde{A}_0 \tilde{Q}^W + \tilde{Q}^W \tilde{A}_0^T + \delta^2 \tilde{Q}^W + \tilde{B}\tilde{B}^T \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) \tilde{A}_i \tilde{Q}^W \tilde{A}_i^T \tag{3.86}$$

where δ_i is defined from Equation (3.78) and $\delta \in \mathbb{R}$.

Proof: The result can be obtained by substituting into Prop. 3.3.1 the definitions for the $\mathbf{L}_0$ and $\mathbf{L}_1(\alpha)$ operators given in Prop. 3.1.2, and substituting for $\mathbf{N}$ the definition for the linear bound given in Prop. 3.3.2. □

The addition of the bounding term in effect produces a modified Lyapunov equation, (3.86) whose solution bounds the average cost. The solution of Eq. (3.86) also bounds all of the solutions of the parameterized Lyapunov equation, (3.41), and is therefore called the worst-case bound. The modification to the Lyapunov equation makes it more difficult to solve. Two possible solution methods are presented next.

Remark 3.3.1 (Solution) *The Lyapunov equation represented by Eq. (3.86) can be solved by representing it as a single linear equation using Kronecher notation [113].*

$$\operatorname{vec}\left\{\tilde{Q}^W\right\} = \left[I + \left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1} \left(\delta^2 I + \sum_{i=1}^{r} \frac{\delta_i^2}{\delta^2} \left(\tilde{A}_i \otimes \tilde{A}_i\right)\right)\right]^{-1} \operatorname{vec}\left\{\tilde{Q}^0\right\} \tag{3.87}$$

where "vec" is the column stacking operation defined in [113]. It can also be solved by iterative solution for $\tilde{Q}^W$ using $\tilde{Q}^0$ as an initial guess.

For large order systems the size of the matrices involved in the Kronecher notation solution become prohibitively large and so the iterative solution technique becomes more attractive. The Kronecher notation solution does give some insight into the conditions under which the modified Lyapunov equation has unique positive definite solutions.

Proposition 3.3.3 *A sufficient condition for the existence of the inverse in Eq. (3.87) and resulting uniqueness and positive definiteness of the solution is that*

$$\left\| \left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1} \left(\delta^2 I + \sum_{i=1}^{r} \frac{\delta_i^2}{\delta^2} \left(\tilde{A}_i \otimes \tilde{A}_i\right)\right) \right\| < 1 \qquad (3.88)$$

Proof: This is a direct consequence of Proposition (22.10) in Ref. [112] and is given in Ref. [74]. □

Existence of a solution to the worst case bound thus guarantees simultaneous stability over the model set. This also indicates that the worst case bound is more conservative than the condition given in Prop. 3.1.4 for simultaneous stability. In the next section, a bound will be developed which bounds the average but not all of the solutions of (3.41).

3.3.2 Average Bound

In this section a function which bounds the average cost will be presented. This function, called the average bound, yields solutions which bound the average cost but not necessarily all of the possible costs on the set. In this manner it is distinct from the worst-case bound presented in the previous section. This new bound is derived in an attempt to take advantage of the property that finite average cost guarantees stability on the model set. By bounding only the average and not all possible solutions of the parameterized Lyapunov equation, it is hypothesized that compensators can be derived which guarantee stability without the high gains and control effort characteristic of worst-case bound synthesis. This hypothesis will be tested in Chapter 5. The analytical underpinnings for this average bound will be presented in this section for later use in control synthesis. To start, the operator equation associated with the bound will be presented.

Theorem 3.3.2 (Average Bound Equation) *Given a parameter independent bound* $\mathbf{N}$ *of* $\mathbf{L}_1(\alpha)$ *and a nominal function,* $\mathbf{L}_0^{-1}$*, which is negative definite and preserves orderings, the solution to*

$$\bar{y}^A = y_0 + \mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-1}\mathbf{N}\bar{y}^A \tag{3.89}$$

bounds the exact average solution, $\bar{y}$*, and in addition is itself bounded by the worst case bound given by Eq. (3.73):*

$$\bar{y} \leq \bar{y}^A \leq \bar{y}^W \tag{3.90}$$

Proof: First it will be shown that the solution to Eq. (3.89) bounds the exact average. It is shown that the series for the exact average, Eq. (3.32) is bounded term by term by the series for the average bound implicit in Eq. (3.89).

$$\bar{y}^A = y_0 + \mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-1}\mathbf{N}y_0 + \cdots = \sum_{i=0}^{\infty}\left(\mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-1}\mathbf{N}\right)^i y_0 \tag{3.91}$$

Considering the equivalent terms of the two series:

$$\mathbf{A}\left(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)\right)^{2i}\mathbf{A}y_0 < \left(\mathbf{L}_0^{-1}\mathbf{N}\right)^{2i} y_0 \tag{3.92}$$

This is true because $\mathbf{L}_1(\alpha) < \mathbf{N} \quad \forall\, \alpha \in \Omega$ and $\mathbf{L}_0^{-1}$ preserves orderings and the regularity assumption on the averaging operator.

To prove that the solution to Eq. (3.89) is itself bounded by the solution to Eq. (3.73), it must first be noted that all of the terms of the series for the average bound, Eq. (3.89,) are contained in the series for the worst-case bound Eq. (3.73)

$$\bar{y}^W = y_0 - \mathbf{L}_0^{-1}\mathbf{N}y_0 + \mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-1}\mathbf{N}y_0 - \cdots = \sum_{i=0}^{\infty}\left(-\mathbf{L}_0^{-1}\mathbf{N}\right)^i y_0 \tag{3.93}$$

$$\bar{y}^A = y_0 \qquad + \mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-1}\mathbf{N}y_0 + \cdots = \sum_{i=0}^{\infty}\left(\mathbf{L}_0^{-1}\mathbf{N}\mathbf{L}_0^{-1}\mathbf{N}\right)^i y_0 \tag{3.94}$$

so what remains is to examine the other terms of the worst-case bound, Eq. (3.73). Since $\mathbf{L}_0^{-1} < 0 \ \forall \alpha \in \Omega$ each odd term is positive definite and have the form

$$-\left(\mathbf{L}_0^{-1}\mathbf{N}\right)^i > 0 \quad i = 1, 3, 5, \ldots \tag{3.95}$$

Since all of the extra terms in Eq. (3.73) are nonnegative, the assertion holds. □

An interesting comparison is possible between the average and the worst case bounds. As shown in the proof in Eqs. (3.93) and (3.94), the series for the average bound essentially skips every other term in the series for the worst-case bound. The skipped terms are those with odd powers of the parameter dependent operator, $\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha)$. These odd power terms do not contribute to the average since $\mathbf{L}_1(\alpha)$ is assumed centered (zero mean). The average bound can therefore be thought of as the worst-case bound with all of the terms which would have been averaged out neglected. By neglecting these terms, it is evident that the value of the solution of the average bound equation will therefore be itself bounded by the worst-case bound solution.

The operator equation presented above can be applied to the problem of computing a bound for the average cost of the structured set of systems. The linear bounding function which was used in the worst case bound will be used for $\mathbf{N}$ in the average bound.

Theorem 3.3.3 (Average Bound) *Given a specified compensator, G_c, and the linear bound presented in Prop. 3.3.2, if the nominal Lyapunov function is negative definite and preserves orderings then*

$$J(G_c) \leq \operatorname{tr}\left\{\tilde{Q}^A\tilde{C}^T\tilde{C}\right\} \leq \operatorname{tr}\left\{\tilde{Q}^W\tilde{C}^T\tilde{C}\right\} \tag{3.96}$$

where $\tilde{Q}^A$ is the unique positive definite solution to the following system of Lyapunov Equations

$$0 = \tilde{A}_0\tilde{Q}^A + \tilde{Q}^A\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}_1 + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}_1\tilde{A}_i^T \tag{3.97}$$

$$0 = \tilde{A}_0\tilde{Q}_1 + \tilde{Q}_1\tilde{A}_0^T + \delta^2\tilde{Q}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^A\tilde{A}_i^T \tag{3.98}$$

and δ_i is defined from Eq. (3.78) and $\tilde{Q}^W$ is the solution to Eq. (3.86).

Proof: The result can be obtained by substituting into Prop. 3.3.2 the definitions for the $\mathbf{L}_0$ and $\mathbf{L}_1(\alpha)$ operators given in Prop. 3.1.2, and substituting for $\mathbf{N}$ the definition for the linear bound given in Prop. 3.3.2. □

Remark 3.3.2 (Solution) *The system of Lyapunov equations represented by Eqs. (3.97-3.98) can be represented by a single linear equation for the bound using Kronecher notation [113].*

$$\text{vec}\left\{\tilde{Q}_O\right\} = \left[I - \left(\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\delta^2 I + \sum_{i=1}^{r}\frac{\delta_i^2}{\delta^2}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)\right)^2\right]^{-1} \text{vec}\left\{\tilde{Q}^0\right\} \quad (3.99)$$

where "vec" is the column stacking operation defined in [113].

The form of this Kronecher notation equation for the average bound is quite similar to the equation for the Bourret approximation to the average cost as given in Eq. (3.99). The equations have similar structure since both can be generally put in the form

$$y = (1 - x^2)^{-1} y_0 \quad (3.100)$$

with the difference between the two appears in the form of the x term which involves the $\tilde{A}_i$ matrices:

$$\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left[\delta^2 I + \sum_{i=1}^{r}\frac{\delta_i^2}{\delta^2}\left(\tilde{A}_i \otimes \tilde{A}_i\right)\right] \quad : \quad \text{Average Bound} \quad (3.101)$$

$$\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left[\sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right] \quad : \quad \text{Bourret Approximation} \quad (3.102)$$

Since (3.101) is larger than (3.102), the average bound will "blow up" for smaller values of the uncertainty than the Bourret equation. This property is precisely what gives the average bound its guaranteed stability property, but it also contributes to the conservatism of the bound. The Kronecher notation solution can be reinterpreted to give conditions for the existence of positive definite solutions for the average bound equation.

Proposition 3.3.4 *A sufficient condition for the existence of the inverse in Eq. (3.99) and resulting uniqueness and positive definiteness of the solution is that*

$$\left\| \left(\left(\tilde{A}_0 \oplus \tilde{A}_0 \right)^{-1} \left(\delta^2 I + \sum_{i=1}^{r} \frac{\delta_i^2}{\delta^2} \left(\tilde{A}_i \oplus \tilde{A}_i \right) \right) \right)^2 \right\| < 1 \tag{3.103}$$

Proof: This is a direct consequence of Proposition (22.10) in Ref. [112] which states that the inverse in (3.100) exists if $\|x^2\| < 1$. □

Comparison of Eq. (3.103) to the equivalent condition for the worst case bound given in Eq. (3.88) reveals that the conditions are identical since the condition $\|X\| < 1$ is equivalent to $\|X^2\| < 1$. This means that the average bound asymptopes to infinity at the same value of uncertainty as the worst case bound. Although the worst case bound is always larger than the average bound, the bounds are equally conservative in the sense that the sufficient conditions for solution existence are identical. The bounds asymptope to There are therefore two issues associated with bounding functions. One is the size of the set over which the bound will have finite value and the other is the value over this set. While the average bound and the worst-case bound have the same size set, the average bound has lower value. This may lead to less control effort to achieve stability. These issues will be explored more in examples in the next section.

3.4 Second Order System Example

In this section the exact average cost and its approximations and bounds will be compared on a second order system with resonant poles with uncertain natural frequency or damping ratio. The purpose is to gain insight into the structure of the equations for exact averge cost and its approximations and bounds presented in the previous sections. The open-loop costs for the system will be derived as a function of the natural frequencies and damping ratios. To start consider the second order system represented by Figure 3.3

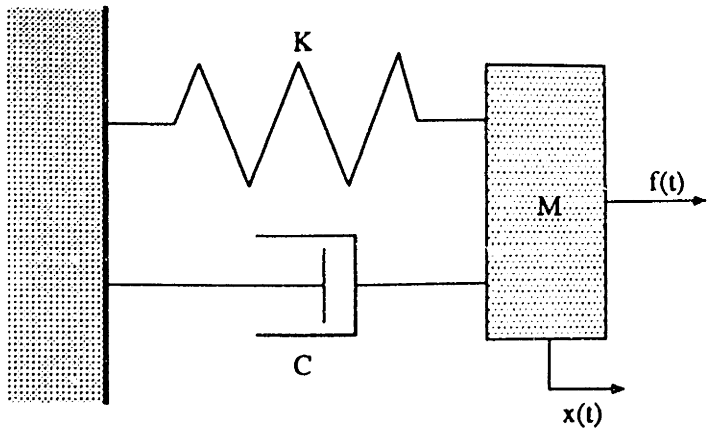

Figure 3.3: Second Order System

which has dynamics

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2 x(t) = bf(t) \tag{3.104}$$

where $x(t)$ is the mass displacement, $f(t)$ is the input force, ω is the system natural frequency, ζ is the system damping ratio, and b is the forcing coefficient. The natural frequency, damping ratio can be defined by

$$\omega^2 = k/m \quad 2\zeta\omega = c/m \quad b = 1/m \tag{3.105}$$

The system can be represented in state space form as:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{3.106}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad C = \begin{bmatrix} \sqrt{\nu} & 0 \\ 0 & \sqrt{\beta} \end{bmatrix} \tag{3.107}$$

ν is the position penalty and β is the velocity penalty. In this problem the natural frequency and damping ratios of the the system will be assumed uncertain and of the form

$$\omega^2 = {\omega_0}^2 + \tilde{\omega}^2, \quad -\delta_{\omega^2} \le \tilde{\omega}^2 \le \delta_{\omega^2} \tag{3.108}$$

$$\zeta = \zeta_0 + \tilde{\zeta}, \quad -\delta_\zeta \le \tilde{\zeta} \le \delta_\zeta \tag{3.109}$$

and the values are distributed uniformly.

The first case to be considered is the $\mathcal{H}_2$cost associated with the nominal plant with no parametric uncertainty. Some algebraic manipulation gives the nominal system cost as

$$J^0 = \frac{(\nu + {\omega_0}^2\beta)\sigma^2}{4\zeta_0{\omega_0}^3} \tag{3.110}$$

where $\sigma = bb^T$.

The exact average cost is computed by averaging over the costs in the parameter domain. It can be expressed as

$$J^E = \frac{\sigma^2}{4\zeta_0{\omega_0}^3}\left(\frac{\tanh^{-1}\bar{\zeta}}{\bar{\zeta}}\right)\left(\nu\left(\frac{\tanh^{-1}\bar{\omega}^2}{\bar{\omega}^2}\right) + \beta{\omega_0}^2\right) \tag{3.111}$$

where

$$\bar{\omega}^2 = \delta_{\omega^2}/{\omega_0}^2, \quad \bar{\zeta} = \delta_\zeta/\zeta_0 \tag{3.112}$$

The exact average cost is essentially the same as the nominal cost with the exception of the two terms involving $\bar{\omega}^2$ and $\bar{\zeta}$ which take the form, $\tanh^{-1} x/x$. It should be noted that in the limiting case of no uncertainty these terms assume unity value.

$$\lim_{x\to\infty}\left\{\frac{\tanh^{-1} x}{x}\right\} = 1 \tag{3.113}$$

The nominal cost, Eq. (3.110), is recovered in the limit. Infinite exact average cost indicates instability somewhere in the model set by Prop. 3.1.1. One can see in (3.111) that infinite cost is associated with the uncertain terms going to infinity. This occurs when either $\bar{\omega}^2 = 1$ or $\bar{\zeta} = 1$. For this system these conditions indicate that either the range of possible damping ratios is larger than the nominal value and therefore there can be unstable systems in the model set or that the natural frequency can vary to zero and the system can thus exhibit rigid body behavior.

These asymptotes for the exact average cost can be seen in Figure 3.4 for damping ratio uncertainty, and Figure 3.5 for natural frequency uncertainty. These curves where generated assuming that

$$\omega^2 = 1 \quad \zeta = 0.1 \tag{3.114}$$

and taking only a single uncertainty at a time. Note especially that the frequency must vary to zero before it starts to radically effect the cost. This is a very large frequency variation. The exact average cost's insensitivity to frequency uncertainty is a reflection of the fact that frequency uncertainties for nominally stable systems will not lead to unstable elements of the model set and therefore will not tend to drive the average cost.

When considering the uncertainties simultaneously, an important property of Eq. (3.111) is the *uncertainty independence property.* This property states that the amount of parameter uncertainty which will give infinite costs is independently effected by each uncertain parameter. By this, it is meant that the value of $\bar{\zeta}$ for which the $\bar{\zeta}$ term goes to infinity is independent of the value of $\bar{\omega}^2$ and similarly for the uncertain frequency. This property, while generally found in the approximations to the exact average, does not hold with the bounding costs in this example.

The perturbation expansion approximate average cost is given by:

$$J^P = \frac{\sigma^2}{4\zeta_0\omega_0{}^3}\left(\left(\nu + \beta\omega_0{}^2\right) + \frac{\bar{\zeta}^2}{3}\left(\nu + \beta\omega_0{}^2\right) + \frac{\bar{\omega}^4}{3}\nu\right) \tag{3.115}$$

The relationship between the exact average and the perturbation expansion approximate average can be seen by examining the expansion of the uncertainty terms in Eq. (3.111)

$$\frac{\tanh^{-1} x}{x} = 1 + \frac{x^2}{3} + \text{H.O.T.} \tag{3.116}$$

The perturbation expansion approximate average cost can be obtained from the exact average by substituting the series expansions in Eqs. (3.116) into Eq. (3.111) and retaining only the first order terms. In contrast to the exact average, the perturbation approximation retains only a quadratic dependence on the uncertain parameters and therefore a finite cost is associated with all finite values of the parameter bounds, i.e., the cost never asymptotes to infinity. This property is reflected in the quadratic form of the curves for the perturbation expansion approximation shown in Figures 3.4 and 3.5. This characteristic limits the perturbation expansion's effectiveness as an approximation to the average cost for systems with large uncertainties.

The Bourret approximate cost has a form very similar to the exact average cost:

$$J^B = \frac{\sigma^2}{4\zeta_0\omega_0{}^3}\left(\frac{1}{1-\frac{\zeta^2}{3}}\right)\left(\nu\left(\frac{1}{1-\frac{\omega^4}{3}}\right)+\beta\omega_0{}^2\right) \tag{3.117}$$

The only difference between Eq. (3.111) and Eq. (3.117) is the substitution of a new function for the $\tanh^{-1} x/x$ terms:

$$\frac{\tanh^{-1} x}{x} \longmapsto \frac{1}{1-\frac{x^2}{3}} \tag{3.118}$$

The Bourret cost is a better approximation to the exact average cost than is the perturbation expansion approximate cost because

$$\left|(\frac{\tanh^{-1} x}{x}) - (\frac{1}{1-\frac{x^2}{3}})\right| < \left|(\frac{\tanh^{-1} x}{x}) - (1+\frac{x^2}{3})\right| \quad \forall x \in \mathbb{R} \tag{3.119}$$

The Bourret approximate average cost also shares the uncertainty independence property of the exact average. As shown in Figures 3.4 and 3.5 the Bourret approximate cost does indeed asymptote to infinity but at a larger value of the uncertainty bound than the exact average. Examination of Eqs. (3.115) and (3.117) indicate that these two approximations for the exact average cost are always less then the exact average cost. This property is illustrated in Figures 3.4 and 3.5.

We turn now to the bounding functions. The worst case bound cost is given by

$$J^W = \frac{\sigma^2(\nu+\beta\omega_0{}^2)}{4\zeta_0\omega_0{}^3}\left(\frac{1}{1-\left(p+\frac{\zeta^2}{p}+\frac{\omega^4}{4p\zeta_0{}^2}\right)}\right) \tag{3.120}$$

where

$$p = \frac{\delta^2}{2\zeta_0\omega_0} \qquad \delta \in \mathbb{R} \tag{3.121}$$

is the free parameter used in the bound in Eq. (6.20) and it has been assumed that

$$\delta_1^2 = 2(\delta_{\omega^2})^2 \qquad \delta_2^2 = 2(\delta_\zeta)^2 \tag{3.122}$$

Eq. (3.120) is essentially the nominal cost modified by a term which is dependent on the parameter uncertainty bounds and p. Eq. (3.120) reveals several important properties of the bounding functions for the average cost. The first is that $J_i^W \to \infty$ when

$$p + \frac{\bar{\zeta}^2}{p} + \frac{\bar{\omega}^4}{4p{\zeta_0}^2} = 1 \tag{3.123}$$

The characteristics of this bound can be obtained by examining Eq. (3.123). The worst case cost does not have the uncertainty independence property since both uncertainties ($\bar{\zeta}^2$ and $\bar{\omega}^4$) contribute to (3.123) and their effects are additive. That is, increasing uncertainty in the damping ratio will decrease the allowable uncertainty in the natural frequency.

The value of the left hand side of Eq. (3.123) is a function of the parameter, p, as well as the uncertainty bounds. The bounding function can be made less conservative by finding the value of p which minimizes the left hand side of Eq. (3.123). For this system, this value of p is given by

$$p_{opt} = \sqrt{\bar{\zeta}^2 + \frac{\bar{\omega}^4}{4{\zeta_0}^2}} \tag{3.124}$$

If the bound is not chosen to be the optimal but is instead arbitrary, then even with no uncertainty ($\bar{\zeta}^2 = 0$ and $\bar{\omega}^4 = 0$) the cost does not reduce to the nominal. The bound contains a term which shifts the apparent system eigenvalues to the right and thereby increases the cost. To see this more clearly, we can examine Eq. (6.20) with no uncertainty

$$0 = \tilde{A}_0\tilde{Q}^W + \tilde{Q}^W\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}^W \tag{3.125}$$

The last term of Eq. (3.86) can be factored and included in the $\tilde{A}_0$ terms to give

$$0 = \tilde{A}_\delta\tilde{Q}^W + \tilde{Q}^W\tilde{A}_\delta + \tilde{B}\tilde{B}^T \tag{3.126}$$

where

$$\tilde{A}_\delta = \tilde{A}_0 + \frac{\delta^2}{2}I \tag{3.127}$$

This apparent right shift of the worst-case linear bound and the lack of uncertain parameter independence increases the linear bounds conservatism. Another factor contributing to this bound's conservatism is the structure of the frequency uncertainty term. The frequency uncertainty bound, $\bar{\omega}^4$, is scaled by a term containing ${\zeta_0}^2$. This scaling greatly increases the bounds sensitivity to frequency variation when the system has light damping. This effect is evident in Figure 3.5 where the small amount of frequency uncertainty which can be tolerated is evident. The maximum frequency uncertainty that the bound allows is relatively smaller than the amount of damping uncertainty, Fig. 3.4. The actual dimensional amount of the frequency uncertainty is on the order of the system nominal damping ratio, around 0.1. The bound doesn't reflect the fact that frequency uncertainties will not lead to unstable elements of the set when the nominal system is itself stable.

The average bound cost is given by

$$J^A = \frac{\sigma^2 \left(\nu + \beta {\omega_0}^2\right)}{4\zeta_0 {\omega_0}^3} \left(\frac{1}{1 - \left(p + \frac{\bar{\zeta}^2}{p} + \frac{\bar{\omega}^4}{4p{\zeta_0}^2}\right)^2} \right) \tag{3.128}$$

This cost function is always less than the worst case bound since the parameter dependent modifier term has the form $1/(1-x^2)$ which is always less than $1/(1-x)$, the form of the parameter dependent term of the worst case bound. Because of the form of this term, however, the conditions for infinite cost are identical for the two bounds and they are thus equally conservative. This property is illustrated in Figures 3.4 and 3.5 by the identical asymptotes of the worst-case and the average bound.

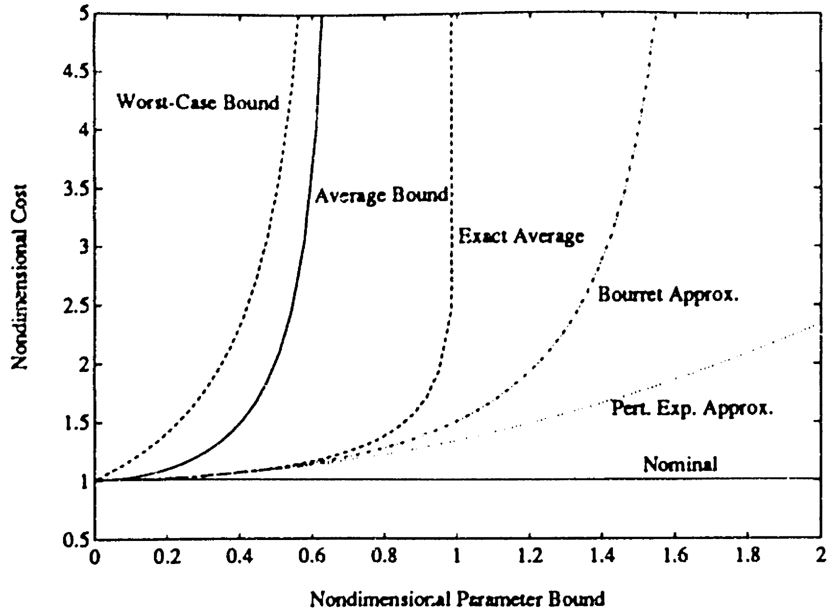

Figure 3.4: Damping Ratio Uncertainty: Nondimensional Cost (J/J^0) as a Function of Nondimensional Damping Ratio Bound, $\bar{\zeta} = \delta_\zeta/\zeta_0$

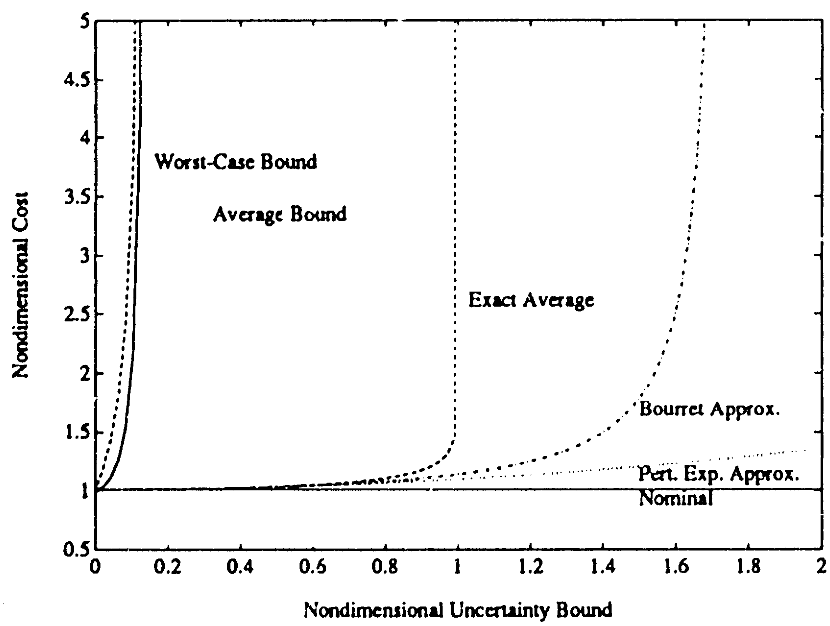

Figure 3.5: Natural Frequency Uncertainty: Nondimensional Cost (J/J^0) as a Function of Nondimensional Frequency Bound, $\bar{\omega}^2 = \delta_{\omega^2}/{\omega_0}^2$

3.5 Summary

In this chapter, the fundamentals of the average cost analysis of linear time-invariant systems with real parameterized uncertainties has been presented. This was motivated by showing that bounded average $\mathcal{H}_2$-norm implies stability throughout the model set. therefore minimization of the average cost will guarantee stability without having to resort to bounding the worst case over the set as is typically done.

Insight and conditions for the existence of the average (and therefore stability over the model set) were gained by examining the problem in terms of parameterized linear operators. The equation for the average cost involves the average solution of a parameterized Lyapunov equation which is shown to be a parameterized linear operator. Two techniques for finding the average solution of a parameterized operator equation were presented. The first has to simply average the perturbation expansion term by term. This could lead to series with poor convergence. The second method for finding the average was a technique borrowed from the random wave propagation literature, known as the Dyson Equation. The primary utility of these methods was in the area of approximate solutions to the exact average. The exact average is calculable only in the simplest of cases. There is therefore a strong motivation to produce good approximations and bounds to the average.

The two techniques for finding the average solution were applied to the problem of deriving approximate average costs for a parameterized set of systems. The first, called the perturbation expansion approximation, was based on a truncation of the infinite series comprising the formal perturbation expansion. The second, called the Bourret approximation, was based on a truncation of the Dyson equation. Two bounds were also derived. One, based on the perturbation expansion was shown to bound the worst case cost over the set; while another based on the Bourret equation was shown to bound the average but not necessarily the worst case. The properties of these approximations and bounds were discussed in the context of a simple second order system example.

Chapter 4

Synthesis of Controllers for Parameterized Systems

In this chapter the formulation of controllers based on the average cost and its approximations and bounds will be presented. A minimization problem can be formulated around the exact average $\mathcal{H}_2$ norm as well as each of its approximations and bounds. For each, necessary conditions can be derived for the respective cost minimization. In this section we will deal with two problems. The first is static output feedback compensation and its simplification to full state feedback. This will be compared to standard LQR theory. The other is dynamic output feedback compensation which will be compared to the standard LQG theory.

In the sections that follow fixed-form compensation, either static or dynamic, will be assumed and used to derive the necessary conditions based on that form. This type of compensator design has been used in Refs. [66, 69, 71, 72, 93, 94, 96] in the area of robust control and in Refs. [97, 98] in the area of reduced order controller design. The general steps in the derivation of the necessary conditions are as follows. First the cost is augmented with the appropriate set of equations appended with symmetric matrices of Lagrange multipliers. The appropriate equations are given by the problem. The minimization problems yields necessary conditions for the controller matrices. In the cases of exact average and bound minimization, a solution of these necessary conditions can be shown to be sufficient conditions for stability over the model set.

The chapter concludes with a discussion of the computational aspect of controller synthesis. The controllers are derived using the method of homotopic continuation for solving systems of coupled matrix equations. In this method the controllers are derived with successively larger values of uncertainty starting with the standard LQR or LQG controllers. In addition, the minimization algorithms and solution procedures for the various systems of coupled Lyapunov equations are presented. First, however, the necessary conditions for the problems must be derived. The discussion starts with the static output feedback problem.

4.1 Static Output Feedback Problems

In this section we will investigate five static output feedback problems. These are:

(i) Exact average cost minimization

(ii) Perturbation expansion approximate average cost minimization

(iii) Bourret approximate average cost minimization

(iv) Worst-case bound minimization

(v) Average bound minimization

The problems and assumptions will be stated, followed by the necessary conditions and properties of the resulting compensators.

Before proceeding further, the class of systems considered in all the static output feedback problems will be defined. The systems are essentially the same as the general model set defined in Definition 2.2.2 and the structured model set defined in Definition 2.2.4 with the addition of a few additional assumptions. For the static output feedback problem we will also assume the $D_{21} = 0$, i.e. there is no direct feedthrough from the disturbance to the output. In addition we will assume in the case of the structured set for any given i, $C_{2_i} = 0$. While this assumption is unnecessary for the exact average cost minimization problem, it is necessary for the approximations and bounds for the exact average to be calculable and will therefore be assumed for purposes of comparison. All other assumptions for the model sets, such as complete observability and controllability over the set, are as described in Chapter 2. These new assumptions modify the general and structured sets of systems used for all static output feedback problems.

Definition 4.1.1 (General Set of Systems) *The set $\mathcal{G}_g$ of systems used in static output feedback is parameterized as follows*

$$\mathcal{G}_g = \{G_g(\alpha)\ \forall\, \alpha\ \in\ \Omega\} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^r$ is defined in Def. 2.2.1 and each element of the set is described in the state space as

$$G_g(\alpha) = \left[\begin{array}{c|cc} A(\alpha) & B_1(\alpha) & B_2(\alpha) \\ \hline C_1(\alpha) & 0 & D_{12}(\alpha) \\ C_2(\alpha) & 0 & 0 \end{array}\right] \tag{4.2}$$

where $A(\alpha) \in \mathbb{R}^{n\times n}$, $B_2(\alpha) \in \mathbb{R}^{n\times m}$, $C_2(\alpha) \in \mathbb{R}^{l\times n}$, $B_1(\alpha) \in \mathbb{R}^{n\times p}$, $C_1(\alpha) \in \mathbb{R}^{q\times n}$, $\forall \alpha \in \Omega$.

Definition 4.1.2 (Structured Set of Systems) *The set $\mathcal{G}_s$ of systems is parameterized as follows*

$$\mathcal{G}_s = \{G_s(\alpha)\ : \alpha \in \Omega_s\} \tag{4.3}$$

where Ω_s is the structured set of parameter vectors defined

$$\Omega_s = \{\alpha\ : \alpha \in \mathbb{R}^r, -\delta_i \leq \alpha_i \leq \delta_i \quad i = 1, \cdots, r\} \tag{4.4}$$

and where each element of $\mathcal{G}_s$ is described in the state space as

$$\cdot G_s(\alpha) = \left[\begin{array}{c|cc} A_0 + \sum_{i=1}^{r} \alpha_i A_i & B_1 & B_{2_0} + \sum_{i=1}^{r} \alpha_i B_{2_i} \\ \hline C_1 & 0 & D_{12} \\ C_{2_0} & 0 & 0 \end{array}\right] \tag{4.5}$$

where for $i = 0, \ldots, r$; $A_i \in \mathbb{R}^{n \times n}, B_{2_i} \in \mathbb{R}^{n \times m}, C_{2_i} \in \mathbb{R}^{l \times n}$, *and* $B_1 \in \mathbb{R}^{n \times p}, C_1 \in \mathbb{R}^{q \times n}, C_{2_C} \in \mathbb{R}^{l \times n}$.

It is useful at this point to consider the set of closed-loop systems. The general arrangement of the control loop is shown in Figure 4.1.

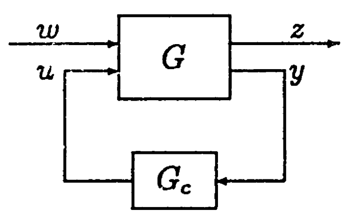

Figure 4.1: The Standard Control Problem

A fixed-form static output feedback compensator can be represented by

$$G_c = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_c \end{array}\right] \tag{4.6}$$

with input y and output u. The compensator is thus just a static direct feedthrough gain from y to u. Using this static compensator adn the set of open-loop transfer functions, the set of closed-loop transfer functions from w to z, $\mathcal{G}_{zw}$, can be defined. Each element of $\mathcal{G}_{zw}$ can be expressed is state space form for static output feedback as:

$$\begin{aligned} G_{zw}(\alpha) &= \left[\begin{array}{c|c} A(\alpha) + B_2(\alpha) D_c C_2(\alpha) & B_1(\alpha) \\ \hline C_1(\alpha) + D_{12}(\alpha) D_c C_2(\alpha) & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \tilde{A}(\alpha) & \tilde{B}(\alpha) \\ \hline \tilde{C}(\alpha) & 0 \end{array}\right] \end{aligned} \tag{4.7}$$

Using the structured set of systems

$$G_{zw}(\alpha) = \left[\begin{array}{c|c} A_0 + B_{2_0} D_c C_{2_0} + \sum_{i=1}^{r} \alpha_i (A_i + B_{2_i} G_c C_{2_0}) & B_1 \\ \hline C_1 + D_{12} D_c C_{2_0} & 0 \end{array}\right] \quad (4.8)$$

$$= \left[\begin{array}{c|c} \tilde{A}_0 + \sum_{i=1}^{r} \alpha_i \tilde{A}_i & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] \quad (4.9)$$

Because of the form assumed for the uncertainty in the structured set of systems, only the resulting closed-loop A matrix is parameter dependent; and the closed-loop system is strictly proper.

With the sets of systems established, a general performance problem can be stated. The general model set average performance problem is the basis of the other auxiliary minimization problems used to derive controllers in Sections 4.1.2, and 4.1.3.

Problem 4.1.1 (Average Performance Problem) *Given the set $\mathcal{G}_g$ or $\mathcal{G}_s$ of systems, determine the static feedback compensator,*

$$G_c = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_c \end{array}\right] \quad (4.10)$$

with $D_c \in \mathbb{R}^{m \times l}$, which minimizes the the closed-loop $\mathcal{H}_2$-norm averaged over the model set.

$$J(G_c) = \int_{\Omega} \|G_{zw}(\alpha)\|_2^2 \, \mathrm{d}\mu(\alpha) \quad (4.11)$$

In the next sections, we will take a closer look at the minimization of both the average $\mathcal{H}_2$-norm and its approximations and bounds.

4.1.1 Average Cost Minimization

In this section the formulation for the necessary conditions for the minimization of the exact average cost will be presented. The first step is to use the result of Proposition 3.1.1 to define the auxiliary minimization problem for the exact average cost.

Problem 4.1.2 (Auxiliary Minimization Problem) *Given the general set of systems in $\mathcal{G}_g$ described in Eq. (4.2), determine the static compensator G_c, Eq. (4.10), which minimizes*

$$\mathcal{J}^E(G_c) = \mathrm{tr}\left\{\left\langle \tilde{Q}(\alpha)\,\tilde{C}^T(\alpha)\tilde{C}(\alpha)\right\rangle\right\} \tag{4.12}$$

subject to the parameterized Lyapunov equation.

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\,\tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha) \tag{4.13}$$

for each $\alpha \in \Omega$.

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the stability of the plant.

Proposition 4.1.1 *if $(G_c, \tilde{Q}(\alpha))$ satisfies Eq. (4.13), $\tilde{A}(\alpha)$ is decomposed into $\tilde{A}(\alpha) = \tilde{A}_0 + \tilde{A}_1(\alpha)$, with $\tilde{A}_0$ asymptotically stable, and the norm constraint*

$$\left\|\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_1(\alpha) \oplus \tilde{A}_1(\alpha)\right)\right\| < 1 \quad \forall\, \alpha \in \Omega \tag{4.14}$$

is satisfied, then $\tilde{A}(\alpha)$ is asymptotically stable $\forall\, \alpha \in \Omega$ and

$$J(G_c) = \mathcal{J}^E(G_c) \tag{4.15}$$

Proof: From Prop. 3.1.4, Eq. (4.14) implies stability over the set of systems. From Prop. 3.1.1, stability over the set of systems allows the average cost to be given by the average solution to a parameterized Lyapunov equation. □

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The general method is to use the Lagrange multipliers technique. Rigorous application of the Lagrange multiplier techniques requires that G_c be restricted to the set of always stabilizing controls. The conditions for simultaneous stability have been discussed in Refs. [27-41]. In pratice this condition is enforced using the sufficient condition for simultaneous stability associated with the given robust design problem. This condition is difficult to use for the exact average cost minimization problem and so exhaustive stbility testing must be preformed.

The first step is to append Eq. (4.13) to the cost using a parameter dependent, symmetric matrix of Lagrange multipliers, $\tilde{P}(\alpha) \in \mathbb{R}^{n \times n}$. The matrix of Lagrange multipliers must be parameter dependent because the appended equations are parameter dependent. The appended cost becomes

$$\begin{aligned}
\mathcal{J}^E(G_c) = & \\
& \operatorname{tr}\left\{\int_\Omega \tilde{Q}(\alpha)\, \tilde{C}^T(\alpha)\tilde{C}(\alpha)\, \mathrm{d}\mu(\alpha)\right\} \\
+ & \operatorname{tr}\left\{\int_\Omega \left[\tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\, \tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha)\right] \tilde{P}(\alpha)\, \mathrm{d}\mu(\alpha)\right\} \qquad (4.16)
\end{aligned}$$

where $\tilde{A}(\alpha)$, $\tilde{B}(\alpha)$, and $\tilde{C}(\alpha)$ are defined in Def. (4.1.1). The necessary conditions for minimization of the exact average cost can now be stated by taking the derivatives with respect to G_c, $\tilde{P}(\alpha)$, and $\tilde{Q}(\alpha)$. A table of matrix derivatives can be found in Ref. [125].

$$\frac{\partial \mathcal{J}}{\partial \tilde{P}(\alpha)} = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\, \tilde{A}^T(\alpha) + \tilde{B}\tilde{B}^T = 0 \quad \forall\, \alpha \in \Omega \qquad (4.17)$$

$$\frac{\partial \mathcal{J}}{\partial \tilde{Q}(\alpha)} = \tilde{A}^T(\alpha)\tilde{P}(\alpha) + \tilde{P}(\alpha)\, \tilde{A}(\alpha) + \tilde{C}^T\tilde{C} = 0 \quad \forall\, \alpha \in \Omega \qquad (4.18)$$

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial D_c} = & \left\langle D_{12}^T(\alpha) D_{12}(\alpha) D_c C_2(\alpha)\tilde{Q}(\alpha)\, C_2^T(\alpha)\right\rangle \\
+ & \left\langle B_2^T(\alpha)\tilde{P}(\alpha)\, \tilde{Q}(\alpha)\, C_2^T(\alpha)\right\rangle = 0 \qquad (4.19)
\end{aligned}$$

The necessary conditions for local minimization of the exact average cost can now be stated using Eqs. (4.17)-(4.19)

Theorem 4.1.1 (Necessary Conditions) *Suppose G_c, Eq. (4.10) solves the average cost minimization problem (4.1.2), then there exist matrices, $\tilde{Q}(\alpha), \tilde{P}(\alpha) \geq 0 \in \mathbb{R}^{n \times n}$ such that*

$$0 = \left\langle D_{12}^T(\alpha) D_{12}(\alpha) D_c C_2(\alpha)\tilde{Q}(\alpha)\, C_2^T(\alpha)\right\rangle + \left\langle B_2^T(\alpha)\tilde{P}(\alpha)\, \tilde{Q}(\alpha)\, C_2^T(\alpha)\right\rangle \qquad (4.20)$$

where $\tilde{Q}(\alpha)$ satisfies the parameterized Lyapunov equation

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\, \tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha) \qquad (4.21)$$

and $\tilde{P}(\alpha)$ satisfies the Adjoint Lyapunov equation

$$0 = \tilde{A}^T(\alpha)\tilde{P}(\alpha) + \tilde{P}(\alpha)\tilde{A}(\alpha) + \tilde{C}^T(\alpha)\tilde{C}(\alpha) \tag{4.22}$$

Proof: The proof is a direct consequence of the differentiation of the cost with respect to D_c, $\tilde{Q}(\alpha)$, and $\tilde{P}(\alpha)$ in Eq. (4.19). □

Remark 4.1.1 *Equation (4.20) was derived previously in Ref. [108] under more restrictive uncertainty assumptions of parameter independence of D_{12}. In this case, D_c can be solved for explicitly*

$$D_c = -\left(D_{12}^T D_{12}\right)^{-1}\left\langle B_{2_0}^T(\alpha)\tilde{P}(\alpha)\tilde{Q}(\alpha)C_2^T(\alpha)\right\rangle\left\langle C_2(\alpha)\tilde{Q}(\alpha)C_2^T(\alpha)\right\rangle^{-1} \tag{4.23}$$

Remark 4.1.2 *Equations (4.20)-(4.22) have a form very similar to the necessary conditions derived for the output feedback problem in Refs. [126–128]. This is especially evident in the case of parameter independent B_2, C_2, and D_{12} matrices,*

$$D_c = -\left(D_{12}^T D_{12}\right)^{-1} B_2^T\left\langle\tilde{P}(\alpha)\tilde{Q}(\alpha)\right\rangle C_2(C_2\left\langle\tilde{Q}(\alpha)\right\rangle C_2^T)^{-1} \tag{4.24}$$

Remark 4.1.3 *In the case of parameter independent B_2 and D_{12} matrices, and full state feedback, $C_2 = I$, equation simplifies to*

$$D_c = -\left(D_{12}^T D_{12}\right)^{-1} B_2^T\left\langle\tilde{P}(\alpha)\tilde{Q}(\alpha)\right\rangle\left\langle\tilde{Q}(\alpha)\right\rangle^{-1} \tag{4.25}$$

and it is evident that the averaging operator couples the solutions of the two parameterized Lyapunov equations for $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$.

Remark 4.1.4 *For the case of a parameter independent systems with full state feedback, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are not functions of alpha and therefore neither are $\tilde{Q}$ or $\tilde{P}$. In this case, $\tilde{Q}$ cancels in (4.20) and the equations simplify to the traditional LQR results.*

$$G_c = -R^{-1}B_2^T\tilde{P} \tag{4.26}$$

where $R = D_{12}^T D_{12}$. Upon substitution of (4.26) into (4.21) we obtain a Riccati equation for $\tilde{P}$.

$$A_0^T\tilde{P} + \tilde{P}A_0 + C_{1_0}^T C_{1_0} - \tilde{P}B_{2_0}R^{-1}B_{2_0}^T\tilde{P} = 0 \tag{4.27}$$

The difficulty inherent in Eq. (4.20) for the optimal gain is that it involves the average of the product of the solution of two Lyapunov equations for $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$. These matrices are only given as implicit functions of α in Equations (4.21) and (4.22) respectively. Only in the simplest of cases can the average of the product be solved for exactly. The options available for the solution of such equations are presented in Section 4.3. The approximate solution can be obtained by Monte-Carlo techniques or the explicit α dependence can be found by symbolic manipulations and the expressions averaged numerically or symbolically. All of these techniques are computationally intensive. In the next sections, the approximations and bounds to the cost will be minimized in an attempt to reach the optimal average minimization solution by using computable expressions for the cost.

4.1.2 Approximate Average Cost Minimization

Perturbation Expansion Approximate Cost Minimization

In this section the formulation of the necessary conditions for the minimization of the perturbation expansion approximate cost will be presented. The first step is to use the result of Proposition 3.2.1 to define the auxiliary minimization problem.

Problem 4.1.3 (Auxiliary Minimization Problem) *Given the set $\mathcal{G}_s$ of systems described in Def. 4.1.2 determine the static compensator G_c, Eq. (4.10), which minimizes*

$$\mathcal{J}^P(G_c) = \operatorname{tr}\left\{\left(\tilde{Q}^0 + \tilde{Q}^P\right)\tilde{C}^T\tilde{C}\right\} \tag{4.28}$$

where the nominal cost, $\tilde{Q}^0$, and the parameter dependent cost, $\tilde{Q}^P$, are the unique positive definite solutions to the following system of Lyapunov equations

$$0 = \tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T \tag{4.29}$$

$$0 = \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{4.30}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right) \quad i = 1,\ldots,r \tag{4.31}$$

where σ_i is defined from Equation (3.47).

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.2 establishing the solution of the perturbation expansion equation as an approximation for the exact average solution. The first step in deriving necessary conditions to the auxiliary minimization problem is to append Eqs. (4.29)-(4.31) to the cost using parameter independent, symmetric matrices of Lagrange multipliers, $\tilde{P}^0$, $\tilde{P}^P$, and $\tilde{P}^i$, $i = 1\ldots r \in \mathbb{R}^{n\times n}$. The appended cost is given by

$$\begin{aligned}
\mathcal{J}(G_c) &= \operatorname{tr}\left\{\left(\tilde{Q}^0+\tilde{Q}^P\right)\tilde{C}^T\tilde{C}\right\} \\
&+ \operatorname{tr}\left\{\left[\tilde{A}_0\tilde{Q}^0+\tilde{Q}^0\tilde{A}_0^T+\tilde{B}\tilde{B}^T\right]\tilde{P}^0\right\} \\
&+ \operatorname{tr}\left\{\left[\tilde{A}_0\tilde{Q}^P+\tilde{Q}^P\tilde{A}_0^T+\sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i+\tilde{Q}^i\tilde{A}_i^T\right)\right]\tilde{P}^P\right\} \\
&+ \operatorname{tr}\left\{\sum_{i=1}^{r}\left[\tilde{A}_0\tilde{Q}^i+\tilde{Q}^i\tilde{A}_0^T+\sigma_i\left(\tilde{A}_i\tilde{Q}^0+\tilde{Q}^0\tilde{A}_i^T\right)\right]\tilde{P}^i\right\} \qquad (4.32)
\end{aligned}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are defined in Def. (4.1.2). Taking the derivatives with respect to $D_c, \tilde{P}^0$, $\tilde{P}^P$, $\tilde{P}^i$ and $\tilde{Q}^0$, $\tilde{Q}^P$, $\tilde{Q}^i$ gives the necessary conditions for minimization of the perturbation expansion approximation to the exact average cost.

Theorem 4.1.2 (Necessary Conditions) *Suppose G_c the static compensator from Eq. (4.10)*

$$G_c = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_c \end{array}\right] \qquad (4.33)$$

solves the perturbation expansion approximate cost minimization problem (4.1.3), then there exist matrices, $\tilde{P}^0$, $\tilde{P}^P$, $\tilde{P}^i$ and $\tilde{Q}^0$, $\tilde{Q}^P$, $\tilde{Q}^i \geq 0 \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}
D_c = &-\left(D_{12}^TD_{12}\right)^{-1}\left[B_{2_0}^T\tilde{P}^0\tilde{Q}^0+B_{2_0}^T\tilde{P}^P\tilde{Q}^P\right. \\
&+ \left.\sum_{i=1}^{r}\left(B_{2_0}^T\tilde{P}^i\tilde{Q}^i+\sigma_iB_{2_i}^T\left(\tilde{P}^P\tilde{Q}^i+\tilde{P}^i\tilde{Q}^0\right)\right)\right]C_2^T\left(C_2(\tilde{Q}^0+\tilde{Q}^P)C_2^T\right)^{-1} \qquad (4.34)
\end{aligned}$$

where $\tilde{Q}^0$, $\tilde{Q}^P$, and $\tilde{Q}^i$ *satisfy the Lyapunov equations*

$$
\begin{aligned}
0 &= \tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T \\
0 &= \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \\
0 &= \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right) \quad i = 1,\ldots,r
\end{aligned}
\tag{4.35}
$$

and $\tilde{P}^0$, $\tilde{P}^P$, and $\tilde{P}^i$ *satisfy the adjoint Lyapunov equations*

$$
\begin{aligned}
0 &= \tilde{A}_0^T\tilde{P}^P + \tilde{P}^P\tilde{A}_0 + \tilde{C}^T\tilde{C} \\
0 &= \tilde{A}_0^T\tilde{P}^0 + \tilde{P}^0\tilde{A}_0 + \tilde{C}^T\tilde{C} + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i^T\tilde{P}^i + \tilde{P}^i\tilde{A}_i\right) \\
0 &= \tilde{A}_0^T\tilde{P}^i + \tilde{P}^i\tilde{A}_0 + \sigma_i\left(\tilde{A}_i^T\tilde{P}^P + \tilde{P}^P\tilde{A}_i\right) \quad i = 1,\ldots,r
\end{aligned}
\tag{4.36}
$$

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.32), with respect to $D_c, \tilde{P}^0, \tilde{P}^P, \tilde{P}^i$ and $\tilde{Q}^0, \tilde{Q}^P, \tilde{Q}^i$. □

Remark 4.1.5 *For full state feedback, the condition for local minima, Eq. (4.34) becomes*

$$
\begin{aligned}
D_c = -\left(D_{12}^T D_{12}\right)^{-1} \Big[& B_{2_0}^T\tilde{P}^0\tilde{Q}^0 + B_{2_0}^T\tilde{P}^P\tilde{Q}^P \\
+ & \sum_{i=1}^{r}\left(B_{2_0}^T\tilde{P}^i\tilde{Q}^i + \sigma_i B_{2_i}^T\left(\tilde{P}^P\tilde{Q}^i + \tilde{P}^i\tilde{Q}^0\right)\right)\Big]\left(\tilde{Q}^0 + \tilde{Q}^P\right)^{-1}
\end{aligned}
\tag{4.37}
$$

Remark 4.1.6 *The traditional LQR results are recovered in the case of no uncertainty and full state feedback.*

Bourret Approximate Cost Minimization

In this section the formulation for the necessary conditions for the minimization of the Bourret approximate cost will be presented. The first step is to use the result of Proposition 3.2.2 to define the auxiliary minimization problem.

Problem 4.1.4 (Auxiliary Minimization Problem) *Given the set $\mathcal{G}_s$ of systems defined in Def. 4.1.2, determine the static compensator G_c, Eq. (4.10), which minimizes*

$$\mathcal{J}^B(G_c) \;=\; \mathrm{tr}\left\{\tilde{Q}^B\tilde{C}^T\tilde{C}\right\} \tag{4.38}$$

where $\tilde{Q}^B$ is the unique positive definite solutions to the following system of coupled Lyapunov equations:

$$0 \;=\; \tilde{A}_0\tilde{Q}^B + \tilde{Q}^B\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{4.39}$$

$$0 \;=\; \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}_i^T\right) \quad i = 1,\ldots,r \tag{4.40}$$

and σ_i is defined from Equation (3.47).

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.2 relating the solution of the Bourret equation to the Exact Average solution. It will be restated here for clarity.

Proposition 4.1.2 *If $(G_c, \tilde{Q}^B)$ satisfies Eqs. (4.39) and (4.40) and the norm constraint given in Proposition 3.2.3*

$$\left\|\sum_{i=1}^{r}\left(\sigma_i\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)^2\right\| < 1 \tag{4.41}$$

is satisfied, then $\tilde{A}(\alpha)$ is almost always stable $\forall\, \alpha \in \hat{\Omega}$ where $\hat{\Omega}$ is defined

$$\hat{\Omega} = \left\{\alpha \;:\; \sum_{i=1}^{r}|\alpha_i|\left\|\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right\| \le \left\|\sum_{i=1}^{r}\sigma_i^2\left(\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)^2\right\|\right\} \tag{4.42}$$

In addition, the unique positive definite solution to (4.39)-(4.40), $\tilde{Q}^B$, gives

$$J(G_c) \cong \mathcal{J}^B(G_c, \tilde{Q}) \tag{4.43}$$

Proof: From Proposition 3.2.3, Eq. (4.41) guarantees uniqueness and positive definiteness of the solution of (4.39)-(4.40). Eq. (4.41) also guarantees closed-loop stability $\forall\, \alpha \in \hat{\Omega}$ by Prop. 3.2.1. □

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eqs. (4.39)-(4.40) to the cost using parameter independent, symmetric matrices of Lagrange multipliers, $\tilde{P}^B$ and $\tilde{P}^i$, $i = 1 \ldots r \in \mathbb{R}^{n \times n}$. The appended cost becomes.

$$\begin{aligned} \mathcal{J}^B(G_c) &= \operatorname{tr}\left\{\tilde{Q}\tilde{C}^T\tilde{C}\right\} \\ &+ \operatorname{tr}\left\{\left[\tilde{A}_0\tilde{Q}^B + \tilde{Q}^B\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right)\right]\tilde{P}^B\right\} \\ &+ \operatorname{tr}\left\{\sum_{i=1}^{r}\left[\tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}_i^T\right)\right]\tilde{P}^i\right\} \end{aligned} \tag{4.44}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are defined in Def. (4.1.2). Taking the derivatives with respect to $D_c, \tilde{P}^B$, $\tilde{P}^i$ and $\tilde{Q}^B$, $\tilde{Q}^i$ gives the necessary conditions for minimization of the Bourret approximate cost.

Theorem 4.1.3 (Necessary Conditions) *Suppose G_c solves the Bourret approximate cost minimization problem (4.1.4), then there exist matrices, $\tilde{Q}$ and $\tilde{P} \geq 0 \in \mathbb{R}^{n \times n}$ such that*

$$\begin{aligned} D_c = -\left(D_{12}^T D_{12}\right)^{-1} & \left[B_{2_0}^T\tilde{P}^B\tilde{Q}^B\right. \\ + & \left.\sum_{i=1}^{r}\left(B_{2_0}^T\tilde{P}^i\tilde{Q}^i + \sigma_i B_{2_i}^T\left(\tilde{P}^B\tilde{Q}^i + \tilde{P}^i\tilde{Q}^B\right)\right)\right] C_2^T\left(C_2\tilde{Q}C_2^T\right)^{-1} \end{aligned} \tag{4.45}$$

where $\tilde{Q}^B$ satisfies the Bourret equation

$$0 = \tilde{A}_0\tilde{Q}^B + \tilde{Q}^B\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{4.46}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}_i^T\right) \quad i = 1, \ldots, r \tag{4.47}$$

and $\tilde{P}^B$ satisfies the Adjoint Bourret equation

$$0 = \tilde{A}_0^T\tilde{P}^B + \tilde{P}^B\tilde{A}_0 + \tilde{C}^T\tilde{C} + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i^T\tilde{P}^i + \tilde{P}^i\tilde{A}_i\right) \tag{4.48}$$

$$0 = \tilde{A}_0^T\tilde{P}^i + \tilde{P}^i\tilde{A}_0 + \sigma_i\left(\tilde{A}_i^T\tilde{P}^B + \tilde{P}^B\tilde{A}_i\right) \quad i = 1, \ldots, r \tag{4.49}$$

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.44), with respect to $D_c, \tilde{P}^B, \tilde{P}^i$ and $\tilde{Q}^B, \tilde{Q}^i$. □

Remark 4.1.7 *In the case of full state feedback, $C_2 = I$, the condition for local minima, Eq. (4.45), is given by*

$$G_c = -\left(D_{12}^T D_{12}\right)^{-1}\left[B_{2_0}^T \tilde{P}^B \tilde{Q}^B + \sum_{i=1}^{r}\left(B_{2_0}^T \tilde{P}^i \tilde{Q}^i + \sigma_i B_{2_i}^T\left(\tilde{P}^B \tilde{Q}^i + \tilde{P}^i \tilde{Q}^B\right)\right)\right]\tilde{Q}^{B^{-1}} \tag{4.50}$$

Remark 4.1.8 *The traditional LQR results presented in Eqs. (4.26) and (4.27) are recovered in the case of no uncertainty and full state feedback.*

4.1.3 Bound Minimization

Worst-Case Bound Minimization

In this section the formulation for the necessary conditions for the minimization of the worst-case bound will be presented. The first step is to use the result of Theorem 3.3.1 to define the auxiliary minimization problem.

Problem 4.1.5 (Auxiliary Minimization Problem) *Given a set $\mathcal{G}_s$ of systems described in Def. 4.1.2, determine the static feedback compensator G_c, Eq. (4.10), which minimizes*

$$\mathcal{J}^W(G_c) = \operatorname{tr}\left\{\tilde{Q}^W \tilde{C}^T \tilde{C}\right\} \tag{4.51}$$

where $\tilde{Q}^W$ is the unique positive definite solution to the following system of Lyapunov Equations

$$0 = \tilde{A}_0 \tilde{Q}^W + \tilde{Q}^W \tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2 \tilde{Q}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i \tilde{Q}^W \tilde{A}_i^T \tag{4.52}$$

where δ_i is defined from Equation (3.78) and $\delta \in \mathbb{R}$.

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.3.1 relating the solution of the worst-case bound to the exact average solution. It will be restated here for clarity.

Proposition 4.1.3 *If the norm constraint given in Proposition 3.3.3*

$$\left\| \left(\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1} \left(\delta^2 I + \sum_{i=1}^{r} \frac{\delta_i^2}{\delta^2} \left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)\right) \right\| < 1 \tag{4.53}$$

is satisfied and $\tilde{A}_0$ is asymptotically stable, then $\tilde{A}(\alpha)$ is asymptotically stable $\forall\, \alpha \in \Omega$; and $\tilde{Q}^W$, the unique positive definite solution to (4.52), gives

$$J(G_c) \leq \mathcal{J}^W(G_c, \tilde{Q}^W) \tag{4.54}$$

Proof: From Proposition 3.3.3, Eq. (4.53) and stability of $\tilde{A}_0$ guarantees uniqueness and positive definiteness of the solution of (4.52). Eq. (4.53) also guarantees existence of the average cost $\forall\, \alpha \in \Omega$ since (4.53) implies (3.21). By Prop. 3.1.1 bounded average cost implies that the closed-loop systems, $G_{zw}(\alpha)$, are stable in the sense of Lyapunov $\forall\, \alpha \in \Omega$. □

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eq. (4.52) to the cost using a parameter independent, symmetric matrix of Lagrange multipliers, $\tilde{P}^W \in \mathbb{R}^{n\times n}$. The appended cost becomes

$$\begin{aligned} \mathcal{J}^W(G_c) &= \operatorname{tr}\left\{\tilde{Q}^W \tilde{C}^T \tilde{C}\right\} \\ &+ \operatorname{tr}\left\{\left[\tilde{A}_0 \tilde{Q}^W + \tilde{Q}^W \tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta \tilde{Q}^W + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta}\right) \tilde{A}_i \tilde{Q}^W \tilde{A}_i^T\right] \tilde{P}^W\right\} \end{aligned} \tag{4.55}$$

where $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ are defined in Def. (4.1.2). Taking the derivatives with respect to $D_c, \tilde{P}^W$ and $\tilde{Q}^W$ gives the necessary conditions for optimization.

Theorem 4.1.4 (Necessary Conditions) *Suppose G_c solves the average bound minimization problem (4.1.5), then there exist matrices, $\tilde{Q}^W$ and $\tilde{P}^W \geq 0 \in \mathbb{R}^{n\times n}$ such that*

$$\begin{aligned} D_c = -\left(D_{12}^T D_{12} + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) B_{2_i}^T \tilde{P}^W B_{2_i}\right)^{-1} &\left[B_{2_0}^T \tilde{P}^W \tilde{Q}^W \right. \\ + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) &\left.\left(B_{2_i}^T \tilde{P}^W A_i \tilde{Q}^W\right)\right] C_2^T \left(C_2 \tilde{Q}^W C_2^T\right)^{-1} \end{aligned} \tag{4.56}$$

where $\tilde{Q}^W$ satisfies the worst-case bound equation

$$0 = \tilde{A}_0\tilde{Q}^W + \tilde{Q}^W\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^W\tilde{A}_i^T \tag{4.57}$$

and $\tilde{P}^W$ satisfies the adjoint worst-case bound equation

$$0 = \tilde{A}_0^T\tilde{P}^W + \tilde{P}^W\tilde{A}_0 + \tilde{C}^T\tilde{C} + \delta^2\tilde{P}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i^T\tilde{P}^W\tilde{A}_i \tag{4.58}$$

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.55), with respect to D_c, $\tilde{P}^W$ and $\tilde{Q}^W$. □

Remark 4.1.9 *In the case of full state feedback, $C_2 = I$. In this case, Eq. (4.56) is given by*

$$D_c = -\left(D_{12}^T D_{12} + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta}\right)B_{2_i}^T\tilde{P}^W B_{2_i}\right)^{-1}\left[B_{2_0}^T\tilde{P}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta}\right)\left(B_{2_i}^T\tilde{P}^W A_i\right)\right] \tag{4.59}$$

The gain is thus only and function of $\tilde{P}^W$ and not $\tilde{Q}^W$.

Remark 4.1.10 *The traditional LQR results presented in Eqs. 4.26 and 4.27 are recovered in the case of no uncertainty and full state feedback.*

Average Bound Minimization

In this section the formulation for the necessary conditions for the minimization of the average bound will be presented. The first step is to use the result of Proposition 3.3.3 to define the auxiliary minimization problem.

Problem 4.1.6 (Auxiliary Minimization Problem) *Given a set $\mathcal{G}_s$ of systems described above in Def. 4.1.2, determine the static feedback compensator G_c, Eq. (4.10), which minimizes*

$$\mathcal{J}^A(G_c, \tilde{Q}^A) = \operatorname{tr}\left\{\tilde{Q}^A\tilde{C}^T\tilde{C}\right\} \tag{4.60}$$

where $\tilde{Q}^A$ is the unique positive definite solutions to the following system of Lyapunov equations

$$0 = \tilde{A}_0\tilde{Q}^A + \tilde{Q}^A\tilde{A}_0^T + +\tilde{B}\tilde{B}^T + \delta^2\tilde{Q}_1 + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}_1\tilde{A}_i^T \tag{4.61}$$

$$0 = \tilde{A}_0\tilde{Q}_1 + \tilde{Q}_1\tilde{A}_0^T + \delta^2\tilde{Q}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^A\tilde{A}_i^T \tag{4.62}$$

and δ_i is defined from Eq. (3.78) and $\delta \in \mathbb{R}$.

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.3 relating the solution of the bound equation to the exact average solution. It will be restated here for clarity.

Proposition 4.1.4 *If the norm constraint given in Proposition 3.3.4*

$$\left\|\left(\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\delta^2 I + \sum_{i=1}^{r}\tfrac{\delta_i^2}{\delta^2}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)\right)^2\right\| < 1 \tag{4.63}$$

is satisfied and $\tilde{A}_0$ is asymptotically stable, then $\tilde{A}(\alpha)$ is asymptotically stable $\forall\, \alpha \in \Omega$; and $\tilde{Q}^A$, the unique positive definite solution to (4.61) and (4.62), gives

$$J(G_c) \leq \mathcal{J}^A(G_c, \tilde{Q}^A) \tag{4.64}$$

Proof: From Proposition 3.3.4, Eq. (4.63) and stability of $\tilde{A}_0$ guarantees uniqueness and positive definiteness of the solution of (4.61) and (4.62). Eq. (4.63) also guarantees existence of the average cost $\forall\, \alpha \in \Omega$ since (4.63) implies (3.21). By Prop. 3.1.1 bounded average cost implies that the closed-loop systems, $G_{zw}(\alpha)$ are stable in the sense of Lyapunov $\forall\, \alpha \in \Omega$. □

We proceed now to the problem .f deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eqs. (4.61) and (4.62) to the cost using parameter independent, symmetric matrices of Lagrange multipliers, $\tilde{P}^A$ and $\tilde{P}_1 \in \mathbb{R}^{n\times n}$. The appended cost becomes.

$$\mathcal{J}^A(G_c) = \operatorname{tr}\left\{\tilde{Q}^A\tilde{C}^T\tilde{C}\right\}$$

$$+ \quad \mathrm{tr}\left\{\left[\tilde{A}_0\tilde{Q}^A + \tilde{Q}^A\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}_1 + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}_1\tilde{A}_i^T\right]\tilde{P}^A\right\}$$

$$+ \quad \mathrm{tr}\left\{\left[\tilde{A}_0\tilde{Q}_1 + \tilde{Q}_1\tilde{A}_0^T + \delta^2\tilde{Q}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^A\tilde{A}_i^T\right]\tilde{P}_1\right\} \tag{4.65}$$

where $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ are defined in Def. (4.1.2). Taking the derivatives with respect to $D_c, \tilde{P}^A$, $\tilde{P}_1$ and $\tilde{Q}^A$, $\tilde{Q}_1$ give the necessary conditions for optimization.

Theorem 4.1.5 (Necessary Conditions) *Suppose G_c solves the average bound minimization problem (4.1.6), then there exist matrices, $\tilde{Q}^A, \tilde{Q}_1$ and $\tilde{P}^A, \tilde{P}_1 \geq 0 \in \mathbb{R}^{n\times n}$ such that*

$$D_c = -\left(D_{12}^T D_{12} + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)B_{2_i}^T\left(\tilde{P}^A + \tilde{P}_1\right)B_{2_i}\right)^{-1}\left[B_{2_0}^T\left(\tilde{P}^A\tilde{Q}^A + \tilde{P}_1\tilde{Q}_1\right)\right.$$

$$+ \left.\sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)B_{2_i}^T\left(\tilde{P}^A A_i\tilde{Q}_1 + \tilde{P}_1 A_i\tilde{Q}^A\right)\right]C_2^T\left(C_2\tilde{Q}^A C_2^T\right)^{-1} \tag{4.66}$$

where $\tilde{Q}^A$ satisfies the bound equation

$$0 = \tilde{A}_0\tilde{Q}^A + \tilde{Q}^A\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}_1 + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}_1\tilde{A}_i^T \tag{4.67}$$

$$0 = \tilde{A}_0\tilde{Q}_1 + \tilde{Q}_1\tilde{A}_0^T + \delta^2\tilde{Q}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^A\tilde{A}_i^T \tag{4.68}$$

and $\tilde{P}^A$ satisfies the adjoint bound equation

$$0 = \tilde{A}_0^T\tilde{P}^A + \tilde{P}^A\tilde{A}_0 + \tilde{C}^T\tilde{C} + \delta^2\tilde{P}_1 + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i^T\tilde{P}_1\tilde{A}_i \tag{4.69}$$

$$0 = \tilde{A}_0^T\tilde{P}_1 + \tilde{P}_1\tilde{A}_0 + \delta^2\tilde{P}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i^T\tilde{P}^A\tilde{A}_i \tag{4.70}$$

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.65), with respect to $D_c, \tilde{P}^A$, $\tilde{P}_1$ and $\tilde{Q}^A$, $\tilde{Q}_1$. □

Remark 4.1.11 *In the case of full state feedback, $C_2 = I$. In this case, Eq. (4.66) is given by*

$$D_c = -\left(D_{12}^T D_{12} + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta}\right)B_{2_i}^T\left(\tilde{P}^A + \tilde{P}_1\right)B_{2_i}\right)^{-1}\left[B_{2_0}^T\left(\tilde{P}^A\tilde{Q}^A + \tilde{P}_1\tilde{Q}_1\right)\right.$$

$$+ \left.\sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta}\right)B_{2_i}^T\left(\tilde{P}^A A_i\tilde{Q}_1 + \tilde{P}_1 A_i\tilde{Q}^A\right)\right]\left(\tilde{Q}^A\right)^{-1} \tag{4.71}$$

Remark 4.1.12 *The traditional LQR results presented in Eqs. 4.26 and 4.27 are recovered in the case of no uncertainty and full state feedback.*

4.2 Dynamic Output Feedback Problems

In this section five dynamic output feedback problems will be investigated. They are:

(i) Exact average cost minimization

(ii) Perturbation expansion approximate average cost minimization

(iii) Bourret approximate average cost minimization

(iv) Worst-case bound minimization

(v) Average bound minimization

The problem and assumptions will be stated, followed by the necessary conditions and properties of the resulting compensators.

The sets of systems used for dynamic output feedback are presented in Chapter 2 as either $\mathcal{G}_g$ for the exact average cost minimization or $\mathcal{G}_s$ for the approximate average cost minimizations. For clarity these definitions are restated here.

Definition 4.2.1 (General Set of Systems) *The set $\mathcal{G}_g$ of systems is parameterized as follows:*

$$\mathcal{G}_g = \{G_g(\alpha) \,\forall\, \alpha \in \Omega\} \tag{4.72}$$

where $\Omega_g \subset \mathbb{R}^r$ is defined in Def. 2.2.1 and each element of the set is described in the state space as

$$G_g(\alpha) = \left[\begin{array}{c|cc} A(\alpha) & B_1(\alpha) & B_2(\alpha) \\ \hline C_1(\alpha) & 0 & D_{12}(\alpha) \\ C_2(\alpha) & D_{21}(\alpha) & 0 \end{array}\right] \tag{4.73}$$

where $A(\alpha) \in \mathbb{R}^{n\times n}$, $B_2(\alpha) \in \mathbb{R}^{n\times m}$, $C_2(\alpha) \in \mathbb{R}^{l\times n}$, $B_1(\alpha) \in \mathbb{R}^{n\times p}$, $C_1(\alpha) \in \mathbb{R}^{q\times n}$, $\forall\alpha \in \Omega_g$ *and teh elements of the matrices are continuous functions of the parameters over* Ω_g.

Definition 4.2.2 (Structured Set of Systems) *The set* $\mathcal{G}_s$ *of systems is parameterized as follows*

$$\mathcal{G}_s = \{G_s(\alpha)\ : \alpha \in \Omega_s\} \tag{4.74}$$

where Ω_s *is the structured set of parameter vectors defined in Def. 2.2.3 and each element of* $\mathcal{G}_s$ *is described in the state space as*

$$G_s(\alpha) = \left[\begin{array}{c|cc} A_0 + \sum_{i=1}^{r}\alpha_i A_i & B_1 & B_{2_0} + \sum_{i=1}^{r}\alpha_i B_{2_i} \\ \hline C_1 & 0 & D_{12} \\ C_{2_0} + \sum_{i=1}^{r}\alpha_i C_{2_i} & D_{21} & 0 \end{array}\right] \tag{4.75}$$

where for $i = 0,\dots,r;$ $A_i \in \mathbb{R}^{n\times n}, B_{2_i} \in \mathbb{R}^{n\times m}, C_{3_i} \in \mathbb{R}^{l\times n}$, *and* $B_1 \in \mathbb{R}^{n\times p} C_1 \in \mathbb{R}^{q\times n}$.

The general model set average performance problem is the basis of the other auxiliary minimization problems used to derive controllers in Sections 4.2.1 and 4.2.2.

Problem 4.2.1 (Model Set Average Performance Problem) *Given a set* $\mathcal{G}_g$ *or* $\mathcal{G}_s$ *of systems, determine the dynamic compensator or order,* n_c,

$$G_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{4.76}$$

which minimizes the the closed-loop $\mathcal{H}_2$*-norm averaged over the model set.*

$$J(G_c) \ = \ \int_{\Omega} \|G_{zw}(\alpha)\|_2^2 \, d\mu(\alpha) \tag{4.77}$$

The structures of the closed-loop systems for the various model sets, general or structured are presented in Eqs. (2.31) and (2.35). In the following sections, we will take a closer look at the calculation of the average $\mathcal{H}_2$-norm using the equations for the exact average cost and its approximations presented in Section 3.2 and bounds presented in Section 3.3.

4.2.1 Average Cost Minimization

In this section the formulation for the necessary conditions for the minimization of the exact average cost will be presented. The first step is to use the result of Proposition 3.1.1 to define the auxiliary minimization problem for the exact average cost.

Problem 4.2.2 (Auxiliary Minimization Problem) *Given the general set of systems described in Def. (2.2.2), determine the dynamic compensator of order n_c, given by*

$$G_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{4.78}$$

which minimizes

$$\mathcal{J}^E(G_c) = \operatorname{tr}\left\{\left\langle \tilde{Q}\,(\alpha)\,\tilde{C}^T(\alpha)\tilde{C}(\alpha)\right\rangle\right\} \tag{4.79}$$

subject to

$$0 = \tilde{A}(\alpha)\tilde{Q}\,(\alpha) + \tilde{Q}\,(\alpha)\,\tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha) \tag{4.80}$$

for each $\alpha \in \Omega$.

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the stability of the plant.

Proposition 4.2.1 *If $(G_c, \tilde{Q}\,(\alpha))$ satisfy Eq. (4.80), $\tilde{A}(\alpha)$ is decomposed into $\tilde{A}(\alpha) = \tilde{A}_0 + \tilde{A}_1(\alpha)$, and the norm constraint given by*

$$\left\|\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\tilde{A}_1(\alpha) \oplus \tilde{A}_1(\alpha)\right)\right\| < 1 \quad \forall\, \alpha\, \in\, \Omega \tag{4.81}$$

is satisfied, then $\tilde{A}(\alpha)$ is asymptotically stable $\forall\, \alpha\, \in\, \Omega$ and

$$J(G_c) = \mathcal{J}^E(G_c) \tag{4.82}$$

Proof: From Prop. 3.1.4, Eq. (4.81) implies stability over the set of systems. From Prop. 3.1.1, stability over the set of systems allows the average cost to be given by the average solution to a parameterized Lyapunov equation. □

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eq. (4.80) to the cost using a parameter dependent, symmetric matrix of Lagrange multipliers, $\tilde{P}(\alpha) \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$. The matrix of Lagrange multipliers must be parameter dependent because the appended equations are parameter dependent. The appended cost becomes

$$\begin{aligned}
\mathcal{J}^E(G_c) = & \\
& \operatorname{tr}\left\{\int_\Omega \tilde{Q}(\alpha)\,\tilde{C}^T(\alpha)\tilde{C}(\alpha)\,\mathrm{d}\mu(\alpha)\right\} \\
+ \; & \operatorname{tr}\left\{\int_\Omega \left[\tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\,\tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha)\right]\tilde{P}(\alpha)\;\mathrm{d}\mu(\alpha)\right\}
\end{aligned} \tag{4.83}$$

where $\tilde{A}(\alpha)$, $\tilde{B}(\alpha)$, and $\tilde{C}(\alpha)$ are defined in Def. (2.2.2). The necessary conditions for minimization of the exact average cost can now be stated by taking the derivatives with respect to G_c, $\tilde{P}(\alpha)$, and $\tilde{Q}(\alpha)$. A table of matrix derivatives can be found in Ref. [125].

Theorem 4.2.1 (Necessary Conditions) *Suppose G_c the dynamic compensator of order n_c, Eq. (4.78) solves the exact average cost minimization problem (4.2.2), then there exist matrices, $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha) \geq 0 \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ such that*

$$0 = \left\langle \tilde{P}_{21}(\alpha)\tilde{Q}_{12}(\alpha) + \tilde{P}_{22}(\alpha)\tilde{Q}_{22}(\alpha)\right\rangle \tag{4.84}$$

$$\begin{aligned}
0 = \; & \left\langle \tilde{P}_{22}(\alpha)B_c D_{21}(\alpha)D_{21}^T(\alpha)\right\rangle \\
+ \; & \left\langle \tilde{P}_{21}(\alpha)\tilde{Q}_{11}(\alpha)C_2^T(\alpha) + \tilde{P}_{22}(\alpha)\tilde{Q}_{21}(\alpha)C_2^T(\alpha)\right\rangle
\end{aligned} \tag{4.85}$$

$$\begin{aligned}
0 = \; & \left\langle D_{12}^T(\alpha)D_{12}(\alpha)C_c\tilde{Q}_{22}(\alpha)\right\rangle \\
+ \; & \left\langle B_2^T(\alpha)\tilde{P}_{11}(\alpha)\tilde{Q}_{12}(\alpha) + B_2^T(\alpha)\tilde{P}_{12}(\alpha)\tilde{Q}_{22}(\alpha)\right\rangle
\end{aligned} \tag{4.86}$$

where $\tilde{Q}(\alpha)$ satisfies the parameterized Lyapunov equation

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\,\tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha) \tag{4.87}$$

$\tilde{P}(\alpha)$ satisfies the Adjoint Lyapunov equation

$$0 = \tilde{A}^T(\alpha)\tilde{P}(\alpha) + \tilde{P}(\alpha)\,\tilde{A}(\alpha) + \tilde{C}^T(\alpha)\tilde{C}(\alpha) \tag{4.88}$$

and $\tilde{Q}(\alpha)$ *and* $\tilde{P}(\alpha)$ *are partitioned*

$$\tilde{Q}(\alpha) = \begin{bmatrix} \tilde{Q}_{11}(\alpha) & \tilde{Q}_{12}(\alpha) \\ \tilde{Q}_{21}(\alpha) & \tilde{Q}_{22}(\alpha) \end{bmatrix}, \quad \tilde{P}(\alpha) = \begin{bmatrix} \tilde{P}_{11}(\alpha) & \tilde{P}_{12}(\alpha) \\ \tilde{P}_{21}(\alpha) & \tilde{P}_{22}(\alpha) \end{bmatrix} \tag{4.89}$$

with $\tilde{Q}_{11}$, $\tilde{P}_{11} \in \mathbb{R}^{n \times n}$, *and* $\tilde{Q}_{22}$, $\tilde{P}_{22} \in \mathbb{R}^{n_c \times n_c}$.

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.83), with respect to $A_c, B_c, C_c, \tilde{P}(\alpha)$, and $\tilde{Q}(\alpha)$. Note that the necessary conditions that result from differentiation of the cost with respect to $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$, (4.88) and (4.87) respectively, are parameter dependent because $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$ are parameter dependent. □

Remark 4.2.1 *The traditional LQG results are recovered in the case of no uncertainty and* $n_c = n$.

The difficulty inherent in Eqs. (4.84)-(4.86) for the optimal gains is that that they involve the average of the product of the solution of two Lyapunov equations, $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$. These matrices are only given as implicit functions of α in Equations (4.87) and (4.88). Only in the simplest of cases can the average of the product be solved for exactly. In the next sections, the approximations and bounds to the average cost will be minimized in an attempt to approach the optimal solution by minimizing computable expressions for the cost.

4.2.2 Approximate Average Cost Minimization

Perturbation Expansion Approximate Cost Minimization

In this section the formulation of the necessary conditions for the minimization of the perturbation expansion approximate cost will be presented. The first step is to use the result of Proposition 3.2.1 to define the auxiliary minimization problem.

Problem 4.2.3 (Auxiliary Minimization Problem) *Given the set $\mathcal{G}_s$ of systems described in Def. 2.2.4 determine the dynamic compensator of order n_c*

$$G_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{4.90}$$

which minimizes

$$\mathcal{J}^P(G_c) \;=\; \mathrm{tr}\left\{\left(\tilde{Q}^0 + \tilde{Q}^P\right)\tilde{C}^T\tilde{C}\right\} \tag{4.91}$$

where the nominal cost, $\tilde{Q}^0$, and the parameter defendant cost, $\tilde{Q}^P$, are the unique positive definite solutions to the following system of Lyapunov equations:

$$0 \;=\; \tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T \tag{4.92}$$

$$0 \;=\; \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{4.93}$$

$$0 \;=\; \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right), \quad i = 1,\ldots,r \tag{4.94}$$

where σ_i is defined from Equation (3.47).

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.2.1 where the perturbation expansion equation is derived as an approximate solution for the exact average cost.

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eqs. (4.92)-(4.94) to the cost using parameter independent, symmetric matrices of Lagrange multipliers, $\tilde{P}^0$, $\tilde{P}^P$, and $\tilde{P}^i$, $i = 1\ldots r \;\in \mathbb{R}^{\tilde{n}\times\tilde{n}}$.

$$\begin{aligned}
\mathcal{J}^P(G_c) \;=\;& \mathrm{tr}\left\{\left(\tilde{Q}^0 + \tilde{Q}^P\right)\tilde{C}^T\tilde{C}\right\} \\
+\;& \mathrm{tr}\left\{\left[\tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T\right]\tilde{P}^0\right\} \\
+\;& \mathrm{tr}\left\{\left[\tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right)\right]\tilde{P}^P\right\} \\
+\;& \mathrm{tr}\left\{\sum_{i=1}^{r}\left[\tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right)\right]\tilde{P}^i\right\}
\end{aligned} \tag{4.95}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are defined in Def. (2.2.4). Taking the derivatives with respect to $G_c, \tilde{P}^0$, $\tilde{P}^P$, $\tilde{P}^i$ and $\tilde{Q}^0$, $\tilde{Q}^P$, $\tilde{Q}^i$ gives the necessary conditions for minimization of the perturbation expansion approximation to the exact average cost.

Theorem 4.2.2 (Necessary Conditions) *Suppose G_c the dynamic compensator or order n_c, Eq. (4.78), solves the perturbation expansion approximate cost minimization problem (4.2.3), then there exist matrices, $\tilde{P}^0$, $\tilde{P}^P$, $\tilde{P}^i$ and $\tilde{Q}^0$, $\tilde{Q}^P$, $\tilde{Q}^i \geq 0 \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ such that*

$$\begin{aligned}
0 &= \tilde{P}^0_{21}\tilde{Q}^0_{12} + \tilde{P}^0_{22}\tilde{Q}^0_{22} + \tilde{P}^P_{21}\tilde{Q}^P_{12} + \tilde{P}^P_{22}\tilde{Q}^P_{22} \\
&+ \sum_{i=1}^{r} \tilde{P}^i_{21}\tilde{Q}^i_{12} + \tilde{P}^i_{22}\tilde{Q}^i_{22} \qquad (4.96) \\
B_c &= -\tilde{P}^{0\,-1}_{22}\Big[\Big(\tilde{P}^0_{21}\tilde{Q}^0_{11} + \tilde{P}^0_{22}\tilde{Q}^0_{21} + \tilde{P}^P_{21}\tilde{Q}^P_{11} + \tilde{P}^P_{22}\tilde{Q}^P_{21}\Big)C^T_{2_0} \\
&+ \sum_{i=1}^{r}\Big(\Big(\tilde{P}^i_{21}\tilde{Q}^i_{11} + \tilde{P}^i_{22}\tilde{Q}^i_{21}\Big)C^T_{2_0} \\
&+ \Big(\tilde{P}^P_{21}\tilde{Q}^i_{11} + \tilde{P}^i_{21}\tilde{Q}^0_{11} + \tilde{P}^P_{22}\tilde{Q}^i_{21} + \tilde{P}^i_{22}\tilde{Q}^0_{21}\Big)\sigma_i C^T_{2_i}\Big)\Big]\Big(D_{21}D^T_{21}\Big)^{-1} \qquad (4.97) \\
C_c &= -\Big(D^T_{12}D_{12}\Big)^{-1}\Big[B^T_{2_0}\Big(\tilde{P}^0_{11}\tilde{Q}^0_{12} + \tilde{P}^0_{12}\tilde{Q}^0_{22} + \tilde{P}^P_{11}\tilde{Q}^P_{12} + \tilde{P}^P_{12}\tilde{Q}^P_{22}\Big) \\
&+ \sum_{i=1}^{r}\Big(B^T_{2_0}\Big(\tilde{P}^i_{11}\tilde{Q}^i_{12} + \tilde{P}^i_{12}\tilde{Q}^i_{22}\Big) \\
&+ \sigma_i B^T_{2_i}\Big(\tilde{P}^P_{11}\tilde{Q}^i_{12} + \tilde{P}^i_{11}\tilde{Q}^0_{12} + \tilde{P}^P_{12}\tilde{Q}^i_{22} + \tilde{P}^i_{12}\tilde{Q}^0_{22}\Big)\Big)\Big]\Big(\tilde{Q}^0_{22} + \tilde{P}^P_{22}\Big)^{-1} \qquad (4.98)
\end{aligned}$$

where $\tilde{Q}^0$, $\tilde{Q}^P$, and *$\tilde{Q}^i$ satisfy the coupled Lyapunov equations*

$$\begin{aligned}
0 &= \tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T \\
0 &= \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \sum_{i=1}^{r}\sigma_i\Big(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\Big) \\
0 &= \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\Big(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\Big) \quad i = 1,\ldots,r \qquad (4.99)
\end{aligned}$$

and $\tilde{P}^0$, $\tilde{P}^P$, and *$\tilde{P}^i$ satisfy the coupled adjoint Lyapunov equations*

$$\begin{aligned}
0 &= \tilde{A}_0^T\tilde{P}^P + \tilde{P}^P\tilde{A}_0 + \tilde{C}^T\tilde{C} \\
0 &= \tilde{A}_0^T\tilde{P}^0 + \tilde{P}^0\tilde{A}_0 + \tilde{C}^T\tilde{C} + \sum_{i=1}^{r}\sigma_i\Big(\tilde{A}_i^T\tilde{P}^i + \tilde{P}^i\tilde{A}_i\Big) \\
0 &= \tilde{A}_0^T\tilde{P}^i + \tilde{P}^i\tilde{A}_0 + \sigma_i\Big(\tilde{A}_i^T\tilde{Q}^P + \tilde{Q}^P\tilde{A}_i\Big) \quad i = 1,\ldots,r \qquad (4.100)
\end{aligned}$$

and the $\tilde{Q}$ and $\tilde{P}$ matrices have been partitioned according to Eq. (4.89).

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.95), with respect to $A_c, B_c, C_c, \tilde{P}^0, \tilde{P}^P, \tilde{P}^i$ and $\tilde{Q}^0, \tilde{Q}^P, \tilde{Q}^i$. □

Remark 4.2.2 *The traditional LQG results are recovered in the case of no uncertainty.*

Bourret Approximate Cost Minimization

In this section the formulation for the necessary conditions for the minimization of the Bourret approximate cost will be presented. The first step is to use the result of Proposition 3.2.2 to define the auxiliary minimization problem.

Problem 4.2.4 (Auxiliary Minimization Problem) *Given the set $\mathcal{G}_s$ of systems defined in Def. 2.2.4, determine the dynamic compensator or order, n_c,*

$$G_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{4.101}$$

which minimizes

$$\mathcal{J}^B(G_c) \;=\; \operatorname{tr}\left\{\tilde{Q}^B\tilde{C}^T\tilde{C}\right\} \tag{4.102}$$

where $\tilde{Q}^B$ is the unique positive definite solutions to the following system of coupled Lyapunov equations:

$$0 \;=\; \tilde{A}_0\tilde{Q}^B + \tilde{Q}^B\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{4.103}$$

$$0 \;=\; \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}_i^T\right) \quad i = 1,\ldots,r \tag{4.104}$$

and σ_i is defined from Equation (3.47).

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.2.2 relating the solution of the Bourret equation to the Exact Average solution. It will be restated here for clarity.

Proposition 4.2.2 *if $(G_c, \tilde{Q}^B)$ satisfies Eqs. (4.103) and (4.104) and the norm constraint given in Proposition 3.2.3 as*

$$\left\| \sum_{i=1}^{r} \left(\sigma_i \left(\tilde{A}_0 \oplus \tilde{A}_0 \right)^{-1} \left(\tilde{A}_i \oplus \tilde{A}_i \right) \right)^2 \right\| < 1 \tag{4.105}$$

is satisfied, then $\tilde{A}(\alpha)$ is almost always stable $\forall\, \alpha \in \hat{\Omega}$ where $\hat{\Omega}$ is defined

$$\hat{\Omega} = \left\{ \alpha \ : \sum_{i=1}^{r} |\alpha_i| \left\| \left(\tilde{A}_0 \oplus \tilde{A}_0 \right)^{-1} \left(\tilde{A}_i \oplus \tilde{A}_i \right) \right\| \leq \left\| \sum_{i=1}^{r} \sigma_i^2 \left(\left(\tilde{A}_0 \oplus \tilde{A}_0 \right)^{-1} \left(\tilde{A}_i \oplus \tilde{A}_i \right) \right)^2 \right\| \right\} \tag{4.106}$$

In addition, the unique positive definite solution to (4.103)-(4.104), $\tilde{Q}^B$, gives

$$J(G_c) \cong \mathcal{J}^B(G_c, \tilde{Q}) \tag{4.107}$$

Proof: From Proposition 3.2.3, Eq. (4.105) guarantees uniqueness and positive definiteness of the solution of (4.103)-(4.104). Eq. (4.105) also guarantees closed-loop stability $\forall\, \alpha \in \hat{\Omega}$ by Prop. 3.2.1. □

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eqs. (4.103) and (4.104) to the cost using parameter independent, symmetric matrices of Lagrange multipliers, $\tilde{P}^B$ and $\tilde{P}^i$, $i = 1 \ldots r \in \mathbb{R}^{\tilde{n} \mathrm{x} tilde n}$.

$$\begin{aligned} \mathcal{J}^B(G_c) \;=\; & \operatorname{tr}\left\{ \tilde{Q}^B \tilde{C}^T \tilde{C} \right\} + \\ & \operatorname{tr}\left\{ \left[\tilde{A}_0 \tilde{Q}^B + \tilde{Q}^B \tilde{A}_0^T + \tilde{B} \tilde{B}^T + \sum_{i=1}^{r} \sigma_i \left(\tilde{A}_i \tilde{Q}^i + \tilde{Q}^i \tilde{A}_i^T \right) \right] \tilde{P}^B \right\} + \\ & \operatorname{tr}\left\{ \sum_{i=1}^{r} \left[\tilde{A}_0 \tilde{Q}^i + \tilde{Q}^i \tilde{A}_0^T + \sigma_i \left(\tilde{A}_i \tilde{Q}^B + \tilde{Q}^B \tilde{A}_i^T \right) \right] \tilde{P}^i \right\} \end{aligned} \tag{4.108}$$

where $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ are defined in Def. (2.2.4). Taking the derivatives with respect to $G_c, \tilde{P}^B$, $\tilde{P}^i$ and $\tilde{Q}^B$, $\tilde{Q}^i$ gives the necessary conditions for minimization of the Bourret approximation to the exact average.

Theorem 4.2.3 (Necessary Conditions) *Suppose G_c the dynamic compensator of order n_c, Eq. (4.78) solves the Bourret approximate cost minimization problem*

(4.2.4), then there exist matrices, $\tilde{P}^B$, $\tilde{P}^i$ and $\tilde{Q}^B$, $\tilde{Q}^i \geq 0 \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ such that

$$0 = \tilde{P}^B_{21}\tilde{Q}^B_{12} + \tilde{P}^B_{22}\tilde{Q}^B_{22} + \sum_{i=1}^{r}\tilde{P}^i_{21}\tilde{Q}^i_{12} + \tilde{P}^i_{22}\tilde{Q}^i_{22} \tag{4.109}$$

$$\begin{aligned} B_c &= -\tilde{P}^{B^{-1}}_{22}\Big[\left(\tilde{P}^B_{21}\tilde{Q}^B_{11} + \tilde{P}^B_{22}\tilde{Q}^B_{21}\right)C^T_{2_0} \\ &+ \sum_{i=1}^{r}\Big(\left(\tilde{P}^i_{21}\tilde{Q}^i_{11} + \tilde{P}^i_{22}\tilde{Q}^i_{21}\right)C^T_{2_0} \\ &+ \left(\tilde{P}^B_{21}\tilde{Q}^i_{11} + \tilde{P}^i_{21}\tilde{Q}^B_{11} + \tilde{P}^B_{22}\tilde{Q}^i_{21} + \tilde{P}^i_{22}\tilde{Q}^B_{21}\right)\sigma_i C^T_{2_i}\Big)\Big]\left(D_{21}D^T_{21}\right)^{-1} \end{aligned} \tag{4.110}$$

$$\begin{aligned} C_c &= -\left(D^T_{12}D_{12}\right)^{-1}\Big[B^T_{2_0}\left(\tilde{P}^B_{11}\tilde{Q}^B_{12} + \tilde{P}^B_{12}\tilde{Q}^B_{22}\right) \\ &+ \sum_{i=1}^{r}\Big(B^T_{2_0}\left(\tilde{P}^i_{11}\tilde{Q}^i_{12} + \tilde{P}^i_{12}\tilde{Q}^i_{22}\right) \\ &+ \sigma_i B^T_{2_i}\left(\tilde{P}^B_{11}\tilde{Q}^i_{12} + \tilde{P}^i_{11}\tilde{Q}^B_{12} + \tilde{P}^B_{12}\tilde{Q}^i_{22} + \tilde{P}^i_{12}\tilde{Q}^B_{22}\right)\Big)\Big]\tilde{Q}^{B^{-1}}_{22} \end{aligned} \tag{4.111}$$

where $\tilde{Q}^B$ and $\tilde{Q}^i$ *satisfy the Bourret equation*

$$\begin{aligned} 0 &= \tilde{A}_0\tilde{Q}^B + \tilde{Q}^B\tilde{A}^T_0 + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}^T_i\right) \\ 0 &= \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}^T_0 + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}^T_i\right) \quad i = 1,\ldots,r \end{aligned} \tag{4.112}$$

and $\tilde{P}^B$ and $\tilde{P}^i$ *satisfy the adjoint Bourret equation*

$$\begin{aligned} 0 &= \tilde{A}^T_0\tilde{P}^B + \tilde{P}^B\tilde{A}_0 + \tilde{C}^T\tilde{C} + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}^T_i\tilde{P}^i + \tilde{P}^i\tilde{A}_i\right) \\ 0 &= \tilde{A}^T_0\tilde{P}^i + \tilde{P}^i\tilde{A}_0 + \sigma_i\left(\tilde{A}^T_i\tilde{P}^B + \tilde{P}^B\tilde{A}_i\right) \quad i = 1,\ldots,r \end{aligned} \tag{4.113}$$

and the $\tilde{Q}$ and $\tilde{P}$ matrices have been partitioned according to Eq. (4.89).

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.108), with respect to $A_c, B_c, C_c, \tilde{P}^B, \tilde{P}^i$ and $\tilde{Q}^B, \tilde{Q}^i$. □

Remark 4.2.3 *The traditional LQG results are recovered in the case of no uncertainty and $n_c = n$.*

Now that the conditions for existence of local minima to the exact average and approximate average cost have been presented, the necessary conditions for bound-based cost minimization will be presented.

4.2.3 Bound Minimization

Worst-Case Bound Minimization

In this section the formulation for the necessary conditions for the minimization of the worst-case bound will be presented. The first step is to use the result of Theorem 3.3.1 to define the auxiliary minimization problem.

Problem 4.2.5 (Auxiliary Minimization Problem) *Given a set $\mathcal{G}_s$ of systems described in Def. 2.2.4, determine the dynamic feedback compensator G_c, Eq. (4.78), which minimizes*

$$\mathcal{J}^W(G_c) = \mathrm{tr}\left\{\tilde{Q}^W\tilde{C}^T\tilde{C}\right\} \tag{4.114}$$

where $\tilde{Q}^W$ is the unique positive definite solution to the following system of Lyapunov equations

$$0 = \tilde{A}_0\tilde{Q}^W + \tilde{Q}^W\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^W\tilde{A}_i^T \tag{4.115}$$

where δ_i is defined from Equation (3.78) and $\delta \in \mathbb{R}$.

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.3.1 relating the solution of the worst-case bound to the Exact Average solution. It will be restated here for clarity.

Proposition 4.2.3 *If the norm constraint given in Proposition 3.3.3*

$$\left\|\left(\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1}\left(\delta^2 I + \sum_{i=1}^{r}\tfrac{\delta_i^2}{\delta^2}\left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)\right)\right\| < 1 \tag{4.116}$$

is satisfied and $\tilde{A}_0$ is asymptotically stable, then $\tilde{A}(\alpha)$ is asymptotically stable $\forall\ \alpha \in \Omega$; and $\tilde{Q}^W$, the unique positive definite solution to (4.115), gives

$$J(G_c) \leq \mathcal{J}^W(G_c, \tilde{Q}^W) \tag{4.117}$$

Proof: From Proposition 3.3.3, Eq. (4.116) and stability of $\tilde{A}_0$ guarantees uniqueness and positive definiteness of the solution of (4.115). Eq. (4.116) also guarantees existence of the average cost $\forall\, \alpha \in \Omega$ since (4.116) implies (3.21). By Prop. 3.1.1 bounded average cost implies that the closed-loop systems, $G_{zw}(\alpha)$ are stable in the sense of Lyapunov $\forall\, \alpha \in \Omega$. □

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eq. (4.115) to the cost using a parameter independent, symmetric matrix of Lagrange multipliers, $\tilde{P}^W \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$. The appended cost becomes.

$$\begin{aligned}\mathcal{J}^W(G_c) &= \operatorname{tr}\left\{\tilde{Q}^W\tilde{C}^T\tilde{C}\right\} \\ &+ \operatorname{tr}\left\{\left[\tilde{A}_0\tilde{Q}^W + \tilde{Q}^W\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta\tilde{Q}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta}\right)\tilde{A}_i\tilde{Q}^W\tilde{A}_i^T\right]\tilde{P}^W\right\}\end{aligned} \tag{4.118}$$

where $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ are defined in Def. (2.2.4). Taking the derivatives with respect to $D_c, \tilde{P}^W$ and $\tilde{Q}^W$ gives the necessary conditions for optimization.

Theorem 4.2.4 (Necessary Conditions) *Suppose G_c solves the average bound minimization problem (4.2.5), then there exist matrices, $\tilde{Q}^W$ and $\tilde{P}^W \geq 0 \in \mathbb{R}^{n\times n}$ such that*

$$0 = \tilde{P}_{21}^W\tilde{Q}_{12}^W + \tilde{P}_{22}^W\tilde{Q}_{22}^W \tag{4.119}$$

$$\begin{aligned}B_c &= -\tilde{P}_{22}^{W^{-1}}\left[\left(\tilde{P}_{21}^W\tilde{Q}_{11}^W + \tilde{P}_{22}^W\tilde{Q}_{21}^W\right)C_{2_0}^T + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\left(\tilde{P}_{21}^W A_i\tilde{Q}_{11}^W + \tilde{P}_{21}^W B_{2_i}C_c\tilde{Q}_{21}^W\right)C_{2_i}^T\right] \\ &\left(D_{21}D_{21}^T + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)C_{2_i}\tilde{Q}_{11}^W C_{2_i}^T\right)^{-1}\end{aligned} \tag{4.120}$$

$$\begin{aligned}C_c &= -\left(D_{12}^T D_{12} + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)B_{2_i}^T\tilde{P}_{11}^W B_{2_i}\right)^{-1}\left[B_{2_0}^T\left(\tilde{P}_{11}^W\tilde{Q}_{12}^W + \tilde{P}_{12}^W\tilde{Q}_{22}^W\right)\right. \\ &+ \left.\sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)B_{2_i}^T\left(\tilde{P}_{11}^W A_i\tilde{Q}_{12}^W + \tilde{P}_{12}^W B_c C_{2_i}\tilde{Q}_{12}^W\right)\right]\tilde{Q}_{22}^{W^{-1}}\end{aligned} \tag{4.121}$$

where $\tilde{Q}^W$ satisfies the worst-case bound equation

$$0 = \tilde{A}_0\tilde{Q}^W + \tilde{Q}^W\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^W\tilde{A}_i^T \tag{4.122}$$

and $\tilde{P}^W$ satisfies the adjoint worst-case bound equation

$$0 = \tilde{A}_0^T \tilde{P}^W + \tilde{P}^W \tilde{A}_0 + \tilde{C}^T \tilde{C} + \delta^2 \tilde{P}^W + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) \tilde{A}_i^T \tilde{P}^W \tilde{A}_i \tag{4.123}$$

and $\tilde{P}^W$ and $\tilde{Q}^W$ are partitioned as in Eq. (4.89)

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.118), with respect to $A_c, B_c, C_c, \tilde{P}^W$ and $\tilde{Q}^W$. □

Remark 4.2.4 *The traditional LQG results are recovered in the case of no uncertainty and full state feedback.*

Average Bound Minimization

In this section the formulation for the necessary conditions for the minimization of the average bound will be presented. The first step is to use the result of Proposition 3.3.3 to define the auxiliary minimization problem.

Problem 4.2.6 (Auxiliary Minimization Problem) *Given a set $\mathcal{G}_s$ of systems described above in Def. 2.2.4, determine the static feedback compensator G_c, Eq. (4.78), which minimizes*

$$\mathcal{J}^A(G_c, \tilde{Q}^A) = \mathrm{tr}\left\{\tilde{Q}^A \tilde{C}^T \tilde{C}\right\} \tag{4.124}$$

where $\tilde{Q}^A$ is the unique positive definite solutions to the following system of Lyapunov equations

$$0 = \tilde{A}_0 \tilde{Q}^A + \tilde{Q}^A \tilde{A}_0^T + + \tilde{B}\tilde{B}^T + \delta^2 \tilde{Q}_1 + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) \tilde{A}_i \tilde{Q}_1 \tilde{A}_i^T \tag{4.125}$$

$$0 = \tilde{A}_0 \tilde{Q}_1 + \tilde{Q}_1 \tilde{A}_0^T + \delta^2 \tilde{Q}^A + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2}\right) \tilde{A}_i \tilde{Q}^A \tilde{A}_i^T \tag{4.126}$$

and δ_i is defined from Eq. (3.78) and $\delta \in \mathbb{R}$.

The relation between the Auxiliary Minimization Problem and the Average Performance Problem is based on the results of Section 3.3 relating the solution of the bound equation to the exact average solution. It will be restated here for clarity.

Proposition 4.2.4 *If the norm constraint given in Proposition 3.3.4*

$$\left\| \left(\left(\tilde{A}_0 \oplus \tilde{A}_0\right)^{-1} \left(\delta^2 I + \sum_{i=1}^{r} \tfrac{\delta_i^2}{\delta^2} \left(\tilde{A}_i \oplus \tilde{A}_i\right)\right)\right)^2 \right\| < 1 \tag{4.127}$$

is satisfied and $\tilde{A}_0$ is asymptotically stable, then $\tilde{A}(\alpha)$ is asymptotically stable $\forall\, \alpha \in \Omega$; and $\tilde{Q}^A$, the unique positive definite solution to (4.125) and (4.126), gives

$$J(G_c) \leq \mathcal{J}^A(G_c, \tilde{Q}^A) \tag{4.128}$$

Proof: From Proposition 3.3.4, Eq. (4.127) and stability of $\tilde{A}_0$ guarantees uniqueness and positive definiteness of the solution of (4.125) and (4.126). Eq. (4.127) also guarantees existence of the average cost $\forall\, \alpha \in \Omega$ since (4.127) implies (3.21). By Prop. 3.1.1 bounded average cost implies that the closed-loop systems, $G_{zw}(\alpha)$ are stable in the sense of Lyapunov $\forall\, \alpha \in \Omega$. □

We proceed now to the problem of deriving necessary conditions for the Auxiliary Minimization Problem. The first step is to append Eqs. (4.125) and (4.126) to the cost using parameter independent, symmetric matrices of Lagrange multipliers, $\tilde{P}^A$ and $\tilde{P}_1 \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$. The appended cost becomes.

$$\begin{aligned} \mathcal{J}^A(G_c) &= \operatorname{tr}\left\{\tilde{Q}^A \tilde{C}^T \tilde{C}\right\} \\ &+ \operatorname{tr}\left\{\left[\tilde{A}_0\tilde{Q}^A + \tilde{Q}^A\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta\tilde{Q}_1 + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta}\right)\tilde{A}_i\tilde{Q}_1\tilde{A}_i^T\right]\tilde{P}^A\right\} \\ &+ \operatorname{tr}\left\{\left[\tilde{A}_0\tilde{Q}_1 + \tilde{Q}_1\tilde{A}_0^T + \delta\tilde{Q}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta}\right)\tilde{A}_i\tilde{Q}^A\tilde{A}_i^T\right]\tilde{P}_1\right\} \end{aligned} \tag{4.129}$$

where $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ are defined in Def. (2.2.4). Taking the derivatives with respect to $D_c, \tilde{P}^A$, $\tilde{P}_1$ and $\tilde{Q}^A$, $\tilde{Q}_1$ give the necessary conditions for optimization.

Theorem 4.2.5 (Necessary Conditions) *Suppose G_c solves the average bound minimization problem (4.2.6), then there exist matrices, $\tilde{Q}^A, \tilde{Q}_1$ and $\tilde{P}^A, \tilde{P}_1 \geq 0 \in \mathbb{R}^{n+n_c \times n+n_c}$ such that*

$$0 = \tilde{P}_{21}^A \tilde{Q}_{12}^A + \tilde{P}_{22}^A \tilde{Q}_{22}^A + \tilde{P}_{1_{21}} \tilde{Q}_{1_{12}} + \tilde{P}_{1_{22}} \tilde{Q}_{1_{22}} \tag{4.130}$$

$$
\begin{aligned}
0 &= \tilde{P}^A_{22} B_c D_{21} D^T_{21} + \left(\tilde{P}^A_{21} \tilde{Q}^A_{11} + \tilde{P}^A_{22} \tilde{Q}^A_{21} + \tilde{P}_{1_{21}} \tilde{Q}_{1_{11}} + \tilde{P}_{1_{22}} \tilde{Q}_{1_{21}} \right) C^T_{2_0} \\
&+ \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2} \right) \left[\left(\tilde{P}^A_{22} B_c C_{2_i} \tilde{Q}_{1_{11}} + \tilde{P}_{1_{22}} B_c C_{2_i} \tilde{Q}^A_{11} \right) \right. \\
&+ \left. \left(\tilde{P}^A_{21} \left(A_i \tilde{Q}_{1_{11}} + B_{2_i} C_c \tilde{Q}_{1_{21}} \right) + \tilde{P}_{1_{21}} \left(A_i \tilde{Q}^A_{11} + B_{2_i} C_c \tilde{Q}^A_{21} \right) \right) \right] C^T_{2_i} \qquad (4.131) \\
0 &= D^T_{12} D_{12} C_c \tilde{Q}^A_{22} + B^T_{2_0} \left(\tilde{P}^A_{11} \tilde{Q}^A_{12} + \tilde{P}^A_{12} \tilde{Q}^A_{22} + \tilde{P}_{1_{11}} \tilde{Q}_{1_{12}} + \tilde{P}_{1_{12}} \tilde{Q}_{1_{22}} \right) \\
&+ \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta^2} \right) B^T_{2_i} \left[\left(\tilde{P}^A_{11} B_{2_i} C_c \tilde{Q}_{1_{22}} + \tilde{P}_{1_{11}} B_{2_i} C_c \tilde{Q}^A_{22} \right) \right. \\
&+ \left. \left(\left(\tilde{P}^A_{11} A_i + \tilde{P}^A_{12} B_c C_{2_i} \right) \tilde{Q}_{1_{12}} + \left(\tilde{P}_{1_{11}} A_i + \tilde{P}_{1_{12}} B_c C_{2_i} \right) \tilde{Q}^A_{12} \right) \right] \qquad (4.132) \\
& \qquad (4.133)
\end{aligned}
$$

where $\tilde{Q}^A$ satisfies the bound equation

$$
0 = \tilde{A}_0 \tilde{Q}^A + \tilde{Q}^A \tilde{A}_0^T + \tilde{B} \tilde{B}^T + \delta \tilde{Q}_1 + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta} \right) \tilde{A}_i \tilde{Q}_1 \tilde{A}_i^T \qquad (4.134)
$$

$$
0 = \tilde{A}_0 \tilde{Q}_1 + \tilde{Q}_1 \tilde{A}_0^T + \delta \tilde{Q}^A + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta} \right) \tilde{A}_i \tilde{Q}^A \tilde{A}_i^T \qquad (4.135)
$$

and $\tilde{P}^A$ satisfies the adjoint bound equation

$$
0 = \tilde{A}_0^T \tilde{P}^A + \tilde{P}^A \tilde{A}_0 + \tilde{C}^T \tilde{C} + \delta \tilde{P}_1 + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta} \right) \tilde{A}_i^T \tilde{P}_1 \tilde{A}_i \qquad (4.136)
$$

$$
0 = \tilde{A}_0^T \tilde{P}_1 + \tilde{P}_1 \tilde{A}_0 + \delta \tilde{P}^A + \sum_{i=1}^{r} \left(\frac{\delta_i^2}{\delta} \right) \tilde{A}_i^T \tilde{P}^A \tilde{A}_i \qquad (4.137)
$$

and the $\tilde{P}^A$ and $\tilde{Q}^A$ matrices are partitioned as in Eq. (4.89).

Proof: The proof is a direct consequence of the differentiation of the cost, Eq. (4.129), with respect to $A_c, B_c, D_c, \tilde{P}^A$, $\tilde{P}_1$ and $\tilde{Q}^A$, $\tilde{Q}_1$. □

Remark 4.2.5 *The traditional LQG results are recovered in the case of no uncertainty.*

4.3 Solution of the Necessary Conditions

In the previous sections five different cost functionals were considered for static and dynamic output feedback compensation. Necessary conditions were derived for each of the associated minimization problems. In this section the techniques used to compute controllers based on the five cost functionals will be presented. The general technique used for computing the minimum cost controllers is parameter optimization. Since the controllers are fixed-form, the optimal controller can be found by minimizing the cost with respect to each of the parameters in the controller matrices. It should be noted that the parameter minimization is non-convex and the resulting minima can only be considered local minima. With this caveat, the discussion of the numerical minimization technique can be begun. The first step of deriving the necessary conditions has been accomplished. In general, finding analytical expressions for the controller matrices which satisfy the necessary conditions is impossible due to the complexity of the necessary conditions. In the following sections numerical methods for obtaining the optimal controller parameters will be presented.

The problem of obtaining the controller parameters can be divided into two distinct computational processes. The first process, common to all of the problems considered, is the general minimization scheme which uses the necessary conditions to guide its search for the optimal parameters. In this work, this minimization scheme is embedded within another process which gradually increases the amount of uncertainty considered in the minimization problem. The details of this *homotopic continuation* method will be discussed in more detail in the next section. The second computational process is the procedure for computing the gradient of the respective costs using the necessary conditions. Typically in control design this process is trivial once the necessary conditions have been derived. It usually involves solving uncoupled Lyapunov equations. For the average-related costs, however, the problem of computing the costs and gradients is non-trivial since even the cost computation involves solving systems of coupled Lyapunov equations. The details of the solution procedures for the gradients of the respective problems will be presented in Section 4.3.2.

4.3.1 General Algorithm

The general algorithm used to compute the controller matrices is parameter optimization. The closed-loop cost is a function of the controller parameters and can be minimized by appropriate choice of these. The cost can be any of the five considered: the exact average, the approximations, or the bounds. The gradient of the cost with respect to the controller parameters is given by the necessary conditions derived in the previous sections. These gradients can be used in standard numerical optimization routines, such as a quasi-Newton minimization, to find the optimal controllers. In general the conditions for optimality for each of the problems considered previously can be expressed as

$$\frac{\partial J(q)}{\partial q} = g(q) = 0 \tag{4.138}$$

where q is the vector of controller parameters and g is the vector function which gives the gradients of the cost with respect to the controller parameters. The quasi-Newton minimization scheme seeks the solution of Eq. (4.138).

The numerical minimization is complicated by the fact that it may be non-convex, i.e., there can be many local minima and correspondingly many possible solutions, x_i, such that $g(x_i) = 0$. The solution will therefore be a function of the initial guess used in the optimization. This initial guess must also be a stabilizing compensator. This can be difficult to find for large values of uncertainty. These problems are overcome by first assuming little or no uncertainty and using the resulting controller as a starting point for calculating controllers at successively larger values of uncertainty, i.e, larger parameter sets. Standard LQR or LQG techniques can be used to find stabilizing compensators for systems with no uncertainty. The amount of uncertainty used in the design is gradually increased until the desired amount is reached.

This solution technique is known as homotopic continuation and has been applied to the solution of coupled systems of Riccati and Lyapunov equations in Ref. [95]. If the set of parameters used in the design is described

$$\Omega_\gamma = \left\{\alpha \ : \alpha \in \mathbb{R}^r, \gamma\delta_i^L \leq \alpha_i \leq \gamma\delta_i^U \ \ i = 1, \cdots, r\right\} \tag{4.139}$$

where δ_i^L and δ_i^U are the lower and upper bounds for the i^{th} uncertain parameter and γ is a positive real scaling parameter, then the optimal solution is a function of the value of γ used in the design. The scaling parameter, γ is known as the *homotopic parameter.* The locus of optimal solutions as a function of γ is called the *solution manifold*, denoted by

$$\mathcal{M} = \{q^*(\gamma) \ : g[q^*(\gamma)] = 0\} \tag{4.140}$$

The homotopy essentially finds elements of this solution manifold progressively as the uncertainty bounds are increased.

Definition 4.3.1 (Controller Solution Algorithm) *The general algorithm used to compute the controllers can be written.*

(i) **Initialize** *the homotopy with a stabilizing compensator (with parameters denoted q_0) for the system with $\gamma = 0$, i.e., no uncertainty.*

(ii) **Increment** *the value of $\gamma_i = \gamma_{i-1} + \delta\gamma$ and thereby increase the size of the uncertainty used in the design.*

(iii) *(Optional)***Reduce** *the number of uncertain parameters to be retained in the minimization process using the algorithm discussed in Section 6.3.*

(iv) **Minimize** *the cost at the new value of γ to derive a new compensator, $x_i(\gamma_i)$ using a Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton scheme [144].*

(v) **Evaluate** *the resulting compensator to 1) check the set of retained parameters and 2) check the homotopy termination conditions.*

(vi) **Iterate**

The homotopy is discrete because discrete steps are taken in the value of the homotopic parameter. In the following paragraphs these steps will be discussed in more detail.

The homotopy is begun by assuming no uncertainty in the plant ($\gamma = 0$) and assuming an initial stabilizing compensator. In the case of full state static compensation, the standard LQR gain can be used. For static output feedback, the homotopy can be started with the feedback compensator gains described in [126, 127]. For dynamic full order compensation, the LQG compensator can be used. If the compensator is of reduced order, optimal projection or a heuristic compensator reduction procedure can be used to find stabilizing compensators. If the compensator initially doesn't minimize the $\mathcal{H}_2$-norm of the closed-loop system, then an initial minimization step can be performed (with $\gamma = 0$) to find the optimal compensator for the parameter independent plant.

Once the initial guess at the compensator has been made, a small amount of uncertainty is introduced into the problem and a new controller is found by minimization of the appropriate cost functional starting from the initial guess. The problem considered here is how to determine the step size. If the amount of uncertainty is increased too much in the step the initial guess will not be near the new optimal solution and may be difficult to locate. Since at each step a new cost minimization is performed, taking too small of a step is computationally wasteful. One useful measure of degree to which a given compensator is nonoptimal is the norm of the gradient vector, $\|g(x,\gamma)\|$. If the compensator is optimal for a given value of the homotopic parameter, γ, then the gradient is exactly zero. As the uncertainty is increased, the previous optimal solution no longer satisfies the necessary conditions for the new problem and thus the magnitude of the gradient increases. A tolerance can be placed on how large the gradient is allowed to grow before the cost is reminimized. For a given step i, this bound can be expressed

$$\|g(x_{i-1},\gamma_i)\| \leq \epsilon \tag{4.141}$$

When the norm of the gradient exceeds the tolerance, the cost is reminimized to find a new compensator which satisfies the necessary conditions. If ϵ is chosen sufficiently small then small steps are taken and the minimization is begun near the optimal. If however, the bound is chosen too small, then many unnecessary minimizations will be performed.

Before the minimization, techniques for reducing the number of uncertain parameters which will be developed in Chapter 6 can be applied. The closed-loop cost at the current level of uncertainty can be decomposed into its respective parameter contributions. The parameters which do not influence the cost can be discard in the minimization to follow. This is a tricky step, however, because the designer doesn't know if the minimization will yield a compensator which changes the relative importance of a parameter, perhaps emphasizing one which was discarded. To avoid this problem, the ranking of the parameters must be again performed after the minimization to check to see that the truncated parameters were indeed unimportant. If the set of important parameters has changed, then the set must be modified and the minimization repeated until the correct set for the current level of uncertainty is reached.

The minimization step is relatively straightforward. The appropriate cost is minimized with respect to the controller parameters using the necessary conditions for gradient information. The minimization technique used to derive the controllers presented in the next chapter was the popular BFGS quasi-Newton method with a modification to constrain the parameter minimization to the set of stabilizing compensators. In addition, to jump start the minimization, the second derivatives (Hessian) matrix at the initial controller parameters are calculated numerically by finite differences. This Hessian matrix is used in the minimization to calculate the initial step size. If the minimization starts in the vicinity of a minimum because the norm of the gradient was held small, then the fist step based on the Hessian will give near optimal controller parameter values. If the cost function's parameter dependence is in fact not quadratic at this point then further minimization must be performed.

After the minimization, the resulting compensator must be evaluated to determine if the appropriate conditions for termination are met. For instance, the designer can choose to terminate when the design has reached a given level of uncertainty. Another possible termination condition would be a stability test using the controller on the evaluation plant. As stated above, the set of retained uncertain parameters must also be checked to be sure that the dominant parameters where included in the design. If the set of dominant parameters has changed due to the neglected coupling then the minimization must be performed using a more complete set of parameters. If the termination conditions on the homotopy are not met then the homotopy parameter is increased and the process is restarted using the newly derived compensator as an initial guess for further minimization at the incremented value of γ.

This γ-homotopy is useful for determining both the family of controllers as a function of the uncertainty bound as well as for deriving the final controller. In the next section, the details of the calculation of the function gradient will be presented.

4.3.2 Cost and Gradient Calculation

Central to the numerical minimization set of the homotopy is the computation of the cost and associated gradient for a particular value of the controller parameters. The computation of the cost and gradient is problem dependent. In this section, the details of these calculations will be presented for each of the five cost functionals considered in the static and dynamic compensation problems.

It is appropriate to first discuss the general form of the problem for the various cost functionals. The minimization scheme requires the cost and gradient information at a particular value of the controller parameters. The cost is usually given by either the average value of a parameterized Lyapunov equation in the exact average case or by the solution of a set of coupled Lyapunov equations as for the approximation and bounding cost functionals. The function for each of the cases can be denoted $\mathcal{Q}(\gamma, G_c)$, which is a function of the size of the uncertain parameter set and the controller parameters. The gradient of the cost with respect to the compensator parameters is usually a function of the compensator parameters as well as the solution to $\mathcal{Q}$ and its adjoint, $\mathcal{P}$. The necessary conditions for optimality can therefore be written.

$$g[G_c, \gamma, \mathcal{Q}(G_c, \gamma), \mathcal{P}(G_c, \gamma)] = 0 \tag{4.142}$$

The problem of calculating the cost is essentially the problem of calculating $\mathcal{Q}$ while the problem of calculating the gradient can be decomposed into first solving for the solutions of $\mathcal{Q}$ and $\mathcal{P}$ and then solving for g. Since the calculation of g is usually trivial (except in the exact average case) and the computational difficulties are usually the same for $\mathcal{Q}$ and $\mathcal{P}$, the following paragraphs will concentrate on the solution to $\mathcal{Q}$ for the various average-related cost functionals.

The exact average cost is calculated by numerical integration over the parameter domain using a 32 point Gaussian quadrature. If more than three uncertain parameters must be retained in the design, then Monte-Carlo integration is the only feasible method of computing the averages needed for the cost and gradient calculations. In addition to averages of the solution to a parameterized Lyapunov equation, the gradient functions also require averages of the product of the solutions of the parameterized Lyapunov equation and its adjoint. For speed, these averages can be computed at the same time as the average cost. The number of points needed in the integration is dictated by the rate at which the cost changes as a function of the uncertain parameters and should be chosen so that the gradient is accurate to the level of the tolerance used in the minimization procedure.

The solution of the approximations and the bounding functions are discussed in Chapter 3. The perturbation expansion approximate average is computed by utilizing a standard Lyapunov solver and solving the equations hierarchically as mentioned in Remark 3.2.1. The other three equations: the Bourret approximation, the worst-case bound and the average bound are either solved iteratively or by Kronecker math, [113]. In the examples presented in the next chapter they were solved using a modified version of the Kronecker math which takes advantage of the symmetry of the solutions. This greatly reduces the computational burden associated with the size of the matrices involved.

The reduced-order Kronecker math will be demonstrated on the standard Lyapunov equation for simplicity. Consider the equation

$$0 = AQ + QA^T + BB^T \tag{4.143}$$

which can be written using standard Kronecker math as

$$0 = (A \oplus A)\,\text{vec}\{Q\} + \text{vec}\left\{BB^T\right\} \tag{4.144}$$

$$0 = \mathcal{A}q + b \tag{4.145}$$

which can be solved to yield

$$q = -\mathcal{A}^{-1}b \tag{4.146}$$

where $\mathcal{A} = \left(\tilde{A} \oplus \tilde{A}\right)$, $q = \text{vec}\{Q\}$, and $b = \text{vec}\left\{BB^T\right\}$. The size of $\mathcal{A}$ makes the inversion and solution expensive. The size of $\mathcal{A}$ can be reduced by taking advantage of the symmetry in the problem. The vec$\{\cdot\}$ operation stacks the columns of the argument so that a $n \times n$ matrix becomes a $n^2 \times 1$ column vector. Because Q is symmetric, however, many of the elements of q are identical. If we consider only the elements above or on the diagonal as independent, a new vector containing only this portion of the matrix can be defined

$$\tilde{q} = Uq \tag{4.147}$$

The matrix U picks off the appropriate independent elements of q and therefore is a matrix of ones and zeros. It is also useful to define another matrix which reconstitutes the reduced vector

$$T\tilde{q} = q \tag{4.148}$$

These two matrices can be applied to Eq. (4.144) to obtain a reduced order version of the equation.

$$0 = \tilde{\mathcal{A}}\tilde{q} + \tilde{b} \tag{4.149}$$

where

$$\tilde{\mathcal{A}} = U(A \oplus A)T \tag{4.150}$$

Since $\tilde{\mathcal{A}}$ has dimension $(\frac{n^2}{2}+n)\times(\frac{n^2}{2}+n)$, it is almost half the size of $\mathcal{A}$. The inversion for the solution is thus almost eight time faster since inversion is a $\mathcal{O}(n^3)$ process. This general technique can be applied to the solution of the Kronecker math equations for the Bourret approximation and the worst-case and average bounds. Since the U and T matrices in Eq. (4.150) are only full of ones and zeros it can be faster to simply eliminate the unnecessary rows of $\mathcal{A}$ and add the columns corresponding to the redundant elements of q.

4.4 Summary

In this chapter the average performance problem has been formulated for two general cases; static and dynamic output feedback. For each of these cases, the cost minimized was represented by either the exact average, the approximations to the average, or the bounds to the average presented in Chapter 3. Each cost minimization yields different necessary conditions and different properties for the resulting controllers. When the exact average or its bounds are minimized they yield controllers which guarantee stability throughout the model set. When the approximations to the average is minimized, robustness is increased over the non-augmented cost minimization, (LQG or LQR), but stability is not necessarily guaranteed throughout the design model set. For the case of Bourret approximate average cost minimization, however, stability is guaranteed over a smaller model set.

The necessary conditions for the case of fixed order compensation where derived using Lagrange multiplier techniques. The necessary conditions for the exact average minimization problems requires averages of the product of the solutions of two parameterized Lyapunov functions. The necessary conditions for the Bourret approximate average cost minimization gave fixed order compensator gains which depended on the product of the solutions of two Bourret equations. The parameter independent Bourret equations take the place of the parameterized Lyapunov equations found in the exact average minimization. This is a general trend found when using the modified Lyapunov equations for the various costs. The uncertainty couples the modified Lyapunov equation and its adjoint in the solution process for the compensator gains.

Chapter 5

Examples of Controllers for Parameterized Systems

5.1 Controller Design Philosophy

The goal of this section is to present some general thoughts on robust controller design. This discussion is intended to unify the various techniques developed in the previous chapters and help focus them on the examples presented in this chapter. The discussion will center around the tasks involved in the robust controller design process and how these tasks are influenced by the choice of analytical framework.

The robust control process can be divided into four distinct tasks

(i) Model generation: realize the uncertainties in a parameterized evaluation model

(ii) Model reduction (optional): reduce the evaluation model order or number of uncertainties for design purposes

(iii) Controller synthesis: generate a controller based on the design model

(iv) Controller evaluation: check the controller on the evaluation model for stability and performance robustness.

with the process being iterative if necessary.

All four steps in the robust control design process are influenced by the analytical techniques used to derive them. In the case of this report this analytical framework is centered about real parametric uncertainty and average cost related measures of it.

The choice of analytical framework influences how errors are represented in the model. Real parametric uncertainties represent a limited class of the possible plant uncertainties and do not well represent unmodeled dynamics. If a model of the system dynamics does exist but the values of the interactions are uncertain then parametric error modeling is appropriate. The choice of how errors are to be considered affects how they are most easily represented. For instance, unmodeled dynamics are most commonly represented in an input-output frequency-domain specification while parametric uncertainties are usually represented in the time domain as errors in the matrix components. Real parametric error *can* be represented in the frequency domain as input output error and unmodeled dynamics *can* be represented by parametric error but in both cases the error models would be inappropriate.

To begin the robust control design process for real parametric uncertainties, the structure of the component interactions and the range of possible values must be obtained through experimentation, analytical modeling, or a combination of the two. The structure can be obtained using analytical assumptions, and the error ranges can be obtained through comparison between the analytical prediction and the experimental data. The output of this step in the process is a complete (and hopefully accurate) error model which can be used for evaluating the robustness of the controllers.

The model reduction step is presented in Chapter 6. The complete model must sometimes be reduced before control design is computationally feasible. There are two types of reduction presented in this report, model order reduction and model uncertainty reduction. For the simple systems presented in this chapter, model reduction for control design is unnecessary. For more complex system however it will be a necessity due to the large computational costs associated with large numbers of uncertainties. The output of this step is a reduced error model which is suitable for control design.

The control design step centers around the algorithm presented in Section 4.3.2. Before the controller can be designed the reduced order model must be modified with input-output weights to produce a suitable design model. An example of this weighting procedure is presented for LQG design in Section 2.1.2. The weights represent the design variables and how they are chosen is problem dependent. In the following examples, emphasis will not be placed on weight selection since the intent is to present the relative merits and ramifications of using the cost functionals for control synthesis. The control design step consists of minimizing one of the five average-related cost functionals as applied to the design plant. The mechanisms of controller design are presented in Chapter 4. The output of the controller design step is a set of static or dynamic compensator matrices which minimize the closed-loop cost when applied to the design plant.

The final step in the robust control design process is controller evaluation on the complete error model. The controller should be evaluated on the the basis of nominal performance and performance robustness, as well as stability robustness in the presence of modeled parametric error and unmodeled dynamics. The performance specification can be different than the one used for controller design. For instance, transient response can be used to evaluate controllers derived by minimizing system $\mathcal{H}_2$ -norm. In addition, the control design methodology needn't be based on the stability robustness analysis technique used for evaluation. For instance, majorant-Lyapunov stability analysis [77] can be used for stability analysis of the closed-loop evaluation model even though the controller was derived using the worst-case bound. An analysis tool can be used for stability evaluation which was too complex to be used for control design. If the design doesn't meet the stability and performance criteria of a given specification set, then the design process must be iterated with a different set of retained uncertain parameters and design weights.

In the following sections, some examples of control design and evaluation will be presented that use the average-cost related analysis tools developed in the previous chapters. Particular emphasis will be placed on the tradeoff between nominal performance and stability and performance robustness. It is also recognized that higher stability adn performance robustness is usually associated with higher control effort so that careful attention will be directed at the cost of the robustness of the various designs.

The five average-related cost functionals will be compared on some simple examples. To streamline discussion in these sections it is convenient to define a series of acronyms for the various designs.

PEACM Perturbation Expansion Approximate Cost Minimization

BACM Bourret Approximate Cost Minimization

EACM Exact Average Cost Minimization

ABM Average Bound Minimization

WBM Worst-Case Bound Minimization

these acronyms will be used extensively in subsequent sections.

It should be noted that the PEACM design is essentially equivalent the the sensitivity system cost minimization presented in [103] and discussed in Remark 3.2.2. The WBM design has been previously presented in [69] and is here presented for comparison purposes.

5.2 Example 1: The Robust-Control Benchmark Problem

In this section, static and dynamic output feedback compensators based on the five techniques presented in Chapter 4 will be designed for the robust-control benchmark problem presented in Ref. [129]. The problem considered is a two-mass/spring/damper system shown in Figure 5.1, which is a generic model of an uncertain dynamic system with noncolocated sensor and actuator. The uncertainty stems from either an uncertain spring connecting the two masses or an uncertain damper. First the system represented in Figure 5.1 will be presented. From Ref. [129] the system matrices can be represented in state space form as

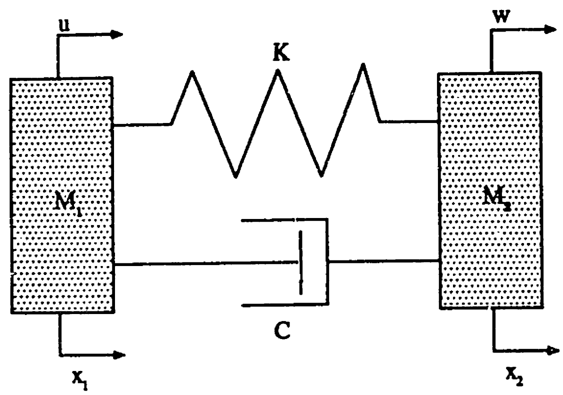

Figure 5.1: The Robust-Control Benchmark Problem

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \\ z \\ y \end{bmatrix} = \left[\begin{array}{cccc|cc|c} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -k/m_1 & k/m_1 & -c/m_1 & c/m_1 & 0 & 0 & 1/m_1 \\ k/m_2 & -k/m_2 & c/m_2 & -c/m_2 & 0 & 1/m_2 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \\ v \\ w \\ u \end{bmatrix} \tag{5.1}$$

where

x_1 = position of body 1
x_2 = position of body 2
u = control force input
w = plant force disturbance
y = sensor measurement
v = sensor noise
z = performance variable

Within the system described in Eqs. (5.1), the uncertain spring, k, and damper, c, are decomposed into a nominal value and a bounded variable parameter

$$k = k_0 + \tilde{k}, \quad k_0 = 1.25, \quad |\tilde{k}| \leq \delta_k \tag{5.2}$$

$$c = c_0 + \tilde{c}, \quad c_0 = 0, \quad |\tilde{c}| \leq \delta_c \tag{5.3}$$

With this factorization the set of systems can be defined in the notation from Definition 2.2.4. In particular, only the A matrix is uncertain. It can be factored as

$$A(\tilde{k}, \tilde{c}) = A_0 + \tilde{k}A_k + \tilde{c}A_c \tag{5.4}$$

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & 0 & 0 \\ 1.25 & -1.25 & 0 & 0 \end{bmatrix}$$

$$A_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \quad A_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

With this factorization, the robust control design methodologies presented in the previous sections can be applied. To design the controller, the controls engineer would like more flexibility in weighing the outputs and controls in the cost and specifying the disturbance and sensor noise intensities. First the evaluation plant outputs, z, are augmented to include all the states and the control, $z^T = \begin{bmatrix} x & u \end{bmatrix}^T$. Next, the relative magnitudes of the input disturbances and output variables can be explicitly weighted. The method of weighting the system that was presented in Section 2.1.2 which is based on the standard LQG design weights will be adopted in the next two sections. In this method the designer selects the state weighting matrix, Q, the control weighting matrix, R, and the sensor and plant noise intensity matrices, Θ and Ξ respectively. The evaluation plant is modified as in Eq. (2.25) to give the design plant. The control is designed on the design plant and implemented on the evaluation plant.

5.2.1 Full State Feedback: Uncertain Damping

In this section, the five control design methodologies will be applied to the problem of determining full state feedback gains for the benchmark problem. For this section only the damper is uncertain and the spring is assumed known with a value of $k = 1.25$. The uncertain parameter c has the form:

$$c = c_0 + \tilde{c}, \quad c_0 = 0, \quad |\tilde{c}| \leq 0.5 \tag{5.5}$$

Thus the parameter design bound, $\delta_c = 0.5$, is large enough to allow unstable elements in the open-loop model set. For the case of full state feedback there is no sensor noise and the plant noise intensity was assumed equal to unity. In addition, only the position of the second mass was penalized. The weighting values used in the design are

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = 1 \tag{5.6}$$

The five designs and the corresponding LQR design are presented in Figures 5.2 - 5.8.

The stability and performance robustness of the designs can be seen compared in Fig. 5.2. In Fig. 5.2 the system closed loop $\mathcal{H}_2$-norm given by the various designs using $\delta_c = 0.5$ is plotted as a function of the actual value of $\tilde{c}$. The curves show the variation in system nominal performance as well as the performance robustness for each design. Infinite cost is associated with unstable systems so the asymptotes of the cost curves give the achieved stability bounds for the given value of the design bound, $\delta_c = 0.5$. The lack of an asymptote for positive values of the damper parameter reinforces the physical intuition that additional system damping isn't destabilizing.

The relative amount of stability robustness gained by the designs is consistent with the relative ordering of the approximations established in Fig. 3.4. The LQR design is shown to be the least robust in part because of the relatively high control weight assumption. The high control penalty keeps the undamped pole from being pushed far enough into the left hand plane to guarantee stability over the parameter range. The PEACM design gives the smallest stability range followed by the BACM and EACM designs. Both the BACM and EACM designs achieve stability throughout the design set. As expected the ABM and WBM designs, being bounds, achieved stability throughout the design set. It is interesting to note that both bounds lead to identical closed-loop system $\mathcal{H}_2$-norms as a function of the parameter. The bound-based designs are essentially identical. The similarity between the bound-based designs arises from the fact that both bounds give the same parameter stability regions when they are used for analysis. In control synthesis, they therefore behave similarly when the design is stability driven (as is this one).

Figure 5.3 shows the achieved design C bound, as a function of the bound used in the design, δ_c. The ordering of the approximations and the bounds relative to the average is again visible. In all the designs, increasing the design bound increases the achieved bound. For this problem, only the PEACM design doesn't always achieve stability throughout the design set. As the design bound is increased the ABM and WBM designs quickly respond by dramatically increasing the achieved bound over and above what is needed for stability. This is an indication of the conservatism of the designs. The increased stability range is associated with increased control effort and higher nominal costs for a given design bound.

It is perhaps more important to examine the system closed-loop cost as a function of the achieved stability bound rather than the parameter bound used in the control design. Figure 5.4 shows the closed-loop $\mathcal{H}_2$ cost for the nominal system as a function of the limits of the range of damping parameter over which the closed-loop systems are stable, called the achieved stability bounds. The stability range is characterized by

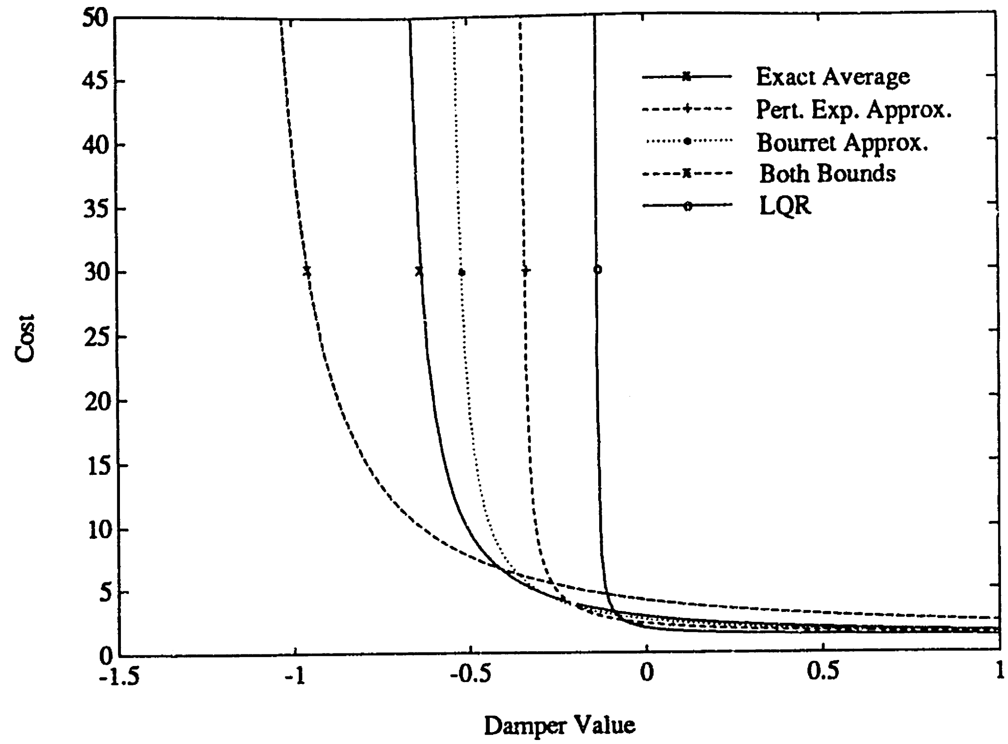

Figure 5.2: System Closed-Loop $\mathcal{H}_2$-norm as the Damping Parameter, $\tilde{c}$, is varied from -1 to 1 for $\delta_c = 0.5$.

an upper and lower limit. Since the lower limit was closer to the nominal parameter value (the parameter was limited in its variation below nominal), this limit was chosen as a characterization of the amount of parameter uncertainty a given robust design can accomodate. This type of plot will be called the ***design efficiency plot*** since it is an indicator of how much the nominal cost must be increased to achieve a given level of robustness. The design which achieves the most robustness with the least increase in nominal cost is the most efficient robust control design. This type of plot easily captures the relevent characteristics of the performance-robustness trade for the various designs. It does not however capture teh off-nominal performance adn performanc robustness issues.

Expressing the cost in terms of achieved stabilty range rather that parameter range used in the design reveals connections between the designs that are difficult

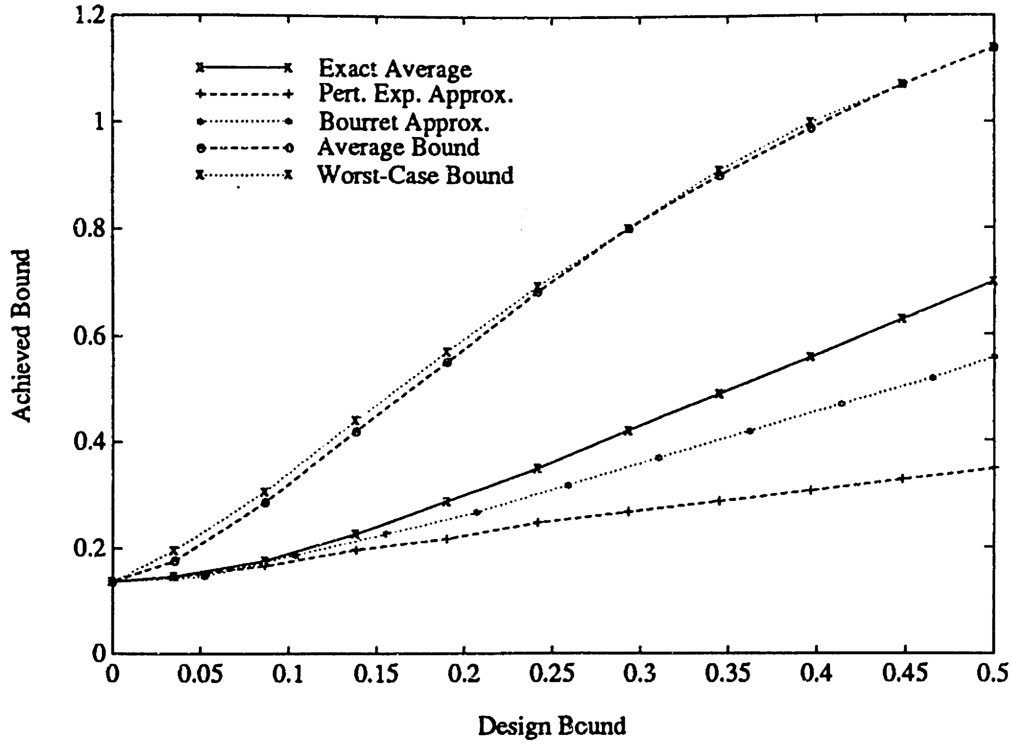

Figure 5.3: Achieved Closed-Loop Stability Bounds as the Design Bound, δ_c, is increased from $0 \longrightarrow 0.5$.

to recognize otherwise. The lower limit of the achieved stability range will be called the *achieved bound* and the bound on the parameter variation considered in a given design will be called the *design bound.* All of the designs converge to the inherent LQR robustness properties when the design bound is zero (no parameter uncertainty used in the design). There is marked similarity in the relative design efficiencies for this problem. This is due to the fact that there is only a single uncertainty and the problem is low order with full state feedback. In this case, the stability range guarantee associated with the bounds, the exact average, and the Bourret approximation can be made equal by only varying the design bounds. This is not true for the PEACM design which doesn't "blow up" for any design bounds. The PEACM design is therefore the least efficient. As expected from the analysis in Chapter 3, the EACM design is most efficient and is closely matched by the Bourret approximation design and the average bound design.

The decomposition of the total cost into the cost associated with the output and the cost associated with the control is shown in the two smaller plots of Fig. 5.4. These plots reveal that higher state cost is associated with lower control cost in all of the designs. Thus, the PEACM design has the highest state cost and the lowest control cost of the designs. In, general the efficiency ranking given by the total cost is also that given by examining only the state cost.

The designs were generated by slowly increasing the uncertainty in the design process as described in Definition 4.3.1. This homotopy can be examined by plotting the closed-loop poles as a function of the uncertainty bound used in the design. This is done in Fig. 5.5 for the five design procedures. The curves in all of the graphs start at the LQR pole locations. The LQR poles are denoted by X. Because of the high control penalty, the LQR poles at 1.6 rad/sec are relatively lightly damped. The designs add robustness by shifting the poles to the left, i.e., making them more damped. All of the designs shift the poles left. This left-shift of the poles is associated with higher control gains and can thus degrade robustness to high frequency dynamics. The approximation based designs, PEACM and BACM designs, shift them less than the exact average based design. The EACM design in turn shifts them less than the bound-based designs which critically damp the poles.

Figure 5.6 shows the closed-loop pole locations for designs with $\delta_c = 0.5$ as a function of the actual damper parameter value as the damper is varied from -0.5 to 0.5. The curves represent the locus of closed loop poles. If the line intersects the imaginary axis then some of the elements of the set of systems are unstable. The locus of the high frequency resonant poles has been placed on the real axis by the bound designs. This gives good performance robustness for the bound based designs.

The bound based designs also lead to higher loop gains and correspondingly higher bandwidth solutions. Figure 5.7 shows that the loop gains progressively increase as one progresses from the PEACM design to the WBM design, i.e., as the design

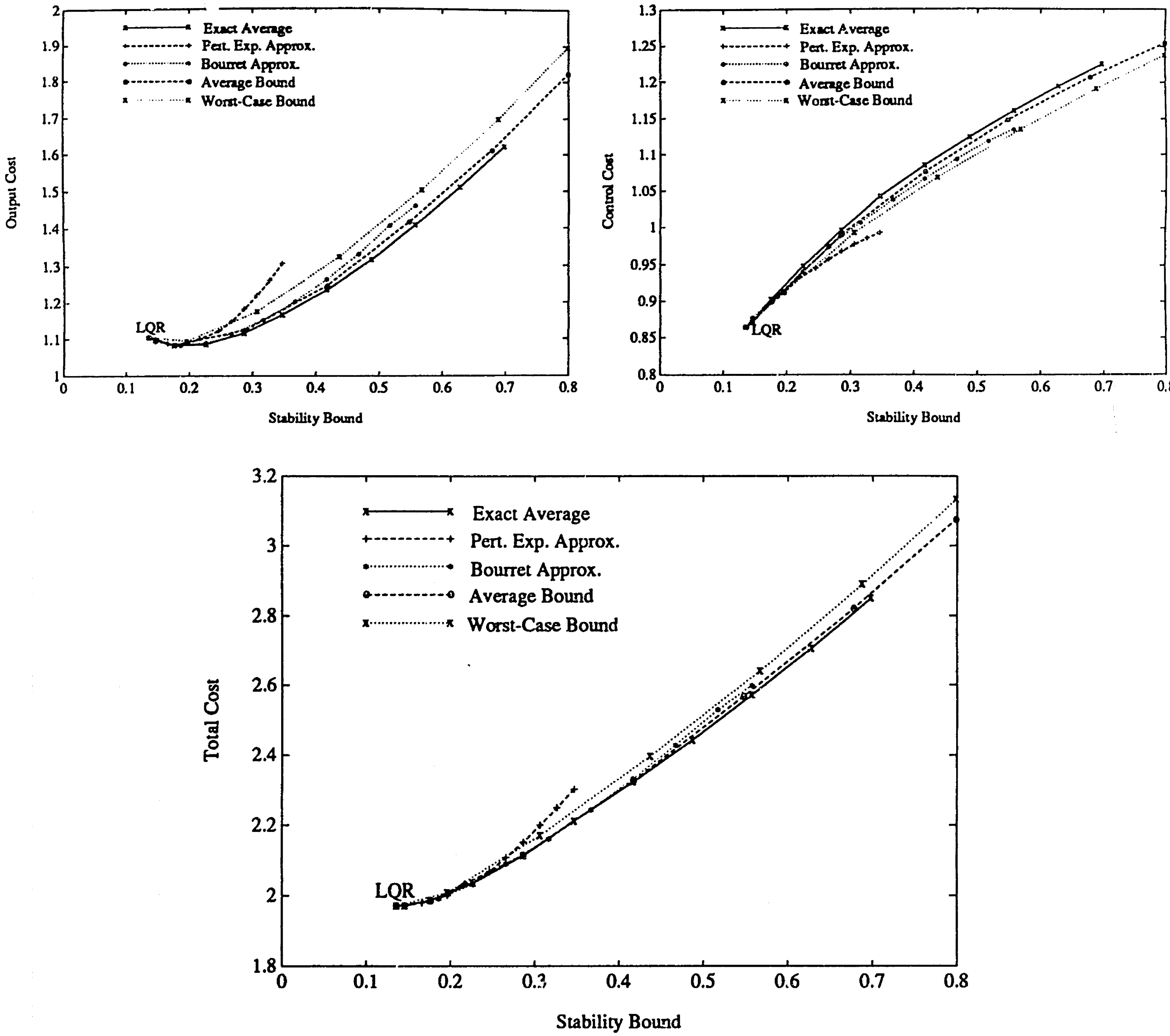

Figure 5.4: Total Cost, State Cost, and Control Cost as a Function of the Achieved Stability Bound.

conservatism and stability radius increases. The higher gain is tolerable for this full state feedback design because the phase remains between -180 and +180. The loop

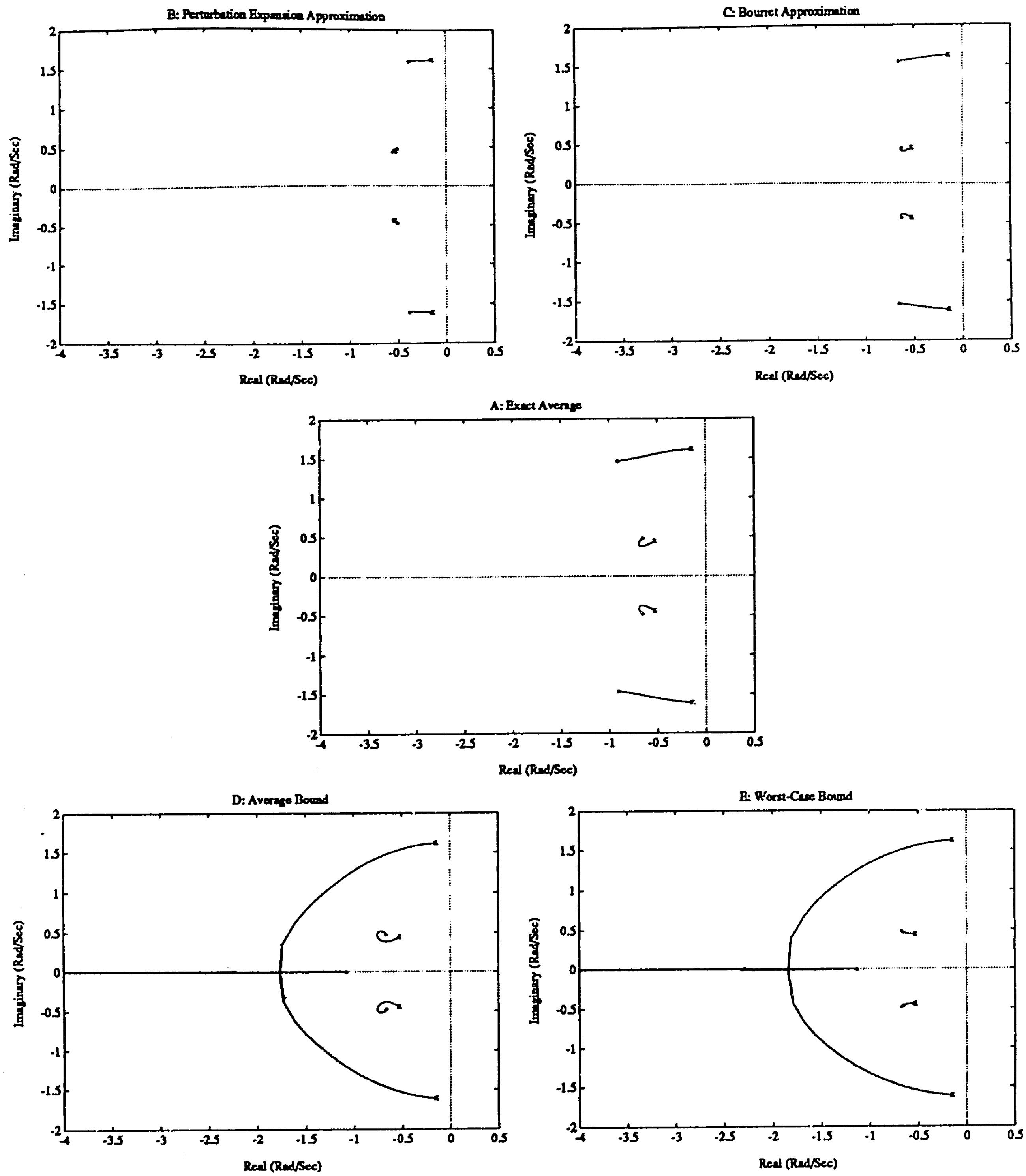

Figure 5.5: Closed-Loop Pole and Zero Locations as a Function of Uncertainty Design Bound as δ_c is Varied from 0 (Poles X, Zeros O) to 0.5 (Poles $*$, Zeros $+$)

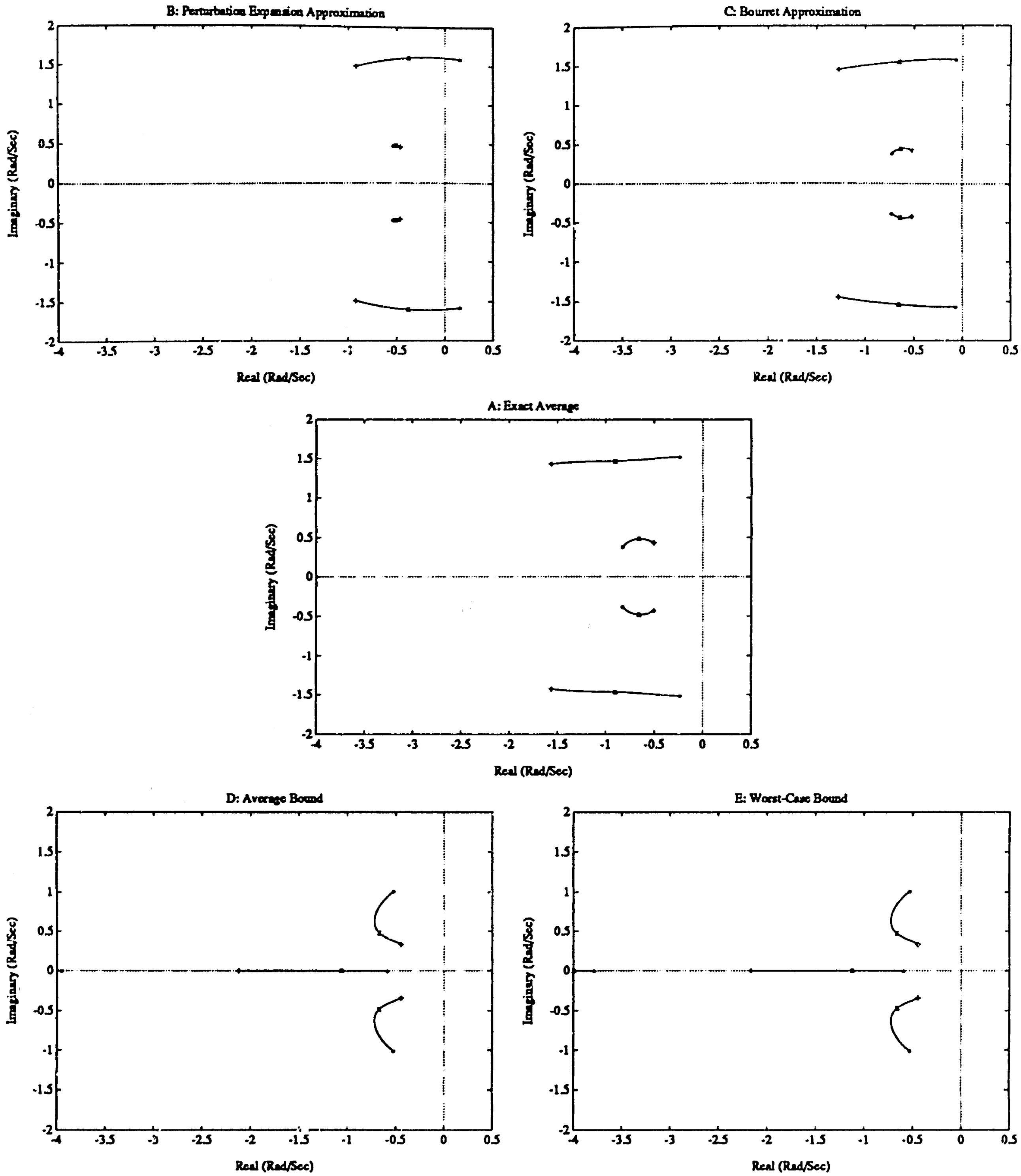

Figure 5.6: Closed-Loop Pole Locations as the Parameter, $\tilde{c}$, is Varied from -0.5 (*) to 0 (x) to 0.5 (+) with $\delta_c = 0.5$

gain for this system can thus be increased indefinitely without instability. All of the designs tend to have at least 90 degrees of phase margin.

The impulse response transients for the various designs are shown in Figure 5.8. As expected the PEACM and BACM deigns show less performance robustness to variations of the damper value than the bound-based designs. The nominal performance of the designs are actually quite similar however. The peak position and control amplitudes are higher for the bound-based designs, but not by much. In general the slow, highly damped pole at 0.5 Hertz dominates the response for all of the designs.

The design results for this problem agreed well with the analytical expectations. The bound-based designs used higher gains to drive the uncertain system pole further left than the exact average-based design while the approximations used lower gains and moved the poles less. The design efficiency chart revealed that the perturbation based compensator was the least efficient of those examined while the exact average-based design was the most efficient.

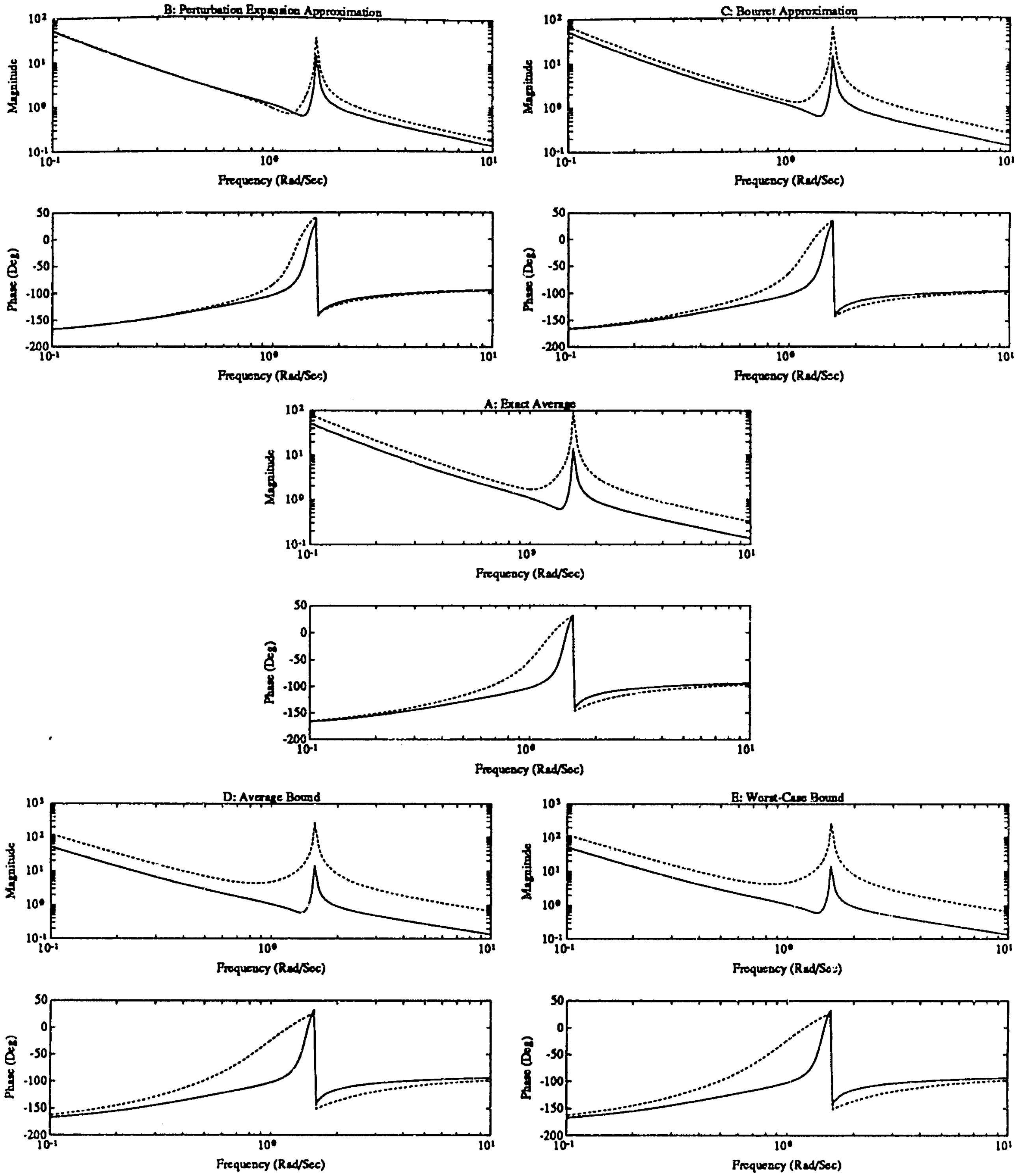

Figure 5.7: Loop Transfer Functions from Controller Input, y, to Plant Output, y, for the Various Designs (Dashed) Compared to LQR (Solid)

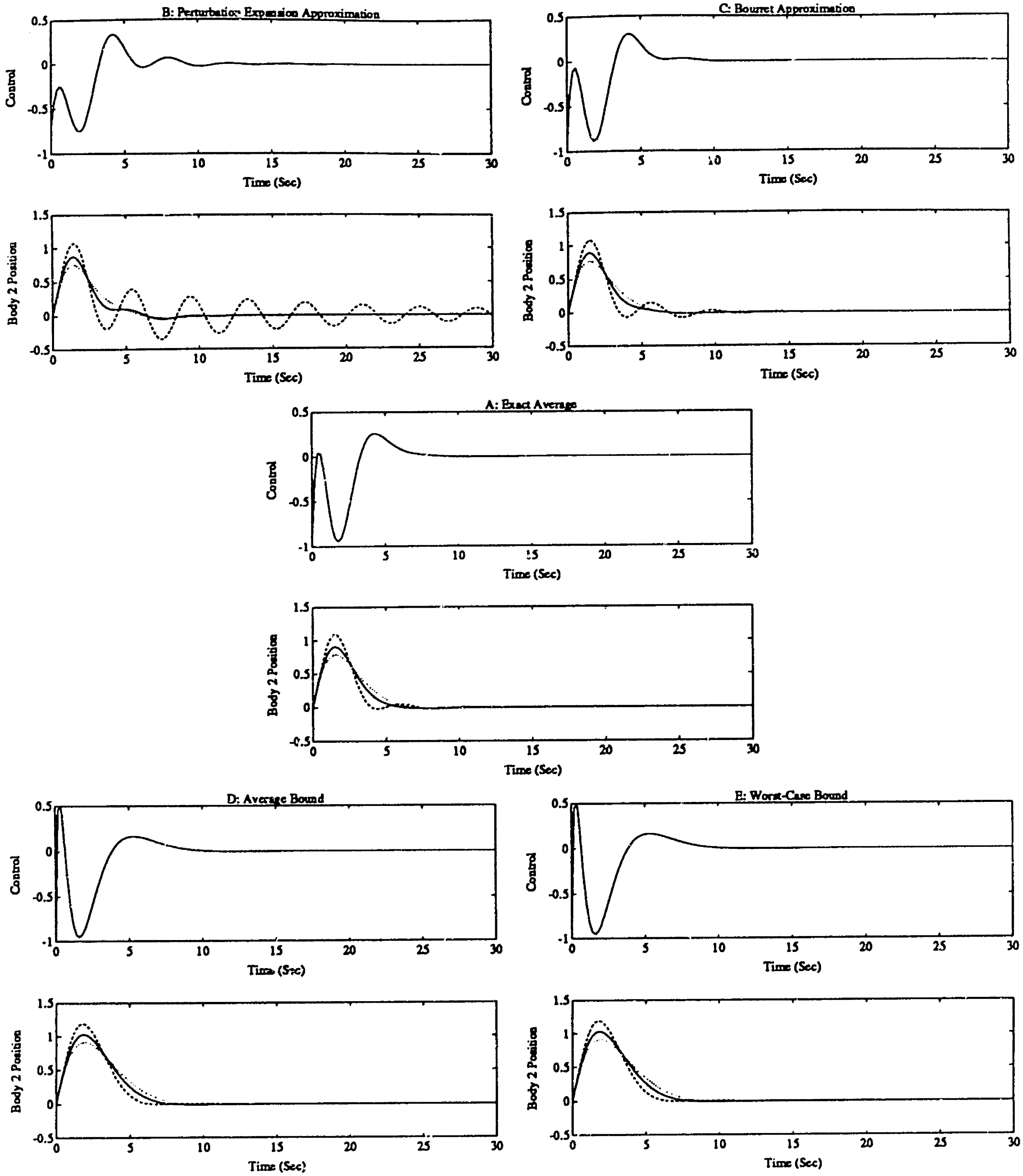

Figure 5.8: Closed-Loop Impulse Response Time Histories: $\tilde{c} = 0$ (solid), $\tilde{c} = -0.3$ (dash), $\tilde{c} = 0.3$ (dotted)

5.2.2 Dynamic Compensation: Uncertain Spring

In this section, the five control design methodologies will be applied to the problem of determining full-order SISO dynamic output feedback compensators for the benchmark problem. In this section only the spring is uncertain and the damper is set to zero. The uncertain parameter k has the form

$$k = k_0 + \tilde{k}, \quad k_0 = 1.25, \quad |\tilde{k}| \le \delta_k = 0.75 \tag{5.7}$$

Thus the parameter design bound, $\delta_k = 0.75$, allows the stiffness to vary in the range from 0.5 to 2. The LQG problem statement presented in Section 2.1.2 was adopted to specify the design weights. Only the position of the second mass was penalized. The weighting values used in the design are

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = 0.0005 \tag{5.8}$$

The control weighting was chosen to be low to examine high performance designs which meet a settling time requirement of 15 seconds as specified in Ref. [129]. Although each design is not optimized to meet this specification, the weight was chosen so that the closed-loop performance of the EACM design met this specification. The other designs will have performance in the area of 15 second settling time. This consideration was not treated as a hard design constraint. In addition to the state and control penalties, the plant noise and the plant noise intensity were assumed to be

$$\Xi = 1, \quad \Theta = .0005 \tag{5.9}$$

The signal noise intensity was chosen low to give a high gain Kalman filter in the LQG design.

The robustness properties of the five control designs are compared to those of the standard LQG design in Figures 5.9-5.16. Figure 5.9 compares the closed-loop

$\mathcal{H}_2$-norm resulting from the various designs using $\delta_k = 0.4$ as a function of the deviation from the nominal spring constant, $\tilde{k}$. Thus the controllers were designed to accomodate a stiffness variation, $0.85 \leq k \leq 1.65$. Instability regions are indicated by unbounded closed-loop $\mathcal{H}_2$-norm. The LQG results clearly indicate the well-known loss of robustness associated with high-gain LQG solutions. The LQG cost curve achieves a minimum at the nominal spring constant, $k = 1.25$, but tolerates almost no lower values of k. The stability region is increased by the PEACM and BACM designs at the cost of increasing nominal system closed-loop $\mathcal{H}_2$-norm. Although both the PEACM and the BACM designs increase robustness they do not achieve stability throughout the whole design set, $-0.4 \leq \tilde{k} \leq 0.4$. Of the approximate methods, the Bourret approximation more nearly achieves stability throughout the set. The EACM design does achieve stability throughout the set as was indicated by the analysis. The cost of this stability guarantee is loss of nominal system performance.

This performance-stability trade is especially evident for the bound-based designs. The nominal costs for these designs are two orders of magnitude higher than for the average-based design and its approximations. The bound-based designs extend the upper stability bound for the stiffness much more than the lower stability bound. It is also interesting to note that although the stability bounds for the ABM and WBM designs are identical the ABM design achieves this with 30 % lower nominal performance. The relative flatness of the bottoms of the performance "buckets" indicates that the performance of the various designs are also relatively robust to changes in the spring constant.

As for the static compensation case, the range over which a given design is stable can be ploted as a function of the parameter range used in the design. The parameter range over which a particualar design maintains stability is characterized by the *achieved bound* which is chosen to be the lower limit of teh stability range. The paramter range actually considered in the design is characterized by the *design bound*, denoted δ_k, which specifies the upper and lower limit of $\tilde{k}$. Figure 5.10 shows the

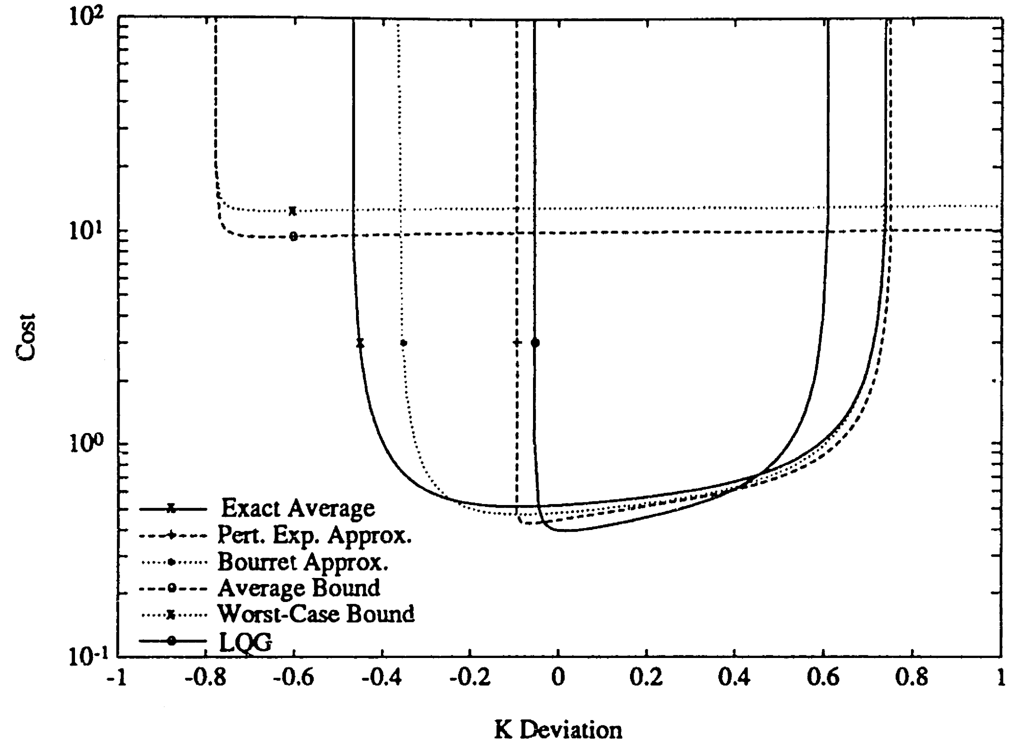

Figure 5.9: System Closed-Loop $\mathcal{H}_2$-norm as a Function of the Deviation about the Nominal Spring Constant, $\tilde{k}$, for Controllers Designed Using $\delta_k = 0.4$.

achieved lower $\tilde{k}$ stability bounds as a function of the design bound, δ_k. With no design uncertainty all five techniques converge to the stability range achieved by the standard LQG design ($|\tilde{k}| \leq 0.06$). As the uncertainty used in the design process is increased the achieved robustness is also increased. Again, the EACM design always increases robustness enough to guarantee stability throughout the design set, while the approximate cost minimization techniques don't provide this guarantee. The BACM design does come closer to guaranteeing stability than the PEACM design which does particularly poorly in decreasing the lower stability bound. The bound-based designs achieve much larger stability bounds for a given design bound. The interesting point is that the robustness is incrementally increased much more for small values of uncertainty than for larger values. The curve starts very steep and then level off to the same slope as the EACM design. The ABM and WBM designs' achieved bounds also converge for the higher design values. The spacing maintained between the exact average and the bound-based designs is indicative of the conservatism of the bounds.

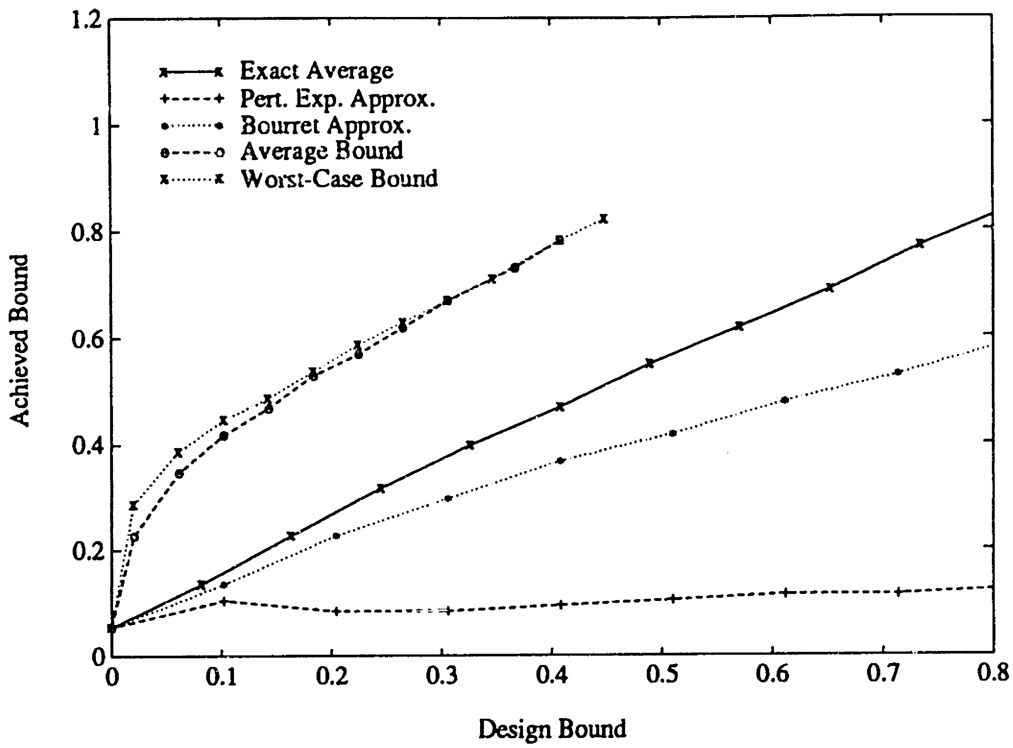

Figure 5.10: Achieved Closed-Loop Stability Bounds as a Function of the Design Bound, δ_k

In Figure 5.11, the closed-loop $\mathcal{H}_2$-norm of the nominal plant ($k = 1.25$) is examined as a function of the achieved stability bound. This is the design efficiency plot mentioned in the previous section. The closed-loop cost ($\mathcal{H}_2$-norm) is also shown decomposed into the component associated with the output weighting, called the output cost, and the component associated with the control weighting, called the control cost. Both output and control costs clearly increase as the achieved robustness increases. The EACM design achieves a given level of robustness with the least increase in the nominal cost and is therefore considered the most efficient design. The BACM design also has good efficiency, almost matching that of the EACM design.

The PEACM design is clearly the least efficient of the five. It cannot achieve a design bound of more than 0.2. The bound-based designs also asymptote to infinite cost but at a higher value of the achieved stability bound. For low levels of achieved stability, the bound-based designs are actually as efficient as the average and the Bourret approximation based designs.

The compensator and closed-loop poles and zeros as a function of the design uncertainty bound are plotted in Figures 5.12 and 5.13. All of the compensator designs (including LQG) use non-minimum phase compensation. The compensators thus introduce a nonminimum phase zero into the closed loop plant. The differences in the compensators arise in how the uncertainty affects the poles. In the EACM and the approximation based designs one pair of poles actually move in toward the imaginary axis. The other pair of poles is critically damped. The EACM designs and the approximations based designs also have similar closed-loop pole loci as seen in Fig. 5.6. The general trends from the PEACM design to the BACM design and finally the EACM design is to move the higher frequency poles more toward the imaginary axis while damping the lower frequency poles at 1.4 rad/sec. The nonminimum phase zeros resulting from the compensator are unaffected by uncertainty in the PEACM design. They are moved closer to the imaginary axis in the EACM design.

The general trend for the bound-based designs is to push the poles away from the imaginary axis. The LQG poles at 1.4 rad/sec are almost critically damped while the LQG poles at $-1 \pm 3i$ follow Butterworth patterns away from the imaginary axis. Pushing the poles in this manner causes the bound based designs to have higher gain and bandwidth than the approximation and average based designs.

The loci of closed-loop poles as a function of the uncertain parameter are shown in Figure 5.14. The stiffness deviation, $\tilde{k}$, is shown varying in the range from -0.75 to 0.75 which results in open-loop pole locations varying in the range from 0.5 to 2 rad/sec. Only the lower frequency poles are shown since these are the poles and zeros most influenced by the variable stiffness. If any of the loci intersect the imaginary

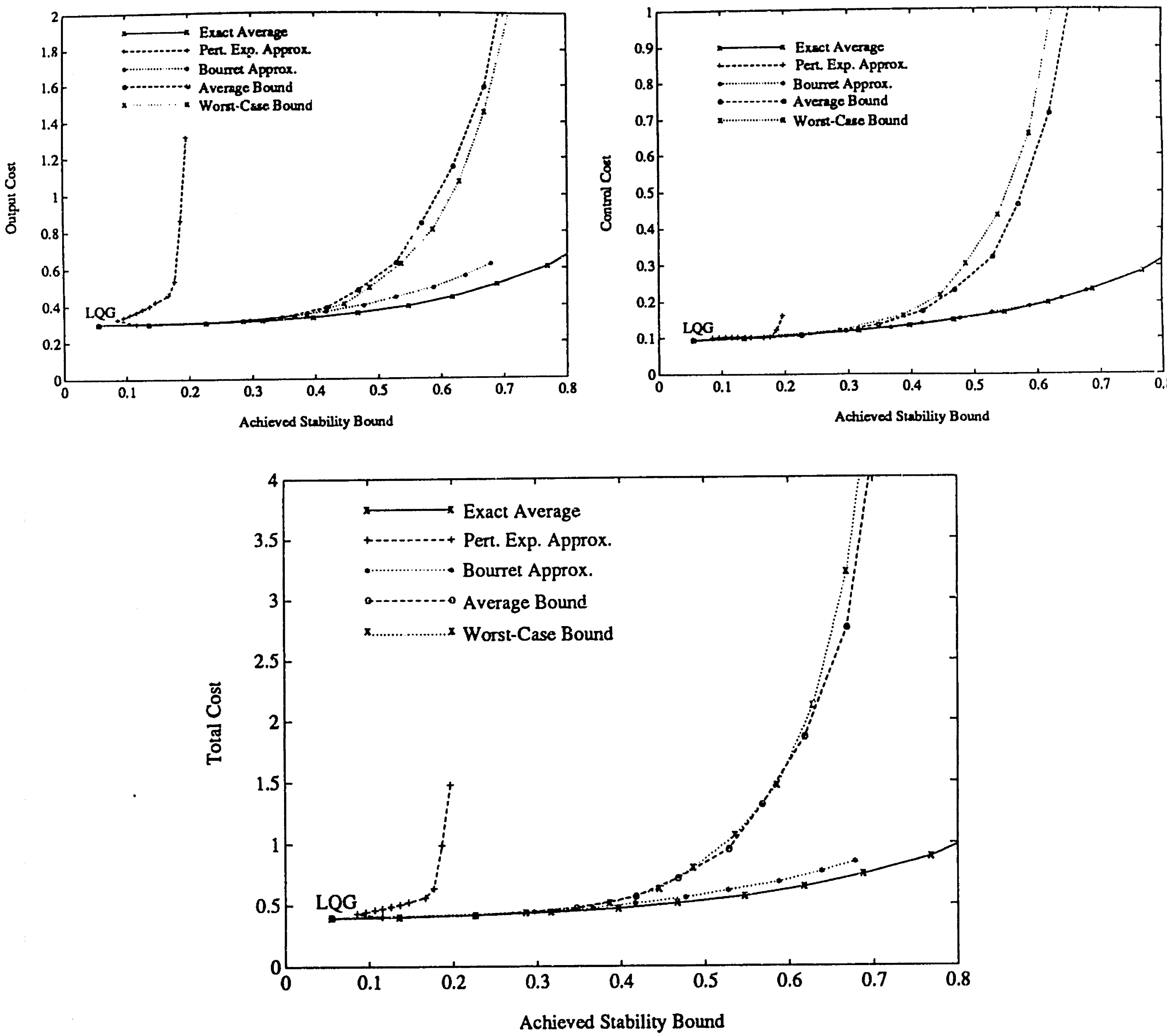

Figure 5.11: Total Cost, Output Cost, and Control Cost as a Function of the Achieved Stability Bound.

axis then an element of the closed-loop set of systems is unstable for that design. Clearly the PEACM and BACM designs allow instabilities in the design set. The

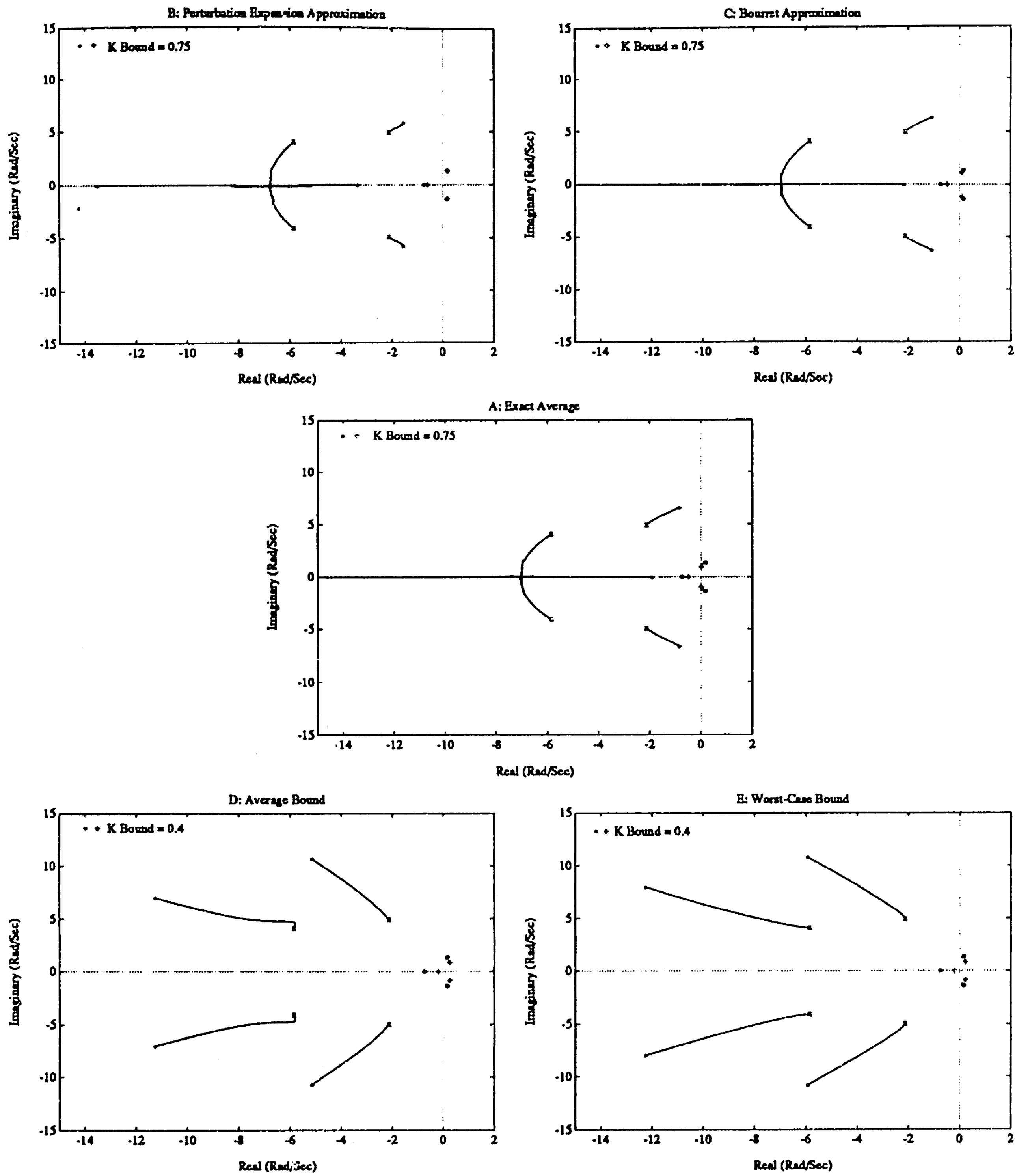

Figure 5.12: Compensator Pole and Zero Loci as a Function of Uncertainty Design Bound as δ_k is Increased from 0 (LQG poles-x, zeros-o) to the Indicated Amount (poles-*, zeros-+).

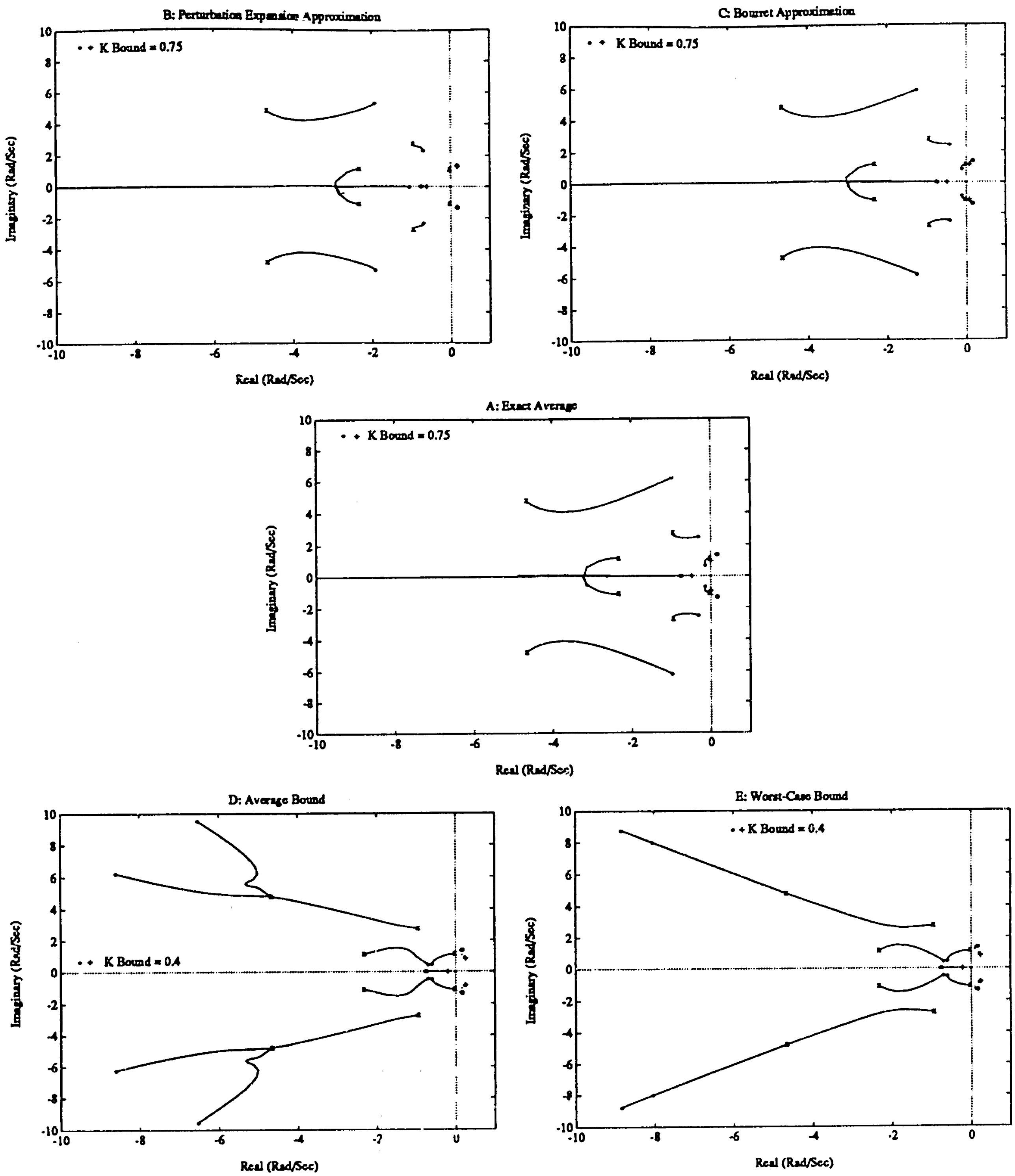

Figure 5.13: Closed-Loop Pole and Zero Loci as a Function of Uncertainty Design Bound as δ_k is Increased from 0 (LQG poles-x, zeros-o) to the Indicated Amount (poles-*, zeros-+).

EACM design poles move primarily along the imaginary axis instead of crossing it. Since control effort is expended to move poles further from the axis this design is the least expensive for the achieved stability bound. This pole locus is another indicator of the EACM design's efficiency. The zeros of all of the designs are unaffected by the variable stiffness.

From Fig. 5.14 the bound-based designs are shown to have nearly identical closed-loop pole loci. Even though the designs were computed using a bound of 0.4 the designs still maintain stability over a parameter range of ± 0.75. The nominal pole locations (indicated by the x's) are nearly coalesced in the bound-based designs. They are also further left than in the average and approximation-based designs.

The loop transfer functions of the various designs are compared to the LQG loop transfer function in Figure 5.15. The robust designs are shown by the dashed lines while the LQG design is shown with a solid line. The general characteristics of all of the design transfer functions are similar to the LQG design's. The curves start with a -40 db/decade slope caused by the two rigid-body poles. The magnitude dips at about 1.1 rad/sec at the location of the two damped nonminimum-phase compensator zeros. The nonminimum phase zeros cause the magnitude to dip below 1 and the phase to increase (not decrease as for normal zeros) past 180 degrees. The transfer function dips under 1 just as it roles past the 180 degree mark. The gain and phase margins for the LQG design are thus very poor. This is in part why the LQG design is so sensitive to parameter errors. The LQG curve continues with the resonant pole at 1.4 rad/sec. The resonant is followed by a frequency range of -10 db/decade slope until the LQG compensator starts to rolloff at about 5 rad/sec.

The approximation and bound-based designs are distinguished on the basis of how they handle the nonminimum phase zero and how they roll off. The PEACM design gains little robustness over the LQG as seen by the similarity in the transfer functions. The BACM and EACM designs increase the robustness by lowering the frequency and damping of the nonminimum phase zeros. This has the effect of decreasing the low

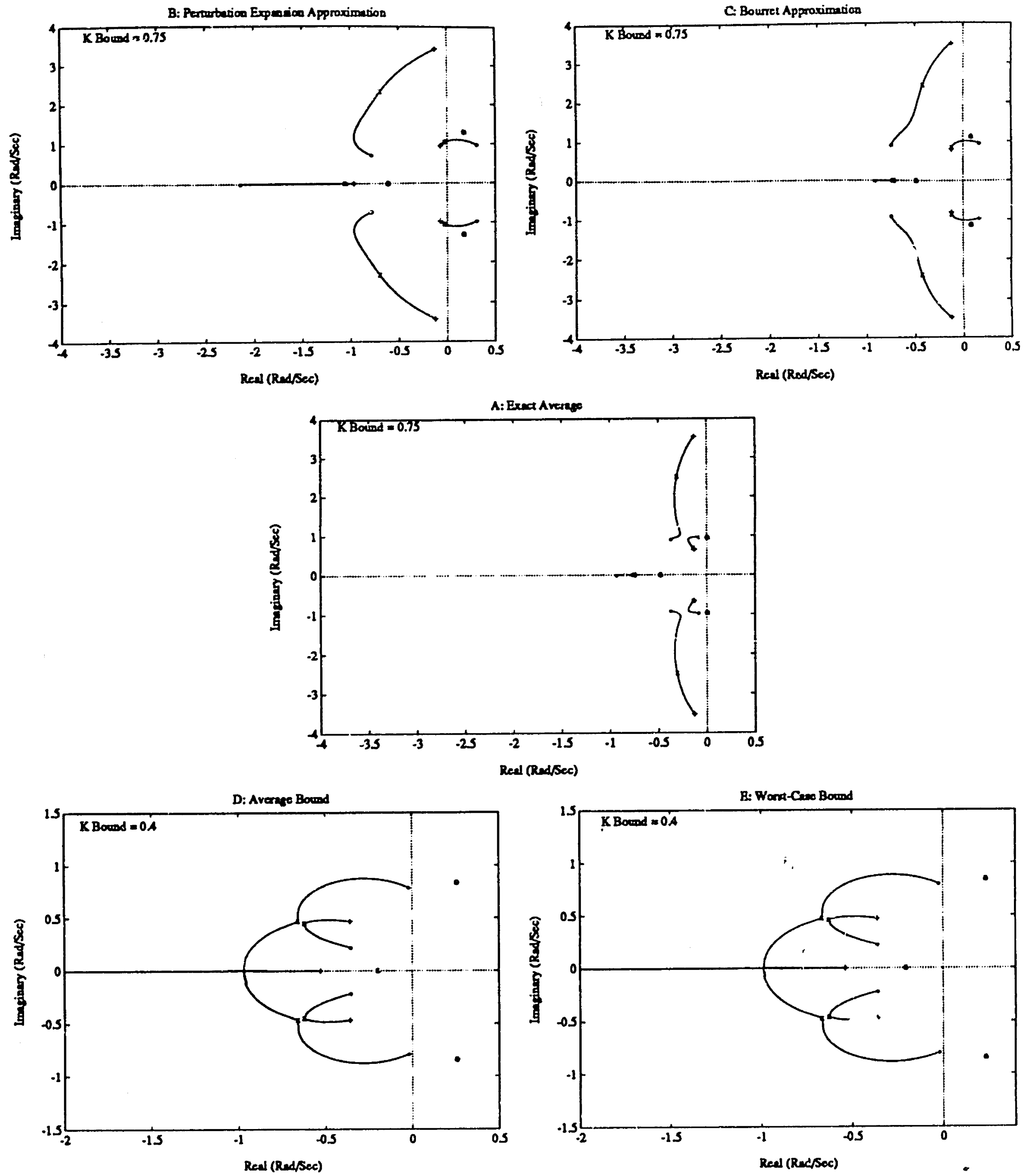

Figure 5.14: Closed-Loop Pole Locations as the Parameter, $\tilde{k}$, is Varied from -0.75 (*) to 0 (x) to 0.75 (+)

frequency magnitude of the loop transfer function and thereby increasing the range over which the magnitude drops below one. These designs also decrease the damping in the high frequency rolloff pole at 8 rad/sec. The magnitude of the transfer function is kept below one, however, to avoid instability.

The bound-based designs lower the frequency of the nonminimum phase zero to about .8 rad/sec but tend to damp it more heavily than the exact average design. This broadens the range of frequency over which the loop transfer function has phase less than one and thereby greatly increases the design robustness. The bound-based designs also increase the bandwidth of the dynamic compensator which would make the system more susceptible to high frequency unmodeled dynamics.

The response transients of the closed-loop system to an impulse disturbance at w are shown in Figure 5.16. The performance robustness of the design can be shown by the dotted and dashed curves corresponding to off nominal parameter values. The PEACM design was not stable at $\tilde{k} = -0.5$ and this curve is omitted. The general trend as one progresses from the approximate to the average to the bound-based designs is for better performance (settling time) and performance robustness at the cost of higher maximum control signals. The EACM and approximation-based designs tend to have more oscillatory behavior than the bound-based designs. The bound based designs show time histories quite similar to those for $\mathcal{H}_\infty$ designs for this benchmark problem [131]. The highly damped low frequency dynamics dominate for these designs. All of the designs except the PEACM meet the 15 second settling time specification stated in [129].

The two-mass benchmark problem demonstrates the relative characteristics of the five designs. The bound-based designs yield nearly identical compensators for the cases considered. This reduces the motivation for using the computationally more intensive average bound. The exact average minimization has predictably good characteristics but can really only be applied to simple systems or systems that can be decomposed into simple systems. The Bourret approximate average is clearly

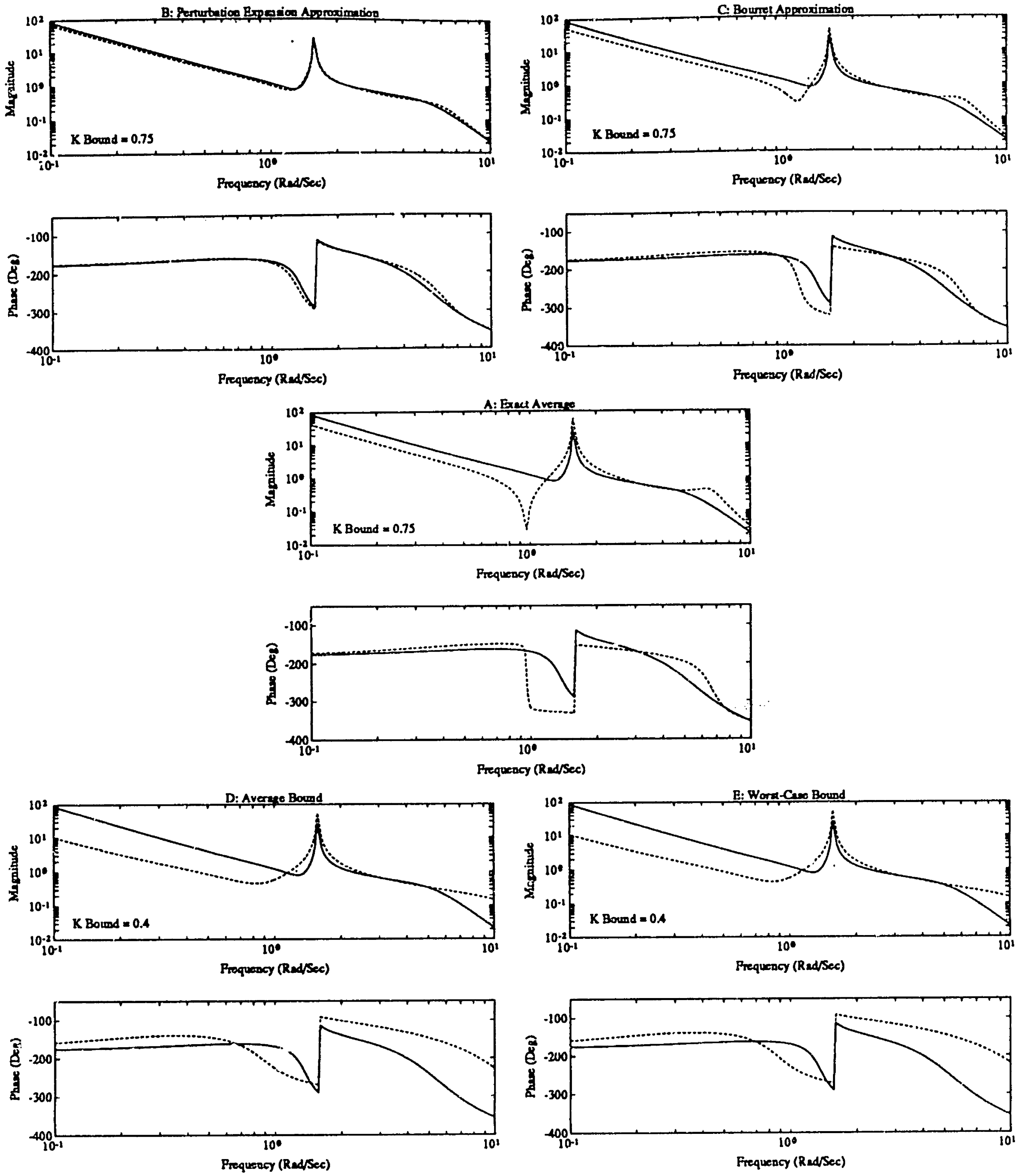

Figure 5.15: Open-Loop Transfer Functions of the Various Designs (dashed) compared to LQG (solid)

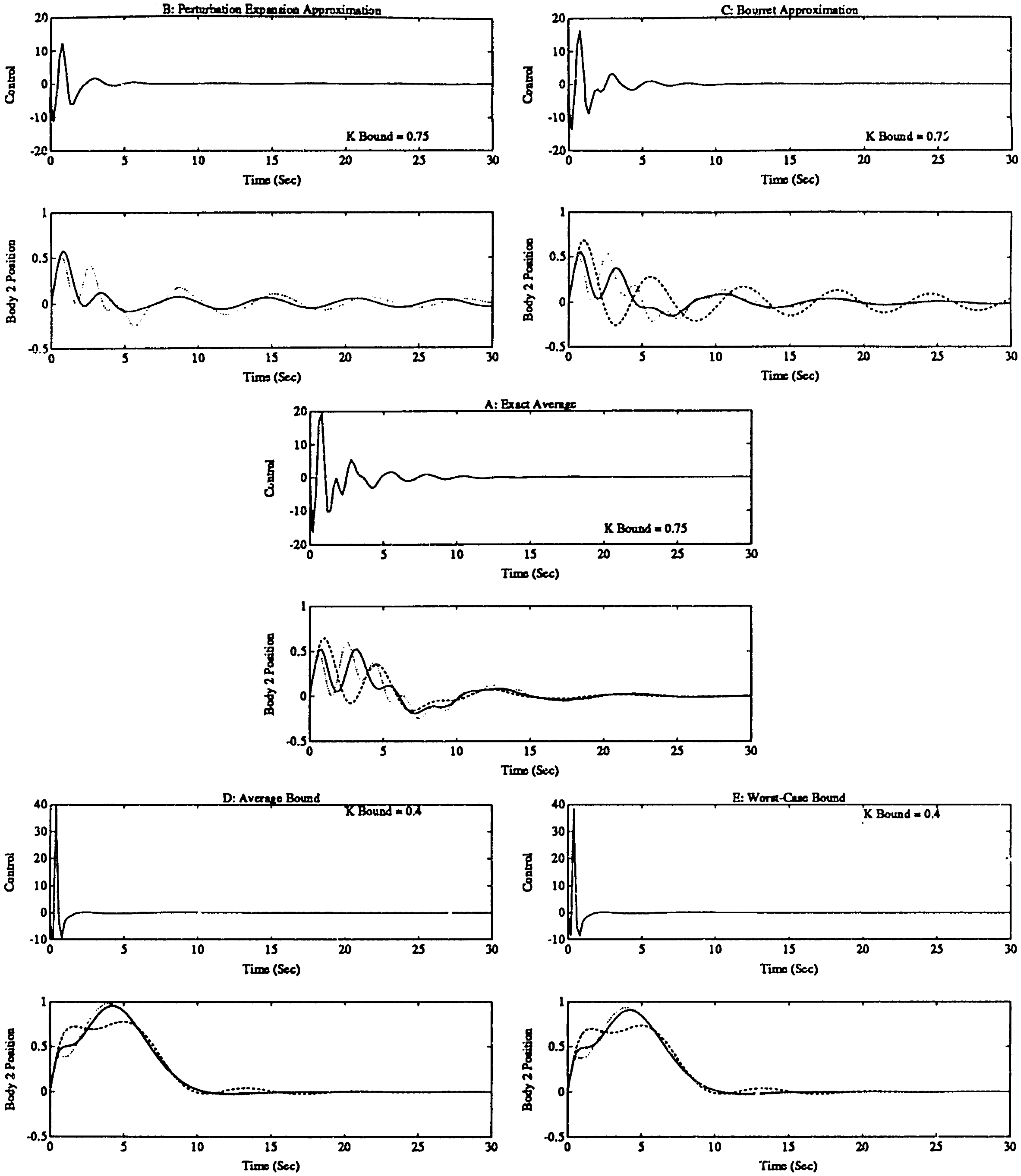

Figure 5.16: Closed-Loop Impulse Response Time Histories: $\tilde{k} = 0$ (solid), $\tilde{k} = -0.5$ (dash), $\tilde{k} = 0.5$ (dotted)

superior for design since it has much better efficiency than the perturbation expansion approximation design and nearly recovers the guaranteed stability properties of the exact average without the associated computational burden. The relative times for the computation of the respective costs will be presented in a later section.

5.3 Example 2: The Cannon-Rosenthal Problem

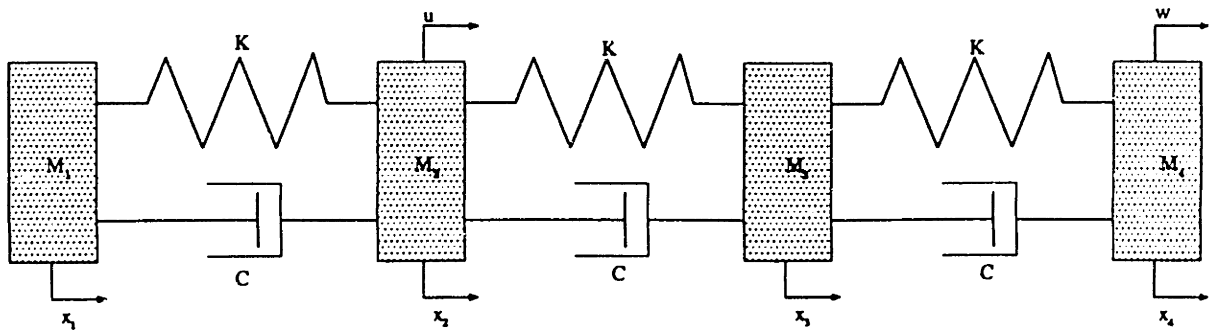

Figure 5.17: The Cannon-Rosenthal Problem

In this section, a four mass/spring/damper problem will be examined which was presented first in [140] and examined as a typical uncertain flexible structure in [141], [100], and in the context of passive damping in [143]. The layout of the system is shown in Fig. 5.17. The system consists of four masses connected by springs and viscous dampers. The uncertainty enters into the problem through a variable body-1 mass. The system can be represented in state space as

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & D_{21} & 0 \end{array}\right] \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{5.10}$$

where the matrices are defined

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 & -c/m_1 & c/m_1 & 0 & 0 \\ k/m_2 & -2k/m_2 & k/m_2 & 0 & c/m_2 & -2c/m_2 & c/m_2 & 0 \\ 0 & k/m_3 & -2k/m_3 & k/m_3 & 0 & c/m_3 & -2c/m_3 & c/m_3 \\ 0 & 0 & k/m_4 & -k/m_4 & 0 & 0 & c/m_4 & -c/m_4 \end{bmatrix} \tag{5.11}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1/m_4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/m_2 \\ 0 \\ 0 \end{bmatrix} \tag{5.12}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{5.13}$$

where

x = plant states (positions and velocities of the four masses)
u = control input
w = plant disturbance
y = sensor measurement
v = sensor noise
z = performance variable

For this problem the nominal values of the springs, dampers and masses were chosen to be $k = 1$, $c = .01$, $m_2 = m_3 = m_4 = 1$, and $m_1 = 0.5$. With these choices the nominal system has poles and zeros as given in Table 5.3

The uncertain mass enters into the problem in an interesting way. Within the system described in Eqs. (5.10), the uncertain mass, m_1, enters into the equations through its inverse. The inverse of the mass will therefore be used as the uncertain parameter called $\tilde{m}$. If the nominal value of m_1 is 0.5, then the uncertainty can be represented as

$$1/m_1 = 1/m_{1_0} + \tilde{m}, \quad m_{1_0} = 0.5, \quad |\tilde{m}| \leq \delta_m \tag{5.14}$$

Thus m_1 varies from 1 to 0.25 as $\tilde{m}$ varies from -1 to 2. With this factorization the set of systems can be defined in the notation from Definition 2.2.4. In particular, only the A matrix is uncertain. It can be decomposed as

$$A(\tilde{m}) = A_0 + \tilde{m}A_m$$

	$m_1 = 0.5$		$m_1 = 0.25$	
Poles	ω_n	ζ	ω_n	ζ
1	0		0	
2	0.86	0.43%	0.93	0.43%
3	1.56	0.78%	1.67	0.79%
4	1.95	0.97%	2.31	0.77%
Zeros				
1	1.41	0.71%	2.00	0.50%

Table 5.1: Poles and Zeros for the Cannon-Rosenthal Problem

in a manner analogous to the factorization for the uncertain spring and damper in the robust-control benchmark problem. This problem was considered because of a pole-zero flip caused by the uncertain mass. In addition to changing the natural frequencies of all of the modes, as the mass is decreased from its nominal value of 0.5 to 0.25, an undamped zero between the first and second modes moves to between the second and third modes as shown in Table 5.3. The second pole and zero flip relative positions along the imaginary axis. This effect is illustrated by the two open-loop transfer functions shown in Figure 5.18. This type of uncertainty is especially difficult to deal with since in effect the phase of the second mode can vary by ± 180 degrees between elements of the model set. A control which adds damping to the second mode for one element of the set will destabilize another. This pole-zero flip makes the robust control design problem difficult. In addition if there is little damping, then the system effectively becomes uncontrollable or unobservable when the pole and zero cancel. This can cause problems with the solution algorithm.

The robust control design methodologies presented in the previous sections can be applied to this problem. Just as in the Robust Control Benchmark Problem, the evaluation plant outputs, z, are augmented to include all the states and the control,

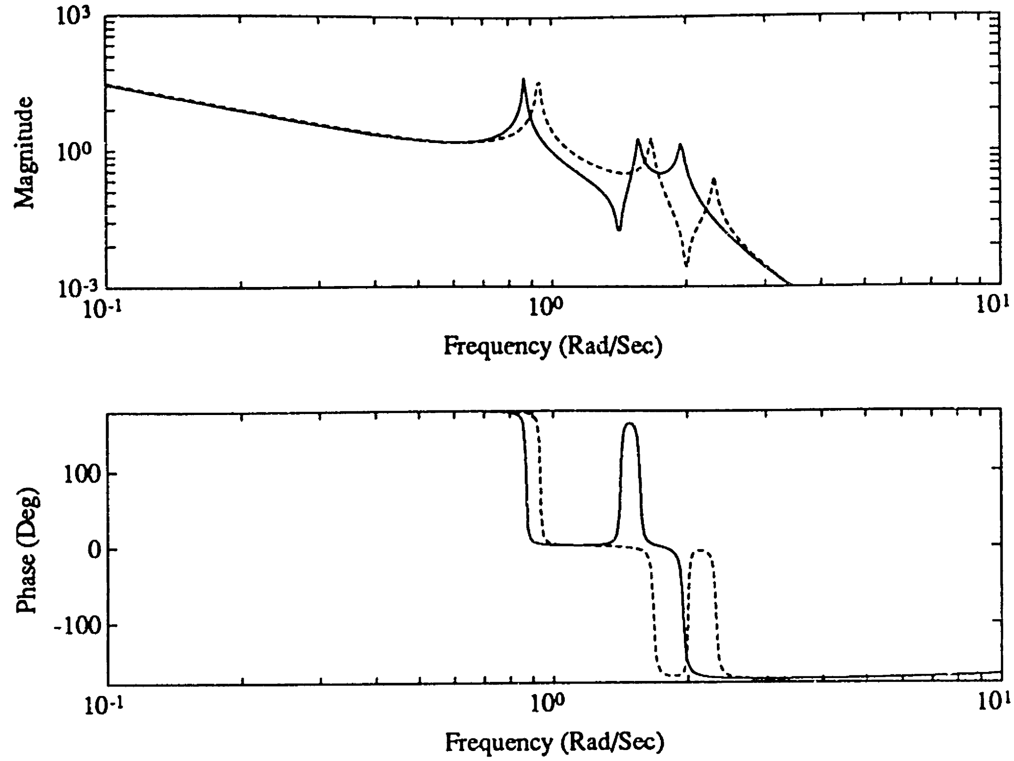

Figure 5.18: The Open-Loop u-y Transfer Functions for the Cannon-Rosenthal Problem for $m_1 = 0.5$ (solid) and $m_1 = 0.25$ (dashed).

$z^T = \begin{bmatrix} x & u \end{bmatrix}^T$. The method of weighting the system that was presented in Section 2.1.2 which is based on the standard LQG design weights will be used for the control design. The evaluation plant given in Eq. (5.10) is modified as in Eq. (2.25) to give the design plant. The control is designed on the design plant and implemented on the evaluation plant.

5.3.1 Dynamic Compensation

In this section, the five control design methodologies will be applied to the problem of determining full-order SISO dynamic output feedback compensators for the Cannon-Rosenthal Problem with noncolocated sensor and actuator and uncertain mass, m_1

which varies in the range

$$0.25 \leq m_1 \leq 1.0 \qquad (5.15)$$

Only the position of the fourth mass was penalized. The weighting values used in the design are

$$Q(4,4) = 1, \quad R = 0.05 \qquad (5.16)$$

In addition to the state and control penalties, the plant noise and the plant noise intensity were assumed to be

$$\Xi = 1, \quad \Theta = 0.05 \qquad (5.17)$$

The signal noise intensity was chosen low to give a relatively high gain Kalman filter in the LQG design. This choice of penalties makes the LQG controller very sensitive to m_1 varaition and thus presents a challenging robustness problem for the average-based methods.

The robustness properties of the control designs are compared to those of the standard LQG design in folowing discussions. Figure 5.19 compares the closed-loop $\mathcal{H}_2$-norm resulting from the various designs using $\delta_m = 0.1$ as a function of the deviation. $\tilde{m}$, from the nominal system mass. Thus as $\tilde{m}$ varies in the range, $-0.1 \leq \tilde{m} \leq 0.1$, m_1 varies in the range, $2.5 \geq m_1 \geq 1.6$. Instability regions are indicated by unbounded closed-loop $\mathcal{H}_2$-norm. The designs can thus be considered stable inside the region descibed by the upper and lower asymptopes. These asymptopes will be called the upper and lower achieved stability bounds for the particular problem.

The LQG results clearly indicate the well-known loss of robustness associated with high-gain LQG solutions. The LQG cost curve achieves a minimum at the nominal mass value, $\tilde{m} = 0$, but tolerates almost no variation in $\tilde{m}$. The stability region is increased by the PEACM and BACM designs at the cost of increasing nominal system closed-loop $\mathcal{H}_2$-norm. The PEACM design increases robustness, but it does not achieve stability throughout the whole design set. The Bourret approximation does achieve stability throughout the set. The EACM design also achieves stability throughout the set as was indicated by the analysis. The cost of this stability guarantee is loss of nominal system performance, although for this small amount of uncertainty the performance loss is negligible. The stability robustness-performance trade will be discussed in more detail with Figure 5.22.

This performance-stability trade is especially evident for the bound-based designs. The nominal costs for these designs are much higher than for the average-based design and its approximations. The bound-based designs extend the upper stability bound for $\tilde{m}$ much more than the lower stability bound. It is also interesting to note that although the stability bounds for the ABM and WBM designs are identical the ABM design achieves this with lower nominal performance. As for the Robust-Control Benchmark Problem, the relative flatness of the bottoms of the performance "buckets" indicates that the performance of the various designs are also relatively robust to changes in the mass.

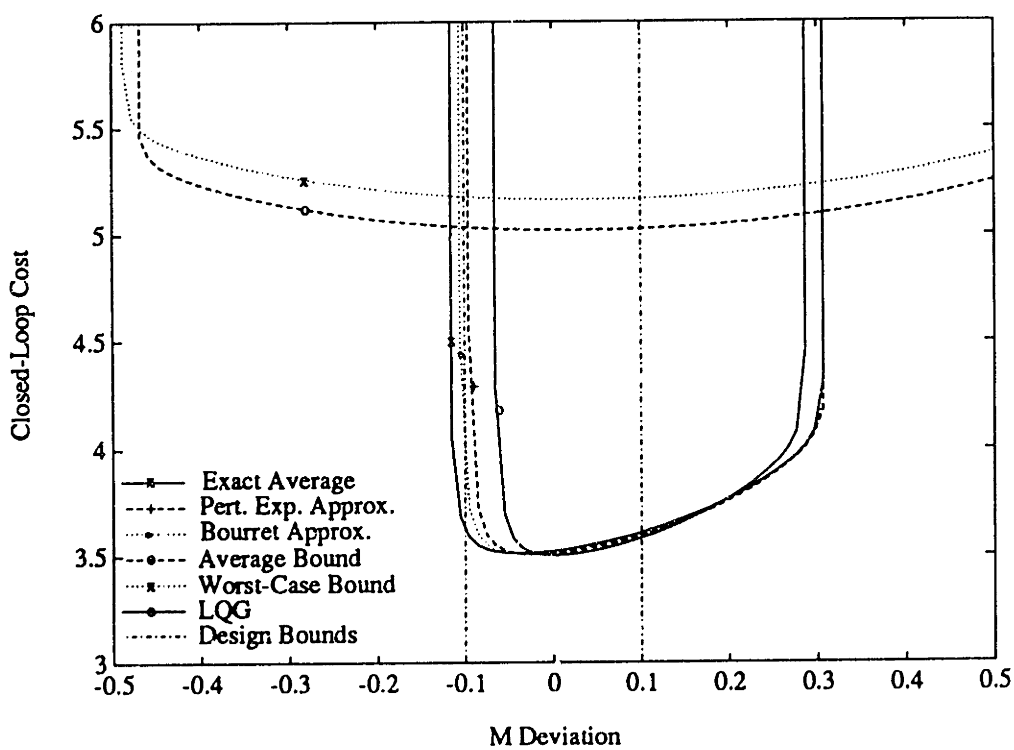

Figure 5.19: System Closed-Loop $\mathcal{H}_2$-norm as a Function of $\tilde{m}$, the Deviation about $1/m_1$, for Controllers Designed Using $\delta_m = 0.1$.

Figure 5.20 shows the upper and lower values of $\tilde{m}$ beyond which the respective designs are unstable as a function of the bound on the parameter variation used in the design, δ_m. Figure 5.20 is thus a plot of the actual stability range achieved as a function of the the parameter bound used in the design. The system is thus stable in the range

$$-\delta_L \leq \tilde{m} \leq \delta_U$$

where δ_L is the lower sability bound and δ_U is the upper stability bound. For the designs considered, the lower $\tilde{m}$ bound was always smaller than the upper indicating that the design proceedures had more difficulty extending the stability range for negative $\tilde{m}$ (large mass) than for positive $\tilde{m}$ (smaller mass). The difficulty in increasing the lower bound is also illustrated by the fact that the lower bound usuallly is monotonic with the design bound while the upper bound is generally non-monotonic. This is especially true of the bound-based designs which decrease the upper bound to achieve greater lower bound. For the exact average and approximation-based designs the upper bound increases rapidly with increasing design bound in the region of $\delta_m = 0.5$ while the rate of lower bound growth remains constant. The lower bound in the limiting factor in estabilishing a stable range about the nominal and will thus be used for comparison in the performance-robustness trade.

With no design uncertainty all five techniques converge to the stability bounds achieved by the standard LQG design ($|\tilde{m}| \leq 0.06$). Just as for the Robust-Control Benchmark Problem, as the uncertainty used in the design process is increased the achieved robustness is also increased. Again, the EACM design always increases robustness enough to guarantee stability throughout the design set, while the approximate cost minimization techniques don't provide this guarantee. Their curves lie below the EACM design's. The EACM design curve has unity slope indicating that the EACM design achieves nonconservative stability over the parameter set used in the design as was predicted by the analysis. The EACM design only achieves stability over parameter range used in the design. The BACM design does come closer to guaranteeing stability than the PEACM design which has difficulty extending the stability range. In particular, for the PEACM design, increasing the design bound above $\delta_m = 0.5$ yields no increase in the achieved stability bound. The bound-based designs achieve a much larger stability range for a given design bound. The ABM and WBM designs were again shown to be nearly identical.

The design costs associated with the nominal system ($\tilde{m} = 0$) are ploted as a function of the achieved lower stability bound in Figure 5.21. The total cost curves shown in Fig. 5.21 exhibit the same general trend as for the Robust Control Benchmark Problem. Figure 5.21 is an indicator of the efficiency of the robust design proceedure. The EACM design is most efficient followed by the BACM design. In this problem the PEACM design exhibited much better relative efficiency than in the previous section. It cannot however yield controllers with stability bounds larger than 0.2. Increasing the design bound has no effect on the achieved bound. In essence the EACM design "stalls" out. This is possible because there are no stability guarantees associated with a given design bound. The bound based designs fared particularly poorly in relative efficiency in this problem.

The output costs are the chief contributors to the total cost as shown in Fig. 5.22. The control cost shown in Fig. 5.23 are lowered in all of the designs methods so as to increase the achieved stability robustness. Lowering the control cost is indicative of lower gain controllers. This is the opposite trend as the one observed in the benchmark problem where the control cost increased with greater achieved stability range. The Benchmark Problem has only a single open-loop resonant pole which can be easily phase stabilized. For the Cannon-Rosenthal there are modes which cannot be phase stabilized due to the large phase uncertainty caused by the pole-zero flip. The only alternative left to the robust design proceedure is gain stabilization. The lowering of the control cost is contrary to the traditional expectation that bound-based designs are associated with high loop gain. In all of the designs, the loop gain and control cost must be lowered to achieve larger stability ranges.

The level of inherent system damping played a critical role in enabling robust

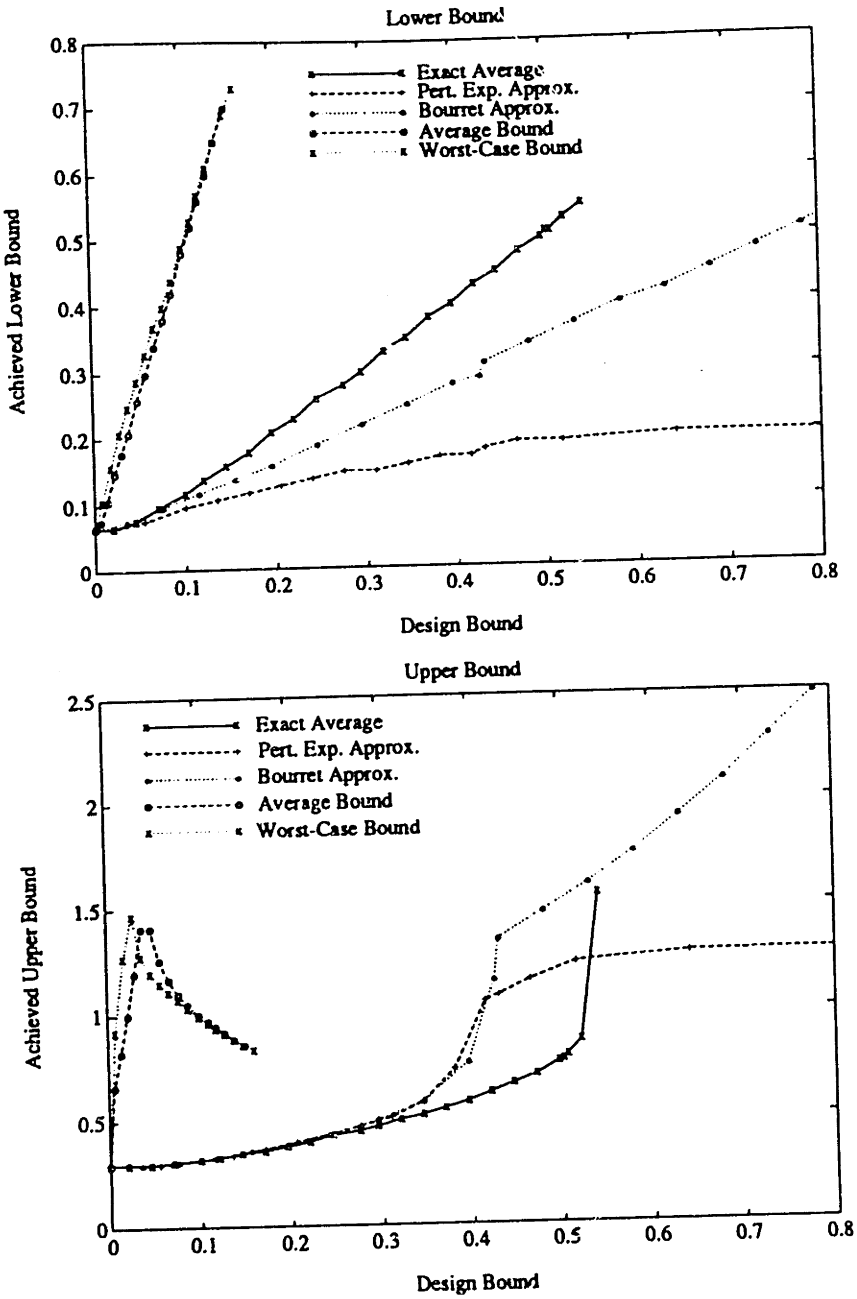

Figure 5.20: Achieved Upper and Lower Closed-Loop Stability Bounds as a Function of the Bound Used in the Design, δ_m

control for this problem as was indicated in [143]. If there was no system damping, no design was found which could extend the stability range to include the pole-zero flip. This numerical result is verifiable by simple Nyquist arguments. Figure 5.22

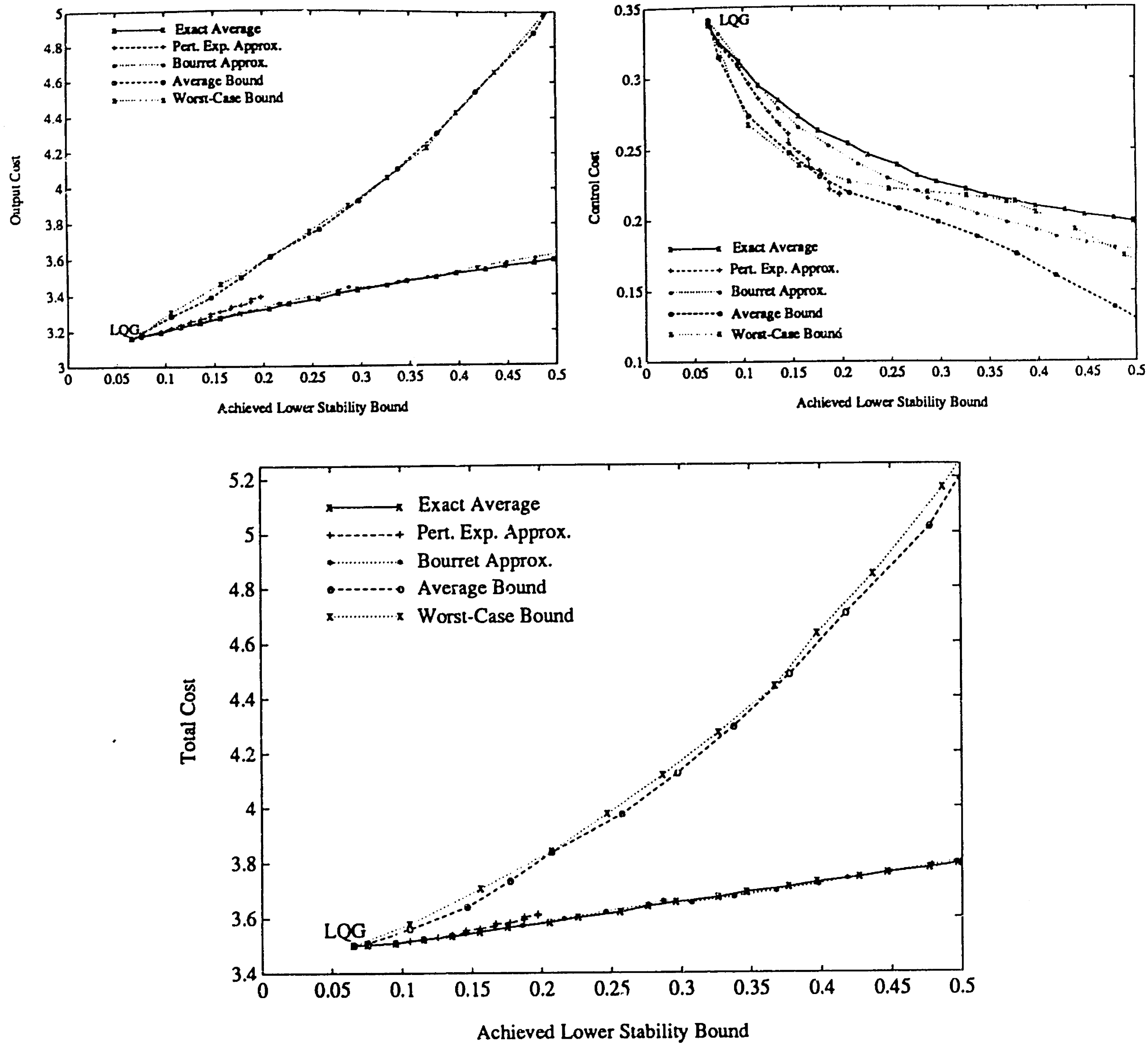

Figure 5.21: Total Cost, Output Cost, and Control Cost as a Function of the Achieved Stability Bound.

shows how increasing damping helped the Bourret control designs. The nominal system costs are normalized by the LQG costs associated with that level of damping. Thus the effect that damping has on nominal performance has been removed from the curves and what remains is the effect that damping has on the robust design process. With no damping the cost curve asymptotes to infinity at the bound value which corresponds to the onset of pole-zero flipping. By adding only a small amount of damping ($C = 0.01, \zeta \approx 0.01$) the design procedure can stabilize the system past the flip, although there were some numerical difficulties as discussed in the next section. Adding yet more damping ($C = 0.1, \zeta \approx 0.10$) not only extends the nominal bounds to include the flip but also aides the robust design process. Adding damping also reduces the relative cost associated with a given stability bound since higher damper values give lower nondimensional cost.

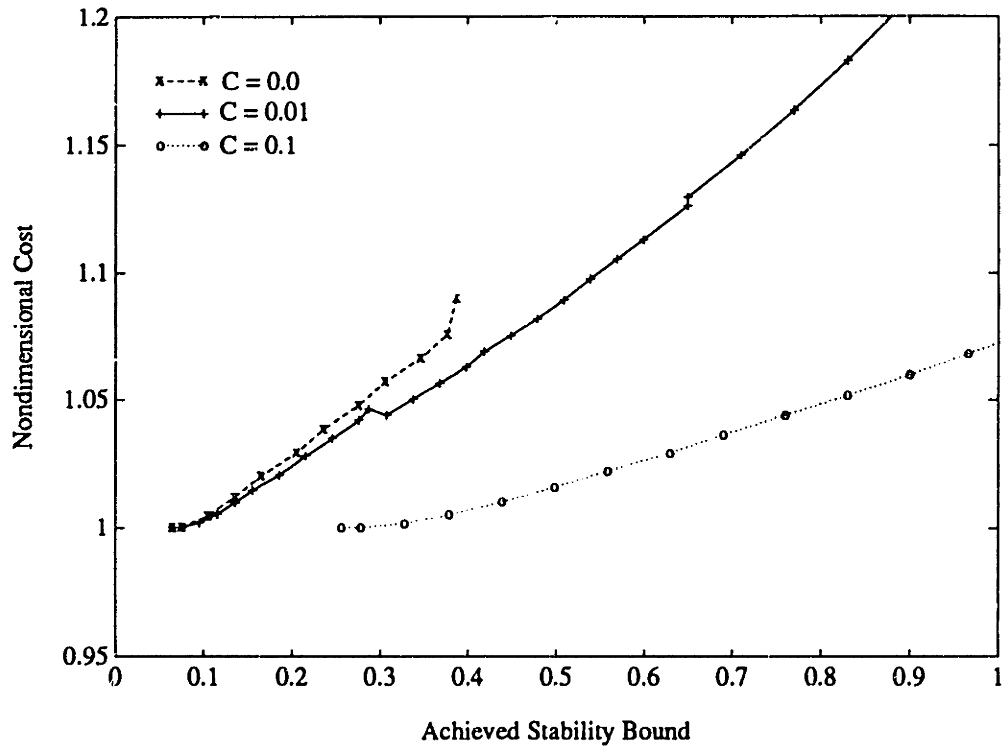

Figure 5.22: Nondimensional Nominal System Cost (J/J_{LQG}) vs. the Achieved Stability Bounds Given by the Bourret Designs for Various Damper Values.

The characteristics of some of the controllers derived for the Cannon-Rosenthal Problem will be presented in the following paragraphs. For the design bound values considered the PEACM and ABM design have characteristics very similar to the

BACM and WBM designs, respectively. The PEACM and ABM designs will therefore not be shown for clarity of presentation.

Figures 5.23 and 5.24 show the compensator and closed loop poles as a function of the bound on $\tilde{m}$ used in the design, δ_m. The general trend is for the robust compensator poles and zeros to be detuned from the uncertain plant poles and zeros. As in the benchmark problem, the LQG design gives nonminimal phase compensation. The LQG compensator places zeros very near to the second and third flexible modes of the system and a pole very near the plant zero at 1.41 rad/sec. To some extent all of the designs move these "plant inversion" poles and zeros away from the actual plant poles and zeros, therby detuning the inversion. For instance the undamped nonminimum phase zero at 1.6 rad/sec (which is near the second plant resonance) is moved away from the imaginary axis. The LQG design uses almost perfect pole-zero cancellation at the third mode. In the average and Bourret-based designs the pole-zero cancellation at the third mode is replace by a closely spaced pole-zero-pole configuration. For the WBM based design the pole-zero cancellation is removed by making the compensator pole and zero shift away from the imaginary axis. The zero is made nonminimum phase.

The closed-loop pole and zero loci shown in Figure 5.24 demonstrate the same general detuning trends as for the compensator poles and zeros. The poles and zeros in the range from 1 to 2.5 rad/sec corresponding to the open-loop second and third modes are the ones which effect stability. The LQG design effectively cancels the open-loop plant zero with a compensator pole at 1.45 rad/sec. The robust designs detune this zero-pole cancellation either by lowering the pole frequency as for the Bourret and exact average-based designs or by damping the offending pole as for the WBM design. This detuning has the effect of allowing greater zero location variation without having a pole-zero flip. If the pole and zero were very closely spaced, a slight zero variation would radically change the lightly damped zero's phase contribution at the pole and therby drive the pole unstable. The detuning avoids this problem by increasing the space between the open loop plant zero and the inverting pole.

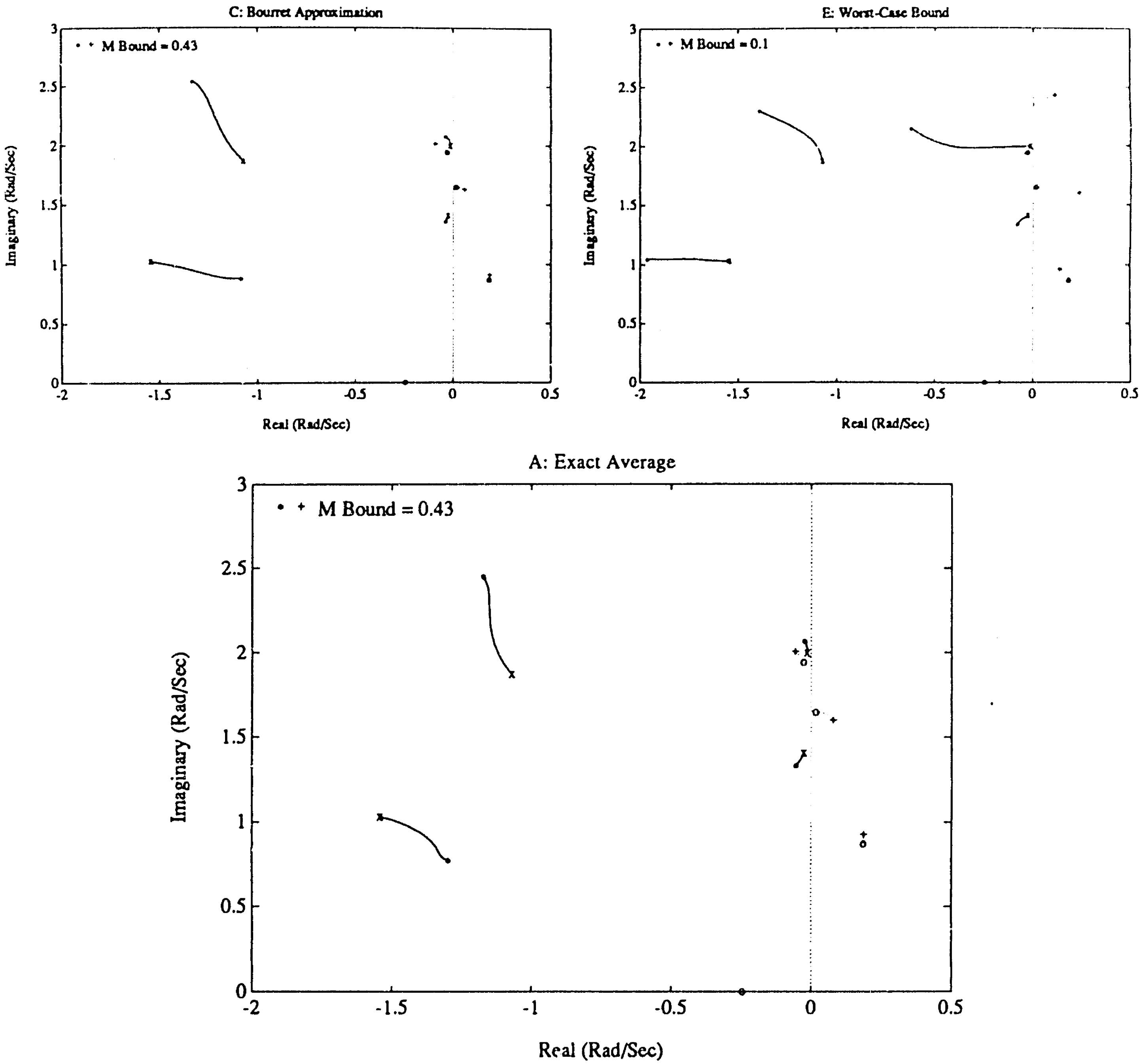

Figure 5.23: Compensator Pole and Zero Loci as a Function of Uncertainty Design Bound as δ_m is Increased from 0 (LQG poles-x, zeros-o) to the Indicated Amount (poles-*, zeros-+).

Figure 5.25 shows the movement of the closed-loop poles and zeros of the robust compensator designs as the uncertain parameter, $\tilde{m}$, is varied over the range from -0.5

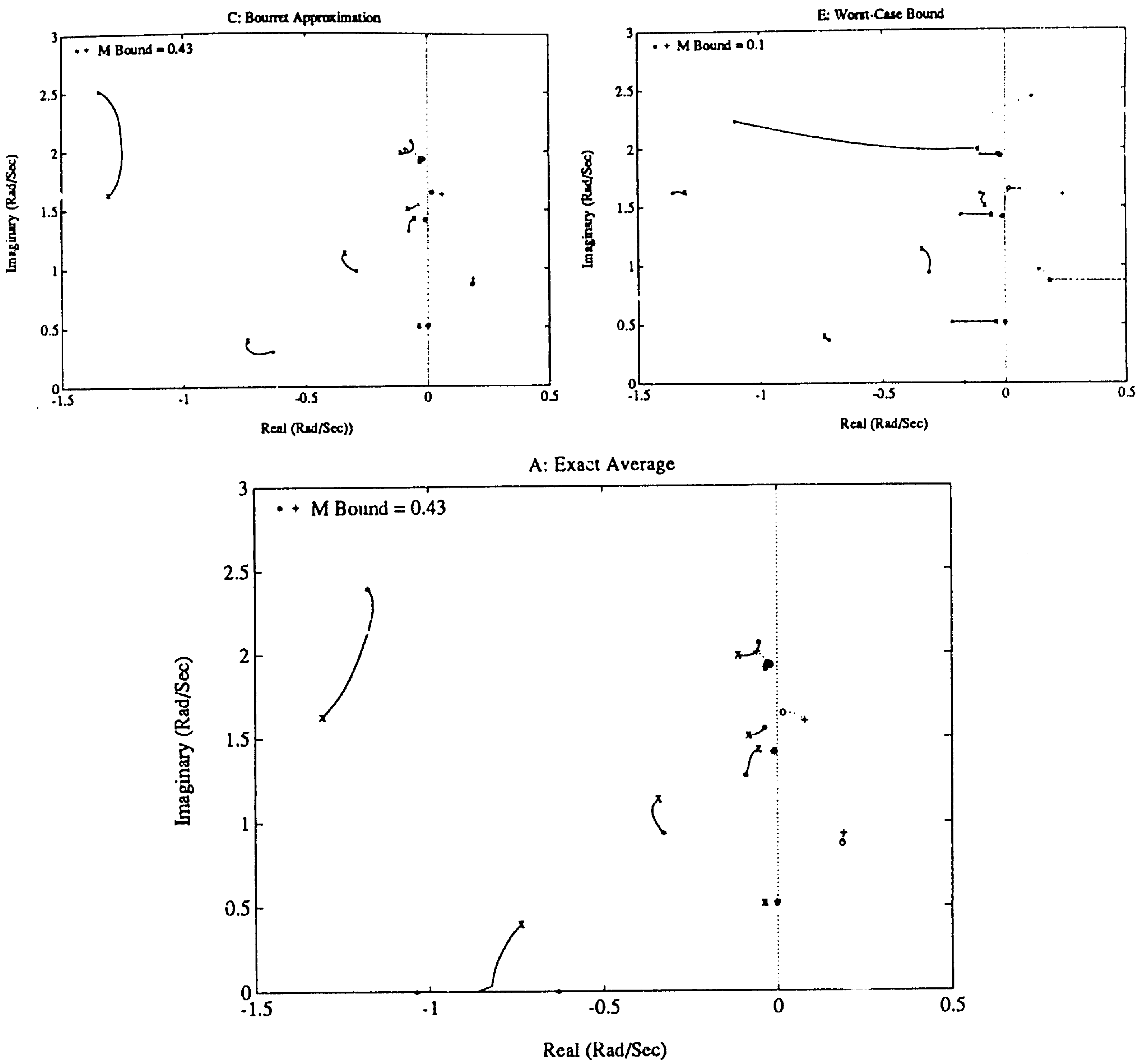

Figure 5.24: Closed-Loop Pole and Zero Loci as a Function of Uncertainty Design Bound as δ_m is Increased from 0 (LQG poles-x, zeros-o) to the Indicated Amount (poles-*, zeros-+).

to 0.5. It shows that the pole which the LQG design placed near the open-loop plant zero at 1.41 rad/sec and which was "detuned" by the robust designs is the pole which limits the achieved stability range. This pole will be called the "offending pole". The parameter $\tilde{m}$ cannot be decreased beyond the point where this pole goes unstable. Thus it is not the pole-zero flip associated with the higher frequency second mode which limits stability robustness but rather a pole-zero flip between the open-loop zero an a lower frequency compensator mode. The path which the offending pole takes toward the right-half plane is dependent on the design technique used. The BACM and ACM designs increase the stability range by causing the offending mode to interact with the second system open-loop resonance. The WBM design increases the stability range by increasing the damping of the offending mode and thus moving it further from instabiliity.

Figure 5.26 shows the loop transfer functions associated with teh various designs compared to the LQG loop. The WBM design generally lowers loop gain. In the exact average and Bourret approximation-based designs, the unusual interaction at the third mode is evident. The pole-zero structure is required to maintain stability in the third mode as the mass is varied. The dynamics in the vicinity of the second mode are especially important because Figure 5.25 showed that these dynamics are critical for stability considerations. In all of the robust designs the zero between the first and second modes is made more visible. The LQG design cancels this zero with a pole. Because the zero location is uncertain, the robust designs detune this offending pole. By removing the pole-zero cancelation the zero is made more visible. This in turn lowers the loop gain in the vicinity of the second mode.

Figures 5.27 and 5.28 show the Nyquist diagrams for the designs at two different values of the parameter, $\tilde{m}$. Figure 5.27 shown the Nyquist diagram at $\tilde{m} = 0$ while Figure 5.28 shows it at $\tilde{m} = -0.43$ (near instability). Instability is indicated by encirclement of the -1 point. One dominant feature of the plots is the initial swing by the -1 point when the magnitude of the loop gain drops below one before the first mode. The phase can wrap around at this point. This trait was common to the Benchmark Problem as well as all of the cases considered for the Cannon-

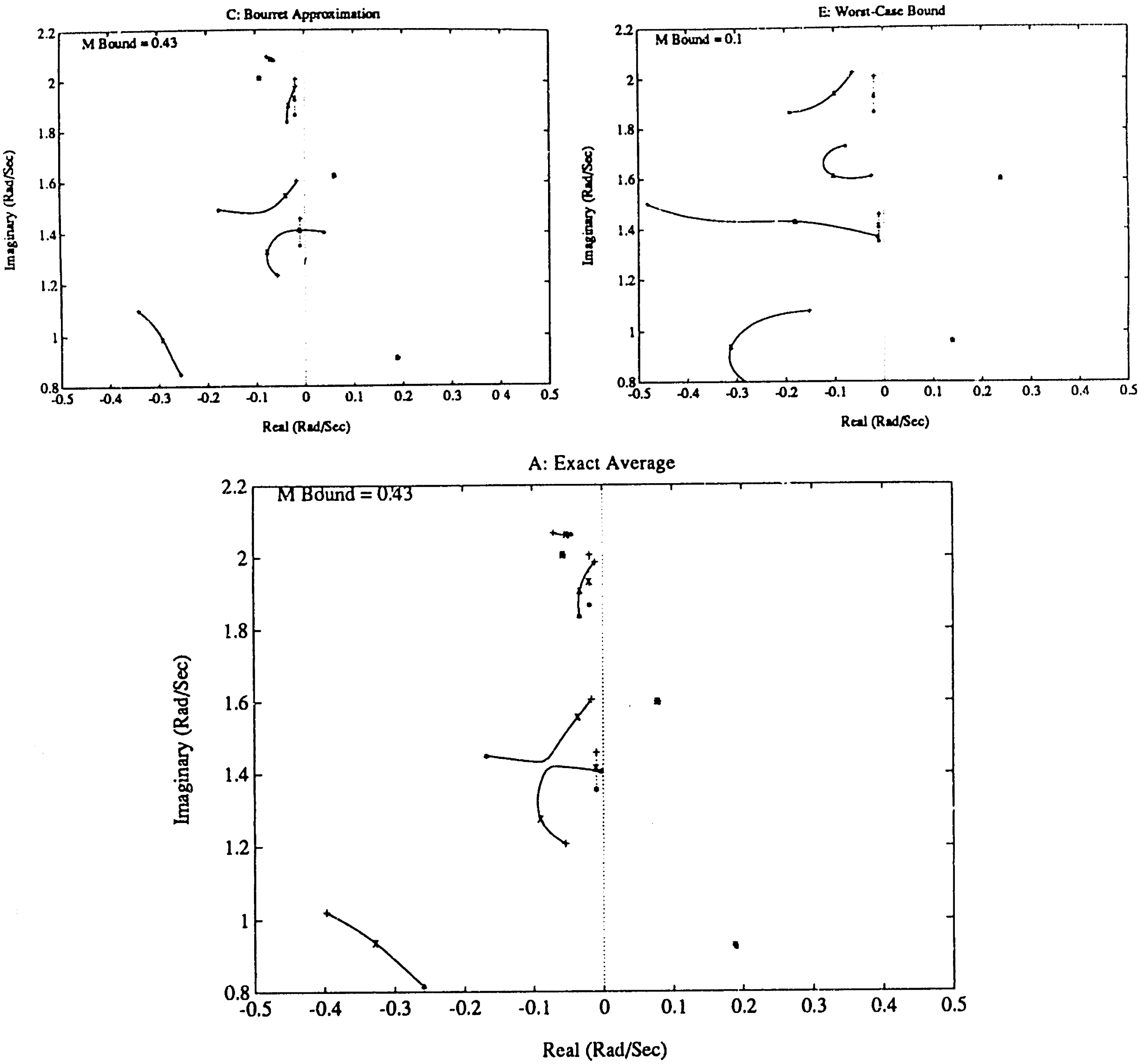

Figure 5.25: Closed-Loop Pole Locations as the parameter, $\tilde{m}$, is varied from -0.5 (*) to 0 (x) to 0.5 (+)

Rosenthal Problem. For the nominal case ($\tilde{m} = 0$), after the first mode the curve goes toward its first zero with an intervening highly damped mode. The "offending mode" causes the curve to loop around an extra 180 degrees. For the perturbed case

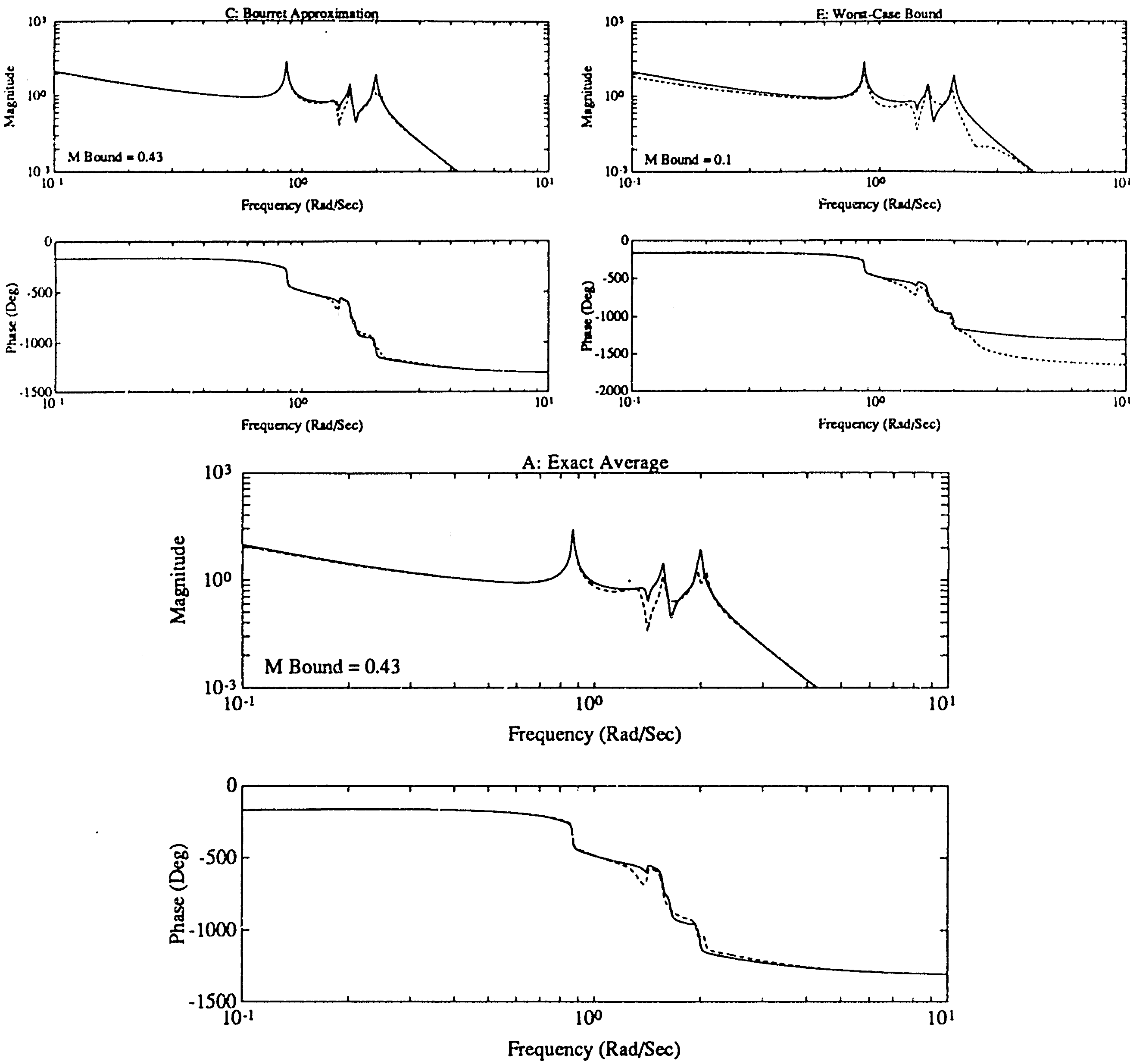

Figure 5.26: Open-Loop Transfer Functions of the Various Designs (dashed) compared to LQG (solid)

($\tilde{m} = -0.43$) the zero occurs before the offending mode and thus the curve is shifted toward encirclement of the -1 point. For the BACM design the -1 point is encircled by the loop associated with the offending pole when $\tilde{m}$ is decreased to -0.43. The EACM and WBM designs are robust to this amount of parameter variation.

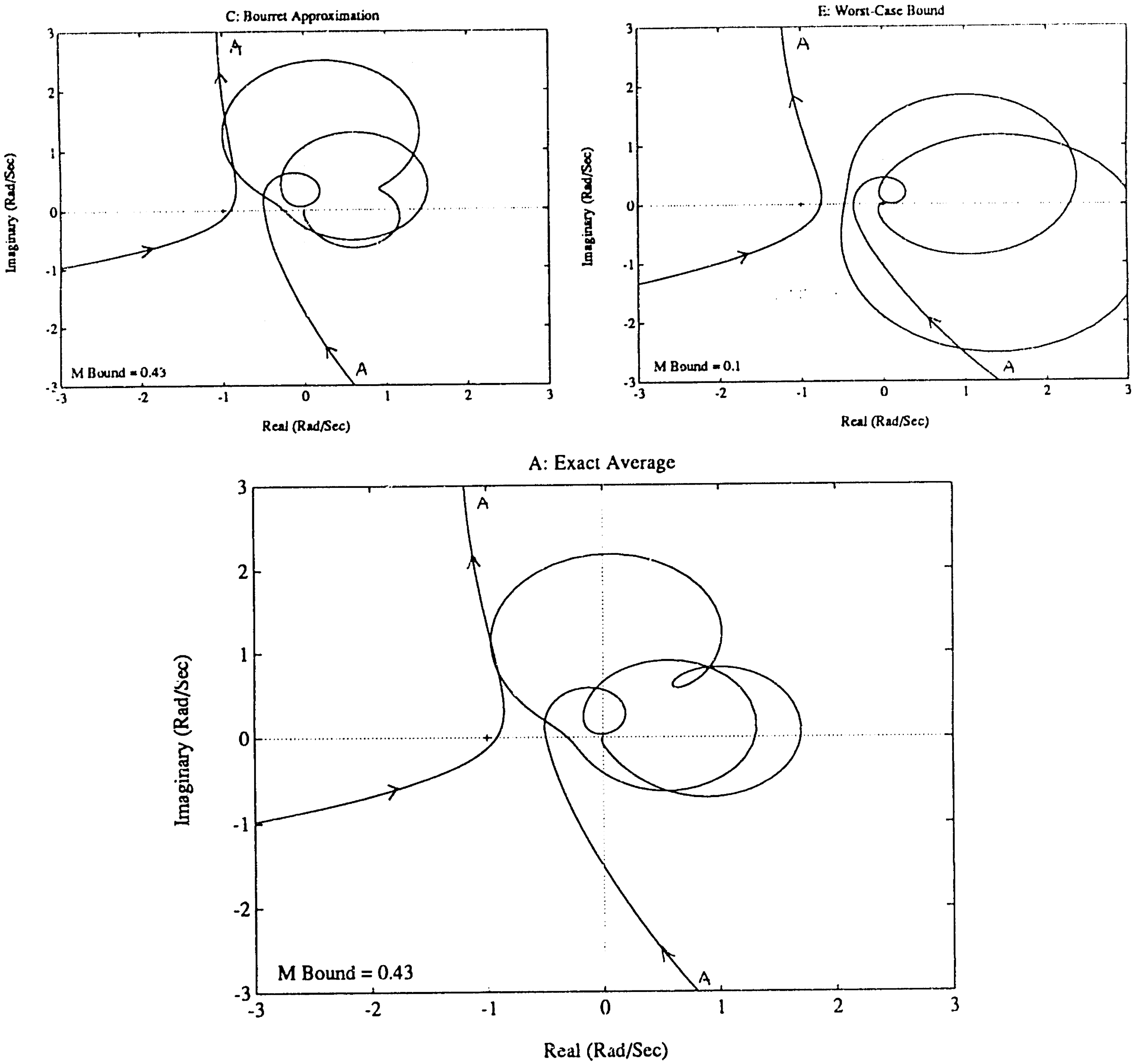

Figure 5.27: Nyquist Diagram for the Various Designs for $\tilde{m} = 0$

Figure 5.29 shows the system transient responses for the various designs. The

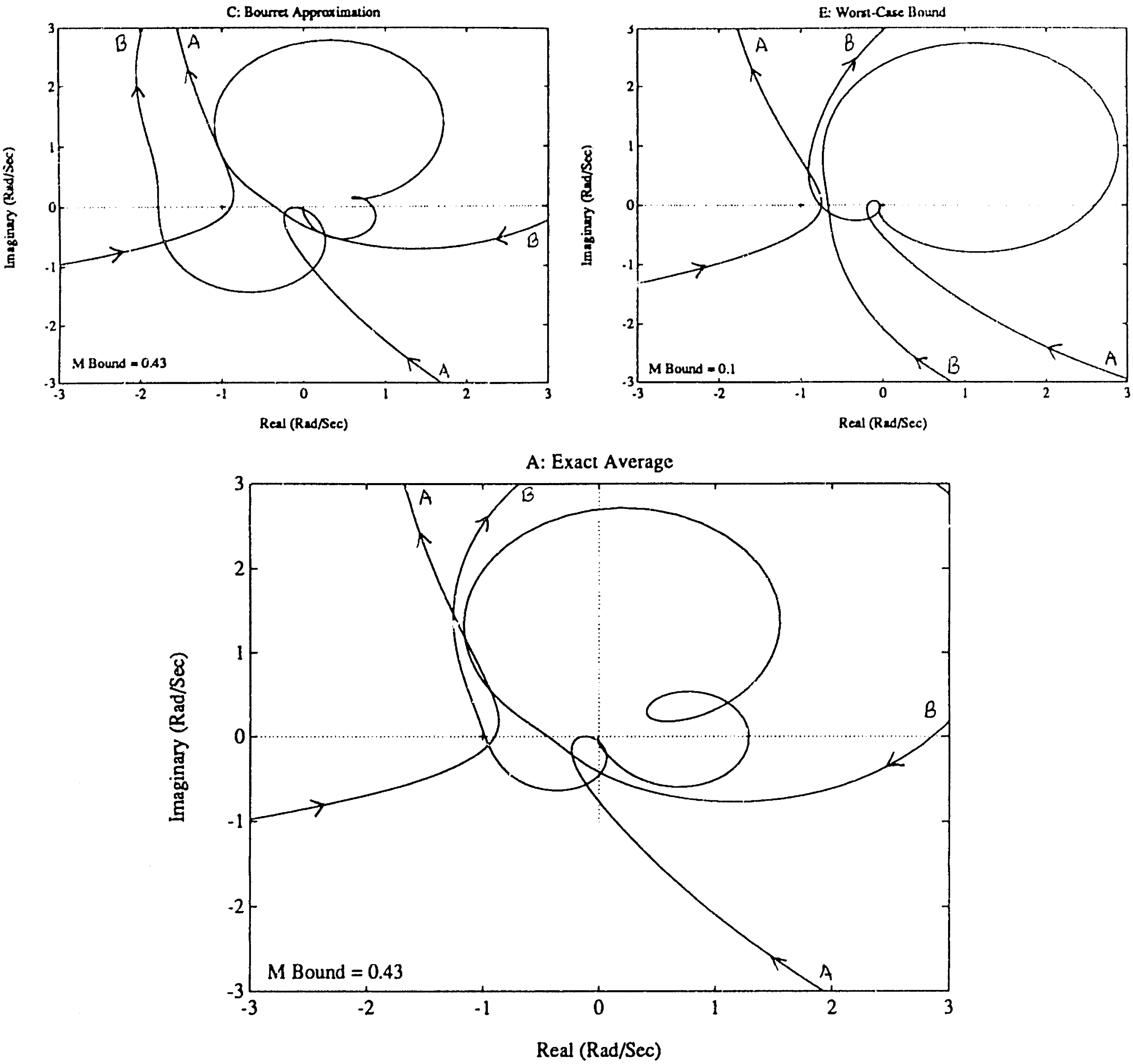

Figure 5.28: Nyquist Diagram for the Various Designs for $\bar{m} = -0.43$

control magnitudes are much lower than in the Robust-Control Benchmark Problem because of the higher control weighting in this problem. In general the designs exhibit good performance reobustness. For all the designs, the transients are dominated by the highly damped pole at 1 rad/sec.

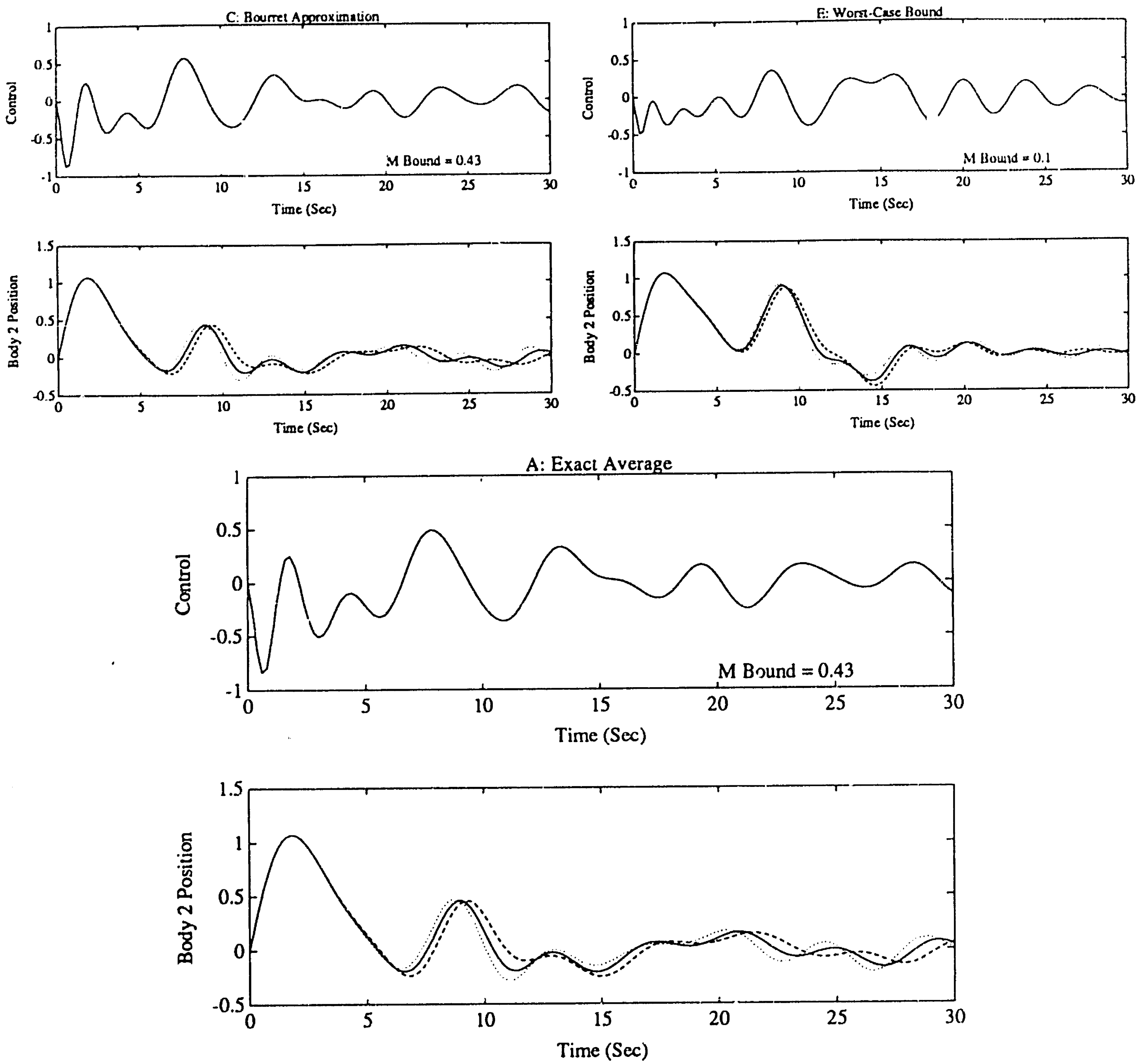

Figure 5.29: Closed-Loop Impulse Response Time Histories: $\tilde{m} = 0$ (solid), $\tilde{m} = -0.25$ (dash), $\tilde{m} = 0.25$ (dotted)

Design	Mflops
EACM	25.8
PEACM	2.5
BACM: Kronecker	45.2
BACM: Iteration	12.3
BACM: Lyapunov	6.1
ABM: Kronecker	43.2
ABM: Iteration	12.3
WBM: Kronecker	34.8
WBM: Iteration	6.3

Table 5.2: Number of Floating Point Operations required for a Single Cost and Gradient Calulation for the Respective Techniques and Solution Methods.

5.3.2 Computational Issues

In this section the computational issues associated with the robust controller design techniques will be discussed. One issue is the computational burden associated with each design method. One measure of this computational burden is the number of floating point operations reguired for a single cost and gradient calculation for each of the five design techniques. These results are presented in Table 5.3.2.

The EACM design cost equation was solved using 32 point Gaussian quadrature to integrate the cost and gradients over the single uncertain parameter. Even with only a single ucertain parameter the computational burden is high and will increase geometrically for multiple parameters. The PEACM design was the least computationally intensive requiring less then a tenth of the CPU power of the EACM design. The solution technique adopted for the PEACM design was the direct heirarchical solution described in Remark 3.2.1. There were three methods presented for computation of the Bourret equation, and the computational cost is dependent on the method adopted. The most computationally intensive was the use of Kronecker notation and associated high dimensional matrix manipulation. This was followed by the computational cost associated with computing the Bourret equations by iteration on the equation as is discussed in Remark 3.2.4. The operation count is given for 5 iterations of the equations. The least computationally intensive method which can only be applied for the case of decoupled uncertainties was the double-order Lyapunov solution proceedure presented in Remark 3.2.5. The bound-based designs can be solved using either the Kronecker notation or repetitive iteration. In general for this problem, the iteration techniques were several times faster than the Kronecker techniques.

The number of operations for a cost and gradient calculation is only one method to measure the computational burden of the design techniques. Some designs are better behaved and therefore require fewer cost calulations. In practice, the solution times for the PEACM and BACM (Lyapunov method) approximation-based techniques were similar and ran in the range of 10 hours of Sun Sparcstation CPU time for high-uncertainty problems with little damping. A small amount of problem damping reduced the computation time 80%. It is believed that this can again be decreased an order of magnitude by more efficient coding. The EACM, WBM (Kronecker method) and ABM (Kronecker method) designs presented in this section required on the order of 100 CPU hours to compute compensators for large levels of uncertainty. To achieve stability in the range of $-0.2 \leq \tilde{m} \leq 0.2$ required only about 10 CPU hours. Adding more robustness past this level was progressively more numerically difficult.

It is also usefull to examine the nature of the homotopy path for the various designs. Figure 5.30 shows the $\mathcal{H}_2$-norm of the robust compensators as a funcion of the parameter bound used in the design, δ_m. The curves associated with the exact average and bound-based designs are smooth functions of the design bound while the curves associated with the approximation based designs have a discontinuity in the region of $\delta_m = 0.43$. This discontinutiy represents a radical change in the actual compensator matrices associated with the PEACM nad BACM designs. They represent a discontinuity in the solution manifold associated with the homotopy and illustrate a shortcoming of the hootopy solution method since a small change in the design bound is supposed to keep the sclution near the optimal. The discontinuity represents a change in the nature of sclution used to add robutness for these design techniques. It also interesting to note that the compensator $\mathcal{H}_2$-norm generally decreases (lower gain solutions) as the design bound is increased. The notable exceptions are the infinite gain asymptopes of the bound-based designs associated with these designs reaching the limit of their capability to stabilize the system.

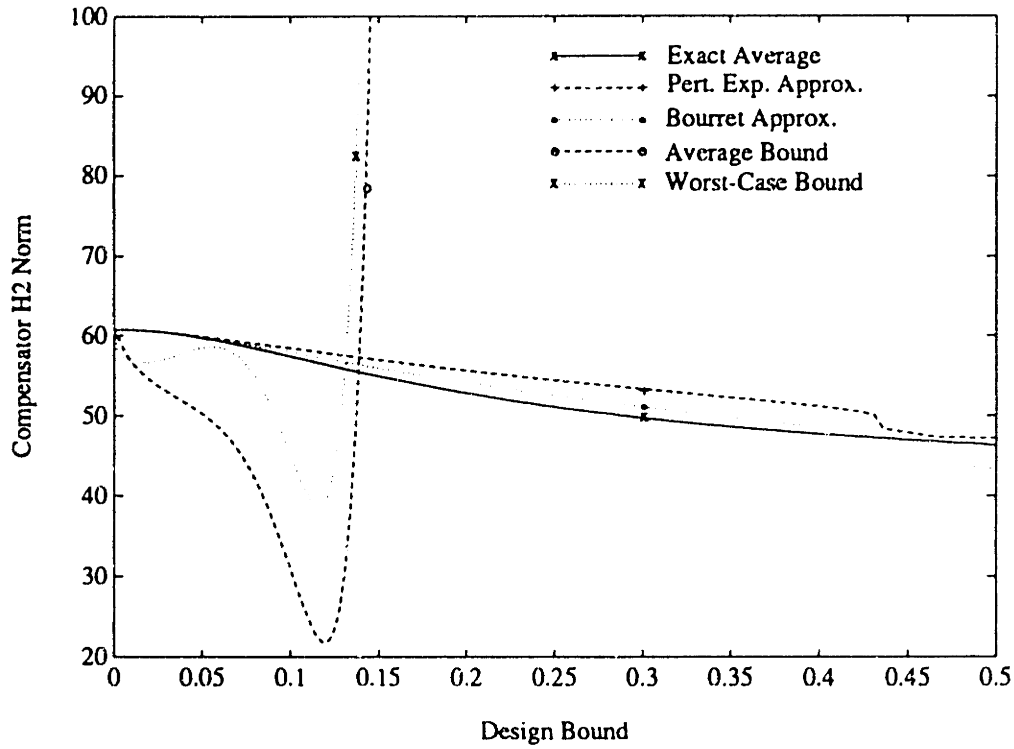

Figure 5.30: Compensator $\mathcal{H}_2$-norm as a Function of the $\tilde{m}$ Bound used in the Various Design Methods

5.4 Summary

In this chapter, examples of control synthesis for parametrically uncertain systems have been presented. The control design methodology presented in Chapter 5 was applied to two systems representing flexible structural plants. These systems were the fourth-order Robust Control Benchmark Problem and the eighth-order Cannon-Rosenthal Problem. The exact average and bound based designs were shown to provide stability over the design model set. The Bourret approximation-based design, while not guaranteeing stability over the design set, did posess some useful properties. Among these are low nominal cost increase for a given achieved uncertainty bound. This property is called design efficiency. In all the examples, the exact average-based design was most efficient with the Bourret approximation based design a close second. The perturbation expansion-based design was shown to have particularly poor efficiency in these examples and in many cases couldn't yield controllers to stabilize a system which the other design methods could. The bound based designs were shown to be essentially equivalent in the three cases considered. Both bound-based designs guaranteed stability over the design set and in general over a much larger set than the design set. This conservatism resulted in lower design efficiency for large uncertainty levels.

Chapter 6

Reduction of Parameterized Systems

6.1 Introduction

This section addresses the problem of reducing the size of the uncertain model for the purposes of control design. As demonstrated in the simple examples in the previous chapter, the robust control design techniques examined in this thesis are computationally intensive. A necessary step to applying these techniques on more realistic high order systems with multiple uncertain parameters is the process of model reduction. In this chapter, techniques for reducing the model order and number of uncertain parameters are presented to enable the application of the robust control design techniques to higher order systems. The discussion is intended to address the important issues in uncertain model reduction and present some usefull tools for the reduction process.

In the preceding chapters, the exact average cost and related approximations and bounds were presented as possible performance metrics for systems with parameterized uncertainty structures. These performance metrics were applied to the problem of robust control synthesis in previous chapters. The cost functionals presented can also provide a framework for comparing parameterized sets of systems for the purposes of model reduction. Each cost functional defines a measure which can be used to determine the relative importance of elements of the uncertain system. The cost functionals can thus be used to help decide which elements must be included for effective control design.

The fundamental problem in model reduction is to determine which system elements contribute most to the chosen performance metric and should therefore be included in the model used for control design. The elements which do not contribute significantly to the chosen performance metric can be ignored for the purposes of control, i.e., the model can be reduced for control design purposes. This uncertain model reduction can greatly alleviate the computational burden associated with determining the optimal compensator.

There are two type of elements to be considered for parameterized systems. The first are the system *components* or *states.* Reduction of the number of components in a system effects the order of the system. The word "components" is more appropriate than states because several states can be associated with a single component (subsystem) of a given system. When the system is uncertain, the uncertainties must be considered in criteria by which model order is reduced. A component which is unimportant at one set of parameter values may be critical at another. A performance metric which reflects the effects of model uncertainty is therefore essential in the process of model order reduction.

The second type of elements considered are the *uncertain parameters.* The number of uncertain parameters is independent of order of the uncertain system. Each uncertain parameter retained in the control design greatly complicates the process of finding the optimum controller. The problem of reducing the number of uncertain parameters is to find the minimum set of parameters which must be included in the control design to achieve the desired level of stability robustness. This is done by determining which parameters contribute most to the performance metric and must therefore be retained. In the following section, the problems of component and uncertain parameter truncation will be addressed using the methods of cost decomposition. In both cases, the cost can be decomposed into a sum over either the components or the uncertain parameters. This decomposition will then be used to rank the importance of the elements for truncation of the least important. The first model reduction problem considered is model order reduction.

6.2 Uncertain Model Order Reduction

6.2.1 Introduction and Problem Definition

The problem considered in this section is the reduction of the order of a system with parameter uncertainty. A system can be described as a collection of interacting components. If the system is represented in state space, each component has associated with it some subset of the states of the system. Model reduction can then be considered the process of removing the states associated with a given component. The problem is to decide which states should be removed. One method which has been proposed is Component Cost Analysis, Refs. [134, 135], and its application in modal coordinates, Refs. [132, 133, 136]. In this method the overall system performance is given by some function J, called the cost or performance metric. The problem then is to determine what fraction of the overall system performance metric can be attributed to each system component.

Component Cost Analysis (CCA) has been previously applied to the problem of certain system reduction in both open-loop for model reduction, Ref. [134, 135], and closed-loop for controller reduction, Ref. [133,137]. It has also been used to determine critical parameters [101], (addressed in Section 6.3). In this section it will be extended to the problem of uncertain system model order reduction. The uncertain system cost functionals presented in the previous chapter will be used to explicitly measure the importance of a component *in the presence of uncertainties.* Before developing this further CCA for unparameterized systems will be examined.

Component Cost Analysis relies on the idea of cost decomposition. Consider a dynamic system:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{6.1}$$

with $y \in \mathbb{R}^l$, $w \in \mathbb{R}^m$, and the states, $x \in \mathbb{R}^n$. The system above can represent either an open-loop or closed loop system for the purposes of model reduction. Define the performance metric as the squared $\mathcal{H}_2$-norm of the system.

$$J = \|G(s)\|_2^2 \tag{6.2}$$

The *cost decomposition property* asserts that this performance metric can be decomposed into a sum of contributions, J_i, associated with each component.

$$J = \sum_{i=1}^{n} J_i \tag{6.3}$$

These components can be associated with physical coordinates of the model such as sensing, actuation, or structural subsystems or with mathematical components such as modal coordinates as in modal cost analysis, Ref. [132, 133, 136] or balance coordinates, Ref. [138,139]. For the unparameterized system problem, the component costs for the components associated with the states, x_i, can be defined as

$$J_i = \frac{1}{2}\left\{\frac{\partial \|G(s)\|_2^2}{\partial x_i} x_i\right\} \tag{6.4}$$

This function has the properties that, if any modes are simultaneously controllable, observable, and unstable: $J_i = \infty$, and if the component cost is finite,

$$J_i = \operatorname{tr}\left\{QC^TC\right\}_{ii} \tag{6.5}$$

where Q is the solution to the Lyapunov equation:

$$0 = AQ + QA^T + BB^T \tag{6.6}$$

The components can now be ranked on the basis of the magnitude of their component costs.

$$|J_i| \leq |J_2| \leq \cdots \leq |J_n| \tag{6.7}$$

Those components with the largest component costs are the most critical while those with smallest component cost are the least critical. The sign of the component cost indicates the effect of interaction with that component. If the component cost is negative, the component is labeled beneficial; while if the component cost is positive, it is labeled costly.

With this introduction to unparameterized system model reduction, we can now address the problem of uncertain system model reduction. The key to generalizing component cost analysis to uncertain model reduction is to include the effects of the uncertainty in the performance metric. The focus of the next section will be to consider the average cost and its bounds and approximations as possible performance metrics with which to rank component costs.

6.2.2 Parameterized System Component Cost Decomposition

The need to consider the effects of uncertain parameters in the performance metric used to decompose the cost and reduce the system order can be clearly illustrated in a simple example. Consider a two mode system

$$\ddot{\eta}_i + 2\zeta_i\dot{\eta}_i + \omega_i^2\eta = \gamma_i(t) \qquad i = 1,2 \tag{6.8}$$

where η_i is the modal coordinate and γ_i is the modal Gaussian white noise forcing with intensity $E[\gamma_i^2(t)] = \sigma_i^2$. In addition consider a standard quadratic cost for the system defined

$$J = \lim_{t\to\infty} E\left[\sum_{i=1}^{2} \nu\eta_i^2 + \beta\dot{\eta}_i^2\right] \tag{6.9}$$

Modal cost analysis gives the cost attributable to each mode as

$$J_i = \frac{(\nu + \omega_i^2 \beta)\sigma_i^2}{4\zeta_i \omega_i^3} \tag{6.10}$$

If we assume for simplicity that each mode has identical natural frequency and forcing but that $\zeta_1 = .1$ and $\zeta_2 = .2$ then clearly $J_1 > J_2$. If we now assume that the damping ratio of mode 2 is uncertain and can vary in the range $0.4 \leq \zeta_2 \leq 0.0$ then for some values of ζ_2, $J_2 > J_1$ and in fact J_2 can be unbounded. Therefore removing mode 2 based on the nominal values of the parameters alone ignores the possibly critical importance of the second mode at parameter values away from the nominal.

A natural solution to this problem is to include the system's parameter dependence in the performance metric to be decomposed. In this report, that dependence is included by considering the average cost and similarly its approximations and bounds. Decomposing the average cost explicitly incorporates into the reduction process the system's performance away form the nominal.

We are primarily interested in model reduction for the purposes of control design and will therefore consider closed-loop model reduction. closed-loop model reduction is essential because the reduction of the model must be performed based on the performance in the regime of operation. This point is clearly presented in Ref. [1]. Through the remainder of this section, it will be assumed that the loop from y to u has been closed by some fixed-form static or dynamic compensator, G_c, which is subsequently incorporated into the system transfer function from w to z. The set of closed-loop systems, $\mathcal{G}_{zw}$, will therefore be the model upon which order reduction is performed. It will also be assumed that only the open-loop system states are candidates for truncation since the assumption of fixed-form compensation essentially circumvents the need for controller reduction. To begin consider a decomposition of the exact average cost.

Proposition 6.2.1 (Exact Average Cost Decomposition) *Given a compensator, G_c, the exact average cost of a general parameterized set of systems,*

$\mathcal{G}_{zw}$, which is stable for almost all $\alpha \in \Omega$, presented in Eq. (3.5) as

$$J^E = \operatorname{tr}\left\{\left(\int_\Omega \tilde{Q}(\alpha)\tilde{C}_T(\alpha)\tilde{C}(\alpha)\,\mathrm{d}\mu(\alpha)\right)\right\} = \operatorname{tr}\left\{\left\langle \tilde{Q}(\alpha)\tilde{C}^T(\alpha)\tilde{C}(\alpha)\right\rangle\right\} \tag{6.11}$$

can be decomposed

$$J^E = \sum_{i=1}^{\tilde{n}} J_i^E(G_c) \tag{6.12}$$

with the properties that $J^E = \infty$ if there are any unstable, observable and controllable modes in Ω and

$$J_i^E = \operatorname{tr}\left\{\left\langle \tilde{Q}(\alpha)\tilde{C}^T(\alpha)\tilde{C}(\alpha)\right\rangle_{ii}\right\} \tag{6.13}$$

otherwise, and where $\tilde{Q}(\alpha)$ is the unique positive definite solution to

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\tilde{A}^T(\alpha) + \tilde{B}(\alpha)\tilde{B}^T(\alpha) \qquad \alpha \in \Omega \tag{6.14}$$

The exact average cost decomposition possesses all of the properties necessary to be an effective indicator of component importance in the presence of parameter uncertainty. By averaging over the set instead of decomposing the cost at a particular parameter value, a composite indicator of component importance is developed. Just as in the previous chapter, however, there are many important situation where the exact average cost decomposition is hard to calculate due to the difficulties in averaging the solution to (6.14) over the parameter set. Again we must turn to the approximations and bounds for the exact average cost to derive computable expressions for cost decomposition. For the approximations and bounds, the set of systems is restricted to be the structured set of systems, $\mathcal{G}^s_{zw}$. Once this restriction is made, the respective costs can be decomposed

$$J_i = \operatorname{tr}\left\{\tilde{Q}\tilde{C}^T\tilde{C}\right\}_{ii} \tag{6.15}$$

where $\tilde{Q}$ is given by the particular approximation or bound used. For the perturbation expansion approximate average cost, given in Prop. (3.2.1), the state covariance matrix, $\tilde{Q}$, can be set equal to $\tilde{Q}^P$, where $\tilde{Q}^P$ is given by the solution to

$$0 = \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{6.16}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right) \quad i = 1, \ldots, r \tag{6.17}$$

$\tilde{Q}^0$ is the nominal Lyapunov equation solution, and σ_i is defined in (3.47).

For the Bourret approximate average cost decomposition, Prop. (3.2.2), the state covariance matrix, $\tilde{Q}$, can be set equal to $\tilde{Q}^B$, where $\tilde{Q}^B$ is given by the solution to

$$0 = \tilde{A}_0\tilde{Q}^B + \tilde{Q}^B\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{6.18}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}_i^T\right) \quad i = 1,\ldots,r \tag{6.19}$$

and σ_i is defined in (3.47).

For the worst case cost bound decomposition, Proposition (3.3.2), the state covariance matrix, $\tilde{Q}$, can be set equal to $\tilde{Q}^W$, where $\tilde{Q}^W$ is given by the solution to

$$0 = \tilde{A}_0\tilde{Q}^W + \tilde{Q}^W\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}^W + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^W\tilde{A}_i^T \tag{6.20}$$

where $\delta \in \mathbb{R}$ and δ_i is defined from Equation (3.78).

Finally, for the average cost bound decomposition, Theorem (3.3.3), the state covariance matrix, $\tilde{Q}$, can be set equal to $\tilde{Q}^A$, where $\tilde{Q}^A$ is given by the solution to

$$0 = \tilde{A}_0\tilde{Q}^A + \tilde{Q}^A\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \delta^2\tilde{Q}_1 + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}_1\tilde{A}_i^T \tag{6.21}$$

$$0 = \tilde{A}_0\tilde{Q}_1 + \tilde{Q}_1\tilde{A}_0^T + \delta^2\tilde{Q}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^A\tilde{A}_i^T \tag{6.22}$$

where $\delta \in \mathbb{R}$ and δ_i is defined from Equation (3.78).

The covariance matrix, $\tilde{Q}$, for each case can be used to decompose the component costs. The resulting ordering reflects the nature of the approximation or bounding method and can give component costs orderings which may not be identical to the orderings derived using the exact average cost decomposition. To make further progress in understanding the nature of a particular approximation or bound, the modal costs for a second order system will be derived in Section 6.2.4 for each of the cases. Before this is done, however, some possible indicators of the quality of the uncertain model truncation will be derived.

6.2.3 Model Reduction and Reduction Indices

Having defined some possible parameter sensitive cost decompositions, the generalized component cost analysis (GCCA) model order reduction procedure can now be defined

Definition 6.2.1 (GCCA Model Order Reduction Procedure)

Step 1 *Pick one of the performance metrics given above in Eq. (6.11) for the exact average or Eq. (6.15) for the approximations and bounds.*

Step 2 *Compute the component costs, J_i.*

Step 3 *Rank the component costs for the closed-loop system by magnitude*

$$|J_1| \leq |J_2| \leq \cdots \leq |J_{\tilde{n}}| \tag{6.23}$$

Step 4 *Remove those open-loop model components which are associated with the smallest component costs. Only the original states associated with the open-loop plant are candidates for truncation.*

There are some clear shortcomings of the above algorithm. The first is that the components are coupled so that removing a component effects the relative ordering of all of the other components. The coupling is neglected in the algorithm and therefor the reduced order system will be in no way optimal. The proceedure is necessarily hueristic in nature. Thbe second shortcoming is that the ranking of the components is dependent on the compensator used to close the loop (assuming closed-loop model reduction). Since the compensator is designed using the reduced model it is not yet available for reduction purposes. This problem can be ameliorated by incorporating the reduction step into the controller solution algorithm discussed in Definition 4.3.1.

By performing this procedure, a reduced order open-loop system can be obtained which combined with the given controller produces a reduced order closed-loop system. If it is assumed that k components are retained from the open-loop model, this closed-loop system can be described in state space for the general set of systems as

$$G_{zw}^K(\alpha) = \left[\begin{array}{cc|c} A^K(\alpha) & B_2^K(\alpha)C_c & B_1^K(\alpha) \\ B_cC_2^K(\alpha) & A_c & B_cD_{21}^K(\alpha) \\ \hline C_1^K(\alpha) & D_{12}^K(\alpha)C_c & 0 \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}^K(\alpha) & \tilde{B}^K(\alpha) \\ \hline \tilde{C}^K(\alpha) & 0 \end{array}\right] \tag{6.24}$$

and for the structured set of systems described by

$$G_{zw}^K(\alpha) = \left[\begin{array}{cc|c} A_0^K + \sum_{i=1}^{r}\alpha_i A_i^k & B_{2_0}^K C_c + \sum_{i=1}^{r}\alpha_i B_{2_i}^K C_c & B_1^K \\ B_cC_{2_0}^K + \sum_{i=1}^{r}\alpha_i B_c C_{2_i}^K & A_c & B_cD_{21}^K \\ \hline C_1^K & D_{12}^K C_c & 0 \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}_0^K + \sum_{i=1}^{r}\alpha_i \tilde{A}_i^K & \tilde{B}^K \\ \hline \tilde{C}^K & 0 \end{array}\right] \tag{6.25}$$

where the superscript $(\cdot)^K$ indicates that the rows and columns associated with the truncated states have been removed from the respective matrices.

Having defined a reduced model from Eq. (6.24) or (6.25), a model order reduction index can be defined to measure the size of the reduction error. This *model order reduction index* can be defined as the relative error between the cost associated with the full order model and the cost associated with the fixed order model. It can be written

$$\mathcal{I} = \frac{|J - J^K|}{J} \tag{6.26}$$

where J^K is the cost associated with the reduced order model calculated using the chosen performance metric.

It is important to note that the GCCA algorithm for model reduction of parametrically uncertain systems given in Def. 6.2.1 does not necessarily minimize the model reduction index, (6.26). It does however minimize the predicted model order reduction index

$$\hat{\mathcal{I}} = \frac{|J - \hat{J}^K|}{J} \tag{6.27}$$

where, if $\mathcal{R}$ is defined as the set of retained components, $\hat{J}^K$ is defined as

$$\hat{J}^K = \sum_{i \in \mathcal{R}} J_i \tag{6.28}$$

The difference between the predicted and actual model reduction indices is caused by the coupling between the components. The component costs are not necessarily independent. If the components were uncoupled the two indices would be identical. If in fact $J^K = \hat{J}^K$, this property is known as the *Cost Superposition Property* since equivalence between (6.26) and (6.27) implies that the total cost is just a superposition of the costs found when considering only the respective subsystems.

When there is no uncertainty, several coordinate transformations can be found which decouple the component costs. These can be modal coordinates, balanced coordinates, and cost-decoupled coordinates. For parametrically uncertain systems, however, the parameter dependence can make it impossible to find a coordinate transformation which decouples the component costs for each model in the parameterized set. In general, a decoupled coordinate system cannot be found for parameterized systems. In some cases, for instance modal coordinates with modal parameter uncertainties, the Cost Superposition Property holds and the GCCA algorithm minimizes the model reduction index. This special case will be examined in more detail in the next section.

6.2.4 Modal Costs for Uncertain Systems

In this section, the modal costs of a system with uncertain natural frequencies and damping will be computed using the equations for the exact average and its approximations and bounds. In the following derivations, the system will be assumed to be in modal form. The costs for the system will be derived as a function of the modal natural frequencies, damping ratios, and input and output weights. These can be open-loop quantities or the closed-loop quantities as long as the modal decomposition remain intact. To start consider the n mode system in modal form

$$\ddot{\eta}_i + 2\zeta_i\dot{\eta}_i + \omega_i^2\eta = b_i w, \qquad i = 1, \cdots, n \tag{6.29}$$

where η_i is the modal coordinate and w is the $m \times 1$ Gaussian white noise forcing vector with unity intensity matrix $E[w(t)w^T(\tau)] = I\delta(t-\tau)$, and b_i is the $1xm$ modal input vector. The system can be represented in state space form as:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{6.30}$$

where

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix} \quad C = \begin{bmatrix} C_1 & \cdots & C_N \end{bmatrix} \tag{6.31}$$

and

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix} \quad B_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix} \quad C_i = \begin{bmatrix} \sqrt{\nu_i} & 0 \\ 0 & \sqrt{\beta_i} \end{bmatrix} \tag{6.32}$$

where ω_i and ζ_i are the natural frequency and damping ratio of the i^{th} mode and ν_i and β_i are the position and velocity penalties for that mode. In this problem the natural frequency and damping ratios of the modes will be assumed uncertain and of the form

$$\omega_i^2 = {\omega_0}_i^{\,2} + \tilde{\omega}^2, \quad -\delta_{\omega_i^2} \leq \tilde{\omega}_i^2 \leq \delta_{\omega_i^2} \tag{6.33}$$

$$\zeta_i = \zeta_{0_i} + \tilde{\zeta}_i, \quad -\delta_{\zeta_i} \leq \tilde{\zeta}_i \leq \delta_{\zeta_i} \tag{6.34}$$

and the values are distributed uniformly.

The system dynamics for each mode are identical to those of the second order example considered in Section 3.4. The modal costs will therefore have the same form as the open-loop costs for that system. Much of this comparison discussion is drawn from that section with special emphasis placed here on the role of the parameter uncertainty.

If it is assumed that all of the modes are nominally stable as well as observable and controllable, the component cost for each mode can be defined for each of the cases as

$$J = \mathrm{tr}\left\{\tilde{Q}\tilde{C}^T\tilde{C}\right\}_{ii} \tag{6.35}$$

where the matrix, $\tilde{Q}$, is calculated using the appropriate equations for each case.

The first case to be considered is the modal cost of the nominal plant with no parametric uncertainty. From Section 3.4

$$J_i^0 = \frac{(\nu_i + {\omega_{0_i}}^2\beta_i)\sigma_i^2}{4\zeta_{0_i}{\omega_{0_i}}^3} \tag{6.36}$$

where $\sigma_i = b_i b_i^T$. The nominal cost is thus composed of three parts: the modal observability through ν_i and β_i, the modal controllability through σ_i, and the modal time constant as reflected in $4\zeta_{0_i}{\omega_{0_i}}^3$.

The exact average modal cost is computed by averaging over the modal costs in the parameter domain. It is given by

$$J_i^E = \frac{\sigma^2}{4\zeta_{0_i}{\omega_{0_i}}^3}\left(\frac{\tanh^{-1}\bar{\zeta}_i}{\bar{\zeta}_i}\right)\left(\nu_i\left(\frac{\tanh^{-1}\bar{\omega}_i^2}{\bar{\omega}_i^2}\right) + \beta_i{\omega_{0_i}}^2\right) \tag{6.37}$$

where

$$\bar{\omega}_i^2 = \delta_{\omega_i^2}/{\omega_{0_i}}^2 \quad \bar{\zeta}_i = \delta_{\zeta_i}/\zeta_{0_i} \tag{6.38}$$

are the non-dimensional uncertainty bounds. The exact average modal cost is essentially the same as the nominal cost with the exception of the two terms involving $\bar{\omega}_i^2$ and $\bar{\zeta}_i$. As is noted in Section 3.4, in the limiting case of no uncertainty these terms assume unity value, so the nominal modal cost is recovered in the limit. Infinite modal cost occurs when either $\bar{\omega}_i^2 = 1$ or $\bar{\zeta}_i = 1$. Note that the frequency uncertainty must be on the order of the natural frequency itself before it starts to effect the cost. The exact average cost will be insensitive to frequency uncertainty if the modal position penalty, ν_i, is zero. This is an expression of the *uncertainty independence property*. This property states that the amount of parameter uncertainty which will give infinite modal costs is independently effected by each uncertain parameter.

The perturbation expansion approximate average modal cost is given by:

$$J_i^P = \frac{\sigma^2}{4\zeta_{0_i}\omega_{0_i}{}^3}\left(\left(\nu_i + \beta_i\omega_{0_i}{}^2\right) + \frac{\bar{\zeta}_i^2}{3}\left(\nu_i + \beta_i\omega_{0_i}{}^2\right) + \frac{\hbar\omega_i^4}{3}\nu_i\right) \tag{6.39}$$

The relationship between the exact average and the perturbation expansion approximate average is established in Section 3.4. The perturbation expansion approximate average modal cost can be obtained from the exact average by retaining only the first order terms in the expansion for $\tanh^{-1}$. In contrast to the exact average, the perturbation approximation retains only a quadratic dependence on the uncertain parameters and therefore a finite cost is associated with all finite values of the parameter bounds. It also retains the uncertainty independence property.

The Bourret approximate modal cost has a form very similar to the exact average modal cost.

$$J_i^B = \frac{\sigma^2}{4\zeta_{0_i}\omega_{0_i}{}^3}\left(\frac{1}{1-\frac{\bar{\zeta}_i^2}{3}}\right)\left(\nu_i\left(\frac{1}{1-\frac{\bar{\omega}_i^4}{3}}\right) + \beta_i\omega_{0_i}{}^2\right) \tag{6.40}$$

As discussed in Section 3.4, the Bourret cost is a better approximation to the exact average cost than the perturbation expansion approximate cost is. The Bourret approximate average cost also shares the uncertainty independence property of the exact average.

The worst case bound modal cost is given by

$$J_i^W = J_i^0\left(\frac{1}{1-\left(p + \frac{\bar{\zeta}_i^2}{p} + \frac{\bar{\omega}_i^4}{4p\zeta_{0_i}{}^2}\right)}\right) \tag{6.41}$$

where

$$p = \frac{\delta^2}{2\zeta_{0_i}\omega_{0_i}}, \quad \delta \in \mathbb{R} \tag{6.42}$$

is the free parameter used in the bound in Eq. (6.20) and it has been assumed that

$$\delta_1^2 = 2(\delta_{\omega_i^2})^2, \quad \delta_2^2 = 2(\delta_{\zeta_i})^2 \tag{6.43}$$

Eq. (6.41) is essentially the nominal cost modified by a term which is dependent on the parameter uncertainty bounds and p. This term has the form

$$p+\frac{\bar{\zeta}_i^2}{p}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2} \qquad (6.44)$$

The characteristics of this bound can be obtained by examining Eq. (6.44). The worst case modal cost does not have the uncertainty independence property since both uncertainties ($\bar{\zeta}_i^2$ and $\bar{\omega}_i^4$) contribute to (6.44) and their effects are additive. That is, increasing uncertainty in the damping ratio will decrease the allowable uncertainty in the natural frequency. This frequency uncertainty also enters into the cost even if there is only a rate penalty, unlike the exact average and the approximations. In addition, the frequency uncertainty bound, $\bar{\omega}_i^4$, is scaled by a term containing ${\zeta_{0_i}}^2$. This scaling greatly increases the bounds sensitivity to frequency variation when the system has light damping. Even with no uncertainty, $\bar{\zeta}_i^2 = 0$ and $\bar{\omega}_i^4 = 0$, the cost does not reduce to the nominal because the bound contains a term which shifts the apparent system eigenvalues to the right and thereby increases the cost.

The average bound modal cost is given by:

$$J_i{}^A = J_i{}^0\left(\frac{1}{1-\left(p+\frac{\bar{\zeta}_i^2}{p}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}\right)^2}\right) \qquad (6.45)$$

This modal cost function is always less than the worst case bound. It also lacks the uncertainty independence property and shares the worst-case bound's sensitivity to frequency uncertainty.

This concludes the discussion of the open-loop modal costs derived by using the exact average and its approximations and bounds. These costs are more illustrative than useful since the real desire is for closed loop costs with dynamic compensators coupling the modal dynamics. They do, however, illustrate the properties of the various possible average related cost functionals and can thus help guide the designer to the correct state truncation for reducing the order of a particular problem. For instance, it is evident that the bounds are more sensitive to frequency uncertainty than are the exact average and its approximations.

Although the cost decomposition technique for model reduction has some clear shortcomings in terms of ignoring component coupling and requiring foreknowledge knowledge of controller, it can stil provide a powerfull tool for including the effects of parametric uncertainties in the reduction decision.

6.3 Reducing the Number of Uncertain Parameters

6.3.1 Problem Statement

In this section the problem of uncertain parameter truncation will be addressed. This problem is distinct from the model order reduction problem considered in the previous section. The model order reduction problem considered the importance of uncertain system components or subsystems. The uncertainties entered the problem through their effect on making a certain component more or less important. The average cost was decomposed as an *explicit* sum over the components with the effect of the uncertain parameters being *implicit.*

The uncertain parameter truncation problem involves explicitly considering the costs due to the parameter uncertainties and using these as a basis for removing parameter uncertainties from the system model. The number of uncertain parameters is completely independent of the number of components. For example, a second-order SISO system could have uncertainties in each of the 8 matrix elements. Each element could also have multiple uncertain parameters contributing to it. The uncertain parameters also enter the model in a different manner than the components. The components represent dynamic systems which interact through time with the other system components. The parameter uncertainties are time invariant elements which specify the nature of these interactions. The problem of removing uncertain parameters from the model is therefore distinct from the problem of removing model components or states.

The need for reducing the number of uncertain parameter retained in the system model used for control design is motivated by the large computational burden associated with each uncertainty. For the computation of the exact average cost, each parameter uncertainty contributes an extra dimension to the parameter space over which the integration is performed. This curse of dimensionality limits the number of uncertain parameters which can in practice be considered in average cost evaluation and minimization for control design. While the problem is not so severe for approximate average cost or bounding function evaluation (the computational burden increases linearly with additional parameters for these cases rather than geometrically as in the exact average case) retaining unnecessary parameters in the control design is computationally wasteful.

Although model order reduction is distinct from uncertain parameter truncation, both can be performed using cost decomposition. Instead of decomposing the cost along the lines of component contributions, the cost can be decomposed into the cost associated with the nominal system and the addition cost associated with the uncertain parameters. This uncertain parameter cost decomposition can then be used as a measuring stick for evaluating the relative importance of the parameters and truncating the least important parameter uncertainties from the model. Removing a parameter uncertainty does not involve removing the parameter from the model as in order reduction; but rather setting the parameter to its nominal value for later analysis, order reduction, or control synthesis.

6.3.2 Decomposition by Uncertain Parameter

In this section, the decomposition of the chosen performance metric in terms of contributions from the uncertain parameters will be presented. The decomposition will be restricted to the approximations and bounds because it is difficult to decompose the exact average cost into portions attributable to individual parameters. The calculation of the exact average cost involves averaging the solution to a parameterized Lyapunov equation, (6.14). The average is performed over the uncertain parameters which enter into the solution procedure nonlinearly. The sums of the costs calculated using only one parameter at a time would therefore not equal the average cost considering all of the parameters. In addition, the average over the parameters to obtain the average solution to (6.14) implicitly defines a function from the parameter domain to the average cost. Since the function is implicit, there is no way to explicitly solve for a given parameter's contribution to the total average cost.

The approximation and bound equations involve explicit terms for the effects of uncertain parameters on the respective costs, in contrast to the implicit mapping from the parameter domain to the exact average cost. This is because the approximation and bound equations have already encapsulated the parameter dependence either by incorporating the average over the parameter explicitly or by bounding the parameter with an explicit function. These functions are therefore easily decomposable into cost contributions due to the nominal system and due to the uncertain parameters.

As in the model order reduction problem, we are primarily interested in uncertain parameter truncation for the purposes of control design and will therefore consider closed-loop parameter truncation. The importance of closed-loop model reduction has been previously discussed in Section 6.2.2. Throughout the remainder of this section, it will be assumed that the loop from y to u has been closed by some fixed-form static or dynamic compensator, G_c, which is subsequently incorporated into the system transfer function from w to z. The structured set of closed-loop systems, $\mathcal{G}_{zw}$, will therefore be the model upon which parameter truncation is performed. The respective costs can be decomposed

$$J = J_0 + \sum_{i=1}^{r} J_i \tag{6.46}$$

where J_0 is the cost associated with the nominal plant

$$J_0 = \mathrm{tr}\left\{\tilde{Q}^0 \tilde{C}^T \tilde{C}\right\} \tag{6.47}$$

and $\tilde{Q}^0$ is given by the solution to the nominal Lyapunov equation

$$0 = \tilde{A}_0\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_0^T + \tilde{B}\tilde{B}^T \tag{6.48}$$

The uncertain parameter cost contribution, J_i, depends on the particular approximation or bounding function. The equations for J_i will be presented in the following paragraphs.

For the perturbation expansion approximate average cost, Proposition (3.2.1), the uncertain parameter cost contribution, J_i, is given by

$$J_i = \mathrm{tr}\left\{\tilde{Q}_\alpha\tilde{C}^T\tilde{C}\right\} \tag{6.49}$$

where $\tilde{Q}_\alpha$ is given by the solution to

$$0 = \tilde{A}_0\tilde{Q}_\alpha + \tilde{Q}_\alpha\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{6.50}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right) \tag{6.51}$$

$\tilde{Q}^0$ is defined in (6.48), and σ_i is defined in (3.47).

For the Bourret approximate average cost decomposition, Proposition (3.2.2), $\tilde{Q}_\alpha$, is given by the solution of

$$0 = \tilde{A}_0\tilde{Q}_\alpha + \tilde{Q}_\alpha\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \tag{6.52}$$

$$0 = \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}_i^T\right) \quad i = 1, \ldots, r \tag{6.53}$$

where $\tilde{Q}^B$is the Bourret approximate average, Eqs. (6.18) and (6.19), and σ_i is defined as before.

For the worst case cost bound decomposition, Proposition (3.3.2), $\tilde{Q}_\alpha$ is given by the solution to

$$0 = \tilde{A}_0\tilde{Q}_\alpha + \tilde{Q}_\alpha\tilde{A}_0^T + \frac{\delta^2\tilde{Q}^W}{r} + \frac{\delta_i^2}{\delta^2}\tilde{A}_i\tilde{Q}^W\tilde{A}_i^T \tag{6.54}$$

where $\tilde{Q}^W$is the worst case cost bound, Eq. (6.20), and δ_i is defined from Equation (3.78).

Finally, for the average cost bound decomposition, Theorem (3.3.3), $\tilde{Q}_\alpha$ is given by the solution to

$$0 = \tilde{A}_0\tilde{Q}_\alpha + \tilde{Q}_\alpha\tilde{A}_0^T + \frac{\delta^2\tilde{Q}_1}{r} + \frac{\delta_i^2}{\delta^2}\tilde{A}_i\tilde{Q}_1\tilde{A}_i^T \tag{6.55}$$

$$0 = \tilde{A}_0\tilde{Q}_1 + \tilde{Q}_1\tilde{A}_0^T + \delta^2\tilde{Q}^A + \sum_{i=1}^{r}\left(\frac{\delta_i^2}{\delta^2}\right)\tilde{A}_i\tilde{Q}^A\tilde{A}_i^T \tag{6.56}$$

where $\tilde{Q}^A$ is the average cost bound, Eqs. (6.21) and (6.22), δ_i is defined from Equation (3.78), and r is the number of uncertain parameters.

Each definition for the uncertain parameter cost decomposition defines a ranking of the relative importance of the parameters. The resulting orderings can be used as a basis for removing unimportant parameters from the design process. Each ordering reflects the nature of the approximation or bounding method. To investigate the ramifications of using a particular approximation or bound, the uncertain parameter costs for a second order system with uncertain natural frequency, damping ratio, and forcing constant will be considered in Section 6.3.4. First some possible indicators of the quality of the uncertain parameter truncation will be derived.

6.3.3 Parameter Truncation and Truncation Indices

Having defined some possible uncertain parameter cost decompositions, an uncertain parameter cost analysis (UPCA) model reduction procedure can be defined.

Definition 6.3.1 (UPCA Model Reduction Procedure)

Step 1 *Pick one of the performance metrics given above for the approximations and bounds.*

Step 2 *Compute the uncertain parameter costs, J_i.*

Step 3 *Rank the parameter costs by magnitude*

$$|J_1| \leq |J_2| \leq \cdots \leq |J_r| \tag{6.57}$$

Step 4 *Remove those parameters from the open-loop model which are associated with the smallest uncertain parameter costs by setting these parameters equal to their nominal values.*

If it is assumed that k uncertain parameters are retained from the original r in the open-loop model, this closed-loop set of system can be described for the general set of systems as

$$\mathcal{G}_{zw} = \{G_{zw}(\alpha) \ \ \forall \, \alpha \, \in \, \Omega\} \tag{6.58}$$

where $\Omega \subset \mathbb{R}^k$ and each system is described in the state space as defined in (2.28). For the structured case this set is composed of elements of the form

$$\begin{aligned} G^K_{zw}(\alpha) &= \left[\begin{array}{cc|c} A_0 + \sum_{i=1}^{k} \alpha_i A_i & B_{2_0} C_c + \sum_{i=1}^{k} \alpha_i B_{2_i} C_c & B_1 \\ B_c C_{2_0} + \sum_{i=1}^{k} \alpha_i B_c C_{2_i} & A_c & B_c D_{21} \\ \hline C_1 & D_{12} C_c & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \tilde{A}_0 + \sum_{i=1}^{k} \alpha_i \tilde{A}_i & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] \end{aligned} \tag{6.59}$$

where there are now only k uncertain parameters.

A model reduction index can be defined to measure the size of the error incurred by assuming the nominal values of the truncated uncertain parameters. This *parameter truncation index* can be defined as the relative error between the cost associated with using the full set of uncertain parameters and the cost associated with using the reduced set. It can be written as in the case of model order reduction

$$\mathcal{I} = \frac{|J - J^K|}{J} \tag{6.60}$$

where J^K is the cost associated with the reduced uncertainty model calculated using the chosen performance metric.

As in the case for model order reduction, it is important to note that the UPCA algorithm for uncertain parameter truncation given in Def. 6.3.1 does not necessarily minimize the parameter truncation index, (6.60). It does however minimize the predicted parameter truncation index

$$\hat{\mathcal{I}} = \frac{|J - \hat{J}^K|}{J} \tag{6.61}$$

where, if $\mathcal{R}$ is defined as the set of retained uncertain parameters, $\hat{J}^K$ is defined as

$$\hat{J}^K = \sum_{i \in \mathcal{R}} J_i \tag{6.62}$$

The difference between the predicted and actual parameter truncation indices is caused by the coupling between the parameters. The uncertain parameter costs are not necessarily independent. If in fact $J^K = \hat{J}^K$, this property is known as the *Cost Superposition Property* since equivalence between (6.60) and (6.61) implies that the total cost is just a superposition of the costs found when considering only a single parameter uncertainty.

A simple system will be considered in the next section to further investigate the nature of the uncertain parameter decomposition in the context of the various approximations and bounds to the exact average. The system is identical to the one considered in Section 6.2.4. In the next section, however, the costs will be decomposed into uncertain parameter contributions rather than modal contributions.

6.3.4 Costs for Uncertain Modal Parameters

In this section, the modal costs presented in Section 6.2.4 will be further decomposed into cost associated with the nominal system and the cost associated with the uncertain natural frequencies and damping ratios.

$$J_i = J_{i0} + J_{\zeta_i} + J_{\omega_i^2} \tag{6.63}$$

The intent is to establish the relative importance of the uncertainties in the modal parameters to help in the uncertain parameter truncation process. The system considered in this section is the same as the one considered in Section 6.2.4. The natural frequencies and damping ratios of the various modes have been assumed to be independent. The uncertain parameter costs are only calculated for the approximating and bounding functions since the exact average cost can not be simply decomposed.

The perturbation expansion approximate costs can be decomposed

$$J^P_{\zeta_i} = \frac{\sigma^2}{4\zeta_{0_i}{\omega_{0_i}}^3}\left(\frac{\bar{\zeta}_i^2}{3}\right)\left(\nu+\beta\omega_i^2\right) \tag{6.64}$$

$$J^P_{\omega_i^2} = \frac{\sigma^2}{4\zeta_{0_i}{\omega_{0_i}}^3}\left(\frac{\bar{\omega}_i^4}{3}\nu\right) \tag{6.65}$$

which are quadratic functions of the uncertainty bounds. Notice that each uncertain parameter cost is independent of the bound on the other parameter.

The Bourret approximate costs can be decomposed

$$J^B_{\zeta_i} = \frac{\sigma^2}{4\zeta_{0_i}{\omega_{0_i}}^3}\left(\frac{\frac{\bar{\zeta}_i^2}{3}}{1-\frac{\bar{\zeta}_i^2}{3}}\right)\left(\nu+\beta\omega_i^2\right) \tag{6.66}$$

$$J^B_{\omega_i^2} = \frac{\sigma^2}{4\zeta_{0_i}{\omega_{0_i}}^3}\left(\frac{1}{1-\frac{\bar{\zeta}_i^2}{3}}\right)\left(\frac{\frac{\bar{\omega}_i^4}{3}}{1-\frac{\bar{\omega}_i^4}{3}}\right)\nu \tag{6.67}$$

The structure of the uncertainty dependent terms in these equations is the same as in Eq. (6.40). The uncertain damping ratio cost is independent of the uncertain frequency bound, but the uncertain frequency cost is not independent of the damping ratio bound. The values of the uncertainty bounds where these cost go to infinity is however unchanged from Eq. (6.40).

The worst-case bound can be decomposed into the contributions from the uncertain parameters

$$J^W_{\zeta_i} = \frac{\sigma^2\left(\nu+\beta\omega_i^2\right)}{4\zeta_i\omega_i^3}\left(\frac{\frac{p}{2}+\frac{\bar{\zeta}_i^2}{p}}{1-\left(p+\frac{\bar{\zeta}_i^2}{p}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}\right)}\right) \tag{6.68}$$

$$J^W_{\omega_i^2} = \frac{\sigma^2\left(\nu+\beta\omega_i^2\right)}{4\zeta_i\omega_i^3}\left(\frac{\frac{p}{2}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}}{1-\left(p+\frac{\bar{\zeta}_i^2}{p}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}\right)}\right) \tag{6.69}$$

These equations have the same asymptotes and properties as Eq. (6.41). This can be seen by substituting Eq. (6.41) into Eqs. (6.68) and (6.69).

$$J^W_{\zeta_i} = J_i^W\left(\frac{p}{2}+\frac{\bar{\zeta}_i^2}{p}\right) \tag{6.70}$$

$$J^W_{\omega_i^2} = J_i^W\left(\frac{p}{2}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}\right) \tag{6.71}$$

The dependence on $p/2$ rather than p is due to an arbitrary factoring of the p term in the denominator of (6.41).

The average bound function can be decomposed in a fashion similar to the worst case bound

$$J^A_{\zeta_i} = J_i^A\left(p+\frac{\bar{\zeta}_i^2}{p}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}\right)\left(\frac{p}{2}+\frac{\bar{\zeta}_i^2}{p}\right) \tag{6.72}$$

$$J^A_{\omega_i^2} = J_i^A\left(p+\frac{\bar{\zeta}_i^2}{p}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}\right)\left(\frac{p}{2}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2}\right) \tag{6.73}$$

The average bound parameter costs are always less than the worst case linear bound parameter costs since

$$p+\frac{\bar{\zeta}_i^2}{p}+\frac{\bar{\omega}_i^4}{4p{\zeta_{0_i}}^2} \leq 1 \tag{6.74}$$

These parameter costs can be used to rank relative importance of the parameter uncertainties for the uncertain parameter reduction process. The general Lyapunov equation forms of these parameter costs have been applied to this simple system in modal form to better understand the ramifications of a particular choice of cost functional.

6.4 Summary

This chapter has dealt with two model reduction problems, the problem of reducing the order of a system with parameter uncertainties and the problem of reducing the number of parameter uncertainties in the model. These model reduction problems are motivated by the need to reduce the computational burden associated with the calculation of the exact average cost and its approximations and bounds. Procedures similar to Component Cost Analysis were proposed to help solve these model reduction problems. The open-loop modal costs and modal parameter costs were derived to illustrate the properties of the exact average cost decomposition in relation to the certain system cost decomposition and to compare the decompositions of the approximations and bounds to the average. The cost decomposition model reduction schemes presented in this chapter have some clear weaknesses in that they essentially ignore the component and uncertain parameter cost coupling and assume knowledge of the controller. In spite of these weaknesses, the methods presented provide a powerfull tool for initial model reduction for parametrically uncertain plants.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

This work has addressed the problem of synthesizing robust controllers for linear time-invariant systems with real parameter uncertainty. This problem is motivated by the types of uncertainty encountered in control of flexible structures, where the natural frequencies, damping ratios, stiffnesses, and other model parameters can be uncertain. This work has examined the trades between stability robustness, performance, and control effort in the context of systems with real parameter errors.

Present frequency-domain, input-output techniques for dealing with such uncertainties, such as $\mathcal{H}_\infty$ robust design or μ-synthesis, treat the uncertainties as complex quantities and are conservative for real parameter errors. These types of error models are better suited for characterizing high-frequency unmodelled dynamics. When used for parametric errors, design techniques employing unstructured error representations can require higher control effort than that necessary for achieving stability. When designing robust controllers for parametric errors, care must be taken to avoid sacrificing stability robustness to high-frequency unmodelled dynamics. In this sense the two error representations are complementary.

Present time domain techniques can be divided into those which guarantee stability and those which do not. Those which guarantee robust stability sometimes guarantee robust performance as well as. Chief among these are those which rely on Lyapunov stability theory. While desirable, the guarantee for robust performance is in most cases associated with higher control effort. The other class of time domain techniques typically involve minimization of cost functionals that approximate the parametric dependence of the system performance. This has been accomplished by minimizing cost sensitivity or by considering multiple models and minimizing the average cost.

The present work has extended the methods of multi-model robust control design by considering performance metrics related to the quadratic ($\mathcal{H}_2$) cost averaged over a set of possible systems. The set of possible systems, called the model set, has system matrices which are functions of a number of real parameters varying over a bounded region. The average cost is defined as the average of the $\mathcal{H}_2$-norms of the elements of the model set. Bounded average cost over a continuous set was shown to be a sufficient condition for set stability. Average cost analysis tools were developed for modeling, analysis, and control design for systems with parametric uncertainty.

For modeling of uncertain systems, the key concept considered was one of appropriateness of the uncertainty representation. For high frequency unmodeled dynamics, frequency domain input-output representations were considered most appropriate because the structure of the interactions between the system components is not known. Parametric uncertainty modeling was most appropriate when the structure of component interactions was known but the properties of these interactions were uncertain. It was shown that a given parametric uncertainty could be represented in different forms; for instance, a single uncertain stiffness could be modeled in modal coordinates as many independent uncertain natural frequencies and mode shapes. In general, parameter uncertainties should be represented in their most primitive forms to reduce model representation conservatism.

The average cost was shown to have useful properties for analyzing the stability of the model set. It was shown that bounded average cost implies that there are no elements of the model set that have eigenvalues with positive real parts, i.e., all the elements are stable. This new result was the foundation and motivation for developing average cost techniques for robust control design. The average cost could be written as the average solution of a parameterized Lyapunov equation. A sufficient condition for simultaneous stability was developed based upon guaranteeing existence of a solution for this equation.

One of the principle thesis contributions is the analysis of the average cost of a set of systems using operator decomposition techniques. These techniques have been borrowed from the fields of wave propagation in random media and turbulence modeling. This was their first application to the analysis and control of parameterized systems. The techniques entailed decomposing a parameter dependent operator into a known nominal operator and a parameter dependent remainder. This technique was applied to find expressions for the average solution of the parameterized Lyapunov equation. The techniques were most usefull in deriving computable approximations to the average cost.

Computing the exact average cost for systems with large numbers of uncertainties was shown to be impractical. The operator decomposition techniques were used to derive computable expressions for approximations and bounds to the average cost. Two types of approximations and two types of bounds were presented. The first type of approximation was based on the truncation of the perturbation expansion for the average solution. This approximation, though easily solved, was shown to be accurate for only small amounts of uncertainty. It was shown to be essentially identical to the sensitivity system cost used by Skelton. The second type of approximation was generated by the Bourret equation, which is based on the truncation of an operator series for the exact average. The Bourret approximation possessed some useful properties for control design and was shown to be a better approximation to the average cost of a single spring/mass system than the perturbation expansion. In addition, existence of a positive definite solution to the Bourret equation was shown to guarantee stability on a specifiable set of parameters. No such stability guarantee was available for the perturbation expansion approximation. The structure of the Bourret equation made it more difficult to solve than the perturbation expansion approximation, however.

Existence of a solution to the two bounds was shown to guarantee stability over the set of systems. The first bound discussed was based on the linear bound developed by Bernstein. It was called "worst-case" because its solution bounded all of the possible $\mathcal{H}_2$-norms of the elements of the model set including the set with the highest cost. Since bounded average (and not worst case) is all that is needed for stability robustness, another bound was developed which bounded the average but not necessarily the worst case cost. This bound, called the average bound, was developed in an attempt to guarantee robust stability without the cost increases associated with worst-case design. A simple mass spring example validated the relative orderings of the approximations and bounds. The bounds were found to be more sensitive than necessary to frequency uncertainties and less representative of the physical structure of the uncertainties. They did, however, guarantee stability over their design set.

The synthesis of controllers based on minimizing either the exact average cost or its approximations and bounds was presented. The necessary conditions for the five cost functionals were derived for static and dynamic output feedback. Controllers derived using either the Bourret approximation, the exact average, or the two bounds were shown to guarantee stability over a specifiable set. In the case of the Bourret approximation, this set is smaller than the design set while the others guranteed stability over the design set *a priori.*. Because of the complexity of the cost equations, closed form solution for the controller gain matrices was impractical. Instead, the necessary conditions were used for slope information in a quasi-Newton numerical minimization scheme. Since it is difficult to find stabilizing starting compensators for the minimization scheme, a homotopy on the uncertain parameter design bounds was utilized to derive controllers at progressively larger values of the uncertainty.

This homotopy framework also provided a useful structure for performing closed-loop uncertain model reduction.

The utility of this homotopy algorithm was demonstrated with several numerical examples. These were static and dynamic compensation for the robust control sample problem and dynamic compensation for the Cannon-Rosenthal problem. The homotopy algorithm for solution computation worked well in general. There were some problems with homotopy path discontinuities in the higher order problem, but these are easily attributable to the nature of the uncertainty. The central concept used to distinguish the controllers designs was design efficiency. An efficient design is one which achieves large stability bounds while sacrificing little nominal performance. The exact average design was found to be most efficient as expected from the analysis. The Bourret approximation design was the next best with efficiency very similar to the exact average. The perturbation expansion approximation, which is structurally similar to Skelton's cost sensitivity design, was shown to have the worst efficiency for the cases considered. In general it was also more numerically finicky and less capable of stabilization for high levels of uncertainty.

The average bound and the worst case bound were shown to be functionally identical for the cases considered. Since the average bound is more difficult to calculate than the worst case bound, its utility was deemed questionable. It did not achieve its desired objective of lower cost for guaranteed stability than the worst case bound, in part because the stable regions given by the two bound equations are identical. As predicted by the analysis, the design using the bounds guaranteed stability over the design set. This stability guarantee was associated with control and output costs that were higher for a given achieved bound than those resulting from the exact average and approximation-based designs. Since the purpose of using bounds is *a priori* stability guarantees, it is more appropriate to judge the bound-based designs on design rather than achieved stability bound. In this case the bounds fare even worse in comparison to the other design techniques.

During the design process, damping was shown to be important for stabilization in the face of uncertainties resulting in pole-zero flips. This characteristic was found in the Cannon-Rosenthal problem. Stabilization of poles-zero flips was found to be impossible without some system damping. The problem became progressively easier as the damping was increased This result supported the claims made in [142] concerning the need for passive damping for robust structural control.

Finally, the problem of model reduction was briefly considered to enable the application of these average-based robust control design techniques to higher order systems with multiple uncertainties. It was shown that model reduction techniques which explicitly incorporate the "cost" of the uncertainty must be used when dealing with parametrically uncertain systems. This was necessary to avoid truncation of nominally benign but highly uncertain modes and parameters. The model reduction problem could be divided into the problem of reducing the model order and that of reducing the number of uncertain parameters. Both reduction problems were motivated by the need for simplifying the controller computation. An algorithm similar to Component Cost Analysis has be applied to both model order and parameter number reduction. General formulae for component and parameter costs have been developed for each of the approximations and bounds. Explicit modal costs and uncertain modal parameter costs were derived which possessed the same properties as the single mass/spring example considered above.

In general, all five design techniques were shown to be capable of increasing the system robustness to some extent. The perturbation based method was limited in the amount of uncertainty it could accommodate. The bound based designs were also limited and resulted in high-cost controllers in the cases considered. The exact average-based design, though most efficient, is not computable for more complex cases than those considered here. Based on the above considerations and its high efficiency, the Bourret approximation based design method was shown to be the best overall tool for robust control design.

7.2 Future Work

The results of the present work suggest several new avenues of investigation. They are generally in the areas of improving robust controller computation, expanding the average based analysis tools, and further applications of these tools.

There is clear motivation for improving the computation of the controllers. Faster robust controller computation greatly increases a design technique's usefulness. The present work has barely scratched the surface of the controller computation issues focusing instead on comparing the various types of controllers. This investigation has indicated that the Bourret approximation based design method was the best and future work should be devoted to its more efficient computation.

There is good reason to believe that the structure of the Bourret equation can be utilized to develop a solution algorithm more efficient than the Kronecker math solution presented. Under certain conditions the Bourret equation can be represented by a double order Lyapunov equation, and it can always be represented by a higher order modified Lyapunov equation. This structure suggests psooibilities for more efficient solution. The controller computation can be speeded by using a more efficient cost minimization algorithm and by using a continuous rather than discrete homotopy in the solution process.

The average bound failed to live up to its promise of low cost guaranteed stability because it was as conservative as the worst-case bound, which also guarantees robust stability. One potentially fruitful avenue of research would be to develop a better (less conservative) computable bound for the average. This average would guarantee *a priori* stability without the high cost associated with robust performance. One approach to deriving such a bound would be to find a function which bounds the Dyson equation from which the Bourret approximation is derived rather than bound the perturbation expansion series as do the present bounds.

An exciting area of future work is related to further applications of these techniques. The problem of uncertain model reduction was briefly presented in Chapter

4. It would be interesting to apply these reduction procedures to an uncertain large order structure to better evaluate the role of uncertainty in mode retention. This is particularly feasible since the component and parameter cost ranking require only a single cost calculation and not the thousands required for controller calculation. The effects and importance of component and parameter cost coupling can be evaluated to test the practicallity of the model reduction techniques presented in Chapter 6.

Another applications related area is the quantitative assessment of the relationship between passive damping and robust control of uncertain systems. The average cost analysis tools presented in this work provide a framework in which the effects of the uncertainties on the system cost can be measured. It is a straightforward extension to use these tools to measure the effects that varying passive system parameters has on the uncertainty cost. Thus, the tools developed in Chapter 3 can be applied to the design of structures for robust control by varying such passive parameters as system stiffness distribution and damping. The Bourret approximation can also be used to investigate more fully the robustness-performance-control effort trades analytically on simple structural sample problems and numerically on more complex problems.

The final area of extension of this work is in the area of experimental design verification. A design methodology was presented and the elements of this methodology were discussed in detail. This methodology needs to be applied to the robust control of a realistic structure whose uncertainties cannot necessarily all be represented within the framework presented. This particular application will most help focus the process of model development for robust control which was briefly discussed in Chapter 2, and help further assess the limitations of the cost averaging techniques.

Bibliography

[1] Skelton, R. E., "A Tutorial on Model Error Concepts in Control Design," to appear in the *International Journal of Control.*

[2] Skelton, R. E., and Owens, D. H., (eds.), *Model Error Concepts and Compensation*, Pergammon, New York, 1986.

[3] Doyle, J. C., Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Trans. Autom. Contr.*, Vol. AC–26, No. 1, February 1981, pp. 4-16.

[4] M. G. Safonov, *Stability and Robustness of Multivariable Feedback Systems*, MIT Press, Cambridge, MA, 1980.

[5] Vidyasagar M., *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985, pp. 229–317.

[6] Gran, R. and Rossi, M., "A Survey of the Large Structures Control Problem," *Proceedings of the 18th IEEE Conference on Decision and Control*, Ft. Lauderdale, FL, Dec. 7–9, 1979, pp. 1002–1006.

[7] Balas, M. J., "Trends in Large Space Structural Control Theory: Fondest Hopes, Wildest Dreams," *IEEE Trans. Autom. Contr.*, Vol. AC–27, No. 3, June 1982, pp. 522–535.

[8] Kosut, R. L., Salzwedel H., Emami-Naeini, A., "Robust Control of Flexible Spacecraft," *Journal of Guidance Control and Dynamics*, Vol. 6, No. 2, March–April 1983, pp. 104–111.

[9] Gran, R., "Control of Flexible Structures: A Systematic Overview of the Problem," *Proceedings of the Workshop on the Applications of Distributed Systems Theory to the Control of Large Space Structures*, JPL Publication 83–46, July 1983, pp. 49–61.

[10] Atluri, S. N., Amos, A. K., *Large Space Structures: Dynamics and Control*, Springer Verlag, 1988.

[11] Wada, B. K., Fanson, J. L., and Crawley, E. F., "Adaptive Structures," *Adaptive Structures*, AD–Vol. 15, American Society of Mechanical Engineers, New York, New York, 1989, pp. 1–8.

[12] Blackwood, G. H., Chu, C. –C., Fanson, J. L., Sirlin, S. W. "Uncertainty Modeling for the Control of an Active Structure," *Adaptive Structures*, AD–Vol. 15, American Society of Mechanical Engineers, New York, New York, 1989, pp. 43–52.

[13] Safanov, M. G., Flashner H., "Modeling and Robustness Issues in Control Design for Flexible Structures," *Proc. American Control Conference*, pp. 2527–2531, 1987.

[14] Skelton, R. E., Hu, A., "Modeling Structures for Control Design," *Computers & Structures*, Vol. 20, No. 1–3, pp. 303–309.

[15] Doyle, J. C., "Synthesis of Robust Controllers and Filters," *Proc. IEEE Conference on Decision and Control*, pp. 109–114, 1983.

[16] Francis, B., *A Course in $\mathcal{H}_\infty$ Control Theory*, Springer Verlag, 1987.

[17] Morari, M., and Zafiriou, E., *Robust Process Control*, Prentice-Hall, Inc., Englewood Cliffs, 1989.

[18] G. Tadmor, "$\mathcal{H}_\infty$ in the Time Domain: The Standard Four Blocks Problem," *Math. Contr. Syst. Signal Processing*, preprint.

[19] Doyle, J. C., Glover, K., Khargonekar, P. P., Francis, B. A., "State Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems" *IEEE Trans. Autom. Contr.*, Vol. AC–34, No. 8, August 1989, pp. 831–847.

[20] Bernstein, D. S., Haddad, W. M., "LQG Control with an $\mathcal{H}_\infty$ Performance Bound: A Riccati Equation Approach," *IEEE Trans. Autom. Contr.*, Vol. AC–34, No. 3, March 1989, pp. 293–305.

[21] Zhou, K., Khargonekar, P. P., "An Algebraic Riccati Equation Approach to $\mathcal{H}_\infty$ Optimization" *Systems and Control Letters*, Vol. 11, 1988, pp. 85–91.

[22] Morton, B. G., McAfoos, R. M., "A Mu-Test for Robustness Analysis of Real Parameter Variation Problem" *Proc. ACC Conf.*,June 1985, pp. 135-138

[23] Doyle, J. C., "Analysis of Feedback Systems with Structured Uncertainty," *IEE Proc.*, Vol. 120, Pt. D, No. 6, November 1982, pp. 242–251,.

[24] Doyle, J. C., "Structured Uncertainty in Control System Design," *Proceedings of the 24th conference on Decision and Control*, Ft. Lauderdale, FL, December 1985, pp. 260–265.

[25] Foo, Y. K., "Stability and Performance Robustness Analysis with Highly Structured Uncertainties: Necessary and Sufficient Test," *IEEE Trans. Autom. Contr.*, Vol. AC–35, No. 3, March 1990, pp. 350–352

[26] Siljak, D. D., "Parameter Space Methods for Robust Control Design: A Guided Tour," *IEEE Trans. Autom. Contr.*, Vol. AC–34, No. 7, July 1989, pp. 674–688.

[27] Saeks, R., and Murray, J., "Fractional representations, algebraic geometry and the simultaneous stabilization problem," *IEEE Trans. Autom. Contr.*, Vol. AC-27, No. 4, Aug. 1982, pp. 895–903.

[28] Vidyasagar, M. and Viswanadham, N. "Algebraic Design Techniques for Reliable Stabilization," *IEEE Trans. Autom. Contr.*, Vol. AC-27, Oct. 1982, pp. 1085–1095.

[29] Barmish, B. R. and Wei, K. H., "Simultaneous Stabilizability on Single Input Single Output Systems," *in Proc. 7th Int. Symp. Math. Theory of Networks and Sys.*, Stockholm, Sweden, 1985, and *Modeling Identification and Robust Control*, Amsterdam, the Netherlands, North-Holland, 1986.

[30] Wei, K. H. and Barmish, B. R., "An Iterative Design Procedure for Simultaneous Stabilization of MIMO Systems," *Automatica*, Vol. 24, pp. 643–652, 1988.

[31] Kwakernaak, H., "A Condition for Robust Stabilizability," *Syst. and Contr. Letters*, Vol. 2, No. 1, pp 1–5, 1982.

[32] Wei, K. H., and Yedavalli, R. K., "Robust Stabilizability for Linear Systems with Both Parameter Variation and Unstructured Uncertainty," *IEEE Trans. Autom. Contr.*, Vol. AC-34, No. 2, February 1989, pp. 149–156.

[33] Willems, J. L., Willems, J. C., "Robust Stabilization of Uncertain Systems," *SIAM J. of Contr. Opt.*, Vol. 21, pp. 352–374, 1983.

[34] Ackerman, J. "Parameter Space Design of Robust Control Systems," *IEEE Trans. Autom. Contr.*, Vol. AC-25, No. 5, 1980, pp. 1058–1072.

[35] Patel, R. V., Toda, M. and Sridhar, B., "Robustness of Linear Quadratic State Feedback design in the presence of system uncertainty," *IEEE Trans. Autom. Contr.*, Vol. AC-22, No. 6, 1977, pp. 945–949.

[36] Yedavalli, R. K., "Perturbation Bounds for Robust Stability in Linear State Space Models," *Int. Journal of Control*, Vol. 42, No. 6, 1985, pp. 1507–1517.

[37] Ghosh, B. K. and Byrnes, C. I., "Simultaneous stabilization and simultaneous pole placement by non-switching dynamic compensation," *IEEE Trans. Autom. Contr.*, Vol. AC–28, No. 6, June 1983, pp. 735–741.

[38] Ghosh, B. K. and Byrnes, C. I., "Simultaneous pole placement – a new approach to multi-mode system design," *IEEE Trans. Autom. Contr.*, Vol. AC–31, 1986, pp. 440–443.

[39] Ghosh, B. K. and Byrnes, C. I., "An approach to simultaneous system design, Part I, Semialgebraic Geometric Methods," *SIAM Journal of Control Optm.*, Vol. 24, No. 3, May 1986, pp. 480–496.

[40] Ghosh, B. K., "An approach to simultaneous system design, Part II: Nonswitching gain and dynamic Feedback compensation by algebraic geometric methods," *SIAM Journal of Control Optm.*, Vol. 26, No. 4, July 1988, pp. 919–963.

[41] Ghosh, B. K. and Byrnes, C. I., "Transcendental and Interpolation methods in Simultaneous stabilization and simultaneous partial pole placement problems," *SIAM Journal of Control Optm.*, Vol. 24, No. 6, Nov. 1986, pp. 1091–1109.

[42] Vidyasagar, M., "The Graph Metric for Unstable Plants and Robustness Estimates for Feedback Stability," *IEEE Trans. Autom. Contr.*, Vol. AC–29, 1984, pp. 403–417.

[43] Schwartz, R. J. and Friedland, B., *Linear Systems*, McGraw Hill, New York, 1965.

[44] Chapellat, H., Dahleh, M., and Bhattacharyya, S. P., "Robust Stability of Interval Plants: A Review," *Proceedings of the 1990 American Control Conference*, pp. 2548–2553.

[45] Kharitonov, V. L., "Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations," *Differential'nye Uraveniya*, Vol. 14, No. 11, pp. 1483–1485.

[46] Bartlet, A. C., C. V. Hollot and H. Lin, "Root Locations of an Entire Polytope of Polynomials: It Suffices to Check the Edges," *Mathematics of Control, Signals and Systems*, Vol. 1, pp. 61–71.

[47] Barmish, R. B., "New Tools for Robustness Analysis," *Proceedings of the 27th Conference on Decision and Control*, Austin Texas, December 1988, pp. 1–6.

[48] Michael, G. S. and Merriam, C. W., "Stability of Parametrically Disturbed Linear Optimal Control Systems," *J. Math. Anal. Appl.*, Vol. 28, 1969, pp. 294–302.

[49] Chang, S. S. L. and Peng, T. K. C., "Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters," *IEEE Trans. Autom. Contr.*, Vol. AC–17, 1972, pp. 474–483.

[50] Leitmann, G. "Guaranteed Asymptotic Stability for a Class of Uncertain Linear Dynamical Systems," *J. Optim. Thy. Appl.*, Vol. 27, 1979, pp. 99–106.

[51] Vinkler, A. and Wood, L. J., "Multistep Guaranteed Cost Control of Linear Systems with Uncertain Parameters," *AIAA J. Guid. Contr. and Dyn.*, Vol. 2, 1979, pp. 449–456.

[52] Eslami, M. and Russell, D. L., "On Stability with Large Parameter Variations Stemming from the Direct Method of Lyapunov," *IEEE Trans. Autom. Contr.*, Vol. AC–25, 1980, pp. 1231–1234.

[53] Patel, R. V. and Toda, M., "Quantitative Measure of Robustness for Multivariable Systems," *Proc. Joint. Autom. Contr. Conf.*, paper TP8–A, San Francisco, CA, June 1980.

[54] Corless, M. and Leitmann, G., "Continuous State Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Dynamical Systems," *IEEE Trans. Autom. Contr.*, Vol. AC–26, 1981, pp. 1139–1144.

[55] Thorp, J. S. and Barmish, B. R., "On Guaranteed Stability of Uncertain Linear Systems via Linear Control," *J. Optim. Thy. Appl.*, Vol. 35, 1981, pp. 559–579.

[56] Noldus, E., "Design of Robust State Feedback Laws," *Int. J. Contr.*, Vol. 35, 1982, pp. 935–944.

[57] Barmish, B. R., Corless, M., and Leitmann, G., "A New Class of Stabilizing Controllers for Uncertain Dynamical Systems," *SIAM J. Contr. Optim.*, Vol. 21, 1983, pp. 246–255.

[58] Barmish, B. R., Petersen, I. R., and Feuer, A., "Linear Ultimate Boundedness Control of Uncertain Dynamic Systems," *Automatica*, Vol. 19, 1983, pp. 523–532.

[59] Barmish, B. R., "Necessary and Sufficient Conditions for Quadratic Stabilizability of an Uncertain Linear System," *J. Optim. Thy. Appl.*, Vol. 46, 1985, pp. 399–408.

[60] Yedavalli, R. K., Banda, S. S., and Ridgely, D. B., "Time–Domain Stability Robustness Measures for Linear Regulators," *AIAA J. Guid. Contr. Dyn.*, Vol. 8, 1985, pp. 520–524.

[61] Yedavalli, R. K., "Improved Measures of Stability Robustness for Linear State Space Models," *IEEE Trans. Autom. Contr.*, Vol. AC–30, 1985, pp. 577–579.

[62] Galimidi, A. R., and Barmish, B. R., "The Constrained Lyapunov Problem and its Application to Robust Output Feedback Stabilization," *IEEE Trans. Autom. Contr.*, Vol. AC–31, 1986, pp. 410–419.

[63] Petersen, I. R. and Hollot, C. V., "A Ricatti Equation Approach to the Stabilization of Uncertain Systems," *Sys. Contr. Lett.*, Vol. 8, 1987, pp. 351–357.

[64] Petersen, I. R., "A Stabilization Algorithm for a Class of Uncertain Linear Systems," *Automatica*, Vol. 22, 1986, pp. 397–411.

[65] Hinrichsen, D. and Pritchard, A. J., "Stability Radii and Structured Perturbations and the Algebraic Riccati Equation," *Sys. Contr. Lett.*, Vol. 8, 1987, pp. 105–113.

[66] Bernstein, D. S., and Haddad, W. M., "The Optimal Projection Equations with Petersen–Hollot Bounds: Robust Controller Synthesis with Guaranteed Structured Stability Radius," *Proc. IEEE Conf. Dec. Contr.*, Los Angeles, CA., Dec. 1987, pp. 1308–1318,

[67] Zhou, K. and Khargonekar, P. P., "Stability Robustness Bounds for Linear State–Space Models with Structured Uncertainty," *IEEE Trans. Autom. Contr.*, Vol. AC–32, 1987, pp. 621–623.

[68] Bhattacharyya, S. P., *Robust Stabilization Against Structured Perturbations*, Springer–Verlag, 1987.

[69] Bernstein, D. S., "Robust Static and Dynamic Output–Feedback Stabilization: Deterministic and Stochastic Perspectives," *IEEE Trans. Autom. Contr.*, Vol. AC–32, No. 12, December 1987, pp. 1076–1084.

[70] Kosmidou, O. I. and Bertrand, P., "Robust–Controller Design for Systems with Large Parameter Variations," *Int. J. Contr.*, Vol. 45, 1987, pp. 927–938.

[71] Bernstein, D. S., and Haddad, W. M., "Robust Stability and Performance for Fixed–Order Dynamic Compensation via the Optimal Projection Equations with Guaranteed Cost Bounds," *Proc. Amer. Contr. Conf.*, Atlanta, GA., June 1988, pp. 2471-2476.

[72] Bernstein, D. S., and Haddad, W. M., "The Optimal Projection Equations with Petersen–Hollot Bounds: Robust Stability and Performance via Fixed–Order Dynamic Compensation for Systems with Structured Real–Valued Parameter Uncertainty," *IEEE Trans. Autom. Contr.*, Vol. AC–33, 1988, pp. 578–582.

[73] Bernstein, D. S., and Haddad, W. M., "Robust Stability and Performance Analysis for Linear Dynamic Systems," *IEEE Trans. Autom. Contr.*, Vol. AC–34, 1989, pp. 751-758.

[74] Bernstein, D. S., and Haddad, W. M., "Robust Stability and Performance Analysis for State Space Systems via Quadratic Lyapunov Bounds," *SIAM J. Matrix Anal. Appl.*, Vol. 11, pp. 239-271, 1990.

[75] Rouche, N., Habets, P. and Laloy, M., *Stability Theory by Lyapunov's Direct Method*, New York, Springer Verlag, 1977.

[76] Siljak, D. D., *Large Scale Dynamic Systems: Stability and Structure*, Amsterdam, the Netherlands, North–Holland, 1978.

[77] Hyland, D. C. and Bernstein, D. S., "The Majorant Lyapunov Equation: A Nonnegative Matrix Equation for Guaranteed Robust Stability and Performance of Large Scale Systems," *IEEE Trans. Autom. Contr.*, Vol. AC–32, 1987, pp. 1005–1013.

[78] Collins, E. G. Jr. and Hyland, D. C., "Improved Robust Performance Bounds in Covariance Majorant Analysis," *Int. J. Contr.*, Vol. 50, pp. 495-509,1989.

[79] Bar–Shalom, Y. and Sivan, R., "On the Optimal Control of Discrete–Time Varying Linear Systems with Random Parameters," *IEEE Trans. Autom. Contr.*, Vol. AC–14, 1969, pp. 3–8.

[80] Wonham, W. M., "Random Differential Equations in Control Theory," *Probalistic Methods in Applied Mechanics*, Vol. 2, A. T. Bharucha–Reid, Ed., New York, Academic, 1970, pp. 131–212.

[81] Sworder, D. D., "Control of Jump Parameter Systems with Discontinuous State Trajectories," *IEEE Trans. Autom. Contr.*, Vol. AC–17, 1972, pp. 740–741.

[82] Wonham, W. M., "Optimal Stationary Control of a Linear System with State–Dependent Noise," *SIAM J. Contr.*, Vol. 5, 1967, pp. 486–500.

[83] Wonham, W. M., "On a Matrix Riccati Equation for Stochastic Control," *SIAM J. Contr.*, Vol. 6, 1968, pp. 681–697.

[84] Kleinman, D., "Optimal Stationary Control of Linear Systems with Control–Dependent Noise," *IEEE Trans. Autom. Contr.*, Vol. AC–14, 1969, pp. 673–677.

[85] P. McLane, "Optimal Stochastic Control of Linear Systems with State and Control–Dependent Disturbances," *IEEE Trans. Autom. Contr.*, Vol. AC–16, 1971, pp. 793–798.

[86] McLane, P., "Linear Optimal Stochastic Control using Instantaneous Output Feedback," *Int. J. Contr.*, Vol. 13, 1971, pp. 383–396.

[87] Haussmann, U., "Optimal Stationary Control with State and Control Dependent Noise," *SIAM J. Contr.*, Vol. 9, 1971, pp. 184–198.

[88] Merriam III, C. W., *Automated Design of Control Systems*, New York, Gordon and Breach, 1974.

[89] Bismut, J., "Linear–Quadratic Optimal Stochastic Control with Random Coefficients," *SIAM J. Contr.*, Vol. 14, 1976, pp. 419–444.

[90] Mil'stein, G., "Design of Stabilizing Controller with Incomplete State Data for Linear Stochastic System with Multiplicative Noise," *Automat. Remote Contr.*, Vol. 43, 1982, pp. 653–659.

[91] Hyland, D. C., "Application of the Maximum Entropy/Optimal Projection Control Design Approach for Large Space Structures," *Proc. Large Space Antenna Systems Technology Conf.*, pp. 617-654, NASA Langley, Dec. 4–6, 1984.

[92] Phillis, Y. A. "Controller Design of Systems with Multiplicative Noise," *IEEE Trans. Autom. Contr.*, Vol. AC–30, 1985, pp. 1017–1019.

[93] Bernstein, D. S. and Hyland, D, C., "Optimal Projection/Maximum Entropy Stochastic Modeling and Reduced Order Design Synthesis," *Proc. IFAC Workshop on Model Error Concepts and Compensation*, Boston, MA, June 1985, pp. 47–54.

[94] Bernstein, D. S. and Hyland, D, C., "The Optimal Projection/Maximum Entropy approach to Designing Low–Order, Robust Controllers for Flexible Structures," *Proceedings of the 24th Conference on Decision and Control*, Ft. Lauderdale, FL, Dec. 1985, pp. 745–752.

[95] Bernstein, D. S, Davis, L. D., Greeley, S. W., and Hyland, D. C., "Numerical Solution of the Optimal Projection/Maximum Entropy Design Equations for Low Order, Robust Controller Design," *Proceedings of the 24th Conference on Decision and Control*, Ft. Lauderdale, FL, Dec. 1985, pp. 1795–1798.

[96] Bernstein, D. S. and Greely, S. W., "Robust Controller Synthesis Using the Maximum Entropy Design Equations," *IEEE Trans. Autom. Contr.*, Vol. AC–31, No. 4, 1986, pp. 362–364.

[97] Bernstein, D. S. and Hyland, D, C., "The Optimal Projection Equations for Reduced–Order Modeling, Estimation, and Control of linear systems with Multiplicative White Noise," *J. Optim. Theory Appl.*, Vol. 58, pp.387–409, 1988.

[98] Bernstein, D. S. and Hyland, D, C., "The Optimal Projection Equations for Fixed Order Dynamic Compensation," *IEEE Trans. Autom. Contr.*, Vol. AC–29, 1984, pp. 1034–1037.

[99] Bernstein, D. S. and Haddad, W. M., "Robust Stability and Performance via Fixed-Order Dynamic Compensation with Guaranteed Cost Bounds," *Math. Contr. Sig. Sys.*, Vol. 3, 1990, pp. 139–163.

[100] Tahk, M. and Speyer, J., "Parameter Robust Linear–Quadratic Gaussian Design Synthesis with Flexible Structure Control Applications," *AIAA J. Guidance, Navigation and Control*, Vol. 12, No. 4, 1988, pp. 460–468.

[101] Yedavalli, R. K. and Skelton, R. E., "Controller Design for Parameter Sensitivity Reduction in Linear Regulators," *Optimal Control Applications and Methods*, Vol. 3, No. 3, 1982, pp. 221–240.

[102] Wagie, D. A. and Skelton, R. E. "Model Reduction and Controller Synthesis in the Presence of Parameter Uncertainty," *Automatica*, Vol. 22, 1986, pp. 295–308.

[103] Okada, K., Skelton, R. E., "Sensitivity Controller for Uncertain Systems," *AIAA Journal of Guidance, Control and Dynamics*, Vol. 13, No. 2, March–April 1990, pp. 321–329.

[104] Ashkenazi, A. and Bryson, A. E., "Control Logic for Parameter Insensitivity and Disturbance Attenuation," *AIAA J. of Guidance and Control*, Vol. 5, No. 4, 1982, pp. 383–388.

[105] Miyazawa, Y., "Robust Flight Control System Design with the Multiple Model Approach," *Proc. AIAA Guid. Nav. Contr. Conf.*, Portland OR, Aug. 1990, pp. 874–882.

[106] P. M. Makila, "Multiple Models, Multiplicative Noise, and Linear Quadratic Control Algorithm Aspects," *preprint*.

[107] MacMartin, D., Hall, S. R., and Bernstein, D. S., "Fixed order Multi–Model Estimation and Control," *preprint, accepted to 1991 American Control Conference.*

[108] Hopkins, W. E. Jr., "Optimal Control of Linear Systems with Parameter Uncertainty," *IEEE Trans. Autom. Contr.*, Vol. AC–31, No. 1, January 1986, pp. 72–74.

[109] Yedavalli, R. K., and Skelton, R. E., "Determination of Critical Parameters in Large Flexible Space Structures with Uncertain Modal Data," *Journal of Dynamic Systems, Measurement, and Control*, Vol. 105, December 1983, pp. 238–244.

[110] Kato, T. *Perturbation Theory for Linear Operators*, Springer Verlag, 1980.

[111] Stewart, G. W., "Stochastic Perturbation Theory," *SIAM Review*, Vol. 32, No. 4, Dec. 1990, pp.579-610. Springer Verlag, 1980.

[112] Depree, J. D., Swartz, C. W., *Introduction to Real Analysis*, John Wiley & Sons, New York, New York, 1988.

[113] Graham, A., *Kronecker Products and Matrix Calculus: with Applications*, Ellis Horwood Limited, 1981.

[114] Bharucha-Reid, A. T., "On the Theory of Random Equations," *Proceedings of the Symposia in Applied Mathematics, Vol. XVI, Stochastic Processes in Mathematical Physics and Engineering* American Mathematical Society, Providence, RI, 1964, pp. 40-69.

[115] Adomian, G., "Stochastic Green's Functions," *Proceedings of the Symposia in Applied Mathematics, Vol. XVI, Stochastic Processes in Mathematical Physics and Engineering* American Mathematical Society, Providence, RI, 1964, pp. 1-39.

[116] Adomian, G., *Stochastic Systems*, Mathematics in Science and Engineering, Vol. 169, Academic Press, NY, 1983.

[117] Adomian, G., *Nonlinear Stochastic Operator Equations* Academic Press, NY, 1986.

[118] Frisch, U., "Wave Propagation in Random Media" *Probabilistic Methods in Applied Mathematics* Vol. 1, Edt. A. T. Bharucha-Reid, Academic Press, NY, 1968, pp 75-197.

[119] Hoffman, W. C., "Wave Propagation in a General Random Continuous Medium," *Proceedings of the Symposia in Applied Mathematics, Vol. XVI, Stochastic Processes in Mathematical Physics and Engineering* American Mathematical Society, Providence, RI, 1964, pp. 117-144.

[120] Keller, J. B., "Wave Propagation in Random Media," *Proceedings of the Symposia in Applied Mathematics, Vol. XIII, Hydrodynamic Instability* American Mathematical Society, Providence, RI, 1962, pp. 227-246.

[121] Keller, J. B., "Stochastic Equations and Wave Propagation in Random Media," *Proceedings of the Symposia in Applied Mathematics, Vol. XVI, Stochastic Processes in Mathematical Physics and Engineering* American Mathematical Society, Providence, RI, 1964, pp. 145-170.

[122] Hopf, E., "Remarks on the Functional-Analytic Approach to Turbulence," *Proceedings of the Symposia in Applied Mathematics, Vol. XIII, Hydrodynamic Instability* American Mathematical Society, Providence, RI, 1962, pp. 157-163.

[123] Kraichnan, R. H., "The Closure Problem of Turbulence Theory," *Proceedings of the Symposia in Applied Mathematics, Vol. XIII, Hydrodynamic Instability* American Mathematical Society, Providence, RI, 1962, pp. 199-225.

[124] Richardson J. M., "The Application of Truncated Hierarchy Techniques in the Solution of Stochastic Linear Differential Equations," *Proceedings of the Symposia in Applied Mathematics, Vol. XVI, Stochastic Processes in Mathematical Physics and Engineering*, American Mathematical Society, Providence, RI, 1964, pp. 290-302.

[125] Athans, M., "The Matrix Minimum Principle," *Information and Control*, Vol. 11, 1968, pp. 592-606.

[126] Levine, W. S., Athans, M., "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems," *IEEE Trans. Auto. Contr.*, Vol. AC-15, No. 1, February 1970, pp. 44-48.

[127] Levine, W.S., Johnson, T.L., Athans, M., "Optimal Limited State Variable Feedback Controllers for Linear Systems," *IEEE Trans. Auto. Contr.*, Vol. AC-16, No. 6, December 1971, pp. 785-791.

[128] Mendel J. M., "A Concise Derivation of the Optimal Constant Limited State Feedback Gains," *IEEE Trans. Auto. Contr.*, Vol. AC-19, August, 1974, pp. 447-448.

[129] Wie, B., and Bernstein, D. S., "Benchmark Problems for Robust Control Design," *Proc. 1991 American Control Conference*, June, 1991.

[130] Byun, K. -W., Wie, B. and Sunkel, J., "Robust Control Synthesis for Uncertain Dynamical Systems," AIAA paper # 89-3516, *Proc. AIAA GN&C Conference*, pp. 792-801, Boston, MA, August, 1989.

[131] Wie, B., Liu, Q., and Byun, K. -W., "Robust $\mathcal{H}_\infty$ Control Synthesis Method and Its Applications to a Benchmark Problem," copies distributed at the American Control Conference, San Diego, CA, May, 1990.

[132] Skelton, R. E., Gregory, C. V., "Measurement Feedback and Model Reduction by Modal Cost Analysis," *Proceedings of the Joint Automatic Control Conference*, Denver, 1979. pp. 211-219.

[133] Skelton, R. E., Hughes, P. C., "Modal Cost Analysis for Linear Matrix-Second Order Systems," *Journal of Dynamic Systems Measurement and Control*, Vol. 102, Sept. 1980, pp. 151-163.

[134] Skelton, R. E., "Cost Decomposition of Linear Systems with Application to Model Reduction," *International Journal of Control*, Vol. 32, Dec. 1980, pp. 1031-1055.

[135] Skelton, R. E., Yousuff, A., "Component Cost Analysis of Large Scale Systems," *International Journal of Control*, Vol. 37, No. 2, pp. 285-304.

[136] Skelton, R. E., Hughes, P. C., Hablani, H. B., "Order Reduction for Models of Space Structures Using Modal Cost Analysis," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 4, July-Aug. 1982, pp. 351-357.

[137] Yousuff, A., Skelton R. E., "Controller Reduction by Component Cost Analysis," *IEEE Trans. Autom. Contr.*, Vol. AC–29, No. 6, 1984, pp. 520–530.

[138] Moore, B. C., "Principle Component Analysis of Linear Systems: Controlability, Observability, and Model Reduction," *IEEE Trans. Autom. Contr.*, Vol. AC–26, February, 1981, pp. 17–32.

[139] Gregory, C. Z. Jr., "Reduction of Large Flexible Spacecraft Models Using Internal Balancing Theory," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 6, Nov.-Dec. 1984, pp. 725–732.

[140] Cannon, R. H., Rosenthal, D. E., "Experiments in Control of Flexible Structures with Noncolocated Sensors and Actuators," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 5, Sept.-Oct. 1984, pp. 546–553.

[141] Wie, B., Byun, K.W., "New Generalized Structural Filtering Concept for Active Vibation Control Synthesis," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 2, 1989, pp. 147–154.

[142] von Flotow, A. H., Vos, D. W., "The Need for Passive Damping in Feedback Controlled Flexible Structures," *Air Force Damping '91 Conference*, San Diego, California, April 1991.

[143] Gueler, R., "The Virtues of Passive Damping for Feedback Controlled Flexible Structures," M. S. Thesis, Massachusetts Institute of Technology, Dept. of Aeronautics and Astronautics, Cambridge, Massachusetts, June 1991.

[144] Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., *Numerical Recipies: the Art of Scientific Computing*, Cambridge University Press, Cambridge, 1986, pp. 307-312.