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**Citation:** Lusztig, G. A Survey of Total Positivity. Milan j. math. 76, 125–134 (2008)

**As Published:** 10.1007/s00032-008-0083-2

**Publisher:** Springer Nature America, Inc

**Persistent URL:** <https://hdl.handle.net/1721.1/140228>

**Version:** Original manuscript: author's manuscript prior to formal peer review

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# A SURVEY OF TOTAL POSITIVITY

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The theory of totally positive matrices originated in the 1930's in the work of I.J.Schoenberg and Gantmacher-Krein. Since then this theory has found applications to such diverse areas as statistics, game theory, mathematical economics, stochastic processes (see Karlin [Ka]). This survey covers the following topics:

(i) an exposition of the early work of Schoenberg and Gantmacher-Krein and the work of A.Whitney on totally positive matrices (see §1);

(ii) a generalization of the known reduction to a canonical form of a symplectic nondegenerate bilinear form to non-symplectic bilinear forms based on ideas from total positivity (see §2);

(iii) an exposition of results in [L1] which extend the notion of total positivity to a Lie group context based on the theory of quantum groups and canonical bases (see §3);

(iv) a generalization of (ii) placing it in a Lie group context (see §5);

(v) an exposition of recent results of Fock and Goncharov [FG] which provide a beautiful application of the theory of total positivity in Lie groups to the study of homomorphisms of the fundamental group of a closed surface into a Lie group (see §6, §7).

**1.** Let  $V$  be a real vector space with a totally ordered basis  $e_1, e_2, \dots, e_n$ . For any  $k \in [1, n]$  the  $k$ -th exterior power  $\Lambda^k V$  has a basis  $(e_{r_1} \wedge e_{r_2} \wedge \dots \wedge e_{r_k})$  indexed by the sequences  $r_1 < r_2 < \dots < r_k$  in  $[1, n]$ . Let  $G = GL(V)$ . Let  $G_{\geq 0}$  (resp.  $G_{> 0}$ ) be the set of all  $A \in G$  such that for any  $k \in [1, n]$  the coefficients of  $\Lambda^k A : \Lambda^k V \rightarrow \Lambda^k V$  with respect to the basis above are  $\geq 0$  (resp.  $> 0$ ); one then also says that  $A$  is totally  $\geq 0$  (resp. totally  $> 0$ ). This definition was given by I. J. Schoenberg [S] in 1930 in connection with his solution of a problem of Pólya. For  $v \in V$ ,  $v = \sum_r v_r e_r$ ,  $v_r \in \mathbf{R}$ , let  $\vartheta_v$  be the largest integer  $k \geq 0$  for which there exists a sequence  $r_1 < r_2 < \dots < r_k$  in  $[1, n]$  such that  $v_{r_1} v_{r_2} < 0, \dots, v_{r_{k-1}} v_{r_k} < 0$ ; in other words,  $\vartheta_v$  is the number of sign changes in the sequence  $v_1, v_2, \dots, v_n$  with 0's removed. Pólya asked for a characterization of those  $A \in G$  which diminish  $\vartheta_v$  that is, are such that  $\vartheta_{A(v)} \leq \vartheta_v$  for any  $v \in V$ . Schoenberg showed that these  $A$  are exactly those such that for any  $k \in [1, n]$ ,  $\Lambda^k A : \Lambda^k V \rightarrow \Lambda^k V$  does not have two coefficients of opposite signs with respect to the basis above (in particular any  $A \in G_{\geq 0}$  has the required diminishing property).

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A second source of the idea of totally positivity appeared in the work of Gantmacher and Krein [GK] in 1935. Motivated by the study of vibrations of a mechanical system, Gantmacher and Krein were looking for a condition on  $A \in G$  which guarantees that

(a)  $A$  has distinct real positive eigenvalues;

and

-is preserved by a small perturbation of  $A$ ;

-can be easily tested in terms of the coefficients of  $A$ .

They showed that the condition that  $A \in G_{>0}$  has the required properties.

We now sketch a proof of the Gantmacher-Krein theorem asserting that if  $A \in G_{>0}$  then (a) holds. Let  $c_1, c_2, \dots, c_n$  be the eigenvalues of  $A$  arranged so that  $|c_1| \geq |c_2| \geq \dots \geq |c_n|$ . For any  $k \in [1, n]$  the eigenvalues of  $\Lambda^k A$  are  $c_{r_1} c_{r_2} \dots c_{r_k}$  for various  $r_1 < r_2 < \dots < r_k$  in  $[1, n]$ ; if  $k < n$ , the first two eigenvalues of  $\Lambda^k A$  in the decreasing order of absolute value are

$$c_1 c_2 \dots c_k, c_1 c_2 \dots c_{k-1} c_{k+1}.$$

By Perron's theorem [P] on square matrices with all entries in  $\mathbf{R}_{>0}$  applied to  $\Lambda^k A$ , we have  $c_1 c_2 \dots c_k \in \mathbf{R}_{>0}$  and  $c_1 c_2 \dots c_{k-1} c_k > |c_1 c_2 \dots c_{k-1} c_{k+1}|$ . Using this we see by induction on  $k$  that all  $c_r$  are in  $\mathbf{R}_{>0}$  and that  $c_1 > c_2 > \dots > c_n$ .

Let  $I = [1, n-1]$ . For  $i \in I$ ,  $a \in \mathbf{R}$  we define  $x_i(a)$  (resp.  $y_i(a)$ ) in  $G$  by  $x_i(a)e_i = e_i + ae_{i+1}$ ,  $x_i(a)e_h = e_h$  for  $h \neq i$  (resp. by  $y_i(a)e_{i+1} = e_{i+1} + ae_i$ ,  $y_i(a)e_h = e_h$  for  $h \neq i+1$ ). Let  $T$  (resp.  $T_{>0}$ ) be the group of all  $A \in G$  such that  $Ae_i = a_i e_i$  for all  $i$  where  $a_i \in \mathbf{R} - \{0\}$  (resp.  $a_i \in \mathbf{R}_{>0}$ ).

In 1952, Ann Whitney [W] (a Ph.D. student of Schoenberg) proved that  $G_{>0}$  is dense in  $G_{\geq 0}$  using the statements (b),(c) below.

(b)  $G_{\geq 0}$  is the submonoid with 1 of  $G$  generated by  $x_i(a), y_i(a)$  with  $i \in I$ ,  $a \in \mathbf{R}_{\geq 0}$  and by  $T_{>0}$ ;

(c)  $G_{>0}$  consists of all products

$$x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_N}(a_N)ty_{i_1}(b_1)y_{i_2}(b_2)\dots y_{i_N}(b_N)$$

where  $i_1, i_2, \dots, i_N$  is the sequence  $1, 2, \dots, n-1, 1, 2, \dots, n-2, \dots, 1, 2, 1$ ;  $N = n(n-1)/2$ ;  $a_s > 0, b_s > 0$  for all  $s$  and  $t \in T_{>0}$ .

(Actually (c) is only implicit in [W]; it is stated explicitly in Loewner [Lo].) Note that

(d) in (c) we can replace  $1, 2, \dots, n-1, 1, 2, \dots, n-2, \dots, 1, 2, 1$  by  $n-1, n-2, \dots, 1, n-1, n-2, \dots, 2, \dots, n-1, n-2, n-1$  and we get a true statement.

Indeed if we define  $G_{>0}$  in terms of the reverse order  $e_n, e_{n-1}, \dots, e_1$  of the basis we get the same  $G_{>0}$  as before.

**2.** We preserve the notation of §1. Define an involution  $i \mapsto i^*$  of  $[1, n]$  by  $i^* = n+1-i$ . Let  $\langle, \rangle : V \times V \rightarrow \mathbf{R}$  be a bilinear form. When  $\langle, \rangle$  is symplectic, nondegenerate, a standard result shows that there exists a basis  $(v_r)_{r \in [1, n]}$  of  $V$  such that  $\langle v_r, v_s \rangle = 0$  if  $r \neq s^*$  and  $\langle v_r, v_{r^*} \rangle = (-1)^r$  for all  $r$ . We want to prove a similar result without the assumption that  $\langle, \rangle$  is symplectic. Instead we shall

assume that  $\langle, \rangle$  is totally  $> 0$  in the following sense: for any  $k \in [1, n]$  and any two sequences  $r_1 < r_2 < \dots < r_k$ ,  $s_1 < s_2 < \dots < s_k$  in  $[1, n]$ , the determinant of the matrix

$$((-1)^{r_m} \langle e_{r_m}^*, e_{s_{m'}} \rangle)_{m, m' \in [1, k]}$$

is  $> 0$ . Note that the totally  $> 0$  bilinear forms form a nonempty open set in the space of nondegenerate bilinear forms on  $V$ .

We now fix a totally  $> 0$  bilinear form  $\langle, \rangle$ . We prove for it the following result.

(a) *There exists a basis  $v_1, v_2, \dots, v_n$  of  $V$  such that  $\langle v_r, v_s \rangle = 0$  if  $r \neq s^*$ ,  $\langle v_r, v_{r^*} \rangle = (-1)^r z_r \in \mathbf{R} - \{0\}$  for all  $r$  and*

$$0 < z_1 z_n^{-1} < z_2 z_{n-1}^{-1} < \dots < z_n z_1^{-1}.$$

If  $X : E \rightarrow E'$  is an isomorphism of finite dimensional vector spaces let  $\check{X} : E^* \rightarrow E'^*$  be its transpose inverse.

There is a unique isomorphism  $C : V^* \rightarrow V$  such that  $\langle v, v' \rangle = C^{-1}(v)(v')$  for all  $v, v'$  in  $V$ . Let  $e'_1, e'_2, \dots, e'_n$  be the basis of  $V^*$  dual to the basis  $e_1, e_2, \dots, e_n$  of  $V$ . Define an isomorphism  $C_0 : V^* \rightarrow V$  by  $C_0(e'_r) = (-1)^r e_{r^*}$ . We have  $\check{C}_0 = (-1)^{n+1} C_0^{-1}$ . Define an involution  $A \mapsto \check{A}$  of  $G$  by  $\check{A} = C_0 \check{A} C_0^{-1}$ . For  $1 \leq i \leq n-1, a \in \mathbf{R}$  we have  $\widetilde{x_i(a)} = x_{n-i}(a)$ ,  $\widetilde{y_i(a)} = y_{n-i}(a)$ ; moreover  $\tilde{t} \in T_{>0}$  for  $t \in T_{>0}$ . Using this and 1(d) we see that  $A \mapsto \check{A}$  maps  $G_{>0}$  into itself.

Define  $A \in G$  by  $C = C_0 A$ . We can write  $A(e_r) = \sum_{r,s} a_{rs} e_s$  with  $a_{rs} \in \mathbf{R}$ . Then  $\check{A}^{-1}(e'_r) = \sum_{r,s} a_{sr} e'_s$ . For  $r, s$  in  $[1, n]$  we have

$$(-1)^r \langle e_{r^*}, e_s \rangle = (-1)^r \check{A}^{-1} C_0^{-1}(e_{r^*})(e_s) = \check{A}^{-1}(e'_r)(e_s) = \sum_h a_{hr} e'_h(e_s) = a_{sr}.$$

This and our hypothesis on  $\langle, \rangle$  shows that  $A \in G_{>0}$ . We have

$$C\check{C} = (-1)^{n+1} C_0 \check{A} C_0^{-1} A.$$

Since  $A$  and  $C_0 \check{A} C_0^{-1}$  are in  $G_{>0}$ , so is their product. Thus,  $(-1)^{n+1} C\check{C} \in G_{>0}$ . By the Gantmacher-Krein theorem for  $(-1)^{n+1} C\check{C} \in G_{>0}$  there exists a basis  $v_1, v_2, \dots, v_n$  of  $V$  and real numbers  $c_1 > c_2 > \dots > c_n > 0$  such that  $(-1)^{n+1} C\check{C} v_r = c_r v_r$  for all  $r$ . Define  $v'_r \in V^*$  by  $v'_r(v_s) = \delta_{rs}$ . We have  $\check{C} C v'_r = (-1)^{n+1} c_r^{-1} v'_r$ . Let  $v''_r = C(v'_r) \in V$ . Note that  $(v''_r)$  is a basis of  $V$  and

$$C\check{C} v''_r = C\check{C} C(v'_r) = C((-1)^{n+1} c_r^{-1} v'_r) = (-1)^{n+1} c_r^{-1} v''_r.$$

Thus  $v''_r$  is an eigenvector of  $C\check{C}$  that is  $v''_r = \xi_r v_{\tau(r)}$  for some  $\xi_r \in \mathbf{R} - \{0\}$  and some permutation  $\tau$  of  $[1, n]$  such that  $c_{\tau(r)} = c_r^{-1}$ . Since  $c_1 > c_2 > \dots > c_n > 0$  it follows that  $c_{\tau(1)} < c_{\tau(2)} < \dots < c_{\tau(n)}$  hence  $\tau(r) = r^*$  and  $c_{r^*} = c_r^{-1}$ . Thus we have  $C(v'_r) = \xi_r v_{r^*}$  so that  $\check{C} v_r = \xi_r^{-1} v'_{r^*}$  and  $C\check{C} v_r = \xi_{r^*} \xi_r^{-1} v_r$ . We see that  $\xi_{r^*} \xi_r^{-1} = (-1)^{n+1} c_r$ . We have

$$\langle v_r, v_s \rangle = C^{-1}(v_r)(v_s) = \xi_{r^*}^{-1} v'_{r^*}(v_s) = \xi_{r^*}^{-1} \delta_{r,s^*}$$

and (a) follows.

**3.** It is natural to ask whether the definitions of  $G_{\geq 0}, G_{>0}$  given in §1 are specific to  $GL(V)$  or have a Lie theoretical meaning. Thus we wish to replace  $G$  in §1 by an arbitrary split reductive connected algebraic group  $G$  defined over  $\mathbf{R}$ , such as a general linear group, a symplectic group or a group of type  $E_8$ , with a fixed pinning. (We shall identify an algebraic variety defined over  $\mathbf{R}$  with its set of  $\mathbf{R}$ -rational points. Then  $G$  is a not necessarily connected Lie group.) The pinning is a substitute for the choice of a basis for  $V$  in §1; it is a family of homomorphisms  $x_i, y_i$  from  $\mathbf{R}$  into  $G$  indexed by a finite set  $I$ . (For a precise definition of a pinning see [L1, §1]; here we note only that for  $G = GL(V)$ ,  $x_i, y_i$  may be taken as in §1.) Let  $U^+$  (resp.  $U^-$ ) be the subgroup of  $G$  generated by  $x_i(a)$  (resp. by  $y_i(a)$ ) for various  $i \in I, a \in \mathbf{R}$ . Let  $T$  be the unique (split) maximal torus of  $G$  which normalizes both  $U^+$  and  $U^-$ . Let  $B^+ = TU^+, B^- = TU^-$  so that  $T = B^+ \cap B^-$ . Let  $T_{>0}$  be the "identity component" of  $T$  (when we say "identity component" we mean this in the sense of Lie groups, not algebraic groups). This agrees with the notation in §1 for  $G = GL(V)$ .

The results of Ann Whitney for  $GL(V)$  suggest a definition in the general case for the monoid  $G_{\geq 0}$  (by repeating 1(b)) and for  $G_{>0}$  as follows (see [L1]). Let  $N = \dim U^+ = \dim U^-$ . We can find a sequence  $\mathbf{i} = (i_1, i_2, \dots, i_N)$  in  $I$  such that the map  $\mathbf{R}_{>0}^N \rightarrow U^+$ ,

$$(a_1, a_2, \dots, a_N) \mapsto x_{i_1}(a_1)x_{i_2}(a_2) \dots x_{i_N}(a_N)$$

is injective; the image of this map is independent of the choice of  $\mathbf{i}$ . It is an open submonoid of  $U^+$  denoted by  $U_{>0}^+$ . We define an open submonoid  $U_{>0}^-$  of  $U^-$  in a similar way (replacing  $x_i$  by  $y_i$ ). Let  $G_{>0} = U_{>0}^+ T_{>0} U_{>0}^-$ . We have also  $G_{>0} = U_{>0}^- T_{>0} U_{>0}^+$  and  $G_{>0}$  is an open submonoid of  $G$ . We see that  $G_{\geq 0}, G_{>0}$  do indeed have a meaning for an arbitrary  $G$ . As in [L1] we set (imitating the definition for  $GL(V)$  given in [GK])  $G_{\geq 0}^{\text{osc}} = \{g \in G_{\geq 0}; g^m \in G_{>0} \text{ for some } m \geq 1\}$ . We have  $G_{>0} \subset G_{\geq 0}^{\text{osc}} \subset G_{\geq 0}$  and  $G_{\geq 0}^{\text{osc}}$  is closed under multiplication (see [L1, 2.19]).

The following analogue of the Gantmacher-Krein theorem for a general  $G$  is contained in [L1, 5.6], [L1, 8.10].

(a) *Let  $g \in G_{>0}$ . Then  $g$  is contained in a unique  $G$ -conjugate of  $T_{>0}$ .*

We will now make some comments on the proof of (a). The key case in the proof is that where  $G$  is simply connected as an algebraic group and "simply-laced" (the last condition means that  $G$  is a product of groups of type  $A, D, E$  in the Cartan-Killing classification); there are standard techniques by which various statements for a general  $G$  can be reduced to this case.

As in the proof of the classical Gantmacher-Krein theorem which was based on Perron's theorem we find that we need a definition of  $G_{>0}$  along the lines of the original definition of Schoenberg in terms of minors of a matrix. In the case of  $GL(V)$  that definition exploits the fact that the basic representations of  $GL(V)$  (the exterior powers of the standard representation) have a simple natural basis. For general (simply connected, simply laced)  $G$  we replace these exterior power

representations by the finite dimensional irreducible algebraic representations of  $G$ . Quite surprisingly it turns out that these representations admit canonical bases with respect to which the elements  $x_i(a), y_i(a)$  with  $a \geq 0$  act by matrices with all entries in  $\mathbf{R}_{\geq 0}$ . This allows us to give a Schoenberg-style definition of  $G_{\geq 0}$  and  $G_{>0}$  using these entries instead of minors. Then Perron's theorem is applicable for each of these representations and (a) follows.

Note that there is no known definition of the canonical bases which is purely in terms of  $G$ . The only known definition [L2] uses the fact that the irreducible representations of  $G$  considered above are limits as  $q \rightarrow 1$  of irreducible representations of a "quantum group" which are some entities depending on a parameter  $q$ . The canonical bases are first defined at the level of quantum group and then one takes their limit as  $q \rightarrow 1$  to obtain the canonical bases for representations of  $G$ . Moreover the positivity properties of the action of  $x_i(a), y_i(a)$  with respect to the canonical basis come from a stronger property which holds for the generators of the quantum group. This stronger property is established using a geometric interpretation of the quantum group and the associated canonical basis in terms of the theory of perverse sheaves and it ultimately depends on the theory around the Weil conjectures for algebraic varieties over a finite field (proved by Deligne). Thus the statement (a) which is elementary in the case of  $GL(V)$  is very far from elementary in the general case.

**4.** Let  $\mathcal{B}$  the set of all Borel subgroups of  $G$  that is subgroups that are conjugate to  $B^+$  (or equivalently to  $B^-$ ). This is a compact manifold with a transitive  $G$ -action (conjugation) called the flag manifold. In [L1, §8] the positive part  $\mathcal{B}_{>0}$  is defined. It is the set of all  $B \in \mathcal{B}$  such that  $B = uB^+u^{-1}$  for some  $u \in U_{>0}^-$  or equivalently such that  $B = u'B^-u'^{-1}$  for some  $u' \in U_{>0}^+$ . In the case where  $G = GL(V)$ ,  $\dim V = 2$ ,  $\mathcal{B}$  is a circle with two distinguished points  $B^+, B^-$  and  $\mathcal{B}_{>0}$  is one of the two connected components of the complement of  $\{B^+, B^-\}$  (a half circle). In general,  $\mathcal{B}_{>0}$  is an open ball in  $\mathcal{B}$ .

For  $i \in I, a \in \mathbf{R}$  we set  $'x_i(a) = x_i(-a), 'y_i(a) = y_i(-a)$ . Note that  $'x_i, 'y_i$  define a new pinning for  $G$ . The objects attached to this new pinning in the same way as

$$B^+, B^-, U^+, U^-, T, T_{>0}, U_{>0}^+, U_{>0}^-, G_{>0}, \mathcal{B}_{>0}$$

were attached to  $x_i, y_i$  are:

$$B^+, B^-, U^+, U^-, T, T_{>0}, 'U_{>0}^+ = (U_{>0}^+)^{-1}, 'U_{>0}^- = (U_{>0}^-)^{-1}, 'G_{>0} = G_{>0}^{-1}, '\mathcal{B}_{>0}.$$

Recall that  $B, B'$  in  $\mathcal{B}$  are said to be opposed if their intersection is a maximal torus. We show:

(a) *If  $B \in \mathcal{B}_{>0}, B' \in '\mathcal{B}_{>0}$ , then  $B, B'$  are opposed.*

We have  $B = uB^+u^{-1}$  where  $u \in U_{>0}^-$  and  $B' = u'B^+u'^{-1}$  where  $u' \in 'U_{>0}^-$ . It is enough to show that  $u'^{-1}B'u' = \tilde{u}B^+\tilde{u}^{-1}$  is opposed to  $B^+$  where  $\tilde{u} = u'^{-1}u$ . Since  $u, u'^{-1}$  are in  $U_{>0}^-$  and  $U_{>0}^-$  is closed under multiplication, we have  $\tilde{u} \in U_{>0}^-$ . Therefore  $\tilde{u}B^+\tilde{u}^{-1}, B^+$  are opposed by [L1, 2.13(a)].

**5.** In this section we describe a strengthening of 3(a) which also places 2(a) in a Lie theoretic setting.

In the setup of §3 let  $\sigma : G \rightarrow G$  be an automorphism of finite order (as an algebraic group) which preserves the pinning that is, for some permutation  $\tau : I \rightarrow I$  we have  $\sigma(x_i(a)) = x_{\tau(i)}(a)$ ,  $\sigma(y_i(a)) = y_{\tau(i)}(a)$  for all  $i \in I, a \in \mathbf{R}$ . We have the following strengthening of 3(a).

(a) *Let  $g \in G_{\geq 0}^{\text{osc}}$ . Define  $\alpha : G \rightarrow G$  by  $h \mapsto g\sigma(h)g^{-1}$ . There is a unique  $B \in \mathcal{B}_{>0}$  and a unique  $B' \in {}'\mathcal{B}_{>0}$  such that  $B$  and  $B'$  are  $\alpha$ -stable. Then  $B, B'$  are opposed. Moreover,  $\alpha$  induces a dilation on  $\text{Lie}(B)/\text{Lie}(B \cap B')$ , a contraction on  $\text{Lie}(B')/\text{Lie}(B \cap B')$  and an automorphism of finite order of  $\text{Lie}(B \cap B')$ .*

(An endomorphism  $A : E \rightarrow E$  of a finite dimensional  $\mathbf{R}$ -vector space is said to be a dilation (resp. contraction) if all its eigenvalues  $\lambda$  satisfy  $|\lambda| > 1$  (resp.  $|\lambda| < 1$ ).

Let  $m \geq 1$  be such that  $\sigma^m = 1$ . Let  $g' = g\sigma(g)\sigma^2(g) \dots \sigma^{m-1}(g)$ . From the definitions we see that  $\sigma(G_{\geq 0}) = G_{\geq 0}$ ,  $\sigma(G_{>0}) = G_{>0}$ ,  $\sigma(G_{\geq 0}^{\text{osc}}) = G_{\geq 0}^{\text{osc}}$ . Since  $G_{\geq 0}^{\text{osc}}$  is closed under multiplication it follows that  $g' \in G_{\geq 0}^{\text{osc}}$ . Replacing  $m$  by a multiple we can assume that we have  $g' \in G_{>0}$ .

Define uniquely  $B \in \mathcal{B}_{>0}$  by  $g' \in B$ , see [L1, 8.9]; define uniquely  $B' \in {}'\mathcal{B}_{>0}$  by  $g'^{-1} \in B'$ , see [L1, 8.9] for  ${}'G_{>0}$  instead of  $G_{>0}$ . Let  $B_1 = g\sigma(B)g^{-1} \in \mathcal{B}$ ,  $B'_1 = \sigma^{-1}(g^{-1}B'g) \in \mathcal{B}$ . From the definitions we see that  $\sigma(\mathcal{B}_{>0}) = \mathcal{B}_{>0}$ ,  $\sigma({}'\mathcal{B}_{>0}) = {}'\mathcal{B}_{>0}$ . Since  $g \in G_{\geq 0}$  we have  $\text{Ad}(g)(\mathcal{B}_{>0}) \subset \mathcal{B}_{>0}$ , see [L1, 8.12]. Similarly since  $g^{-1} \in {}'G_{\geq 0}$  we have  $\text{Ad}(g^{-1})({}'\mathcal{B}_{>0}) \subset {}'\mathcal{B}_{>0}$ . We see that  $B_1 \in \mathcal{B}_{>0}$ ,  $B'_1 \in {}'\mathcal{B}_{>0}$ . Applying  $\sigma$  to  $g' \in B$  and  $\sigma^{-1}$  to  $g'^{-1} \in B'$  we obtain  $\sigma(g') \in \sigma(B)$ ,  $\sigma^{-1}(g'^{-1}) \in \sigma^{-1}(B')$ . Since  $\sigma(g') = g^{-1}g'g$ ,  $\sigma^{-1}(g'^{-1}) = \sigma^{-1}(g)g'^{-1}\sigma^{-1}(g^{-1})$  we see that  $g^{-1}g'g \in \sigma(B)$ ,  $\sigma^{-1}(g)g'^{-1}\sigma^{-1}(g^{-1}) \in \sigma^{-1}(B')$  that is  $g' \in B_1$ ,  $g'^{-1} \in B'_1$ . By the uniqueness of  $B, B'$  it follows that  $B_1 = B, B'_1 = B'$ . Thus we have  $B = \alpha(B)$ ,  $B' = \alpha^{-1}(B')$ . Hence  $B' = \alpha(B')$ . Note that  $B \in \mathcal{B}_{>0}$  is uniquely determined by the condition  $B = \alpha(B)$ . Indeed this condition implies that  $B = \alpha^m(B)$  that is  $g' \in B$  and this condition is known to determine  $B$  uniquely. Similarly  $B' \in {}'\mathcal{B}_{>0}$  is uniquely determined by the condition  $B' = \alpha(B')$ . Thus the first assertion of (a) is established. The second assertion of (a) follows from 4(a). The third assertion follows from the analogous assertion where  $\alpha$  is replaced by  $\alpha^m = \text{Ad}(g')$ ; hence to prove it we may assume that  $m = 1$  and  $g \in G_{>0}$ . In this case, by [L1, 8.10] we can find unique  $u \in U_{>0}^-$ ,  $\tilde{u} \in U_{>0}^+$ ,  $t \in T_{>0}$  such that  $g = u'\tilde{u}tu^{-1}$ ; moreover,  $\text{Ad}(t)$  is a dilation on  $\text{Lie}(U^+)$  and a contraction on  $\text{Lie}(U^-)$ . It follows that  $\tilde{u}t$  is conjugate to  $t$  under an element in  $U^+$  hence  $\text{Ad}(\tilde{u}t)$  is a dilation on  $\text{Lie}(U^+)$ . We have  $uB^+u^{-1} \in \mathcal{B}_{>0}$ ,  $g \in uB^+u^{-1}$  hence  $uB^+u^{-1} = B$  and  $u^{-1}gu = \tilde{u}t$ . It follows that  $\text{Ad}(g)$  is a dilation on  $\text{Lie}(uU^+u^{-1}) = \text{Lie}(B)/\text{Lie}(B \cap B')$ . Since  $B, B'$  are opposed and  $g \in B \cap B'$ , the linear map  $\text{Ad}(g)$  on  $\text{Lie}(B')/\text{Lie}(B \cap B')$  may be identified with the transpose inverse of  $\text{Ad}(g)$  on  $\text{Lie}(B)/\text{Lie}(B \cap B')$  hence is a contraction. Since  $g \in B \cap B'$ ,  $\text{Ad}(g)$  acts trivially on  $\text{Lie}(B \cap B')$ . This completes the proof of (a).

We now show:

(b) *Let  $g \in G_{>0}$ . Let  $B, B'$  be as in (a). Then  $g$  belongs to the "identity*

component" of the torus  $B \cap B'$ .

Let  $u, \tilde{u}, t$  be as in the proof of (a). Since  $u^{-1}gu = \tilde{u}t$ ,  $\tilde{u}t$  is contained in the maximal torus  $B^+ \cap u^{-1}B'u$  of  $B^+$  and it is enough to show that it is contained in the "identity component" of  $B^+ \cap u^{-1}B'u$  or equivalently its image in  $B^+/U^+$  is contained in the "identity component" of  $B^+/U^+$ . But that image is the same as the image of  $t$  and it remains to use the fact that  $t \in T_{>0}$ .

**6.** We now assume that  $G$  is adjoint (that is with trivial centre) as an algebraic group. Let  $S$  be a closed Riemann surface of genus  $g \geq 2$ . Let  $\pi_1$  be the fundamental group of  $S$  at some point of  $S$ . In this section we review some recent results of Fock and Goncharov [FG] which show that total positivity can be used to understand some features of the real algebraic variety  $\text{Hom}(\pi_1, G)$  of homomorphisms  $\pi_1 \rightarrow G$ .

Let  $X$  be a 4-elements set. A dihedral order on  $X$  is a partition of  $X$  into two 2-element sets. A map  $f : X \rightarrow \mathcal{B}$  is said to be *positive* (see [FG, §5] if for some/any numbering  $x_1, x_2, x_3, x_4$  of  $X$  such that the dihedral order is  $\{x_1, x_3\}, \{x_2, x_4\}$  and some  $h \in G$  we have

$$hf(x_1)h^{-1} = B^+, hf(x_3)h^{-1} = B^-, hf(x_2)h^{-1} \in \mathcal{B}_{>0}, hf(x_4)h^{-1} \in {}' \mathcal{B}_{>0}.$$

(The equivalence of "some" and "any" is proved in [FG, Thm. 5.3].)

Let  $C$  be the unit circle in  $\mathbf{C}$ . A map  $F : C \rightarrow \mathcal{B}$  is said to be a *positive curve* (see [FG, Def. 6.4] if for any 4-element subset  $X$  of  $C$  the restriction of  $F$  to  $X$  is positive with respect to the dihedral order on  $X$  given by the partition  $X = X' \sqcup X''$  such that the line spanned by  $X'$  meets the line spanned by  $X''$  in a point of the (open) unit disk. Let  $\mathcal{C}(\mathcal{B})$  be the set of all positive curves  $C \rightarrow \mathcal{B}$ .

Now there is a free holomorphic action of  $\pi_1$  on the open unit disk whose orbit space is  $S$ ; this extends continuously to an action of  $\pi_1$  on the boundary  $C$  of the open unit disk. This last action is such that for any  $\gamma \in \pi_1$  and any 4-element subset  $X$  of  $C$  the bijection  $X \rightarrow \gamma(X)$  induced by  $\gamma : C \rightarrow C$  is compatible with the dihedral order on  $X$  and  $\gamma(X)$  defined as above. Hence if  $F : C \rightarrow \mathcal{B}$  is a positive curve then  $F \circ \gamma : C \rightarrow \mathcal{B}$  is a positive curve. Thus  $\gamma : F \mapsto F \circ \gamma^{-1}$  is an action of  $\pi_1$  on  $\mathcal{C}(\mathcal{B})$ . On the other hand  $G$  acts on  $\mathcal{C}(\mathcal{B})$  by  $h : F \mapsto {}^h F$  where  ${}^h F(z) = hF(z)h^{-1}$  for any  $z \in C$ . This  $G$ -action commutes with the  $\pi_1$ -action and is free: if  $h \in G, F \in \mathcal{C}(\mathcal{B})$  satisfy  ${}^h F = F$  then  $h$  is contained in three Borel subgroup any two of which are opposed; hence  $h = 1$ . Let  $\mathcal{C}_{\pi_1}(\mathcal{B})$  be the set of all  $F \in \mathcal{C}(\mathcal{B})$  such that for any  $\gamma \in \pi_1, F \circ \gamma$  is in the  $G$ -orbit of  $F$  that is,  $F \circ \gamma = \chi_F(\gamma) F$  for some  $\chi_F(\gamma) \in G$  which is unique (by the freeness of the  $G$ -action). Note that  $\chi_F : \pi_1 \rightarrow G$  is a homomorphism. The map  $\mathcal{C}_{\pi_1}(\mathcal{B}) \rightarrow \text{Hom}(\pi_1, G), F \mapsto \chi_F$  is compatible with the  $G$ -actions (where  $G$  acts on  $\text{Hom}(\pi_1, G)$  by conjugation). Let  $\text{Hom}^{\text{pos}}(\pi_1, G)$  be the image of this map (the set of "positive homomorphisms"). By passage to  $G$ -orbits we get a map  $\Phi : G \backslash \mathcal{C}_{\pi_1}(\mathcal{B}) \rightarrow G \backslash \text{Hom}^{\text{pos}}(\pi_1, G)$ .

According to [FG],  $\Phi$  is a bijection and  $G \backslash \text{Hom}^{\text{pos}}(\pi_1, G)$  is a ball of dimension  $(2g - 2) \dim G$ ; moreover, any positive homomorphism  $\chi : \pi_1 \rightarrow G$  is injective



with discrete image and for any  $\gamma \in \pi_i - \{1\}$ ,  $\chi(\gamma)$  is contained in a  $G$ -conjugate of  $G_{>0}$  hence, by 3(a), it is contained in a unique  $G$ -conjugate of  $T_{>0}$ .

This sheds a new light on a result of Hitchin [H]. Let  $\text{Hom}^{cr}(\pi_1, G)$  be the space of all  $\chi \in \text{Hom}(\pi_1, G)$  such that the induced action of  $\pi_1$  on  $\text{Lie}(G)$  is completely reducible. In [H], Hitchin shows using techniques of analysis that there is a canonical connected component of  $G \backslash \text{Hom}^{cr}(\pi_1, G)$  which is a ball of dimension  $(2g-2) \dim G$ . As a consequence of [FG], this "Hitchin component" coincides with  $G \backslash \text{Hom}^{\text{pos}}(\pi_1, G)$ .

**7.** Let  $P^m$  be the  $m$ -dimensional real projective space. Following Schoenberg we say that an imbedding  $f : C \rightarrow P^m$  is a *convex curve* if for any hyperplane  $H$  in  $P^m$  the intersection  $f(C) \cap H$  has at most  $n$  points. In [FG, Thm. 9.4] it is shown that if  $G = GL(\mathbf{R}^{m+1})$  then the image of a positive curve in  $\mathcal{B}$  under the natural map  $\mathcal{B} \rightarrow P^m$  is a convex curve; moreover, with suitable smoothness assumptions, this gives a bijection between positive curves in  $\mathcal{B}$  and convex curves in  $P^m$ .

*Acknowledgement.* I thank A. Goncharov for useful discussions.

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