Analysis and Interventions in Large Network Games

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Abstract

We review classic results and recent progress on equilibrium analysis, dynamics, and optimal interventions in network games with both continuous and discrete strategy sets. We study strategic interactions in deterministic networks as well as networks generated from a stochastic network formation model. For the former case, we review a unifying framework for analysis based on the theory of variational inequalities. For the latter case, we highlight how knowledge of the stochastic network formation model can be used by a central planner to design interventions for large networks in a computationally efficient manner when exact network data is not available.

1 INTRODUCTION

In many social or economic settings, decisions of individuals are affected by the actions of their friends, colleagues, peers, or competitors. For example, the decision of an individual to buy a new product, adopt new ideas or innovations, commit a crime, find a job, or contribute to a public good is influenced by the choice of his friends or acquaintances. Network games have emerged as a powerful framework for the formal analysis of such interactions by providing a model for settings where a large number of agents interact with each other according to an underlying network represented by a graph. The restriction with respect to a general game formulation is that in a network game, a player's payoff does not depend on the strategy of all other players, but only on the aggregated actions of the agents in his neighborhood. Such a framework is general enough to nest many setups of practical interest and allows for diverse heterogeneities in the way agents interact with each other.

In this chapter we provide a systematic review and unified analysis of network games for both deterministic and stochastic networks. In each case, we characterize the equilibrium, study dynamics and provide results on targeted interventions from the literature. We focus on games with both continuous and discrete strategy sets and show that their analysis requires different methodologies and tools.

Specifically, we begin the chapter by reviewing some classical network game models and their connections to well known dynamics over deterministic networks. After presenting some widely used models with specific structure from the literature,[1] we discuss a *unifying framework* based on the theory of variational inequalities. Such a framework enables the analysis of a broader class of network games with vector strategies and mixed strategic interactions.

We then turn to analysis of interactions in very large networks. In this scenario two problems emerge from the perspective of a central planner. First collecting data about the exact network

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[1] More details on games with special structure, either in terms of payoff or strategic interactions with monotonicity properties, can be found in Jackson (2010); Easley et al. (2010); Jackson and Zenou (2014); Bramoullé and Kranton (2016); Bullo (2019).

of interactions becomes very expensive or not at all possible because of privacy and proprietary concerns. Second, methods for designing optimal interventions that rely on the exact network structure typically do not scale well with the population size. To obviate these issues, we consider networks generated from a *stochastic network formation model*, which we assumed is known to the central planner. By discussing several results from the literature that use such a framework, we emphasize how models based on stochastic networks provide an effective framework for policy intervention by a central planner that has limited information about exact network interactions and we stress the computational advantages of policy analysis in stochastic network environments.

2 DETERMINISTIC NETWORKS

In this section we consider games where there is a fixed number of agents N which interact according to a given network of interactions. From here on we denote with $P \in \mathbb{R}^{N\times N}$ the adjacency matrix of such a network, with the interpretation that $P_{ij} \in [0,1]$ denotes the influence that agent j has on agent i. We assume no self loops, hence $P_{ii} = 0$ for all i, we denote by $\mathcal{N}^i := \{j \mid P_{ij} \neq 0\}$ the set of neighbors of agent i and by $d^i := \sum_j P_{ij}$ his degree. We say that the network is undirected if $P_{ij} = P_{ji}$ for all i, j, otherwise it is directed.

Each agent has a strategy space $\mathcal{S}^i$, which could be either a finite set or a subset of a Euclidean space (we discuss both cases in the next subsections). The objective of each agent is to select a feasible (pure) strategy $s^i \in \mathcal{S}^i$ to maximize a payoff function[2]

$$U^i(s^i, z^i(s)) \tag{1}$$

which depends on the strategy of agent i and the *network aggregate*

$$z^i(s) = \sum_{j=1}^{N} P_{ij} s^j,$$

where $s = [s^1; \dots; s^N]$. Note that the payoff of each agent depends on the choices of the other agents, hence each agent aims at computing his best response to others actions[3]

$$B^i(z^i(s)) := \arg\max_{s^i \in \mathcal{S}^i} \quad U^i(s^i, z^i(s)), \tag{2}$$

where we recall that $z^i(s)$ does not depend on s^i since $P_{ii} = 0$. A set of strategies where each agent is playing a best response to others agents' strategies is a Nash equilibrium.

Definition 1 (Nash equilibrium) *A set of strategies $\{\bar{s}^i\}_{i=1}^N$ is a Nash equilibrium if for all $i \in \{1, \dots, N\}$, $\bar{s}^i \in \mathcal{S}^i$ and $U^i(\bar{s}^i, z^i(\bar{s})) \geq U^i(s^i, z^i(\bar{s}))$ for all $s^i \in \mathcal{S}^i$.*

Intuitively, a Nash equilibrium is a strategy profile where no agent has a unilateral profitable deviation. To motivate our analysis, we next introduce two examples of network games.

Example 1 (Local public good game) *Consider a game where each agent needs to decide how much effort to invest in a local public good. This could represent for example a consumer's research into new products, farmer investments in new agricultural techniques or research and development efforts in industry. Following Bramoullé and Kranton (2007), we assume that each agent selects a non-negative effort level s^i, so that $\mathcal{S}^i = \mathbb{R}_{\geq 0}$, and has a constant marginal cost which we denote by b. We also assume that effort from an agent is perfectly substitutable with efforts from his neighbors, so that $P_{ij} = 1$ for all $j \in \mathcal{N}^i$ and the return on investment depends on $s^i + z^i(s)$. Each agent's payoff is then $U^i(s^i, z^i(s)) = f(s^i + z^i(s)) - bs^i$ where f is a return of investment function, typically strictly increasing and concave.*

[2]In this chapter we focus always on pure strategies, therefore from here on we refer to pure strategies simply as strategies.

[3]We use the term strategy and action interchangeably.

Example 2 (Homogeneous coordination game) *Consider a game where* $P \in \{0,1\}^{N\times N}$ *and each agent needs to select among two strategies, so that* $\mathcal{S}^i = \{0,1\}$.

This framework could model settings where an agent needs to decide whether to adopt a certain innovation or behavior, buy a certain product or participate in a certain activity. We assume that neighboring agents have an incentive to match their strategies. Specifically, neighbors i, j *play a coordination game according to the payoff matrix on the right, for some* $a, b > 0$.

		i: 1	*i*: 0
j	1	a,a	0,0
	0	0,0	b,b

Note that $z^i = \sum_j P_{ij} s^j$ *in this case corresponds to the number of neighbors of* i *playing strategy* 1. *The overall payoff experienced by agent* i *is then* $U^i(s^i, z^i) = s^i a z^i + (1-s^i) b(d^i - z^i)$, *where we recall that* d^i *denotes agent* i*'s degree. Agent* i*'s best response is then to choose strategy* 1 *if and only if* $U^i(1, z^i) \geq U^i(0, z^i)$ *leading to*

$$az^i \geq b(d^i - z^i) \quad \Leftrightarrow \quad \tfrac{z^i}{d^i} \geq \tfrac{b}{a+b} =: \theta \tag{3}$$

In other words, the best response of an agent is a threshold policy: agent i *selects strategy* 1 *if and only if at least* θ *fraction of his neighbors select strategy* 1.

Note that in Example 1 each agent selects a strategy from the Euclidean set $\mathbb{R}_{\geq 0}$, while in Example 2 each agent selects a strategy from the discrete set $\{0,1\}$. We show how we can analyze network games with continuous and discrete strategies in the next two subsections. Before focusing on network games, we review in Sidebar 1 two widely used structural properties for generic games that enable tractable equilibrium analysis, we refer to Dubey et al. (2006); Jackson and Zenou (2014); Menache and Ozdaglar (2011); Marden and Shamma (2018) for more details.

2.1 Continuous strategies

We first consider *"linear-quadratic"* network games where each agent has a payoff that is quadratic in his own strategy and linear in the network aggregate.[4] Indeed, this simple parametric form enables an explicit characterization of equilibrium strategies and allows a tractable analysis of the impact of network structure. The public good game in Example 1 with quadratic return function is a special case of such games.

Linear quadratic games:

Consider a network game where each agent chooses a scalar non-negative strategy $s^i \in \mathcal{S}^i = \mathbb{R}_{\geq 0}$ (representing for instance how much effort he exerts on a specific activity) to maximize the linear quadratic payoff function

$$U^i(s^i, z^i(s)) = -\frac{1}{2}(s^i)^2 + [a^i z^i(s) + b^i] s^i, \tag{4}$$

where $a^i \in \mathbb{R}$ captures the impact of the network aggregate $z^i(s)$ on the payoff of agent i and $b^i \in \mathbb{R}$ is the standalone marginal return. For these games, the best response of agent i is given by a (truncated) affine function of the network aggregate $z^i(s)$:

$$B^i(z^i(s)) := \max\{0, a^i z^i(s) + b^i\}. \tag{5}$$

An interesting class of games is obtained when the sign of the payoff parameters $\{a^i\}_{i=1}^N$ is positive (or negative), since in this case the game exhibits increasing (decreasing) payoff differences, as discussed in Sidebar 1, and the best response is a monotone increasing (decreasing) function

[4] We here present a selection of results needed to build intuition: A more detailed survey can be found in Jackson and Zenou (2014); Bramoullé and Kranton (2016).

SIDEBAR 1: GAMES WITH SPECIAL STRUCTURE

In this sidebar we consider generic games (not necessarily network games) hence we consider payoffs of the form $U^i(s^i, s^{-i})$, where s^{-i} denotes the strategies of all agents except for agent i, instead of $U^i(s^i, z^i(s))$.

Supermodular games

When the strategy sets are lattices, one can study properties of equilibria by using lattice theory and monotonicity results in lattice programming, as discussed in detail in Vives (2005). We quickly review such a framework for the case of scalar strategies so that $\mathcal{S}^i$ is a subset of $\mathbb{R}$. To this end, we introduce the following monotonicity property of the payoff function.

Definition 2 (Increasing (decreasing) differences) *A game exhibits increasing (decreasing) payoff differences if for all $i \in \{1, \ldots, N\}$, $\tilde{s}^i \geq s^i$ and $\tilde{s}^{-i} \geq s^{-i}$ it holds*

$$U^i(\tilde{s}^i, \tilde{s}^{-i}) - U^i(s^i, \tilde{s}^{-i}) \ \geq (\leq) \ U^i(\tilde{s}^i, s^{-i}) - U^i(s^i, s^{-i}),$$

that is, an increase in the strategies of the other players raises (decreases) the desirability of playing a higher strategy for player i.

If U^i is twice continuously differentiable an equivalent condition for increasing payoff differences is $\frac{\partial^2 U^i(s^i, s^{-i})}{\partial s^i, s^j} \geq 0$. Games that exhibit increasing (decreasing) payoff differences are also termed games of **strategic complements (substitutes)** and, in the scalar case, are **supermodular (submodular)** games. Note that Example 1 is a game of strategic substitutes, while Example 2 is a game of strategic complements.

Supermodular games are of particular interest because, under suitable continuity assumptions, existence of a pure strategy equilibrium can be obtained without requiring the quasi-concavity of the payoff functions, the equilibrium set is a lattice with a smallest and a largest element and the best response has underlying monotonicity properties that enable sharp comparative statics results, see Milgrom and Roberts (1994); Milgrom and Shannon (1994). While we here considered the scalar case, we quickly note that such a theory can be extended to vector strategies (under the assumption that $\mathcal{S}^i$ is a complete lattice). In this case to define a supermodular game one needs to also guarantee complementarity among components of an agent strategy, see (Fudenberg and Tirole, 1991, Section 12.3), Topkis (2011).

Potential games

A game admits an exact potential if there exists a function $\Phi(s)$ such that $U^i(s^i, s^{-i}) - U^i(\tilde{s}^i, s^{-i}) = \Phi(s^i, s^{-i}) - \Phi(\tilde{s}^i, s^{-i})$ for all $i \in \{1, \ldots, N\}$, $s^i, \tilde{s}^i \in \mathcal{S}^i$ and $s^{-i} \in \mathcal{S}^{-i}$. In this case, one can connect Nash equilibria to the stationary points of the potential function and use optimization theory results to study equilibria and dynamics. For example, the Nash equilibrium exists and is unique if the $\mathcal{S}^i$ are non-empty, closed and convex and Φ is strongly concave, since in this case Φ has a unique stationary point (the maximum). We refer the interested reader to Monderer and Shapley (1996) for more details.

of other agents actions. Much of the literature has exploited such monotonicity properties to study equilibrium properties and dynamics as a function of the network structure. We next review results specific to network games with linear quadratic structure. We focus for simplicity on *homogeneous games* for which $a^i = a, b^i = b$ for all $i \in \{1, \dots, N\}$.

Linear quadratic network games of strategic complements ($a > 0$) have been studied for example by Ballester et al. (2006). A key result of this work is that under strategic complements and if $b > 0$, the equilibrium is internal, that is, $\bar{s}^i > 0$ for all $i \in \{1, \dots, N\}$. It then follows from (5) that $\bar{s}^i = az^i(\bar{s}) + b^i$ and the equilibrium condition can be expressed in matrix form as $\bar{s} = aP\bar{s} + b1_N$ or equivalently, $(I - aP)\bar{s} = b1_N$, where $1_N \in \mathbb{R}^N$ is the vector of all ones. Under the assumption

$$a\lambda_{\max}(P) < 1, \tag{6}$$

$(I - aP)$ is invertible and we obtain $\bar{s} = b(I - aP)^{-1}1_N$, that is the unique Nash equilibrium is proportional to the vector of *Bonacich centralities*, see Bonacich (1987), which is defined as $\bar{c} := (I - aP)^{-1}1_N$ and is a measure of importance of a node in a network in terms of number and (discounted) length of outgoing walks. To see this connection, note that under (6) the Bonacich centrality of agent i can be expressed as

$$\bar{c}_i = \sum_{j=1}^{N} \sum_{k=0}^{\infty} (a^k P^k)_{ij}$$

and P^k_{ij} is the weight of walks from i to j of length k. This is thus a simple case where the network position determines equilibrium play.

Linear quadratic network games of strategic substitutes ($a < 0$) have been studied for example by Bramoullé et al. (2014), using the theory of potential games, see Sidebar 1. As derived in Monderer and Shapley (1996), a necessary condition for the existence of an exact potential function is

$$\frac{\partial^2 U^i(s^i, s^{-i})}{\partial s^i s^j} = \frac{\partial^2 U^j(s^j, s^{-j})}{\partial s^j s^i}. \tag{7}$$

For linear quadratic homogeneous network games, this amounts to the restriction that $P_{ij} = P_{ji}$, which holds if and only if the network is undirected. Under this assumption Bramoullé et al. (2014), show that if

$$|a||\lambda_{\min}(P)| < 1 \tag{8}$$

the potential function $\Phi(s) = -\frac{s^\top (I - aP)s}{2} + bs$ is strongly concave, guaranteeing existence and uniqueness of the Nash equilibrium. We note that condition (8) is less restrictive than condition (6) since by Perron-Frobenious theorem $|\lambda_{\min}(P)| < \lambda_{\max}(P)$.

The results discussed so far focused on games with affine best responses. A few recent papers extended these results to games with nonlinear best responses while maintaining the assumption of strategic complements, see e.g., Belhaj and Deroïan (2014); Acemoglu et al. (2015), or strategic substitutes, see e.g., Allouch (2015). In the following subsections, we instead review a general approach based on the theory of variational inequalities, that enables the analysis of more general network games with nonlinear best responses, vector strategies and mixed strategic effects. We divide the exposition into three main sections: analysis of equilibria, dynamics and interventions.

2.1.1 Equilibria

We start our discussion with two motivating examples. First we describe a variation of the linear quadratic model that features a non monotone best response.

Example 3 ***(Races and tournaments)*** *Consider a network game where each player has a scalar strategy $s^i \in [s_L, s_H]$ (with $0 < s_L < s_H$) and a nonlinear best response*

$$B^i(z^i(s)) = \min\{s_L + \phi(z^i(s)), s_H\},$$

with $\phi(0) = 0$ and $\phi(z^i) \geq 0$. Special cases of this model have been considered in the literature for example in Belhaj and Deroïan (2014) with the additional assumption $\phi' \geq 0$ (so that the best response is increasing in other agents actions) or in Allouch (2015) with $\phi' \leq 0$ (so that the best response is decreasing in other agents actions).

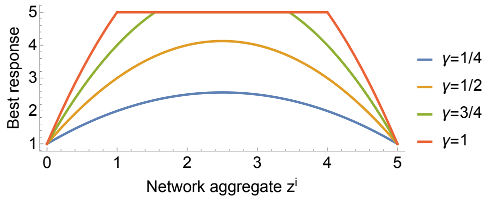

Figure 1: $\phi(z^i) = \gamma z^i(s_H - z^i)$, $s_L = 1$, $s_H = 5$ and different values of γ.

We instead do not impose any monotonicity assumption on ϕ and focus on cases where the sign of ϕ' may change. For example one could consider ϕ as in Figure 1. The corresponding best response function is non-monotone and can be used to model races and tournaments: In the initial phase, when $z_i \leq \frac{s_H}{2}$, the player's increasing effort motivates a neck-to-neck race which is then followed by a second phase, when $z_i \geq \frac{s_H}{2}$, where agent's effort level declines capturing a discouragement effect. This is an example of network games with mixed strategic interactions.

In the examples discussed so far agents had scalar strategies. Our second motivating example features agents that engage in multiple activities, as studied in Chen et al. (2018); Belhaj and Deroïan (2014).

Example 4 ***(Multiple activities in networks)*** *Consider a network game where each player i has a strategy vector $s^i = [s^i_A, s^i_B] \in \mathbb{R}^2_{\geq 0}$ with s^i_A, s^i_B representing his level of engagement in two interdependent activities A and B, such as crime and education. Each agent selects his level of engagement in activities A and B to maximize the following payoff*

$$\underbrace{b^i_A s^i_A - \frac{1}{2}(s^i_A)^2 + \delta s^i_A z^i_A(s)}_{\text{net proceeds from activity A}} + \underbrace{b^i_B s^i_B - \frac{1}{2}(s^i_B)^2 + \delta s^i_B z^i_B(s)}_{\text{net proceeds from activity B}} + \underbrace{\mu s^i_A z^i_B(s) + \mu s^i_B z^i_A(s) - \beta^i s^i_A s^i_B}_{\text{interdependence of activities}},$$

where the parameter δ represents the effect of the network aggregate within each activity, μ represents the effect of the network aggregate across different activities and β^i captures the interdependence of the two activities for each agent i. Chen et al. (2018), focus on the case $\delta > 0$ so that the effort of each agent and his neighbors are strategic complements within each activity, while $\beta^i \in (-1, 1)$ can be negative (modeling two complementary activities such as crime and drug use) or positive (modeling two substitutable activities such as crime and education).

To study these more general interactions a number of recent papers suggested the use of variational inequalities (see e.g., Ui (2016); Melo (2017); Naghizadeh and Liu (2017) which apply this approach to special classes of network games). We here review the unifying framework presented in Parise and Ozdaglar (2019b) under the following assumption.

Assumption 1 *The strategy set $\mathcal{S}^i \subseteq \mathbb{R}^n$ is nonempty, closed and convex for all $i \in \{1 \ldots, N\}$. The function $U^i(s^i, z^i(s))$ is continuously differentiable and concave in s^i for all $i \in \{1 \ldots, N\}$ and for all $s^j \in \mathcal{S}^j$, $j \in \mathcal{N}^i$. Moreover, $U^i(s^i, z^i)$ is twice differentiable in $[s^i; z^i]$ and $\nabla_{s^i} U^i(s^i, z^i)$ is Lipschitz in $[s^i; z^i]$.*

At a high level variational inequalities can be used to capture simultaneously a set of optimization problems belonging to multiple agents through their optimality conditions. To see this, consider the optimization problem faced by each agent at a Nash equilibrium:

$$\bar{s}^i = \arg\max_{s^i \in \mathcal{S}^i} \quad U^i(s^i, z^i(\bar{s})). \tag{9}$$

If U^i is concave in s^i, using first order optimality conditions, (9) is equivalent to

$$F^i(\bar{s})^\top (s^i - \bar{s}^i) \geq 0, \quad \forall s^i \in \mathcal{S}^i$$

where we defined $F^i(s) := -\nabla_{s^i} U^i(s^i, z^i(s))$. These conditions can be aggregated over all agents and admit the following compact representation:

$$F(\bar{s})^\top (s - \bar{s}) \geq 0, \quad \forall s \in \mathcal{S}, \tag{10}$$

where $F(s) := [F^1(s); \ldots; F^N(s)]$ is the so-called *Game Jacobian* and $\mathcal{S} := \mathcal{S}^1 \times \ldots \times \mathcal{S}^N$ is the cartesian product of the strategy sets. As detailed in Sidebar 2, condition (10) is equivalent to requiring that the vector $\bar{s}$ solves the variational inequality VI$(\mathcal{S}, F)$. Under Assumption 1 this is a well known equivalent characterization of a Nash equilibrium , see e.g., Scutari et al. (2010).

Before delving into equilibrium analysis, it is interesting to highlight a connection between variational inequalities and the potential game approach highlighted in Sidebar 1. To this end, note that if $F(s)$ is integrable, that is, if there exists $\Theta(s)$ such that $-\nabla_s \Theta(s) = F(s)$, then according to (10), under Assumption 1, $\bar{s}$ is a Nash equilibrium if and only if $-\nabla_s \Theta(\bar{s})^\top (s - \bar{s}) \geq 0$ for all $s \in \mathcal{S}$, equivalently $\bar{s}$ is a stationary point of the potential $\Theta(s)$. The variational approach can therefore be seen as a generalization of potential games to cases when the game jacobian $F(s)$ is not integrable.

We mentioned in Sidebar 1 that, for potential games, a sufficient condition for existence and uniqueness of equilibria is the potential function being strongly concave (i.e., there exists $\alpha > 0$ such that $-\nabla^2_s \Theta(s) \succeq \alpha I$ or equivalently $\nabla_s F(s) \succeq \alpha I$). The extension of such a condition to variational inequalities is that the game jacobian is strongly monotone, as detailed in Sidebar 2. Since an equivalent condition for strong monotonicity is $\frac{\nabla_s F(s) + \nabla_s F(s)^\top}{2} \succeq \alpha I$, strongly convex potential games are a special case of strongly monotone games (recall that $F(s)$ is symmetric when the game is potential due to condition (7)). In the case of network games, $\nabla_s F(s)$ can be rewritten as

$$\nabla_s F(s) = D(s) + K(s) T, \tag{13}$$

where $T := P \otimes I_n$,

$$D(s) := \text{blkd}[D^i(s)]_{i=1}^N := -\text{blkd}[\nabla^2_{s^i s^i} U^i(s^i, z^i) \mid_{z^i = z^i(s)}]_{i=1}^N$$
$$K(s) := \text{blkd}[K^i(s)]_{i=1}^N := -\text{blkd}[\nabla^2_{s^i z^i} U^i(s^i, z^i) \mid_{z^i = z^i(s)}]_{i=1}^N.$$

Let us define $\kappa_1 := \min_i \min_s \ \lambda_{\min}(D^i(s)), \kappa_2 := \max_i \max_s \ \|K^i(s)\|_2$. For a linear quadratic game as in (4), we obtain $D^i(s) = 1$, $K^i(s) = -a^i$ for all $i \in \{1, \ldots, N\}$, $\kappa_1 = 1$ and $\kappa_2 = \max_i |a^i|$.

Using the gradient structure highlighted in (13) and the equivalent condition for strong monotonicity of F given in Sidebar 2, one obtains that for network games

$$\frac{\nabla_s F(s) + \nabla_s F(s)^\top}{2} = D(s) + \frac{K(s)T + T^\top K(s)^\top}{2} \succeq \lambda_{\min}(D(s)) I + \frac{K(s)T + T^\top K(s)^\top}{2}. \tag{14}$$

It is shown in Parise and Ozdaglar (2019b) that the right hand side of (14) can be further lower bounded by $(\kappa_1 - \kappa_2 \|P\|_2) I$. Together with the results of Sidebar 2, this leads to the following theorem.

Theorem 1.A *Suppose Assumption 1 holds and*

$$\kappa_1 - \kappa_2 \|P\|_2 > 0 \tag{Assumption 2a}$$

then the game Jacobian is strongly monotone and satisfies the P_Υ condition (as detailed in Sidebar 2). Consequently, there exists a unique Nash equilibrium.

SIDEBAR 2: VARIATIONAL INEQUALITIES

A vector $\bar{x} \in \mathbb{R}^d$ solves the variational inequality $\mathrm{VI}(\mathcal{X}, F)$ with set $\mathcal{X} \subseteq \mathbb{R}^d$ and operator $F : \mathcal{X} \to \mathbb{R}^d$ if and only if

$$F(\bar{x})^\top (x - \bar{x}) \geq 0, \quad \text{for all } x \in \mathcal{X}.$$

There is a vast literature on properties of variational inequalities and their applications to game theory, we refer the interested reader to Facchinei and Pang (2003) and Scutari et al. (2010). We here comment on two results that will be useful for network game analysis. Suppose that F is continuously differentiable and $\mathcal{X}$ is nonempty, closed and convex. Then $\mathrm{VI}(\mathcal{X}, F)$ admits a unique solution under any of the following:

1. The operator F is *strongly monotone*, that is, there exists $\alpha > 0$ such that

$$(F(x) - F(y))^\top (x - y) \geq \alpha \|x - y\|_2^2 \quad \text{for all} \quad x, y \in \mathcal{X}. \tag{11}$$

2. The set $\mathcal{X}$ is a cartesian product $\mathcal{X}^1 \times \ldots \times \mathcal{X}^N$ and F is a *uniform block P-function* with respect to the same partition, that is, there exists $\eta > 0$ such that

$$\max_i [F^i(x) - F^i(y)]^\top [x^i - y^i] \geq \eta \|x - y\|_2^2 \quad \text{for all} \quad x, y \in \mathcal{X}. \tag{12}$$

As discussed in (Facchinei and Pang, 2003, Proposition 2.3.2(c)), an equivalent condition for (11) is

$$\frac{\nabla_x F(x) + \nabla_x F(x)^\top}{2} \succeq \alpha I \quad \text{for all} \quad x \in \mathcal{X}.$$

Following (Scutari et al., 2014, Proposition 5(e)), a sufficient condition for (12) can be obtained by constructing an auxiliary matrix Υ as follows

$$\Upsilon := \begin{bmatrix} \kappa_{1,1} & -\kappa_{1,2} & \ldots & -\kappa_{1,N} \\ -\kappa_{2,1} & \kappa_{2,2} & \ldots & -\kappa_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa_{N,1} & -\kappa_{N,2} & \ldots & \kappa_{N,N} \end{bmatrix} \quad \text{where} \quad \begin{cases} \kappa_{i,j} = \sup_{x \in \mathcal{X}} \|\nabla_{x_i x_j} F(x)\|_2 & \text{if } i \neq j, \\ \kappa_{i,i} = \inf_{x \in \mathcal{X}} \lambda_{\min}(\nabla^2_{x_i} F(x)) & \text{otherwise} \end{cases}$$

We say that F satisfies the P_Υ *condition* if Υ is a P-matrix (see Fiedler and Ptak (1962)). This is a sufficient (but not necessary) condition for (12) to hold.
The way the two uniqueness results above are proven is by exploiting an equivalent reformulation of the variational inequality into a fixed point problem: $\bar{x}$ solves $\mathrm{VI}(\mathcal{X}, F)$ if and only if $\bar{x} = \Pi_{\mathcal{X}}(\bar{x} - \tau \nabla_x F(\bar{x}))$, for some $\tau > 0$, where $\Pi_{\mathcal{X}}$ denotes the projection into the set $\mathcal{X}$. The way uniqueness is proven is by showing that $\Pi_{\mathcal{X}}(x - \tau \nabla_x F(x))$ is a contraction when F is strongly monotone and a block-contraction when F satisfies the P_Υ condition. The conclusion then follows from standard fixed point arguments, see Bertsekas and Tsitsiklis (1997). We finally note that there is an interesting connection between monotonicity and incremental passivity of the static map $y = F(u)$, as discussed e.g. in Gadjov and Pavel (2018) in the context of gradient dynamics.

It is interesting to note the relation of this theorem to results on linear quadratic games. If the network is undirected, by Perron-Frobenious theorem, $\|P\|_2 = \lambda_{\max}(P)$. Recalling that for homogeneous linear quadratic games $\kappa_1 = 1, \kappa_2 = |a|$ it is easy to see that Theorem 1.A is a generalization of the result in Ballester et al. (2006). Moreover, in the previous subsection we showed that for linear quadratic games of strategic substitutes, uniqueness conditions may be refined in terms of $\lambda_{\min}(P)$ instead of $\lambda_{\max}(P)$. It is proven in Parise and Ozdaglar (2019b) that a similar argument can be extended also to games with nonlinear best responses and scalar strategies or games with linear best responses but vector strategies. Whether this result can be extended to games with both nonlinear best responses and vector strategies is an open problem.

Theorem 1.B *Suppose that Assumption 1 holds, $P = P^\top$,*

$$\kappa_1 - \kappa_2|\lambda_{\min}(P)| > 0, \qquad \text{(Assumption 2b)}$$

and either: i) $n = 1$ and $K^i(s) > 0$ for all s or ii) $n \geq 1$, $K^i(s) = \tilde{K}^i$ for all i, s with $\tilde{K} + \tilde{K}^\top \succeq 0$, then there exists a unique Nash equilibrium.

Finally, in Parise and Ozdaglar (2019b) a third condition in terms of $\|P\|_\infty$ is discussed. Recall that $\|P\|_\infty$ is the maximum row sum of P, hence this condition has an interpretation in terms of the maximum aggregate influence that the neighbors have on each agent. One can show uniqueness of the Nash equilibrium, in this case by using the P_Υ condition discussed in Sidebar 2.

Theorem 1.C *Suppose Assumption 1 holds and*

$$\kappa_1 - \kappa_2\|P\|_\infty > 0, \qquad \text{(Assumption 2c)}$$

then the game Jacobian satisfies the P_Υ condition and there exists a unique Nash equilibrium.

It is important to note that the while the condition in terms of $\lambda_{\min}(P)$ is always less restrictive than the condition in terms of $\|P\|_2$, conditions in terms of $\|P\|_2$ and $\|P\|_\infty$ provide results for different sets of directed networks (instead if the network is undirected, then $\|P\|_2 \leq \|P\|_\infty$).

To illustrate the results above we consider again the multi-activity game in Example 4. For such a game, Assumptions 2a/2b/2c can be reformulated as $\min_i(1 - |\beta^i|) - (|\delta| + |\mu|)\eta(P) > 0$ for $\eta(P) = \|P\|_2, |\lambda_{\min}(P)|, \|P\|_\infty$ respectively. One can then see that Theorem 1.A and 1.B admit as special cases results in Chen et al. (2018), Bramoullé et al. (2014) and Ballester et al. (2006), as detailed in the following table. Theorem 1.C leads to novel conditions in terms of $\|P\|_\infty$.

	Complements ($\delta > 0$)	Substitutes ($\delta < 0$)
Single activity ($\beta = 0$)	$\|P\|_2 < \frac{1}{\delta}$ Ballester et al. (2006)	$\|\lambda_{\min}(P)\| < \frac{1}{\|\delta\|}$ Bramoullé et al. (2014)
Multiple activities ($\beta \neq 0$)	$\|P\|_2 < \frac{1-\|\beta\|}{\delta}$ Chen et al. (2018)	$\|\lambda_{\min}(P)\| < \frac{1-\|\beta\|}{\|\delta\|}$ Parise and Ozdaglar (2019b)

Table 1: Sufficient conditions for uniqueness for different linear quadratic games, seen as special cases of Example 4 for $\mu = 0$, $\beta^i = \beta$, $P = P^\top$ and $\mathcal{S}^i = \mathbb{R}^2_{\geq 0}$.

2.1.2 Dynamics

Establishing a connection between the network and properties of the game Jacobian is useful not only for deducing existence and uniqueness of the Nash equilibrium, but also for studying convergence of learning dynamics. In the following we focus on the Best Response (BR) dynamics where agents update their strategies iteratively by taking a best response (as defined in (5)) to the current actions of their neighbors.[5] We organize our exposition based on whether agents update their strategies continuously or at discrete instants of time.

Continuous BR dynamics:

For continuous time we consider the scheme introduced in Bramoullé et al. (2014) and reported in Algorithm 1.

Algorithm 1 Continuous best response dynamics

Set: $s^i(0) \in \mathcal{S}^i$, $z^i(0) = \sum_{j=1}^N P_{ij} s^j(0)$
Dynamics:

$$\dot{s}^i(t) = B^i(z^i(t)) - s^i(t) = \left[\arg\max_{s^i \in \mathcal{S}^i} U^i(s^i, z^i(t))\right] - s^i(t) \qquad \forall i \in \{1, \dots, N\}.$$

It is well known that if the game has a strongly convex potential then the continuous BR dynamics globally converge to the unique Nash equilibrium, see e.g., Monderer and Shapley (1996) and (Sandholm, 2010, Theorem 7.1.3). The intuition behind this result is simple: since each strategy update leads to an increase of the potential function, the BR dynamics must converge to the unique maximum of such potential which is the unique Nash equilibrium. We have already established that strongly convex potential games are a subclass of strongly monotone games. This motivates the question of whether the preceding convergence result can be generalized to strongly monotone games that are not potential.

Note that, in a potential game, the potential function $\Theta(x)$ acts as a Lyapunov function (Sandholm, 2010, Section 7.1.1). In fact let $B(s)$ be the vector of best responses to the strategy vector s (i.e., $[B(s)]_i = B^i(z^i(s))$) and define $d(s) := B(s) - s$, then

$$\dot{\Theta}(s(t)) = \nabla_s \Theta(s(t))^\top \dot{s}(t) = -F(s(t))^\top d(s(t))$$

and it follows from the definition of best response, see e.g., (Parise and Ozdaglar, 2019b, Lemma E.1), that for any $s \in \mathcal{S}$

$$F(s)^\top d(s) \leq -\kappa_1 \|d(s)\|_2^2. \tag{15}$$

In other words, $d(s)$ is an ascent direction for the potential function $\Theta(s)$.

If the game is not potential then $F(s)$ cannot be seen as a gradient of any function. Hence while (15) still holds, the interpretation of $d(s)$ as an ascent direction is lost. In Parise and Ozdaglar (2019b) it is shown, however, that if $F(s)$ is strongly monotone and the payoff function has the following form (covering many cases of economic and engineering interest)

$$U^i(s^i, z^i) := -\frac{1}{2}(s^i)^\top Q^i s^i - f^i(z^i)^\top s^i \tag{16}$$

for $Q^i = (Q^i)^\top \succ 0$ and $f^i : \mathbb{R}^n \to \mathbb{R}^n$, then a similar argument can be made by using the Lyapunov function $\tilde{\Theta}(s) := -F(s)^\top(s - B(s)) + \frac{1}{2}(s - B(s))^\top Q(s - B(s))$ where $Q := \text{blkd}[Q^i]_{i=1}^N$, leading to the following result, see Parise and Ozdaglar (2019b).

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[5] As noted in Ghaderi and Srikant (2014), the best response dynamics of network games with payoff $U^i(s^i, z^i) = -\frac{1}{2}\left[(1-\eta)(s^i - z^i)^2 + \eta(s^i - s_0^i)^2\right]$ coincide with the Friedkin-Johnsen model of opinion dynamics, which for $\eta = 0$ reduces to the consensus dynamics introduced in DeGroot (1974). We refer to Bullo (2019) for a detailed analysis of opinion dynamics models.

Theorem 2 *Consider a network game satisfying Assumption 1 and with agent payoff functions as in (16). Suppose that F is strongly monotone. Then for any $s_0 \in \mathcal{S}$, the sequence $\{s(t)\}_{t\geq 0}$ generated by Algorithm 1 converges to the unique Nash equilibrium.*

Discrete BR dynamics:

For the case of discrete dynamics, we index by $k \in \mathbb{N}$ the time instants at which at least one agent is updating his strategy and we denote by $\mathcal{T}_i \subseteq \mathbb{N}$ the subset of time instants at which agent i updates his strategy. In Algorithm 2 we consider two variants of the discrete BR dynamics depending on whether the agents update their strategies simultaneously or sequentially.

Algorithm 2 Discrete best response dynamics

Set: $k = 0$, $s_0^i \in \mathcal{S}^i$, $z_0^i = \sum_{j=1}^N P_{ij} s_0^j$. Set either $\mathcal{T}_i = \mathbb{N}$ for all i (simultaneous BR dynamics) or $\mathcal{T}_i = N(\mathbb{N}-1)+i$ for all i (sequential BR dynamics).
Iterate:

$$s_{k+1}^i = \begin{cases} B^i(z_k^i) = \arg\max_{s^i \in \mathcal{S}^i} U^i(s^i, z_k^i), & \text{if } k \in \mathcal{T}_i \\ s_k^i, & \text{otherwise} \end{cases} \qquad \forall i \in \mathbb{N}[1, N].$$

An interesting fact to note is that strong monotonicity is not sufficient to guarantee convergence of the discrete BR dynamics. To see this, note that, when the cost function is as in (16), the discrete BR dynamics coincide with the projection algorithm

$$s_{k+1} = \Pi_{\mathcal{S}}^Q [s_k - \tau Q^{-1} F(s_k)]$$

for the step choice $\tau = 1$. If F is strongly monotone the projection algorithm converges when τ is small but not necessarily for $\tau = 1$, see e.g., Facchinei and Pang (2003).

In (Scutari et al., 2014, Theorem 10) it is shown that convergence of the discrete simultaneous BR dynamics can instead be guaranteed by using the P_Υ property introduced in Sidebar 2. The reason why this result holds is fundamentally different from the argument used in potential games. In fact it is not based on the existence of a Lyapunov function, but convergence is instead guaranteed by showing that the best response mapping is a block-contraction. The latter property indeed guarantees that the conditions of the asynchronous convergence theorem in Bertsekas and Tsitsiklis (1997) are satisfied.

Theorem 3 *Consider a network game satisfying Assumption 1 and suppose that $F(x)$ satisfies the P_Υ condition. Then for any s_0, the sequence $\{s(t)\}_{t\geq 0}$ generated by Algorithm 1 and the sequence $\{s_k\}_{k=0}^\infty$ generated by Algorithm 2 converge to the unique Nash equilibrium.*

We note that convergence of the discrete BR dynamics follows immediately from (Scutari et al., 2014, Theorem 10) where convergence is proven for general games (i.e., not necessarily network games) and for both simultaneous and sequential updates (in fact, also random updates with delays can be considered, see (Scutari et al., 2014, Algorithm 1)). The result on convergence of the continuous BR dynamics under the P_Υ condition is proven in Parise and Ozdaglar (2019b).

2.1.3 Interventions

From the perspective of a central planner (CP) it is often of interest to exploit the characterization of equilibria developed above to understand how to best intervene to optimize a system level objective. As an example, we here focus on the model of targeted interventions in linear quadratic games studied by Galeotti et al. (2017).

The goal of the CP in this case is to maximize the *social welfare* (defined as the sum of the agents payoffs at equilibrium) through interventions that directly modify the standalone marginal return for an arbitrary agent i from b^i to $b^i + \hat{b}^i$, leading to the modified payoff function

$$U(s^i, z^i; \hat{b}^i) = -\frac{1}{2}(s^i)^2 + s^i[az^i + b^i + \hat{b}^i], \tag{17}$$

where we assume for simplicity that $a^i = a$ for all $i \in \{1, \dots, N\}$.

The planner has a budget constraint which penalizes interventions in a convex form (to capture the fact that interventions are increasingly costly), leading to $\sum_{i=1}^{N}(\hat{b}^i)^2 \leq C$. By using the characterization of equilibrium in linear quadratic games,[6] i.e., $\bar{s}^i = a\bar{z}^i + b^i + \hat{b}^i$ with $\bar{z}^i = \sum_j P_{ij}\bar{s}^j$, the objective function of the central planner can be rewritten as

$$T(\hat{b}) := \sum_{i=1}^{N} U(\bar{s}^i, \bar{z}^i; \hat{b}^i) = \sum_{i=1}^{N}\left(-\frac{1}{2}(\bar{s}^i)^2 + \bar{s}^i[a\bar{z}^i + b^i + \hat{b}^i]\right) = \frac{1}{2}\sum_{i=1}^{N}(\bar{s}^i)^2 = \frac{1}{2}\|\bar{s}\|^2$$

where $\hat{b} := [\hat{b}^i]_{i=1}^N$. This leads to the following optimization problem

$$\begin{aligned} T_{\text{opt}} := \max_{\hat{b}\in\mathbb{R}^N} \quad & \frac{1}{2}\|\bar{s}\|^2, \\ \text{s.t.} \quad & \bar{s} = (I - aP)^{-1}(b + \hat{b}), \\ & \|\hat{b}\|^2 \leq C. \end{aligned} \tag{18}$$

Solving (18) is in general not computationally tractable for large populations.[7] It is however shown in Galeotti et al. (2017) that, under some regularity assumptions, one can approximate its solution by exploiting a decomposition of interventions into the principal components of the network. Let us assume that P is symmetric and its eigenvalues are distinct so that $P = \Psi\Lambda\Psi^\top$ where Λ is a diagonal matrix of eigenvalues of P (ordered from greatest to smallest: $\lambda_1 > \lambda_2 > \dots > \lambda_N$) and Ψ is an orthogonal matrix whose l^{th} column, ψ^l, is the eigenvector of P associated to the eigenvalue λ_l and normalized so that $\|\psi^l\| = 1$. Let $\hat{b}^*$ be an optimizer of (18) and define the similarity ratio $r_l^* := \frac{\rho(\hat{b}^*, \psi^l)}{\rho(b, \psi^l)}$, where $\rho(z, y) = \frac{z^\top y}{\|z\|\|y\|}$ is the cosine similarity of two nonzero vectors y, z. The following theorem is proven in Galeotti et al. (2017).

Theorem 4 *Suppose that $P = P^\top$, all eigenvalues of P are distinct and $|a| < \frac{1}{\lambda_{\max}(P)}$. Then the similarity ratio r_l^* is proportional to $\frac{0.5\alpha_l}{\mu - 0.5\alpha_l}$ with μ solution to $\sum_l \left(\frac{0.5\alpha_l}{\mu - 0.5\alpha_l}\right)^2 \underline{b_l}^2 = C$, where $\alpha_l = \frac{1}{(1-a\lambda_l)^2}$ and $\underline{b_l}$ is the projection of b onto ψ^l. Moreover, the following hold:*

1. *As $C \to 0$: for any l, l', $\frac{r_l^*}{r_{l'}^*} \to \frac{\alpha_l}{\alpha_{l'}}$;*
2. *As $C \to \infty$: (a) If $a > 0$ then $\rho(\hat{b}^*, \psi^1) \to 1$; (b) If $a < 0$ then $\rho(\hat{b}^*, \psi^N) \to 1$.*

Hence if C is large, an almost optimal intervention can be obtained by allocating the budget according to the principal component ψ^1 (i.e., the dominant eigenvector of P) for games of strategic complements ($a > 0$) and ψ^N for substitutes ($a < 0$). The intuition is simple, recall that in linear quadratic games the equilibrium after the intervention is $(I - aP)^{-1}(b + \hat{b})$ and note that $(I - aP)^{-1}$ has eigenvectors ψ^l with corresponding eigenvalue $\frac{1}{1-a\lambda_l}$. For large budget the status-quo marginal return b is irrelevant and one should intuitively maximize the norm of $(I - aP)^{-1}\hat{b}$, recalling that the dominant eigenvector is the vector that is most amplified by the network, it is then clear that one should allocate the budget proportionally to the dominant eigenvector of $(I - aP)^{-1}$, which is ψ^1 for game of strategic complements and ψ^N for game of strategic substitutes. More details and extensions are given in Galeotti et al. (2017).

[6]For simplicity we assume here that $\mathcal{S}^i = \mathbb{R}$ instead of $\mathbb{R}_{\geq 0}$, see Galeotti et al. (2017) for more details.

[7]Problem (18) can be reformulated as an SDP with two variables and an inequality constraint involving a matrix of dimension $N + 1$, see Galeotti et al. (2017) and (Boyd and Vandenberghe, 2004, Appendix B.1).

While we here focused on one specific type of intervention, we want to highlight that many different types of interventions have been considered in the network game literature. To mention just a few, Ballester et al. (2006), showed that to reduce overall activity levels the central planner should remove the agent with highest key player centrality, while Candogan et al. (2012), suggested an optimal pricing scheme for a monopolist selling a good subject to network externalities based on a variation of Bonacich centrality. These are just some examples, we remark that the problem of relating optimal interventions to network centrality measures is a very active and interesting area of research.

2.2 Discrete strategies

In this section we consider games in which each agent needs to select a strategy among a finite set of actions. Our focus will be on pure strategy Nash equilibria, as described in Definition 1. The two classic approaches used to study properties of pure strategy Nash equilibria are to assume that the game posseses either a supermodular or potential structure, as defined in Sidebar 1. Instead, we here focus on a generalization of Example 2 to heterogenous settings and we review results specific to this class of games.

Example 5 (Coordination games) *Consider a generalization of Example 2 where agents need to select a strategy in $\mathcal{S}^i = \{0, 1\}$ and their best response is given by equation (3) with heterogeneous thresholds. In particular agent i selects strategy 1 if and only if at least θ^i fraction of his neighbors do so.*

Coordination games are pervasive and model a wide range of economic and social behaviors, including the choice of conforming to a social norm, buying a new product or adopting a certain innovation or technology. Moreover, as we illustrate next, the best response dynamics of such games coincide with a well studied model of contagion, further motivating our interest in this class of games.

2.2.1 Equilibria

To study equilibria of the coordination game in Example 5, it is useful to introduce the notion of a cohesive set.

Definition 3 (Cohesive set) *A set $S \subseteq \{1, \ldots, N\}$ is θ-cohesive if $\frac{|\mathcal{N}^i \cap S|}{d^i} \geq \theta^i, \forall i \in S$.*

In simple terms, a set is cohesive if each agent has at least θ^i fraction of his neighbors within the set. For example, if $\theta^i = \theta$ for all $i \in \{1, \ldots, N\}$, the blue and magenta sets in the third plot of Figure 2 are cohesive sets for any $\theta \leq \frac{2}{3}$.

Let S^* be the set of agents that select strategy 1 at equilibrium. If the thresholds are homogeneous (i.e., $\theta^i = \theta$ for all $i \in \{1, \ldots, N\}$) then clearly it must be that each agent in S^* has at least a fraction θ of neighbors selecting strategy 1, while each agent outside S^* has less than θ fraction of neighbors selecting strategy 1, leading to the following result from Morris (2000).

Proposition 1 *Consider a coordination game as in Example (2), with homogeneous threshold $\theta \in [0, 1]$. There exists an equilibrium where strategy 1 is played by agents in S^* and 0 by agents in $\{1, \ldots, N\} \backslash S^*$ if and only if S^* is θ-cohesive and $(S^*)^c$ is $(1 - \theta)$-cohesive.*

Note that any coordination game has multiple equilibria (also referred to as *conventions*). For example all agents playing strategy 1 or all agents playing strategy 0 are both equilibria (see first two plots in Figure 2). It is then interesting to understand if there are sets of agents that play the same strategy in any such equilibria.

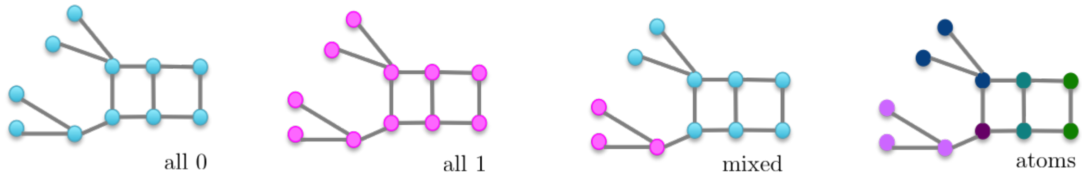

Figure 2: The first three figures show some of the equilibria of a homogeneous coordination games with threshold $\theta = 0.4$. The last figure shows the corresponding partition in atoms.

Definition 4 (Atoms) *Let $\mathcal{C}(\theta, P)$ denote the σ-algebra generated by all the subsets of $\{1, \ldots, N\}$ that represent equilibria of a coordination game with homogeneous threshold θ as introduced in Example 2 (i.e., in the equilibrium all agents of the subset play strategy one and all other agents play zero). The atoms of $\mathcal{C}(\theta, P)$ (i.e., the nonempty sets on $\mathcal{C}(\theta, P)$ that do not contain any other nonempty set of $\mathcal{C}(\theta, P)$) exist and form a partition of the set of nodes. Agents within the same atom play the same action in any equilibrium.*

Such an atom partition can be used to represent the concept of "behavioral community" as a set of agents who behave identically in any equilibrium, as detailed in Jackson and Storms (2019) (see Figure 2, right).

2.2.2 Dynamics

The best response dynamics of the coordination game in Example 5 are particularly interesting, as they lead to a well-known model of contagion dynamics over networks called the *linear threshold model*, Granovetter (1978). In the following we consider two variants of such best response dynamics; alternative models of contagion are discussed in Sidebar 3.

Multiple switches:

Consider a setting where initially all agents have selected strategy 0 except for an initial set S_0 of agents playing strategy 1. The set S_0 is called the *seed set* and represents early adopters (e.g., first adopters of a new product or innovation, initial bank failures, etc.). At every iterative step k each agent takes a best response action according to the coordination game in Example 5 and therefore selects strategy 1 if at least a fraction θ^i of his neighbors have selected strategy 1 at step $k-1$ and 0 otherwise, see Algorithm 3.

Algorithm 3 Best response dynamics for the coordination game: Multiple switches

Set: $k = 0$, Initialize $s_0^i = 1$ for all $i \in S_0$ and $s_0^i = 0$, otherwise.
Iterate:

$$s_{k+1}^i = \begin{cases} 1 & \text{if } \frac{z^i}{d^i} \geq \theta^i \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in \{1, \ldots, N\}.$$

These dynamics are studied for example in Adam et al. (2012), which shows that agent behavior may cycle among different actions in the limit. Limit cycles consist of at most two action profiles for any network and any threshold distribution. Hence either each agent sticks with one strategy in the limit or he switches action at every time step. Moreover, the limiting cycle is reached in at most $2N$ steps. Noisy versions of Algorithm 3 have also been considered in the literature, see e.g., Young (2006); Montanari and Saberi (2010); Liggett (2012); Durrett (1988).

One switch:

In Algorithm 3 an agent is allowed to switch back from 1 to 0. If we additionally impose that $\theta^i = 0$ for all agents in the seed set S_0, no agent (including the initial adopters) will ever switch back from 1 to 0, that is, the sequence $|S_k|$ is non-decreasing. In such one-switch-only dynamics, sometimes referred to as the *progressive* linear threshold model, at every iterative step k an agent selects action 1 if either at least a fraction θ^i of his neighbors has selected strategy 1 at step $k-1$ or if he was playing action 1 at step $k-1$, see Algorithm 4. Let S_k be the set of adopters at iteration k.

Algorithm 4 Best response dynamics for the coordination game: One switch

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Set: $k = 0$, Initialize $s_0^i = 1$ for all $i \in S_0$ and $s_0^i = 0$, otherwise.
Iterate:

$$s_{k+1}^i = \begin{cases} 1 & \text{if} \quad \frac{z^i}{d^i} > \theta^i \text{ or } s_k^i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in \{1, \ldots, N\}.$$

Contrary to Algorithm 3, Algorithm 4 is guaranteed to converge to a final set S^* of adopters in at most N iterations since at every iteration either the set of adopters monotonically increases by at least one agent or it stops growing from that point onwards. Since there are N agents, after at most N steps the final set S^* is reached. A characterization of the limit set S^* in terms of network properties is provided in Acemoglu et al. (2011). To present this result we start by introducing the definition of a fixed point of Algorithm 4.

Definition 5 *$\bar{S}$ is a fixed point of Algorithm 4 if $S_0 = \bar{S}$ implies $S_k = \bar{S}$ for all $k \geq 0$.*

Following the same argument as in Morris (2000), it is immediate to see that an adopter set $\bar{S}$ is a fixed point if and only if $(\bar{S})^c$ is a $(1-\theta)$-cohesive set. In this case in fact members of $(\bar{S})^c$ cannot adopt the innovation unless there exists at least one adopter inside the set itself. Exploiting these arguments Acemoglu et al. (2011), derive a characterization of the final set of adopters for any given graph, seed set and threshold values.

Lemma 1 *Let $\{\bar{S}_h\}_{h=1}^H$, $H \geq 1$, be the set of fixed points of Algorithm 4 for which $S_0 \subseteq \bar{S}_h$. Then, $S^* = \cap_{h=1}^H \bar{S}_h$ is the final set of adopters.*

This result can be used to upper bound the expected number of final adopters when the initial seed set is a random subset of $\{1, \ldots, N\}$ of fixed cardinality k.

Lemma 2 *Let $\{\bar{S}_1, \ldots, \bar{S}_r\}$ be a partition of $\{1, \ldots, N\}$ made of $(1-\theta)$-cohesive sets ordered with descending cardinalities. The expected number of final adopters over all initial seed set of cardinality $k \leq r$ is upper bounded by $\mathbb{E}[|S^*|] \leq \sum_{h=1}^k |\bar{S}_h|$.*

This result is used by Acemoglu et al. (2011), to argue that highly clustered networks might have smaller expected number of final adopters, since one can construct cohesive partitions with sets of smaller size. This is against the classic intuition that innovation spreads faster in highly connected networks. To understand why this is the case, note that highly cohesive clusters will adopt quickly the innovation if one of their members is an adopter, but at the same time are very resistant to influence from outside members.

In the discussion so far we assumed that thresholds are fixed a priori and deterministic. Following Kempe et al. (2003), Lim et al. (2016) consider an alternative model where instead thresholds are sampled uniformly at random from $[0, 1]$. Note that the expected number of final adopters given the initial seed set S_0 is $\mathbb{E}[|S^*|] = \sum_{i=1}^N \mathbb{P}_i(S_0)$ where $\mathbb{P}_i(S_0)$ is the probability that agent i adopts if the initial seed set is S_0; Lim et al. (2016) connect this probability

SIDEBAR 3: CONTAGION MODELS

The linear threshold model described above is one type of contagion model, which can be derived from the best response dynamics of a coordination game. We here highlight other contagion models considered in the literature. A commonly studied model is that of *independent cascades*, see e.g., Goldenberg et al. (2001); Liggett (2012); Durrett (1988); Goldenberg et al. (2010). As in the linear threshold model, suppose that there is an initial set of adopters S_0 and that contagion spreads in discrete time steps. However, whenever an agent i becomes active (i.e., selects strategy 1), he is given a single chance to activate each currently inactive neighbor j; each of these neighbors is activated with communication probability p_{ij} independently of the others. Contrary to the linear threshold model where an agent needs enough active neighbors to switch, in the independent cascade model one active neighbor is sufficient to spread contagion. For this reason the linear threshold model is referred to as a "complex contagion" model, while independent cascade is referred to as a model of "simple contagion", see e.g., Centola and Macy (2007); Centola (2018); Young (2009). In Acemoglu et al. (2011) an extension of the linear threshold model is discussed where agents make stochastic decisions of adoption when activated instead of switching deterministically. Game-theoretic models of diffusion, where agents make strategic decisions on whether to spread information or rumors, have also been intensively investigated, see e.g., Galeotti and Goyal (2009); Bloch et al. (2018); Sadler (2020) to mention just a few, as well as competitive contagion models, see e.g., Goyal et al. (2019); Fazeli and Jadbabaie (2012); Draief et al. (2014); Tzoumas et al. (2012); Fazeli et al. (2016); Mei and Bullo (2017). Finally, other well studied models of contagion are *epidemic models*, as surveyed for example in Nowzari et al. (2016).

to paths in the network. Specifically, given a path p with initial node i_0 the *degree sequence product* is defined as $\chi_p := \sum_{i\in p} \frac{d^i}{d^{i_0}}$ and $\mathbb{P}_i(S_0) = \sum_{j\in S_0} \sum_{p\in\mathcal{P}^*_{ji}} \frac{1}{\chi_p}$ where $\mathcal{P}^*_{ji}$ is the set of paths beginning at $j \in S_0$ and ending at $i \in \{1,\ldots,N\} \setminus S_0$. This result can be used to study the *cascade centrality* $\mathcal{C}_i$ of a node i, that is, the expected number of final adopters when $S_0 = \{i\}$ as well as *contagion centrality* $\mathcal{K}_i$, that is, the probability that the seed set $S_0 = \{i\}$ induces a cascade reaching every node. For example, Lim et al. (2016) show that if the network P is a tree, $\mathcal{C}_i = d^i + 1$ and $\mathcal{K}_i = \frac{1}{\Pi_{j\neq i} d_j}$, if P is a cycle of order N, $\mathcal{C}_i = 3 - \frac{1}{2^{N-2}}$ and $\mathcal{K}_i = \frac{N}{2^{N-1}}$, and for complete graphs $\lim_{N\to\infty} \frac{\mathcal{C}_i}{\sqrt{N}} = \sqrt{\frac{\pi}{2}}$ and $\lim_{N\to\infty} N\mathcal{K}_i = e$.

2.2.3 Interventions

In the previous section we reviewed properties of the final set of adopters for a given seed set. An important question from the point of view of a central planner is how to select the seed set that maximizes the number of final adopters given a budget on the seed set cardinality. This corresponds to the following optimization problem

$$\begin{aligned} \max_{S_0} \quad & |S^*| \\ \text{s.t.} \quad & |S_0| = k, \end{aligned} \tag{19}$$

where S^* is the set of final adopters when the initial seed set is S_0.

This optimization problem is not computationally tractable for large populations. For example, when agent's thresholds are selected from $[0,1]$ uniformly at random (and one aims at maximizing the expected size of the final set given the randomness on thresholds) this problem in NP-hard, as shown by Kempe et al. (2003). In such a setting, it is however possible to obtain an approximate solution by exploiting the fact that the optimization problem in (19) has the following properties.

Definition 6 *A function $f(S)$ is monotone if for all sets $S_1 \subseteq S_2$, $f(S_1) \leq f(S_2)$, that is larger sets lead to higher function values, and is submodular if for $s \notin S_2$, $f(S_1 \cup \{s\}) - f(S_1) \geq f(S_2 \cup \{s\}) - f(S_2)$, that is, adding element s to a smaller set S_1 is more beneficial than adding it to the larger set S_2.*

Theorem 5 *$\sigma(S_0) := \mathbb{E}[|S^*|]$ is a non-negative, monotone, submodular function.*

Theorem 5 is proven in Kempe et al. (2003). Therein, using such submodularity property, it is additionally proven that a set $\hat{S}_0$ that approximates the solution of 19 within a factor of $(1 - 1/e - \epsilon)$, where e is the base of the natural logarithm and ϵ is any desired positive number[8], can be computed with a natural hill-climbing greedy algorithm by starting with the empty set and adding one-by-one the node that best complements the current seed set (i.e., the node that if added increases $\mathbb{E}[|S^*|]$ the most).

Influence maximization problems of this type have been the subject of a large number of works, we refer to the survey Liu-Thompkins (2012) for a review. Interestingly, for networks with homogeneous deterministic thresholds, a seeding heuristic based on the concept of atoms, as introduced in Section 2.2.1, is suggested in Jackson and Storms (2019). Running such heuristic requires computing the atoms first; an heuristic to do so is provided in Jackson and Storms (2019).

3 SAMPLED NETWORKS

In the discussion so far we assumed that both the agents and the central planner have perfect knowledge of the underlying pattern of interactions, that is, they know exactly the network P. When considering large networks, two problems emerge from the perspective of the central planner. First, collecting exact network information in large networks is extremely costly and in many cases not at all possible due to privacy and proprietary data concerns. Second, even if the central planner has full network information, when the number of agents is very large, planning optimal interventions may become computationally intractable. We have already seen for example that the computational requirement for targeted interventions (discussed in Section 2.1.3) or optimal seeding problems (discussed in Section 2.2.3) do not scale well with the population size. In this section we illustrate how working with stochastic network formation models instead of deterministic networks may alleviate both of these problems.

First, while the central planner may not know the exact network, in many cases he may have a probabilistic description of how agents interact, that is, he might have information about the stochastic network formation model that generated the network. For example, Breza et al. Breza et al. (2018) shows that the use of aggregated relational data collected by asking questions regarding aggregated neighbors' characteristics (e.g., how many neighbors belong to a given community or have a certain trait) instead of exact data about all neighbors' identity is sufficient to obtain reasonable estimates of many network features of economic interest. In Parise and Ozdaglar (2019a) and Mele et al. (2019), it is argued that such aggregated data can be used to recover network formation models such as stochastic block models.

Second, knowledge of the stochastic network formation model can help obtain a robust analysis of equilibria and computationally efficient planning of interventions. To illustrate this, we start by introducing the key concept of a sampled network game.

[8]Note that $\sigma(S_0)$ is an expectation over the random thresholds; for any ϵ there exists $\gamma > 0$ such that if one uses $(1+\gamma)$-approximate values of $\sigma(\cdot)$, obtained for example by simulating the random process, the result above holds.

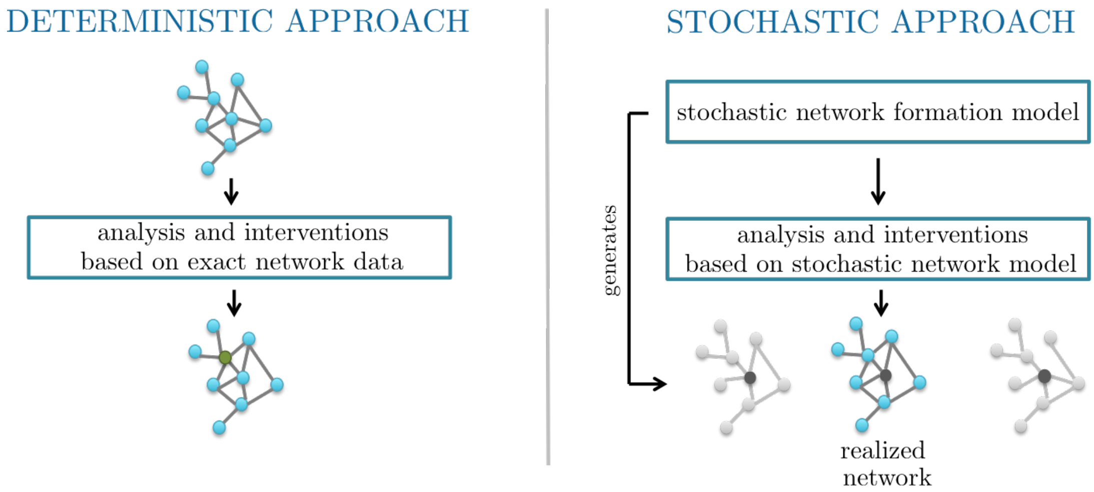

Figure 3: Left: The deterministic approach. Analysis and interventions are based on exact network data. Right: The stochastic approach. Networks are generated from a stochastic network formation model. Analysis and interventions are conducted based on the probabilistic model and used to approximate behavior and intervention strategies in the realized network. In this illustration the deterministic intervention selects the agent with highest degree given the exact network (green), the stochastic approach selects the agent with highest expected degree based on the network formation model (dark grey). The two interventions do not necessarily coincide since the node with highest expected degree might or might not be the node with highest realized degree. In many cases however one can derive bounds on the performance of such interventions that improve with the network size.

Definition 7 (Sampled network game) *A sampled network game is a network game as defined in the previous section where the network of interactions, denoted by P^N to stress the dependence on the population size N, is sampled from a given stochastic network formation model.*

The difference between the approach based on exact versus stochastic network information is illustrated in Figure 3. In the next section we review some commonly used stochastic network formation models, more details and extensions can be found in Bollobás and Béla (2001); Jackson (2010); Pin and Rogers (2015); Newman et al. (2001); Newman (2010). We then illustrate in Sections 3.2 and 3.3 how these models have been used to study sampled network games with continuous and discrete action sets, respectively. We remark that this is a very active area of research; our objective is to provide some representative examples more than an exhaustive list of papers adopting this approach.

3.1 A review on stochastic network formation models

Erdös-Rényi model:

One of the first and simplest stochastic network formation model is the Erdös-Rényi model, Erdős and Rényi (1959). In this model each pair of agents is connected with independent probability $p^N \in [0,1]$, possibly dependent on the population size N. A large literature has focused on identifying threshold conditions on p^N such that the realized network has desirable properties with probability converging to 1 for $N \to \infty$. For example the realized network becomes connected when p^N is of order greater than $\frac{\log(N)}{N}$ (i.e., $\lim_{N\to\infty} \frac{\log(N)/N}{p^N} = 0$), see Jackson (2010). Let us define a network to be sparse if $\lim_{N\to\infty} \frac{\text{number of edges}}{N^2} = 0$ and dense otherwise. We then note that if $p^N \equiv p \in [0,1]$ is independent on the population size, networks generated from a Erdös-Rényi model are dense since each agent has in expectation $p(N-1)$ neighbors.

Stochastic block model:

A generalization of the Erdös-Rényi model is obtained by assuming that agents are not completely homogeneous in how they form connections, but are instead partitioned into H different communities. Agents in community h and h' form a link with independent probability $p^N_{hh'}$ that depends on the communities to which they belong to. This model is used often to encode homophily, that is, the propensity of agents to form more links with agents that are "similar" to them along various social characteristics (e.g., gender, race or age group) as detailed in McPherson et al. (2001); Currarini et al. (2009). Homophily can be modeled in stochastic block models by assuming $p^N_{hh} > p^N_{hh'}$ for all $h' \neq h$, see e.g Golub and Jackson (2012).

Configuration model:

In a stochastic block model each agent belonging to the same community has the same expected degree. Another commonly used stochastic network formation model is the configuration model, presented in Bender and Canfield (1978); Newman et al. (2001), where one instead can specify the degree of each node. Specifically given a desired degree vector $d \in \mathbb{N}^N$ (either assigned deterministically or sampled from a degree distribution), a realized network is obtained by assigning degree d^i to each node i. The degrees of the vertices are represented as half-links or stubs. The algorithm then proceeds by choosing two stubs uniformly at random and connecting them to form an edge, iteratively until all the stubs are connected into an edge. Conditioned on the obtained realization being a simple graph (i.e., without self-loops or multi-links), this procedure generates a uniformly distributed random graph with the given degree sequence, Janson and Luczak (2009). One important property of configuration models is that, under suitable assumptions on the degree distribution, they are locally tree-like, see Bordenave (2012). While this feature simplifies the analysis, networks generated from such configuration models typically lack the clustering and correlation patterns observed in social networks.

Graphon model:

We finally consider a stochastic network formation model based on the theory of graphons. Graphons were originally introduced in Lovász and Szegedy (2006); Lovász (2012); Borgs et al. (2008) as the limit of a sequence of dense networks when the number of nodes tends to infinity. In this case, one can assume that the set of nodes converges to the continuum interval $[0,1]$. Interactions can then be described with a integrable function $W : [0,1]^2 \to [0,1]$, where the value $W(x,y)$ represents the level of interaction between infinitesimal node x and y. For simplicity, we focus here on symmetric interactions so that $W(x,y) = W(y,x)$.

Besides their interpretation as the limit of growing networks, graphons can also be used as a stochastic network formation model: a network with N nodes can be generated by sampling N labels $\{t^i\}_{i=1}^N$ uniformly at random from the $[0,1]$ interval and connecting them with independent probability $W(t^i, t^j)$. Note that this procedure generates dense networks. To obtain sparse networks, Borgs et al. (2019) suggest to connect agents with independent probability $\rho_N W(t^i, t^j)$ where $\rho_N \to 0$ as $N \to \infty$ (but with ρ_N of order greater than $\frac{\log N}{N}$, similar to the requirement needed in Erdös-Rényi model for connectedness). In the following when reviewing results for graphon models we will focus on dense networks, but most of the results discussed can be generalized to the sparser case just presented.

Finally, it is important to remark that both the Erdös-Rényi model and dense stochastic block models are special cases of graphons, obtained for $W \equiv p$ for some $p \in [0,1]$ and W a piecewise constant function. Specifically for the stochastic block model, if we let π_h be the probability that an agent belongs to community h, we can define a partition of $[0,1]$ in intervals $\mathcal{C}_h$ of measure π_h and define $W(x,y) = p_{hh'}$ for all $x \in \mathcal{C}_h, y \in \mathcal{C}_{h'}$. Other sampled networks that can be generated from a graphon are discussed in Borgs et al. (2011).

3.2 Continuous strategies

3.2.1 Centrality measures

In Section 2.1 we showed that the equilibrium of a linear quadratic game is proportional to the Bonacich centrality of the agents. It is then interesting to understand whether one can approximate such a measure of centrality based only on information about the stochastic network formation model. This question is answer affirmatively in Dasaratha (2017) for sampled networks generated from stochastic block models as well as a spatial model and in Avella-Medina et al. (2018) for sampled networks generated from a graphon.

More specifically, Avella-Medina et al. (2018) define several centrality measures for graphons, including Bonacich centrality, eigenvector centrality, Page-rank and degree centrality (for example the degree of the infinitesimal agent $x \in [0,1]$ can be defined as $d(x) := \int_0^1 W(x,y)dy$). Note that such centrality measures are functions that map $x \in [0,1]$ to the corresponding centrality. Centralities in sampled networks are instead vectors in $\mathbb{R}^N$. To compare these two objects, one can define a one-to-one correspondence between vectors and functions using a uniform partition $\mathcal{U}^N = \{\mathcal{U}_1^N, \mathcal{U}_2^N, \ldots, \mathcal{U}_N^N\}$ of $[0,1]$. Pairing each agent i with the interval $\mathcal{U}_i^N$, one can then define the *step function* $v(x)$ corresponding to any vector $v \in \mathbb{R}^N$ as $v(x) := v_i, \ \forall x \in \mathcal{U}_i^N$, for all $i \in \{1, \ldots, N\}$.

Using this correspondence it is shown in Avella-Medina et al. (2018) that, for the centrality measures mentioned above and under suitable regularity conditions on the graphon, the L^2 distance between the step function corresponding to the normalized vector of centralities in any sampled network and the graphon centrality function can be upper bounded, with high probability, by a quantity that converges to zero when the population size increases. This result for example suggests that if the optimal intervention is to target the agent with highest degree then targeting the agent i for which $d(t^i)$ is maximum is asymptotically optimal (recall that $t^i \in [0,1]$ is the label of agent i in the sampled network).

3.2.2 Graphon games

In the previous subsection we focused on Bonacich centrality, which corresponds to the equilibrium of a linear quadratic network game. Parise and Ozdaglar (2019a) extend these convergence results to more general classes of network games.[9] To this end, Parise and Ozdaglar (2019a) introduce a new class of infinite population games, called *graphon games*, modeling strategic interactions among a continuum of agents mapped to $[0,1]$. As in the finite population case, agents select their strategies $s(x)$ from a set $\mathtt{S}(x)$. For any graphon W, define the local aggregate experienced by agent x as the "weighted average" of the other agents actions according to the graphon, that is,

$$z(x \mid s) := \int_0^1 W(x,y)s(y)dy.$$

Note that for graphon games, a strategy profile $s : [0,1] \to \mathbb{R}$ is a *function*. As in network games, the goal of each agent in a graphon game is to select the strategy $s(x) \in \mathtt{S}(x)$ that maximizes his payoff given by

$$U(s(x), z(x \mid s), \theta(x)), \tag{20}$$

which depends on his strategy $s(x)$, the local aggregate $z(x \mid s)$ and on a parameter $\theta(x)$ (modeling heterogeneity in payoffs). Note that such a payoff function has the same structural form as in network games, see 1. The difference in the two setups is the way in which the local

[9] We here discuss games with scalar strategies, but the extension to the vector case is immediate.

aggregate ($z^i(s)$ for agent i in network games[10] and $z(x \mid s)$ for agent x in graphon games) is evaluated. The concept of Nash equilibrium extends easily to graphon games.

Definition 8 (Graphon equilibrium) *A function $\bar{s} \in L^2([0,1])$ with associated local aggregate $\bar{z}(x) := z(x \mid \bar{s}) = \int_0^1 W(x,y)\bar{s}(y)dy$ is a Nash equilibrium if for all $x \in [0,1]$, we have $\bar{s}(x) \in \mathtt{S}(x)$ and $U(\bar{s}(x), \bar{z}(x), \theta(x)) \geq U(\tilde{s}, \bar{z}(x), \theta(x))$ for all $\tilde{s} \in \mathtt{S}(x)$.*

The key result in Parise and Ozdaglar (2019a) is to relate equilibria in network games sampled from a graphon to the equilibrium of the corresponding graphon game. To this end, Parise and Ozdaglar (2019a) proceeds in 2 steps.

First, conditions for existence and uniqueness of the graphon equilibrium are derived by using a reformulation as fixed point of a best response operator. Such reformulation relies on the notion of graphon operator, see (Lovász, 2012, Section 7.5).

Definition 9 (Graphon operator) *For a given graphon W, we define the associated graphon operator $\mathbb{W}$ as the integral operator $\mathbb{W} : L^2([0,1]) \mapsto L^2([0,1])$ given by $f(x) \mapsto (\mathbb{W}f)(x) = \int_0^1 W(x,y)f(y)\mathrm{d}y$.*

Note that if W is symmetric then all the eigenvalues of the graphon operator $\mathbb{W}$ are real. We next show that if the maximum eigenvalue $\lambda_{\max}(\mathbb{W})$ is not too large then the best response operator is a contraction, guaranteeing uniqueness of the graphon equilibrium.

Theorem 6 *Suppose that the function $U(s,z,\theta)$ in (20) is continuously differentiable and strongly concave in s with uniform constant α_U for each value of z, θ and that $\nabla_s U(s,z,\theta)$ is uniformly Lipschitz in $[z]$ with constant ℓ_U. Moreover, suppose that for each $x \in [0,1]$ the set $\mathtt{S}(x)$ is convex and closed and there exists a compact set $\mathcal{S}$ such that $\mathtt{S}(x) \subseteq \mathcal{S}$ for all $x \in [0,1]$. Finally, suppose that*

$$\frac{\ell_U}{\alpha_U} \cdot \lambda_{\max}(\mathbb{W}) < 1. \tag{21}$$

Then the graphon equilibrium exists and is unique.

The condition in 21 is similar to Assumption 2a used to obtain uniqueness in finite network games. The only difference is that while in the network game literature the effect of the network is captured by the maximum eigenvalue of the finite network P^N, in the case of graphon games the corresponding role is played by the dominant eigenvalue of the graphon operator, that is, $\lambda_{\max}(\mathbb{W})$.

Second, it is shown in Parise and Ozdaglar (2019a) that under suitable regularity conditions on the graphons, one can bound with high probability the L^2 distance between the step function corresponding to the Nash equilibrium in any network game sampled from the graphon and the equilibrium of the corresponding graphon game with a quantity that converges to zero as the population size increases. An illustration is given in Figure 4.

One can exploit this converge result to study properties of equilibria in sampled network games by looking at properties of the corresponding graphon equilibrium. This analysis becomes particularly interesting when turning to interventions. Indeed, in parallel with Galeotti et al. (2017), one can define a graphon game with payoff function

$$U(s(x), z(x); \hat{b}(x)) = -\frac{1}{2}(s(x))^2 + s(x)[az(x) + b(x) + \hat{b}(x)], \tag{22}$$

[10] In this section we define $z^i(s) := \frac{1}{N}\sum_{j=1}^N P_{ij}^N s^j$, that is, we add a $\frac{1}{N}$ normalization to the definition given in Section 2.1. Since Parise and Ozdaglar (2019a) study the behavior when N changes it is useful to consider this factor explicitly. A different normalization in terms of agents degree instead of population size is also considered in Parise and Ozdaglar (2019a).

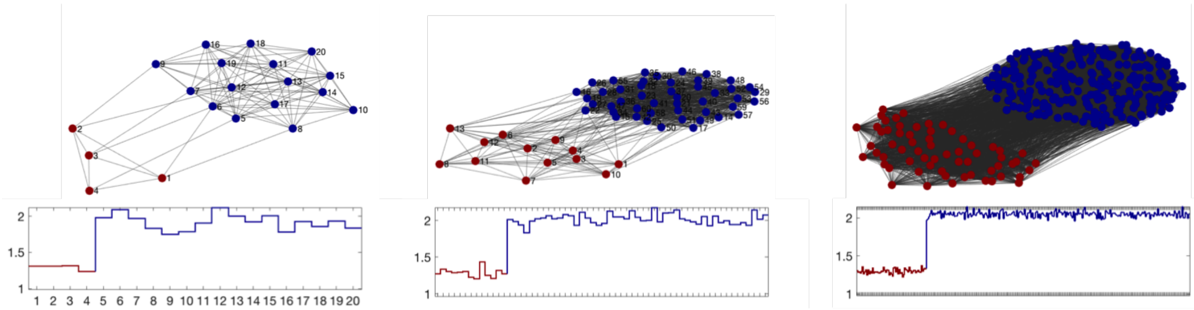

Figure 4: Three networks sampled from a stochastic block model with two communities of size $\pi_{red} = 0.2, \pi_{blue} = 0.8$ with intra-community connection probability 0.8 and inter-community probability 0.1 for $N = 20, 60, 300$. Note that the corresponding equilibria in a sampled linear quadratic network game (with $a = 0.8, b = 1$) converge to the graphon equilibrium where all red agents select strategy 1.3 and all blue agents 2.1.

where $b(x)$ is the status-quo marginal return of agent $x \in [0,1]$, $\hat{b}(x)$ is a targeted intervention, subject to the convex budget constraint $\|\hat{b}\|^2_{L^2} \leq C$, and we assume $a > 0$ for simplicity. Let $\bar{s}_{\hat{b}}$ be the graphon equilibrium corresponding to intervention $\hat{b}(x)$. A targeted intervention problem similar to Problem (18) can be defined in the graphon domain as follows:

$$\begin{aligned} \hat{b}^* \in \arg \max_{\hat{b} \in L^2([0,1])} \quad & \frac{1}{2}\|\bar{s}_{\hat{b}}\|^2_{L^2}, \\ \text{s.t.} \quad & \|\hat{b}\|^2_{L^2} \leq C. \end{aligned} \tag{23}$$

It is shown in Parise and Ozdaglar (2019a) that when the graphon has suitable regularity properties (e.g., it is of finite rank R) then problem (23) can be solved by solving an optimization problem with $R+1$ variables (recall that Problem (18) instead depends on $N+1$ variables and requires exact network information). The optimal solution of (23) can be used to define the sampled graphon intervention

$$[\hat{b}^N_{\text{graphon}}]_i = \frac{\hat{b}^*(t^i)}{\eta^N},$$

where η^N is a normalization to guarantee that the budget constraint in the sampled network is met with equality (i.e., $\frac{1}{N}\|\hat{b}^N_{\text{graphon}}\|^2 = C$, note that we here assume the budget scales with the population size). It is proven in Parise and Ozdaglar (2019a) that such graphon based intervention leads to asymptotically optimal performance for the sampled network.

3.3 Discrete strategies

3.3.1 Independent cascade in Erdos-Renyi models

We discussed in Section 2.2.3 that optimal seeding in large networks may become computationally intractable. Akbarpour et al. (2018) investigate the question of how much gain one would obtain from computing the optimal seeding strategy versus a much simpler strategy where agents are seeded randomly. Therein authors found that for simple contagion models such as the independent cascade model described in Sidebar 3, the advantage is actually marginal. For example they show that if the network is sampled from a Erdős-Rényi model, it is not too sparse and the communication probability among agents is not too low, then randomly seeding a few nodes more leads to a larger cascade than optimal seeding based on detailed network information. This result holds also for a generalized version of Erdős-Rényi graphs with high clustering, against the classic intuition that seeding very central agents (e.g., agents with high degree) should lead to larger gains. The reason for this is that random seeding is likely to seed some of the connections of those highly central agents, precisely because such individuals have many connections. In a simple contagion model, random seeding is therefore sufficient for central agents to be infected at a later iteration. The assumption of simple contagion is crucial for the discussion above, complex contagion models (where a fraction of the neighbors needs to be active for contagion to propagate) require careful placement of the initial seeds even for large populations, see e.g., Jackson and Storms (2019); Moharrami et al. (2016); Erol et al. (2020).

3.3.2 Behavioral communities in stochastic block models

Jackson and Storms (2019) study homogeneous coordination games and the corresponding linear threshold dynamics in networks sampled from a stochastic block model. In this case, coming back to the concept of "behavioral communities", it is interesting to understand what is the connection between atoms in the realized network and communities (blocks) as defined by the stochastic block model.

Jackson and Storms (2019) derive sufficient conditions on the threshold θ for behaviors to result in atoms that remain within blocks or instead produce atoms that spread across multiple blocks. For example, when the threshold θ is low, atoms may cover multiple blocks (i.e., different blocks may behave identically in all equilibria). Under different conditions on θ (which dependen on link probabilities) one can instead ensure that there will always be equilibria where all agents in a block adopt the innovation and all agents outside of that block do not, which immediately implies that atoms will be weakly finer than blocks.

3.3.3 Linear threshold model contagion in configuration models

Several papers looked at linear threshold dynamics in configuration models. Among the first results of this type are Watts (2002); Amini (2010); Lelarge (2012); Baxter et al. (2010), which derived conditions for the contagion to spread to a positive fraction of the network (a "global cascade") when started from a single node or a fraction of nodes that is sublinear in N. Besides characterizing the asymptotic behavior; one may be interested also in understanding the path of contagion. To this point, Rossi et al. (2017) shows that the entire transient of the contagion dynamics on networks sampled from a configuration model with bounded degree can be approximated with a nonlinear, one-dimensional, recursive equation that depends only on the distribution of degrees and thresholds. Specifically, define the absolute threshold of a node as $r^i := \theta^i d^i$, that is, the minimum number of active neighbors needed for an agent to adopt. Let $p_{k,r}$ be the fraction of nodes with out-degree k and absolute threshold r and $q_{k,r}$ be the fraction of edges pointing to agents of out-degree k and absolute threshold r. Rossi et al. (2017) shows that for all but a vanishingly small (as the network size N grows large) fraction of networks, the fraction of state-1 adopters in Algorithm 3 can be approximated, to an arbitrary small tolerance level, by the solution $y(t)$ of the recursion

$$x(t+1) = \phi(x(t)), \quad y(t+1) = \psi(x(t)) \tag{24}$$

where $\phi(x) := \sum_{k\geq 0}\sum_{r\geq 0} q_{k,r}\varphi_{k,r}(x)$ and $\psi(x) := \sum_{k\geq 0}\sum_{r\geq 0} p_{k,r}\varphi_{k,r}(x)$, with $\varphi_{k,r}(x) := \sum_{u=r}^{k}\binom{k}{u}x^u(1-x)^{k-u}$. Intuitively, $y(t)$ represents the expected fraction of active nodes while $x(t)$ represents the expected fraction of edges pointing to an active node. To understand equation (24), suppose that the network is rewired according to the procedure detailed in the configuration model *at every iteration*. Then at time $t+1$ any node with degree k, connects to k half-links at random and the probability that each of this half-links points to an active agent is by definition $x(t)$. If the node has absolute threshold r then he will become active at time $t+1$ with probability $\varphi_{k,r}(x)$. The fraction of expected active nodes at $t+1$ is then $y(t+1) = \psi(x(t))$. The recursion $x(t+1) = \phi(x(t))$ follows similarly using $q_{k,r}$ instead of $p_{k,r}$. This intuition is formalized in (24), which also discusses an analogous result for the one-switch dynamics discussed in Algorithm 4.

Extensions of the results above to generalizations of the configuration model have been discussed for example in Amini et al. (2016), which considers weighted directed networks in the context of cascading failures in financial networks, or in Moharrami et al. (2016), which focuses on the impact of community structure on the cascade dynamics by considering the weak interconnection of multiple graphs with bounded average degree generated from the configuration model. The paper Sadler (2020) considers diffusion games over standard and multi-type configuration models.

3.3.4 Linear threshold model contagion in graphon models

Finally, the behavior of linear threshold models over networks sampled from a graphon has been recently characterized by Erol et al. (2020). Similar to the analysis of graphon games, the first step is to define a contagion process for the continuum of agents in $[0,1]$. Let $S_0 \subseteq [0,1]$ be the integrable set of initial adopters and let $S_k \subseteq [0,1]$ be the set of adopters at iteration k. Contagion proceeds as detailed in Algorithm 5.

Algorithm 5 Linear threshold model with one switch

Set: $k=0$, Initialize $s_0(x)=1$ for all $x \in S_0 \subseteq [0,1]$ and $s_0(x)=0$, otherwise.
Iterate:

$$s_{k+1}(x) = \begin{cases} 1 & \text{if } \frac{\int_{S_k} W(x,y)dy}{\int_0^1 W(x,y)dy} > \theta(x) \text{ or } s_k(x)=1 \\ 0 & \text{otherwise} \end{cases} \qquad \forall x \in [0,1].$$

The main objective of Erol et al. (2020) is to show that, under suitable regularity conditions, the set of asymptotic adopters $S^* := \lim_{k\to\infty} S_k$ in the graphon contagion process gives, with high probability, information about the set of final adopters $S^{\star N}$ in a large network sampled from a graphon according to the procedure described in Section 3.1, under the assumption that the sampled agent i has threshold $\theta(t^i)$ and the initial seed set is $S_0^N := \{i \mid t^i \in S_0\}$. We refer to Erol et al. (2020) for the exact convergence statement. We instead here comment on how such a result could be used to plan asymptotically optimal seeding in dense stochastic block models (recall these are a special case of graphons) when the central planner can select μN seeds, for some $\mu \in [0,1]$.

To this end, we introduce an optimal seeding problem for the graphon process

$$\begin{aligned} S_0^* := \arg\max_{S_0} \quad & |S^*| \\ \text{s.t.} \quad & |S_0| = \mu. \end{aligned} \tag{25}$$

It is shown in Erol et al. (2020) that solving (25) is equivalent to solving a problem that has dimension given by the number of blocks, leading to a computationally tractable task even for large networks. The solution of (25) suggests a heuristic strategy where in any sampled network the central planner seeds $|S_0^* \cap \mathcal{C}_h| N$ agents at random from community h (recall that $\mathcal{C}_h \subset [0,1]$ is the interval associated with community h). It is shown in Erol et al. (2020), via a simulation example with deterministic homogeneous thresholds, that for large networks such a graphon intervention can outperform both random and greedy seeding policies.[11]

4 CONCLUSION

Network games model a wide range of social and economic settings where heterogeneous agent interactions have significant impact on individual behavior. This chapter present a review of

[11] Note that the approximation guarantees in Kempe et al. (2003) for the greedy policy only apply when agents have random uniform thresholds.

classical results and recent developments that provide a key theoretical underpinning for the study of network games. We organize our exposition around games over deterministic and stochastic networks and covered models and methodologies to study games with both continuous and discrete strategy sets. For continuous strategies, we presented a unifying framework based on variational inequalities to study a broad set of network games with mixed strategic interactions. For discrete strategies, we focused on network coordination games and the resulting threshold-type dynamics. We reviewed key recent results for both analysis of equilibria and design of targeted interventions. For stochastic networks we highlighted the informational and computational gains obtained by a central planner designing intervention policies based on knowledge of the stochastic network formation model instead of the realized network.

We conclude our exposition by listing several research directions motivated by the presented framework as well as other important topics in network games that we have not been able to cover in this chapter.

1. *Partial information games:* There are several different extensions based on what information is available to the agents and the central planner. The first direction relaxes the assumption that agents have *full network and/or payoff information* and focuses on Bayesian Nash equilibria, see e.g., Sadler (2020); Galeotti et al. (2010); Kalai (2004); Eksin et al. (2013). In particular, it is shown in Parise and Ozdaglar (2019a) that the graphon equilibrium introduced in Section 3.2 can also be used to approximate the Bayesian Nash equilibria of an incomplete information version of network games sampled from a graphon. Another important direction relates to *identification of peer effects* and involves estimation of game primitives, in particular payoff parameters, from the perspective of the planner by using complete or partial network information, see e.g., Bramoullé et al. (2009); Chandrasekhar and Lewis (2016); De Paula et al. (2018); Boucher and Houndetoungan (2019); Lewbel et al. (2019). A final interesting direction in the context of optimal interventions, in particular seeding, with incomplete information is when the central planner can access the network through *queries*, see e.g., Stein et al. (2017); Wilder et al. (2018); Eckles et al. (2019); Chin et al. (2018); Banerjee et al. (2019).

2. *Population, mean field, aggregative and graphical games:* Graphon games as described in Section 3.2 are infinite population games. Other widely used infinite population models include population games Sandholm (2010); Quijano et al. (2017) and mean field games Lasry and Lions (2007); Huang et al. (2007). In the context of population games, for example, asymptotic stability of the Nash equilibrium set has been established for a widely-used set of protocols and population game classes, such as potential and contractive games, see Sandholm (2001), Hofbauer and Sandholm (2009). Such results have been extended to games with dynamically modified payoffs and higher order dynamics by exploiting the tool of passivity in Fox and Shamma (2013); Gao and Pavel (2020), we refer the interested reader to Park et al. (2019) for a recent survey. In the context of mean-field games, we note that graphons have been recently used to suggest extensions of mean-field games to heterogeneous settings, see e.g., Caines and Huang (2018, 2019); Carmona et al. (2019).

   The behavior of infinite but countable populations has also been studied in aggregative games where each agent is influenced by the same aggregate of the strategies of the rest of the population, as discussed in Kukushkin (2004); Jensen (2010); Cornes and Hartley (2012); Jensen (2005); Acemoglu and Jensen (2013); Jensen (2018). Motivated by technological applications such as demand-response energy markets or communication networks, several papers have studied distributed dynamics for convergence to the equilibria in aggregative games, see e.g., Ma et al. (2013); Chen et al. (2014); Koshal et al. (2016); Grammatico et al. (2016); Paccagnan et al. (2016); Liang et al. (2017); Grammatico (2017a,b); Paccagnan et al. (2018); De Persis and Grammatico (2019); Parise et al. (2019); Gadjov and Pavel (2019); Belgioioso et al. (2020); Parise et al. (2020). Finally, we here focused on games where the strategies of the neighbors appear in aggregated form, graphical games allow for more general network dependence, see e.g., Kearns et al. (2001); Kakade et al. (2004).

3. *Network formation and dynamic contagion games:* In this chapter, we assumed that the network is exogenously given. There is a large literature on endogenous network formation games, where link formation is a strategic decision and agents trade off benefits of connecting to other agents with the cost of establishing a link, see e.g., Jackson (2010); Jackson and Zenou (2014); Bala and Goyal (2000); Pagan and Dörfler (2019); Erol (2018); Chasparis and Shamma (2012, 2013). A related class of games considers strategic decisions related to information or infections traveling through a network via a contagion process, with each agent or a central planner responding ex-ante or ex-post (i.e., before contagion starts or dynamically in real time), see e.g., Acemoglu et al. (2016); Blume et al. (2013); Acemoglu et al. (2017); Eksin et al. (2017); Ajorlou et al. (2018). While almost all works in this area take the network structure as given, in reality strategic link formation has a crucial impact on agents' behavior. These games typically have elements of both complements and substitutes, see e.g., Acemoglu et al. (2017), hence the tools presented here may be relevant in these settings.

References

Acemoglu, D. and Jensen, M. K. (2013). Aggregate comparative statics. *Games and Economic Behavior*, 81:27–49.

Acemoglu, D., Makhdoumi, A., Malekian, A., and Ozdaglar, A. (2017). Privacy-constrained network formation. *Games and Economic Behavior*, 105:255–275.

Acemoglu, D., Malekian, A., and Ozdaglar, A. (2016). Network security and contagion. *Journal of Economic Theory*, 166:536–585.

Acemoglu, D., Ozdaglar, A., and Tahbaz-Salehi, A. (2015). Networks, shocks, and systemic risk. In Bramoullé, Y., Galeotti, A., and Rogers, B., editors, *The Oxford Handbook of the Economics of Networks*. Oxford University Press.

Acemoglu, D., Ozdaglar, A., and Yildiz, E. (Dec 2011). Diffusion of innovations in social networks. In *Proceedings of the IEEE Conference on Decision and Control and European Control Conference*, pages 2329–2334.

Adam, E. M., Dahleh, M. A., and Ozdaglar, A. (2012). On the behavior of threshold models over finite networks. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pages 2672–2677. IEEE.

Ajorlou, A., Jadbabaie, A., and Kakhbod, A. (2018). Dynamic pricing in social networks: The word-of-mouth effect. *Management Science*, 64(2):971–979.

Akbarpour, M., Malladi, S., and Saberi, A. (2018). Just a few seeds more: value of network information for diffusion. *Available at SSRN 3062830*.

Allouch, N. (2015). On the private provision of public goods on networks. *Journal of Economic Theory*, 157:527–552.

Amini, H. (2010). Bootstrap percolation and diffusion in random graphs with given vertex degrees. *the electronic journal of combinatorics*, 17(1):25.

Amini, H., Cont, R., and Minca, A. (2016). Resilience to contagion in financial networks. *Mathematical finance*, 26(2):329–365.

Avella-Medina, M., Parise, F., Schaub, M., and Segarra, S. (Aug 2018). Centrality measures for graphons: Accounting for uncertainty in networks. *arXiv preprint, arXiv:1707.09350*.

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2006). Who's who in networks. Wanted: The key player. *Econometrica*, 74(5):1403–1417.

Banerjee, A., Chandrasekhar, A. G., Duflo, E., and Jackson, M. O. (2019). Using gossips to spread information: Theory and evidence from two randomized controlled trials. *The Review of Economic Studies*, 86(6):2453–2490.

Baxter, G. J., Dorogovtsev, S. N., Goltsev, A. V., and Mendes, J. F. (2010). Bootstrap percolation on complex networks. *Physical Review E*, 82(1):011103.

Belgioioso, G., Nedich, A., and Grammatico, S. (2020). Distributed generalized nash equilibrium seeking in aggregative games on time-varying networks. *IEEE Transactions on Automatic Control*.

Belhaj, M. and Deroïan, F. (2014). Competing activities in social networks. *The B.E. Journal of Economic Analysis & Policy*, 14(4):1431–1466.

Bender, E. A. and Canfield, E. R. (1978). The asymptotic number of labeled graphs with given degree sequences. *Journal of Combinatorial Theory, Series A*, 24(3):296–307.

Bertsekas, D. P. and Tsitsiklis, J. N. (1997). *Parallel and distributed computation: Numerical methods*. Athena Scientific.

Bloch, F., Demange, G., and Kranton, R. (2018). Rumors and social networks. *International Economic Review*, 59(2):421–448.

Blume, L., Easley, D., Kleinberg, J., Kleinberg, R., and Tardos, É. (2013). Network formation in the presence of contagious risk. *ACM Transactions on Economics and Computation (TEAC)*, 1(2):1–20.

Bollobás, B. and Béla, B. (2001). *Random graphs*. Number 73. Cambridge university press.

Bonacich, P. (1987). Power and centrality: A family of measures. *American journal of sociology*, 92(5):1170–1182.

Bordenave, C. (2012). Lecture notes on random graphs and probabilistic combinatorial optimization.

Borgs, C., Chayes, J., Cohn, H., and Zhao, Y. (2019). An L^p theory of sparse graph convergence I: Limits, sparse random graph models, and power law distributions. *Transactions of the American Mathematical Society*, 372(5):3019–3062.

Borgs, C., Chayes, J., Lovász, L., Sós, V., and Vesztergombi, K. (2011). Limits of randomly grown graph sequences. *European Journal of Combinatorics*, 32(7):985–999.

Borgs, C., Chayes, J. T., Lovász, L., Sós, V. T., and Vesztergombi, K. (2008). Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. *Advances in Mathematics*, 219(6):1801–1851.

Boucher, V. and Houndetoungan, A. (2019). Estimating peer effects using partial network data. *Working paper*.

Boyd, S. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Bramoullé, Y., Djebbari, H., and Fortin, B. (2009). Identification of peer effects through social networks. *Journal of econometrics*, 150(1):41–55.

Bramoullé, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, 135(1):478–494.

Bramoullé, Y. and Kranton, R. (2016). Games played on networks. In Bramoullé, Y., Galeotti, A., and Rogers, B., editors, *The Oxford Handbook of the Economics of Networks*. Oxford University Press.

Bramoullé, Y., Kranton, R., and D'Amours, M. (2014). Strategic interaction and networks. *The American Economic Review*, 104(3):898–930.

Breza, E., Chandrasekhar, A. G., McCormick, T. H., and Pan, M. (2018). Using aggregated relational data to feasibly identify network structure without network data. *arXiv preprint arXiv:1703.04157*.

Bullo, F. (2019). *Lectures on Network Systems*. Kindle Direct Publishing, 1.3 edition. With contributions by J. Cortes, F. Dorfler, and S. Martinez.

Caines, P. E. and Huang, M. (2018). Graphon mean field games and the GMFG equations. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 4129–4134. IEEE.

Caines, P. E. and Huang, M. (2019). Graphon mean field games and the gmfg equations: ε-nash equilibria. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 286–292. IEEE.

Candogan, O., Bimpikis, K., and Ozdaglar, A. (2012). Optimal pricing in networks with externalities. *Operations Research*, 60(4):883–905.

Carmona, R., Cooney, D., Graves, C., and Lauriere, M. (2019). Stochastic graphon games: I. the static case. *arXiv preprint arXiv:1911.10664*.

Centola, D. (2018). *How behavior spreads: The science of complex contagions*, volume 3. Princeton University Press.

Centola, D. and Macy, M. (2007). Complex contagions and the weakness of long ties. *American journal of Sociology*, 113(3):702–734.

Chandrasekhar, A. and Lewis, R. (2016). Econometrics of sampled networks.

Chasparis, G. C. and Shamma, J. S. (2012). Distributed dynamic reinforcement of efficient outcomes in multiagent coordination and network formation. *Dynamic games and applications*, 2(1):18–50.

Chasparis, G. C. and Shamma, J. S. (2013). Network formation: Neighborhood structures, establishment costs, and distributed learning. *IEEE Transactions on Cybernetics*, 43(6):1950–1962.

Chen, H., Li, Y., Louie, R. H. Y., and Vucetic, B. (2014). Autonomous demand side management based on energy consumption scheduling and instantaneous load billing: An aggregative game approach. *IEEE Transactions on Smart Grid*, 5(4):1744–1754.

Chen, Y.-J., Zenou, Y., and Zhou, J. (2018). Multiple activities in networks. *American Economic Journal: Microeconomics*, 10(3):34–85.

Chin, A., Eckles, D., and Ugander, J. (2018). Evaluating stochastic seeding strategies in networks. *arXiv preprint arXiv:1809.09561*.

Cornes, R. and Hartley, R. (2012). Fully aggregative games. *Economics Letters*, 116(3):631–633.

Currarini, S., Jackson, M. O., and Pin, P. (2009). An economic model of friendship: Homophily, minorities, and segregation. *Econometrica*, 77(4):1003–1045.

Dasaratha, K. (2017). Distributions of centrality on networks. *arXiv preprint arXiv:1709.10402*.

De Paula, A., Rasul, I., and Souza, P. (2018). Recovering social networks from panel data: identification, simulations and an application.

De Persis, C. and Grammatico, S. (2019). Continuous-time integral dynamics for a class of aggregative games with coupling constraints. *IEEE Transactions on Automatic Control*, 65(5):2171–2176.

DeGroot, M. H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121.

Draief, M., Heidari, H., and Kearns, M. (2014). New models for competitive contagion. In *Twenty-Eighth AAAI Conference on Artificial Intelligence*.

Dubey, P., Haimanko, O., and Zapechelnyuk, A. (2006). Strategic complements and substitutes, and potential games. *Games and Economic Behavior*, 54(1):77–94.

Durrett, R. (1988). *Lecture notes on particle systems and percolation*. Wadsworth Publishing Company.

Easley, D., Kleinberg, J., et al. (2010). *Networks, crowds, and markets*, volume 8. Cambridge university press Cambridge.

Eckles, D., Esfandiari, H., Mossel, E., and Rahimian, M. A. (2019). Seeding with costly network information. *Available at SSRN 3386417*.

Eksin, C., Molavi, P., Ribeiro, A., and Jadbabaie, A. (2013). Learning in network games with incomplete information: Asymptotic analysis and tractable implementation of rational behavior. *IEEE Signal Processing Magazine*, 30(3):30–42.

Eksin, C., Shamma, J. S., and Weitz, J. S. (2017). Disease dynamics in a stochastic network game: a little empathy goes a long way in averting outbreaks. *Scientific reports*, 7:44122.

Erdős, P. and Rényi, A. (1959). On random graphs. *Publicationes Mathematicae Debrecen*, 6:290–297.

Erol, S. (2018). Network hazard and bailouts. *Available at SSRN 3034406*.

Erol, S., Parise, F., and Teytelboym, A. (2020). Contagion in graphons. In *Proceedings of the 2020 ACM Conference on Economics and Computation, SSRN link: 3674691*.

Facchinei, F. and Pang, J.-S. (2003). *Finite-dimensional variational inequalities and complementarity problems*. Operations Research. Springer.

Fazeli, A., Ajorlou, A., and Jadbabaie, A. (2016). Competitive diffusion in social networks: Quality or seeding? *IEEE Transactions on Control of Network Systems*, 4(3):665–675.

Fazeli, A. and Jadbabaie, A. (2012). Game theoretic analysis of a strategic model of competitive contagion and product adoption in social networks. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pages 74–79. IEEE.

Fiedler, M. and Ptak, V. (1962). On matrices with non-positive off-diagonal elements and positive principal minors. *Czechoslovak Mathematical Journal*, 12(3):382–400.

Fox, M. J. and Shamma, J. S. (2013). Population games, stable games, and passivity. *Games*, 4(4):561–583.

Fudenberg, D. and Tirole, J. (1991). *Game theory*. MIT press.

Gadjov, D. and Pavel, L. (2018). A passivity-based approach to nash equilibrium seeking over networks. *IEEE Transactions on Automatic Control*, 64(3):1077–1092.

Gadjov, D. and Pavel, L. (2019). Single-timescale distributed gne seeking for aggregative games over networks via forward-backward operator splitting. *arXiv preprint arXiv:1908.00107*.

Galeotti, A., Golub, B., and Goyal, S. (2017). Targeting interventions in networks. *arXiv preprint arXiv:1710.06026*.

Galeotti, A. and Goyal, S. (2009). Influencing the influencers: a theory of strategic diffusion. *The RAND Journal of Economics*, 40(3):509–532.

Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. (2010). Network games. *The review of economic studies*, 77(1):218–244.

Gao, B. and Pavel, L. (2020). On passivity, reinforcement learning and higher-order learning in multiagent finite games. *IEEE Transactions on Automatic Control*.

Ghaderi, J. and Srikant, R. (2014). Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate. *Automatica*, 50(12):3209–3215.

Goldenberg, A., Zheng, A. X., Fienberg, S. E., and Airoldi, E. M. (2010). A survey of statistical network models. *Foundations and Trends in Machine Learning*, 2(2):129–233.

Goldenberg, J., Libai, B., and Muller, E. (2001). Talk of the network: A complex systems look at the underlying process of word-of-mouth. *Marketing letters*, 12(3):211–223.

Golub, B. and Jackson, M. O. (2012). How homophily affects the speed of learning and best-response dynamics. *The Quarterly Journal of Economics*, 127(3):1287–1338.

Goyal, S., Heidari, H., and Kearns, M. (2019). Competitive contagion in networks. *Games and Economic Behavior*, 113:58–79.

Grammatico, S. (2017a). Dynamic control of agents playing aggregative games with coupling constraints. *IEEE Transactions on Automatic Control*, 62(9):4537–4548.

Grammatico, S. (2017b). Proximal dynamics in multiagent network games. *IEEE Transactions on Control of Network Systems*, 5(4):1707–1716.

Grammatico, S., Parise, F., Colombino, M., and Lygeros, J. (2016). Decentralized convergence to Nash equilibria in constrained deterministic mean field control. *IEEE Transactions on Automatic Control*.

Granovetter, M. (1978). Threshold models of collective behavior. *American journal of sociology*, 83(6):1420–1443.

Hofbauer, J. and Sandholm, W. H. (2009). Stable games and their dynamics. *Journal of Economic theory*, 144(4):1665–1693.

Huang, M., Caines, P. E., and Malhamé, R. P. (2007). Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria. *IEEE Transactions on Automatic Control*, 52(9):1560–1571.

Jackson, M. O. (2010). *Social and economic networks*. Princeton university press.

Jackson, M. O. and Storms, E. (2019). Behavioral communities and the atomic structure of networks. *Available at SSRN 3049748*.

Jackson, M. O. and Zenou, Y. (2014). Games on networks. In Young, P. and Zamir, S., editors, *Handbook of game theory*, volume 4. Elsevier.

Janson, S. and Luczak, M. J. (2009). A new approach to the giant component problem. *Random Structures & Algorithms*, 34(2):197–216.

Jensen, M. K. (2005). Existence, comparative statics and stability in games with strategic substitutes. *Birmingham University*.

Jensen, M. K. (2010). Aggregative games and best-reply potentials. *Economic theory*, 43(1):45–66.

Jensen, M. K. (2018). Aggregative games. In *Handbook of Game Theory and Industrial Organization, Volume I*. Edward Elgar Publishing.

Kakade, S. M., Kearns, M., and Ortiz, L. E. (2004). Graphical economics. In *International Conference on Computational Learning Theory*, pages 17–32. Springer.

Kalai, E. (2004). Large robust games. *Econometrica*, 72(6):1631–1665.

Kearns, M., Littman, M. L., and Singh, S. (2001). Graphical models for game theory. In *Proceedings of the Seventeenth conference on Uncertainty in artificial intelligence*, pages 253–260.

Kempe, D., Kleinberg, J., and Tardos, É. (2003). Maximizing the spread of influence through a social network. In *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 137–146. ACM.

Koshal, J., Nedić, A., and Shanbhag, U. V. (2016). Distributed algorithms for aggregative games on graphs. *Operations Research*, 64(3):680–704.

Kukushkin, N. S. (2004). Best response dynamics in finite games with additive aggregation. *Games and Economic Behavior*, 48(1):94–10.

Lasry, J.-M. and Lions, P.-L. (2007). Mean field games. *Japanese Journal of Mathematics*, 2:229–260.

Lelarge, M. (2012). Diffusion and cascading behavior in random networks. *Games and Economic Behavior*, 75(2):752–775.

Lewbel, A., Qu, X., and Tang, X. (2019). Social networks with misclassified or unobserved links.

Liang, S., Yi, P., and Hong, Y. (2017). Distributed nash equilibrium seeking for aggregative games with coupled constraints. *Automatica*, 85:179–185.

Liggett, T. M. (2012). *Interacting particle systems*, volume 276. Springer Science & Business Media.

Lim, Y., Ozdaglar, A., and Teytelboym, A. (2016). A simple model of cascades in networks.

Liu-Thompkins, Y. (2012). Seeding viral content: The role of message and network factors. *Journal of advertising research*, 52(4):465–478.

Lovász, L. (2012). *Large networks and graph limits*, volume 60. American Mathematical Society Providence.

Lovász, L. and Szegedy, B. (2006). Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, 96(6):933–957.

Ma, Z., Callaway, D. S., and Hiskens, I. A. (2013). Decentralized charging control of large populations of plug-in electric vehicles. *IEEE Transactions on Control Systems Technology*, 21(1):67–78.

Marden, J. R. and Shamma, J. S. (2018). Game theory and control. *Annual Review of Control, Robotics, and Autonomous Systems*, 1:105–134.

McPherson, M., Smith-Lovin, L., and Cook, J. M. (2001). Birds of a feather: Homophily in social networks. *Annual review of sociology*, 27(1):415–444.

Mei, W. and Bullo, F. (2017). Competitive propagation: Models, asymptotic behavior and quality-seeding games. *IEEE Transactions on Network Science and Engineering*, 4(2):83–99.

Mele, A., Hao, L., Cape, J., and Priebe, C. E. (2019). Spectral inference for large stochastic blockmodels with nodal covariates. *arXiv preprint arXiv:1908.06438*.

Melo, E. (2017). A variational approach to network games.

Menache, I. and Ozdaglar, A. (2011). Network games: Theory, models, and dynamics. *Synthesis Lectures on Communication Networks*, 4(1):75.

Milgrom, P. and Roberts, J. (1994). Comparing equilibria. *The American Economic Review*, pages 441–459.

Milgrom, P. and Shannon, C. (1994). Monotone comparative statics. *Econometrica*, pages 157–180.

Moharrami, M., Subramanian, V., Liu, M., and Lelarge, M. (2016). Impact of community structure on cascades. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 635–636. ACM.

Monderer, D. and Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, 14(1):124–143.

Montanari, A. and Saberi, A. (2010). The spread of innovations in social networks. *Proceedings of the National Academy of Sciences*, 107(47):20196–20201.

Morris, S. (2000). Contagion. *The Review of Economic Studies*, 67(1):57–78.

Naghizadeh, P. and Liu, M. (2017). On the uniqueness and stability of equilibria of network games. *Proceedings of the 55th Allerton conference*.

Newman, M. E. J. (2010). *Networks: An Introduction.* Oxford University Press.

Newman, M. E. J., Strogatz, S. H., and Watts, D. J. (2001). Random graphs with arbitrary degree distributions and their applications. *Physical review E*, 64(2):026118.

Nowzari, C., Preciado, V. M., and Pappas, G. J. (2016). Analysis and control of epidemics: A survey of spreading processes on complex networks. *IEEE Control Systems Magazine*, 36(1):26–46.

Paccagnan, D., Gentile, B., Parise, F., Kamgarpour, M., and Lygeros, J. (2018). Nash and wardrop equilibria in aggregative games with coupling constraints. *IEEE Transactions on Automatic Control*, 64(4):1373–1388.

Paccagnan, D., Kamgarpour, M., and Lygeros, J. (2016). On aggregative and mean field games with applications to electricity markets. In *European Control Conference.*

Pagan, N. and Dörfler, F. (2019). Game theoretical inference of human behavior in social networks. *Nature communications*, 10(1):1–12.

Parise, F., Gentile, B., and Lygeros, J. (2019). A distributed algorithm for almost-nash equilibria of average aggregative games with coupling constraints. *IEEE Transactions on Control of Network Systems.*

Parise, F., Grammatico, S., Gentile, B., and Lygeros, J. (2020). Distributed convergence to nash equilibria in network and average aggregative games. *Automatica*, 117:108959.

Parise, F. and Ozdaglar, A. (2019a). Graphon games. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 457–458.

Parise, F. and Ozdaglar, A. (2019b). A variational inequality framework for network games: Existence, uniqueness, convergence and sensitivity analysis. *Games and Economic Behavior*, 114:47–82.

Park, S., Martins, N. C., and Shamma, J. S. (2019). Payoff dynamics model and evolutionary dynamics model: Feedback and convergence to equilibria. *arXiv preprint arXiv:1903.02018.*

Pin, P. and Rogers, B. (2015). Stochastic network formation and homophily. In Bramoullé, Y., Galeotti, A., and Rogers, B., editors, *The Oxford Handbook of the Economics of Networks.* Oxford University Press.

Quijano, N., Ocampo-Martinez, C., Barreiro-Gomez, J., Obando, G., Pantoja, A., and Mojica-Nava, E. (2017). The role of population games and evolutionary dynamics in distributed control systems: The advantages of evolutionary game theory. *IEEE Control Systems Magazine*, 37(1):70–97.

Rossi, W. S., Como, G., and Fagnani, F. (2017). Threshold models of cascades in large-scale networks. *IEEE Transactions on Network Science and Engineering.*

Sadler, E. (2020). Diffusion games. *American Economic Review*, 110(1):225–70.

Sandholm, W. H. (2001). Potential games with continuous player sets. *Journal of Economic theory*, 97(1):81–108.

Sandholm, W. H. (2010). *Population games and evolutionary dynamics.* The MIT press.

Scutari, G., Facchinei, F., Pang, J.-S., and Palomar, D. P. (2014). Real and complex monotone communication games. *IEEE Transactions on Information Theory*, 60(7):4197–4231.

Scutari, G., Palomar, D. P., Facchinei, F., and Pang, J.-S. (2010). Convex optimization, game theory, and variational inequality theory. *IEEE Signal Processing Magazine*, 27(3):35–49.

Stein, S., Eshghi, S., Maghsudi, S., Tassiulas, L., Bellamy, R. K., and Jennings, N. R. (2017). Heuristic algorithms for influence maximization in partially observable social networks. In *SocInf@ IJCAI*, pages 20–32.

Topkis, D. M. (2011). *Supermodularity and complementarity.* Princeton university press.

Tzoumas, V., Amanatidis, C., and Markakis, E. (2012). A game-theoretic analysis of a competitive diffusion process over social networks. In *International Workshop on Internet and Network Economics*, pages 1–14. Springer.

Ui, T. (2016). Bayesian nash equilibrium and variational inequalities. *Journal of Mathematical Economics*, 63:139–146.

Vives, X. (2005). Complementarities and games: New developments. *Journal of Economic Literature*, 43(2):437–479.

Watts, D. J. (2002). A simple model of global cascades on random networks. *Proceedings of the National Academy of Sciences*, 99(9):5766–5771.

Wilder, B., Immorlica, N., Rice, E., and Tambe, M. (2018). Maximizing influence in an unknown social network. In *Thirty-Second AAAI Conference on Artificial Intelligence*.

Young, H. P. (2006). The diffusion of innovations in social networks. *The economy as an evolving complex system III: Current perspectives and future directions*, 267:39.

Young, H. P. (2009). Innovation diffusion in heterogeneous populations: Contagion, social influence, and social learning. *American economic review*, 99(5):1899–1924.