Extensions of Maps from Suspensions of Finite Projective Spaces

by

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Submitted on April 25, 1988 to the Department of Mathematics at the
Massachusetts Institute of Technology in partial fulfillment of the
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Abstract

H. Miller's proof of the Sullivan Conjecture shows that if X is a nilpotent space and $H^*(X; \mathbf{Z}/2\mathbf{Z})$ is bounded, then $[\Sigma^t RP^\infty, X] = 0$. We prove a finite version of this theorem. Specifically, if $t \geq 2$ we give a number k such that the restriction map $[\Sigma^t RP^{2^k}, X] \rightarrow [\Sigma^t RP^n, X]$ is the zero map. The number k depends on n , t , and the connectivity and dimension of $\overline{H}^*(X; \mathbf{Z}/2\mathbf{Z})$. The proof factors the restriction map through another group of homotopy classes of maps, $[\Sigma^{t-1} T_1(2^k), X]$, and studies the filtrations in the Bousfield-Kan spectral sequences for $[\Sigma^{t-1} T_1(2^k), X]$ and $[\Sigma^t RP^n, X]$.

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1 Introduction

Let $\text{map}_*(RP^\infty, X)$ be the space of pointed maps from RP^∞ to a topological space X . We call X nilpotent if X is path-connected, $\pi_1 X$ is nilpotent, and $\pi_1 X$ acts nilpotently on $\pi_n X$ for $n > 1$ [B-K, p.59]. H. Miller proves in [M, Theorem C] that if X is nilpotent and $\overline{H}^*(X; \mathbf{Z}/2\mathbf{Z})$ is nonzero in finitely many dimensions, then $\text{map}_*(RP^\infty, X)$ has the weak homotopy type of a point:

$$\pi_* \text{map}_*(RP^\infty, X) = 0.$$

Since $\pi_t \text{map}_*(RP^\infty, X) \cong [\Sigma^t RP^\infty, X]$, Miller's theorem implies that no essential map $f : \Sigma^t RP^n \rightarrow X$ can be extended to a map $\bar{f} : \Sigma^t RP^\infty \rightarrow X$. This paper gives information about the question, does f extend to a map $\Sigma^t RP^{n+j} \rightarrow X$? We will prove the following theorem.

Theorem 1.1 *Let X be a nilpotent space such that $\overline{H}^i(X; \mathbf{Z}/2\mathbf{Z}) = 0$ for $i < c$ and $i > d$. Let $t \geq 2$ and let n be even. Then the restriction map*

$$[\Sigma^t RP^{2^k}, X] \longrightarrow [\Sigma^t RP^n, X]$$

is the zero map if $k \geq \max(n, n + t + \log_2 d - c)$ and $c \geq 3$, or if $k \geq n + t + \log_2 d$ and $c = 2$.

If n is odd, we still get a result by applying Theorem 1.1 to $n + 1$.

We can reduce immediately to the case that X is a CW complex, since we are mapping out of $\Sigma^t RP^{2^k}$, which is a finite CW complex. Furthermore, we may assume that X is a finite complex, because the image of a map $\Sigma^t RP^{2^k} \rightarrow X$ will lie in a finite subcomplex of X .

The organization of the rest of the paper is as follows. Section 2 gives the proof of Theorem 1.1 except for the proofs of Lemmas 2.2 and 2.5. Section 3 gives background to the proof of Lemma 2.2, and Section 4 gives the proof. Section 5 gives the proof of Lemma 2.5. Finally, Section 6 describes work of H. Miller [M] and unpublished work of P. Goerss [G] on left derived functors of indecomposables and proves a technical result which was used in Section 5 (Theorem 5.7).

All cohomology is taken with $\mathbf{Z}/2\mathbf{Z}$ coefficients. We write $\mathcal{A}_2$ for the mod 2 Steenrod algebra, $\underline{\mathbf{U}}$ for the category of unstable modules over $\mathcal{A}_2$, and $\underline{\mathbf{UA}}$ for the category of unstable algebras over $\mathcal{A}_2$. We assume that X is a finite CW complex unless otherwise stated, and that n is even.

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2 Proof of Theorem 1.1

The method of proof of Theorem 1.1 is to factor the restriction map through another group of homotopy classes of maps. The motivation for attempting to find such a factorization is algebraic. Consider the projection map $\overline{H}^* RP^{2^k} \rightarrow \overline{H}^* RP^n$, where $k \geq n$. Miller proved in [M] that, as a morphism in $\underline{\mathbf{U}}$, this map factors through a module $J(2^k)$ which is an injective object in $\underline{\mathbf{U}}$:

$$\begin{array}{ccc} \overline{H}^* RP^{2^k} & \longrightarrow & J(2^k) \\ \downarrow & & \downarrow \\ \overline{H}^* RP^n & \xrightarrow{=} & \overline{H}^* RP^n. \end{array} \quad (2.1)$$

(This can also be recovered from earlier work of Carlsson [C].) Background on the module $J(2^k)$ and on the factoring (2.1) is discussed in Section 3.

The essential step in the proof of Theorem 1.1 is that diagram (2.1) can be realized topologically after two suspensions.

Lemma 2.2 *Let $k \geq n$. There is a simply-connected space $T_1(2^k)$ such that $\overline{H}^* T_1(2^k) \cong \Sigma J(2^k)$, and there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma^2 RP^n & \longrightarrow & \Sigma T_1(2^k) \\ \downarrow \simeq & & \downarrow \\ \Sigma^2 RP^n & \longrightarrow & \Sigma^2 RP^{2^k} \end{array}$$

which on cohomology is the double suspension of diagram (2.1).

This lemma gives us a corresponding factoring of the restriction map of Theorem 1.1 (recall $t \geq 2$):

$$\begin{array}{ccc} [\Sigma^t RP^{2^k}, X] & \longrightarrow & [\Sigma^{t-1} T_1(2^k), X] \\ \downarrow & & \downarrow \\ [\Sigma^t RP^n, X] & \xrightarrow{\cong} & [\Sigma^t RP^n, X]. \end{array}$$

We study these groups by means of the unstable Adams spectral sequence constructed by Bousfield and Kan. Let $X^\wedge$ denote the $\mathbf{F}_2$ -completion of the space X [B-K].

Theorem 2.3 ([B-K]) *Let W be a finite CW complex and let X be a nilpotent space of finite type. Then for $t \geq 1$ there is a spectral sequence*

$$E_2^{s,t} = \text{Ext}_{\underline{\mathbf{UA}}}^s(H^*X, \Sigma^{s+t}H^*W) \implies [\Sigma^t W, X^\wedge].$$

The Bousfield-Kan spectral sequence comes from a decreasing filtration of $[\Sigma^t W, X^\wedge]$, and the filtration degree is s . If $g : \Sigma^t W_2 \rightarrow X^\wedge$ has filtration degree s , and $f : W_1 \rightarrow W_2$, then $(\Sigma^t f)^\# g$ has filtration degree at least s . The E_2 -term of the spectral sequence is a derived functor in the non-abelian category $\underline{\mathbf{UA}}$. It is described briefly in Section 5. For a more complete discussion of the definition of derived functors in a nonabelian category such as $\underline{\mathbf{UA}}$, we refer the reader to [M], Section 2.

We recall a theorem from [M] showing that in the cases to which we will apply the Bousfield-Kan spectral sequence, namely $W = RP^n$, RP^{2^k} , or $T_1(2^k)$, we do not need to complete the target X .

Theorem 2.4 ([M, Theorem 1.5]) *Let W be a finite CW complex such that $\overline{H}^*(W; \mathbf{Z}[1/2]) = 0$, and let X be a nilpotent space. Then the completion map $X \rightarrow X^\wedge$ induces a weak homotopy equivalence*

$$\text{map}_*(W, X) \rightarrow \text{map}_*(W, X^\wedge).$$

The hypotheses of this theorem hold for $W = RP^n$ when n is even and for $W = RP^{2^k}$, and we check in the proof of Corollary 4.3 that they hold for $T_1(2^k)$. In these cases we do not have to complete the target in Theorem 2.3, and the Bousfield-Kan spectral sequence will actually converge to $[\Sigma^t W, X]$.

Theorem 1.1 is proved by studying the filtrations in the Bousfield-Kan spectral sequences for $[\Sigma^{t-1} T_1(2^k), X]$ and $[\Sigma^t RP^n, X]$. To this end, we calculate a range in which the E_2 -terms are zero. Recall that by hypothesis, $\overline{H}^i X = 0$ for $i < c$ and $i > d$, and that n is even.

Lemma 2.5 *Under the hypotheses of Theorem 1.1,*

1. $\text{Ext}_{\underline{\mathbf{U}}\mathbf{A}}^s(H^* X, \Sigma^{s+t} J(2^k)) = 0$ for $s \leq k - \log_2 d$;
2. $\text{Ext}_{\underline{\mathbf{U}}\mathbf{A}}^s(H^* X, \Sigma^{s+t} H^* RP^n) = 0$ for $s \geq \begin{cases} n + t - c & \text{if } c \geq 3 \\ n + t & \text{if } c = 2. \end{cases}$

Proof of Theorem 1.1: Let $c \geq 3$. By Lemma 2.5(1), the filtration degree of elements of $[\Sigma^{t-1} T_1(2^k), X]$ in the Bousfield-Kan spectral sequence is at least $k - \log_2 d$, so maps in the image of

$$[\Sigma^{t-1} T_1(2^k), X] \longrightarrow [\Sigma^t RP^n, X]$$

also have filtration at least $k - \log_2 d$. On the other hand, Lemma 2.5(2) implies that elements of $[\Sigma^t RP^n, X]$ with filtration at least $n + t - c$ are zero. Thus the restriction map is zero if $k - \log_2 d \geq n + t - c$, which is exactly the conclusion of Theorem 1.1. If $c = 2$, we use the same argument with $n + t$ in place of $n + t - c$. □

3 Injectives in $\underline{\mathbf{U}}$

This section describes a family of injective modules in the category $\underline{\mathbf{U}}$ of unstable left $\mathcal{A}_2$ -modules ($\underline{\mathbf{U}}$ -injectives, for short). This background information, especially Theorem 3.5, is the motivation for the method of proof of Theorem 1.1 described in Section 2.

Let $\underline{\mathbf{E}}$ denote the category of $\mathbf{F}_2$ -vector spaces.

Proposition 3.1 ([M, Lemma 6.2]) *Let $T : \underline{\mathbf{U}} \rightarrow \underline{\mathbf{E}}$ be the functor*

$$T(M) = (M^n)^*,$$

the vector space dual of M in dimension n . Then T is representable, that is, there exists $J(n) \in \underline{\mathbf{U}}$ and a natural transformation such that

$$T(M) \cong \text{Hom}_{\underline{\mathbf{U}}}(M, J(n)).$$

Remark: Since T is an exact functor, $\text{Hom}_{\underline{\mathbf{U}}}(-, J(n))$ is exact and so $J(n)$ is injective in $\underline{\mathbf{U}}$.

Sketch of Proof [L-Z, Section 2]: Let $F(k) \in \underline{\mathbf{U}}$ be the primitive elements in $H^*K(\mathbf{Z}_2, k)$. Then $F(k)$ is a free unstable $\mathcal{A}_2$ -module on a k -dimensional element in the sense that for $M \in \underline{\mathbf{U}}$,

$$\text{Hom}_{\underline{\mathbf{U}}}(F(k), M) \cong M^k.$$

Thus we must define $J(n)$ in dimension k to be $(F(k)^n)^*$. The action of $\theta \in \mathcal{A}_2$ on $J(n)$ is given by the dual of the map $F(k + |\theta|) \rightarrow F(k)$ sending $\iota_{k+|\theta|}$ to $\theta \iota_k$. Lemma 3.1 certainly holds for free modules. Now apply T to the beginning of a free resolution for a general M and use the right exactness of $\text{Hom}_{\underline{\mathbf{U}}}(-, J(n))$ and the five lemma. $\square$

From the construction of $J(n)$, we see that $J(n)^n$ is a vector space of dimension one and that $J(n)^{n+k} = 0$ for $k > 0$. Thus there is a unique top class of $J(p) \otimes J(q)$ in dimension $p + q$, and by the defining property of $J(p + q)$ there is a unique nonzero map $J(p) \otimes J(q) \rightarrow J(p + q)$. The structure of the family $\{J(n)\}_{n \geq 0}$ is given by Theorem 3.2.

Theorem 3.2 ([M, Theorem 6.17]) *Let $J(\bullet) = \bigoplus_{n \geq 0} J(n)$ be bigraded by $J(\bullet)^{n,k} = J(n)^k$. Then $J(\bullet)$ has the structure of a bigraded algebra with multiplication given by the unique nonzero maps*

$$J(p) \otimes J(q) \rightarrow J(p + q) \tag{3.3}$$

and this multiplication makes $J(\bullet)$ a polynomial algebra on generators x_i of bidegree $(2^i, 1)$ for $i \geq 0$.

Sketch of Proof: Notice that $J(\bullet)$ can also be thought of as $\oplus_{k \geq 0} F(k)^*$. One can check that the dual of the unique nonzero $\mathcal{A}_2$ -linear map $F(k+m) \rightarrow F(k) \otimes F(m)$ gives the same multiplication as (3.3). A calculation shows that $F(1)^* \otimes F(k)^* \rightarrow F(k+1)^*$ is onto, so the generators of $J(\bullet)$ all lie in $F(1)^*$, which has a vector space basis of elements x_i in bidegree $(2^i, 1)$. The structure of the dual Steenrod algebra gives a count of the elements in $F(k)^*$ and shows that there can be no multiplicative relations between the x_i .

□

We now give an omnibus proposition collecting important properties of the modules $J(n)$. We will use a map $(Sq^n)^* : J(2n) \rightarrow J(n)$ which is given by the following $f \in (J(2n)^n)^*$: for $a \in J(2n)^n$, define $f(a) = Sq^n(a) \in J(2n)^{2n} \cong \mathbf{F}_2$. Also, if $\beta \in \mathcal{A}_2$ is the Bockstein operator, then $\beta^2 = 0$ and we can define the homology of $M \in \underline{\mathbf{U}}$ with respect to β . We call M Bockstein acyclic if $H(M; \beta) = 0$.

Proposition 3.4 1. $\Sigma J(2n) \cong J(2n+1)$.

2. There is an “EHP” exact sequence

$$0 \longrightarrow \Sigma J(2n-1) \longrightarrow J(2n) \xrightarrow{(Sq^n)^*} J(n) \longrightarrow 0.$$

3. The element of smallest dimension in $J(n)$ has dimension $\alpha(n)$, where $\alpha(n)$ is the sum of the digits in the dyadic expansion of n .

4. [G-L, Lemma 1.1] $J(n)$ is Bockstein acyclic for $n \geq 2$.

Properties (1)-(3) can be proved using the basis of monomials in the x_i given by Theorem 3.2. See [M, Corollary 6.2] and [Da, Section 2]. Property (4) can be proved by induction using the “EHP” sequence (2).

We will be interested in a $\underline{\mathbf{U}}$ -injective $K(1)$, which has the property of having $\overline{H}^* RP^\infty$ as a direct summand, thus showing that $\overline{H}^* RP^\infty$ is $\underline{\mathbf{U}}$ -injective. Define this module $K(1)$ as the inverse limit of the system

$$K(1) \rightarrow \dots \xrightarrow{(Sq^{2^n})^*} J(2^n) \dots \xrightarrow{(Sq^2)^*} J(2) \xrightarrow{(Sq^1)^*} J(1).$$

It is not obvious that $K(1)$ is a $\underline{\mathbf{U}}$ -injective, because in an arbitrary category an inverse limit of injectives is not always injective. However, Lannes and Zarati show in [L-Z] that an inverse limit of $\underline{\mathbf{U}}$ -injectives of finite type is $\underline{\mathbf{U}}$ -injective. Thus $K(1)$ is $\underline{\mathbf{U}}$ -injective since each $J(n)$ is of finite type by its construction.

By the defining property of $J(2^n)$ there are unique nonzero homomorphisms $\overline{H}^*RP^\infty \rightarrow J(2^n)$. They are compatible, and give a unique nonzero homomorphism $\overline{H}^*RP^\infty \rightarrow K(1)$. Carlsson, and then Miller, showed that this was a split monomorphism. Thus $K(1)$ has $\overline{H}^*RP^\infty$ as a summand, and $\overline{H}^*RP^\infty$ is also $\underline{\mathbf{U}}$ -injective. Furthermore, the unique nonzero map $\overline{H}^*RP^{2^n} \rightarrow J(2^n)$ has a “splitting” to $\overline{H}^*RP^{n+1}$, which is what is required to obtain diagram (2.1).

Theorem 3.5 ([C], [M, Section 6]) *There are morphisms Θ and θ_n in $\underline{\mathbf{U}}$ such that the following diagram commutes:*

$$\begin{array}{ccccc}
\overline{H}^*RP^\infty & \longrightarrow & \overline{H}^*RP^{2^{n+1}} & \longrightarrow & \overline{H}^*RP^{2^n} \\
\downarrow i & & \downarrow & & \downarrow \\
K(1) & \longrightarrow & J(2^{n+1}) & \longrightarrow & J(2^n) \\
\downarrow \Theta & & \downarrow \theta_{n+1} & & \downarrow \theta_n \\
\overline{H}^*RP^\infty & \longrightarrow & \overline{H}^*RP^{n+2} & \longrightarrow & \overline{H}^*RP^{n+1}.
\end{array}$$

Remark 3.6: Since $\Sigma J(2^n) \cong J(2^n + 1)$ (Proposition 3.4(1)), we can form $\Sigma K(1)$ as the inverse limit of the injectives $\Sigma J(2^n)$. This means that $\Sigma K(1)$, and hence $\Sigma \overline{H}^*RP^\infty$, are $\underline{\mathbf{U}}$ -injective.

4 Realizations of Morphisms in $\underline{\mathbf{U}}$

For $k \geq n$, we compose the splitting θ_k of Theorem 3.5 with the projection $\overline{H}^*RP^{k+1} \rightarrow \overline{H}^*RP^n$ to obtain the factoring of diagram (2.1):

$$\begin{array}{ccc}
\overline{H}^*RP^{2^k} & \longrightarrow & J(2^k) \\
\downarrow & & \downarrow \\
\overline{H}^*RP^n & \xrightarrow{=} & \overline{H}^*RP^n.
\end{array}$$

In this section we discuss to what extent this algebraic factoring can be reflected topologically. Theorem 4.1, Corollary 4.3, and Proposition 4.4 together prove Lemma 2.2.

The first remark to make is that $J(2^n)$ is not always the cohomology of a space. This can be shown by noticing that $J(2^n)$ cannot always be made into an algebra over $\mathcal{A}_2$ ($J(8)$ is such an example). However, $\Sigma J(2^n)$ can be made into an algebra over $\mathcal{A}_2$ by giving it the trivial algebra structure where all products are zero, and we can now ask whether this algebra is the cohomology of a space. This question is answered by the following theorem of Goerss and Lannes.

Theorem 4.1 ([G-L, Theorem A]) *Let $M \in \underline{\mathbf{U}}$ be injective and of finite type. Then there exists a simply-connected, $\mathbf{F}_2$ -complete space X such that:*

1. $\overline{H}^* X \cong \Sigma M$ where ΣM is given trivial algebra structure.
2. If $\sigma : \Sigma \Omega X \rightarrow X$ is the counit of the adjunction, then

$$\sigma^* : \overline{H}^* X \rightarrow \overline{H}^* \Sigma \Omega X$$

is a monomorphism.

Remark. If Y is the suspension of a space, then $\sigma : \Sigma \Omega Y \rightarrow Y$ must induce a monomorphism in cohomology. Thus the second part of Theorem 4.1 says that although X may not be the suspension of a space, it looks like a suspension in that σ^* is a monomorphism.

Theorem 4.1 provides a space $T_1(2^k)$, which we will take to be a CW complex, such that $\overline{H}^* T_1(2^k) \cong \Sigma J(2^k)$. One might hope for a factoring of $\Sigma RP^n \rightarrow \Sigma RP^{2^k}$ through $T_1(2^k)$, but obtaining such maps seems to be difficult. However, technology of Goerss and Lannes allows us to construct a factoring after one more suspension using the following theorem. Recall that $Z^\wedge$ denotes the $\mathbf{F}_2$ -completion of the space Z [B-K].

Theorem 4.2 ([G-L, Corollary 2.8]) *Let Y be a nilpotent, connected space and X a space so that $H^* X$ has trivial algebra structure and $\overline{H}^* X$ is*

injective in $\underline{\mathbf{U}}$. If $\phi : H^*Y \rightarrow H^*X$ is a morphism in $\underline{\mathbf{UA}}$, then there exists a map $f : \Sigma X \rightarrow (\Sigma Y)^\wedge$ so that $H^*f = \Sigma\phi$.

The proof uses obstruction theory of A. K. Bousfield [B], who proves that the obstructions to the existence of the adjoint of f lie in the groups

$$\mathrm{Ext}_{\underline{\mathbf{UA}}}^s (H^*\Omega\Sigma Y, \Sigma^{s-1}H^*X).$$

These groups are then shown to be zero using the composite functor spectral sequence of Theorem 5.1.

Corollary 4.3 *There exists $f : \Sigma T_1(2^k) \rightarrow \Sigma^2 RP^{2^k}$ such that $\overline{H}^*f$ is the unique non-zero map $i : \Sigma^2 \overline{H}^* RP^{2^k} \rightarrow \Sigma^2 J(2^k)$.*

Proof: Proposition 3.4 (1) tells us that $\overline{H}^*T_1(2^k) \cong \Sigma J(2^k)$ is $\underline{\mathbf{U}}$ -injective, so by Theorem 4.2 there is a map $\hat{f} : \Sigma T_1(2^k) \rightarrow (\Sigma^2 RP^{2^k})^\wedge$ such that $\overline{H}^*\hat{f} \cong i$. We apply Theorem 2.4, which says that if $\overline{H}^*(T_1(2^k); \mathbf{Z}[1/2]) = 0$, then $\mathbf{F}_2$ -completion of $\Sigma^2 RP^{2^k}$ induces an isomorphism

$$[\Sigma T_1(2^k), \Sigma^2 RP^{2^k}] \xrightarrow{\cong} [\Sigma T_1(2^k), \Sigma^2 RP^{2^k}^\wedge].$$

This would complete the proof, for if $[f]$ is the preimage of $[\hat{f}]$ under the isomorphism, then $\overline{H}^*f \cong i$. Now $T_1(2^k)$ is given by Theorem 4.1 as an $\mathbf{F}_2$ -complete, therefore 2-local space. Hence $\overline{H}^*(T_1(2^k); \mathbf{Z}[1/2]) = 0$ provided that $\overline{H}^*(T_1(2^k); \mathbf{Z})$ is all torsion. But this last is true, since $\Sigma J(2^k)$ is Bockstein acyclic.

□

Proposition 4.4 *For $k \geq n$, there exists a map $g : \Sigma^2 RP^n \rightarrow \Sigma T_1(2^k)$ such that $\overline{H}^*g$ is the splitting map $\Sigma^2 J(2^k) \rightarrow \Sigma^2 \overline{H}^* RP^n$ of diagram (2.1).*

Proof: We cannot apply Theorem 4.2 directly, as in Corollary 4.3, since $\overline{H}^*\Sigma RP^n$ is not $\underline{\mathbf{U}}$ -injective. However, it is possible to sneak around this problem by considering the infinite case. Let $T_1(K(1))$ be the space given by Theorem 4.1 such that $\overline{H}^*T_1(K(1)) \cong \Sigma K(1)$, and recall from Remark 3.6

that $\Sigma \overline{H}^* RP^\infty$ is $\underline{U}$ -injective. By an argument similar to that of Corollary 4.3, there is a map

$$\Sigma^2 RP^\infty \rightarrow \Sigma T_1(K(1))$$

realizing the splitting $\Theta : \Sigma^2 K(1) \rightarrow \Sigma^2 RP^\infty$. Similarly, there are maps

$$\Sigma T_1(2^k) \rightarrow \Sigma T_1(K(1))$$

realizing the projection from the inverse limit $\Sigma^2 K(1) \rightarrow \Sigma^2 J(2^k)$. Now recall that there is a short exact sequence

$$0 \longrightarrow \Sigma J(2^{k+1} - 1) \longrightarrow J(2^{k+1}) \xrightarrow{(Sq^{2^k})^*} J(2^k) \longrightarrow 0$$

and that the first non-zero element of $\Sigma J(2^{k+1} - 1)$ lies in dimension $k + 2$ (Proposition 3.4). This means that $(Sq^{2^k})^*$ is an isomorphism to dimension $k + 1$. Because the module $K(1)$ is the inverse limit of the maps $(Sq^{2^k})^*$, the projection $K(1) \rightarrow J(2^k)$ will also be an isomorphism up to dimension $k + 1$. Thus $\Sigma T_1(2^k) \rightarrow \Sigma T_1(K(1))$ is a $(k + 2)$ -equivalence, and since $k \geq n$, we can now complete the following diagram:

$$\begin{array}{ccc} \Sigma^2 RP^n & \dashrightarrow & \Sigma T_1(2^k) \\ \downarrow & & \downarrow \\ \Sigma^2 RP^\infty & \longrightarrow & \Sigma T_1(K(1)). \end{array}$$

□

5 Proof of Lemma 2.5

This section gives the proof of Lemma 2.5 which we restate here for convenience. The hypothesis of Theorem 1.1 is that $\overline{H}^i X = 0$ if $i < c$ and $i < d$ and that n is even.

Lemma 2.5 *Under the hypotheses of Theorem 1.1,*

1. $\text{Ext}_{\underline{U}\mathbf{A}}^s(H^* X, \Sigma^{s+t} J(2^k)) = 0$ for $s \leq k - \log_2 d$;

$$2. \text{Ext}_{\underline{\mathbf{UA}}}^s (H^*X, \Sigma^{s+t} H^*RP^n) = 0 \text{ for } s \geq \begin{cases} n+t-c & \text{if } c \geq 3 \\ n+t & \text{if } c = 2. \end{cases}$$

In a non-abelian category like $\underline{\mathbf{UA}}$, it is necessary to define derived functors such as Ext with respect to a triple or a cotriple. We refer the reader to Section 2 of [M] for details, and confine ourselves to defining the cotriples that we will use. Let $\underline{\mathbf{nF}_2}$ denote the category of graded $\mathbf{F}_2$ -vector spaces, let $I : \underline{\mathbf{UA}} \rightarrow \underline{\mathbf{nF}_2}$ take the augmentation ideal, and let $G : \underline{\mathbf{UA}} \rightarrow \underline{\mathbf{UA}}$ be the composition of I with its left adjoint. Then G has the structure of a cotriple, and this is the cotriple we will use to take derived functors in $\underline{\mathbf{UA}}$. Our notation is as follows: if T is a cotriple on a category $\underline{\mathbf{B}}$ and $F : \underline{\mathbf{B}} \rightarrow \underline{\mathbf{A}}$ is a functor to an abelian category $\underline{\mathbf{A}}$, then we write $L_r^T FB$ for the r th left derived functor of F with respect to T applied to $B \in \underline{\mathbf{B}}$. (The exception is the notation for derived functors of a Hom or tensor functor, which we write in the usual way as Ext or Tor .)

We now describe functors of the form $\text{Hom}_{\underline{\mathbf{UA}}}(-, \Sigma M)$ as composite functors, so that we can prove Lemma 2.5 by applying a spectral sequence for the derived functors of a composite. Let $M \in \underline{\mathbf{U}}$, and let $\Sigma : \underline{\mathbf{U}} \rightarrow \underline{\mathbf{UA}}$ be the functor taking M to the algebra which is ΣM as an $\mathcal{A}_2$ module and has trivial algebra structure, i.e. all products are zero. Let $Q : \underline{\mathbf{UA}} \rightarrow \underline{\mathbf{U}}$ be the indecomposables functor sending $K \in \underline{\mathbf{UA}}$ to $I(K)/I(K)^2$. The module $QK \in \underline{\mathbf{U}}$ desuspends, and the functor $\Sigma^{-1}Q : \underline{\mathbf{UA}} \rightarrow \underline{\mathbf{U}}$ is left adjoint to Σ . Thus there is a factorization

$$\text{Hom}_{\underline{\mathbf{UA}}}(-, \Sigma M) \cong \text{Hom}_{\underline{\mathbf{U}}}(-, M) \circ \Sigma^{-1}Q.$$

H. Miller constructs a spectral sequence for the derived functors of this composite functor.

Theorem 5.1 ([M, Theorem 2.5(i)]) *There is a convergent homological spectral sequence*

$$E_{p,q}^2 = \text{Ext}_{\underline{\mathbf{U}}}^p (L_q^G(\Sigma^{-1}Q)K, M) \implies \text{Ext}_{\underline{\mathbf{UA}}}^{p+q} (K, \Sigma M)$$

natural in $M \in \underline{\mathbf{U}}$ and $K \in \underline{\mathbf{UA}}$.

To prove Lemma 2.5 we will study the composite functor spectral sequence in the two cases of the lemma:

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^{s-r} (L_r^G(\Sigma^{-1}Q)H^*X, \Sigma^{s+t-1}J(2^k)) \Rightarrow \mathrm{Ext}_{\underline{\mathbf{UA}}}^s (H^*X, \Sigma^{s+t}J(2^k))$$

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^{s-r} (L_r^G(\Sigma^{-1}Q)H^*X, \Sigma^{s+t-1}\overline{H}^*RP^n) \Rightarrow \mathrm{Ext}_{\underline{\mathbf{UA}}}^s (H^*X, \Sigma^{s+t}H^*RP^n).$$

Our arguments from this point on will depend on $L_r^G(\Sigma^{-1}Q)H^*X$ only as a graded vector space, and this information is independent of the $\mathcal{A}_2$ action in the following way. Let $\underline{\mathbf{A}}$ be the category of commutative, augmented algebras over $\mathbf{F}_2$, and let $\Phi : \underline{\mathbf{UA}} \rightarrow \underline{\mathbf{A}}$ be the forgetful functor. Let $S : \underline{\mathbf{A}} \rightarrow \underline{\mathbf{A}}$ be the composition of the augmentation ideal functor $I : \underline{\mathbf{A}} \rightarrow \underline{\mathbf{nF}_2}$ with its left adjoint. The functor S has the structure of a cotriple, and we have the following proposition of H. Miller.

Theorem 5.2 ([M, Theorem 2.5(ii)]) *There is for each r an isomorphism of graded vector spaces*

$$L_r^G(\Sigma^{-1}Q)K \cong \Sigma^{-1}L_r^S Q(\Phi K)$$

which is natural in $K \in \underline{\mathbf{UA}}$.

The proof of Lemma 2.5(1) will follow directly from the following two theorems.

Theorem 5.3 ([Da, Theorem 2.13]) *Let $M(k, n, s) = 2 + [(n + k - s - 1)/2^s]$. If $M \in \underline{\mathbf{U}}$ is such that $M^i = 0$ for $i \geq M(k, n, s)$, then*

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^s (M, \Sigma^k J(n)) = 0.$$

The proof of Theorem 5.3 uses the exact sequence of Proposition 3.4(3),

$$0 \rightarrow \Sigma J(2n - 1) \rightarrow J(2n) \rightarrow J(n) \rightarrow 0,$$

and the corresponding long exact sequence of Ext groups to get an induction. Davis actually states this theorem for $M = \overline{H}^*S^m$, but his proof goes through as written for a general M . In order to apply this result to the proof of Lemma 2.5(1), we need an estimate of the top nonzero dimension of $L_r^G(\Sigma^{-1}Q)H^*X$, which is the same as the top nonzero dimension of $\Sigma^{-1}L_r^S Q(\Phi H^*X)$.

Theorem 5.4 ([M, Corollary 5.2]) *Let $\Lambda \in \underline{\mathbf{A}}$ and let $\Lambda^i = 0$ for $i > d$. Then $(L_r^S Q \Lambda)^i = 0$ for $i > d2^r$.*

The proof of Theorem 5.4 uses methods described in Section 6.

Proof of Lemma 2.5(1): Consider the composite functor spectral sequence of Theorem 5.1:

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^{s-r} (L_r^G(\Sigma^{-1}Q)H^*X, \Sigma^{s+t-1}J(2^k)) \Rightarrow \mathrm{Ext}_{\underline{\mathbf{U}}\underline{\mathbf{A}}}^s (H^*X, \Sigma^{s+t}J(2^k)).$$

We will show that the E_2 -term is zero if $s \leq k - \log_2 d$. Recall $\overline{H}^i X = 0$ for $i > d$. By Theorem 5.3,

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^{s-r} (L_r^G(\Sigma^{-1}Q)H^*X, \Sigma^{s+t-1}J(2^k)) = 0$$

if $(L_r^G(\Sigma^{-1}Q)H^*X)^i = 0$ for $i \geq M(2^k, s+t, s-r)$. A little arithmetic shows that $M(2^k, s+t, s-r) > 2^{k-s+r}$, so in fact it is sufficient that $(L_r^G(\Sigma^{-1}Q)H^*X)^i = 0$ for $i \geq 2^{k-s+r}$. From Theorems 5.2 and 5.4, we have $(L_r^G(\Sigma^{-1}Q)H^*X)^i = 0$ for $i > d2^r - 1$ (the -1 comes from the desuspension). The E^2 -term of the composite functor spectral sequence will be zero provided that

$$d2^r - 1 < 2^{k-s+r}, \quad \text{or} \quad s \leq k - \log_2 d,$$

which is what was to be proved. □

To prove Lemma 2.5(2) we study the E^2 -term of the composite functor spectral sequence

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^{s-r} (L_r^G(\Sigma^{-1}Q)H^*X, \Sigma^{s+t-1}\overline{H}^*RP^n) \Rightarrow \mathrm{Ext}_{\underline{\mathbf{U}}\underline{\mathbf{A}}}^s (H^*X, \Sigma^{s+t}H^*RP^n)$$

by considering a minimal resolution of $\Sigma^{s+t-1}\overline{H}^*RP^n$ by $\underline{\mathbf{U}}$ -injectives. Recall that $\beta \in \mathcal{A}_2$ is the Bockstein operator, and that we call $M \in \underline{\mathbf{U}}$ Bockstein acyclic if $H(M; \beta) = 0$. Notice that $\Sigma^{s+t-1}\overline{H}^*RP^n$ is Bockstein acyclic since we have assumed that n is even.

Lemma 5.5 *Let $N \in \underline{\mathbf{U}}$ be Bockstein acyclic, and let $N^i = 0$ for $i = 0$ and $i > d$. Then there is a resolution of N by $\underline{\mathbf{U}}$ -injectives*

$$0 \rightarrow N \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$$

such that $(I_s)^i = 0$ for $i > d - 2s$.

Corollary 5.6 *Let $M \in \underline{\mathbf{U}}$ be $c-1$ connected, and let $N \in \underline{\mathbf{U}}$ be as above. Then if $c > d-2s$,*

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^s(M, N) = 0.$$

Proof of Lemma 5.5: Since N is nonzero in finitely many degrees, we can make a resolution of N using sums of modules $J(i)$. The defining property of $J(i)$ shows that $J(i)^i$ and $J(i)^{i-1}$ have unique nonzero elements, and these are connected by the Bockstein operator β . Now all elements of N^d are in the image of β (they cannot support β), so it is possible to choose a monomorphism $j : N \hookrightarrow I_0 = \bigoplus J(n_\alpha)$ which is an isomorphism in dimensions d and $d-1$. Furthermore, we can guarantee that I_0 is Bockstein acyclic by choosing all $n_\alpha \geq 2$: it is not necessary to use any copies of $J(1)$ because if $N \rightarrow I_0$ is one to one in dimension 2, it will be dimension 1 also. Looking at the short exact sequence

$$0 \rightarrow N \xrightarrow{j} I_0 \rightarrow \mathrm{coker}(j) \rightarrow 0,$$

we see that $(\mathrm{coker}(j))^i = 0$ if $i > d-2$. Furthermore, since N and I_0 are both Bockstein acyclic, $\mathrm{coker}(j)$ is also. The lemma now follows by induction. $\square$

In order to apply Corollary 5.6 to the calculation of the E^2 -term of $\mathrm{Ext}_{\underline{\mathbf{U}}}^{s-r}(L_r^G(\Sigma^{-1}Q)H^*X, \Sigma^{s+t-1}\overline{H}^*RP^n) \Rightarrow \mathrm{Ext}_{\underline{\mathbf{UA}}}^s(H^*X, \Sigma^{s+t}H^*RP^n)$, we need to know the connectivity of $L_r^G(\Sigma^{-1}Q)H^*X$. By Theorem 5.2 it is the same as the connectivity of $\Sigma^{-1}L_r^S Q(\Phi H^*X)$. We will prove the following result in Section 6. Recall that $\underline{\mathbf{A}}$ is the category of commutative, augmented, graded $\mathbf{F}_2$ -algebras.

Theorem 5.7 *Let $\Lambda \in \underline{\mathbf{A}}$ be $c-1$ connected. Then $L_r^S Q\Lambda$ has connectivity*

$$\begin{array}{ll} rc-1 & \text{if } 4 \leq r \\ (r+1)c-1 & \text{if } 0 \leq r \leq 3. \end{array}$$

Proof of Lemma 2.5(2): We apply Corollary 5.6 and Theorem 5.7 to show that the E^2 -term is zero in the composite functor spectral sequence:

$$\mathrm{Ext}_{\underline{\mathbf{U}}}^{s-r}(L_r^G(\Sigma^{-1}Q)H^*X, \Sigma^{s+t-1}\overline{H}^*RP^n) \Rightarrow \mathrm{Ext}_{\underline{\mathbf{UA}}}^s(H^*X, \Sigma^{s+t}H^*RP^n).$$

It is sufficient to show the following two conditions:

$$\begin{aligned} rc - 1 &> (n + s + t - 1) - 2(s - r) && \text{for } 4 \leq r \leq s \\ (r + 1)c - 1 &> (n + s + t - 1) - 2(s - r) && \text{for } 0 \leq r \leq 3. \end{aligned}$$

Rearranging, we can see that it is sufficient that

$$\begin{aligned} s &> n + t + (2 - c)r && \text{for } 4 \leq r \leq s \\ s &> n + t + (2 - c)r - c && \text{for } 0 \leq r \leq 3. \end{aligned}$$

If $c = 2$, then the first condition is the most strict, and we must require $s > n + t$. If $c \geq 3$, then $r = 0$ gives the most strict condition, and we must require $s > n + t - c$. This completes the proof of Lemma 2.5. $\square$

6 Left Derived Functors of Q

Let $\underline{\mathbf{A}}$ be the category of commutative graded augmented $\mathbf{F}_2$ -algebras and let $\underline{\mathbf{nF}}_2$ be the category of graded $\mathbf{F}_2$ -vector spaces. Let $Q : \underline{\mathbf{A}} \rightarrow \underline{\mathbf{nF}}_2$ be the indecomposables functor. The purpose of this section is to discuss the left derived functors of the functor Q . Since $\underline{\mathbf{A}}$ is a non-abelian category, we must take derived functors of Q with respect to a cotriple. (See [M, Section 2] for details on derived functors in a nonabelian category.) The cotriple we will use is S , which is the composite of the augmentation ideal functor $I : \underline{\mathbf{A}} \rightarrow \underline{\mathbf{nF}}_2$ with its left adjoint. We write $L_r^S Q\Lambda$ for the r th derived functor of Q with respect to S applied to $\Lambda \in \underline{\mathbf{A}}$. Our goal is to prove Theorem 5.7, which gives the connectivity of $L_r^S Q\Lambda$ in terms of the connectivity of Λ .

The plan of the section is as follows. In Section 6.1 we define the Quillen homology $H_*^Q X_\bullet$ of a simplicial algebra $X_\bullet$ and show that we can recover $L_*^S Q\Lambda$ as a special case. We then describe a spectral sequence converging to $H_*^Q X_\bullet$. In Section 6.2 we identify the E_2 -term as derived functors of a functor Q_D to be defined below. In Section 6.3 we show that the spectral sequence applied in a special case has the form

$$E_2^{s,t} = \mathrm{Tor}_{\underline{\mathbf{UD}}}^s(\mathbf{F}_2, Q\mathrm{Tor}_{\Lambda}^t(\mathbf{F}_2, \mathbf{F}_2)) \implies L_{s+t-1}^S Q\Lambda.$$

(This form of the spectral sequence first appeared in [M3].) In Section 6.4 we give a Koszul resolution for computing E_2 in this case, and give the proof of Theorem 5.7.

The exposition of Sections 6.1 - 6.3 follows the account given by P. Goerss in [G, Section 10] of results of H. Miller in [M, Sections 3-5]. The chain complex of Section 6.4 is from [G, Section 11].

6.1 A Spectral Sequence for $L_*^S Q\Lambda$

In this section we generalize the definition of $L_*^S Q\Lambda$ to a notion of homology of a simplicial algebra. We first describe carefully the definition of $L_*^S Q\Lambda$. Let $Sy : \mathbf{nF}_2 \rightarrow \mathbf{A}$ take the symmetric algebra on a graded vector space and let $I : \mathbf{A} \rightarrow \mathbf{nF}_2$ take the augmentation ideal. These are adjoint functors, and thus the composition $S = Sy \circ I$ comes with natural transformations $\psi : S \rightarrow S^2$ and $\eta : S \rightarrow 1$ giving S the structure of a cotriple [M]. Let $S^\bullet \Lambda$ be the simplicial algebra with

$$\begin{aligned} (S^\bullet \Lambda)_r &= S^r \Lambda \\ d_i &= S^i \eta S^{r-i} \\ s_i &= S^i \psi S^{r-i}. \end{aligned}$$

The homotopy of $S^\bullet \Lambda$ can be shown to be

$$\begin{aligned} \pi_r S^\bullet \Lambda &= 0 & (r > 0) \\ \pi_0 S^\bullet \Lambda &= \Lambda \end{aligned}$$

and so $S^\bullet \Lambda$ should be thought of as a “projective resolution” of Λ with respect to S . We define the derived functors with respect to S of $Q : \mathbf{A} \rightarrow \mathbf{nF}_2$ by applying Q to each level of the “resolution” $S^\bullet \Lambda$ and taking homotopy:

$$L_r^S Q\Lambda = \pi_r Q S^\bullet \Lambda.$$

We now put the computation of $L_r^S Q\Lambda$ into the setting of the category $\mathbf{sA}$ of simplicial algebras by defining the Quillen homology of $X_\bullet \in \mathbf{sA}$; we will recover the definition of $L_*^S Q\Lambda$ as a special case. The advantage of this approach is that $\mathbf{sA}$ is a closed model category in the sense of Quillen

[Q], which allows us to do “homotopy theory,” and Quillen homology is a “homotopy functor,” *i.e.* an invariant of homotopy type.

Our first problem in defining the Quillen homology of $X_\bullet \in \underline{\mathbf{sA}}$ is that it is important to work with the cofibrant objects in a category to avoid pathologies. (For a topological space X , for example, it is usual to take a CW complex weak homotopy equivalent to X before beginning to do homotopy theory.) We refer the reader to [M2] for a discussion of how to recognize cofibrant objects in $\underline{\mathbf{sA}}$, and proceed with the definition of a particular cofibrant object which is homotopy equivalent to $X_\bullet$.

Let $S^\bullet X_\bullet$ be the bisimplicial algebra

$$(S^\bullet X_\bullet)_{p,q} = S^{p+1} X_q$$

and let $PX_\bullet$ be the diagonal of $S^\bullet X_\bullet$. Then $PX_\bullet$ is cofibrant [M2, Theorem 3.4(ii)], and the augmentation $S^\bullet X_\bullet \rightarrow X_\bullet$ gives a weak homotopy equivalence of $PX_\bullet$ with $X_\bullet$. We define the Quillen homology of $X_\bullet$ by

$$H_r^Q X_\bullet = \pi_r QPX_\bullet.$$

Notice that if we let $\Lambda_\bullet$ be the constant simplicial algebra which is Λ in each degree and has the identity maps for faces and degeneracies, then $P\Lambda_\bullet = S^\bullet \Lambda$ and we recover $L_r^S Q\Lambda$ as $H_r^Q \Lambda_\bullet$.

Consider the bicomplex obtained from $S^\bullet X_\bullet$ by applying Q to each $S^{p+1} X_q$ and the maps between them, and then taking the alternating sum of the face maps. This bicomplex gives rise to a spectral sequence by filtering by degree in p :

$$E_0^{p,q} = QS^{p+1} X_q \implies \pi_* QPX_\bullet = H_*^Q X_\bullet. \quad (6.1)$$

When $X_\bullet = \Lambda_\bullet$, the E_1 -term is $S^\bullet \Lambda$ in $p = 0$ and zero elsewhere. Computing E_2 is the same problem as computing $L_r^S Q\Lambda$, so we have not advanced far! However, we can identify the E_2 -term in general as derived functors of a functor $\mathcal{Q}_D$ (defined below) applied to $X_\bullet$. Furthermore, the closed model category $\underline{\mathbf{sA}}$ has a suspension functor $\overline{W} : \underline{\mathbf{sA}} \rightarrow \underline{\mathbf{sA}}$ ([M, Corollary 5.6] and [M2]) with the property

$$H_r^Q \overline{W} X_\bullet = H_{r-1}^Q X_\bullet.$$

The identification of the E_2 -term in Section 6.2 as derived functors of $\mathcal{Q}_D$ will give the following form to the spectral sequence when it is applied to $X_\bullet = \overline{W}\Lambda_\bullet$:

$$E_2^{s,t} = \text{Tor}_{\underline{\mathbf{UD}}}^s(\mathbf{F}_2, Q\text{Tor}_\Lambda^t(\mathbf{F}_2, \mathbf{F}_2)) \implies H_{s+t}^Q \overline{W}\Lambda_\bullet.$$

Then since $H_r^Q \overline{W}\Lambda_\bullet \cong H_{r-1}^Q \Lambda_\bullet \cong L_{r-1}^S Q\Lambda_\bullet$, we will be able to study $L_*^S Q\Lambda$ by studying a chain complex for $\text{Tor}_{\underline{\mathbf{UD}}}^s(\mathbf{F}_2, Q\text{Tor}_\Lambda^t(\mathbf{F}_2, \mathbf{F}_2))$. This is carried out in Section 6.4.

6.2 Identification of the E_2 -Term

In this section we identify the E_1 and E_2 -terms of (6.1).

The homotopy of a simplicial algebra is a commutative (bigraded) $\mathbf{F}_2$ -algebra equipped with operations defined by A. K. Bousfield and W. Dwyer. It is possible to express E_1 and E_2 in terms of $\pi_* X_\bullet$ and these operations. We begin by stating some properties of the operations and by setting up appropriate categories in which to study them.

Theorem 6.2 ([G, Theorem 5.1 and Section 6] or [Dw]) *Let $X_\bullet \in \underline{\mathbf{sA}}$. There exist natural operations*

$$\delta_i : \pi_n X_\bullet \rightarrow \pi_{n+i} X_\bullet$$

for $2 \leq i \leq n$ which double internal degree and satisfy the following properties:

1. δ_i is a homomorphism for $2 \leq i < n$ and $\delta_n(x + y) = \delta_n x + \delta_n y + xy$.
2. $\delta_i(xy) = x^2 \delta_i y$ if $x \in \pi_0 X_\bullet$. If $x \notin \pi_0 X_\bullet$, $y \notin \pi_0 X_\bullet$, then $\delta_i(xy) = 0$.
3. For $j < 2i$ and $x \in \pi_* X_\bullet$,

$$\delta_j \delta_i(x) = \sum_{k \leq i+j/3} \binom{i-j+k-1}{i-k} \delta_{i+j-k} \delta_k(x)$$

Remarks:

1. Theorem 6.2(1) implies $x^2 = 0$ if $x \in \pi_n X_\bullet$ where $n \geq 2$. This is also true if $x \in \pi_1 X_\bullet$, but for a different reason. [G, Lemma 6.2]
2. Studying $\pi_* X_\bullet$ as a commutative algebra with Bousfield-Dwyer operations is analagous to studying the cohomology of a topological space as a commutative algebra with Steenrod operations. The definitions which follow parallel the definitions of the Steenrod algebra and the categories of unstable modules and algebras over it.

Definition 6.3 *Let D be the associative graded $\mathbf{F}_2$ -algebra generated by the symbols δ_i ($i \geq 2$) of degree i with the relations*

$$\delta_j \delta_i = \sum_{k \leq i+j/3} \binom{i-j+k-1}{i-k} \delta_{i+j-k} \delta_k.$$

Let $\underline{\mathbf{UD}}$, the category of unstable D -modules, be the category of bigraded $\mathbf{F}_2$ -vector spaces with a left action of δ_i doubling internal degree such that for $M \in \underline{\mathbf{UD}}$,

1. $\delta_i : M_{n,t} \rightarrow M_{n+i,2t}$ is a homomorphism;
2. δ_{n+i} is zero on $M_{n,t}$ for $i > 0$.

Let $\underline{\mathbf{DA}}$, the category of unstable algebras over D , be the category of commutative bigraded $\mathbf{F}_2$ -algebras with a left action of D satisfying the properties of Theorem 6.2 and such that if $A \in \underline{\mathbf{DA}}$ then δ_{n+i} is zero on $A_{n,t}$ for $i > 0$. Morphisms in $\underline{\mathbf{DA}}$ and $\underline{\mathbf{UD}}$ are required to commute with the action of D , and in $\underline{\mathbf{DA}}$ with the algebra structure.

Let $\underline{\mathbf{nnF}}_2$ be the category of bigraded $\mathbf{F}_2$ -vector spaces, and let $\Theta : \underline{\mathbf{DA}} \rightarrow \underline{\mathbf{DA}}$ be the composition of the augmentation ideal functor $I : \underline{\mathbf{DA}} \rightarrow \underline{\mathbf{nnF}}_2$ with its left adjoint. There is a cotriple structure on Θ , since it arises from a pair of adjoint functors. We will identify the E_2 -term of (6.1) as a derived functor with respect to Θ . Let $Q : \underline{\mathbf{DA}} \rightarrow \underline{\mathbf{UD}}$ take indecomposables, and let $\mathbf{F}_2$ denote the unstable D -module which is $\mathbf{F}_2$ concentrated in bidegree $(0,0)$. Let $\mathcal{Q}_D$ be the composite

$$\mathcal{Q}_D = (\mathbf{F}_2 \otimes_D -) \circ Q : \underline{\mathbf{DA}} \rightarrow \underline{\mathbf{nnF}}_2.$$

Then the spectral sequence of (6.1) has the following form.

Theorem 6.4 ([M, Remark 4.7], [G, Theorem 10.4]) *There is a convergent spectral sequence natural in $X_\bullet \in \underline{\mathbf{sA}}$:*

$$E_2^{p,q} = L_p^\Theta \mathcal{Q}_D(\pi_q X_\bullet) \implies H_*^Q X_\bullet.$$

We give a brief justification of this result. Recall that the spectral sequence was obtained by filtering the bicomplex $E_0^{p,q} = QS^{p+1}X_q$ by degree in p . Thus we can identify the E_1 -term as

$$E_1^{p,*} = \pi_* QS^{p+1}X_\bullet.$$

Theorem 6.4 now follows by application of the following two lemmas.

Lemma 6.5 ([G, 10.3]) *Let V be a simplicial vector space, and let $Sy(V)$ be the simplicial algebra which is the symmetric algebra on each level of V . Then*

$$\pi_* Q(Sy(V)) \cong \mathcal{Q}_D \pi_*(Sy(V))$$

Application of this lemma gives the following form of E_1 :

$$E_1^{p,*} = \mathcal{Q}_D \pi_* S^{p+1}X_\bullet.$$

Lemma 6.6 [Do, 4.2, 5.6] *Let $X_\bullet \in \underline{\mathbf{sA}}$ and let $SX_\bullet$ be the simplicial algebra obtained by applying $S = Sy \circ I$ to $X_\bullet$ levelwise. Then*

$$\pi_* SX_\bullet \cong \Theta(\pi_* X_\bullet).$$

Theorem 6.4 now follows by the definition of $L_*^\Theta \mathcal{Q}_D$, since we can use the preceding lemma to write

$$E_1^{p,*} = \mathcal{Q}_D \Theta^{p+1}(\pi_* X_\bullet).$$

6.3 A Spectral Sequence for $L_*^\ominus \mathcal{Q}_D$

This section describes the composite functor spectral sequence for $L_*^\ominus \mathcal{Q}_D$ and explains how it comes to have a particularly nice form when applied to the suspension of a constant algebra, $\overline{W}\Lambda_\bullet$.

Let $A \in \underline{\mathbf{DA}}$. The functor $\mathcal{Q}_D$ is a composite,

$$\mathcal{Q}_D : \underline{\mathbf{DA}} \xrightarrow{Q} \underline{\mathbf{UD}} \xrightarrow{\mathbf{F}_2 \otimes_D -} \underline{\mathbf{nnF}_2},$$

and satisfies the conditions of [M, Proposition 2.13], so there is a composite functor spectral sequence

$$\mathrm{Tor}_{\underline{\mathbf{UD}}}^i(\mathbf{F}_2, L_t^\ominus \mathcal{Q}_D A) \implies L_{s+t}^\ominus \mathcal{Q}_D A.$$

It turns out that this composite functor spectral sequence has only one line in the case that the commutative algebra A is the algebra underlying a Hopf algebra.

Proposition 6.7 ([G, Proposition 10.8]) *Let $A \in \underline{\mathbf{DA}}$ be the algebra underlying a Hopf algebra, and suppose that $A_0 = \mathrm{Sym}(W)$ for some graded vector space W . Let $Q : \underline{\mathbf{DA}} \rightarrow \underline{\mathbf{UD}}$ take indecomposables. Then for $t > 0$,*

$$L_t^\ominus \mathcal{Q}_D A = 0.$$

The reason for this, briefly, is that an algebra in $\underline{\mathbf{DA}}$ which is the underlying algebra of a Hopf algebra is exterior, and hence free in the category of algebras with $x^2 = 0$. The D -action turns out not to matter to the calculation of $L_s^\ominus \mathcal{Q}_D A$, and hence the freeness of the algebra structure makes the higher derived functors of $Q : \underline{\mathbf{DA}} \rightarrow \underline{\mathbf{UD}}$ vanish.

If $A = \pi_* X_\bullet \in \underline{\mathbf{DA}}$ satisfies the hypotheses of Proposition 6.7, the Miller spectral sequence of Theorem 6.4 becomes

$$\mathrm{Tor}_{\underline{\mathbf{UD}}}^i(\mathbf{F}_2, Q\pi_* X_\bullet) \implies H_{s+t}^Q X_\bullet. \quad (6.8)$$

None of this helps very much in the computation of $L_*^S \mathcal{Q}_D \Lambda = H_*^Q \Lambda_\bullet$, since $\pi_* \Lambda_\bullet$ is not a Hopf algebra. However, the category $\underline{\mathbf{sA}}$ comes equipped with

a suspension functor defined as follows for $X_\bullet \in \underline{\mathbf{sA}}$:

$$\begin{aligned}
(\overline{W}X_\bullet)_n &= X_{n-1} \otimes X_{n-2} \otimes \dots \otimes X_0. \\
d_i(x_{n-1} \otimes \dots \otimes x_0) &= \begin{cases} d_0 x_{n-1} \otimes \dots \otimes d_0 x_2 \otimes (d_0 x_1)(\eta \epsilon x_0) & \text{if } i = 0; \\
d_i x_{n-1} \otimes \dots \otimes d_i x_{i+1} \otimes (d_i x_i) x_{i-1} & \text{if } 0 < i < n; \\
\otimes x_{i-2} \otimes \dots \otimes x_0 & \\
0 & \text{if } i = n. \end{cases} \\
s_i(x_{n-1} \otimes \dots \otimes x_0) &= s_i x_{n-1} \otimes \dots \otimes s_i x_i \otimes 1 \otimes x_{i-1} \otimes \dots \otimes x_0.
\end{aligned}$$

The simplicial algebra $\overline{W}X_\bullet$ deserves to be called a suspension because

$$H_n^Q \overline{W}X_\bullet \cong H_{n-1}^Q X_\bullet.$$

[M, Corollary 5.6]. When $X_\bullet = \Lambda_\bullet$ then $\overline{W}\Lambda_\bullet$ is the reduced bar construction on Λ , and thus $\pi_* \overline{W}\Lambda_\bullet = \text{Tor}_\Lambda^*(\mathbf{F}_2, \mathbf{F}_2)$ is a Hopf algebra. Hence form (6.8) of Miller's spectral sequence applies, and

$$E_2^{s,t} = \text{Tor}_{\underline{\mathbf{UD}}}^s(\mathbf{F}_2, Q\text{Tor}_\Lambda^t(\mathbf{F}_2, \mathbf{F}_2)) \implies H_{s+t}^Q \overline{W}\Lambda_\bullet \cong L_{s+t-1}^S Q\Lambda.$$

This form of the spectral sequence appeared in [M3].

The following section descusses a Koszul resolution to compute the E_2 -term and gives the proof of Theorem 5.7.

Remark: The definition of $\overline{W}X_\bullet$ given in [M, Section 5] was incompatible with the corrections made to [M] in [M2]. The definition of $\overline{W}X_\bullet$ we give here is the “opposite” construction and is called $\overline{W}'X_\bullet$ in [M2].

6.4 Proof of Theorem 5.7

In Section 6.3 we explained that when the spectral sequence for $H_*^Q X_\bullet$ is applied to $X_\bullet = \overline{W}\Lambda_\bullet$, the suspension of the constant simplicial algebra $\Lambda_\bullet$, it has the form:

$$E_2^{s,t} = \text{Tor}_{\underline{\mathbf{UD}}}^s(\mathbf{F}_2, Q\text{Tor}_\Lambda^t(\mathbf{F}_2, \mathbf{F}_2)) \implies H_{s+t}^Q \overline{W}\Lambda_\bullet \cong L_{s+t-1}^S Q\Lambda.$$

In this section we use this to prove Theorem 5.7, which we restate here.

Theorem 5.7 *Let $\Lambda \in \underline{\mathbf{A}}$ be $c-1$ connected. Then $L_r^S Q\Lambda$ has connectivity*

$$\begin{array}{ll} rc - 1 & \text{if } 4 \leq r \\ (r+1)c - 1 & \text{if } 0 \leq r \leq 3. \end{array}$$

The proof goes by computing a lower bound on the connectivity of $E_2^{s,t}$ and taking the minimum over $s+t = r+1$ to get a lower bound on the connectivity of $L_r^S Q\Lambda$. To compute the connectivity of $E_2^{s,t}$ we use the fact that the algebra D of Bousfield-Dwyer is a Koszul algebra [P]. For $M \in \underline{\mathbf{UD}}$, work of S. Priddy [P] applied by P. Goerss [G, Section 11] gives an explicit chain complex whose homology is $\text{Ext}_{\underline{\mathbf{UD}}}^*(M, \mathbf{F}_2)$, the $\mathbf{F}_2$ -dual of $\text{Tor}_{\underline{\mathbf{UD}}}^*(\mathbf{F}_2, M)$. We describe this chain complex and calculate its connectivity for $M = Q\text{Tor}_{\Lambda}^*(\mathbf{F}_2, \mathbf{F}_2)$ below. We refer the reader to [G, Section 11] for proofs regarding the chain complex. Define the bigraded algebra Γ to be the tensor algebra on symbols γ_i ($i \geq 2$) of bidegree $(1, i)$ modulo the relations for $j \geq 2i$

$$R(j, i) = \gamma_j \gamma_i + \sum_{j-i+1}^{j+i-2} \binom{2s-j-1}{s-i} \gamma_{i+j-s} \gamma_s.$$

Call a monomial $\gamma_I = \gamma_{i_1} \dots \gamma_{i_s}$ allowable if $i_t < 2i_{t+1}$ for $1 \leq t \leq s-1$.

Lemma 6.9 ([G, Lemma 11.3]) *The set of allowable monomials in the γ_i is a vector space basis for Γ .*

If $M \in \underline{\mathbf{UD}}$, then M has a left action of D raising degree

$$\delta_i : M_{n,t} \rightarrow M_{n+i,2t}$$

which is zero if $i > n$. The vector space dual M^* has a right action of D which lowers degree

$$\delta_i : M_{n,t}^* \rightarrow M_{n-i,t/2}^*$$

which is zero if $2i > n$. Define a chain complex $\Gamma \otimes M^*$

$$(\Gamma \otimes M^*)_s = (\text{Span } \{\gamma_{i_1} \dots \gamma_{i_s}\}) \otimes M^*$$

with the differential $d : (\Gamma \otimes M^*)_s \rightarrow (\Gamma \otimes M^*)_{s+1}$

$$d(\gamma_I \otimes x) = \sum_{i \geq 2} \gamma_I \gamma_i \otimes x \delta_i.$$

A calculation shows that $d^2 = 0$, so d really is a differential [G, Lemma 11.4]. Now we need to make $\Gamma \otimes M^*$ unstable. If $x \in M_{n,t}^*$, we say $\deg(x) = n$. Define

$$(\Gamma \underline{\otimes} M^*)_s = \text{Span} \{ \gamma_{i_1} \cdots \gamma_{i_s} \otimes x \mid i_s > \deg(x) \} \subseteq (\Gamma \otimes M^*)_s.$$

Then $\Gamma \underline{\otimes} M^*$ is a subcomplex of $\Gamma \otimes M^*$ [G, Lemma 11.6], and we let $\Gamma(M^*)$ be the quotient chain complex $\Gamma \otimes M^* / \Gamma \underline{\otimes} M^*$.

Proposition 6.10 ([G, Proposition 11.8]) *There is a natural isomorphism*

$$H^* \Gamma(M^*) \cong \text{Ext}_{\underline{\mathbf{UD}}}^*(M, \mathbf{F}_2).$$

Since we are interested in the internal grading of $\text{Ext}_{\underline{\mathbf{UD}}}^*(M, \mathbf{F}_2)$ for $M = Q\text{Tor}_{\Lambda}^t(\mathbf{F}_2, \mathbf{F}_2)$, we make explicit the gradings in $\Gamma(M^*)$. If $m \in (M^*)_{p,q}$, then $\gamma_{i_1} \cdots \gamma_{i_s} \otimes m$ lies in degree $(s, t, 2^s q)$ where $t = p + i_1 + \cdots + i_s$, and thus contributes to $\text{Ext}_{\underline{\mathbf{UD}}}^*(M^t, \mathbf{F}_2)$ an element of internal degree $2^s q$. Notice that the differential raises the homological degree by one, and leaves the other degrees unchanged.

Proposition 6.11 *Let $M \in \underline{\mathbf{UD}}$ have the property that $M_{p,q} = 0$ for $q < cp$ for a fixed constant c . Then*

$$\text{Ext}_{\underline{\mathbf{UD}}}^s(M^t, \mathbf{F}_2)$$

has connectivity $(t - s + 2^s - 1)c - 1$.

Corollary 6.12 *Let $\Lambda \in \underline{\mathbf{A}}$ be $c - 1$ connected. Then*

$$\text{Ext}_{\underline{\mathbf{UD}}}^s(Q\text{Tor}_{\Lambda}^t(\mathbf{F}_2, \mathbf{F}_2), \mathbf{F}_2)$$

is $(t - s + 2^s - 1)c - 1$ connected.

Proof of Proposition 6.11: Consider a non-zero element $\gamma_{i_1} \dots \gamma_{i_s} \otimes m$ of $\Gamma(M^*)_{s,t}$, where $\gamma_{i_1} \dots \gamma_{i_s}$ is allowable and $m \in (M^*)_{p,q}$. Since $i_s \leq p$ and each $i_j < 2i_{j+1}$,

$$\begin{aligned} \Sigma i_j &\leq (2^{s-1}p - 2^{s-1} + 1) + \dots + (4p - 3) + (2p - 1) + p \\ &= (2^s - 1)p - (2^s - 1) + s. \end{aligned}$$

Also, since $\gamma_{i_1} \dots \gamma_{i_s} \otimes m \in \Gamma(M^*)_{s,t}$ we know that

$$\begin{aligned} t &= p + \Sigma i_j \\ &\leq p + (2^s p - p - 2^s + 1 + s) \\ &= 2^s p - 2^s + s + 1. \end{aligned}$$

Solving for p we get $p \geq 1 + (t - s - 1)/2^s$. But $q \geq cp$ by hypothesis, and the internal degree of $\gamma_{i_1} \dots \gamma_{i_s} \otimes m$ is

$$2^s q \geq 2^s pc \geq (t - s - 1 + 2^s)c.$$

□

Proof of Theorem 5.7: Recall the spectral sequence

$$E_2^{s,t} = \text{Tor}_{\underline{\mathbf{UD}}}^s(\mathbf{F}_2, Q\text{Tor}_{\Lambda}^t(\mathbf{F}_2, \mathbf{F}_2)) \implies H_{s+t}^Q \overline{W}\Lambda_{\bullet} \cong L_{s+t-1}^S Q\Lambda_{\bullet}.$$

By Corollary 6.12, $E_2^{s,t}$ is $(t - s + 2^s - 1)c - 1$ connected, for its $\mathbf{F}_2$ -dual is. Minimizing over $s + t = r + 1$ gives the result.

□

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