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Gapped domain walls between 2+1D topologically ordered states

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The (2+1)-dimensional (2 + 1D) topological order can be characterized by the mapping-class-group representations for Riemann surfaces of genus 1, genus 2, etc. In this paper, we use those representations to determine the possible gapped boundaries of a 2 + 1D topological order, as well as the domain walls between two topological orders. We find that mapping-class-group representations for both genus-1 and genus-2 surfaces are needed to determine the gapped domain walls and boundaries. Our systematic theory is based on the fixed-point partition functions for the walls (or the boundaries), which completely characterize the gapped domain walls (or the boundaries). The mapping-class-group representations give rise to conditions that must be satisfied by the fixed-point partition functions, which leads to a systematic theory. Such conditions can be viewed as bulk topological order determining the (noninvertible) gravitational anomaly at the domain wall, and our theory can be viewed as finding all types of the gapped domain wall given a (noninvertible) gravitational anomaly. We also developed a systematic theory of gapped domain walls (boundaries) based on the structure coefficients of condensable algebras.

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I. INTRODUCTION

Topological order is a new kind of order in gapped quantum states of matter beyond Landau symmetry breaking theory [1,2]. In Refs. [3,4], it was conjectured that *the non-Abelian geometric phases* [5] [both the $U(1)$ part and the non-Abelian part] of degenerate ground states generated by the automorphism of Riemann surfaces can completely characterize different topological orders [3].

The non-Abelian geometric phases contain a universal non-Abelian part [3,4] and a path dependent $U(1)$ part [3,6]. The non-Abelian part carries information about the projective representation of the mapping class group (MCG) of the space manifold. For a torus, the MCG is $\Gamma_1 = SL(2, \mathbb{Z})$, which is generated by a 90° rotation and a Dehn twist. The associated non-Abelian geometric phases for two such generators are denoted by S and T , which are unitary matrices. S and T generate a projective representation of MCG $SL(2, \mathbb{Z})$ for a torus.

The Abelian part of the non-Abelian geometric phases is also important: it is related to the gravitational Chern-Simons term [7–9] in the partition function and carries information about the chiral central charge c for the gapless edge excitations [10,11]. The data (S, T, c) are a quite complete

description of (2+1)-dimensional (2 + 1D) topological orders. However, to obtain a full description of 2 + 1D topological orders, the modular data for a genus-1 surface are not enough [12–14]; we must also use the non-Abelian geometric phases (i.e., the mapping-class-group representations) for genus-2 surfaces [15].

In this paper, we will study the boundary of topological orders, or more generally, the domain wall between two topological orders. We will see how the boundary properties are determined by the bulk topological orders. We would like to consider the following issues: *What are the data that allow us to characterize different gapped domain walls between topologically ordered states? How do we classify gapped domain walls?* References [16–19] studied those issues for the case of 2 + 1D Abelian topological orders, using a condensable set of bosonic topological excitations. References [20–26] considered this problem for the more general case of 2 + 1D non-Abelian topological orders, using boson condensation [27,28] and/or condensable algebra [25]. Some discussions of the boundaries of topological orders beyond 2 + 1D can be found in Refs. [9,29,30]. In particular, it was pointed out that the boundary effective theory of a bulk topological order has a gravitational anomaly [9,31–33]. In fact, the bulk topological order gives a one-to-one classification of a gravitational anomaly in one low dimension (realized by the boundary) [9]. Thus in some sense *types of bosonic topological order = types of bosonic gravitational anomaly in one lower dimension*. (Note that bosonic gravitational anomaly is a property of an effective theory that cannot be realized as a local bosonic system with finite cut-off. Two effective theories are said to have the same type of gravitational anomaly if one effective

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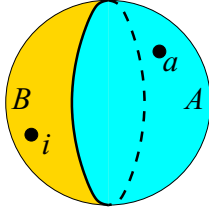


FIG. 1. Two topologically ordered phases \mathcal{A} and \mathcal{B} , each occupying a hemisphere of a 2D space S^2 . There is an excitation of type a in the phase \mathcal{A} and an excitation of type i in the phase \mathcal{B} .

theory can change into another effective theory, possibly via phase transitions in the same dimension [9].)

In this paper, we will rederive the simple results based on the (S, T, c) introduced in Ref. [34], for the boundaries of general $2 + 1$ D non-Abelian topological orders. Our approach also allows us to generalize the approach in Ref. [34] to high genus Riemann surface, which may lead to a complete description of the gapped boundary of $2 + 1$ D topological orders. We try to address the following question: *Given a $2 + 1$ D topological order described by (S, T, c) and the data from higher genus, how do we describe and classify different gapped $1 + 1$ D boundary phases?* If we find that there is no gapped $1 + 1$ D boundary for a type of $2 + 1$ D topological order, then such a type of topological order must have a gapless boundary. It can also be stated in the following way: Given a type of gravitational anomaly, how do we describe different gapped $1 + 1$ D phases? If we find that there is no gapped $1 + 1$ D phase for a type of gravitational anomaly, then such a type of gravitational anomaly will require (or ensure) the $1 + 1$ D phases to be gapless.

Our new approach is based on the spacetime path integral description of topological order. We remark that the “spacetime” in this paper means “spacetime complex with branching structure” (see Appendix A 1). “Spacetime complex with branching structure” may correspond to spacetime manifold with some structure, but it is still an open question. Also our approach can be directly generalized to higher dimensions, to systematically find gapped boundaries of a topological order from the data that describe mapping-class-group action in higher dimensions [35].

II. CHARACTERIZATION OF GAPPED DOMAIN WALLS

A. Dimensions of fusion spaces

To introduce data that can characterize different gapped domain walls, let us consider a 2D space S^2 . Half of S^2 is occupied by the phase \mathcal{A} and the other half by the phase \mathcal{B} . Let us assume the domain wall between the topological phases is gapped. We put a type- a topological excitation in the phase \mathcal{A} and a type- i topological excitation in the phase \mathcal{B} . We denote such a configuration as $S^2_{\mathcal{B}\mathcal{A};ia}$ (see Fig. 1). The ground state space for such a configuration is a fusion space. The dimension of the fusion space (i.e., the ground state degeneracy), denoted as $M_{\mathcal{B}\mathcal{A}}^{ia} \in \mathbb{N}$, is the data that characterizes the gapped domain wall between \mathcal{A} and \mathcal{B} phases. We may also view $M_{\mathcal{B}\mathcal{A}}^{ia}$ as the fusion coefficients for the fusion of type- i particle, type- a particle, and the domain wall BA .

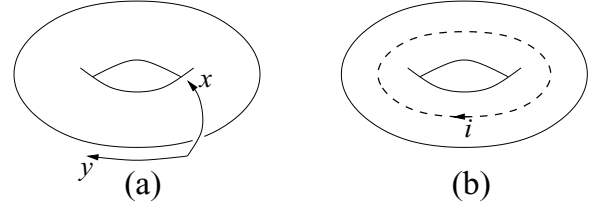


FIG. 2. (a) The ground state $|1\rangle$ on a torus that corresponds to the trivial particle 1 can be represented by an empty solid torus. (b) The other ground state $|i\rangle$ that corresponds to a type- i topological excitation can be represented by a solid torus with a worldline of type i in the center.

B. Weighted wave function overlap

To obtain more data to characterize the domain walls, let us consider the degenerate ground states of the topologically ordered phase \mathcal{A} , described by normalized wave function $|\psi_{I_A}^{\mathcal{A}}\rangle$, where the index I_A labels different ground states on a closed Riemann surface of genus g . Similarly, we have the degenerate ground states of the topologically ordered phase \mathcal{B} : $|\psi_{I_B}^{\mathcal{B}}\rangle$.

The degenerate ground states on a closed surface can be obtained as path integral on the “solid surface” (a three dimensional manifold whose boundary is the surface) with a worldline of a topological excitation (see Fig. 2). Thus we can label different ground states using the label of those topological excitations. Also, the above construction of degenerate ground states using worldlines give rise to a natural basis for the degenerate ground states. We will refer such a basis as the *excitation basis*.

For example, on a genus-1 surface, the degenerate ground states are labeled by $I_B = i$ for the phase \mathcal{B} and $I_A = a$ for the phase \mathcal{A} . Here $i = 1, \dots, N_B$ label the types of the topological excitations in the phase \mathcal{B} and $a = 1, \dots, N_A$ label the types of the topological excitations in the phase \mathcal{A} . For genus-1 surface, those excitation-labeled ground states happen to be orthogonal. We see that the ground state degeneracies on a genus-1 surface are given by N_A for phase \mathcal{A} and N_B for phase \mathcal{B} .

As another example, on a genus-2 surface, the degenerate ground states are labeled by $I_B = (i, j, z, \mu, \nu)$, where $\mu = 1, \dots, N_{B;z}^{\bar{i}}$ and $\nu = 1, \dots, N_{B;\bar{j}}^{j\bar{z}}$ for the phase \mathcal{B} . The degenerate ground states are labeled by $I_A = (a, b, c, \alpha, \beta)$, where $\alpha = 1, \dots, N_{A;c}^{a\bar{a}}$ and $\beta = 1, \dots, N_{A;\bar{c}}^{b\bar{b}}$ for the phase \mathcal{A} (see Fig. 3). Here i, j, z label the types of the topological excitations in the phase \mathcal{B} and a, b, c label the types of the

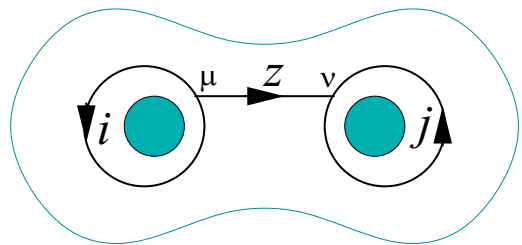


FIG. 3. A filled genus-2 Riemannian surface (denoted as Σ_2^{fill}), which has three worldlines i, j , and z in the interior.

topological excitations in the phase \mathcal{A} . $N_{B,z}^{ij}$ are the fusion coefficients of the topological excitations in the phase \mathcal{A} and $N_{A,c}^{ab}$ are the fusion coefficients of the topological excitations in the phase \mathcal{B} . We see that the ground state degeneracies on a genus-2 surface are given by $\sum_{z=1}^{N_B} N_{B,z}^{i\bar{i}} N_{B,z}^{j\bar{j}}$ for phase \mathcal{B} and $\sum_{c=1}^{N_A} N_{A,c}^{a\bar{a}} N_{A,c}^{b\bar{b}}$ for phase \mathcal{A} .

Motivated by the wave function overlap [36–39] that can characterize different topological orders, here we will use the *weighted wave function overlap* to characterize different domain walls. We conjecture that [9,40] *the weighted overlap of the degenerate ground states on a closed genus- g surface Σ_g for topologically ordered phases \mathcal{A} and \mathcal{B} has the following form:*

$$\langle \psi_{I_B}^{\mathcal{B}} | e^{-H_W} | \psi_{I_A}^{\mathcal{A}} \rangle = e^{-\sigma_{\mathcal{B},\mathcal{A}} A_{\Sigma_g} + o(\frac{1}{A_{\Sigma_g}})} W_{\mathcal{B},\mathcal{A},g}^{I_B I_A} \quad (1)$$

where H_W is local Hermitian operator like a Hamiltonian of a quantum system, A_{Σ_g} is the area of the surface Σ_g , and $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$ is a topological invariant that characterizes the domain between the phases \mathcal{A} and \mathcal{B} .

In general, $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$ depends on the choices of H_W , which correspond to different choices of domain walls. A concrete calculation of the wave function overlap, $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$, for a simple topological order is presented in Sec. IV. We will show in next section that the wave function overlap can also be viewed as partition function of the domain wall.

When \mathcal{B} is a trivial product state, the index I_B is always fixed to be 1, since \mathcal{B} has no ground state degeneracy. In this case we simplify

$$W_{\mathcal{B},\mathcal{A},g}^{1,I_A} = W_{\mathcal{A},g}^{I_A} \quad (2)$$

by dropping the indices for trivial phase \mathcal{B} and I_B . Similarly, we simplify

$$M_{\mathcal{B},\mathcal{A}}^{1a} = M_{\mathcal{A}}^a. \quad (3)$$

Those are data that describe a gapped boundary of topological order \mathcal{A} .

We note that $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$ are in general complex numbers, whose phase can be changed by a change of phases for the ground states:

$$|\psi_{I_A}^{\mathcal{A}}\rangle \rightarrow e^{i\theta_{I_A}} |\psi_{I_A}^{\mathcal{A}}\rangle, \quad |\psi_{I_B}^{\mathcal{B}}\rangle \rightarrow e^{i\phi_{I_B}} |\psi_{I_B}^{\mathcal{B}}\rangle. \quad (4)$$

Also $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$ may depend on some choices of basis that cannot be fixed. So $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$ by themselves are not a topological invariant, and they are not even physical. So we need to find a way made out those phase and basis dependence. When we say $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$ are “topological invariants,” we mean that they are topological invariants up to those phase and basis choices.

For genus-1 surface Σ_1 , we may choose the phases of $|\psi_1^{\mathcal{A}}\rangle$ and $|\psi_1^{\mathcal{B}}\rangle$ to make $W_{\mathcal{B},\mathcal{A},1}^{11}$ real and positive. (Here 1 correspond to the trivial particle.) We then choose the phases of $|\psi_a^{\mathcal{A}}\rangle$ to make $W_{\mathcal{B},\mathcal{A},1}^{1a}$ real and positive. Similarly, we choose the phases of $|\psi_i^{\mathcal{B}}\rangle$ to make $W_{\mathcal{B},\mathcal{A},1}^{i1}$ real and positive. With such a choice, we find that $W_{\mathcal{B},\mathcal{A},1}^{I_B I_A} = M_{\mathcal{B},\mathcal{A}}^{I_B I_A}$, which is the dimension of the fusion space defined in the last subsection (see Sec. III A for an explanation). For higher genus $g > 1$, $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$, after moding out some gauge redundancy, are new topological invariants. We hope $W_{\mathcal{B},\mathcal{A},g}^{I_B I_A}$ carry enough information to fully characterize a gapped domain wall.

III. THE CONDITIONS ON THE DOMAIN-WALL DATA

In this section, we are going to derive some conditions on the data that characterize the gapped domain walls. For example, we show that the dimension of the fusion state $M_{\mathcal{B},\mathcal{A}}^{ia}$ (which is a non-negative integer) and the weighted wave function overlap for a torus $W_{\mathcal{B},\mathcal{A},1}^{ia}$ (which can be a complex number) are actually equal to each other, $M_{\mathcal{B},\mathcal{A}}^{ia} = W_{\mathcal{B},\mathcal{A},1}^{ia}$. This is quite an amazing relation.

To derive those conditions, we need to introduce the topological path integral, i.e., the path integral for triangulated spacetime whose value is retriangulation invariant. This is done in Appendix A. We also need to introduce an algebraic (i.e., a categorical) approach to evaluate those topological path integrals, which is done in Appendix B. Using those results, we can derive the conditions on the domain-wall data.

A. Why $M_{\mathcal{B},\mathcal{A}}^{ia} = W_{\mathcal{B},\mathcal{A},1}^{ia}$?

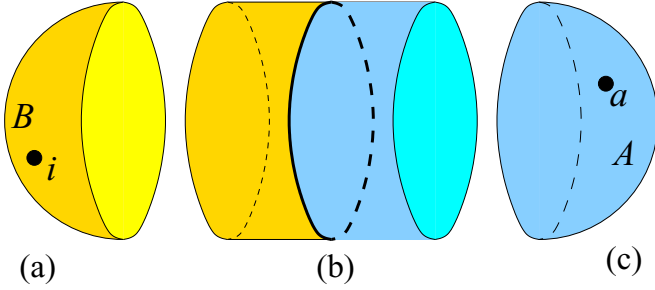
First, we show $M_{\mathcal{B},\mathcal{A}}^{ia} = W_{\mathcal{B},\mathcal{A},1}^{ia}$. Let us consider the topological path integral on spacetime $D_i^2 \times S^1$, where D_i^2 is a disk with a puncture. Such a puncture corresponds to a worldline of a type- i topological excitation wrapping around S^1 . The topological path integral on $D_i^2 \times S^1$ only sums over the degrees of freedom in the bulk, and leaves the boundary degrees of freedom fixed. Thus $Z^{\text{top}}(D_i^2 \times S^1)$ corresponds to a wave function $|\psi_i\rangle$ on $S^1 \times S^1 = \partial(D_i^2 \times S^1)$ (for details, see Ref. [40]). First, we show that such a wave function $|\psi_i\rangle$ is automatically normalized. According to Ref. [40], $\langle \psi_i | \psi_i \rangle$ is given by $Z^{\text{top}}(S_{i,\bar{i}}^2 \times S^1)$, where $S_{i,\bar{i}}^2 \times S^1$ is obtained by gluing two $D_i^2 \times S^1$ together along their boundary. $S_{i,\bar{i}}^2 \times S^1$ contains two worldlines of type- i and type- \bar{i} .

Let us evaluate the above partition function on $S_{i,\bar{i}}^2 \times S^1$, which turns out to be $Z^{\text{top}}(S_{i,\bar{i}}^2 \times S^1) = 1$:

$$\begin{aligned} Z^{\text{top}} \left(\text{diagram of } S_{i,\bar{i}}^2 \times S^1 \right) &= \sum_{\alpha\beta, k=1} Y_{k,\alpha\beta}^{\bar{i}i} Z^{\text{top}} \left(\text{diagram of } S^2 \times S^1 \right) \\ &= \sum_{\alpha\beta, k=1} Y_{k,\alpha\beta}^{\bar{i}i} O_{k,\alpha\beta}^{\bar{i}i} Z^{\text{top}} \left(\text{diagram of } S^2 \times S^1 \right) \\ &= \sum_{\alpha\beta} Y_{1,\alpha\beta}^{\bar{i}i} O_{1,\alpha\beta}^{\bar{i}i} Z^{\text{top}}(S^2 \times S^1) = 1, \end{aligned} \quad (5)$$

where we have used $N_1^{\bar{i}i} = N_1^{ii} = 1$, so $\alpha = \beta = 1$ and the $\sum_{\alpha\beta}$ only contains one term. k has to be the trivial particle 1, since k lives on a sphere S^2 alone. We have also used $Y_{1,11}^{\bar{i}i} O_{1,11}^{\bar{i}i} = 1$ (see Appendix B for the definition of Y -move and O -move) and $Z^{\text{top}}(S^2 \times S^1) = 1$. So the topological path integral on $D_i^2 \times S^1$ gives rise to normalized wave function $|\psi_i\rangle$.

Now, instead of a path integral on $S_{i,\bar{i}}^2 \times S^1$ which gives us $Z^{\text{top}}(S_{i,\bar{i}}^2 \times S^1) = 1$, let us consider a path integral on $S_{\mathcal{B},\mathcal{A};ia}^2 \times S^1$, where $S_{\mathcal{B},\mathcal{A};ia}^2$ is described by Fig. 1. In spacetime, the particles a and i correspond to worldlines wrapping around S^1 which is viewed as a space direction. The boundary between

FIG. 4. Divide S^2 in Fig. 1 into three pieces.

the two hemispheres in Fig. 1 is viewed as another space direction. The topological partition function for spacetime $S^2_{B,A;ia} \times S^1$ corresponds to the weighted wave function overlap $\langle \psi_i^B | e^{-H_W} | \psi_a^A \rangle$ with a fine tune choice of H_W , which has the form

$$\langle \psi_i^B | e^{-H_W} | \psi_a^A \rangle = Z^{\text{top}}(S^2_{B,A;ia} \times S^1) = e^{-\sigma_{B,A} A_{\Sigma_1}} W_{B,A,1}^{ia}.$$

Here $\sigma_{B,A}$ is the energy density of the domain wall between the \mathcal{A} and \mathcal{B} phases, and A_{Σ_1} is the spacetime area occupied by the domain wall. For our topological path integral, the energy density of the domain wall $\sigma_{B,A}$ is fine tuned to zero. Thus we actually have

$$\langle \psi_i^B | e^{-H_W} | \psi_a^A \rangle = Z^{\text{top}}(S^2_{B,A;ia} \times S^1) = W_{B,A,1}^{ia}. \quad (6)$$

In the above, we have regarded the S^1 in $S^2_{B,A;ia} \times S^1$ as a space direction. Now we regard the S^1 in $S^2_{B,A;ia} \times S^1$ as the time direction, and $S^2_{B,A;ia}$ in $S^2_{B,A;ia} \times S^1$ as the space. In this case, the area independent part of the partition function has a different meaning: it becomes the ground state degeneracy on the space $S^2_{B,A;ia}$ which is denoted by $M_{B,A}^{ia}$. Thus $M_{B,A}^{ia} = W_{B,A,1}^{ia}$. It is interesting to see that the weighted wave function overlaps for torus $W_{B,A,1}^{ia}$ (after being fine-tuned into topological ones), are always given by non-negative integers in the excitation basis.

When viewed as partition function, $W_{B,A,1}^{ia}$ is a multicomponent partition function (labeled by i and a) on a torus. Such a multicomponent partition function describes a gapped theory on the torus (i.e., the domain wall) that has a noninvertible gravitational anomaly [35]. Thus the conditions on $W_{B,A,1}^{ia}$ (see next subsection) are a special case of the modular covariance condition discussed in Ref. [35].

B. Invariance under the MCG action

Next, we divide the spacetime $S^2_{B,A;ia} \times S^1$ into three pieces, $D^2_{A;a} \times S^1$ [see Fig. 4(a)], $D^2_{B;i} \times S^1$ [see Fig. 4(c)], and $S^1_{B,A} \times S^1 \times S^1$ [see Fig. 4(b)], where $D^2_{A;a}$ is a disk occupied by the phase \mathcal{A} and the type- a particle, $D^2_{B;i}$ is a disk occupied by the phase \mathcal{B} and the type- i particle, and $S^1_{B,A} \times S^1$ is a cylinder occupied by the \mathcal{A} and \mathcal{B} phases and the domain wall. We can use the three pieces, $D^2_{A;a} \times S^1$, $D^2_{B;i} \times S^1$, and $S^1_{B,A} \times S^1 \times S^1$, to assemble the same closed spacetime in two ways: (1) we glue $D^2_{A;a} \times S^1$ and $S^1_{B,A} \times S^1 \times S^1$ directly along the $S^1 \times S^1$ boundary, and glue $S^1_{B,A} \times S^1 \times S^1$ and $D^2_{B;i} \times S^1$ with a twist $\hat{U} \in \Gamma_1 = SL(2, \mathbb{Z})$ for the boundary $S^1 \times S^1$; (2) we glue $D^2_{A;a} \times S^1$ and $S^1_{B,A} \times S^1 \times S^1$ with a

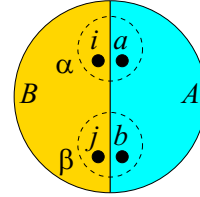


FIG. 5. A 2D space S^2 is occupied by two topologically ordered phases, the phase \mathcal{A} and the phase \mathcal{B} . Fusing an excitation of type a in the phase \mathcal{A} and an excitation of type i in the phase \mathcal{B} produces an excitation of type s on the domain wall between the phase \mathcal{A} and the phase \mathcal{B} .

twist $\hat{U} \in \Gamma_1 = SL(2, \mathbb{Z})$ and glue $S^1_{B,A} \times S^1 \times S^1$ and $D^2_{B;i} \times S^1$ directly.

The two ways to assemble the closed spacetime *only differ by a retriangulation*, and they must produce the same topological partition function (see Appendices A 4 and A 5). This leads to a matrix relation

$$R_B^U W_{B,A,1} = W_{B,A,1} R_A^U. \quad (7)$$

We notice that the path integral on $D^2_{A;a} \times S^1$ produces the basis state $|a; A\rangle$ in the excitation basis for the phase \mathcal{A} on the surface $\partial(D^2_{A;a} \times S^1)$. The action of the MCG transformation \hat{U} on $|a; A\rangle$ is described by the unitary representation R_A^U of \hat{U} for the phase \mathcal{A} . Similarly, the path integral on $D^2_{B;i} \times S^1$ produces the basis state $|i; B\rangle$ in the quasiparticle basis for the phase \mathcal{B} . The action of the MCG transformation \hat{U} on $|i; B\rangle$ is described by the unitary representation R_B^U of \hat{U} for the phase \mathcal{B} . This is how we obtain (7).

C. A consistent condition between the fusions in the bulk and on the wall

In this section, we obtain some additional conditions. Let us consider the following fusion process: we bring a type- a excitation in phase \mathcal{A} and a type- i excitation in phase \mathcal{B} to the domain wall $w_{B,A}$ between the two phases and fuse them into an excitation of type s on the domain wall (see Fig. 5). The corresponding fusion algebra is given by

$$i \otimes_{w_{B,A}} a = M_{B,A;s}^{ia}, \quad M_{B,A;s}^{ia} \in \mathbb{N}. \quad (8)$$

It turns out that

$$M_{B,A}^{ia} = M_{B,A;1}^{ia}, \quad (9)$$

where $s = 1$ represents the trivial type of excitations on the domain wall.

Now let us compute the dimension of the fusion space of $S^2_{B,A}$ with excitations of types a and b in the phase \mathcal{A} and excitations of types i and j in the phase \mathcal{B} . There are two ways to compute the dimension of the fusion space: (1) we may fuse a and b into c and fuse i and j into k , or (2) we may first fuse a and i into s on the domain wall and fuse j and b into \bar{s} on the domain, and then fuse s and \bar{s} into the trivial excitation on the domain wall. The two fusion paths should produce the same result, and we obtain

$$\sum_{c,k} N_{A;c}^{ab} N_{B;k}^{ij} M_{B,A}^{kc} = \sum_s M_{B,A;s}^{ia} M_{B,A;\bar{s}}^{jb}. \quad (10)$$

The two Eqs. (9) and (10) imply the condition (14).

D. The conditions on genus-1 data for a gapped domain wall

Consider two topological orders, phases \mathcal{A} and \mathcal{B} , described by $(S^{\mathcal{A}}, T^{\mathcal{A}}, c^{\mathcal{A}})$ and $(S^{\mathcal{B}}, T^{\mathcal{B}}, c^{\mathcal{B}})$. Suppose there are $N^{\mathcal{A}}$ and $N^{\mathcal{B}}$ types of topological excitations in the phase \mathcal{A} and the phase \mathcal{B} ; then the ranks of their modular matrices are $N^{\mathcal{A}}$ and $N^{\mathcal{B}}$ respectively. We find the following necessary conditions for the phase \mathcal{A} and the phase \mathcal{B} to be connected by a gapped domain wall [34]:

$$c^{\mathcal{A}} = c^{\mathcal{B}}; \quad (11)$$

there exist nonzero

$$M_{\mathcal{B},\mathcal{A}}^{ia} \in \mathbb{N}, \quad (12)$$

such that

$$\sum_j S_{ij}^{\mathcal{B}} M_{\mathcal{B},\mathcal{A}}^{ja} = \sum_b M_{\mathcal{B},\mathcal{A}}^{ib} S_{ba}^{\mathcal{A}}, \quad (13)$$

$$\sum_j T_{ij}^{\mathcal{B}} M_{\mathcal{B},\mathcal{A}}^{ja} = \sum_b M_{\mathcal{B},\mathcal{A}}^{ib} T_{ba}^{\mathcal{A}}. \quad (14)$$

$$M_{\mathcal{B},\mathcal{A}}^{ia} M_{\mathcal{B},\mathcal{A}}^{jb} \leq \sum_{kc} N_{\mathcal{B},k}^{ij} M_{\mathcal{B},\mathcal{A}}^{kc} N_{\mathcal{A},c}^{ab}. \quad (14)$$

Here \mathbb{N} denotes the set of non-negative integers. a, b, c, \dots and i, j, k, \dots are indices for the particle types in phases \mathcal{A} and \mathcal{B} . $N_{\mathcal{A},c}^{ab}$ and $N_{\mathcal{B},k}^{ij}$ are fusion coefficients of phases \mathcal{A} and \mathcal{B} . We may call Eq. (13) the commuting condition, and Eq. (14) the stable condition [34].

In fact the matrix $M_{\mathcal{B},\mathcal{A}}$ labels a gapped domain wall between phases \mathcal{A} and \mathcal{B} . Equations (11), (12), (13), and (14) are a set of necessary conditions a gapped domain wall $M_{\mathcal{B},\mathcal{A}}$ must satisfy, i.e., if there is a gapped domain wall, we will have a nonzero $M_{\mathcal{B},\mathcal{A}}$ that satisfies those conditions. This implies that if there is no nonzero solution of $M_{\mathcal{B},\mathcal{A}}$, the domain wall must be gapless. However, it is not clear if those condition are sufficient for a gapped domain wall to exist. In some simple examples, the solutions $M_{\mathcal{B},\mathcal{A}}$ are in one-to-one correspondence with gapped domain walls. However, for some complicated examples [41], $M_{\mathcal{B},\mathcal{A}}$ may correspond to more than one type of gapped domain wall. This indicates that some additional data are needed to completely characterize gapped domain walls.

E. Wave function overlap on genus-2 Riemannian surfaces and topological path integral

In the above, we have developed a theory on the domain wall based on the datum $W_{\mathcal{B},\mathcal{A};1}^{ia}$, the wave function overlap on torus. However, such a datum is insufficient to fully characterize the domain wall, i.e., there are different domain walls known to produce the same $W_{\mathcal{B},\mathcal{A};1}^{ia}$. So, in this section, we consider the overlap of normalized ground state wave functions, $|\psi_{I_{\mathcal{A}}}^{\mathcal{A}}\rangle$ and $|\psi_{I_{\mathcal{B}}}^{\mathcal{B}}\rangle$, on genus-2 surfaces, which gives rise to data $W_{\mathcal{B},\mathcal{A};2}^{I_{\mathcal{B}}I_{\mathcal{A}}}$ [see Eq. (1)].

To understand the structure of the data $W_{\mathcal{B},\mathcal{A};2}^{I_{\mathcal{B}}I_{\mathcal{A}}}$, let us generate the degenerate ground states on a genus-2 surface Σ_2 via a path integral. We first consider a filled genus-2 surface denoted as $\Sigma_2^{\text{fill}}: \Sigma_2 = \partial \Sigma_2^{\text{fill}}$. The path integral on Σ_2^{fill} will generate a ground state on Σ_2 . To generate other ground states, we need to add three worldlines i, j, z in the interior of the filled

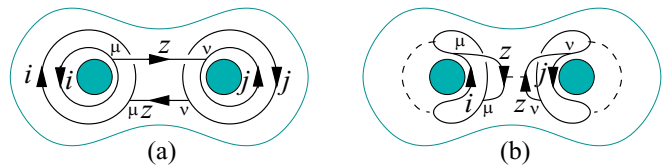


FIG. 6. A path integral on a closed three-dimensional spacetime $\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}}$.

genus-2 surface (see Fig. 3). Thus the linearly independent states on Σ_2 are labeled by i, j, z, μ, ν , where $\mu = 1, \dots, N_z^{\bar{i}}$ and $\nu = 1, \dots, N_z^{\bar{j}}$. Thus, for topological phase \mathcal{A} , the label $I_{\mathcal{A}}$ correspond to $I_{\mathcal{A}} \sim (i_{\mathcal{A}}, j_{\mathcal{A}}, z_{\mathcal{A}}, \mu_{\mathcal{A}}, \nu_{\mathcal{A}})$. For topological phase \mathcal{B} , the label $I_{\mathcal{B}}$ correspond to $I_{\mathcal{B}} \sim (i_{\mathcal{B}}, j_{\mathcal{B}}, z_{\mathcal{B}}, \mu_{\mathcal{B}}, \nu_{\mathcal{B}})$. So the wave function overlap on genus-2 surfaces is given by

$$W_{\mathcal{B},\mathcal{A};2}^{I_{\mathcal{B}}I_{\mathcal{A}}} = W_{\mathcal{B},\mathcal{A};2}^{(i_{\mathcal{B}},j_{\mathcal{B}},z_{\mathcal{B}},\mu_{\mathcal{B}},\nu_{\mathcal{B}}),(i_{\mathcal{A}},j_{\mathcal{A}},z_{\mathcal{A}},\mu_{\mathcal{A}},\nu_{\mathcal{A}})}. \quad (15)$$

The wave functions, $|\Psi_{i,j,z,\mu,\nu}\rangle$, given by the path integral on Σ_2^{fill} may not be normalized. In the following we will consider their normalization. First, we express the normalization via path integral,

$$\langle \Psi_{i,j,z,\mu,\nu} | \Psi_{i,j,z,\mu,\nu} \rangle = Z^{\text{top}}[\Sigma_2^{\text{fill}}(i,j,z,\mu,\nu) \sqcup \Sigma_2^{\text{fill}}(i,j,z,\mu,\nu)] \quad (16)$$

where $\Sigma_2^{\text{fill}}(i,j,z,\mu,\nu) \sqcup \Sigma_2^{\text{fill}}(i,j,z,\mu,\nu)$ is the closed three-dimensional spacetime obtained by gluing two $\Sigma_2^{\text{fill}}(i,j,z,\mu,\nu)$'s along their Σ_2 boundary (see Fig. 6). Also $\Sigma_2^{\text{fill}}(i,j,z,\mu,\nu)$ contain the worldlines i, j, z (Fig. 6). We next deform the worldlines in Fig. 6(a) to those in Fig. 6(b) using Eq. (B6). This gives rise to a factor $Y_{1;11}^{\bar{i}} Y_{1;11}^{\bar{j}} Y_{1;11}^{\bar{z}}$. Note that dashed lines in Fig. 6(b) all go through S^2 and hence the dashed lines must correspond to worldlines of the trivial particles (i.e., type 1). We then drop the three dashed lines in Fig. 6(b). Next, we deform the two clusters of the worldlines into two Θ graphs, which give rise to a phase factor since the worldlines may get twisted (see Fig. 16). Last, we use Eq. (B5) to reduce the two Θ graphs into two loops of the z worldlines, which gives rise to a factor $O_z^{\bar{i};\mu\mu} O_z^{\bar{j};\nu\nu}$ (here the repeated indices $\mu\mu$ and $\nu\nu$ are not summed). The two z loops produces d_z^2 .

Collecting all those factors, we find that $\langle \Psi_{i,j,z,\mu,\nu} | \Psi_{i,j,z,\mu,\nu} \rangle$ is given by (where μ, ν are not summed), up to a phase factor,

$$\begin{aligned} & Y_{1;11}^{\bar{i}} Y_{1;11}^{\bar{j}} Y_{1;11}^{\bar{z}} O_z^{\bar{i};\mu\mu} O_z^{\bar{j};\nu\nu} d_z^2 Z^{\text{top}}(\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}}) \\ &= \sqrt{\frac{d_1}{d_i d_{\bar{i}}}} \sqrt{\frac{d_1}{d_j d_{\bar{j}}}} \sqrt{\frac{d_1}{d_z d_{\bar{z}}}} \sqrt{\frac{d_i d_{\bar{i}}}{d_z}} \sqrt{\frac{d_j d_{\bar{j}}}{d_z}} d_z^2 Z^{\text{top}}(\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}}) \\ &= Z^{\text{top}}(\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}}), \end{aligned} \quad (17)$$

where $Z^{\text{top}}(\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}})$ is the topological partition function on $\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}}$ without worldlines.

Since $\langle \Psi_{i,j,z,\mu,\nu} | \Psi_{i,j,z,\mu,\nu} \rangle$ is positive, the ambiguous phase factor must be 1. We see that the normalized ground state wave function is given by the topological path integral on Σ_2^{fill}

with the worldlines i, j, z :

$$|\psi_{i,j,z,\mu,v}\rangle = \frac{Z^{\text{top}}[\Sigma_2^{\text{fill}}(i, j, z, \mu, v)]}{\sqrt{Z^{\text{top}}(\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}})}}. \quad (18)$$

This result allows us to express the wave function overlap in terms of a path integral on $\Sigma_2^{\text{fill}} \sqcup \Sigma_2^{\text{fill}}$:

$$W_{\mathcal{B},\mathcal{A},2}^{(i_{\mathcal{B}},j_{\mathcal{B}},z_{\mathcal{B}},\mu_{\mathcal{B}},\nu_{\mathcal{B}}),(i_{\mathcal{A}},j_{\mathcal{A}},z_{\mathcal{A}},\mu_{\mathcal{A}},\nu_{\mathcal{A}})} = \frac{Z^{\text{top}}[\Sigma_2^{\text{fill}}(\mathcal{A}; i_{\mathcal{A}}, j_{\mathcal{A}}, z_{\mathcal{A}}, \mu_{\mathcal{A}}, \nu_{\mathcal{A}}) \sqcup \Sigma_2^{\text{fill}}(\mathcal{B}; i_{\mathcal{B}}, j_{\mathcal{B}}, z_{\mathcal{B}}, \mu_{\mathcal{B}}, \nu_{\mathcal{B}})]}{\sqrt{Z_{\Sigma_2;\mathcal{A}}^{\text{top}} Z_{\Sigma_2;\mathcal{B}}^{\text{top}}}}. \quad (19)$$

But now $\Sigma_2^{\text{fill}}(\mathcal{A}; i_{\mathcal{A}}, j_{\mathcal{A}}, z_{\mathcal{A}}, \mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ is occupied by a topological phase \mathcal{A} and the worldlines of $i_{\mathcal{A}}, j_{\mathcal{A}}, z_{\mathcal{A}}$. $\Sigma_2^{\text{fill}}(\mathcal{B}; i_{\mathcal{B}}, j_{\mathcal{B}}, z_{\mathcal{B}}, \mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ is occupied by a topological phase \mathcal{B} and the worldlines of $i_{\mathcal{B}}, j_{\mathcal{B}}, z_{\mathcal{B}}$. Also, $Z_{\Sigma_2;\mathcal{A}}^{\text{top}} \equiv Z^{\text{top}}[\Sigma_2^{\text{fill}}(\mathcal{A}) \sqcup \Sigma_2^{\text{fill}}(\mathcal{A})]$ is the topological partition function on spacetime $\Sigma_2^{\text{fill}}(\mathcal{A}) \sqcup \Sigma_2^{\text{fill}}(\mathcal{A})$ filled with phase \mathcal{A} without worldlines. Similarly, $Z_{\Sigma_2;\mathcal{B}}^{\text{top}} \equiv Z^{\text{top}}[\Sigma_2^{\text{fill}}(\mathcal{B}) \sqcup \Sigma_2^{\text{fill}}(\mathcal{B})]$ is the topological partition function for phase \mathcal{B} .

Using the same reasoning as in Eq. (7), we find that the rectangular matrix $W_{\mathcal{B},\mathcal{A},2}$ satisfies

$$R_{\mathcal{B}}^U W_{\mathcal{B},\mathcal{A},2} = W_{\mathcal{B},\mathcal{A},2} R_{\mathcal{A}}^U, \quad (20)$$

where $R_{\mathcal{A},\mathcal{B}}^U$ are the representations of genus-2 MCG Γ_2 for the phases \mathcal{A} and \mathcal{B} .

The wave function overlap on a genus-2 surface $W_{\mathcal{B},\mathcal{A},2}$ and the wave function overlap on a genus-1 surface $W_{\mathcal{B},\mathcal{A},1}$ are related:

$$\begin{aligned} W_{\mathcal{B},\mathcal{A},2}^{(i_{\mathcal{B}},j_{\mathcal{B}},1,1,1),(i_{\mathcal{A}},j_{\mathcal{A}},1,1,1)} &= \sqrt{Z_{\Sigma_2;\mathcal{A}}^{\text{top}} Z_{\Sigma_2;\mathcal{B}}^{\text{top}}} Z^{\text{top}}[B^3(\mathcal{A}) \sqcup B^3(\mathcal{B})] \\ &= W_{\mathcal{B},\mathcal{A},1}^{i_{\mathcal{B}}i_{\mathcal{A}}} W_{\mathcal{B},\mathcal{A},1}^{j_{\mathcal{B}}j_{\mathcal{A}}}. \end{aligned} \quad (21)$$

Here $W_{\mathcal{B},\mathcal{A},2}^{(i_{\mathcal{B}},j_{\mathcal{B}},1,1,1),(i_{\mathcal{A}},j_{\mathcal{A}},1,1,1)} \sqrt{Z_{\Sigma_2;\mathcal{A}}^{\text{top}} Z_{\Sigma_2;\mathcal{B}}^{\text{top}}}$ is the topological partition function on the spacetime obtained by gluing Figs. 7(a) and 7(b) together, and $Z^{\text{top}}[B^3(\mathcal{A}) \sqcup B^3(\mathcal{B})]$ is the topological partition function on the spacetime obtained by gluing Figs. 7(c) and 7(d) together (which gives rise to an S^3 occupied by phase \mathcal{A} and \mathcal{B}). Also, $W_{\mathcal{B},\mathcal{A},1}^{i_{\mathcal{B}}i_{\mathcal{A}}}$ is the topological partition function on the spacetime obtained by gluing

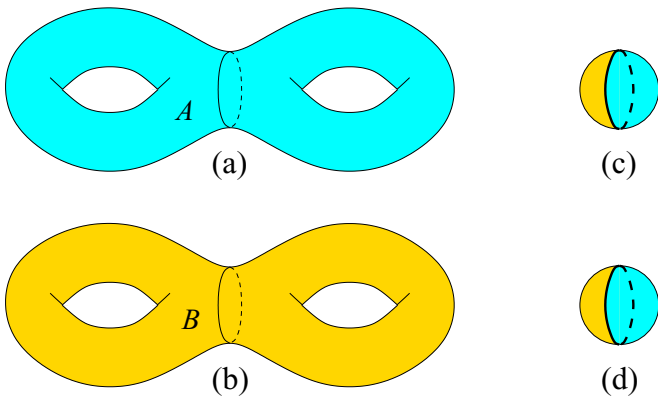


FIG. 7. (a) and (b) are two three-dimensional manifolds with topological orders \mathcal{A} and \mathcal{B} , whose surfaces are genus-2 Riemann surfaces. (c) and (d) are two three-dimensional balls.

Figs. 8(a) and 8(b) together. It is the topological partition function on S^3 where half of S^3 is occupied by phase \mathcal{A} , and the other half by phase \mathcal{B} . $W_{\mathcal{B},\mathcal{A},1}^{i_{\mathcal{B}}j_{\mathcal{A}}}$ is the topological partition function on the spacetime obtained by gluing Figs. 8(c) and 8(d) together.

The equality of (21) is obtained by noticing that the spacetime manifolds in Fig. 7 correspond to the spacetime manifolds in Fig. 8, after some cutting and gluing. To see this, we first cut Figs. 7(a) and 7(b) into left and right pieces. Then gluing Figs. 7(a)(left), 7(b)(left), and 7(c) together along their surfaces, we obtain the spacetime obtained by gluing Figs. 8(a) and 8(b) together along their surfaces. Gluing Figs. 7(a)(right), 7(b)(right), and 7(d) together, we obtain the spacetime obtained by gluing Figs. 8(c) and 8(d) together.

For convenience, we introduce a normalized wave function overlap via

$$\tilde{W}_{\mathcal{B},\mathcal{A},g}^{I_{\mathcal{B}}J_{\mathcal{A}}} \equiv N_{\mathcal{B},\mathcal{A},g} W_{\mathcal{B},\mathcal{A},g}^{I_{\mathcal{B}}J_{\mathcal{A}}}, \quad (22)$$

where we choose the normalization $N_{\mathcal{B},\mathcal{A},g}$ such that $\tilde{W}_{\mathcal{B},\mathcal{A},g}^{I_{\mathcal{B}}J_{\mathcal{A}}} = 1$ when $I_{\mathcal{B}}, J_{\mathcal{A}}$ correspond to trivial wordlines; i.e., for $g = 1, 2$ we have

$$\begin{aligned} \tilde{W}_{\mathcal{B},\mathcal{A},1}^{11} &= 1, \\ \tilde{W}_{\mathcal{B},\mathcal{A},2}^{(1,1,1,1,1),(1,1,1,1,1)} &= 1. \end{aligned} \quad (23)$$

Thus we have

$$\tilde{W}_{\mathcal{B},\mathcal{A},1}^{i_{\mathcal{B}}j_{\mathcal{A}}} = W_{\mathcal{B},\mathcal{A},1}^{i_{\mathcal{B}}j_{\mathcal{A}}} \quad (24)$$

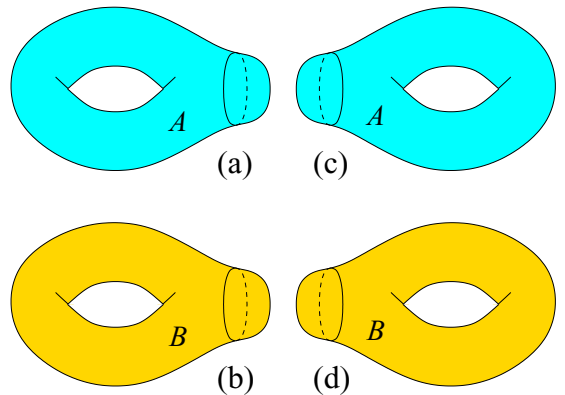


FIG. 8. (a), (b), (c), and (d) are four three-dimensional solid tori whose surfaces are genus-1 Riemann surfaces.

and

$$\begin{aligned} \tilde{W}_{\mathcal{B},\mathcal{A},2}^{(i_{\mathcal{B}},j_{\mathcal{B}},z_{\mathcal{B}},\mu_{\mathcal{B}},\nu_{\mathcal{B}}),(i_{\mathcal{A}},j_{\mathcal{A}},z_{\mathcal{A}},\mu_{\mathcal{A}},\nu_{\mathcal{A}})} \\ = \sqrt{Z_{\Sigma_2;\mathcal{A}}^{\text{top}} Z_{\Sigma_2;\mathcal{B}}^{\text{top}} Z^{\text{top}}[B^3(\mathcal{A}) \sqcup B^3(\mathcal{B})]} \\ \times W_{\mathcal{B},\mathcal{A},2}^{(i_{\mathcal{B}},j_{\mathcal{B}},z_{\mathcal{B}},\mu_{\mathcal{B}},\nu_{\mathcal{B}}),(i_{\mathcal{A}},j_{\mathcal{A}},z_{\mathcal{A}},\mu_{\mathcal{A}},\nu_{\mathcal{A}})} \end{aligned} \quad (25)$$

The normalized wave function overlaps for genus 1 and genus 2 have a simpler relation:

$$\tilde{W}_{\mathcal{B},\mathcal{A},2}^{(i_{\mathcal{B}},j_{\mathcal{B}},1,1,1),(i_{\mathcal{A}},j_{\mathcal{A}},1,1,1)} = \tilde{W}_{\mathcal{B},\mathcal{A},1}^{i_{\mathcal{B}},i_{\mathcal{A}}} \tilde{W}_{\mathcal{B},\mathcal{A},1}^{j_{\mathcal{B}},j_{\mathcal{A}}} \quad (26)$$

IV. THE WAVE FUNCTION OVERLAP IN A LATTICE HAMILTONIAN MODEL

In this section we try to compute the wave function overlap in a concrete lattice Hamiltonian model. Above, in the path integral formulation in Euclidean spacetime, we learned the lesson that a gapped domain wall W between phases \mathcal{A}, \mathcal{B} (a defect along a space direction), after a proper Wick rotation, becomes an operator e^{-H_W} (a defect along the time direction) that determines the weighted wave function overlap. However, in the Hamiltonian formulation there is no natural equivalence between space and time. To apply the previous results, we first try to analyze the physical picture in a lattice model.

Assume that there is a gapped domain wall at $x = 0$, phase \mathcal{A} in the region $x < 0$, and phase \mathcal{B} in the region $x > 0$. Clearly the Hamiltonian near $x = 0$ cannot be the same as those far from $x = 0$. For simplicity, we may assume that for some small positive number ε the Hamiltonian is uniform in the regions $x < -\varepsilon$ and $x > \varepsilon$. On the other hand, in the region $-\varepsilon < x < \varepsilon$, the Hamiltonian is uniform along y and t directions, but changes with x . The Wick-rotated version of this picture is that there are uniform phases \mathcal{A} at time $t < -\varepsilon$ and \mathcal{B} at time $t > \varepsilon$, and during time $-\varepsilon < t < \varepsilon$ the Hamiltonian evolves from that of \mathcal{A} to that of \mathcal{B} . Thus e^{-H_W} should correspond to the accumulated evolution operator during $-\varepsilon < t < \varepsilon$.

For the domain wall Hamiltonian in the region $-\varepsilon < x < \varepsilon$, the requirement is that it should keep the spectrum of the whole system gapped. However, we have no idea how this requirement is translated for e^{-H_W} . There is further another subtlety, that phases \mathcal{A}, \mathcal{B} may not be defined on the same microscopic lattice; in particular, any generalized local unitary transformations can be inserted to deform \mathcal{A} or \mathcal{B} . In the following, we try to propose some reasonable assumptions that allow us to calculate the quantity $W_{\mathcal{B},\mathcal{A},g}^{I_{\mathcal{B}},I_{\mathcal{A}}}$:

(1) First, to remove the ambiguity of generalized local unitary transformations, we assume that proper generalized local unitary transformations have been applied such that \mathcal{A}, \mathcal{B} are already on the same microscopic lattice and their Hamiltonians satisfy the following relation elaborated in (2).

(2) We focus on the gapped domain walls that are directly induced by anyon condensations. Suppose H^{ac} is the Hermitian operator that forces anyon condensation in \mathcal{A} . (If only Abelian anyons are condensed, H^{ac} is simply the sum of hopping string operators of the condensed anyons, that forces a total zero-momentum state of the condensed anyons. We will give more detail later in the toric code example.) We require

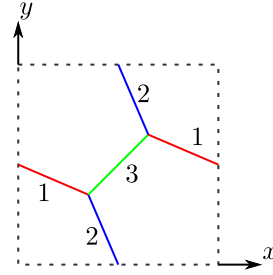


FIG. 9. The simplest triangulated torus with three links and two vertices.

the Hamiltonian $H_{\mathcal{B}}$ of phase \mathcal{B} to be

$$H_{\mathcal{B}} = H_{\mathcal{A}} + hH^{\text{ac}}, \quad h \rightarrow +\infty,$$

where $H_{\mathcal{A}}$ is the Hamiltonian of phase \mathcal{A} .

(3) Then we assume that e^{-H_W} acts on the ground state subspace of $H_{\mathcal{A}}$ (or $H_{\mathcal{B}}$) by multiplying a constant factor that may depend on the system size.

Now we look at the example of the toric code model [42], which realizes a Z_2 topological order [43,44]:

$$H_{\mathcal{A}} = - \sum_v \prod_{i \in \text{star}(v)} \sigma_i^z - \sum_p \prod_{i \in \partial p} \sigma_i^x. \quad (27)$$

The spins are on the links of the lattice. The first term sums over all vertices and $\text{star}(v)$ denotes the legs of the vertex v . The second term sums over all plaquettes and ∂p denotes the boundary edges of the plaquette p . There are three types of nontrivial anyons, e, m, f : e is created by string operators of the form $\prod \sigma_i^x$ along links, m is created by string operators of the form $\prod \sigma_i^z$ along links on the dual lattice, and f is the fusion of e and m . Both e and m can be condensed. The term that forces the condensation of e is $H_e^{\text{ac}} = - \sum_i \sigma_i^x$, and that for m is $H_m^{\text{ac}} = - \sum_i \sigma_i^z$. According to the above assumptions, we next calculate the overlap between the ground states of $H_{\mathcal{A}}$ and $H_{\mathcal{B}} = H_{\mathcal{A}} + hH_{e/m}^{\text{ac}}$. It is easy to see that, under the $h \rightarrow +\infty$ limit, the ground state of $H_{\mathcal{B}}$ is simply the spin polarized state $\otimes_i \frac{|0\rangle_i + |1\rangle_i}{\sqrt{2}} = \otimes_i |+\rangle_i$ or $\otimes_i |0\rangle_i$.

Now we write down the ground states of \mathcal{A} on a torus with the simplest lattice of only three links (see Fig. 9) According to the anyon flux along the y direction (measured by string operators winding around the x direction), the four ground states are

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |101\rangle), \\ |\psi_e\rangle &= \frac{1}{\sqrt{2}}(|110\rangle + |011\rangle), \\ |\psi_m\rangle &= \frac{1}{\sqrt{2}}(|000\rangle - |101\rangle), \\ |\psi_f\rangle &= \frac{1}{\sqrt{2}}(|110\rangle - |011\rangle). \end{aligned} \quad (28)$$

The corresponding ground states of \mathcal{B} are

$$\begin{aligned} e \text{ condenses: } |\psi_e^{\text{ac}}\rangle &= |++++\rangle, \\ m \text{ condenses: } |\psi_m^{\text{ac}}\rangle &= |0000\rangle. \end{aligned} \quad (29)$$

We obtain

$$\begin{aligned}\langle \psi_e^{\text{ac}} | \psi_a \rangle &= \frac{1}{2}(1, 1, 0, 0), \\ \langle \psi_m^{\text{ac}} | \psi_a \rangle &= \frac{1}{\sqrt{2}}(1, 0, 1, 0), \quad a = \mathbf{1}, e, m, f.\end{aligned}\quad (30)$$

which are indeed solutions to (13) and (14) after dropping the prefactors. One reason for the different prefactors of e, m condensations is that the number of excitations the lattice can host is different. e is hosted by vertices while m is hosted by plaquettes, and there are two vertices and only one plaquette in Fig. 9.

Let us generalize the above calculation to arbitrary system size and arbitrary lattice. We note that the four ground states on a torus for the toric code model H_A are sums of all closed strings formed by $|1\rangle$'s on the links, with fixed even-odd winding numbers in the two directions of the torus, which are given by $|ee\rangle, |eo\rangle, |oe\rangle$, and $|oo\rangle$. The four ground states in the quasiparticle basis are given by

$$\begin{aligned}|\psi_1\rangle &= \frac{1}{\sqrt{2}}(|ee\rangle + |oe\rangle), \\ |\psi_e\rangle &= \frac{1}{\sqrt{2}}(|eo\rangle + |oo\rangle), \\ |\psi_m\rangle &= \frac{1}{\sqrt{2}}(|ee\rangle - |oe\rangle), \\ |\psi_f\rangle &= \frac{1}{\sqrt{2}}(|eo\rangle - |oo\rangle).\end{aligned}\quad (31)$$

The ground states for the two possible \mathcal{B} phases are

$$|\psi_e^{\text{ac}}\rangle = \bigotimes_i |+\rangle_i, \quad |\psi_m^{\text{ac}}\rangle = \bigotimes_i |0\rangle_i. \quad (32)$$

To compute the overlap between $|\psi_e^{\text{ac}}\rangle$ with $|\psi_{1,e,m,f}\rangle$, let N_l be the number of links, N_v the number vertices, and N_p the number of plaquettes of the triangulated torus (see Fig. 9). On the torus whose genus $g = 1$, they satisfy

$$N_l = N_v + N_p. \quad (33)$$

We note that each of $|ee\rangle, |eo\rangle, |oe\rangle$, and $|oo\rangle$ is an equal-weight superposition of $2^{N_p-1} = 2^{N_l-N_v-1}$ closed string configurations.

Since $|\psi_m^{\text{ac}}\rangle$ corresponds to a single no-string configuration, it only overlaps with $|ee\rangle$ (which contains a no-string configuration). We find

$$\begin{aligned}\langle \psi_m^{\text{ac}} | \psi_1 \rangle &= \frac{1}{\sqrt{2}\sqrt{2^{N_p-1}}} = 2^{-\frac{N_p}{2}}, \\ \langle \psi_m^{\text{ac}} | \psi_e \rangle &= 0, \\ \langle \psi_m^{\text{ac}} | \psi_m \rangle &= \frac{1}{\sqrt{2}\sqrt{2^{N_p-1}}} = 2^{-\frac{N_p}{2}}, \\ \langle \psi_m^{\text{ac}} | \psi_f \rangle &= 0.\end{aligned}\quad (34)$$

After removing the area term $2^{-\frac{N_p}{2}}$, we see that the wave function overlaps $W_{\mathcal{B}\mathcal{A},1}^{\mathbf{1}a}$ are given by integers $W_{\mathcal{B}\mathcal{A},1}^{\mathbf{1}a} = (1, 0, 1, 0)$, $a = \mathbf{1}, e, m, f$, when the \mathcal{B} phase is given by m -particle condensation.

Also $|\psi_e^{\text{ac}}\rangle$ has the same overlap $2^{-N_l/2}$ with any string configuration. We find

$$\begin{aligned}\langle \psi_e^{\text{ac}} | \psi_1 \rangle &= \frac{2 \times 2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2}\sqrt{2^{N_p-1}}} = 2^{\frac{N_p-N_l}{2}} = 2^{-\frac{N_v}{2}}, \\ \langle \psi_e^{\text{ac}} | \psi_e \rangle &= \frac{2 \times 2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2}\sqrt{2^{N_p-1}}} = 2^{\frac{N_p-N_l}{2}} = 2^{-\frac{N_v}{2}}, \\ \langle \psi_e^{\text{ac}} | \psi_m \rangle &= \frac{2^{N_p-1} \times 2^{-N_l/2} - 2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2}\sqrt{2^{N_p-1}}} = 0, \\ \langle \psi_e^{\text{ac}} | \psi_f \rangle &= \frac{2^{N_p-1} \times 2^{-N_l/2} - 2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2}\sqrt{2^{N_p-1}}} = 0.\end{aligned}\quad (35)$$

After removing the area term $2^{-\frac{N_v}{2}}$, we see that the wave function overlaps $W_{\mathcal{B}\mathcal{A},1}^{\mathbf{1}a}$ are given by integers $W_{\mathcal{B}\mathcal{A},1}^{\mathbf{1}a} = (1, 1, 0, 0)$, $a = \mathbf{1}, e, m, f$, when the \mathcal{B} phase is given by e -particle condensation.

The above example supports our previous results that the area independent parts of wave function overlap on a torus are universal and are given by integers.

From a more experimental point of view, suppose there is a small physical system (say, one realized by a quantum simulator) where we can tune several parameters to force phase transitions, and the wave function overlaps can be measured (by interference for example). If *quantization* is observed in the wave function overlaps, it may relate to the universal integer part as above, and such quantization is a sign of topological order and anyon condensation.

V. GAPPED BOUNDARIES OF A 2 + 1D TOPOLOGICAL ORDER

We can use the results developed in this paper to study the gapped boundaries of 2 + 1D topological order, for example try to find out how many different gapped boundaries a topological order can have. In this section, we study some simple examples.

A. Boundary of Z_2 topological order

Let us choose phase \mathcal{A} to be of Z_2 topological order (denoted by \mathcal{Z}_2) and phase \mathcal{B} to be trivial. A domain wall between them is a boundary of Z_2 topological order. The modular matrices for the Z_2 topological order are given by

$$\begin{aligned}T_{\mathcal{Z}_2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ S_{\mathcal{Z}_2} &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\end{aligned}\quad (36)$$

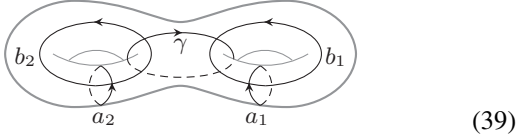
in the basis $(\mathbf{1}, e, m, f)$. The modular matrices for the trivial topological order are $S_B = 1$, $T_B = 1$. From modular transformation for normalized wave function overlap,

$$\tilde{W}_{\mathcal{Z}_2,g=1} = \tilde{W}_{\mathcal{Z}_2,g=1} S_{\mathcal{Z}_2}, \quad \tilde{W}_{\mathcal{Z}_2,g=1} = \tilde{W}_{\mathcal{Z}_2,g=1} T_{\mathcal{Z}_2}, \quad (37)$$

we find two types of gapped boundaries, characterized by two integer vector solutions of the above equation:

$$\begin{aligned}\tilde{W}_{\mathcal{Z}_2, g=1}^e &= (1, 1, 0, 0), \\ \tilde{W}_{\mathcal{Z}_2, g=1}^m &= (1, 0, 1, 0).\end{aligned}\quad (38)$$

Next, let us consider the case of a genus-2 manifold Σ_2 :



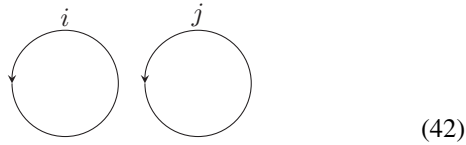
For $\text{MCG}(\Sigma_2)$, there are five generators which are Dehn twists along the closed curves a_1 , b_1 , a_2 , b_2 , and γ , as shown in (39). Denoting the projective representations of these five Dehn twists as T_{a_1} , T_{b_1} , T_{a_2} , T_{b_2} , and T_γ respectively, we can use T_{a_i} and T_{b_i} to construct an S_i matrix that acts on the left (right) half of Σ_2 , with $S_i = T_{b_i} T_{a_i}^{-1} T_{b_i} = T_{a_i}^{-1} T_{b_i} T_{a_i}^{-1}$. Then we have the following projective representations of the five generators of $\text{MCG}(\Sigma_2)$:

$$T_1, S_1, T_2, S_2, T_5, \quad (40)$$

where we have denoted $T_1 := T_{a_1}$, $T_2 := T_{a_2}$, and $T_5 := T_\gamma$. Then, based on Eq.(20), we have

$$\tilde{W}_{\mathcal{Z}_2, g=2} = \tilde{W}_{\mathcal{Z}_2, g=2} R_{\mathcal{Z}_2}, \quad (41)$$

where $R_{\mathcal{Z}_2} = S_1, T_1, S_2, T_2, T_5$. It is noted that for an Abelian theory, the basis in Fig. 3 can be represented as

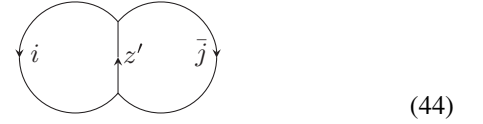


That is, the anyon z in Fig. 3 corresponds to the identity anyon **1** now. Then the two anyon loops in (42) are decoupled. With this basis, it is straightforward to check that for $R_{\mathcal{Z}_2} = S_1, T_1, S_2$, and T_2 the solutions to Eq. (41) are simply tensor products of genus-1 solutions [see Eq. (26)]:

$$\begin{aligned}\tilde{W}_{\mathcal{Z}_2, g=2}^{(1)} &= \tilde{W}_{\mathcal{Z}_2, g=1}^e \otimes \tilde{W}_{\mathcal{Z}_2, g=1}^e, \\ \tilde{W}_{\mathcal{Z}_2, g=2}^{(2)} &= \tilde{W}_{\mathcal{Z}_2, g=1}^e \otimes \tilde{W}_{\mathcal{Z}_2, g=1}^m, \\ \tilde{W}_{\mathcal{Z}_2, g=2}^{(3)} &= \tilde{W}_{\mathcal{Z}_2, g=1}^m \otimes \tilde{W}_{\mathcal{Z}_2, g=1}^e, \\ \tilde{W}_{\mathcal{Z}_2, g=2}^{(4)} &= \tilde{W}_{\mathcal{Z}_2, g=1}^m \otimes \tilde{W}_{\mathcal{Z}_2, g=1}^m,\end{aligned}\quad (43)$$

where $\tilde{W}_{\mathcal{Z}_2, g=1}^e$ and $\tilde{W}_{\mathcal{Z}_2, g=1}^m$ are the genus-1 results as expressed in Eq. (38). However, the solutions $\tilde{W}_{\mathcal{Z}_2, g=2}^{(2)}$ and $\tilde{W}_{\mathcal{Z}_2, g=2}^{(3)}$ in (43) are illegal, since both phase \mathcal{Z}_2 and phase \mathcal{I} are homogeneous here and we do not consider the case that e -condensation and m -condensation boundaries coexist. In the following, we show that $\tilde{W}_{\mathcal{Z}_2, g=2}^{(2)}$ and $\tilde{W}_{\mathcal{Z}_2, g=2}^{(3)}$ are ruled out by considering $R_{\mathcal{Z}_2} = T_5$ in Eq. (41).

For an Abelian theory, since the fusion result of $i \otimes \bar{j}$ is unique, the basis in (42) can be rewritten as



with $i \otimes \bar{j} = z'$. Then applying T_5 to the basis in (42) or equivalently (44) results in a phase $\theta_{z'} = \theta_z = e^{2\pi i s_{z'}}$ (see Fig. 16). In other words, the matrix T_5 in the basis (42) is a diagonal matrix with the diagonal elements being θ_z . For \mathcal{Z}_2 topological order, we have $\theta_1 = \theta_m = \theta_e = 1$ and $\theta_f = -1$. Given a vector $M_{\mathcal{Z}_2, g=2}$, one can find that if the vector element corresponding to $\theta_f = -1$ is nonzero, then it cannot be the solution of $\tilde{W}_{\mathcal{Z}_2, g=2} = M_{\mathcal{Z}_2, g=2} T_5$ in Eq. (41).

More explicitly, let us take the vector $M_{\mathcal{Z}_2, g=2}^{(2)}$ in Eq. (43) for example. Denoting the basis in (42) as

$$\begin{aligned}\{ij\} &= (\mathbf{1}, e, m, f) \otimes (\mathbf{1}, e, m, f) \\ &= (\mathbf{11}, \mathbf{1e}, \mathbf{1m}, \mathbf{1f}; e\mathbf{1}, ee, em, ef; \\ &\quad m\mathbf{1}, me, mm, mf; f\mathbf{1}, fe, fm, ff),\end{aligned}\quad (45)$$

then T_5 is a diagonal matrix of the form

$$\text{diag}(T_5) = (1, 1, 1, -1; 1, 1, -1, 1; 1, -1, 1, 1; -1, 1, 1, 1), \quad (46)$$

and

$$\tilde{W}_{\mathcal{Z}_2, g=2}^{(2)} = (1, 1, 0, 0; 0, 0, 0, 0; 1, 1, 0, 0; 0, 0, 0, 0). \quad (47)$$

One can check explicitly that $\tilde{W}_{\mathcal{Z}_2, g=2}^{(2)} \neq \tilde{W}_{\mathcal{Z}_2, g=2}^{(2)} T_5$. Similarly, one can find that $\tilde{W}_{\mathcal{Z}_2, g=2}^{(3)} \neq \tilde{W}_{\mathcal{Z}_2, g=2}^{(3)} T_5$, $\tilde{W}_{\mathcal{Z}_2, g=2}^{(1)} = \tilde{W}_{\mathcal{Z}_2, g=2}^{(1)} T_5$, and $\tilde{W}_{\mathcal{Z}_2, g=2}^{(4)} = \tilde{W}_{\mathcal{Z}_2, g=2}^{(4)} T_5$. That is, only $\tilde{W}_{\mathcal{Z}_2, g=2}^{(1)}$ and $\tilde{W}_{\mathcal{Z}_2, g=2}^{(4)}$ in Eq. (43) are the true solutions of Eq. (41) by considering all the five generators $R_{\mathcal{Z}_2} = T_1, S_1, T_2, S_2$, and T_5 .

B. Boundary of S_3 topological order

S_3 topological order (described by a quantum double of finite group S_3 with the fusion rule given by Table I) is more interesting, since it is a non-Abelian theory. Let us choose phase \mathcal{A} to be of S_3 topological order (denoted as S_3) and phase \mathcal{B} to be trivial (denoted as \mathcal{I}). The modular matrices for the S_3 topological order are given by [with basis $(\mathbf{1}, a^1, a^2, b, b^1, b^2, c, c^1)]$

$$\text{diag}(T) = (1, 1, 1, 1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, 1, -1), \quad (48)$$

$$S = \frac{1}{6} \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \end{pmatrix}. \quad (49)$$

TABLE I. Fusion rules N_c^{ab} of 2+1D S_3 topological order. Here b and c correspond to pure flux excitations, a^1 and a^2 pure charge excitations, $\mathbf{1}$ the vacuum sector, while b^1 , b^2 , and c^1 are charge-flux composites.

\otimes	$\mathbf{1}$	a^1	a^2	b	b^1	b^2	c	c^1
$\mathbf{1}$	$\mathbf{1}$	a^1	a^2	b	b^1	b^2	c	c^1
a^1	a^1	$\mathbf{1}$	a^2	b	b^1	b^2	c^1	c
a^2	a^2	a^2	$\mathbf{1} \oplus a^1 \oplus a^2$	$b^1 \oplus b^2$	$c \oplus b^2$	$b \oplus b^1$	$c \oplus c^1$	$c \oplus c^1$
b	b	b	$b^1 \oplus b^2$	$\mathbf{1} \oplus a^1 \oplus b$	$b^2 \oplus a^2$	$b^1 \oplus a^2$	$c \oplus c^1$	$c \oplus c^1$
b^1	b^1	b^1	$b \oplus b^2$	$b^2 \oplus a^2$	$\mathbf{1} \oplus a^1 \oplus b^1$	$b \oplus a^2$	$c \oplus c^1$	$c \oplus c^1$
b^2	b^2	b^2	$b \oplus b^1$	$b^1 \oplus a^2$	$b \oplus a^2$	$\mathbf{1} \oplus a^1 \oplus b^2$	$c \oplus c^1$	$c \oplus c^1$
c	c	c^1	$c \oplus c^1$	$c \oplus c^1$	$c \oplus c^1$	$c \oplus c^1$	$\mathbf{1} \oplus a^2 \oplus b \oplus b^1 \oplus b^2$	$a^1 \oplus a^2 \oplus b \oplus b^1 \oplus b^2$
c^1	c^1	c	$c \oplus c^1$	$c \oplus c^1$	$c \oplus c^1$	$c \oplus c^1$	$a^1 \oplus a^2 \oplus b \oplus b^1 \oplus b^2$	$\mathbf{1} \oplus a^2 \oplus b \oplus b^1 \oplus b^2$

Equation (13) has five solutions with $M_{S_3}^{\mathbf{11}} = 1$:

$$\begin{aligned}
\tilde{W}_{S_3, g=1}^{(1)} &= (1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0), \\
\tilde{W}_{S_3, g=1}^{(2)} &= (1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0), \\
\tilde{W}_{S_3, g=1}^{(3)} &= (1 \ 1 \ 0 \ 2 \ 0 \ 0 \ 0 \ 0), \\
\tilde{W}_{S_3, g=1}^{(4)} &= (1 \ 1 \ 2 \ 0 \ 0 \ 0 \ 0 \ 0), \\
\tilde{W}_{S_3, g=1}^{(5)} &= (1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0). \quad (50)
\end{aligned}$$

It is found that the last solution does not satisfy the stable condition in Eq. (14), i.e.,

$$(\tilde{W}_{S_3, g=1}^{(5)})^a (\tilde{W}_{S_3, g=1}^{(5)})^b \not\leq \sum_i N_i^{a^2, b} (\tilde{W}_{S_3, g=1}^{(5)})^i. \quad (51)$$

So the S_3 topological order has only four types of gapped boundaries.

In fact, as discussed in the following, without resorting to the stable condition in (14), we show that the fake solution $\tilde{W}_{S_3, g=1}^{(5)}$ can be ruled out by the genus-2 gapping boundary condition in Eq. (20), i.e.,

$$\tilde{W}_{S_3, g=2} = \tilde{W}_{S_3, g=2} R_{S_3}, \quad (52)$$

which is a linear condition. Since the S_3 topological order is multiplicity free (i.e., $N_k^{ij} \leq 1$), we denote the component of $\tilde{W}_{S_3, g=2}$ as $\tilde{W}_{S_3, g=2}^{i, j, z}$ [see also Eq. (19)]. Here the anyon types i, j, z are used to label the basis vectors in the Hilbert space of degenerate ground states on a genus-2 manifold Σ_2 (see Fig. 3). It is noted that both $\tilde{W}_{S_3, g=2}^{i, j, z}$ and the projective representations $R_{S_3}^U$ of $\text{MCG}(\Sigma_2)$ depend on the choice of basis vectors (see Appendix D). In the following discussion, we use the basis vectors in Fig. 3.

Similarly to the previous subsection on Z_2 topological order, we check whether the genus-1 solutions in Eq. (50) can be embedded in the genus-2 solutions of Eq. (52). Our logic is as follows: First, it is straightforward to find that $\tilde{W}_{S_3, g=1}^{(I)} \otimes \tilde{W}_{S_3, g=1}^{(J)}$ (where $I, J = 1, 2, 3, 4, 5$) exhaust all the solutions $\tilde{W}_{S_3, g=2}^{i, j, z}$ of Eq. (52) with $z = \mathbf{1}$ if we simply consider $R_{S_3} = T_1, S_1, T_2, S_2$. Second, we need to check if $\tilde{W}_{S_3, g=1}^{(I)} \otimes \tilde{W}_{S_3, g=1}^{(J)}$ are solutions of Eq. (52) for $R_{S_3} = T_5$. In general, the operation $R_{S_3} = T_5$ will mix the components $\tilde{W}_{S_3, g=2}^{i, j, z=\mathbf{1}}$ with $\tilde{W}_{S_3, g=2}^{i', j', z \neq \mathbf{1}}$, and it is not apparent that $\tilde{W}_{S_3, g=1}^{(I)} \otimes \tilde{W}_{S_3, g=1}^{(J)}$ are solutions of Eq. (52) for $R_{S_3} = T_5$. In this case, a

careful study of Eq. (52) is needed. Third, we need to solve for the components $\tilde{W}_{S_3, g=2}^{i, j, z}$ with $z \neq \mathbf{1}$ in Eq. (52).

In the following, we solve all the components $\tilde{W}_{S_3, g=2}^{i, j, z}$ that are solutions of Eq. (52). The results can be mainly summarized as follows:

(1) It is found there are in total four sets of independent solutions of the genus-2 condition in Eq. (52), which we denote as $\tilde{W}_{S_3, g=2}^{(i)}$, with $i = 1, 2, 3, 4$. They embed the genus-1 solutions of the form $\tilde{W}_{S_3, g=1}^{(i)} \otimes \tilde{W}_{S_3, g=1}^{(i)}$ with $i = 1, 2, 3, 4$. Other pairings of $\tilde{W}_{S_3, g=1}^{(i)} \otimes \tilde{W}_{S_3, g=1}^{(j)}$ (including $\tilde{W}_{S_3, g=1}^{(5)} \otimes \tilde{W}_{S_3, g=1}^{(5)}$) are ruled out by Eq. (52), as summarized in Table II.

(2) For the 4 sets of independent solutions $\tilde{W}_{S_3, g=2}^{(i)}$ with $i = 1, 2, 3, 4$, all the components can be *uniquely* determined as follows (see Appendix D for details):

(a) For $\tilde{W}_{S_3, g=2}^{(1)}$, which embeds the genus-1 solution of the form $\tilde{W}_{S_3, g=1}^{(1)} \otimes \tilde{W}_{S_3, g=1}^{(1)}$, the condensed anyons are pure flux $\mathbf{1}, b$, and c . With the genus-2 condition in Eq. (52), one can obtain the nonzero components of $\tilde{W}_{S_3, g=2}^{i, j, z}$ with $z \neq \mathbf{1}$ as

$$\begin{aligned}
\tilde{W}_{S_3, g=2}^{b, c, z=b} &= \tilde{W}_{S_3, g=2}^{c, b, z=b} = 1, \\
\tilde{W}_{S_3, g=2}^{c, c, z=b} &= \sqrt{2}, \\
\tilde{W}_{S_3, g=2}^{b, b, z=b} &= \frac{1}{\sqrt{2}}.
\end{aligned}$$

The nonzero components $\tilde{W}_{S_3, g=2}^{i, j, z}$ with $z = \mathbf{1}$ can be expressed as the product of genus-1 results as $\tilde{W}_{S_3, g=2}^{i, j, z=\mathbf{1}} = (\tilde{W}_{S_3, g=1}^{(1)})^i (\tilde{W}_{S_3, g=1}^{(1)})^j$, where $i, j \in \{\mathbf{1}, b, c\}$. For all the nonzero components $\tilde{W}_{S_3, g=2}^{i, j, z}$, one can find that $i, j, z \in \{\mathbf{1}, b, c\}$.

TABLE II. Only four pairings of genus-1 solutions are allowed by the genus-2 condition.

\otimes	$\tilde{W}_{S_3, g=1}^{(1)}$	$\tilde{W}_{S_3, g=1}^{(2)}$	$\tilde{W}_{S_3, g=1}^{(3)}$	$\tilde{W}_{S_3, g=1}^{(4)}$	$\tilde{W}_{S_3, g=1}^{(5)}$
$\tilde{W}_{S_3, g=1}^{(1)}$	✓	×	×	×	×
$\tilde{W}_{S_3, g=1}^{(2)}$		✓	×	×	×
$\tilde{W}_{S_3, g=1}^{(3)}$			✓	×	×
$\tilde{W}_{S_3, g=1}^{(4)}$				✓	×
$\tilde{W}_{S_3, g=1}^{(5)}$					×

(b) For $\tilde{W}_{S_3,g=2}^{(2)}$, which embeds the genus-1 solution of the form $\tilde{W}_{S_3,g=1}^{(2)} \otimes \tilde{W}_{S_3,g=1}^{(2)}$, the condensed anyons are $\mathbf{1}$, a^2 , and c . One can obtain the nonzero component of $\tilde{W}_{S_3,g=2}^{i,j,z}$ with $z \neq \mathbf{1}$ as

$$\begin{aligned}\tilde{W}_{S_3,g=2}^{a^2,c,z=a^2} &= \tilde{W}_{S_3,g=2}^{c,a^2,z=a^2} = -1, \\ \tilde{W}_{S_3,g=2}^{c,c,z=a^2} &= \sqrt{2}, \\ \tilde{W}_{S_3,g=2}^{a^2,a^2,z=a^2} &= \frac{1}{\sqrt{2}}.\end{aligned}$$

The nonzero components $\tilde{W}_{S_3,g=2}^{i,j,z}$ with $z = \mathbf{1}$ can be expressed as the product of genus-1 results as $\tilde{W}_{S_3,g=2}^{i,j,z=\mathbf{1}} = (\tilde{W}_{S_3,g=1}^{(2)})^i (\tilde{W}_{S_3,g=1}^{(2)})^j$, where $i, j \in \{\mathbf{1}, a^2, c\}$. Again, for all the nonzero components $\tilde{W}_{S_3,g=2}^{i,j,z}$, one has $i, j, z \in \{\mathbf{1}, a^2, c\}$.

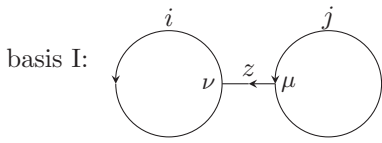
(c) For $\tilde{W}_{S_3,g=2}^{(3)}$, which embeds the genus-1 solution of the form $\tilde{W}_{S_3,g=1}^{(3)} \otimes \tilde{W}_{S_3,g=1}^{(3)}$, the condensed anyons are $\mathbf{1}$, a^1 , and b . As studied in Appendix D, all the components $\tilde{W}_{S_3,g=2}^{i,j,z}$ with $z \neq \mathbf{1}$ are zero. The only nonzero components can be considered as the product of genus-1 results as $\tilde{W}_{S_3,g=2}^{i,j,z=\mathbf{1}} = (\tilde{W}_{S_3,g=1}^{(3)})^i (\tilde{W}_{S_3,g=1}^{(3)})^j$, where $i, j \in \{\mathbf{1}, a^1, b\}$.

(d) For $\tilde{W}_{S_3,g=2}^{(4)}$, which embeds the genus-1 solution of the form $\tilde{W}_{S_3,g=1}^{(4)} \otimes \tilde{W}_{S_3,g=1}^{(4)}$, the condensed anyons are pure charges with $\mathbf{1}$, a^1 , and a^2 . Similar to the case of $\tilde{W}_{S_3,g=2}^{(3)}$, all the components $\tilde{W}_{S_3,g=2}^{i,j,z}$ with $z \neq \mathbf{1}$ are zero, and the only nonzero components correspond to the product of genus-1 results as $\tilde{W}_{S_3,g=2}^{i,j,z=\mathbf{1}} = (\tilde{W}_{S_3,g=1}^{(4)})^i (\tilde{W}_{S_3,g=1}^{(4)})^j$, where $i, j \in \{\mathbf{1}, a^1, a^2\}$.

1. Rule out the fake solution with genus-2 condition

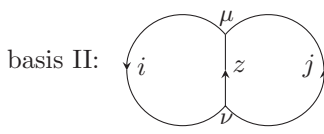
As an illustration of how to determine the solutions of the genus-2 condition in Eq. (52), here we give an example on how $\tilde{W}_{S_3,g=1}^{(5)} \otimes \tilde{W}_{S_3,g=1}^{(5)}$ is ruled out. The details of finding all the solutions of Eq. (52) can be found in Appendix D.

It is convenient to consider the following two choices of basis vectors:



(53)

and



(54)

The concrete expression of Eq. (52) will depend on the choices of basis vectors. For example, choosing $R = T_5$ in Eq. (52), T_5 is a diagonal matrix in the basis II, but in general is not diagonal in basis I. Denoting the wave function on a genus-2 manifold after anyon condensation as $|\Psi_{g=2}^{ac}\rangle$, then the wave function overlap $\tilde{W}_{S_3,g=2}^{i,j,z}$ in Eq. (52) depends on the

choice of bases as follows:

$$\tilde{W}_{S_3,g=2}^{I(II);i,j,z} := \langle \Psi_{g=2}^{ac} | \psi^{I(II);i,j,z} \rangle, \quad (55)$$

Suppose $\tilde{W}_{S_3,g=1}^{(5)} \otimes \tilde{W}_{S_3,g=1}^{(5)}$ is the solution of Eq. (52), then based on the expression in (50), we have $\tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} = 1$. In the following, we will show that the genus-2 condition imposes that $\tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} = 0$, and therefore $\tilde{W}_{S_3,g=1}^{(5)} \otimes \tilde{W}_{S_3,g=1}^{(5)}$ cannot be the solution of Eq. (52).

To study the effect of T_5 , it is convenient to consider the basis vectors II in (54). Let us focus on the components with $i = a^2$ and $j = b$. (Recall that T_5 does not change i and j .) Considering the fusion rule $a^2 \otimes b = b_1 \oplus b_2$, then in the basis vectors $|\psi^{II;a^2,b,z}\rangle$ the anyon type z can only be chosen as b_1 or b_2 , which have nontrivial topological spins [see (48)]. Then, considering $R_{S_3} = T_5$ in Eq. (52), we have

$$\tilde{W}_{S_3,g=2}^{II;a^2,b,z=b^1} = \tilde{W}_{S_3,g=2}^{II;a^2,b,z=b^2} = 0. \quad (56)$$

By inserting a complete set of basis vectors I in the expression (55), $\tilde{W}_{S_3,g=2}^{II;a^2,b,z}$ can be expressed as $\tilde{W}_{S_3,g=2}^{II;a^2,b,z=b^1} = \tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} \langle \psi^{I;a^2,b,\mathbf{1}} | \psi^{II;a^2,b,b^1} \rangle + \tilde{W}_{S_3,g=2}^{I;a^2,b,z=a^1} \langle \psi^{I;a^2,b,a^1} | \psi^{II;a^2,b,b^1} \rangle$, and $\tilde{W}_{S_3,g=2}^{II;a^2,b,z=b^2} = \tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} \langle \psi^{I;a^2,b,\mathbf{1}} | \psi^{II;a^2,b,b^2} \rangle + \tilde{W}_{S_3,g=2}^{I;a^2,b,z=a^1} \langle \psi^{I;a^2,b,a^1} | \psi^{II;a^2,b,b^2} \rangle$, where we have considered the fusion rules $a^2 \otimes a^2 = \mathbf{1} \oplus a^1 \oplus a^2$ and $b \otimes b = \mathbf{1} \oplus a^1 \oplus b$. In the Appendix [see Eq. (D20)], one can explicitly show that $\tilde{W}_{S_3,g=2}^{I;a^2,b,z=a^1} = 0$ by considering the genus-2 condition in Eq. (52) with $R_{S_3} = S_1$ or S_2 , which correspond to the punctured S matrix. Then one arrives at

$$\begin{aligned}\tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} \langle \psi^{I;a^2,b,\mathbf{1}} | \psi^{II;a^2,b,b^1} \rangle &= 0, \\ \tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} \langle \psi^{I;a^2,b,\mathbf{1}} | \psi^{II;a^2,b,b^2} \rangle &= 0.\end{aligned} \quad (57)$$

Furthermore, $\langle \psi^{I;a^2,b,\mathbf{1}} | \psi^{II;a^2,b,b^1} \rangle = \sqrt{d_{a^2} d_b d_{b^1}} / D$ and $\langle \psi^{I;a^2,b,\mathbf{1}} | \psi^{II;a^2,b,b^2} \rangle = \sqrt{d_{a^2} d_b d_{b^2}} / D$, which are nonzero. Then based on Eq. (57), one can immediately obtain

$$\tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} = 0. \quad (58)$$

On the other hand, we know that if $\tilde{W}_{S_3,g=1}^{(5)} \otimes \tilde{W}_{S_3,g=1}^{(5)}$ is the solution of Eq. (52), then we have $\tilde{W}_{S_3,g=2}^{I;a^2,b,z=\mathbf{1}} = 1$, which contradicts (58). Therefore, $\tilde{W}_{S_3,g=1}^{(5)} \otimes \tilde{W}_{S_3,g=1}^{(5)}$ is ruled out as the solution of the genus-2 condition in Eq. (52).

One can refer to Appendix D for all the solutions of Eq. (52) for S_3 topological order.

VI. SUMMARY

In this paper, we develop a systematic approach to the gapped domain walls between two topological orders \mathcal{A} and \mathcal{B} . (If \mathcal{B} is the trivial topological order, the domain wall becomes the boundary of topological order \mathcal{A} .) Our systematic approach is based on the topological partition function $W_{\mathcal{B},\mathcal{A},g}^{I_{\mathcal{B}}I_{\mathcal{A}}}$ of the domain wall Σ_g , which is a Riemann surface of arbitrary genus g , which is a multicomponent partition function labeled by $I_{\mathcal{A}}, I_{\mathcal{B}}$. The multicomponent partition function $W_{\mathcal{B},\mathcal{A},g}^{I_{\mathcal{B}}I_{\mathcal{A}}}$ is expected since the domain wall has a noninvertible gravitational anomaly [35] as characterized by topological orders \mathcal{A} and \mathcal{B} .

The topological partition function $W_{B,A,g}^{I_B I_A}$ can also be viewed as the overlap of the degenerate ground states of \mathcal{A} and \mathcal{B} on Σ_g , where I_A (I_B) labels the ground states of topological order \mathcal{A} on Σ_g . This allows us to derive the following linear conditions on the data $W_{B,A,g}^{I_B I_A}$ that characterized the domain walls:

$$\begin{aligned} R_B^{I_B, J_B} W_{B,A,g}^{J_B I_A} &= W_{B,A,g}^{I_B J_A} R_A^{J_A, I_A}, \\ W_{B,A,1}^{J_B I_A} &\in \mathbb{N} \quad (\text{in quasiparticle basis}), \end{aligned} \quad (59)$$

where R_A (R_B) is the mapping-class-group representation for topological order \mathcal{A} (topological order \mathcal{B}) for a genus- g Riemann surface. Equation (59) is a special case of a more general condition proposed in Ref. [35], for the partition function with noninvertible gravitational anomaly.

In this paper, through some simple examples, we try to demonstrate the validity of the condition (59) (and the condition in Ref. [35]), by showing that the condition gives rise to a classification of gapped domain walls between two topological orders. In particular, we show that the topological partition function $W_{B,A,1}^{I_B I_A}$ for a genus-1 surface, plus the linear condition (59), is not enough [34]. We need to, at least, use the topological partition function $W_{B,A,2}^{I_B I_A}$ for a genus-2 surface and its condition (59) to obtain a correct classification of gapped domain walls.

At moment, we do not know if the topological partition functions $W_{B,A,g}^{I_B I_A}$ for arbitrary genus- g surface can fully characterize the gapped domain wall or not (although we think that, very likely, they do). In Appendix E, using the connection between anyon condensation and domain walls, we develop a classifying theory of a gapped domain wall based on the structure coefficients $M_{ia,jb}^{kc,u}$ that describe the condensable algebra in a topological order. We also give the conditions (E21), (E26), and (E28) on the structure coefficients $M_{ia,jb}^{kc,u}$, so that they can describe a gapped domain wall. However, those conditions are nonlinear and very hard to solve.

We see that we have two systematic ways to describe the gapped domain walls between two topological orders. The first approach is based on topological partition functions $W_{B,A,g}^{I_B I_A}$, which are easier to solve, but not known to fully classify the domain walls. The second approach is based on the structure coefficients $M_{ia,jb}^{kc,u}$, which classify the gapped domain walls, but are hard to solve. Trying to gain a deeper understanding of the two approaches may help us to find an easier way to fully classify gapped domain walls.

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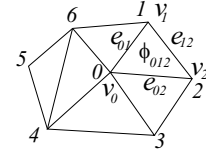


FIG. 10. A two-dimensional complex. The vertices (0-simplices) are labeled by i . The edges (1-simplices) are labeled by $\langle ij \rangle$. The faces (2-simplices) are labeled by $\langle ijk \rangle$. The degrees of freedoms may live on the vertices (labeled by v_i), on the edges (labeled by e_{ij}), and on the faces (labeled by ϕ_{ijk}).

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APPENDIX A: TOPOLOGICAL PATH INTEGRAL ON A SPACETIME WITH DOMAIN WALLS AND WORLDLINES

1. Spacetime lattice and branching structure

To find the conditions on the domain-wall data, we need to use extensively the spacetime path integral. So we will first describe how to define a spacetime path integral. We first triangulate the three-dimensional spacetime to obtain a simplicial complex M^3 (see Fig. 10). Here we assume that all simplicial complexes are of bounded geometry in the sense that the number of edges that connect to one vertex is bounded by a fixed value. Also the number of triangles that connect to one edge is bounded by a fixed value, etc.

In order to define a generic lattice theory on the spacetime complex M^3 , it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure [45–47]. A branching structure is a choice of orientation of each edge in the n -dimensional complex so that there is no oriented loop on any triangle (see Fig. 11).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 11(a) has the following vertex ordering: $0 < 1 < 2 < 3$.

The branching structure also gives the simplex (and its subsimplices) an orientation denoted by $s_{ij\dots k} = 1, *$. Figure 11 illustrates two 3-simplices with opposite orientations $s_{0123} = 1$ and $s_{0123} = *$. The red arrows indicate the orientations of the 2-simplices which are the subsimplices of the 3-simplices. The black arrows on the edges indicate the orientations of the 1-simplices.

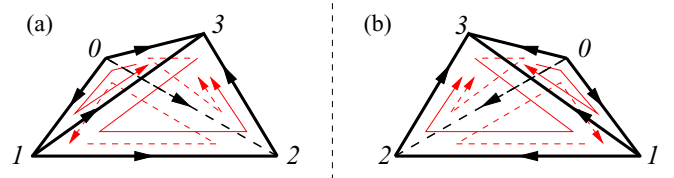


FIG. 11. Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

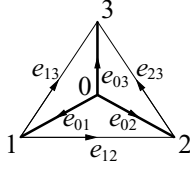


FIG. 12. The tensor $C_{v_0 v_1 v_2 v_3; \phi_{012} \phi_{023} \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}}$ is associated with a tetrahedron, which has a branching structure. If the vertex 0 is above the triangle 123, then the tetrahedron will have an orientation $s_{0123} = *$. If the vertex 0 is below the triangle 123, the tetrahedron will have an orientation $s_{0123} = 1$. The branching structure gives the vertices a local order: the i th vertex has i incoming edges.

The degrees of freedom of our lattice model live on the vertices (denoted by v_i where i labels the vertices), on the edges (denoted by e_{ij} where $\langle ij \rangle$ labels the edges), and on other high dimensional simplices of the spacetime complex (see Fig. 10).

2. Discrete path integral

In this paper, we will only consider a type of 2+1D path integral that can be constructed from a tensor set T of two real and one complex tensors: $T = (w_{v_0}, d_{e_{01}}^{v_0 v_1}, C_{v_0 v_1 v_2 v_3; \phi_{012} \phi_{023} \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}})$. The complex tensor $C_{v_0 v_1 v_2 v_3; \phi_{012} \phi_{023} \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}}$ can be associated with a tetrahedron, which has a branching structure (see Fig. 12). A branching structure is a choice of orientation of each edge in the complex so that there is no oriented loop on any triangle (see Fig. 12). Here the v_0 index is associated with the vertex 0, the e_{01} index is associated with the edge 01, and the ϕ_{012} index is associated with the triangle 012. They represent the degrees of freedom on the vertices, edges, and triangles.

Using the tensors, we can define the path integral on any 3-complex that has no boundary:

$$Z(M^3) = \sum_{v_0, \dots, e_{01}, \dots, \phi_{012}, \dots} \prod_{\text{vertex}} w_{v_0} \prod_{\text{edge}} d_{e_{01}}^{v_0 v_1} \times \prod_{\text{tetra}} [C_{v_0 v_1 v_2 v_3; \phi_{012} \phi_{023} \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}}]^{s_{0123}}, \quad (\text{A1})$$

where $\sum_{v_0, \dots, e_{01}, \dots, \phi_{012}, \dots}$ sums over all the vertex indices, the edge indices, and the face indices; $s_{0123} = 1$ or $*$ depending on the orientation of the tetrahedron (see Fig. 12). We believe such type of path integral can realize any 2+1D topological order.

3. Path integral on spacetime with natural boundary

On the complex M^3 with boundary $B^2 = \partial M^3$, the partition function is defined differently:

$$Z(M^3) = \sum_{\{v_i; e_{ij}; \phi_{ijk}\}} \prod_{\text{vertex} \notin B^2} w_{v_0} \prod_{\text{edge} \notin B^2} d_{e_{01}}^{v_0 v_1} \times \prod_{\text{tetra}} [C_{v_0 v_1 v_2 v_3; \phi_{012} \phi_{023} \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}}]^{s_{0123}}, \quad (\text{A2})$$

where $\sum_{\{v_i; e_{ij}; \phi_{ijk}\}}$ only sums over the vertex indices, the edge indices, and the face indices that are not on the boundary. The resulting $Z(M^3)$ is actually a complex function of v_i 's, e_{ij} 's,

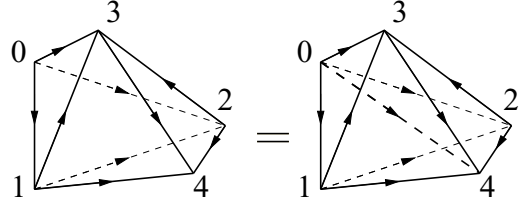


FIG. 13. A retriangulation of a 3D complex.

and ϕ_{ijk} 's on the boundary B^2 : $Z(M^3; \{v_i; e_{ij}; \phi_{ijk}\})$. Such a function is a vector in the vector space \mathcal{V}_{B^2} . [The vector space \mathcal{V}_{B^2} is the space of all complex functions of the boundary indices on the boundary complex B^2 : $\Psi(\{v_i; e_{ij}; \phi_{ijk}\})$.] We will denote such a vector as $|\Psi(M^3)\rangle$. The boundary (A2) defined above is called a natural boundary of the path integral.

We also note that only the vertices and the edges in the bulk (i.e., not on the boundary) are attached with the tensors w_{v_i} and $d_{e_{ij}}^{v_i v_j}$. But when we glue two boundaries together, those tensors w_{v_i} and $d_{e_{ij}}^{v_i v_j}$ are added back. For example, let M^3 and N^3 have the same boundary (with opposite orientations)

$$\partial M^3 = -\partial N^3 = B^2, \quad (\text{A3})$$

which gives rise to wave functions on the boundary $|\Psi(M^3)\rangle$ and $\langle \Psi(N^3)|$ after the path integral in the bulk. Gluing two boundaries together is like doing the inner product $\langle \Psi(N^3) | \Psi(M^3) \rangle$. So the tensors w_{v_i} and $d_{e_{ij}}^{v_i v_j}$ define the inner product in the boundary Hilbert space \mathcal{V}_{B^2} . Therefore, we require w_{v_i} and $d_{e_{ij}}^{v_i v_j}$ to satisfy the unitary condition

$$w_{v_i} > 0, \quad d_{e_{ij}}^{v_i v_j} > 0. \quad (\text{A4})$$

4. Topological path integral

We notice that the above path integral is defined for any spacetime lattice. The partition function $Z(M^3)$ depends on the choices of the spacetime lattice. For example, $Z(M^3)$ depends on the number of cells in the spacetime, which give rise to the leading volume dependent term, in the large spacetime limit (i.e., the thermodynamic limit),

$$Z(M^3) = e^{-\epsilon V} Z^{\text{top}}(M^3), \quad (\text{A5})$$

where V is the spacetime volume, ϵ is the energy density of the ground state, and $Z^{\text{top}}(M^3)$ is the volume independent partition function. It was conjectured that the volume independent partition function $Z^{\text{top}}(M^3)$ in the thermodynamic limit, as a function of closed spacetime M^3 , is a topological invariant that can fully characterize topological order [9,40]. So it is

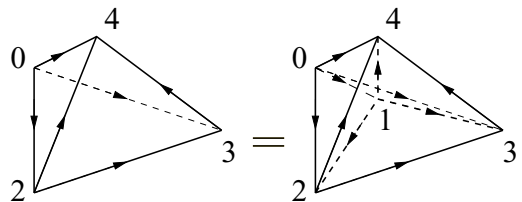


FIG. 14. A retriangulation of another 3D complex.

very desirable to fine tune the path integral to make the energy density $\epsilon = 0$. This can be achieved by fine tuning the tensors w_{v_i} and $d_{e_{ij}}^{v_i v_j}$. But we can do better. We can also choose the tensor $(w_{v_0}, d_{e_{01}}^{v_0 v_1}, C_{v_0 v_1 v_2 v_3; \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}; \phi_{012} \phi_{023}})$ to be the fixed-point tensor set under the renormalization group flow of the tensor network [48,49]. In this case, not only does the volume factor $e^{-\epsilon V}$ disappear, but also the volume independent partition function $Z^{\text{top}}(M^3)$ is retriangulation invariant, for any size of spacetime lattice. In this case, we refer to the path integral as

a topological path integral, and denote the resulting partition function as $Z^{\text{top}}(M^3)$. Z^{top} is also referred to as the volume independent the partition function, which is a very important concept, since only volume independent partition functions correspond to topological invariants. In particular, it was conjectured that such kind of topological path integrals describe all the topological order with gappable boundary. For details, see Refs. [9,40].

The invariance of partition function Z under the re-triangulation in Figs. 13 and 14 requires that

$$\sum_{\phi_{123}} C_{v_0 v_1 v_2 v_3; \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}; \phi_{012} \phi_{023}} C_{v_1 v_2 v_3 v_4; \phi_{124} \phi_{234}}^{e_{12} e_{13} e_{14} e_{23} e_{24} e_{34}; \phi_{123} \phi_{134}} = \sum_{e_{04}} d_{e_{04}}^{v_0 v_4} \sum_{\phi_{014} \phi_{024} \phi_{034}} C_{v_0 v_1 v_2 v_4; \phi_{014} \phi_{124}}^{e_{01} e_{02} e_{04} e_{12} e_{14} e_{24}; \phi_{012} \phi_{024}} \times C_{v_0 v_1 v_3 v_4; \phi_{014} \phi_{134}}^{e_{01} e_{03} e_{04} e_{13} e_{14} e_{34}; \phi_{013} \phi_{034}} C_{v_0 v_2 v_3 v_4; \phi_{024} \phi_{234}}^{e_{02} e_{03} e_{04} e_{23} e_{24} e_{34}; \phi_{023} \phi_{034}}. \quad (\text{A6})$$

$$C_{v_0 v_2 v_3 v_4; \phi_{024} \phi_{234}}^{e_{02} e_{03} e_{04} e_{23} e_{24} e_{34}; \phi_{023} \phi_{034}} = \sum_{e_{01} e_{12} e_{13} e_{14}, v_1} w_{v_1} d_{e_{01}}^{v_0 v_1} d_{e_{12}}^{v_1 v_2} d_{e_{13}}^{v_1 v_3} d_{e_{14}}^{v_1 v_4} \sum_{\phi_{012} \phi_{013} \phi_{014} \phi_{123} \phi_{124} \phi_{134}} C_{v_0 v_1 v_2 v_3; \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}; \phi_{012} \phi_{023}} C_{v_0 v_1 v_2 v_4; \phi_{014} \phi_{124}}^{e_{01} e_{02} e_{04} e_{12} e_{14} e_{24}; \phi_{012} \phi_{024}} \times C_{v_0 v_1 v_3 v_4; \phi_{014} \phi_{134}}^{e_{01} e_{03} e_{04} e_{13} e_{14} e_{34}; \phi_{013} \phi_{034}} C_{v_1 v_2 v_3 v_4; \phi_{124} \phi_{234}}^{e_{12} e_{13} e_{14} e_{23} e_{24} e_{34}; \phi_{123} \phi_{134}}. \quad (\text{A7})$$

We would like to mention that there are other similar conditions for different choices of the branching structures. The branching structure of a tetrahedron affects the labeling of the vertices. For more details, see Ref. [50].

5. Topological path integral with domain walls and worldlines

When the spacetime M^3 has a domain wall in it, we can have different tensor sets, $(w_{A, v_0}, d_{A, e_{01}}^{v_0 v_1}, C_{A, v_0 v_1 v_2 v_3; \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}; \phi_{012} \phi_{023}})$ and $(w_{B, v_0}, d_{B, e_{01}}^{v_0 v_1}, C_{B, v_0 v_1 v_2 v_3; \phi_{013} \phi_{123}}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}; \phi_{012} \phi_{023}})$, on the two sides of the domain wall. Here we will assume that the two tensor sets define topological path integrals in the bulk. The domain wall is defined via a different tensor set for the simplexes that touch the domain wall. Again we can choose the domain wall tensors to make the partition function with domain wall (after summing over the bulk and domain wall degrees of freedom) be retriangulation invariant (even for the retriangulations that involve the domain wall). Therefore, the topological path integral can also be defined for spacetime with domain walls. Different choices of domain wall tensors give rise to different domain walls. Those domain walls can be characterized by the data introduced in Sec II.

To find the conditions on those domain-wall data, we also need to use the spacetime path integral with worldlines of topological excitations. We denote the resulting partition function as

$$Z \left(\begin{array}{c} l \\ \curvearrowright \\ m \\ \curvearrowright \\ i \end{array} \begin{array}{c} n \\ \curvearrowright \\ j \\ \curvearrowright \\ k \end{array} \begin{array}{c} s \\ \curvearrowright \\ \alpha \end{array} \right), \quad (\text{A8})$$

where $i, j, k, \dots \in \{1, 2, \dots, N\}$ label the type of topological excitations, and α, β, γ label the fusion channels (i.e., different choice of actions at the junction of three worldlines). Again, the worldline is defined via a different tensor set for the simplexes that touch the worldline. We can choose the worldline tensors to make the partition function with the worldline to be retriangulation invariant (even for the retriangulations that involve the worldline). Therefore, the topological path integral can also be defined for a spacetime with worldlines. Different choices of worldline tensors give rise to different worldlines, which are labeled by the types of topological excitations. In this paper, we will only consider those topological path integrals with retriangulation invariance.

APPENDIX B: CATEGORICAL APPROACH TO EVALUATE THE TOPOLOGICAL PATH INTEGRAL WITH WORLDLINES

1. Planar worldlines and unitary m -fusion category

The topological path integrals with worldlines (i.e., the retriangulation invariant path integrals with worldlines) Z^{top} can be computed via an algebraic approach (or more precisely a categorical approach). In the following, we will give a brief introduction of such an approach. More details can be found in Ref. [51].

First, let us consider the partition functions with only planar worldline configurations. In this case, we may pretend the spacetime to be two-dimensional (or the spacetime is actually two-dimensional).

The partition functions with different worldline configurations can be related by some linear relations. For example,

$$Z^{\text{top}} \left(\begin{array}{c} i \quad j \quad k \\ \alpha \quad \beta \\ m \quad n \\ l \end{array} \right) = \sum_{n\chi\delta} F_{kln,\chi\delta}^{ijm,\alpha\beta} Z^{\text{top}} \left(\begin{array}{c} i \quad j \quad k \\ \alpha \quad \beta \\ \delta \quad n \\ l \end{array} \right). \quad (\text{B1})$$

This is because the above partition functions describe the amplitude of fusion type- i, j, k topological excitations into degenerate type- l topological excitations. The two sides of the equation just correspond to different orders of fusion which give rise to the same end product. Thus those amplitudes are related.

Let us consider the fusion of two topological excitations of types i, j . From far away, the two topological excitations may be viewed as single topological excitation. But such a single topological excitation may correspond to several different topological excitations which happen to have the same energy. For example, two spin-1/2 excitations can be viewed as spin-0 and spin-1 excitations that happen to have the same energy. So to describe the fusion of i and j , we need to introduce $N_k^{ij} \in \mathbb{N}$, which count the number of the type- k topological excitations which happen to have the same energy that appear in the fusion of type- i, j topological excitations. Note that N_k^{ij} may not be equal to N_k^{ji} .

Therefore, we have

$$F_{kln,\chi\delta}^{ijm,\alpha\beta} = 0 \quad \text{when} \\ N_m^{ij} = 0 \text{ or } N_l^{mk} = 0 \text{ or } N_n^{jk} = 0 \text{ or } N_l^{in} = 0. \quad (\text{B2})$$

When $N_m^{ij} = 0$ or $N_l^{mk} = 0$, the left-hand side of Eq. (B1) is always zero. Thus $F_{kln,\chi\delta}^{ijm,\alpha\beta} = 0$ when $N_m^{ij} = 0$ or $N_l^{mk} = 0$. When $N_n^{jk} = 0$ or $N_l^{in} = 0$, the amplitude on the right-hand side of Eq. (B1) is always zero. So we can choose $F_{kln,\chi\delta}^{ijm,\alpha\beta} = 0$ when $N_n^{jk} = 0$ or $N_l^{in} = 0$.

For fixed i, j, k , and l , the fusion type- i, j, k topological excitations produce N_l^{ijk} type- l topological excitations that happen to have the same energy. We have

$$N_l^{ijk} = \sum_m N_m^{ij} N_l^{mk} = \sum_n N_l^{in} N_n^{jk}. \quad (\text{B3})$$

Thus the matrix F_{kl}^{ij} with matrix elements $(F_{kl}^{ij})_{n,\chi\delta}^{m,\alpha\beta} = F_{kln,\chi\delta}^{ijm,\alpha\beta}$ is a matrix of dimension $N_l^{ijk} \times N_l^{ijk}$. The matrix describes the relation between two different orders of fusion of the same set of degenerate l particles. Thus the matrix F_{kl}^{ij} must be unitary:

$$\sum_{n\chi\delta} F_{kln,\chi\delta}^{ijm',\alpha'\beta'} (F_{kln,\chi\delta}^{ijm,\alpha\beta})^* = \delta_{m,m'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \quad (\text{B4})$$

The second type of linear relations reexpress the amplitude

for $\begin{array}{c} i \\ \alpha \\ \beta \\ i' \end{array}$ in terms of the amplitude for $\begin{array}{c} i \\ \beta \\ i' \end{array}$:

$$Z^{\text{top}} \left(\begin{array}{c} i \\ \alpha \\ \beta \\ i' \end{array} \right) = O_i^{jk,\alpha\beta} \delta_{ii'} Z^{\text{top}} \left(\begin{array}{c} i \\ \beta \\ i' \end{array} \right). \quad (\text{B5})$$

We call such local change of graph an O -move.

For fixed i, j , $\begin{array}{c} i \quad j \\ \alpha \quad \beta \\ k \end{array}$ for different k, α, β describe all the possible processes. So the amplitude for $\begin{array}{c} i \\ \beta \\ i' \end{array}$ amplitudes of $\begin{array}{c} i \quad j \\ \alpha \quad \beta \\ k \end{array}$:

$$\sum_{k,\alpha\beta} Y_{k,\alpha\beta}^{ij} Z^{\text{top}} \left(\begin{array}{c} i \quad j \\ \alpha \quad \beta \\ k \end{array} \right) = Z^{\text{top}} \left(\begin{array}{c} i \\ \beta \\ i' \end{array} \right) \quad (\text{B6})$$

We will call such a local change a Y -move. We can choose

$$Y_{k,\alpha\beta}^{ij} = 0 \quad \text{if } N_k^{ij} < 1. \quad (\text{B7})$$

We can adjust the action at the triple worldline junction to simplify $O_i^{jk,\alpha\beta}$ and $Y_{k,\alpha\beta}^{ij}$. After the simplification,

$$O_k^{ij,\alpha\beta} = \sqrt{\frac{d_i d_j}{d_k}} \delta_k^{ij} \delta_{\alpha\beta}, \\ Y_{k,\alpha\beta}^{ij} = \sqrt{\frac{d_k}{d_i d_j}} \delta_k^{ij} \delta_{\alpha\beta}, \quad d_i > 0, \quad (\text{B8})$$

where $\delta_i^{jk} = 1$ for $N_i^{jk} > 0$ and $\delta_i^{jk} = 0$ for $N_i^{jk} = 0$. Also d_i are the real and positive solutions from

$$\sum_{ij} d_i d_j N_k^{ij} = d_k D^2, \quad D \equiv \sqrt{\sum_l d_l^2}, \quad (\text{B9})$$

which are called the quantum dimensions of type- k topological excitation.

We see that the partition function $A(X)$ for any world-line configurations can be characterized by tensor data $(N, N_k^{ij}, F_{kln,\gamma\lambda}^{ijm,\alpha\beta})$. However, only certain tensor data $(N, N_k^{ij}, F_{kln,\gamma\lambda}^{ijm,\alpha\beta})$ that satisfy some conditions can self-consistently describe partition function $A(X)$. Those conditions form a set of nonlinear equations whose variables are $N_k^{ij}, F_{kln,\gamma\lambda}^{ijm,\alpha\beta}, d_i$ (where d_i can be determined by N_k^{ij} alone):

$$\sum_{m=0}^N N_m^{ij} N_l^{mk} = \sum_{n=0}^N N_n^{jk} N_l^{in} \sum_{jk} (N_i^{jk})^2 \geq 1; \quad (\text{B10})$$

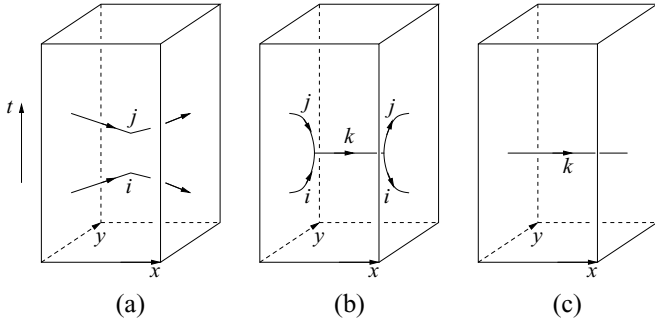


FIG. 15. (a) Two worldlines created by operators W_i^x and W_j^x . (b) The worldlines can be deformed according to the Y -move (c): The O -move can reduce (b) to (c).

$$\sum_{n\chi\delta} F_{kln,\chi\delta}^{ijm',\alpha'\beta'} (F_{kln,\chi\delta}^{ijm,\alpha\beta})^* = \delta_{m,m'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'},$$

$$F_{kln,\chi\delta}^{ijm,\alpha\beta} = 0 \quad \text{when}$$

$$N_m^{ij} < 1 \text{ or } N_l^{mk} < 1 \text{ or } N_n^{jk} < 1 \text{ or } N_l^{in} < 1, \quad (\text{B11})$$

$$\sum_t \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} \sum_{\kappa=1}^{N_s^{tl}} F_{knt,\eta\varphi}^{ijm,\alpha\beta} F_{lps,\kappa\gamma}^{itn,\varphi\chi} F_{lsq,\delta\phi}^{jkt,\eta\kappa}$$

$$= \sum_{\epsilon=1}^{N_p^{mq}} F_{lpq,\delta\epsilon}^{mkn,\beta\chi} F_{qps,\phi\gamma}^{ijm,\alpha\epsilon},$$

$$\sum_{i,j} d_i d_j N_k^{ij} = d_k D^2, \quad D = \sqrt{\sum_l d_l^2}; \quad (\text{B12})$$

$$\sum_{n\chi\delta} d_n F_{jnl,\beta'\delta}^{km'i,\alpha'\chi} (F_{jnl,\beta\delta}^{kmi,\alpha\chi})^* = \frac{d_i d_l}{d_m} \delta_{mm'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}. \quad (\text{B13})$$

We mention that, in the above, we did not assume the existence of trivial particles, which fuse with other particles as an identity. We call such algebraic structure *unitary m -fusion category*.

As an application of the above algebraic structure—the unitary m -fusion category—let us consider two worldlines of type i and type j , wrapping around a torus $S_x^1 \times S_y^1$ in the x direction. Let W_i^x and W_j^x be the string operators that create the worldlines. Applying the Y -move and then the O -move, and using Eq. (B8) (see Fig. 15), we find that

$$W_i^x W_j^x = \sum_{k=1}^N \sum_{\alpha=1}^{N_k^{ij}} Y_{k,\alpha}^{ij} O_k^{ij,\alpha} W_k^x = \sum_k N_k^{ij} W_k^x. \quad (\text{B14})$$

We see that the algebra of the loop operator W_i forms a representation of fusion algebra $i \otimes j = \sum_k N_k^{ij} k$.

2. Presence of trivial particle and unitary fusion category

Now let us assume such a trivial particle type to exist, and denote it by $\mathbf{1}$, which satisfies the following fusion rule:

$$\mathbf{1} \otimes i = i \otimes \mathbf{1} = i. \quad (\text{B15})$$

Thus N_k^{ij} satisfies

$$N_j^{\mathbf{1}i} = N_j^{\mathbf{1}\mathbf{1}} = \delta_{ij}. \quad (\text{B16})$$

We also requires that for every i there exists a unique \bar{i} such that

$$\bar{\bar{i}} = i, \quad \bar{\mathbf{1}} = \mathbf{1}, \quad N_1^{ij} = \delta_{i\bar{j}} \quad (\text{B17})$$

We can represent a type- $\mathbf{1}$ string by a dashed line. By examining the O -move with $k = \mathbf{1}$,

$$Z^{\text{top}} \left(\begin{array}{c} i \\ \uparrow \\ \text{loop} \\ \downarrow \\ i \end{array} \right) = Z^{\text{top}} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right), \quad (\text{B18})$$

we see that we can remove or add any vertex with a dashed line without changing Z^{top} .

With the presence of the trivial particle type, we can determine the amplitude for a loop of $-$ string. Using the rule of adding dashed lines (the trivial strings) and O -move (B5), we find

$$Z^{\text{top}} \left(\begin{array}{c} \text{loop} \end{array} \right) = Z^{\text{top}} \left(\begin{array}{c} \text{loop} \\ \text{dashed line} \end{array} \right) = O_1^{\bar{i}} Z^{\text{top}} \left(\begin{array}{c} \text{loop} \end{array} \right)$$

$$= d_i Z^{\text{top}} \left(\begin{array}{c} \text{loop} \end{array} \right) \quad (\text{B19})$$

Thus a loop of type- i worldline has an amplitude d_i .

3. Nonplanar diagram and braided fusion category

We have been considering planar graphs and the related fusion category theory. In this section we will consider nonplanar graphs. Since the particles now live in two-dimensional space (or higher), the fusion of the particles satisfies

$$i \otimes j = j \otimes i, \quad (\text{B20})$$

and thus

$$N_k^{ij} = N_k^{ji}. \quad (\text{B21})$$

So the fusions of 2D particles are commutative (while the fusions of 1D particles may not be commutative).

Here, we also like point out that a worldline of a particle is always framed (i.e., having a shadow worldline running parallel to it). When we draw a graph on a plane, there is canonical framing, obtain by shifting the graphs perpendicular to the plane (see Fig. 16). We have been using such a canonical framing in our previous discussion, and we have omitted drawing the framing. But if we do not use this canonical

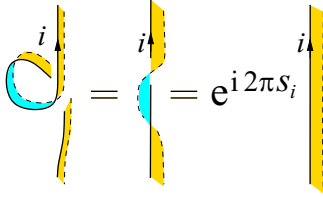


FIG. 16. A “self-loop” with canonical framing corresponds to a twist by 2π . A twist by 2π induces a phase $e^{i2\pi s_i}$ that defines the spin s_i of the particle.

framing, then we need to draw the framing explicitly, as in Fig. 16.

Let us consider simple string configuration with crossing: a “self-loop” with the canonical framing (see Fig. 16). Such a self-loop corresponds to a straight line with a 2π twist, which is equal to an untwisted straight line with a phase $e^{i2\pi s_i}$. Here s_i is the *spin* of the type- i topological excitation, which is defined mod 1. We also note that the handedness of the self-loop determines the direction of the twist (see Fig. 17). As a result, a figure 8 of type- i string has an amplitude $e^{2\pi i s_i} d_i$ (see Fig. 18).

APPENDIX C: THE GAPPED DOMAIN WALLS AS 1+1D ANOMALOUS TOPOLOGICAL ORDER

There is another ways to fully characterize the domain walls. We note that the gapped 1+1D domain walls can be viewed as 1+1D anomalous topological orders [9]. The 1+1D anomalous topological orders are characterized by unitary fusion categories (UFCs) which are described by the following data: (1) $N \in \mathbb{N}$, the number of types (including the trivial type) of topological excitations on the domain wall. We will use i, j, k , etc., to label the types of topological excitations and use 1 to label the trivial type. (2) $N_k^{ij} \in \mathbb{N}$, the fusion coefficients of the topological excitations. (3) $F_{kln, \gamma\lambda}^{ijm, \alpha\beta}$, the unitary relation between different fusion spaces obtained via different fusion paths.

Those data $(N, N_k^{ij}, F_{kln, \gamma\lambda}^{ijm, \alpha\beta})$ satisfy

$$\sum_{m=0}^N N_m^{ij} N_l^{mk} = \sum_{n=0}^N N_n^{jk} N_l^{in},$$

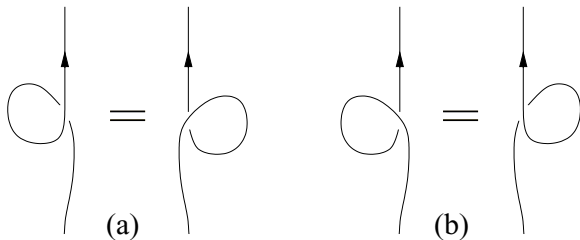


FIG. 17. The two “self-loops” in (a) are “right handed” and correspond to the same twist. The two “self-loops” in (b) are “left handed” and also correspond to the same twist that is opposite to that in (a).

$$\bigcirc_i = e^{i2\pi s_i} \bigcirc_i = e^{i2\pi s_i} d_i$$

FIG. 18. A figure 8 of type- i string has an amplitude $e^{2\pi i s_i} d_i$.

$$N_j^{i1} = N_j^{1i} = \delta_{ij}, \quad \sum_{k=1}^N N_1^{ik} N_1^{kj} = \delta_{ij}; \quad (C1)$$

$$F_{kln, \chi\delta}^{ijm, \alpha\beta} = 0 \quad \text{when}$$

$$N_m^{ij} < 1 \text{ or } N_l^{mk} < 1 \text{ or } N_n^{jk} < 1 \text{ or } N_l^{in} < 1,$$

$$\sum_{n\chi\delta} F_{kln, \chi\delta}^{ijm', \alpha'\beta'} (F_{kln, \chi\delta}^{ijm, \alpha\beta})^* = \delta_{m, m'} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'},$$

$$\sum_{n\chi\delta} d_n F_{jnl, \beta'\delta}^{km'i, \alpha'\chi} (F_{jnl, \beta\delta}^{kmi, \alpha\chi})^* = \frac{d_i d_l}{d_m} \delta_{mm'} \delta_{\alpha\alpha'} \delta_{\beta\beta'},$$

$$\sum_t \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} \sum_{\kappa=1}^{N_s^{tl}} F_{knt, \eta\varphi}^{ijm, \alpha\beta} F_{lps, \kappa\gamma}^{itn, \varphi\chi} F_{lsq, \delta\phi}^{jkt, \eta\kappa}$$

$$= \sum_{\epsilon=1}^{N_p^{mq}} F_{lpq, \delta\epsilon}^{mkn, \beta\chi} F_{qps, \phi\gamma}^{ijm, \alpha\epsilon}. \quad (C2)$$

Here d_i is the largest left eigenvalue of the matrix N_i [defined as $(N_i)_{kj} = N_k^{ij}$].

Fusion category $(N, N_k^{ij}, F_{kln, \gamma\lambda}^{ijm, \alpha\beta})$ and weighted wave function overlap $W_{\mathcal{B}, \mathcal{A}, g}^{I, g, L, A}$ provide two very different ways to characterize the same domain wall. It is amazing that the two sets of data should have a one-to-one correspondence.

APPENDIX D: MORE DETAILS ON THE GAPPED BOUNDARY OF S_3 TOPOLOGICAL ORDER

1. Some preliminaries

In this section, we give the details of solving the gapped boundaries, using the genus-2 condition in (52), for normalized wave function overlap $\tilde{W}_{S_3, g=2}$. For $g = 2$, it is convenient to consider the following two choices of basis vectors:

$$\text{basis I: } \begin{array}{c} \text{Diagram showing two circles, each with a vertical line passing through its center. The left circle is labeled 'i' and the right circle is labeled 'j'. The vertical lines are labeled 'ν' and 'μ' respectively. A horizontal line segment connects the two vertical lines, labeled 'z'. The diagram is labeled (D1). \end{array}$$

and

$$\text{basis II: } \begin{array}{c} \text{Diagram showing two circles, each with a vertical line passing through its center. The left circle is labeled 'i' and the right circle is labeled 'j'. The vertical lines are labeled 'μ' and 'ν' respectively. A horizontal line segment connects the two vertical lines, labeled 'z'. The diagram is labeled (D2). \end{array}$$

In particular, basis I will be useful in making a connection to the genus-1 solution by choosing $z = 1$, and basis II will be useful in studying the effect of Dehn twist operator T_5 (or T_γ) in Eq. (39). Within basis II, T_5 will be a diagonal matrix with the diagonal elements corresponding to the topological spin θ_z of anyon z .

Before any concrete calculation, it is noted that for the solutions of gapped boundaries, if $\tilde{W}_{S_3, g=2}^{1(i); i, j, z} \neq 0$, then the topological spins of i , j , and z in (D1) and (D2) must be trivial. This can be understood by considering $R_{S_3} = T_1, T_2, (S_1)^{-4}, (S_2)^{-4}$, and T_5 in Eq. (41).

We denote basis I and basis II in (D1) and (D2) as

$$|\psi^{I; i, j, z; \mu, \nu}\rangle \quad \text{and} \quad |\psi^{II; i, j, z; \mu, \nu}\rangle, \quad (\text{D3})$$

respectively. Since the S_3 topological order is multiplicity free, we can write the above basis vectors as

$$|\psi^{I; i, j, z}\rangle \quad \text{and} \quad |\psi^{II; i, j, z}\rangle. \quad (\text{D4})$$

These two basis vectors are normalized as follows:

$$\begin{aligned} \langle \psi^{I; i, j, z} | \psi^{I; i', j', z'} \rangle &= \delta_{i, i'} \delta_{j, j'} \delta_{z, z'}, \\ \langle \psi^{II; i, j, z} | \psi^{II; i', j', z'} \rangle &= \delta_{i, i'} \delta_{j, j'} \delta_{z, z'}. \end{aligned} \quad (\text{D5})$$

In addition, they are related to each other as,

$$|\psi^{I; i, j, z}\rangle = \sum_{z'} [F_{ij}^{i\bar{j}}]_{(z, z')} |\psi^{II; i, j, z'}\rangle, \quad (\text{D6})$$

where $[F_{ij}^{i\bar{j}}]_{(z, z')}$ is defined by

$$\begin{array}{c} a \quad b \\ \swarrow \quad \searrow \\ e \quad \beta \\ \swarrow \quad \searrow \\ c \quad d \end{array} = \sum_{f, \mu, \nu} [F_{cd}^{ab}]_{(e, \alpha, \beta), (f, \mu, \nu)} \begin{array}{c} a \quad b \\ \swarrow \quad \searrow \\ f \quad \mu \\ \swarrow \quad \searrow \\ c \quad \nu \quad d \end{array}. \quad (\text{D7})$$

Here $[F_{ij}^{i\bar{j}}]_{(z, z')}$ is related to the conventional F matrix as defined in (B1) through the following relation:

$$[F_{cd}^{ab}]_{(e, \alpha, \beta), (f, \mu, \nu)} = \sqrt{\frac{d_e d_f}{d_a d_d}} [F_{bf d, \mu \nu}^{cea, \alpha \beta}]^*. \quad (\text{D8})$$

It is convenient to study the effect of Dehn twist T_5 in basis II, which simply results in a phase factor, i.e.,

$$T_5 |\psi^{II; i, j, z}\rangle = \theta_z |\psi^{II; i, j, z}\rangle. \quad (\text{D9})$$

Within basis I, one has

$$T_5 |\psi^{I; i, j, z}\rangle = \sum_{z', z''} [F_{ij}^{i\bar{j}}]_{(z, z')} \theta_{z'} [F_{ij}^{i\bar{j}}]_{(z', z'')}^{-1} |\psi^{I; i, j, z''}\rangle. \quad (\text{D10})$$

Apparently, T_5 is in general not diagonal in basis I. However, if the theory is Abelian, then one has $z = \mathbf{1}$ and $[F_{ij}^{i\bar{j}}]_{\mathbf{1} z'} = N_{ij}^{z'}$. Then T_5 is a diagonal matrix with the matrix elements

$$\langle \psi^{I; i, j, z''} | T_5 | \psi^{I; i, j, z} \rangle = \delta_{1, z} \delta_{1, z''} \theta_z N_{ij}^{z'}. \quad (\text{D11})$$

Another interesting case is that if $\theta_{z'} = 1$ for all $[F_{ij}^{i\bar{j}}]_{(z, z')} \neq 0$, then based on Eq. (D10) one has

$$T_5 |\psi^{I; i, j, z}\rangle = |\psi^{I; i, j, z}\rangle \quad \text{if } \theta_{z'} = 1 \text{ for all } [F_{ij}^{i\bar{j}}]_{(z, z')} \neq 0. \quad (\text{D12})$$

For S_3 topological order, all the F matrices have been obtained in Ref. [52]. The so-called punctured S matrix

$$S_{a, \mu; b, \nu}^{(z)} = \frac{1}{D} \frac{1}{\sqrt{d_z}} \quad \begin{array}{c} \text{Diagram: Two circles. The left circle has a clockwise arrow labeled } \mu. \text{ The right circle has a clockwise arrow labeled } \nu. \text{ A line connects the two circles, labeled } a \text{ at the top and } b \text{ at the bottom.} \end{array} \quad (\text{D13})$$

can be expressed in terms of the F -matrix as (in the multiplicity-free case)

$$S_{a, b}^{(z)} = \frac{1}{D} \frac{1}{\sqrt{d_z}} \frac{d_a d_b}{\theta_a \theta_b} \sum_f F_{bbf}^{aaz} \theta_f [F_{bb1}^{aaf}]^*, \quad (\text{D14})$$

based on which we can obtain the punctured $S^{(z)}$ matrix. For our motivation of studying the gapped boundaries, we only need to consider $S^{(z)}$ with $\theta_z = 1$ in (D1). Based on Eq. (48) and the fusion rules of $i \otimes \bar{i}$ in Table I, we only need to check the cases of $z = a^1, a^2, b$.

The results of punctures $S^{(z)}$ and $T^{(z)}$ matrices with $z = a^1, a^2, b$ are summarized as follows:

(i) For $z = a^1$, one has (with the basis a^2, b, b^1 , and b^2)

$$S^{(z=a^1)} = \frac{1}{3}(\omega^2 - \omega) \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \quad (\text{D15})$$

and $T^{(z=a^1)} = \text{diag}(1, 1, \omega, \omega^2)$, where $\omega = e^{\frac{2\pi i}{3}}$.

(ii) For $z = a^2$, one has (with the basis c^1, c , and a^2)

$$S^{(z=a^2)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad (\text{D16})$$

and $T^{(z=a^2)} = \text{diag}(-1, 1, 1)$.

(iii) For $z = b$, one can obtain (in the basis c^1, c , and b):

$$S^{(z=b)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad (\text{D17})$$

and $T^{(z=b)} = \text{diag}(-1, 1, 1)$. For the punctured $S^{(z)}$ and $T^{(z)}$ presented above, one can check explicitly that they satisfied the so-called modular relation

$$(S^{(z)} T^{(z)})^3 = (S^{(z)})^2, \quad (S^{(z)})^4 = \theta_z^*. \quad (\text{D18})$$

Now we are ready to study the solutions of

$$W = WR, \quad (\text{D19})$$

with $R = S^{(z)}$ and $T^{(z)}$.

(iv) For $z = a^1$, since $T^{(z=a^1)}$ is diagonal, one can find that W has the form $W = (x, y, 0, 0)$. Then choosing $R = S^{(z=a^1)}$ in (D19), one can find that $x = y = 0$. That is, there is no nonzero solution of (D19) for $R = S^{(z=a^1)}$ and $T^{(z=a^1)}$. More explicitly, with the basis in (D1), we have

$$\tilde{W}_{S_3, g=2}^{I; i, j, z=a^1} = 0, \quad (\text{D20})$$

where $i, j \in \{a^2, b, b^1, b^2\}$.

TABLE III. Only four sets of solutions are allowed by the genus-2 condition. Related fusion rules are $a^1 \otimes c = c^1$ and $a^2 \otimes b = b^1 \oplus b^2$.

\otimes	$\tilde{W}_{S_3,g=1}^{(1)}$	$\tilde{W}_{S_3,g=1}^{(2)}$	$\tilde{W}_{S_3,g=1}^{(3)}$	$\tilde{W}_{S_3,g=1}^{(4)}$	$\tilde{W}_{S_3,g=1}^{(5)}$
$\tilde{W}_{S_3,g=1}^{(1)}$	✓	$a^2 \otimes b$	$a^1 \otimes c$	$a^1 \otimes c$ $a^2 \otimes b$	$a^1 \otimes c$ $a^2 \otimes b$
$\tilde{W}_{S_3,g=1}^{(2)}$		✓	$a^1 \otimes c$ $a^2 \otimes b$	$a^1 \otimes c$	$a^1 \otimes c$ $a^2 \otimes b$
$\tilde{W}_{S_3,g=1}^{(3)}$			✓	$a^2 \otimes b$	$a^2 \otimes b$
$\tilde{W}_{S_3,g=1}^{(4)}$				✓	$a^2 \otimes b$
$\tilde{W}_{S_3,g=1}^{(5)}$					$a^2 \otimes b$

(v) For $z = a^2$, since $T^{(z=a^2)} = \text{diag}(-1, 1, 1)$, we have $W = (0, x, y)$. Then by choosing $R = S^{(z=a^2)}$ in (D19), one can find that $x = -\sqrt{2}y$. That is,

$$W = (0, -\sqrt{2}y, y), \quad (\text{D21})$$

where y is to be determined by other conditions. With the basis vectors chosen in (D1), Eq. (D21) indicates that

$$\begin{aligned} \tilde{W}_{S_3,g=2}^{I;i,c,z=a^2} &= -\sqrt{2} \times \tilde{W}_{S_3,g=2}^{I;i,a^2,z=a^2}, \\ \tilde{W}_{S_3,g=2}^{I;c,i,z=a^2} &= -\sqrt{2} \times \tilde{W}_{S_3,g=2}^{I;a^2,i,z=a^2}, \end{aligned} \quad (\text{D22})$$

where $i \in \{a^2, c, c^1\}$.

(vi) For $z = b$, since $T^{(z=b)} = \text{diag}(-1, 1, 1)$, we have $W = (0, x, y)$. Then by choosing $R = S^{(z=b)}$ in (D19), one can find that $x = \sqrt{2}y$. That is,

$$W = (0, \sqrt{2}y, y), \quad (\text{D23})$$

where again y is to be determined by other conditions. With the basis vectors chosen in (D1), Eq. (D23) indicates that

$$\begin{aligned} \tilde{W}_{S_3,g=2}^{I;i,c,z=b} &= \sqrt{2} \times \tilde{W}_{S_3,g=2}^{I;i,b,z=b}, \\ \tilde{W}_{S_3,g=2}^{I;c,i,z=b} &= \sqrt{2} \times \tilde{W}_{S_3,g=2}^{I;b,i,z=b}, \end{aligned} \quad (\text{D24})$$

where $i \in \{b, c, c^1\}$.

2. Solutions of genus-2 condition

In this subsection, we solve the genus-2 condition in Eq. (52), and find there are in total four sets of independent solutions, which embed the genus-1 solutions $\tilde{W}_{S_3,g=1}^{(i)} \otimes \tilde{W}_{S_3,g=1}^{(i)}$ with $i = 1, 2, 3, 4$, respectively.

Before solving the genus-2 condition in Eq. (52), it is first noted that the ground state degeneracy of a topological order on genus-2 closed manifold is

$$\text{GSD}(\Sigma_2) = \sum_i \left(\frac{1}{S_{0i}} \right)^2, \quad (\text{D25})$$

where we sum over all the anyon types i . One can check that for the S_3 topological order we have $\text{GSD}(\Sigma_2) = 116$. To obtain the solutions of the genus-2 condition in Eq. (41) means we need to obtain the 116 components of $\tilde{W}_{S_3,g=2}^{I(II);i,j,z}$ if we choose the basis vectors I (II).

Let us start by considering different pairings of genus-1 solutions in Table III. First, it is noted that one common feature of the pairings $\tilde{W}_{S_3,g=1}^{(p)} \otimes \tilde{W}_{S_3,g=1}^{(q)}$ with $(p, q) = (1, 2), (1, 4), (1, 5), (2, 3), (2, 5), (3, 4), (3, 5), (4, 5)$, and $(5, 5)$ is that the component $\tilde{W}_{S_3,g=2}^{I;a^2,b,z=1}$ is nonzero. As illustrated in Sec. VB 1 in the main text, the genus-2 condition imposes that $\tilde{W}_{S_3,g=2}^{I;a^2,b,z=1} = 0$, and therefore the above pairings cannot be the solutions of Eq. (52). Second, the common feature of pairings $\tilde{W}_{S_3,g=1}^{(p)} \otimes \tilde{W}_{S_3,g=1}^{(q)}$ with $(p, q) = (1, 3), (1, 4), (1, 5), (2, 3), (2, 4)$, and $(2, 5)$ is that the component $\tilde{W}_{S_3,g=2}^{I;a^1,c,z=1}$ is nonzero. In the following, we will show that the genus-2 condition imposes that $\tilde{W}_{S_3,g=2}^{I;a^1,c,z=1} = 0$, and therefore the above pairings cannot be the solutions of Eq. (52).

Considering the fusion rule $a^2 \otimes c = c^1$, the only allowed component $|\psi^{II;a^2,c,z}\rangle$ in (D2) is for $z = c^1$. Since $\theta_{c^1} = -1$, we have

$$\langle \Psi_{g=2}^{ac} | \psi^{II;a^2,c,z=c^1} \rangle = 0, \quad (\text{D26})$$

which results from Eq. (52) with $R_{S_3} = T_5$. Now we insert a complete set of basis vectors I into the above equation, and obtain

$$\begin{aligned} &\langle \Psi_{g=2}^{ac} | \psi^{II;a^1,c,c^1} \rangle \\ &= \sum_{i,j,z} \langle \Psi_{g=2}^{ac} | \psi^{I;i,j,z} \rangle \langle \psi^{I;i,j,z} | \psi^{II;a^1,c,c^1} \rangle \\ &= \langle \Psi_{g=2}^{ac} | \psi^{I;a^1,c,z=1} \rangle \langle \psi^{I;a^1,c,z=1} | \psi^{II;a^1,c,c^1} \rangle, \end{aligned} \quad (\text{D27})$$

where we have considered the fusion rules $a^1 \otimes a^1 = \mathbf{1}$ and $c \otimes c = \mathbf{1} \oplus a^2 \oplus b \oplus b^1 \oplus b^2$, and the only allowed component in $|\psi^{I;a^1,c,z}\rangle$ is for $z = \mathbf{1}$. Since $\langle \psi^{I;a^1,c,z=1} | \psi^{II;a^1,c,c^1} \rangle = \sqrt{d_c d_{a^1} d_{c^1}} / D$ which is nonzero in Eq. (D27), then based on Eqs. (D26) and (D27) we obtain

$$\tilde{W}_{S_3,g=2}^{I;a^1,c,z=1} := \langle \Psi_{g=2}^{ac} | \psi^{I;a^1,c,z=1} \rangle = 0. \quad (\text{D28})$$

This means the pairings $\tilde{W}_{S_3,g=1}^{(p)} \otimes \tilde{W}_{S_3,g=1}^{(q)}$ with $(p, q) = (1, 3), (1, 4), (1, 5), (2, 3), (2, 4)$, and $(2, 5)$ cannot be the solutions of genus-2 condition in Eq. (52).

Solutions of genus-2 condition. Now let us check the pairings of genus-1 solutions $\tilde{W}_{S_3,g=1}^{(p)} \otimes \tilde{W}_{S_3,g=1}^{(p)}$ with $p = 1, 2, 3, 4$ respectively in Table III.

$\tilde{W}_{S_3,g=1}^{(1)} \otimes \tilde{W}_{S_3,g=1}^{(1)}$. The genus-1 solution $\tilde{W}_{S_3,g=1}^{(1)}$ in (50) corresponds to flux condensations of anyons $\mathbf{1}$, b , and c . (The relevant fusion rules are $b \otimes b = \mathbf{1} \oplus a^1 \oplus b$, $c \otimes c = \mathbf{1} \oplus a^2 \oplus b \oplus b^1 \oplus b^2$, and $b \otimes c = c \oplus c^1$.) Based on the genus-1 solution, we have

$$\tilde{W}_{S_3,g=2}^{1,i,j,z=1} = 1, \quad \forall i, j \in \{\mathbf{1}, b, c\}, \quad (\text{D29})$$

and $\tilde{W}_{S_3,g=2}^{1,i,j,z=1} = 0$, if $\exists i, j \notin \{\mathbf{1}, b, c\}$.

In the following, we will show that $\tilde{W}_{S_3,g=2}^{1,i,j,z=1}$ are coupled to certain $\tilde{W}_{S_3,g=2}^{1,i,j,z}$ with $z \neq \mathbf{1}$ through the operation $R_{S_3} = T_5$. We will frequently use the basis transformation in Eq. (D6). Let us start from $i = b$ and $j = c$. Then for $|\psi^{1;b,c,z}\rangle = [F_{bc}^{bc}]_{zz'} |\psi^{1;b,c,z'}\rangle$, where $z \in \{\mathbf{1}, b\}$ and $z' \in \{c, c^1\}$, one can find that

$$[F_{bc}^{bc}]_{zz'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (\text{D30})$$

Since the action of T_5 on $|\psi^{1;b,c,z'}\rangle$ is simply $T_5 |\psi^{1;b,c,z'}\rangle = \theta_{z'} |\psi^{1;b,c,z'}\rangle$, then based on the basis transformation in Eqs. (D6) and (D30) one can find that T_5 in the basis $\{|\psi^{1;b,c,\mathbf{1}}\rangle, |\psi^{1;b,c,b}\rangle\}$ has the expression

$$T_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{D31})$$

By solving Eq. (52), one can obtain

$$\tilde{W}_{S_3,g=2}^{1;b,c,b} = \tilde{W}_{S_3,g=2}^{1;b,c,\mathbf{1}} = 1. \quad (\text{D32})$$

For the case of $i = c$ and $j = b$, the discussion is almost the same, and one can find that

$$\tilde{W}_{S_3,g=2}^{1;c,b,b} = \tilde{W}_{S_3,g=2}^{1;c,b,\mathbf{1}} = 1. \quad (\text{D33})$$

Next, let us consider the case $i = j = c$. For $|\psi^{1;c,c,z}\rangle = [F_{cc}^{cc}]_{zz'} |\psi^{1;c,c,z'}\rangle$, where $z, z' \in \{\mathbf{1}, a^2, b, b^1, b^2\}$, one can find that

$$[F_{cc}^{cc}]_{zz'} = \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2 & -1 & -1 & -1 \\ \sqrt{2} & -1 & 2 & -1 & -1 \\ \sqrt{2} & -1 & -1 & 2 & -1 \\ \sqrt{2} & -1 & -1 & 2 & -1 \end{pmatrix}. \quad (\text{D34})$$

Combining with $T_5 |\psi^{1;c,c,z'}\rangle = \theta_{z'} |\psi^{1;c,c,z'}\rangle$, one can find the expression of T_5 in the basis $\{|\psi^{1;c,c,\mathbf{1}}\rangle, |\psi^{1;c,c,a^2}\rangle, |\psi^{1;c,c,b}\rangle, |\psi^{1;c,c,b^1}\rangle, |\psi^{1;c,c,b^2}\rangle\}$ as follows:

$$T_5 = \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2}e^{-\frac{i2\pi}{3}} & \sqrt{2}e^{\frac{i2\pi}{3}} \\ \sqrt{2} & 2 & -1 & e^{\frac{i\pi}{3}} & e^{-\frac{i\pi}{3}} \\ \sqrt{2} & -1 & 2 & e^{\frac{i\pi}{3}} & e^{-\frac{i\pi}{3}} \\ \sqrt{2}e^{-\frac{i2\pi}{3}} & e^{\frac{i\pi}{3}} & e^{\frac{i\pi}{3}} & e^{-\frac{i\pi}{3}} & 2 \\ \sqrt{2}e^{\frac{i2\pi}{3}} & e^{-\frac{i\pi}{3}} & e^{-\frac{i\pi}{3}} & 2 & e^{\frac{i\pi}{3}} \end{pmatrix}. \quad (\text{D35})$$

By solving Eq. (52) with $\tilde{W}_{S_3,g=2}^{1;c,c,z=1} = 1$ and $\tilde{W}_{S_3,g=2}^{1;c,c,b^1} = \tilde{W}_{S_3,g=2}^{1;c,c,b^2} = 0$, one can obtain

$$\tilde{W}_{S_3,g=2}^{1;c,c,a^2} + \tilde{W}_{S_3,g=2}^{1;c,c,b} = \sqrt{2}. \quad (\text{D36})$$

To further determine the concrete value of $\tilde{W}_{S_3,g=2}^{1;c,c,a^2}$ and $\tilde{W}_{S_3,g=2}^{1;c,c,b}$, we resort to Eqs. (D22) and (D24), based on which one can find that

$$\tilde{W}_{S_3,g=2}^{1;c,c,a^2} = -\sqrt{2} \tilde{W}_{S_3,g=2}^{1;c,a^2,a^2} = -\sqrt{2} \tilde{W}_{S_3,g=2}^{1;a^2,c,a^2} = 2 \tilde{W}_{S_3,g=2}^{1;a^2,a^2,a^2} \quad (\text{D37})$$

and

$$\tilde{W}_{S_3,g=2}^{1;c,c,b} = \sqrt{2} \tilde{W}_{S_3,g=2}^{1;c,b,b} = \sqrt{2} \tilde{W}_{S_3,g=2}^{1;b,c,b} = 2 \tilde{W}_{S_3,g=2}^{1;b,b,b}. \quad (\text{D38})$$

Since we have obtained $\tilde{W}_{S_3,g=2}^{1;c,b,b} = 1$ in Eq. (D33), then based on Eq. (D38) we have

$$\tilde{W}_{S_3,g=2}^{1;c,c,b} = \sqrt{2}, \quad \tilde{W}_{S_3,g=2}^{1;c,b,b} = \tilde{W}_{S_3,g=2}^{1;b,c,b} = 1, \quad \tilde{W}_{S_3,g=2}^{1;b,b,b} = \frac{1}{\sqrt{2}}. \quad (\text{D39})$$

Then from Eq. (D36) one has $\tilde{W}_{S_3,g=2}^{1;c,c,a^2} = 0$. Based on Eq. (D37), one has $\tilde{W}_{S_3,g=2}^{1;c,c,a^2} = \tilde{W}_{S_3,g=2}^{1;a^2,c,a^2} = \tilde{W}_{S_3,g=2}^{1;a^2,a^2,a^2} = 0$.

Till now, we have obtained a certain $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ that embeds $\tilde{W}_{S_3,g=1}^{(1)} \otimes \tilde{W}_{S_3,g=1}^{(1)}$. In particular, the nonzero components are as follows:

$$\begin{aligned} \tilde{W}_{S_3,g=2}^{1;b,c,z=b} &= \tilde{W}_{S_3,g=2}^{1;c,b,z=b} = 1, \\ \tilde{W}_{S_3,g=2}^{1;c,c,z=b} &= \sqrt{2}, \\ \tilde{W}_{S_3,g=2}^{1;b,b,z=b} &= \frac{1}{\sqrt{2}}, \end{aligned} \quad (\text{D40})$$

and

$$\tilde{W}_{S_3,g=2}^{1;i,j,z=1} = 1, \quad \forall i, j \in \{\mathbf{1}, b, c\}. \quad (\text{D41})$$

Furthermore, we claim that these are the only nonzero components of $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ that embeds $\tilde{W}_{S_3,g=1}^{(1)} \otimes \tilde{W}_{S_3,g=1}^{(1)}$. That is, for all the nonzero components $\tilde{W}_{S_3,g=2}^{1;i,j,z}$, one can find that $i, j, z \in \{\mathbf{1}, b, c\}$. All the other ($116 - 13 = 103$) components are zero. This can be checked explicitly as follows.

First, for $\tilde{W}_{S_3,g=2}^{1;i,j,z=1}$, which is the tensor product of genus-1 solutions, there are 55 components that are zero. They are $\tilde{W}_{S_3,g=2}^{1;i,j,z=1}$ such that $\exists i, j \notin \{\mathbf{1}, b, c\}$. Second, there are 18 components of $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ which are zero, with $\theta_z \neq 1$. They correspond to $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ with $z = b^1, b^2$.

The rest of the components with $z \neq \mathbf{1}$ and $\theta_z = 1$ need more careful study. Let us check them case by case.

(i) For $z = a^1$, we have shown in Eq. (D20) that $\tilde{W}_{S_3,g=2}^{1;i,j,z=a^1} = 0, \forall i, j \in \{a^2, b, b^1, b^2\}$. There are in total 16 components.

(ii) For $z = a^2$, as we have analyzed above, all the nine components $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ with $z = a^2$ are zero.

(iii) For $z = b$, we have $\tilde{W}_{S_3,g=2}^{1;i,j,z=b} = 0$, if $\exists i, j = c^1$. There are in total five components.

(iv) For $z = c$, this is not allowed by the fusion rules.

Altogether, for all the 116 genus-2 components $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ that embed $\tilde{W}_{S_3,g=1}^{(1)} \otimes \tilde{W}_{S_3,g=1}^{(1)}$, one can find there are indeed 103 components that are zero, and all the 13 nonzero components are listed in Eqs. (D40) and (D41).

$\tilde{W}_{S_3,g=1}^{(2)} \otimes \tilde{W}_{S_3,g=1}^{(2)}$. The genus-1 solution $\tilde{W}_{S_3,g=1}^{(2)}$ in (50) corresponds to the condensation of anyons $\mathbf{1}$, a^2 , and c . (The relevant fusion rules are $a^2 \otimes a^2 = \mathbf{1} \oplus a^1 \oplus a^2$, $c \otimes c = \mathbf{1} \oplus a^2 \oplus b \oplus b^1 \oplus b^2$, and $a^2 \otimes c = c \oplus c^1$.) Based on the genus-1 solution, we have

$$\tilde{W}_{S_3,g=2}^{1;i,j,z=1} = 1, \quad \forall i, j \in \{\mathbf{1}, a^2, c\}, \quad (\text{D42})$$

and $\tilde{W}_{S_3,g=2}^{1;i,j,z=1} = 0$, if $\exists i, j \notin \{\mathbf{1}, a^2, c\}$.

Next, let us consider the components with $z \neq \mathbf{1}$. Let us start from $i = a^2$ and $j = c$. Then for $|\psi^{1;a^2,c,z}\rangle = [F_{a^2c}^{a^2c}]_{zz'} |\psi^{1;a^2,c,z'}\rangle$, where $z \in \{\mathbf{1}, a^2\}$ and $z' \in \{c, c^1\}$, one can find that

$$[F_{a^2c}^{a^2c}]_{zz'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (\text{D43})$$

Recalling that the action of T_5 on $|\psi^{1;i,b,c,z'}\rangle$ is simply $T_5 |\psi^{1;i,b,c,z'}\rangle = \theta_z |\psi^{1;i,b,c,z'}\rangle$, based on the basis transformation in (D43), one can obtain the expression of T_5 in the basis $\{|\psi^{1;a^2,c,\mathbf{1}}\rangle, |\psi^{1;a^2,c,a^2}\rangle\}$ as follows:

$$T_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad (\text{D44})$$

By solving Eq. (52), one can obtain

$$\tilde{W}_{S_3,g=2}^{1;a^2,c,a^2} = -\tilde{W}_{S_3,g=2}^{1;a^2,c,\mathbf{1}} = -1. \quad (\text{D45})$$

With the same procedure, one can find that

$$\tilde{W}_{S_3,g=2}^{1;c,a^2,a^2} = -\tilde{W}_{S_3,g=2}^{1;c,a^2,\mathbf{1}} = -1. \quad (\text{D46})$$

Next, we consider the effect of punctured $S^{(z)}$ matrices. Based on Eq. (D22), we have

$$\tilde{W}_{S_3,g=2}^{1;c,c,a^2} = -\sqrt{2} \tilde{W}_{S_3,g=2}^{1;c,a^2,a^2} = -\sqrt{2} \tilde{W}_{S_3,g=2}^{1;a^2,c,a^2} = 2 \tilde{W}_{S_3,g=2}^{1;a^2,a^2,a^2}. \quad (\text{D47})$$

Then we can obtain

$$\begin{aligned} \tilde{W}_{S_3,g=2}^{1;c,c,a^2} &= \sqrt{2}, \quad \tilde{W}_{S_3,g=2}^{1;c,a^2,a^2} = \tilde{W}_{S_3,g=2}^{1;a^2,c,a^2} = -1, \\ \tilde{W}_{S_3,g=2}^{1;a^2,a^2,a^2} &= \frac{1}{\sqrt{2}}. \end{aligned} \quad (\text{D48})$$

Then we need to solve Eq. (52) with T_5 in (D35). One can find that $\tilde{W}_{S_3,g=2}^{1;c,c,z=1} = 1$, $\tilde{W}_{S_3,g=2}^{1;c,c,b^1} = \tilde{W}_{S_3,g=2}^{1;c,c,b^2} = 0$, and $\tilde{W}_{S_3,g=2}^{1;c,c,a^2} + \tilde{W}_{S_3,g=2}^{1;c,c,b} = \sqrt{2}$. Since $\tilde{W}_{S_3,g=2}^{1;c,c,a^2} = \sqrt{2}$, one can obtain $\tilde{W}_{S_3,g=2}^{1;c,c,b} = 0$. Together with Eq. (D38), we have

$$\tilde{W}_{S_3,g=2}^{1;c,c,b} = \tilde{W}_{S_3,g=2}^{1;c,b,b} = \tilde{W}_{S_3,g=2}^{1;b,c,b} = \tilde{W}_{S_3,g=2}^{1;b,b,b} = 0. \quad (\text{D49})$$

Till now, we have obtained certain $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ that embeds $\tilde{W}_{S_3,g=1}^{(2)} \otimes \tilde{W}_{S_3,g=1}^{(2)}$, with the nonzero components as follows:

$$\tilde{W}_{S_3,g=2}^{1;a^2,c,z=a^2} = \tilde{W}_{S_3,g=2}^{1;c,a^2,z=a^2} = -1,$$

$$\begin{aligned} \tilde{W}_{S_3,g=2}^{1;c,c,z=a^2} &= \sqrt{2}, \\ \tilde{W}_{S_3,g=2}^{1;a^2,a^2,z=a^2} &= \frac{1}{\sqrt{2}}, \end{aligned} \quad (\text{D50})$$

and

$$\tilde{W}_{S_3,g=2}^{1;i,j,z=1} = 1, \quad \forall i, j \in \{\mathbf{1}, a^2, c\}. \quad (\text{D51})$$

It is emphasized that for $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ that embeds $\tilde{W}_{S_3,g=1}^{(2)} \otimes \tilde{W}_{S_3,g=1}^{(2)}$, the components in Eqs. (D50) and (D51) exhaust all the nonzero components. The discussion is almost the same as that for the case of $\tilde{W}_{S_3,g=1}^{(1)} \otimes \tilde{W}_{S_3,g=1}^{(1)}$, except for the following differences:

(i) For $\tilde{W}_{S_3,g=2}^{1;i,j,z=1}$ which is the tensor product of genus-1 solutions, there are 55 components that are zero. They are $\tilde{W}_{S_3,g=2}^{1;i,j,z=1}$ such that $\exists i, j \notin \{\mathbf{1}, a^2, c\}$.

(ii) For $z = b$, all the nine components $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ with $z = b$ are zero.

(iii) For $z = a^2$, we have $\tilde{W}_{S_3,g=2}^{1;i,j,z=a^2} = 0$, if $\exists i, j = c^1$. There are in total five components of this kind.

Altogether, for all the 116 genus-2 components $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ that embed $\tilde{W}_{S_3,g=1}^{(2)} \otimes \tilde{W}_{S_3,g=1}^{(2)}$, one can find there are 103 components that are zero, and all the 13 nonzero components are listed in Eqs. (D50) and (D51).

$\tilde{W}_{S_3,g=1}^{(3)} \otimes \tilde{W}_{S_3,g=1}^{(3)}$. The genus-1 solution $\tilde{W}_{S_3,g=1}^{(3)}$ in (50) corresponds to the condensation of anyons $\mathbf{1}$, a^1 , and b . Relevant fusion rules are $a^1 \otimes a^1 = \mathbf{1}$, $a^1 \otimes b = b$, and $b \otimes b = \mathbf{1} \oplus a^1 \oplus b$. Based on the genus-1 solution, we have

$$\tilde{W}_{S_3,g=2}^{1;\mathbf{1},\mathbf{1},z=1} = \tilde{W}_{S_3,g=2}^{1;a^1,\mathbf{1},z=1} = \tilde{W}_{S_3,g=2}^{1;\mathbf{1},a^1,z=1} = \tilde{W}_{S_3,g=2}^{1;a^1,a^1,z=1} = 1, \quad (\text{D52})$$

$$\tilde{W}_{S_3,g=2}^{1;\mathbf{1},b,z=1} = \tilde{W}_{S_3,g=2}^{1;b,\mathbf{1},z=1} = \tilde{W}_{S_3,g=2}^{1;a^1,b,z=1} = \tilde{W}_{S_3,g=2}^{1;b,a^1,z=1} = 2, \quad (\text{D53})$$

and

$$\tilde{W}_{S_3,g=2}^{1;b,b,z=1} = 4. \quad (\text{D54})$$

One remarkable feature of $|\psi^{1;i,j,z=1}\rangle$ with $i, j \in \{\mathbf{1}, a^1, b\}$ is that, for the fusion results of $i \times j = \sum_z N_{ij}^z z$ with $N_{ij}^z > 0$, all the topological spins are trivial, i.e., $\theta_z = 1$. In this case, based on Eq. (D12), one has

$$T_5 |\psi^{1;i,j,z=1}\rangle = |\psi^{1;i,j,z=1}\rangle, \quad i, j \in \{\mathbf{1}, a^1, b\}. \quad (\text{D55})$$

Then one can find that the solutions $\tilde{W}_{S_3,g=2}^{1;i,j,z=1}$ in Eqs. (D52), (D53), and (D54) are decoupled from other components $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ with $z \neq \mathbf{1}$. In the following, we will show that Eqs. (D52), (D53), and (D54) exhaust all the nonzero components of $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ that embed $\tilde{W}_{S_3,g=1}^{(3)} \otimes \tilde{W}_{S_3,g=1}^{(3)}$.

First, similarly to previous discussions, for $\tilde{W}_{S_3,g=2}^{1;i,j,z=1}$ which is the tensor product of genus-1 solutions, there are 55 components that are zero. They are $\tilde{W}_{S_3,g=2}^{1;i,j,z=1}$ such that $\exists i, j \notin \{\mathbf{1}, a^1, b\}$. Second, there are 18 components of $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ which are zero, with $\theta_z \neq 1$. They correspond to $\tilde{W}_{S_3,g=2}^{1;i,j,z}$ with $z = b^1, b^2$.

The rest of the components with $z \neq \mathbf{1}$ and $\theta_z = 1$ can be checked case by case as follows.

(i) For $z = a^1$, we have shown in Eq. (D20) that $\tilde{W}_{S_3, g=2}^{l,i,j,z=a^1} = 0, \forall i, j \in \{a^2, b, b^1, b^2\}$. There are in total 16 components.

(ii) For $z = a^2$ and b , we need to solve Eq. (52) with T_5 in (D35). Different from the case of $\tilde{W}_{S_3, g=1}^{(1)} \otimes \tilde{W}_{S_3, g=1}^{(1)}$ and $\tilde{W}_{S_3, g=1}^{(2)} \otimes \tilde{W}_{S_3, g=1}^{(2)}$, now we have $\tilde{W}_{S_3, g=2}^{l,c,c,z=1} = 0$, which is set by the genus-1 solutions $(\tilde{W}_{S_3, g=1}^{(3)})^c (\tilde{W}_{S_3, g=1}^{(3)})^c = 0$. Note also that $\tilde{W}_{S_3, g=2}^{l,c,c,b^1} = \tilde{W}_{S_3, g=2}^{l,c,c,b^2} = 0$ because of the nontrivial topological spins of b^1 and b^2 . Then by solving Eq. (52) with T_5 in (D35) we can get

$$\tilde{W}_{S_3, g=2}^{l,c,c,a^2} = \tilde{W}_{S_3, g=2}^{l,c,c,b} = 0. \quad (\text{D56})$$

Then, based on Eqs. (D22) and (D24), we can find that all the 18 components of $\tilde{W}_{S_3, g=2}^{l,i,j,a^2}$ and $\tilde{W}_{S_3, g=2}^{l,i,j,b}$ are zero.

In short, for $\tilde{W}_{S_3, g=2}^{l,i,j,z}$ that embed $\tilde{W}_{S_3, g=1}^{(3)} \otimes \tilde{W}_{S_3, g=1}^{(3)}$, one can find there are in total 107 components that are zero, and 9 nonzero components which are listed in Eqs. (D52), (D53), and (D54).

$\tilde{W}_{S_3, g=1}^{(4)} \otimes \tilde{W}_{S_3, g=1}^{(4)}$. The genus-1 solution $\tilde{W}_{S_3, g=1}^{(4)}$ in (50) corresponds to the condensation of anyons $\mathbf{1}$, a^1 , and a^2 . Relevant fusion rules are $a^1 \otimes a^1 = \mathbf{1}$, $a^1 \otimes a^2 = a^2$, and $a^2 \otimes a^2 = \mathbf{1} \oplus a^1 \oplus a^2$. Based on the genus-1 solution, we have

$$\tilde{W}_{S_3, g=2}^{l,1,1,z=1} = \tilde{W}_{S_3, g=2}^{l,1,a^1,z=1} = \tilde{W}_{S_3, g=2}^{l,1,a^2,z=1} = \tilde{W}_{S_3, g=2}^{l,a^1,a^1,z=1} = 1, \quad (\text{D57})$$

$$\tilde{W}_{S_3, g=2}^{l,1,a^2,z=1} = \tilde{W}_{S_3, g=2}^{l,a^2,1,z=1} = \tilde{W}_{S_3, g=2}^{l,a^1,a^2,z=1} = \tilde{W}_{S_3, g=2}^{l,a^2,a^1,z=1} = 2, \quad (\text{D58})$$

and

$$\tilde{W}_{S_3, g=2}^{l,a^2,a^2,z=1} = 4. \quad (\text{D59})$$

Similarly to the case of $\tilde{W}_{S_3, g=1}^{(3)} \otimes \tilde{W}_{S_3, g=1}^{(3)}$, in the fusion results of $i \times \bar{j} = \sum_z N_{ij}^z z$ with $N_{ij}^z > 0$ and $i, j \in \{\mathbf{1}, a^1, a^2\}$, all the topological spins of anyon z are trivial, i.e., $\theta_z = 1$. In this case, based on Eq. (D12), one has

$$T_5 |\psi^{l,i,j,z=1}\rangle = |\psi^{l,i,j,z=1}\rangle, \quad i, j \in \{\mathbf{1}, a^1, a^2\}. \quad (\text{D60})$$

Then, again, one can find that the solutions $\tilde{W}_{S_3, g=2}^{l,i,j,z=1}$ in Eqs. (D57), (D58), and (D59) are decoupled from other components $\tilde{W}_{S_3, g=2}^{l,i,j,z}$ with $z \neq \mathbf{1}$. We will show that Eqs. (D57), (D58), and (D59) exhaust all the nonzero components of $\tilde{W}_{S_3, g=2}^{l,i,j,z}$ that embed $\tilde{W}_{S_3, g=1}^{(4)} \otimes \tilde{W}_{S_3, g=1}^{(4)}$.

The discussion is the same as that for $\tilde{W}_{S_3, g=1}^{(3)} \otimes \tilde{W}_{S_3, g=1}^{(3)}$ except for the following difference: For $\tilde{W}_{S_3, g=2}^{l,i,j,z=1}$ which is the tensor product of genus-1 solutions, there are 55 components that are zero. They are $\tilde{W}_{S_3, g=2}^{l,i,j,z=1}$ such that $\exists i, j \notin \{\mathbf{1}, a^1, a^2\}$.

Then one can find that for $\tilde{W}_{S_3, g=2}^{l,i,j,z}$ that embed $\tilde{W}_{S_3, g=1}^{(4)} \otimes \tilde{W}_{S_3, g=1}^{(4)}$, there are in total 107 components that are zero, and 9 nonzero components which are listed in Eqs. (D57), (D58), and (D59).

Summary. As a summary of this section, we solve all the components of $\tilde{W}_{S_3, g=2}^{l,i,j,z}$ with genus-2 condition in Eq. (52) with the basis vectors chosen in (D1). There are in total four sets of independent solutions, which embed the genus-1 solutions of the form $\tilde{W}_{S_3, g=1}^{(i)} \otimes \tilde{W}_{S_3, g=1}^{(i)}$ with $i = 1, 2, 3, 4$,

respectively. The genus-1 solution $\tilde{W}_{S_3, g=1}^{(5)}$ in Eq. (50) is ruled out by the genus-2 condition.

In the genus-2 solutions, the only nonzero components are listed in Eqs. (D40) and (D41) for the solutions that embed $\tilde{W}_{S_3, g=1}^{(1)} \otimes \tilde{W}_{S_3, g=1}^{(1)}$, Eqs. (D50) and (D51) for the solutions that embed $\tilde{W}_{S_3, g=1}^{(2)} \otimes \tilde{W}_{S_3, g=1}^{(2)}$, Eqs. (D52), (D53), and (D54) for the solutions that embed $\tilde{W}_{S_3, g=1}^{(3)} \otimes \tilde{W}_{S_3, g=1}^{(3)}$, and Eqs. (D57), (D58), and (D59) for the solutions that embed $\tilde{W}_{S_3, g=1}^{(4)} \otimes \tilde{W}_{S_3, g=1}^{(4)}$.

APPENDIX E: STRUCTURE COEFFICIENTS OF CONDENSABLE ALGEBRA AND GROUND-STATE OVERLAP

It is known that the gapped boundary is closely related to anyon condensation, whose data is fully encoded in the so-called condensable algebra [25] in the unitary modular tensor category (UMTC) that describes the anyons.

Let (\mathcal{C}, c) and (\mathcal{D}, c) be two 2D topological orders with the same chiral central charge c , where \mathcal{C} and \mathcal{D} are two UMTCs. We assume that they are Witt equivalent, in other words, they can be connected by a gapped domain wall. There are two equivalent mathematical descriptions of such a gapped domain wall:

(i) A gapped domain wall between (\mathcal{C}, c) and (\mathcal{D}, c) can be described by a unitary fusion category \mathcal{M} , which is equipped with a unitary braided monoidal equivalence,

$$\phi_{\mathcal{M}} : \mathcal{C} \boxtimes \bar{\mathcal{D}} \xrightarrow{\sim} Z(\mathcal{M}),$$

where $\mathcal{C} \boxtimes \bar{\mathcal{D}}$ is the UMTC describing the topological order obtained by stacking \mathcal{C} with the time reversal of \mathcal{D} , and $Z(\mathcal{M})$ denotes the Drinfeld center of \mathcal{M} .

(ii) A gapped domain wall between (\mathcal{C}, c) and (\mathcal{D}, c) is uniquely determined by a Lagrangian condensable algebra A in $\mathcal{C} \boxtimes \bar{\mathcal{D}}$. Here Lagrangian means that A is “maximal” in the sense

$$(\dim A)^2 = \dim \mathcal{C} \dim \mathcal{D}. \quad (\text{E1})$$

In this case, the gapped domain wall is described by the UFC $(\mathcal{C} \boxtimes \bar{\mathcal{D}})_A$, i.e., the category of right A modules in $\mathcal{C} \boxtimes \bar{\mathcal{D}}$. A Lagrangian algebra A , as an object in $\mathcal{C} \boxtimes \bar{\mathcal{D}}$, can be decomposed as follows:

$$A = \sum_{i \in \text{Irr}(\mathcal{C}), j \in \text{Irr}(\mathcal{D})} (i \boxtimes j)^{\oplus \tilde{W}_{\mathcal{D}\mathcal{C}, 1}^{ji}}, \quad (\text{E2})$$

where $\text{Irr}(\mathcal{C})$ and $\text{Irr}(\mathcal{D})$ denote the sets of isomorphic classes of simple objects.

In particular, when \mathcal{D} is the trivial topological order, the above describes a gapped boundary of \mathcal{C} , and A is a Lagrangian algebra in \mathcal{C} . The existence of a Lagrangian algebra, for example, in terms of the structure coefficients to be introduced in this section, is a necessary and sufficient condition for the existence of a gapped boundary. The approach presented in the main text, using the invariance of wave function overlaps under the modular transformations (or mapping-class-group transformations), however, only constitutes necessary conditions. The advantage of the modular invariance approach is that they are all linear equations and are

much easier to solve, as opposed to the nonlinear equations for the structure coefficients such as the associativity condition. For the examples presented in this paper, each of our solutions does correspond to a Lagrangian algebra.

Physically, a condensable algebra is just a composite anyon that becomes the ground state after condensation (i.e., it is condensed). For this to be possible, it is necessary that this composite anyon has special algebraic structures, which can be thought as a generalization of usual commutative associative algebra. We know that after picking a basis, the multiplication of a usual associative algebra can be expressed by the structure coefficients. It is also the case for the condensable algebra. The structure coefficients encode all the “special algebraic structures” of the condensed composite anyon; in particular, they predict the form of the ground-state overlap.

To explain what a “basis” of a composite anyon means in a unitary modular tensor category, we begin by recalling the (orthonormal) basis of an n -dimensional Hilbert space H , which is a set of state vectors $|\alpha\rangle$ satisfying

$$\langle\alpha|\beta\rangle = \delta_{\alpha\beta}, \quad \sum_{\alpha=1}^n |\alpha\rangle\langle\alpha| = \text{id}_H. \quad (\text{E3})$$

To generalize this notion, note that an n -dimensional Hilbert space can be viewed as a composite anyon, that is composed of n trivial anyons, $H = \oplus \mathbf{1}^{\oplus n}$. The trivial anyon is the tensor unit $\mathbf{1}$, namely the ground field \mathbb{C} , in the tensor category. Moreover, a vector $|\alpha\rangle$ can be identified with the embedding linear map $q_\alpha : \mathbb{C} \rightarrow H$, by $q_\alpha(\lambda) = \lambda|\alpha\rangle$. Similarly, $\langle\alpha|$ can be identified with the projection $p_\alpha : H \rightarrow \mathbb{C}$, by $p_\alpha(|\psi\rangle) = \langle\alpha|\psi\rangle$.

Now, given a composite anyon

$$A = \oplus i^{\oplus N_i^A}, \quad (\text{E4})$$

where i labels the simple anyons, a basis of A is a set of morphisms (“linear maps” in generic tensor category) $p_A^{i,\alpha} : A \rightarrow i$, $q_{i,\alpha}^A : i \rightarrow A$, $\alpha = 1, \dots, N_i^A$,

$$p_A^{i,\alpha} q_{j,\beta}^A = \delta_{ij} \delta_{\alpha\beta} \text{id}_i, \quad \sum_i \sum_{\alpha=1}^{N_i^A} q_{i,\alpha}^A p_A^{i,\alpha} = \text{id}_A. \quad (\text{E5})$$

We usually require that $p_A^{i,\alpha}$, $q_{i,\alpha}^A$ are Hermitian conjugates:

$$p_A^{i,\alpha} = (q_{i,\alpha}^A)^\dagger. \quad (\text{E6})$$

It is intuitive to use the following graphs for the basis:

$$\begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ A \end{array} = p_A^{i,\alpha}, \quad \begin{array}{c} A \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ i \end{array} = q_{i,\alpha}^A. \quad (\text{E7})$$

Our choice of symbol is to remind the reader of the similarity to the basis of Hilbert spaces (rotate the graph by 90° anticlockwise). They satisfy similar orthonormal and complete conditions:

$$\begin{array}{c} \alpha \\ \downarrow \\ i \end{array} \leftarrow A \sim \langle i, \alpha |, \quad A \leftarrow \begin{array}{c} \alpha \\ \downarrow \\ i \end{array} \sim | \alpha, i \rangle, \quad (\text{E8})$$

$$\begin{array}{c} \alpha \\ \downarrow \\ i \end{array} \leftarrow A \leftarrow \begin{array}{c} \beta \\ \downarrow \\ j \end{array} \sim \langle i, \alpha | \beta, j \rangle = \delta_{ij} \delta_{\alpha\beta}, \quad (\text{E9})$$

$$\sum_{i\alpha} \begin{array}{c} \alpha \\ \downarrow \\ i \end{array} \leftarrow A \leftarrow \begin{array}{c} \alpha \\ \downarrow \\ i \end{array} \sim \sum_{i\alpha} |\alpha, i\rangle \langle i, \alpha| = \text{id}_A. \quad (\text{E10})$$

The only subtle part is that the tensor product and braiding involving nontrivial i are different from usual intuitions from vector spaces, which will be explained below. First, we need to take a basis of the tensor product of simple anyons $i \otimes j$:

$$\begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array} = \sum_k \sum_{\alpha=1}^{N_k^{ij}} \begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array} \begin{array}{c} k \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ k \end{array}. \quad (\text{E11})$$

Choosing the orthonormal basis is equivalent to choosing Y and O to be identity matrices. The previous choice of vertex basis is in fact

$$\begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array} = \left(\frac{d_i d_j}{d_k} \right)^{1/4} \begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array}, \quad \begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array} = \left(\frac{d_i d_j}{d_k} \right)^{1/4} \begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array}, \quad (\text{E12})$$

They may be referred to as the “rotatable basis” since, after the above rescaling, rotating a vertex (by bending the legs) leads to at most a phase factor or a unitary matrix, which cancel each other if we are considering a closed graph. Also, as explained in Sec. III E, in the rotatable basis, a closed graph representing a wave function is properly normalized to a constant (17) that does not depend on the anyon and vertex labels. However, in this section we prefer to use the orthonormal basis, indicated by the semicircle in the graph, which is more convenient for open graphs.

The associativity of tensor product is encoded in the F matrix:

$$\begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array} \begin{array}{c} k \\ \downarrow \\ \bigcirc_\beta \\ \uparrow \\ l \end{array} = \sum_{s, \chi, \delta} F_{kls, \chi\delta}^{ijr, \alpha\beta} \begin{array}{c} i \\ \downarrow \\ \bigcirc_\alpha \\ \uparrow \\ j \end{array} \begin{array}{c} k \\ \downarrow \\ \bigcirc_\chi \\ \uparrow \\ s \end{array} \begin{array}{c} r \\ \downarrow \\ \bigcirc_\beta \\ \uparrow \\ l \end{array}, \quad (\text{E13})$$

where we used the definition of F matrix in (B1). The braiding can be similarly represented by the R matrix:

$$= \sum_b R_{k,b}^{ij} \quad (E14)$$

We remark that, in general, the tensors such as F, R matrices do depend on our choice of basis. However, it is easily verified that the two choices of basis (E12) differ by the same overall factor on both sides of (E13) and (E14). Thus, F, R matrices remain the same under the change of basis (E12).

For an algebra A in a tensor category, there is a multiplication morphism $m : A \otimes A \rightarrow A$. First take a basis of A as in (E5):

$$id_A = A = \sum_i \sum_{\alpha=1}^{N_i^A} \quad (E15)$$

The multiplication morphism m is then

$$m = \sum_{ijk, \alpha\beta\chi} \quad (E16)$$

The central part can be expressed in terms of basis vertices $p_{ij}^{k,\mu}$:

$$= \sum_{\mu} M_{i\alpha, j\beta}^{k\chi, \mu} p_{ij}^{k, \mu} \quad (E17)$$

Thus

$$m = \sum_{ijk, \alpha\beta\chi, \mu} M_{i\alpha, j\beta}^{k\chi, \mu} q_{k, \chi}^A p_{ij}^{k, \mu} (p_A^{i, \alpha} \otimes p_A^{j, \beta}) \quad (E18)$$

$M_{i\alpha, j\beta}^{k\chi, \mu}$ are the “structure coefficients” of the algebra. In the category of vector spaces \mathbf{Vec} , the object labels i, j, k and the vertex label μ become trivial, and $M_{i\alpha, j\beta}^{k\chi, \mu}$ reduce to structure coefficients of the usual associative algebra, with α, β, χ the labels of basis vectors. Again, similarly to the usual associative algebra, the structure coefficients depend on the choice of basis, both the basis of A , $p_A^{i, \alpha}, q_{i, \alpha}^A$ and the vertex basis $p_{ij}^{k, \mu}$. It is easy to write down transformations of $M_{i\alpha, j\beta}^{k\chi, \mu}$ under a change of basis. For example, using the rotatable basis (E12), the corresponding structure coefficients are

$$\tilde{M}_{i\alpha, j\beta}^{k\chi, \mu} = \left(\frac{d_k}{d_i d_j} \right)^{1/4} M_{i\alpha, j\beta}^{k\chi, \mu} \quad (E19)$$

Structure coefficients related by a change of basis are considered equivalent and describe the same algebra.

Now we are ready to define the condensable algebra $(A, M_{i\alpha, j\beta}^{k\chi, \mu})$, by listing the defining, i.e., sufficient and necessary conditions of $M_{i\alpha, j\beta}^{k\chi, \mu}$:

(1) Associative:

$$= \quad (E20)$$

$$\sum_{\omega} M_{i\alpha, j\beta}^{r\omega, \mu} M_{r\omega, k\chi}^{l\delta, \nu} = \sum_{s\xi\psi\zeta} F_{kls, \xi\psi}^{ijr, \mu\nu} M_{j\beta, k\chi}^{s\zeta, \xi} M_{i\alpha, s\zeta}^{l\delta, \psi} \quad (E21)$$

(2) Unital: There exists “the unit of the multiplication,” the unit morphism $\eta : \mathbf{1} \rightarrow A$,

$$= A = \quad (E22)$$

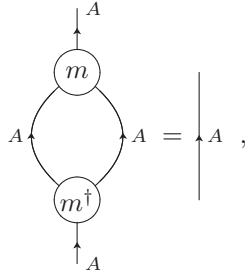
expressed in terms of embedding basis $\eta = \eta_{\alpha} q_{1, \alpha}^A$, and

$$\sum_{\alpha=1}^{N_1^A} \eta_{\alpha} M_{1\alpha, j\beta}^{k\chi, 1} = \sum_{\alpha=1}^{N_1^A} \eta_{\alpha} M_{j\beta, 1\alpha}^{k\chi, 1} = \delta_{jk} \delta_{\beta\chi} \quad (E23)$$

(3) Connected: The range of index α in the above expression is $N_1^A = 1$, thus

$$M_{11,j\beta}^{k\chi,1} = M_{j\beta,11}^{k\chi,1} = \eta_1^{-1} \delta_{jk} \delta_{\beta\chi}. \quad (\text{E24})$$

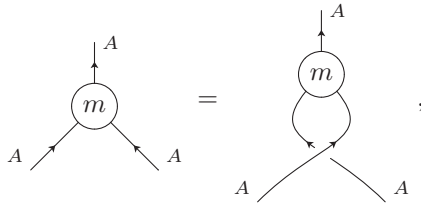
(4) Isometric:



(E25)

$$\sum_{i\alpha j\beta\mu} M_{i\alpha,j\beta}^{k\chi,\mu} (M_{i\alpha,j\beta}^{k'\chi',\mu})^* = \delta_{kk'} \delta_{\chi\chi'}. \quad (\text{E26})$$

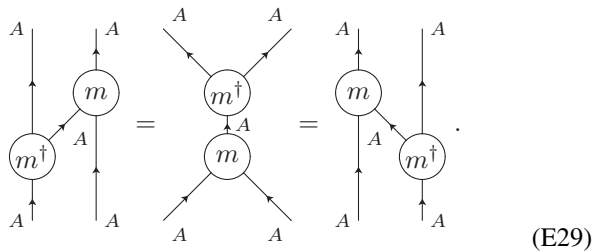
(5) Commutative:



(E27)

$$M_{i\alpha,j\beta}^{k\chi,\mu} = \sum_v R_{k,v\mu}^{ij} M_{j\beta,i\alpha}^{k\chi,v}. \quad (\text{E28})$$

A unital connected isometric algebra in a unitary tensor category automatically satisfies the Frobenius condition [53]:



(E29)

As a direct corollary, A is self-dual and

$$|\eta_1|^2 = \dim A = \sum_i N_i^A d_i, \quad (\text{E30})$$

where η_1 is from the unit morphism $\eta = \eta_1 q_{1,1}^A : \mathbf{1} \rightarrow A$. We choose the phase of η_1 and let

$$\eta_1 = \sqrt{\sum_i N_i^A d_i}. \quad (\text{E31})$$

Also, by attaching counit η^\dagger to top-right A in (E29), we obtain the following useful formula to exchange the upper and lower indices of $M_{i\alpha,j\beta}^{k\chi,\mu}$:

$$M_{i\alpha,j\beta}^{k\chi,\mu} = \sum_{\gamma,v} (M_{k\chi,j\gamma}^{i\alpha,v})^* F_{jk1,11}^{k\bar{j}i,v\mu} M_{j\gamma,j\beta}^{11,1} \eta_1^*, \quad (\text{E32})$$

where $F_{jk1,11}^{k\bar{j}i,v\mu}$ comes from the following:

$$= \sum_{\mu} F_{jk1,11}^{k\bar{j}i,v\mu} \dots \quad (\text{E33})$$

Similarly, attaching η^\dagger to top-left A , we obtain

$$M_{i\alpha,j\beta}^{k\chi,\mu} = \sum_{\gamma,v} (M_{i\gamma,k\chi}^{j\beta,v})^* (F_{kkj,v\mu}^{i\bar{i}1,11})^* M_{i\alpha,i\gamma}^{11,1} \eta_1^*. \quad (\text{E34})$$

Let us compute $M_{i\gamma,i\beta}^{11,1}$. From Eq. (E32), we obtain

$$M_{11,i\beta}^{i\chi,1} = \sum_{\gamma} (M_{i\chi,i\gamma}^{11,1})^* F_{ii1,11}^{i\bar{i}1,11} M_{i\gamma,i\beta}^{11,1} \eta_1^*. \quad (\text{E35})$$

Using $F_{ii1,11}^{i\bar{i}1,11} = d_i^{-1}$, $M_{i\alpha,i\gamma}^{11,1} = M_{i\gamma,i\alpha}^{11,1}$ and Eq. (E24), the above becomes

$$\delta_{\beta\chi} = \frac{\sum_j N_j^A d_j}{d_i} \sum_{\gamma} (M_{i\chi,i\gamma}^{11,1})^* M_{i\gamma,i\beta}^{11,1}. \quad (\text{E36})$$

We see that $M_{i\alpha,i\beta}^{11,1}$ is proportional to a unitary matrix. By choosing a proper basis for Eq. (E8) we can make

$$M_{i\alpha,i\beta}^{11,1} = \frac{d_i \delta_{\alpha\beta}}{\sqrt{\sum_j N_j^A d_j}}. \quad (\text{E37})$$

Together with (E28), one can permute any pair of $i\alpha$ indices.

The structure coefficients can be used to compute the (topological universal part of) overlap between ground states before and after condensation. For simplicity, here we only discuss the case in which the phase after condensation is topologically trivial and has a unique ground state. We start with the ground state on a torus. Before condensation, the loops of simple objects form a basis of the ground space on a torus:

$$(\text{E38})$$

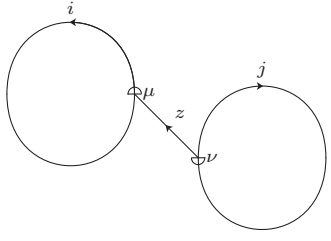
After condensation, the ground state is, up to an overall factor that depends on the system size, the loop of the corresponding condensable algebra A :

$$(\text{E39})$$

Thus we can extract the multiplicity N_i^A from the ground-state overlap on a torus, which is exactly the universal part of wave function overlap [the labels in the trivial phase are omitted; see Eqs. (2) and (3)]:

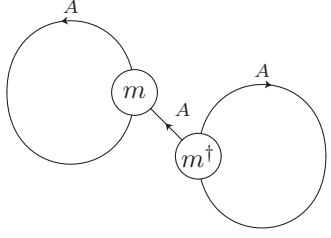
$$\tilde{W}_{C,1}^i = M_C^i = N_i^A. \quad (\text{E40})$$

On a closed two-dimensional surface with genus $g = 2$, one choice of the basis before condensation is given by



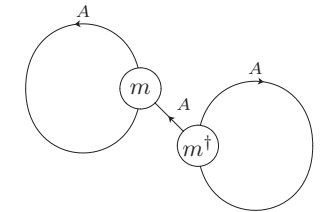
$$(E41)$$

Still, after condensation, the ground state is



$$(E42)$$

Now, the structure coefficients kick in. After straightforward calculation, we find



$$= \sum_{ijz\mu\nu} \sum_{\alpha\beta\chi} M_{i\alpha,z\chi}^{i\alpha,\mu} (M_{z\chi,j\beta}^{j\beta,\nu})^* \text{ (Diagram E41) }$$

$$= \sum_{ijz\mu\nu} \sum_{\alpha\beta\chi} \frac{M_{i\alpha,z\chi}^{i\alpha,\mu} (M_{z\chi,j\beta}^{j\beta,\nu})^*}{d_z} \text{ (Diagram E41) } .$$

$$(E43)$$

Therefore, for genus 2, the wave function overlap, up to an overall factor, is given by (the labels in the trivial phase is omitted)

$$\tilde{W}_{C;2}^{(i,j,z,\mu,\nu)} \propto \frac{1}{d_z} \sum_{\alpha\beta\chi} M_{i\alpha,z\chi}^{i\alpha,\mu} (M_{z\chi,j\beta}^{j\beta,\nu})^* . \quad (E44)$$

Using [see Eqs. (E24) and (26)]

$$M_{11,j\beta}^{k\chi,1} = M_{j\beta,11}^{k\chi,1} = \frac{1}{\sqrt{\sum_i N_i^A d_i}} \delta_{jk} \delta_{\beta\chi} ,$$

$$\tilde{W}_{C;2}^{i,j,1,1,1} = \tilde{W}_{C,1}^i \tilde{W}_{C,1}^j \quad (E45)$$

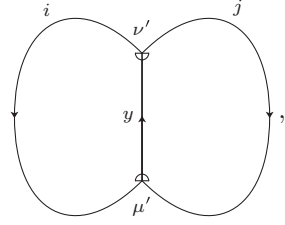
we can fix the overall factor. Notice that [setting $z = \mathbf{1}$ in Eq. (E44)]

$$\sum_{\alpha\beta} M_{i\alpha,11}^{i\alpha,1} (M_{11,j\beta}^{j\beta,1})^* = \frac{N_i^A N_j^A}{\sum_i N_i^A d_i} , \quad (E46)$$

where we have use the fact that $\alpha = 1, \dots, N_i^A$ and $\beta = 1, \dots, N_j^A$. Since $\tilde{W}_{C,1}^1 = N_1^A = 1$, we find

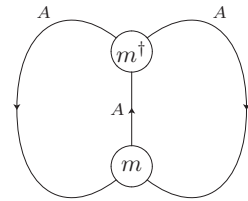
$$\tilde{W}_{C;2}^{(i,j,z,\mu,\nu)} = \frac{\sum_k N_k^A d_k}{d_z} \sum_{\alpha\beta\chi} M_{i\alpha,z\chi}^{i\alpha,\mu} (M_{z\chi,j\beta}^{j\beta,\nu})^* . \quad (E47)$$

Another choice of basis is



$$(E48)$$

whose difference from the first choice is given by the F matrix. Correspondingly,

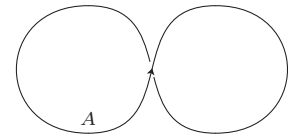


$$= \sum_{ijy\mu'\nu'} \left(\sum_{\alpha\beta\chi} M_{i\alpha,j\beta}^{y\chi,\mu'} (M_{i\alpha,j\beta}^{y\chi,\nu'})^* \right) \text{ (Diagram E48) } .$$

$$(E49)$$

The compatibility between the two choices is guaranteed by the Frobenius condition (E29). It is straightforward to generalize to surfaces with any genus.

A Lagrangian algebra A is *modular invariant*, which was first proposed and proved in [54, Theorem 3.4] for a special modular tensor category. But the proof automatically works for all modular tensor category. Below we explain the property in detail. Modular invariance means T and S invariance. First, the topological spin of A is trivial,

$$\theta_A = \frac{1}{\dim A} \text{ (Diagram E50) } = 1, \quad (E50)$$


which is a direct consequence of the commutative and Frobenius conditions. Thus a twist of string A leaves the graph invariant, which generates a subset a Dehn twists on the manifold. This property is the T invariance. Expressing in terms of simple objects,

$$T_{ii} N_i^A = N_i^A . \quad (E51)$$

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