On the degeneration of asymptotically conical Calabi–Yau metrics

ON THE DEGENERATION OF ASYMPTOTICALLY CONICAL CALABI-YAU METRICS

TRISTAN C. COLLINS, BIN GUO, AND FREID TONG

ABSTRACT. We study the degenerations of asymptotically conical Ricci-flat Kähler metrics as the Kähler class degenerates to a semi-positive class. We show that under appropriate assumptions, the Ricci-flat Kähler metrics converge to a incomplete smooth Ricci-flat Kähler metric away from a compact subvariety. As a consequence, we construct singular Calabi-Yau metrics with asymptotically conical behaviour at infinity on certain quasi-projective varieties and we show that the metric geometry of these singular metrics are homeomorphic to the topology of the singular variety. Finally, we will apply our results to study several classes of examples of geometric transitions between Calabi-Yau manifolds.

1. INTRODUCTION

Following Yau's resolution of the Calabi Conjecture [63] the study of Ricci-flat Kähler metrics has played a central role in geometric analysis. Subsequently, motivated by questions in differential geometry, mathematical physics, and algebraic geometry there has been a great deal of interest in extensions of Yau's theorem to the complete, non-compact setting [53, 54, 18, 31, 32, 61, 62, 28, 15, 16, 17, 35, 25, 2, 3], the degeneration of Calabi-Yau metrics (see, for example, the surveys [57, 56, 58] and the references there in), and the existence of Calabi-Yau metrics on singular spaces (see for example [24, 51]). In this paper we initiate the study of degenerations of *non-compact* Calabi-Yau manifolds, and the existence of Calabi-Yau metrics on certain non-compact singular varieties.

In the compact setting, a special class of Calabi-Yau degenerations are obtained by degenerating the Kähler class. More precisely, fix a compact Calabi-Yau manifold X, and let $\mathcal{K} \subset H^{1,1}(X,\mathbb{R})$ denote the Kähler cone, consisting of all $(1,1)$ cohomology classes admitting a Kähler representative; $\mathcal{K}$ is an open convex cone in $H^{1,1}(X,\mathbb{R})$. For each class $[\omega] \in \mathcal{K}$, Yau's theorem [63] yields the existence of a unique Ricci-flat Kähler metric $\omega_{CY} \in [\omega]$. Choose a family of Kähler classes $[\omega_t] \in \mathcal{K}, t \in (0,1]$ such that $[\omega_t] \to [\alpha] \in \partial\mathcal{K}$ as $t \to 0$. We are interested in understanding the geometry of $(X, \omega_{t,CY})$ as $t \to 0$. Roughly speaking this question can be divided into two cases; the collapsing case, when $\int_X \alpha^n = 0$, and the non-collapsing case, when $\int_X \alpha^n > 0$. The non-collapsing case, is reasonably well understood, thanks to work of Tosatti [59] and the first author and Tosatti [13].

One way to construct a non-collapsed family of Calabi-Yau manifolds is as follows; suppose X_0 is a normal, Gorenstein, projective variety with K_{X_0} trivial. Suppose that $\pi : X \to X_0$ is a crepant resolution of singularities, and let $[\alpha] = \pi^*[\omega_0]$ for some Kähler class $[\omega_0]$ on X_0. A family of Kähler classes on X converging to $[\alpha]$ gives rise to non-collapsed family of Calabi-Yau metrics. In this case, the results of [13] say that the Calabi-Yau metrics $\omega_{t,CY}$ converge in $C^\infty_{loc}(X\backslash \mathrm{Exc}(\pi))$, to an incomplete metric $\omega_{0,CY}$ and $(X, \omega_{t,CY})$ Gromov-Hausdorff converge to the completion $\overline{(X\backslash \mathrm{Exc}(\pi), \omega_{0,CY})}$. $\omega_{0,CY}$ descends to a Ricci-flat metric on X_0^{reg}, and one can ask whether the metric geometry of $\omega_{0,CY}$ is related to the

Date: June 18, 2021.

T.C.C is supported in part by NSF grant DMS-1944952, DMS-1810924 and an Alfred P. Sloan Fellowship.
B. G. is supported in part by NSF grant DMS-1945869.
F. T. is supported in part by NSF grant DMS-1855947.

geometry of the X_0. In this case, assuming that $[\alpha] \in H^{1,1}(X, \mathbb{Q})$, Song [51], proved that $\overline{(X\backslash\mathrm{Exc}(\pi), \omega_{0,CY})} = \overline{(X_0^{reg}, \omega_{0,CY})}$ is homeomorphic to X_0. In particular, this yields the existence of a natural Calabi-Yau metric on the singular variety X_0.

In this paper we study degenerations of Calabi-Yau metrics on complete *non-compact* Calabi-Yau manifolds asymptotic to a cone. Complete, non-compact Calabi-Yau manifolds were first constructed by Tian-Yau in [53, 54], and a plethora of examples are now known to exist. A particular subset of these are Calabi-Yau manifolds which are asymptotic to a cone at infinity, these are sometimes called asymptotically conical Calabi-Yau manifolds. Conical Calabi-Yau manifolds are of fundamental importance, since they arise as blow-up limits at the singular points in the limit of a non-collapsing family of Kähler-Einstein manifolds (or more generally Kähler manifolds with bounded Ricci curvature). The conical asymptotics should be regarded here as akin to the non-collapsing condition in the setting of compact Calabi-Yau manifolds discussed above.

The first analytic construction of asymptotically conical Calabi-Yau manifolds was given in [54] and [2, 3], and the construction has been further refined by the work of many authors, see [32, 61, 62, 28, 15, 16, 17, 35, 25] and the references therein. One nice improvement given by these refinements is that, in analogy with Yau's theorem in the compact case [63], one is able to produce an asymptotically conical Ricci-flat Kähler metric in every suitable Kähler class on an asymptotically conical Kähler manifold X. In particular, this yields families of degenerating asymptotically conical Ricci-flat Käher metrics, and one can then ask what properties limits of these spaces possess.

The motivation for studying these limits is two-fold. First, there is a broad class of non-compact examples which are expected to model the local behavior of Calabi-Yau metrics on compact Calabi-Yau manifolds near certain singular limits. Understanding the behavior of these "local" models through singular transitions will help to sharpen our understanding of the degeneration of Ricci-flat metrics in the compact setting. Secondly, understanding these metric limits allows us to prove the existence of asymptotically conical Calabi-Yau metrics on singular spaces. These metrics can be viewed as interpolating between affine varieties with conical Calabi-Yau metrics (or equivalently, Sasaki-Einstein manifolds).

Let us describe the set-up under consideration and state our main theorems. The terminologies used in this section will be explained in the next section. Let (X, J, ω, Ω) be an open Kähler manifold with trivial canonical bundle, with only one end which is asymptotic to a Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ with rate $\nu > 0$. Consider a linear family of ν-almost compactly supported Kähler classes $[\alpha_t] = (1-t)[\alpha_0] + t[\alpha_1] \in H^{1,1}_\nu(X)$ for $t \in (0, 1]$. Suppose $[\alpha_0]$ satisfies the following assumption.

Assumption 1. *$[\alpha_0]$ contains a semi-positive form α_0, and there exists $\varepsilon_0 > 0$ and a $\psi \in PSH(X, \alpha_0)$ such that $\alpha_0 + i\partial\bar{\partial}\psi \geqslant \varepsilon_0\omega$ for some Kähler form ω on X. Furthermore, assume that ψ is smooth away from a compact analytic subvariety $V \subset X$, and $V = \{\psi = -\infty\}$.*

Remark 1. We expect that Assumption 1 essentially always applies, possibly after weakening the semi-positivity assumption. In fact, in analogy with the main result of [13], we expect that

$$V = \bigcup_{Y \subset X : \int_Y \alpha_0^{\dim Y} = 0} Y$$

where the union is taken over compact, irreducible analytic subvarieties. We will prove this in a large class of examples; see the discussion in Section 3.1.

In [15], it is proved that for $t \in (0,1]$ there exists a unique asymptotically conical Ricci-flat Kähler metric $\omega_{t,CY} \in [\alpha_t]$ satisfying the complex Monge-Ampère equation

$$\omega_{t,CY}^n = i^{n^2}\Omega \wedge \bar{\Omega}$$

Our first theorem is the following,

Theorem 1.1. *Let $0 < \nu < 2n$ and consider a linear family of ν-almost compactly supported Kähler classes $[\omega_t] = (1-t)[\alpha_0] + t[\omega] \in H^{1,1}_\nu(X,\mathbb{R})$ for $t \in (0,1]$. Suppose $[\alpha_0]$ satisfies Assumption 1. Let $\omega_{t,CY}$ be the asymptotically conical Calabi-Yau metrics in $[\omega_t]$. Then, as $t \to 0$, the Ricci-flat Kähler metrics $\omega_{t,CY}$ converge in $C^\infty_{loc}(X \setminus V)$ to an incomplete Ricci-flat Kähler metric $\omega_{0,CY}$ on $X \setminus V$ satisfying*

$$\omega_{0,CY}^n = i^{n^2}\Omega \wedge \bar{\Omega}. \tag{1.1}$$

Moreover, we have

(1) *$\omega_{0,CY}$ extends across V as a positive current with locally bounded potentials and (1.1) holds globally in the sense of Bedford-Taylor [4].*
(2) *$\omega_{0,CY}$ is asymptotically conical at infinity and, outside of a compact set $K \subset X$, $\omega_{0,CY}$ satisfies $|\nabla^k(\omega_{0,CY} - \omega_C)|_{\omega_C} = O(r^{-\nu-k})$, where $r(x) = \operatorname{dist}(x_0, x)$ is the distance to a fixed point with respect to the conical Kähler metric ω_C.*
(3) *$\omega_{0,CY}$ is unique in the sense that, if ω is any closed positive current in the class $[\omega_{0,CY}]$ with locally bounded potentials, which is smooth on $X \setminus V$, asymptotically conical at any rate $\delta > 0$, and satisfying (1.1) on X in the sense of Bedford-Taylor, then $\omega = \omega_{0,CY}$.*

The reader may wish to compare this result with the analogous result in the compact case [13, Theorem 1.6]. As discussed before, a natural way to construct examples where Theorem 1.1 applies is to consider resolutions of singular varieties.

Theorem 1.2. *Let (X_0, Ω) be a normal, log-terminal, Gorenstein variety with K_{X_0} trivial, and suppose that X_0 has compactly supported singularities and admits a crepant resolution of singularities $\pi : (X, \Omega) \to (X_0, \Omega)$. Suppose that $L \to X_0$ is an ample line bundle on X_0 (see Section 5 for the definition of ampleness in this context). Let $[\alpha_0] = \pi^* c_1(L) \in H^2(X, \mathbb{R})$ and suppose that (X, J, ω, Ω) and $[\omega_t] = (1-t)[\alpha_0] + t[\omega] \in H^{1,1}_\nu(X, \mathbb{R})$ is a family of Kähler classes satisfying the same hypothesis as in Theorem 1.1. (In particular $[\alpha_0]$ satisfies Assumption 1) In the situation above the singular Ricci-flat current $\omega_{0,CY}$ descends to a Ricci-flat Kähler current on X_0 and satisfies*

(1) *$\omega_{0,CY}$ is a smooth Ricci-flat Kähler metric on $\pi^{-1}(X_0^{reg})$.*
(2) *$\omega_{0,CY}$ descends to a Kähler current on X_0, (i.e. $\omega_{0,CY} \geqslant \omega$ for some smooth Kähler form ω on X_0)*
(3) *$\overline{(X_0^{reg}, \omega_{0,CY})}$ is homeomorphic to X_0.*
(4) *$(X, \omega_{t,CY}, p)$ pointed Gromov-Hausdorff converges to X_0 with the distance function induced by $\omega_{0,CY}$.*

A couple of remarks are in order concerning the assumptions of Theorem 1.2

Remark 2. (1) Theorem 1.2 requires that Assumption 1 to hold for the class $[\alpha_0]$. As pointed out in Remark 1, we expect that in this situation that we can always take $V = \pi^{-1}(X_0^{sing})$, and we will prove this is a large number of cases in Lemma 3.3. Although we don't actually need to assume this for the proof of Theorem 1.2.
(2) The assumption on the existence of an ample line bundle L may seem at odds with our discussion earlier in the introduction. In many cases where Theorem 1.2 applies, we will take $L = \mathcal{O}_{X_0}$. This can be done, for example, when X_0 is affine which is a natural setting for studying Calabi-Yau varieties with isolated singularities.

We apply these results to study several classes of examples. Let us briefly describe one particular class. Consider the quasi-homogeneous affine variety

$$Y_{p,q} := \{xy + z^p - w^q = 0\} \subset \mathbb{C}^4,$$

where without loss of generality we can assume $p \leqslant q$. $Y_{p,q}$ is normal, Gorenstein and log-terminal, and by [12] $Y_{p,q}$ admits a conical Calabi-Yau metric if and only if $q < 2p$. A result of Katz [33] says that the $Y_{p,p}$ admits p inequivalent small (and hence crepant) resolutions resolutions $\mu_i : Y^i \to Y_{p,p}$ (and if $p \neq q$ then no small resolution exists). We therefore have the following picture

$$\begin{array}{ccccc} Y^1 & Y^2 & \cdots & Y^{p-1} & Y^p \\ & \searrow^{\mu_1} \ \searrow^{\mu_2} & & \swarrow^{\mu_{p-1}} \ \swarrow^{\mu_p} & \\ & & Y_{p,p} & & \end{array}$$

with each pair Y^i, Y^j related by a flop. When $p = 2$, this is the well-known example of the Atiyah flop [1]. In Section 6 we apply our results to this setting.

Corollary 1.3. *Let Y^i be a small resolution of the $Y_{p,p}$ singularity, and let ω_0 denote the Calabi-Yau metric on $Y_{p,p}$. Let $[\omega_t] := (1-t)[\alpha_0] + t[\omega]$ be any linear family of Kähler classes on Y^i, where $[\alpha_0] \in H^{1,1}(Y^i, \mathbb{Q})$ is not Kähler. Then for all $t > 0$ there is an asymptotically conical Calabi-Yau metric $\omega_{t,CY} \in [\omega_t]$. Furthermore, there is a partial resolution $\bar{\mu}_i : \overline{Y} \to Y_{p,p}$ and a map $\nu : Y^i \to \overline{Y}$ such that the following diagram commutes*

$$\begin{array}{ccc} Y^i & \xrightarrow{\nu} & \overline{Y} \\ & \searrow_{\mu_i} & \downarrow{\bar{\mu}_i} \\ & & Y_{p,p} \end{array}$$

As $t \to 0$, $\omega_{t,CY}$ converge in $C^\infty_{loc}(Y^i \backslash \mathrm{Exc}(\nu))$ to an incomplete, asymptotically conical Calabi-Yau metric $\overline{\omega}$ on $\overline{Y}_{reg}$ and (Y^i, ω_t) Gromov-Hausdorff converges to $\overline{(\overline{Y}_{reg}, \overline{\omega})}$ which is homeomorphic to $\overline{Y}$. Furthermore, if $[\alpha_0] = 0$ then $\overline{Y} = Y_{p,p}$, $\bar{\mu}_i$ is the identity, and $\bar{\omega} = \omega_0$ the Calabi-Yau metric on $Y_{p,p}$. In particular, when $[\alpha_0] = 0$, for any i, j the flop from Y^i to Y^j is continuous in the Gromov-Hausdorff topology in the sense that

$$(Y^i, \cdot\omega_{t,CY}) \xrightarrow{GH} (Y_{p,p}, \omega_0) \xleftarrow{GH} (Y^j, \cdot\omega_{t,CY})$$

A second general class of examples we consider gives rise to the following specific example. Let X be a del Pezzo surface of degree $d \geqslant 2$, and let $\tilde{X} = \mathrm{Bl}_p X$ be the blow-up at a point $p \in X$. Assume that $\tilde{X}$ is Fano (and if $d = 8$ assume that $\tilde{X}$ is toric). Then the canonical cone

$$C := \mathrm{Spec} \bigoplus_{m \geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$$

admits a conical Calabi-Yau metric [40, 52, 55, 26, 12]. Then we prove

Corollary 1.4. *In the above setting, there is an asymptotically conical Calabi-Yau metric on the relative spectrum $Z := \underline{\mathrm{Spec}}\, (K_X \otimes \mathfrak{m}_p)$ which is asymptotic at infinity to the conical Calabi-Yau metric on C.*

The metric on Z is constructed as a limit of asymptotically conical Calabi-Yau metrics on a small resolution, and we again obtain a Gromov-Hausdorff covergence statement; see Section 6 for a complete discussion.

We will explain a speculative picture in which that space Z can be viewed as a cobordism between Sasakian manifolds; in this case, the link of the A_1 singularity (topologically $S^2 \times$

S^3) and the link of the cone C (topologically $\#(9-d+1)S^2 \times S^3$). The Calabi-Yau metric on Z upgrades this to a cobordism of Sasaki-Einstein manifolds. In this picture the volume of the geodesic spheres can be viewed as a sort of Morse function.

The examples above all come from (partial-)resolutions of Calabi-Yau cones. Our theorem can also yield examples where the complex structure at ∞ is not biholomorphic to the asymptotic cone.

Let X be an asymptotically conical Calabi-Yau manifold, then by [29], there exist a normal Stein space Y with finitely many isolated singularities and there is a holomorphic map $\pi : X \to Y$ with connected fibers, is an biholomorphim outside the singularities of Y and $\pi^\star \mathcal{O}_Y = \mathcal{O}_X$. The map π contracts the maximal compact analytic subset of X and Y is called the Remmert reduction of X. Since Y is a Stein space, it properly embeds into $\mathbb{C}^N$ for some N sufficiently large. The singularities of Y are rational [15, Theorem A.2], and hence Cohen-Macaulay, and since K_X is trivial and Y is normal, it follows that K_Y is trivial and Y is Gorenstein. Hence π is a crepant resolution of Y.

Corollary 1.5. *Assuming Assumption 1 holds for $[\alpha_0] = 0$, applying our theorem with $[\alpha_0] = 0 \in H^2(X, \mathbb{R})$, $\omega_{0,CY}$ descends to a singular CY current on Y and the AC Calabi-Yau metrics $\omega_{t,CY}$ in the classes $t[\omega] \in H^2(X, \mathbb{R})$ Gromov-Hausdorff coverge to the Remmert reduction Y.*

The outline of this paper is as follows. In Section 2 we discuss some basic properties of asymptotically conical Kähler manifolds, and state two main propositions (Propositions 2.5, and 2.6). We give the proof of Theorem 1.1 assuming these two propositions. In Section 3 we discuss the construction of good background metrics, and prove Proposition 2.5. In Section 4 we prove some a priori estimates and deduce Proposition 2.6, completing the proof of Theorem 1.1. In Section 5 we use L^2 estimates to prove Theorem 1.2. Finally, in Section 6 we explain examples where Theorems 1.1 and 1.2 are applicable, and discuss a speculative Morse theoretic picture.

Acknowledgements:. The authors are grateful to D. H. Phong for his interest and encouragement. The authors are also grateful to R. Conlon and H.-J. Hein for explaining aspects of their papers [15, 16, 17].

2. Preliminaries

2.1. Asymptotically conical Kähler manifolds.

We quote here some basic definitions and an existence theorem for asymptotically conical Calabi Yau metrics from [15].

Definition 2.1. (A) An open *Kähler cone* (C, J_C, ω_C, g_C) is a Riemannian cone (C, g_C) with smooth link L that is additionally equipped with a complex structure J_C such that the Kähler form is $\omega_C = i\partial\bar{\partial} r_C^2$ where r_C is the distance function from the tip of the cone.

(B) A *Calabi-Yau cone* $(C, J_C, \omega_C, g_C, \Omega_C)$ is a Kähler cone with an additional holomorphic volume form Ω_C such that $\omega_C^n = i^{n^2} \Omega_C \wedge \bar{\Omega}_C$.

Definition 2.2. (A) A Kähler manifold (X, J, g, ω) is called *asymptotically conical* if there exist a Kähler cone (C, J_C, g_C, ω_C) and a diffeomorphism $\Phi : C \setminus B_R(o) \to X \setminus K$ for some $K \subset\subset X$ and o is the vertex of the cone C, and $\nu > 0$ such that the following hold

$$|\nabla^k(\Phi^* J - J_C)|_{g_C} + |\nabla^k(\Phi^*\omega - \omega_C)|_{g_C} = O(r_C^{-\nu-k}), \quad \forall k \in \mathbb{N}$$

where the covariant derivatives are taken with respect to g_C. We say that X *asymptotic to C with rate ν.*

(B) We say that an open Calabi-Yau manifold (X, J, ω, Ω) is *asymptotic to the Calabi-Yau cone* $(C, J_C, \omega_C, \Omega_C)$ *with rate* ν if (X, J, g, ω) is asymptotic to the Kähler cone (C, J_C, g_C, ω_C) with rate ν, and, in addition

$$|\nabla^k(\Phi^*\Omega - \Omega_C)|_{g_C} = O(r_C^{-\nu-k})$$

Remark 3. (1) On any asymptotically conical Kähler manifold, we can always find a smooth function $r : X \to \mathbb{R}_{\geqslant 0}$ satisfying $r = r_C \cdot \Phi^{-1}$ away from some compact set K where r_C is the radial distance on the cone C, and furthermore, r satisfies: $|\nabla r| + r|\nabla^2 r| \leqslant C$. We will call such an r a *radius function.*

(2) In fact, it is shown in [15, Lemma 2.14] that $\Phi^*J - J_C$ always decays at the same rate as $\Phi^*\Omega - \Omega_C$, so it suffices just to assume $|\nabla^k(\Phi^*\Omega - \Omega_C)|_{g_C} = O(r^{-\nu-k})$.

(3) We will often say (X, J, g, ω) is an *asymptotically conical Kähler manifold* if it is asymptotic to some Kähler cone (C, J_C, g_C, ω_C) at some rate $\nu > 0$ by some map Φ. We will therefore often suppress the map Φ, with the understanding that all asymptotics are measured with respect to the diffeomorphism Φ. Furthermore, when Φ is implicit, we will often abuse notation and write ω_C, J_C, Ω_C in place of $\Phi^{-1})^*\omega_C$, $(\Phi^{-1})^*J_C$, $(\Phi^{-1})^*\Omega_C$.

(4) On an asymptotically conical Kähler manifold with rate ν we will often refer to a $(1,1)$ form α being asymptotically conical. By this we mean that there is a compact set K such that, on $X \backslash K$ the form α defines an asymptotically conical Kähler metric with rate ν.

We now quote a versions of the $\partial\bar{\partial}$-lemma which hold on asymptotically conical Calabi-Yau manifolds, see [15] for a proof.

Proposition 2.1 ($\partial\bar{\partial}$-lemma, [15], Corollary A.3)**.** *Suppose X is an asymptotically conical Kähler manifold with trivial canonical bundle, then*

(1) If α is an exact real $(1,1)$-form on X, then $\alpha = i\partial\bar{\partial}u$ for some smooth function u.
(2) If $\dim_{\mathbb{C}} X > 2$, then if α is an exact real $(1,1)$-form on $X \setminus K$ for some compact subset K, then there exist a compact set K' containing K such that $\alpha = i\partial\bar{\partial}u$ on $X \setminus K'$.

2.2. **Kähler classes on AC Kähler manifolds.** We recall the definition of a ν-almost compactly supported class, this is defined in [15], but our definition is slightly different.

Definition 2.3. Let X be an asymptotically conical Kähler manifold, then for any class $[\alpha] \in H^2(X, \mathbb{R})$, we say that

(1) $[\alpha]$ is a Kähler class if it contains a positive real $(1,1)$-form $\alpha > 0$
(2) $[\alpha]$ is a ν-almost compactly supported class if it contains a real $(1,1)$-form ξ satisfying $|\nabla^k\xi| = O(r^{-\nu-k})$

and we will denote the set of all ν-almost compactly supported classes by $H^{1,1}_\nu(X)$.

Remark 4. Definition 2.3 is slightly more restrictive than the definition given in [15] where it is only required that the form ξ be defined away from a compact set. But by the second part of Proposition 2.1, the condition in [15] implies our condition in the case when X has trivial canonical bundle and $\dim_{\mathbb{C}} X > 2$.

In [15], it is shown that if $[\alpha]$ is a ν-almost compactly supported and Kähler, then one can always construct an asymptotically conical Kähler form $\omega \in [\alpha]$ with $|\nabla^k(\omega - \omega_C)| = O(r^{-\nu-k})$. We will recall this construction below in Section 3.

2.3. **Weighted Hölder spaces and solvability of Poisson's equation.** Let us recall some useful Holder spaces defined on asymptotically conical manifolds and some basic theorems regarding the solvability of Poisson equations, which will be useful for us later on. For a detailed treatment of these material, see [38, 39].

Definition 2.4. Let X be a AC Kähler manifold as above.

(1) We define the $C^{k,\alpha}_{-\gamma}(X)$ norm of a function as follows

$$\|u\|_{C^{k,\alpha}_{-\gamma}} = \sum_{j=0}^{k} \sup_X |r^{\gamma+j}\nabla^j u| + [\nabla^k u]_{C^{\alpha}_{-\gamma-k-\alpha}}$$

where r is a radius function and

$$[\nabla^k u]_{C^{\alpha}_{-\gamma-k-\alpha}} = \sup_{x\neq y, d(x,y)\leqslant\delta}\left[\min(r(x),r(y))^{\gamma+k+\alpha}\frac{|\nabla^k u(x)-\nabla^k u(y)|}{|d(x,y)|^{\alpha}}\right]$$

where $\delta > 0$ is the convexity radius of X (i.e. balls of radius less than δ are convex), and $|\nabla^k u(x) - \nabla^k u(y)|$ is defined by parallel transporting $\nabla^k u(x)$ along the minimal geodesic from x to y.

(2) We define $C^{\infty}_{-\gamma}(X)$ to be the intersection of $C^{k,\alpha}_{-\gamma}(X)$ over all $k \geqslant 0$.

(3) We will also often use the following space $C^{\infty}_{-\gamma}(X \setminus V)$, which we define to be the space of functions $u \in C^{\infty}_{loc}(X \setminus V)$ such that $(1-\chi)u \in C^{\infty}_{-\gamma}(X)$, where χ is a cutoff function with compact support that is equal to 1 in a neighborhood of V. Where V is the compact analytic subset coming from Assumption 1.

With these definitions, we now recall a quantitative version of the $\partial\partial$-Lemma for asymptotically conical Kähler manifolds with non-negative Ricci curvature, which is proved in [15].

Proposition 2.2 (Quantitative $\partial\bar{\partial}$-lemma, [15], Theorem 3.11)**.** *Suppose X is an asymptotically conical Kähler manifold with $Ric \geqslant 0$, then there exist $\varepsilon_0 > 0$, such that for any η an exact $(1,1)$-form with $\eta \in C^{\infty}_{-\varepsilon}(X)$ for $0 < \varepsilon < \varepsilon_0$, then $\eta = i\partial\bar{\partial}u$ for $u \in C^{\infty}_{2-\varepsilon}$.*

Now we wish to recall some Fredholm theory in the spaces $C^{k,\alpha}_{-\gamma}(X)$, which is a Banach space with the norm $\|\cdot\|_{C^{k,\alpha}_{-\gamma}}$ defined above. In this setting, the Laplace operator $\Delta : C^{k+2,\alpha}_{-\gamma+2}(X) \to C^{k,\alpha}_{-\gamma}(X)$ is a bounded map of Banach spaces, and there is a well-developed Fredholm theory for these spaces on an asymptotically conical manifold (see, e.g. [39]), which we summarize below.

Definition 2.5. Let (C, g_C) be a Riemannian cone of real dimension n over a smooth compact manifold L^{n-1}, then we denote the set of *exceptional weights* of the cone C,

$$P = \left\{-\frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda} : \lambda \text{ is an eigenvalue of } \Delta_{L^{n-1}}\right\}.$$

These correspond to the growth rates of homogenous harmonic functions on the cone (C, g_C).

The following theorem summarizes Fredholm theory on an asymptotically conical manifold

Theorem 2.3 ([39], Theorem 6.10)**.** *Suppose (X, g) is an asymptotically conical Kähler manifold of dimension $2n$. Consider the mapping*

$$\Delta : C^{k+2,\alpha}_{-\gamma}(X) \to C^{k,\alpha}_{-\gamma-2}(X) \tag{2.1}$$

and let P be the set of exceptional weights of the asymptotic cone (C, g_C). Then:

(1) The operator (2.1) Fredholm if $-\gamma \notin P$.
(2) The operator (2.1) is surjective if $-\gamma \in (2-2n, \infty) \setminus P$
(3) The operator (2.1) is injective if $-\gamma \in (-\infty, 0) \setminus P$

Remark 5. We note that $P \cap (2-2n, 0) = \emptyset$, hence (2.1) is an isomorphism for all $-\gamma \in (2-2n, 0)$.

Now we state a general theorem regarding the solvability of the complex Monge-Ampère equation on an asymptotically conical Kähler manifold, which is proved in [15].

Theorem 2.4 ([15], Theorem 2.4)**.** *Let* (X, J, ω) *be a open Kähler manifold asymptotic to a Kähler cone* (C, J_C, ω_C) *with rate* $\nu > 0$, *and suppose* $f \in C^{\infty}_{-\gamma-2}(X)$, *then following Complex Monge-Ampere equation then admits a solution*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^f \omega^n$$

with $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0$ *and*

(1) If $\gamma + 2 > 2n$, *then we can take* $\varphi \in C^{\infty}_{2-2n}$ *and* φ *is the unique solution in* C^{∞}_{2-2n}.
(2) If $\gamma + 2 \in (2, 2n)$ *then we can take* $\varphi \in C^{\infty}_{-\gamma}$ *and* φ *is the unique solution in* $C^{\infty}_{-\gamma}$.
(3) If $\gamma + 2 \in (0, 2)$ *and* $-\gamma$ *is not an exceptional weight, we can take* $\varphi \in C^{\infty}_{-\gamma}$.

2.4. **Proof of Theorem 1.1.** We breakdown the proof of Theorem 1.1 in the following two propositions, and we will give the proof of Theorem 1.1 assuming these results. We will prove Proposition 2.5 in Section 3 and Proposition 2.6 in Section 4. Theorem 1.2 will be proved in section 5.

Proposition 2.5 (Constructing background metrics)**.** *Suppose* $\nu > 0$, *and let* (X, J, ω, Ω) *be an asymptotic to a Calabi-Yau cone* $(C, J_C, \omega_C, \Omega_C)$ *with rate* ν. *Suppose that* $-\nu \in (-2n, 0)$ *and* $-\nu + 2$ *is not an exceptional weight. Suppose* $[\alpha_t] = (1-t)[\alpha_0] + t[\alpha_1] \in H^{1,1}_{\nu}(X)$ *is a linear family of Kähler classes in* $H^{1,1}_{\nu}$ *for* $t \in (0, 1]$, *and suppose that* $[\alpha_0] \in H^{1,1}_{\nu}$ *has a semi-positive representative* α_0. *Then there exists* $\varepsilon > 0$, *a compact set* $K \subset X$ *and a smooth family of real* $(1,1)$*-forms* $\hat{\omega}_t \in [\alpha_t]$ *for* $t \in [0, \varepsilon]$ *satisfying the following:*

(1) $\hat{\omega}_t > 0$ *for all* $t \in (0, \varepsilon]$.
(2) $\hat{\omega}_0 \geqslant 0$ *and* $\hat{\omega}_0 = \alpha_0$ *on a compact set* $K \subset\subset X$. *(In fact, we can choose this compact set* K *to be as large as we like)*
(3) On $X \backslash K$ *there holds* $|\nabla^k(\hat{\omega}_t - \omega_C)|_{g_C} \leqslant Cr^{-\nu-k}$ *for all* $t \in [0, \varepsilon]$ *for a constant* C *independent of* t.
(4) There exist $\gamma > 0$ *such that, on* $X \backslash K$ *the Ricci potentials* $f_t = \log \frac{i^{n^2}\Omega \wedge \bar{\Omega}}{\hat{\omega}_t^n}$ *satisfy the asymptotics* $|\nabla^k f_t| \leqslant Cr^{-\gamma-2-k}$ *uniformly in* t.

Proposition 2.6 (A priori estimates)**.** *Let* (X, J, ω, Ω) *be asymptotic to a Calabi-Yau cone* $(C, J_C, \omega_C, \Omega_C)$ *with rate* $\nu > 0$, *and* $H^{1,1}_{\nu}(X) \ni [\alpha_t] = (1-t)[\alpha_0] + t[\alpha_1]$ *is a linear family of Kähler classes for* $t \in (0, 1]$ *satisfying Assumption 1, and let* $\hat{\omega}_t \in [\alpha_t]$ *be the forms constructed in Proposition 2.5. Let* φ_t *be the solution of the complex Monge-Ampère equations*

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = e^{f_t}\hat{\omega}_t^n (= i^{n^2}\Omega \wedge \bar{\Omega}) \tag{2.2}$$

obtained from Theorem 2.4. Then the following estimates hold uniformly in t

(1) $|\varphi_t| \leqslant C$.
(2) φ_t *is uniformly bounded in* $C^{\infty}_{loc}(X \setminus V)$.
(3) There exist a compact subset $K \subset X$ *containing* V *such that the following estimate hold outside of* K

$$|\nabla^k \varphi_t| \leqslant Cr^{-\gamma-k}$$

for C *independent of* t.

Now we prove Theorem 1.1 given the above two propositions

Proof of Theorem 1.1. Let $[\alpha_t] = (1-t)[\alpha]+t[\varepsilon\omega]$, then by Proposition 2.5, we can construct a sequence of background metrics $\hat{\omega}_t \in [\alpha_t]$ satisfying the properties stated in the Proposition. Then using these as background metrics, we can write down a family of complex Monge-Ampere equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = e^{f_t}\hat{\omega}_t^n (= i^{n^2}\Omega \wedge \bar{\Omega})$$

then by the Theorem 2.4, the equations are solvable for $t > 0$, and Proposition 2.6 applies to the family of solutions φ_t. Once we have the a priori estimate, it's then clear that by taking a subsequence, we can take a limit $\varphi_{t_i} \to \varphi_0$ in $C^\infty_{loc}(X \setminus V)$, which satisfies the equation

$$(\hat{\omega}_0 + i\partial\bar{\partial}\varphi_0)^n = i^{n^2}\Omega \wedge \bar{\Omega} \tag{2.3}$$

smoothly away from the analytic set V. Moreover, φ_0 is a bounded by the uniform C^0 estimate of φ_t, hence $\hat{\omega}_0 + i\partial\bar{\partial}\varphi_0$ extends as a non-negative current on X by [30], and it does not charge any analytic subsets, so the equation (2.3) holds globally. From Proposition 2.5 (2), and Proposition 2.6 (3), we see that ω_{φ_0} is asymptotically conical. It only remains to establish the incompleteness and uniqueness statements of ω_{φ_0} in Theorem 1.1. The incompleteness of ω_{φ_0} follows from the diameter bound in Lemma 4.14, while the uniqueness is established in Theorem 4.15 □

3. Background metrics

The goal of this section is to prove Proposition 2.5, which constructs a family of "good" background metrics $\hat{\omega}_t \in [\alpha_t]$ whose Ricci potentials decay faster than quadratically. Indeed, it is easy to construct $\omega_t \in [\alpha_t]$ satisfying only the first two conditions of Proposition 2.5. However, the proof of the a priori estimates of Proposition 2.6 depends crucially on the additional decay of the Ricci potentials. This idea is used in [15] (see also [14, Prop. 4.2.6]).

From now on we fix an open Calabi-Yau manifold (X, J, Ω) asymptotic to some Calabi-Yau cone $(C, J_C, \Omega_C, \omega_C, g_C)$ at rate $\nu > 0$. In the following proposition, we summarize a construction of asymptotically conical Kähler (semipositive) forms in almost compactly support classes, which is based on [15].

Proposition 3.1. *Suppose $[\alpha] \in H^{1,1}_\nu(X)$ contains a (semi-)positive form α, then there exist a (semi-)positive form $\omega \in [\alpha]$ which agrees with α in a compact set K and satisfies the asymptotics $|\nabla^k(\omega - \omega_C)| = O(r^{-\nu-k})$ for $r \gg 1$.*

Proof. This follows from construction in [15, Theorem 2.4].

□

Proposition 3.2. *Suppose that $(X, J, \Omega, \omega_t, g_t)_{t\in[0,1]}$ are a smooth family of data which is asymptotic to the cone $(C, J_C, \Omega_C, \omega_C, g_C)$ at the rate $-\nu \in (-2, 0)$. Suppose that for $t \in (0, 1]$, ω_t are asymptotically conical Kähler metrics and ω_0 is asymptotically conical and semi-positive $(1,1)$ form. Let f_t, $t \in [0,1]$ be the Ricci potentials of ω_t, defined by $e^{f_t} = \frac{i^{n^2}\Omega\wedge\bar{\Omega}}{\omega_t^n}$, and suppose there is a compact set $K \subset X$ so that on $X \backslash K$, f_t satisfy the following asymptotics:*

(1) $|f_t| \leqslant Cr^{-\beta}$
(2) $|\nabla^k f_t|_{g_C} \leqslant Cr^{-\beta-k}$

where C is independent of t and $\nu \leqslant \beta < 2n-2$ and $-\beta + 2$ is not an exceptional weight.

Then there exist $\varepsilon > 0$ and a family of functions u_t for $t \in [0, \varepsilon]$ such that the following are satisfied

(1) There exist a compact subset $K \subset X$ such that $supp(u_t) \subset X \setminus K$
(2) $\omega_t + i\partial\bar{\partial}u_t > 0$ on $supp(u_t)$
(3) $|\nabla^k u_t|_{g_C} \leqslant Cr^{-\beta+2-k}$
(4) $|\nabla^k \frac{\partial u_t}{\partial t}|_{g_C} \leqslant Cr^{-\beta+2-k}$
(5) Away from a compact set K, we have

$$(\omega_t + i\partial\bar{\partial}u_t)^n = e^{f_t - f_t'}\omega_t^n = e^{-f_t'} i^{n^2}\Omega \wedge \bar{\Omega}$$

where $|\nabla^k f_t'| \leqslant Cr^{-2\beta-k}$ outside a compact set K.

where the constant C is independent of t. In particular, this means if we set $\omega_t' = \omega_t + i\partial\bar{\partial}u_t$, then ω_t' converges to ω_C at the same rate as ω_t, but the Ricci potentials f_t' of ω_t' decays a rate of -2β.

Proof. We can essentially follow the same procedure as in [15, Lemma 2.12]. First we want to solve the equation

$$\Delta_{\omega_t}\hat{u}_t = 2f_t$$

for $t \geqslant 0$, away from a compact set while controlling of the growth of the solutions.

We now fix a standard cutoff function $\chi : \mathbb{R} \to \mathbb{R}$ with

$$\chi(x) = \begin{cases} 0 & \text{for } x \leqslant 1 \\ 1 & \text{for } x \geqslant 2 \end{cases}$$

and satisfy $0 \leqslant \chi \leqslant 1$, $|\chi'| \leqslant 2$, $|\chi''| \leqslant 5$. Then we define $\zeta_R : X \to \mathbb{R}$ by setting $\zeta_R(x) = \chi(\frac{r(x)}{R})$, and let $\hat{g}$ be any metric on X. Then set

$$\bar{g}_t = (1 - \zeta_R)\hat{g} + g_t$$

Since ω_0 is semi-positive and asymptotically conical we can choose R sufficiently large so that $\bar{g}_0$ is an asymptotically conical Riemannian metric. Then for all $t \in [0, 1]$, g_t defines a background metric and for $t \in (0, 1]$, this metric is equal to the ω_t away from a compact set.

If $-\beta + 2$ is not an exceptional weight, then $\Delta_{\bar{g}_t} : C^\infty_{-\beta+2} \to C^\infty_{-\beta}$ is surjective by Theorem 2.3, so we can always solve the equation

$$\Delta_{\bar{g}_t}\hat{u}_t = 2\zeta_R f_t$$

for $\hat{u}_t \in C^\infty_{-\beta+2}$. In fact, by the Implicit Function Theorem [22, Proposition 4.2.19], we can find a family of smoothly varying solutions for $t \in [0, \varepsilon)$, and such that following bounds hold uniformly for small t.

(1) $|\nabla^k \hat{u}_t| \leqslant Cr^{-\beta+2-k}$
(2) $|\nabla^k \frac{\partial \hat{u}_t}{\partial t}| \leqslant Cr^{-\beta+2-k}$

If we set $u_t = \zeta_S \hat{u}_t$ then u_t is supported on $\mathrm{supp}(\zeta_S)$, and then we have

$$\begin{aligned} |i\partial\bar{\partial}u_t| &\leqslant |\zeta_S||\partial\bar{\partial}\hat{u}_t| + |\hat{u}_t||\partial\bar{\partial}\zeta_S| + 2|\partial\zeta_S||\partial u_t| \\ &\leqslant C\zeta_S r^{-\beta} + Cr^{-\beta+2}|\partial\bar{\partial}\zeta_S| + Cr^{-\beta+1}|\nabla\zeta_S| \\ &\leqslant Cr^{-\beta}(\zeta_S + r|\nabla\zeta_S| + r^2|i\partial\bar{\partial}\zeta_S|) \\ &\leqslant Cr^{-\beta}(\zeta_S + S|\nabla\zeta_S| + S^2|i\partial\bar{\partial}\zeta_S|) \end{aligned}$$

but since $\zeta_S(x) = \chi(\frac{r(x)}{S})$, we see that

$$|\nabla\zeta_S| = S^{-1}|\chi'\nabla r| \leqslant CS^{-1}$$

and

$$|i\partial\bar{\partial}\zeta_S| \leqslant S^{-2}|\chi''||\nabla r|^2 + |\chi'|S^{-1}|i\partial\bar{\partial}r| \leqslant CS^{-2}$$

where we used that $r|i\partial\bar{\partial}r|\leqslant C$. So we have

$$|i\partial\bar{\partial}u_t|\leqslant Cr^{-\beta}(\zeta_S+C)$$

and $i\partial\bar{\partial}u_t$ is supported on the support of ζ_S. Hence for S sufficiently large, we can ensure that $\omega_t+i\partial\bar{\partial}u_t>0$ on the $\mathrm{supp}(u_t)$.

Away from the compact set K, we have

$$\begin{aligned}\frac{(\omega_t+i\partial\bar{\partial}u_t)^n}{\omega_t^n}&=1+f_t+O(|i\partial\bar{\partial}u_t|^2)\\&=1+f_t+O(r^{-2\beta})\end{aligned}$$

so setting $f_t'=f_t-\log\frac{(\omega_t+i\partial\bar{\partial}u_t)^n}{\omega_t^n}$, we have

$$(\omega_t+i\partial\bar{\partial}u_t)^n=e^{f_t-f_t'}\omega_t^n$$

and $f_t'=f_t-\log(1+f_t+O(r^{-2\beta}))$ has the desired asymptotics. □

Remark 6. If $-\beta+2$ is an exceptional weight, we can apply the proposition with $\beta+\varepsilon$ in place of β for ε arbitrarily small (since the exceptional weights are discrete). We can then repeatedly apply Proposition 3.2 to improve the decay of Ricci potential for a family of metrics until we obtain the decays we need.

The two previous propositions combined proves Proposition 2.5.

Proof of Proposition 2.5. By Proposition 3.1, we can find a semi-positive form $\omega_0\in[\alpha_0]$ satisfying the asymptotics $|\nabla^k(\omega_C-\omega_0)|=O(r^{-\nu-k})$ and a metric $\omega_1\in[\alpha_1]$ satisfying the same asymptotics, then if we write ω_t by linearly interpolating between ω_0 and ω_1, then clearly ω_t are positive for $t>0$ and satisfy the desired asymptotics, and the Ricci potentials f_t satisfy $|\nabla^k f_t|\leqslant C(1+r)^{-\nu-k}$. If $\nu>2$, then we can take $\gamma=\nu$ and we are done, otherwise, we can apply Proposition 3.2 repeatedly to improve the asymptotics of the Ricci potentials until they decay faster than quadratically. □

3.1. **Kähler currents and Null loci in the asymptotically conical case.** Before proceeding we would like to briefly discuss Assumption 1. Recall that if (X,ω) is compact Kähler and let $\mathcal{K}$ be the Kähler cone of X. Let $[\alpha]\in\overline{\mathcal{K}}$ is a nef class with $\int_X\alpha^n>0$, then, by results of Demailly-Păun [21] there is a function $\psi:X\to\mathbb{R}\cup\{-\infty\}$ such that

$$\alpha+\sqrt{-1}\partial\bar{\partial}\psi\geqslant\varepsilon\omega$$

for some $\varepsilon>0$, ψ is smooth on the complement of an analytic subset Z, and $\{\psi=-\infty\}=Z$. Furthermore, by results of the first author and Tosatti [13] ψ can be chosen so that the analytic subvariety Z is given by

$$\mathrm{Null}(\alpha):=\bigcup_{\int_V\alpha^{\dim V}=0}V$$

where the union is taken over irreducible analyitic subvarieties $V\subset X$. We expect that a similar result holds in the asymptotically conical setting. We make the following conjecture

Conjecture 1. *Suppose $[\alpha]\in H^{1,1}_\nu(X,\mathbb{R})$ is a limit of ν-almost compactly support Kähler classes. Then there is a function $\psi:X\to\mathbb{R}\cup\{-\infty\}$ such that $\alpha+\sqrt{-1}\partial\bar{\partial}\psi\geqslant\varepsilon\omega$ for some asymptotically conical Kähler form ω. Define*

$$Null(\alpha):=\bigcup_{\int_V\alpha^{\dim V}=0}V\qquad(3.1)$$

where the union is taken over all compact, irreducible, analytic subvarieties $V \subset X$. Then Null(α) is an analytic subvariety, and ψ can be chosen so that ψ is smooth on $X \backslash$Null(α) and

$$\{\psi = -\infty\} = \mathit{Null}(\alpha).$$

At a purely moral level, the reason that non-compact analytic subvarieties should not enter into the definition of Null(α) in the asymptotically conical setting is that, at least when $[\alpha]$ admits a semi-positive representative, Proposition 3.1 yields the existence of a form $\hat{\alpha} \in [\alpha]$ which is asymptotically conical. Thus, if V is a non-compact subvariety, then $\int_V \hat{\alpha}^{\dim V} = +\infty$. Of course, this is purely moral reasoning, since the integral $\int_V \hat{\alpha}^{\dim V}$ is not independent of the representative of $[\alpha]$.

Lemma 3.3. *Conjecture 1 holds when, $[\alpha]$ is semi-positive and the cone at infinity is quasi-regular.*

Recall that the cone $(C, J_C, \Omega_C, \omega_C, g_C)$ is quasi-regular if the holomorphic vector field $r_C \frac{\partial}{\partial r_C} - \sqrt{-1} J_C \left(r_C \frac{\partial}{\partial r_C} \right)$ integrates to define a $\mathbb{C}^*$ action.

Proof. By a result of Conlon-Hein [17], building on work of Li [35], if $(X, J, \Omega, \omega, g)$ is asymptotically conical Calabi-Yau with quasi-regular Calabi-Yau cone at infinity, then there is a complex, projective orbifold M without codimension 1 singularities, and a orbidivisor D with positive normal orbibundle such that $M = X \cup D$, and $-K_M = q[D]$ for some $q \geqslant 1$. Furthermore, every Kähler form on X is cohomologous to the restriction of a Kähler form on M, and the restriction map $H^{1,1}(M) \to H^2(X)$ is surjective. Let $[\omega_t] = (1-t)[\alpha_0] + t[\omega_0] \in H^{1,1}(X)$ be a family of ν-almost compactly supported Kähler classes for $t \in (0, 1]$ such that $[\alpha_0]$ is semi-positive. In fact, according to [15, Proposition 2.5] all Kähler classes on X are 2-almost compactly supported, so the assumption of almost compact support can be dropped. Let $[\hat{\omega}], [\hat{\alpha}_0] \in H^{1,1}(M)$ be such that $[\hat{\omega}]$ is Kähler, and $[\hat{\omega}]|_X = [\omega_0], [\hat{\alpha}_0]|_X = [\alpha_0]$. Since α_0 is semi-positive, and D has positive normal bundle, the argument in the proof of [17, Theorem A] shows that we can find a constant $C > 0$ so that $[\hat{\alpha}_0] + C[D]$ is semi-positive, and positive in a neighborhood of D. Furthermore, since $D|_D$ is positive, after possibly increasing C we can assume that

$$\int_M ([\hat{\alpha}_0] + C[D])^n > 0$$

Let $\pi : \overline{M} \to M$ be a resolution of singularities, obtained by blowing up smooth centers. Since X is smooth, and M has only codimension 2 singularities, we can assume that $\pi|_X$ is an isomorphism, and that π is an isomorphism at the generic point of D. Let E denote the exceptional divisor of π, and let $\bar{D} = \pi^{-1}(D)$ be the total transform of D. Now we have

$$\pi^*[\hat{\alpha}_0] + C[\bar{D}]$$

is nef, and big by Demailly-Păun [21]. By the results of [21] and the first author and Tosatti [13] there is a Kähler current in $\pi^*[\hat{\alpha}_0] + C[\bar{D}]$ which is smooth on the complement of Null($\pi^*[\hat{\alpha}_0] + C[\bar{D}]$). Let $Y \subset \bar{M}$ be an irreducible analytic subvariety of dimension $p > 0$. If $Y \cap \pi^{-1}(D) = \emptyset$, then

$$\int_Y (\pi^*[\hat{\alpha}_0] + C[\bar{D}])^p = \int_{\pi(Y)} \alpha_0^p$$

and so $Y \subset$ Null($\pi^*[\hat{\alpha}_0] + C[\bar{D}]$) if and only if $\pi(Y) \subset$ Null($[\alpha_0]$). Now suppose that $Y \cap \pi^{-1}(D) \cap \pi^{-1}(X) \neq \emptyset$. Let $\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u$ be the smooth semi-positive representative of $[\hat{\alpha}_0] + C[D]$ which is positive in a neighborhood of D. Then, since π is an isomorphism at the generic point of Y we have

$$\int_Y (\pi^*[\hat{\alpha}_0]+C[\bar{D}])^p = \int_{Y\setminus(E\cap Y)} [\pi^*(\hat{\alpha}_0+C\beta_D+\sqrt{-1}\partial\bar{\partial}u)]^p \int_{\pi(Y)} (\hat{\alpha}_0+C\beta_D+\sqrt{-1}\partial\bar{\partial}u)^p > 0,$$

where the last inequality follows from the fact that $\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u \geqslant 0$ and there is a neighborhood of $\pi(Y) \cap D$ where $\hat{\alpha}_0 + C\beta_D + \sqrt{-1}\partial\bar{\partial}u > 0$. Thus we have

$$\mathrm{Null}(\pi^*[\hat{\alpha}_0] + C[\bar{D}]) \cap (\pi^{-1}(D))^c = \pi^{-1}\left(\mathrm{Null}([\alpha_0])\right).$$

Since $\pi : \bar{M}\setminus\pi^{-1}(D) \to X$ is an isomorphism, the result follows.

□

4. A priori estimates

In this section, we prove Proposition 2.6. Let us first recall the general setup of the proposition. Let (X, J, ω, Ω) be an asymptotically conical Calabi-Yau manifold which is asymptotic to the Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ with rate $\nu > 0$, and $[\alpha_t] = (1-t)[\alpha_0] + t[\alpha_1] \in H^{1,1}_\nu$ for $t \in [0, 1]$ is a family of ν-almost compactly supported classes such that $[\alpha_t]$ is Kähler for $t > 0$. Suppose $[\alpha_0]$ satisfies Assumption 1. Then let $\hat{\omega}_t \in [\alpha_t]$ for $t \in (0, 1]$ be a family of asymptotically conical Kähler metrics satisfying the conclusion of Proposition 2.5. Then by Theorem 2.4, we can solve the equation

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = i^{n^2}\Omega \wedge \bar{\Omega}(= e^{f_t}\hat{\omega}_t^n)$$

for $\varphi_t \in C^\infty_{-\gamma}(X)$, the our goal in this section is to prove a priori estimates on the potentials φ_t that are uniform in t as $t \to 0$.

4.1. Uniform estimates.

4.1. Uniform estimates. In this section, we prove a uniform bound for φ_t that is independent of t. In the compact case, such an estimate can be proved using pluripotential theory following the seminal work of Kolodziej [34], see [24]. Pluripotential methods allow one to obtain an estimate with a sharper dependence on the data of the right hand side. However, such methods are hard to adapt to the non-compact setting and no proper analogue of such estimates are known. It would be of interest to try to find extensions of the pluripotential estimates to the non-compact setting, as it would give a sharper estimates which would apply more generally to singular Calabi-Yau manifolds not admitting crepant resolutions.

Instead, we will use an idea based on the original argument of Yau [63] using the Moser iteration. However, following an idea of Tosatti [59] we perform the Moser iteration using the Calabi-Yau metrics $\omega_{\varphi_t} := \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$ as background metrics. The advantage of this trick is that since the metrics ω_{φ_t} are Ricci flat and asymptotically conical, they have a uniform Sobolev inequality by results of Croke [19] and Yau [64].

Proposition 4.1. *The metrics ω_{φ_t} satisfy a uniform Sobolev inequality of the form*

$$\left(\int_X |u|^{\frac{2n}{n-1}} i^{n^2}\Omega \wedge \bar{\Omega}\right)^{\frac{n-1}{n}} \leqslant C \int_X |du|^2_{\omega_{\varphi_t}} i^{n^2}\Omega \wedge \bar{\Omega} \tag{4.1}$$

Proof. It suffices to prove the result for compact supported smooth functions. Results of Croke [19] and Yau [64] show that for a compactly supported function u, with $\mathrm{supp}(u) \subset \Omega$ for an arbitrary relatively compact set $\Omega \subset X$, (4.1) holds for a constant C, depending on an upper bound for the diameter of Ω, a lower bound for the volume of Ω, and a lower bound for the Ricci curvature. We only need to exploit the scale invariance of these quantities for asymptotically conical Calabi-Yau metrics. Fix a point $x_0 \in X$. Since ω_{φ_t} are asymptotically conical, for R sufficiently large we have

$$\mathrm{Vol}_{\omega_{\varphi_t}}(B_R(x_0)) \sim R^{2n}\mathrm{Vol}_{\omega_C}(L)$$

where L is the link of the cone, identified with $\{r_C = 1\} \subset C$, and the volume is computed using the conical Calabi-Yau metric ω_C. Therefore, if $\omega_R = R^{-2}\omega_{\varphi_t}$, then with respect to the rescaled metric the diameter is 1, and the volume is $\mathrm{Vol}_{\omega_C}(L)$. Since (4.1) is scale invariant, the result follows. □

Proposition 4.2. *Given solutions φ_t to (2.2), with $|\nabla^k\varphi| = O(r^{-\gamma-k})$ we have the following uniform estimate for the potential*

$$|\varphi_t| \leqslant C\|\varphi_t\|_{L^p(i^{n^2}\Omega\wedge\bar{\Omega})}$$

for any $p > \frac{2n-2}{\gamma} \geqslant 1$ and C depending on n, p, and a uniform bound on $\|e^{-f_t} - 1\|_{L^q}$ for $q \in [p, \infty]$.

Proof. If we set $T_t = \sum_{k=0}^{n-1} \omega_{\varphi_t}^k \wedge \hat{\omega}_t^{n-1-k}$, then we can rewrite the equation as

$$-i\partial\bar{\partial}\varphi_t \wedge T_t = (e^{-f_t} - 1)i^{n^2}\Omega\wedge\bar{\Omega}$$

multiplying both sides by $|\varphi_t|^{p-2}\varphi_t$ and integrating, we get

$$-\int_M |\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t \wedge T_t = \int_M |\varphi_t|^{p-2}\varphi_t(e^{-f_t} - 1)i^{n^2}\Omega\wedge\bar{\Omega}$$

we will integrate by parts on the first term

$$\begin{aligned}
-\int_M |\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t \wedge T_t &= \lim_{R\to\infty}\left(-\int_{B_R} |\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t \wedge T_t\right)\\
&= \lim_{R\to\infty}\left((p-1)\int_{B_r} |\varphi_t|^{p-2} i\partial\varphi_t \wedge \bar{\partial}\varphi_t \wedge T_t - \int_{\partial B_R} |\varphi_t|^{p-2}\varphi_t i\bar{\partial}\varphi_t \wedge T_t\right)\\
&= \frac{4(p-1)}{p^2}\int_M i\partial|\varphi_t|^{\frac{p}{2}} \wedge \bar{\partial}|\varphi_t|^{\frac{p}{2}} \wedge T_t - \underbrace{\lim_{R\to\infty}\int_{\partial B_R} |\varphi_t|^{p-2}\varphi_t i\bar{\partial}\varphi_t \wedge T_t}_{=0 \text{ for } p > \frac{2n-2}{\gamma}}
\end{aligned}$$

Combined with the Sobolev inequality, we have

$$\left(\int_M |\varphi_t|^{p\frac{n}{n-1}} i^{n^2}\Omega\wedge\bar{\Omega}\right)^{\frac{n-1}{n}} \leqslant C\frac{np^2}{4(p-1)}\int_M |\varphi_t|^{p-1}|e^{-f_t} - 1| i^{n^2}\Omega\wedge\bar{\Omega}$$

for any $p > \frac{2n-2}{\gamma}$. By Hölder's inequality, we have (below $\frac{1}{q} + \frac{1}{q'} = 1$)

$$\|\varphi_t\|^p_{L^{p\frac{n}{n-1}}} \leqslant C\frac{np^2}{4(p-1)}\||\varphi_t|^{p-1}\|_{L^q}\|e^{-f_t} - 1\|_{L^{q'}} = C\frac{np^2}{4(p-1)}\|\varphi_t\|^{p-1}_{L^{q(p-1)}}\|e^{-f_t} - 1\|_{L^{q'}} \tag{4.2}$$

picking q such that $q = \frac{p}{p-1} > 1$, we get

$$\begin{aligned}
\|\varphi_t\|^p_{L^{p\frac{n}{n-1}}} &\leqslant \frac{C_S np^2}{4(p-1)}\|\varphi_t\|^{p-1}_{L^p}\|e^{-f_t} - 1\|_{L^p}\\
&\leqslant \frac{CC_S np^2}{4(p-1)}\|\varphi_t\|^{p-1}_{L^p}
\end{aligned}$$

a standard Moser iteration argument gives the result. □

Proposition 4.3. *For any $p > \frac{2n}{\gamma}$, we have a uniform L^p estimate of the form*

$$\|\varphi_t\|_{L^p} \leqslant C$$

for C depending on n, p and $\|e^{-f_t} - 1\|_{L^{\frac{np}{n+p}}}$.

Proof. In equation (4.2), if we pick $q = \frac{p}{p-1}\frac{n}{n-1}$, then $q' = \frac{np}{n+p+1}$ and we get

$$\|\varphi_t\|_{L^{p\frac{n}{n-1}}} \leqslant C\frac{np^2}{4(p-1)}\|e^{-f_t} - 1\|_{L^{\frac{np}{n+p-1}}}$$

relabelling p to be $\frac{np}{n-1}$ gives us our result. □

Corollary 4.4. *The potentials φ_t are bounded in L^p uniformly in t for any $p \in (\frac{2n}{\gamma}, \infty]$,*

$$\|\varphi_t\|_{L^p} \leqslant C_p$$

In particular, the potentials φ_t are uniformly bounded in C^0.

Proof. This follows by combining Proposition 4.2 and Proposition 4.3. Note that since $|f_t| \leqslant Cr^{-\gamma-2}$ outside a fixed compact set, we have an estimate $\|e^{-f_t} - 1\|_{L^{\frac{np}{n+p}}} \leqslant C$ for a constant C independent of p, t for any $p > \frac{2n-2}{\gamma}$. □

4.2. **Convergence of the metric away from the degeneracy locus.** In this section, we prove an estimate for $\partial\bar{\partial}\varphi_t$ away from V, the subvariety coming from Assumption 1. Recall that by Assumption 1, there exist $\psi \in PSH(X, \alpha_0)$ which is smooth outside of V and goes to $-\infty$ near V, the idea is to use this function as a barrier function in the C^2 estimate, and this is first used by Tsuji in in [60] to study Kähler-Ricci flow. We remark that this is the only part of the Theorem that uses the current in Assumption 1.

Before we prove the estimate, we first construct a slightly more better behaved barrier function $\psi_\varepsilon \in PSH(X, \hat{\omega}_0)$ which is compactly supported. Recall that from the construction of $\hat{\omega}_0$, $\hat{\omega}_0$ is equal to α_0 on a large compact set. (which from the construction can be as large as one want)

Lemma 4.5. *There exist $\psi_\varepsilon \in PSH(X, \hat{\omega}_0)$ which is compactly supported and satisfy $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon \geqslant \varepsilon\omega$, and is smooth outside V and goes to $-\infty$ near V.*

Proof. Recall by [15, Lemma 2.15], we know that $r^{2\kappa}$ for $\kappa \in (0, 1)$ is strictly plurisubharmonic for r sufficiently large, and satisfies

$$|\nabla r^{2\kappa}| = O(r^{2\kappa-1}) \qquad |i\partial\bar{\partial}r^{2\kappa}| = O(r^{2\kappa-2})$$

Pick $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ smooth satisfy $\Psi', \Psi'' \geqslant 0$ and

$$\Psi(x) = \begin{cases} T+2 & \text{for } x < T+1 \\ x & \text{for } x > T+3 \end{cases}$$

then as in [15, Lemma 2.15], for $T \gg 1$, $\Psi(r^{2\kappa})$ is plurisubharmonic and equal to $r^{2\kappa}$ for r sufficiently large.

We set

$$\psi_\varepsilon = (1 - \zeta_S)\psi + C(1 - \zeta_R)\Psi(r^{2\kappa})$$

where S, C, R are chosen as follows. First we pick $S \gg 1$ large enough such that $\hat{\omega}_0 = \alpha_0$ on $\{r \leqslant S\}$ and $i\partial\bar{\partial}\Psi(r^{2\kappa}) > 0$ on $\{S \leqslant r \leqslant 2S\}$, which implies that $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon = \alpha_0 + i\partial\bar{\partial}\psi \geqslant \varepsilon_0\omega$ on $\{r \leqslant S\}$. Then pick $C \gg 1$ large enough so that $Ci\partial\bar{\partial}\Psi(r^{2\kappa}) > i\partial\bar{\partial}((1 - \zeta_S)\psi)$ on $\{S \leqslant r \leqslant 2S\}$. Finally, we pick $R \gg S$ such that $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > 0$ on $\{R \leqslant r \leqslant 2R\}$, which is possible since for R large, we have

$$|i\partial\bar{\partial}(1 - \zeta_R)\Psi(r^{2\kappa})| \leqslant |\nabla^2\zeta_R||r^{2\kappa}| + |\nabla^2\Psi(r^{2\kappa})||1 - \zeta_R| + |\nabla\zeta_R||\nabla r^{2\kappa}| \leqslant CR^{2(\kappa-1)} \ll 1$$

Then $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > 0$ and $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > \varepsilon_0\omega$ on the compact set K containing V, hence there exist an $\varepsilon > 0$ such that $\hat{\omega}_0 + i\partial\bar{\partial}\psi_\varepsilon > \varepsilon\omega$ holds. □

Now we prove the main estimate of this section.

Proposition 4.6. *There are uniform constants $B, C > 0$, independent of t such that the following estimate holds:*

$$|\partial\bar{\partial}\varphi_t| \leqslant Ce^{-B\psi_\varepsilon}.$$

Proof. By the well-known computation of Aubin and Yau, we have

$$\Delta_{\varphi_t} \log \mathrm{Tr}_\omega \omega_{\varphi_t} \geqslant -A\mathrm{Tr}_{\varphi_t}\omega$$

where A is a lower bound for the bisectional curvatures of ω. Then if we pick $N \gg B$ sufficiently large, we have

$$\begin{aligned} \Delta_{\varphi_t}\left(\log \mathrm{Tr}_\omega \omega_{\varphi_t} + B\psi_\varepsilon - N\varphi_t\right) &\geqslant (B\varepsilon - A)\mathrm{Tr}_{\varphi_t}\omega - B\mathrm{Tr}_{\varphi_t}\hat{\omega}_0 + N\mathrm{Tr}_{\varphi_t}\hat{\omega}_t - Nn \\ &\geqslant C\left(\frac{\omega^n}{ni^{n^2}\Omega\wedge\bar{\Omega}}\mathrm{Tr}_\omega\omega_{\varphi_t}\right)^{\frac{1}{n-1}} - Bn \end{aligned}$$

since ψ_ε goes to $-\infty$ near V and the function $\log \mathrm{Tr}_\omega\omega_{\varphi_t} + B\psi_\varepsilon - N\varphi_t$ goes to 0 at infinity, either $\log \mathrm{Tr}_\omega\omega_{\varphi_t} + B\psi_\varepsilon - N\varphi_t$ is always non-positive, in which case we are done, or maximum is achieved in the interior, and applying the maximum principle gives

$$\mathrm{Tr}_\omega\omega_{\varphi_t} \leqslant Ce^{B(\sup\psi_\varepsilon - \psi_\varepsilon)}$$

from which the estimate follows. □

Remark 7. This argument is the only place where we used the Kähler current in Assumption 1. In the situation where $[\alpha_0] = \pi^* c_1(L)$ where $\pi : X \to X_0$ is a crepant resolution of a singular Calabi-Yau variety with compactly supported singularities and $L \to X_0$ is an ample line bundle on X_0, the above C^2 estimate can be replaced by the argument in Lemma 5.1, and the convergence holds away from $\pi^{-1}(X_0^{sing})$. In that case we do not need the Kähler current in Assumption 1 to prove Theorem 1.1.

The higher order estimates follow from the standard methods of Yau [63, 45, 50].

Proposition 4.7 (Higher order estimates)**.** *We have a uniform estimate*

$$\|\varphi_t\|_{C^{k,\alpha}_{loc}(K)} \leqslant C(K, k, \alpha)$$

for any $K \subset\subset X \setminus V$ and C independent of t.

Proof. This follows from the local estimates in [50]. □

Corollary 4.8. *The metrics ω_{φ_t} converge after passing to a subsequence in $C^\infty_{loc}(X \setminus V)$ to a possibly incomplete metric ω_{φ_0} on $X \setminus V$, which is uniformly equivalent to ω_C at infinity.*

So far, we've shown the first two parts of Proposition 2.6, in the next section we prove decay estimates for φ_t.

4.3. **Decay estimates.** In this section, we prove uniform decay estimates for φ_t. We use the method of Moser iteration with a weight, similar to the technique used in [32, Chap 8]. However, as in Section 4.1, we use the Ricci flat metrics ω_{φ_t}, exploiting the uniform control of the Sobolev constants.

Recall that $r : X \to \mathbb{R}_{>0}$ is a radius function such that $|\nabla r| + r|i\partial\bar{\partial}r| \leqslant C$, and it's not hard to see that we can also assume that $r = const$ on a compact set K containing the singular set V.

Definition 4.1. We define the following weighted L^p norms,

$$\|u\|_{L^p_\delta(i^{n^2}\Omega\wedge\bar{\Omega})}=\left(\int_X |ur^\delta|^p r^{-2n} i^{n^2}\Omega\wedge\bar{\Omega}\right)^{\frac{1}{p}}$$

Remark 8. Notice if we let $p\to\infty$, then the L^p_δ norms converge to the L^∞_δ norm given by $\|u\|_{L^\infty_\delta}=\sup_X|ur^\delta|$

Proposition 4.9. *For any $\delta<\gamma$, we have a uniform bound of the form*

$$\|\varphi_t\|_{L^p_\delta(\omega^n_{\varphi_t})}\leqslant C$$

for any $p\in(0,\frac{2n}{\delta}]$, and constant depending on p,δ.

Proof. If $p=\frac{2n}{\delta}$, then this is simply the $L^{\frac{2n}{\delta}}$ norm, which is bounded if $\delta<\gamma$ by Proposition 4.3. If $p<\frac{2n}{\delta}$, then

$$\int_X |\varphi r^\delta|^p r^{-2n}\omega^n_{\varphi_t}\leqslant\left(\int_X|\varphi_t|^{pq}\right)^{\frac{1}{q}}\left(\int_X r^{\frac{q}{q-1}(\delta p-2n)}\omega^n_{\varphi_t}\right)^{\frac{q-1}{q}}$$

the first term is bounded if $q>\frac{2n}{\gamma p}$ by Proposition 4.3, and the second term is finite if $q<\frac{2n}{\delta p}$, so we just need to pick $q\in(\frac{2n}{\gamma p},\frac{2n}{\delta p})$ with $q>1$, which is possible since $p<\frac{2n}{\delta}$. □

Proposition 4.10. *For any $\delta<\gamma$, $p>1$, we have*

$$\|\varphi_t r^\delta\|^p_{L^{p\frac{n}{n-1}}(r^{-2n}\omega^n_{\varphi_t})}\leqslant\frac{Cp^2}{p-1}\left(\|\varphi_t r^\delta\|^{p-1}_{L^{p-1}(r^{-2n}\omega^n_{\varphi_t})}+\|\varphi_t r^\delta\|^p_{L^p(r^{-2n}\omega^n_{\varphi_t})}\right)$$

for C depending on the Sobolev constant of ω_{φ_t}, δ and the dimension n.

Proof. We use the same method as in [32, Proposition 8.6.7], but using the Calabi-Yau metrics ω_{φ_t} as the background metrics. The reason is because the metrics ω_{φ_t} are Ricci-flat and hence have a uniform Sobolev inequality. First we set

$$T_t=\sum_{k=0}^{n-1}\omega^k_{\varphi_t}\wedge\hat{\omega}_t^{n-1-k}.$$

If $q-p\gamma<-2n+2$, then Stoke's theorem gives the following two identities

$$\begin{aligned}0&=\int_X i\partial\left(r^q|\varphi_t|^{p-2}\varphi_t\bar{\partial}\varphi_t\wedge T_t\right)\\&=(p-1)\int_X r^q|\varphi_t|^{p-2}i\partial\varphi_t\wedge\bar{\partial}\varphi_t\wedge T_t+q\int_X r^{q-1}|\varphi_t|^{p-2}\varphi_t i\partial r\wedge\bar{\partial}\varphi_t\wedge T_t+\int_X r^q|\varphi_t|^{p-2}\varphi_t i\partial\bar{\partial}\varphi_t\wedge T_t\end{aligned}$$

and

$$\begin{aligned}0&=-\int_X i\bar{\partial}\left(r^{q-1}|\varphi_t|^p i\partial r\wedge T_t\right)\\&=p\int_X r^{q-1}|\varphi_t|^{p-2}\varphi_t i\partial r\wedge\bar{\partial}\varphi_t\wedge T_t+(q-1)\int_X r^{q-2}|\varphi_t|^p i\partial r\wedge\bar{\partial}r\wedge T_t+\int_X r^{q-1}|\varphi_t|^p i\partial\bar{\partial}r\wedge T_t\end{aligned}$$

using these identities, we can obtain through integration by parts

$$\begin{aligned}\int_X |\nabla(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}})|^2_{\omega_{\varphi_t}} \omega^n_{\varphi_t} &= n\int_X i\partial(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge \bar\partial(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge \omega^{n-1}_{\varphi_t} \\ &\leqslant n\int_X i\partial(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge \bar\partial(|\varphi_t|^{\frac{p}{2}} r^{\frac{q}{2}}) \wedge T_t \\ &= -\frac{np^2}{4(p-1)} \int_X \varphi_t|\varphi_t|^{p-2} r^q i\partial\bar\partial\varphi_t \wedge T_t \\ &\quad + \frac{mq}{4(p-1)} \int_X |\varphi_t|^p r^{q-2} [(p+q-2) i\partial r \wedge \bar\partial r - (p-2) r i\partial\bar\partial r] \wedge T_t \\ &= -\frac{np^2}{4(p-1)} \int_X \varphi_t|\varphi_t|^{p-2} r^q (e^{f_t}-1)\omega^n_{\varphi_t} \\ &\quad + \frac{mq}{4(p-1)} \int_X |\varphi_t|^p r^{q-2} [(p+q-2) i\partial r \wedge \bar\partial r - (p-2) r i\partial\bar\partial r] \wedge T_t\end{aligned}$$

where in the last equality, we used the equation $i\partial\bar\partial\varphi_t \wedge T_t = (e^{f_t}-1)\omega^n_{\varphi_t}$. Now we claim there also exist a uniform constant C independent of t and r such that

$$\left|\frac{[(p+q-2) i\partial r \wedge \bar\partial r - (p-2) r i\partial\bar\partial r] \wedge T_t}{i^{n^2}\Omega \wedge \bar\Omega}\right| \leqslant C(p+|q|)$$

recall that we chose r so that $r = const$ on a compact set K containing V, so the left hand side of the expression is 0 on K. By Corollary 4.8 we know that $|T_t| \leqslant C$ on $X \setminus K$, and because r is a radius function we also have $|\nabla r| + r|\partial\bar\partial r| \leqslant C$, hence the expression also holds on $X \setminus K$. Putting it together, we see that the expression holds on all of X.

This then combined with the Sobolev inequality, we conclude that

$$\left(\int_X |\varphi_t|^{p\frac{n}{n-1}} r^{q\frac{n}{n-1}} \omega^n_{\varphi_t}\right)^{\frac{n-1}{n}} \leqslant \frac{Cnp^2}{4(p-1)} \int_X |e^{-f_t}-1||\varphi_t|^{p-1} r^q \omega^n_{\varphi_t} + \frac{Cq(p+q)}{4(p-1)} \int_X |\varphi_t|^p r^{q-2} \omega^n_{\varphi_t}$$

for any $\delta < \gamma$ we can set $q = 2(1-n) + p\delta$ and use the fact that $|e^{f_t} - 1| \leqslant Cr^{-\gamma-2}$ to obtain,

$$\begin{aligned}\left(\int_X |\varphi_t r^\delta|^{p\frac{n}{n-1}} r^{-2n} \omega^n_{\varphi_t}\right)^{\frac{n-1}{n}} &\leqslant C\frac{p^2}{4(p-1)} \left(\int_X |\varphi_t|^{p-1} r^{p\delta-\gamma} r^{-2n} \omega^n_{\varphi_t} + \int_X |\varphi_t r^\delta|^p r^{-2n} \omega^n_{\varphi_t}\right) \\ &= C\frac{p^2}{4(p-1)} \left(\int_X |\varphi_t r^\delta|^{p-1} r^{\delta-\gamma} r^{-2n} \omega^n_{\varphi_t} + \int_X |\varphi_t r^\delta|^p r^{-2n} \omega^n_{\varphi_t}\right)\end{aligned}$$

and since $\delta < \gamma$, which means for any $p > 1$, we have

$$\|\varphi_t r^\delta\|^p_{L^{p\frac{n}{n-1}}(r^{-2n}\omega^n_{\varphi_t})} \leqslant \frac{Cp^2}{p-1} \left(\|\varphi_t r^\delta\|^{p-1}_{L^{p-1}(r^{-2n}\omega^n_{\varphi_t})} + \|\varphi_t r^\delta\|^p_{L^p(r^{-2n}\omega^n_{\varphi_t})}\right)$$

□

Corollary 4.11. *For any $\delta < \gamma$, we have a uniform bound of the form*

$$|\varphi_t| \leqslant Cr^{-\delta}$$

for C depending on δ.

Proof. By Proposition 4.9, we have a weighed L^p bound for any $p \leqslant \frac{2n}{\delta}$, combined with the previous proposition, we can use the standard Moser iteration argument starting from $p = \frac{2n}{\delta} \geqslant \frac{n}{n-1} > 1$. □

Proposition 4.12. *For any $\delta < \gamma$, the derivative of the solutions φ_t satisfy uniform decay estimates on $X \setminus K$,*

$$|\nabla^k \varphi_t| \leqslant C r^{-\delta-k}$$

where $C = C(n, \delta, k)$ which doesn't depend on t.

Proof. This follows from the methods of [32, Theorem 8.6.11] verbatim. The point to note here is that the metrics ω_{φ_t} are uniformly equivalent to ω_C on the region $X \setminus K$, with bounded derivatives as well, hence the Schauder constants are uniformly controlled on far away balls. □

Proposition 4.13. *If $\gamma \in (0, 2n-2)$, then in fact we have*

$$|\nabla^k \varphi_t| \leqslant C r^{-\gamma-k}$$

on $X \setminus K$, and $C = C(n, k)$ independent of t.

Proof. This follows from the same argument as in [32, Chap 8.7, Theorem A2]. □

We can now prove Proposition 2.6, thereby completing the proof of Theorem 1.1.

Proof of Proposition 2.6. Combine Corollary 4.4, Proposition 4.7, Proposition 4.11 and Proposition 4.13.

□

We now prove the local diameter bound, which will play an important role throughout the remainder of the paper.

Lemma 4.14. *In the setting of Theorem 1.1, let $K \subset X$ be a compact subset containing V. Then the diameter of K with respect to the Calabi-Yau metrics $\omega_{t,CY}$ is uniformly bounded from above as $t \to 0$.*

$$Diam_{\omega_{\varphi_t}} K \leqslant C$$

Proof. It suffices to show that the sets $K_R = \{r(x) \leqslant R\}$ have bounded diameters for R sufficiently large. Recall that the metrics ω_{φ_t} are uniformly asymptotic to ω_{cone} for r large and t close to 0 by Proposition 4.12. Fix any two points $x, y \in K_R$, and joint them by a length minimizing geodesic $\gamma : [0, L] \to X$. We claim that γ must lie inside K_{R^2} for R sufficiently large. Note for R large, on the region $\{r(x) \geqslant R\}$ the metric ω_{φ_t} is C^∞ close to a cone metric uniformly in t, and hence for R sufficiently large, the boundary of K_R has diameter bounded by $2\pi R$. However, the distance between the boundary of K_R and K_{R^2} on the order of R^2, so it's clear that any minimizing geodesic between two points in K_R cannot leave K_{R^2}. Now consider $x_i = \gamma(2i+1)$ and disjoint balls $B_1(x_i)$. Note that these balls have a fixed lower bound on the volume, since by Bishop-Gromov volume comparison and the asymptotically conical geometry we have

$$\mathrm{Vol}(B_1(x_i)) \geqslant \lim_{S \to \infty} \frac{\mathrm{Vol}(B_S(x_i))}{S^{2n}} = \mathrm{Vol}_{g_C}(L) =: c > 0$$

where L is the link of the cone at infinity, identified with $\{r_C = 1\}$ and g_C is the conical Calabi-Yau metric. Thus, we have

$$\sum_i \mathrm{Vol}(B_1(x_i)) \geqslant c \frac{\lfloor L \rfloor}{2}$$

where c is the non-collapsing constant. On the other hand, these balls must all lie in K_{2R}, and since the volume form of the Calabi-Yau metrics are fixed, we must have that

$$c \frac{\lfloor L \rfloor}{2} \leqslant \int_{K_{R^2}} i^{n^2} \Omega \wedge \bar{\Omega}$$

which gives us a bound for L, which is $d_{\omega_{\varphi_t}}(x, y)$. □

4.4. **Uniqueness.** In this section, we discuss the uniqueness of the Calabi-Yau currents constructed in the previous sections.

Theorem 4.15. *The current that we constructed ω_{φ_0} above is unique in the sense that if ω is another positive current with locally bounded potentials in the same cohomology class as ω_{φ_0} which is smooth on $X \setminus V$, asymptotically conical at infinity with any rate $\delta > 0$ and satisfies the complex Monge-Ampère equation*

$$\omega^n = \omega_{\varphi_0}^n = i^{n^2}\Omega \wedge \bar{\Omega}$$

in the Bedford-Taylor sense, then $\omega = \omega_{\varphi_0}$.

The proof is modelled after the idea introduced in [15], which relies on the following crucial Lemma proved in [15].

Lemma 4.16. [15, Corollary 3.9] *Suppose (X, ω) is an asymptotically conical Kähler manifold with $Ric \geqslant 0$, then for any $\varepsilon > 0$, any harmonic function $u \in C^{\infty}_{2-\varepsilon}(X)$ is pluriharmonic.*

The idea is to write $\omega = \omega_{\varphi_0} + i\partial\bar{\partial}\psi$ and use this lemma to improve the asymptotics of the potential function ψ by subtracting off pluriharmonic functions from it, until we are left in the case where the potential function is decaying in which case uniqueness follows from a standard integration by parts argument.

Proposition 4.17. *Suppose $\varphi \in PSH(X, \omega_{\varphi_0}) \cap L^{\infty}(X) \cap C^{\infty}_{-\varepsilon}(X \setminus V)$ is a function such that the current $\omega_{\varphi_0} + i\partial\bar{\partial}\varphi$ satisfies*

$$(\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^n = \omega_{\varphi_0}^n = i^{n^2}\Omega \wedge \bar{\Omega}$$

in the Bedford Taylor sense, then $\varphi = 0$.

Proof.

$$\begin{aligned} 0 &= -\int_{B_R} |\varphi|^{p-2}\varphi((\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^n - \omega_{\varphi_0}^n) = -\int_{B_R} |\varphi|^{p-2}\varphi i\partial\bar{\partial}\varphi \wedge \left(\sum_{k=0}^{n-1} \omega_{\varphi_0}^k \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right) \\ &= \frac{4(p-1)}{p^2} \int_{B_R} i\partial(|\varphi|^{\frac{p}{2}}) \wedge \bar{\partial}(|\varphi|^{\frac{p}{2}}) \wedge \left(\sum_{k=0}^{n-1} \omega_{\varphi_0}^k \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right) \\ &\quad - \int_{\partial B_R} |\varphi|^{p-2}\varphi i\bar{\partial}\varphi \wedge \left(\sum_{k=0}^{n-1} \omega_{\varphi_0}^k \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right) \end{aligned}$$

picking $p > \frac{2n-2}{\gamma}$ and letting $R \to \infty$, we get

$$\int_X i\partial(|\varphi|^{\frac{p}{2}}) \wedge \bar{\partial}(|\varphi|^{\frac{p}{2}}) \wedge \left(\sum_{k=0}^{n-1} \omega_{\varphi_0}^k \wedge (\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^{n-1-k}\right) = 0$$

which shows that $\varphi = 0$. □

Lemma 4.18. *Suppose (X, J, g) is an asymptotically conical Calabi-Yau manifold with rate $\nu > 0$, and $\eta = \eta_{i\bar{j}}$ is a asymptotically conical hermitian metric with rate $\nu > 0$ and let $u \in C^{\infty}_{2-\beta}$ such that $\eta^{i\bar{j}}u_{i\bar{j}} \in C^{\infty}_{-\kappa}$, then there exist $\tilde{u} \in C^{\infty}_{2-\beta-\nu}$ such that $i\partial\bar{\partial}\tilde{u} = i\partial\bar{\partial}u$.*

Proof. We have

$$g^{i\bar{j}}u_{i\bar{j}} = (g^{i\bar{j}} - \eta^{i\bar{j}})u_{i\bar{j}} + \eta^{i\bar{j}}u_{i\bar{j}} \in C^{\infty}_{-\min(\kappa, \beta+\nu)}$$

hence we can solve the equation $g^{i\bar{j}}\tilde{u}_{i\bar{j}} = g^{i\bar{j}}u_{i\bar{j}}$ with $\tilde{u} \in C^{\infty}_{2-\min(\kappa,\beta+\nu)}$ and by Lemma 4.16 we have $i\partial\bar{\partial}\tilde{u} = i\partial\bar{\partial}u$. □

Proof of Theorem 4.15. By the $\partial\bar{\partial}$-Lemma (Proposition 2.1), we can write $\omega = \omega_{\varphi_0} + i\partial\bar{\partial}\psi$, for $\psi \in PSH(X, \omega_{\varphi_0}) \cap L^\infty_{loc}(X) \cap C^\infty_{loc}(X \setminus V)$, then choose a cutoff χ such that χ has compact support and $\chi = 1$ on a compact set K containing V, then since $i\partial\bar{\partial}\psi = \omega - \omega_{\varphi_0} \in C^\infty_{-\varepsilon}(X \setminus V)$ for some $\varepsilon > 0$, hence by Proposition 2.2, we can solve $i\partial\bar{\partial}f = i\partial\bar{\partial}[(1-\chi)\psi]$ for $f \in C^\infty_\gamma$, $\gamma = 2 - \varepsilon$. Setting $\varphi = \chi\psi + f$, we have that $\varphi \in L^\infty_{loc}(X) \cap C^\infty_\gamma(X \setminus V)$ and

$$(\omega_{\varphi_0} + i\partial\bar{\partial}\varphi)^n = \omega^n_{\varphi_0} = i^{n^2}\Omega \wedge \bar{\Omega}$$

If $\gamma < 0$, then we are done by Proposition 4.17. If $\gamma > 0$, then we proceed by the following: note that the equation above can be rewritten as

$$\Delta_{\omega_{\varphi_0}}\varphi = -(i\partial\bar{\partial}\varphi)^2 \wedge \left(\sum_{k=2}^{n} \binom{n}{k} \frac{(i\partial\bar{\partial}\varphi)^{k-2} \wedge \omega^{n-k}_{\varphi_0}}{\omega^n_{\varphi_0}}\right) \in C^\infty_{2\gamma-4}(X \setminus V)$$

if χ is the cutoff function as before, then we have $\Delta_{\omega_{\varphi_0}}[(1-\chi)\varphi] \in C^\infty_{2\gamma-4}(X)$ if we let $\eta = \chi\omega_{\varphi_t} + (1-\chi)\omega_{\varphi_0}$, then η is an asymptotically conical hermitian metric which is equal to ω_{φ_0} outside of a compact set, hence $\eta^{i\bar{j}}[(1-\chi)\varphi]_{i\bar{j}} \in C^\infty_{2\gamma-4}(X)$, hence we can apply Lemma 4.18 with $\kappa = 2(2-\gamma)$ and $\beta = 2-\gamma$, so we can solve $i\partial\bar{\partial}v = i\partial\bar{\partial}[(1-\chi)\varphi]$ with $v \in C^\infty_{\gamma-\min(2-\gamma,\nu)}$ now we can set $\tilde{\varphi} = v + \chi\varphi \in C^\infty_{\gamma-\min(2-\gamma,\nu)}(X \setminus C)$ and we can keep repeating this process with $\tilde{\varphi}$ in place of φ and $\gamma - \min(2-\gamma, \nu)$ in place of γ until are in the case where $\gamma < 0$, then we are done by Proposition 4.17.

□

5. Metric geometry of the singular Calabi-Yau

The goal of this section is to prove Theorem 1.2. Let us first begin with some definitions and the general setup.

Definition 5.1. We say that a complex analytic space X_0 is a *singular Calabi-Yau variety with compactly supported, crepant singularities*, if

- X_0 is normal singularities, Gorenstein and log-terminal,
- there is a compact set K so that $X_0 \backslash K$ is smooth,
- there exists a resolution $\pi : X \to X_0$ such that X also has trivial canonical bundle and $\pi^*\Omega$ extends as a non-vanishing global holomorphic $(n,0)$-form on X. (By abuse of notation, we will also denote this holomorphic $(n,0)$-form by Ω)

Let X_0 be a singular Calabi-Yau variety with compactly supported, crepant singularities. Suppose that the resolution (X, J, Ω) is Kähler and it has a Kähler metric ω such that (X, J, ω, Ω) is asymptotic to a Calabi-Yau cone $(C, J_C, \omega_C, \Omega_C)$ at rate ν.

Definition 5.2. A line bundle L on X_0 is *ample* if for some $k > 0$, there exist sections $s_0, \ldots, s_N \in H^0(X_0, L^k)$ such that $[s_0, \ldots, s_N]$ gives an embedding of X_0 into a finite dimensional projective space $\mathbb{C}P^N$, and denote this embedding map by ι, then we have $\frac{1}{k}[\iota^\star\omega_{FS}] = c_1(L)$.

Remark 9. Certainly if X_0 is quasi-projective, then it has an ample line bundle in the above sense. In general, having an ample line bundle in the above sense does not imply X_0 is quasi-projective, however in almost all examples we're interested in, X_0 is a quasi-projective variety.

Let us now fix L an ample line bundle on X_0. If set $[\alpha_0] = \pi^*c_1(L)$, then suppose (X, J, ω, Ω) and $[\alpha_0]$ satisfy the hypothesis of Theorem 1.2. Then from the previous sections, we have on X, a sequence of Calabi-Yau metrics $\omega_{\varphi_t} = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$ with $[\omega_{\varphi_t}] = (1-t)[\alpha_0] + t[\alpha_1]$, which satisfy the equation

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = e^{f_t}\hat{\omega}^n_t (= i^{n^2}\Omega \wedge \bar{\Omega})$$

and $f_t = \log \frac{i^{n^2}\Omega\wedge\bar{\Omega}}{\hat{\omega}_0^n} \in C^\infty_{-\gamma-2}(X)$, and $\varphi_t \in C^\infty_{-\gamma}(X)$.

If we fix a point $p \in \pi^{-1}(X_0^{reg})$, then by Gromov compactness, after passing to a subsequence, the pointed spaces $(X, \omega_{\varphi_{t_i}}, p)$ for $t_i \to 0$ pointed Gromov-Hausdorff converge to a limiting pointed metric space $(X_\infty, d_\infty, p_\infty)$ as $i \to \infty$. By the definition of pointed Gromov-Hausdorff convergence, the convergence can be interpreted in the following sense: If we set $Z = (X_\infty, d_\infty, p_\infty) \sqcup \bigsqcup_{t_i}(X, \omega_{\varphi_{t_i}}, p)$, then there exist a metric d_Z on Z such that

(1) $d_Z|_{X_i} = d_{g_{\varphi_{t_i}}}$
(2) $d_Z(\underbrace{p}_{\in X_i}, p_\infty) \to 0$
(3) $B_{g_{\varphi_{t_i}}}(p, r) \subset X_i \to B_{g_\infty}(p_\infty, r) \subset X_\infty$ in the Hausdorff sense with respect to d_Z.

The asymptotically conical property of ω_{φ_t} implies that the tangent cone at ∞ is independent of t, and by Bishop-Gromov, we have a uniform lower bound on volume of geodesic balls, $\mathrm{Vol}_{\omega_{\varphi_t}} B(p, r) \geqslant cr^{2n}$ where c is the volume ratio of the asymptotic cone C. Hence the regularity theory of Cheeger, Colding and also Tian [6, 7, 8, 9, 10] applies, and the limiting space admits the following structure

(1) All tangent cones of X are metric cones.
(2) $X = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ consists of all the points where all tangent cones are isometric to $\mathbb{R}^{2n}$.
(3) $\mathcal{R}$ is an open dense set in X_∞ with a smooth metric g_∞ and complex structure J_∞ which makes it Ricci-flat Kähler manifold and $(X_\infty, d_\infty) = \overline{(\mathcal{R}, d_{g_\infty})}$. Moreover, the convergence of $(X, J, \omega_{\varphi_t}, p) \to (X_\infty, J_\infty, g_\infty, p)$ is smooth on $\mathcal{R}$ in the sense that for every $K \subset\subset \mathcal{R}$, there exist smooth maps $\eta_i : K \to X$ such that $(\eta_i^\star g_{t_i}, \eta_i^\star J)$ converges to (g_∞, J_∞) smoothly on K. (In fact, we can arrange η_i such that $d_Z(\eta_i(z), z) \to 0$ uniformly in K)
(4) $\mathcal{S}$ is a closed subset of X_∞ with real Hausdorff codimension greater or equal to 4.

5.1. **Properties of the Gromov-Hausdorff limit.** In this section, we prove several preliminary propositions about the relationship between X_∞ and the Kähler current constructed from Theorem 1.1. In particular, we show the following:

(1) ω_{φ_0} is in fact well-defined and smooth on $\pi^{-1}(X_0^{reg})$
(2) There exist a locally isometric embedding of $\iota_\infty : (\pi^{-1}(X_0^{reg}), \omega_{\varphi_0}) \to (\mathcal{R}, g_\infty)$.
(3) X_∞ is isometric to the metric completion $\overline{(\pi^{-1}(X_0^{reg}), \omega_{\varphi_0})}$
(4) ι_∞ is a bijective local isometry between X_0^{reg} and $\mathcal{R}$.

One of the key ingredients is the local diameter bound Lemma 4.14, which we apply with $V = \pi^{-1}(X_0^{sing})$.

Proposition 5.1. *The family of metrics ω_{φ_t} has a uniform lower bound*

$$\omega_{\varphi_t} \geqslant \frac{1}{C}\hat{\omega}_0 \tag{5.1}$$

Proof. By the standard Schwartz lemma calculation, we have

$$\Delta_{\omega_{\varphi_t}} \log \mathrm{Tr}_{\omega_{\varphi_t}} \pi^\star \omega_{FS} \geqslant -4\mathrm{Tr}_{\omega_{\varphi_t}} \pi^\star \omega_{FS}$$

and for any other Kähler metric $\hat{\omega}$, one also has

$$\Delta_{\omega_{\varphi_t}} \log \mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega} \geqslant -C\mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega}$$

with C depending only on the upper bound for the holomorphic bisectional curvature of $\hat{\omega}$. Recall from the construction of $\hat{\omega}_0$ in Propositions 3.1 and 3.2 that $\hat{\omega}_0$ can be taken to be equal to $\frac{1}{k}\pi^\star\omega_{FS}$ on a compact set K containing $\pi^{-1}(X_0^{sing})$, and is a genuine non-degenerate, asymptotically conical Kähler metric outside of K, so we can apply the first inequality inside K and the second outside K to get a uniform estimate

$$\Delta_{\omega_{\varphi_t}} \log \mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega}_0 \geqslant -C\mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega}_0. \tag{5.2}$$

Since $\omega_{\omega_{\varphi_t}} = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$, taking trace gives

$$n = \mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega}_t + \Delta_{\varphi_t}\varphi_t,$$

and we also know that for t reasonably small $\hat{\omega}_t \geqslant c\hat{\omega}_0$ holds for some small constant c uniformly in t as $t \to 0$, which means we have

$$n \geqslant c\mathrm{Tr}_{\varphi_t}\hat{\omega}_0 + \Delta_{\varphi_t}\varphi_t.$$

Combining this with (5.2), we have

$$\Delta_{\omega_{\varphi_t}} \left(\log \mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega}_0 - A\varphi_t\right) \geqslant (\frac{Ac}{2} - C)\mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega}_0 - An$$

since $\log \mathrm{Tr}_{\omega_{\varphi_t}} \hat{\omega}_0 - A\varphi_t$ converges to the constant $\log n$ at spacial infinity, if the maximum is attained at infinity, then we automatically have a uniform bound that we wanted. So we can assume the maximum is achieved in the interior, and applying the maximum principle to the equation above, and we obtain

$$\mathrm{Tr}_{\omega_{\varphi_t}}\hat{\omega}_0 \leqslant Ce^{A(\varphi_t - (\varphi_t)_{min})}$$

which gives a uniform upper bound for $\mathrm{Tr}_{\omega_{\varphi_t}}\hat{\omega}_0$. □

Corollary 5.2. *On $X \setminus \pi^{-1}(X_0^{sing})$, we have*

$$C^{-1}\hat{\omega}_0 \leqslant \omega_{\varphi_t} \leqslant Ce^{f_0}\hat{\omega}_0$$

where $e^{f_0} = \frac{i^{n^2}\Omega\wedge\bar{\Omega}}{\hat{\omega}_0^n}$ is bounded uniformly away from $\pi^{-1}(X_0^{sing})$. In particular, this implies that ω_{φ_0} is smooth on $\pi^{-1}(X_0^{reg})$, and on X_0 it is a Kähler current since it dominates $\hat{\omega}_0$.

Proof. The lower bound on ω_{φ_t} is the content of the previous lemma, and from that and the fact that $\omega_{\varphi_t}^n = i^{n^2}\Omega \wedge \bar{\Omega} = e^{f_0}\hat{\omega}_0^n$, the corollary follows immediately. □

Corollary 5.3. *The maps $\pi_i : (X, \omega_{\varphi_{t_i}}, p) \to (X_0, \hat{\omega}_0, p)$ are has bounded derivative, hence it is uniformly lipschitz and we can pass to a continuous surjective map from the Gromov-Hausdorff limit $\pi_\infty : (X_\infty, d_{X_\infty}, p_\infty) \to X_0$. Furthermore, for any $q \in X_0^{reg}$, the preimage $\pi_\infty^{-1}(q)$ consists of a single point.*

Proof. The fact that the maps have bounded derivative follows from the estimate (5.1), and from this it follows from an Arzela-Ascoli type argument that after passing to a subsequence, the projection maps π_i limit to a continuous surjective map $\pi_\infty : X_\infty \to X_0$. The map π_∞ can be characterized in the following way: if we fix $h_i : (X, \omega_{\varphi_{t_i}}) \to X_\infty$ an ε_i-isometry for $\varepsilon_i \to 0$, then for any sequence of points $q_i \in X$ with $\pi(q_i) \to q \in X_0$, and $h_i(q_i) \to q_\infty \in X_\infty$, we have $\pi_\infty(q) = q_\infty$.

To see that the preimage of $\pi^{-1}(q)$ for $q \in X_0^{reg}$ consists of a single point, suppose for contradiction that it consisted of two points $q_1, q_2 \in X_\infty$ with $d_{X_\infty}(q_1, q_2) = d > 0$ and $\pi_\infty(q_1) = \pi_\infty(q_2) = q \in X_0^{reg}$, then from the construction of π_∞, there exist a sequences of points $q_1^i, q_2^i \in X$ such that $\pi_i(q_1^i) \to q$ and $\pi_i(q_2^i) \to q$ and $h_i(q_1^i) = q_1$ and $h_i(q_2^i) = q_2$. Then from the fact that $\pi_i(q_1^i) \to q$ and $\pi_i(q_2^i) \to q$ and $q \in X_0^{reg}$, we know that $q_1^i \to \pi^{-1}(q)$ and $q_2^i \to \pi^{-1}(q)$ in X since π is a resolution of singularities of X_0, and $g_{t_i} \to g_\infty$ smoothly in a neighborhood of q, it follows that $d_{g_{t_i}}(q_1^i, q_2^i) \to 0$ as $i \to \infty$. But we also have

$$d_{X_\infty}(h_i(q_1^i), h_i(q_2^i)) - \varepsilon_i \leqslant d_{g_{t_i}}(q_1^i, q_2^i)$$

since h_i is an ε_i-isometry. This is a contradiction, because $d_{X_\infty}(h_i(q_1^i), h_i(q_2^i)) - \varepsilon_i \to d > 0$ by our assumption. □

Proposition 5.4. *There is an embedding* $i_\infty : (X_0^{reg}, \omega_{\varphi_0}, p) \hookrightarrow (\mathcal{R}, g_\infty, p)$*, which is a locally isometric embedding, and* $\pi_\infty \circ \iota_\infty = id$*.*

Proof. We can simply take $\iota_\infty = \pi_\infty^{-1}|_{X_0^{reg}}$, which is well-defined by the previous proposition. It's clear that the image of ι_∞ is contained in the regular set $\mathcal{R} \subset X_\infty$ and that it is continuous, so it suffices to show that this map is a local isometry. To see this, we note that if $q \in X_0^{reg}$, then there exist an $\varepsilon > 0$ such that $B_{g_{t_i}}(q, \varepsilon) \subset X_0^{reg}$ for all $i \gg 1$. It follows from the diameter estimate (c.f. Lemma 4.14) that the points $h_i(\pi^{-1}(q))$ are uniformly bounded in X_∞, hence after passing to a subsequence, it converge to some point $q_\infty \in X_\infty$, it's clear that $q_\infty = \iota_\infty(q)$ since $\pi_i(q) = q$. Since the points $\pi^{-1}(q) \in X_i$ have a uniform harmonic radius lower bound, hence $(B_{g_{t_i}}(\pi^{-1}(q), \varepsilon), g_{\varphi_{t_i}}) \xrightarrow{C^\infty} (B_{g_\infty}(q_\infty, \varepsilon), g_\infty)$ and by the smooth convergence of $g_{\varphi_t} \to g_{\varphi_0}$, we also have $(B_{g_{t_i}}(\pi^{-1}(q), \varepsilon), g_{\varphi_{t_i}}) \xrightarrow{C^\infty} (B_{g_{\varphi_0}}(q, \varepsilon), \omega_{\varphi_0})$, it is then clear from the construction of π_∞ that it maps $(B_{g_\infty}(q_\infty, \varepsilon), g_\infty)$ isometrically onto $(B_{g_{\varphi_0}}(q, \varepsilon), \omega_{\varphi_0})$. □

The following Proposition follows from the same arguments as in [48]. We include a proof here for the convenience of the reader.

Proposition 5.5. *The subset* $E = \mathcal{R} \setminus \iota_\infty(X_0^{reg}) \subset \mathcal{R}$ *is an analytic subset, hence of real codimension bigger than or equal to 2, and moreover* $\overline{(X_0^{reg}, g_\infty)} = X_\infty$*.*

Proof. It suffices to show that the holomorphic maps $\pi : (X, \omega_{\varphi_t}, p) \to X_0 \subset (\mathbb{C}P^N, \omega_{FS}, p)$ limits to a holomorphic map $\pi_\infty|_{\mathcal{R}}$: $(\mathcal{R}, J_\infty, g_\infty) \to X_0 \subset \mathbb{C}P^N$. Assuming for now that this is the case, then $\mathcal{R} \setminus \iota_\infty(X_0^{reg}) = \pi_\infty|_{\mathcal{R}}^{-1}(X_0^{sing})$. Since $X_0^{sing} \subset X_0$ is an analytic set, if $\pi_\infty|_{\mathcal{R}}$ is holomorphic, then $\pi_\infty|_{\mathcal{R}}^{-1}(X_0^{sing}) = E \subset \mathcal{R}$ is an analytic subset, and since analytic subsets have real codimension 2, it follows that $X_\infty \setminus X_0^{reg} \subset X_\infty$ has Hausdorff codimension at least 2, and by [8, Theorem 3.7], we have $\overline{(X_0^{reg}, g_\infty)} = X_\infty$.

Now we show that $\pi_\infty|_{\mathcal{R}}$ is holomorphic. Consider the holomorphic maps $\pi : (X, \omega_{\varphi_t}, p) \to X_0 \subset (\mathbb{C}P^N, \omega_{FS}, p)$, since $(X, \omega_{\varphi_t}, p)$ Gromov-Hausdorff converge to X_∞, by Cheeger-Colding theory [7], for any $K \subset\subset \mathcal{R}$ containing p, there exist maps $\iota_{t_i} : K \to (X, \omega_{\varphi_{t_i}})$ such that $\iota_{t_i}^\star g_{t_i} \to g_\infty$ and $\iota_{t_i}^\star J \to J_\infty$ in the smooth topology, and we also get a sequence of holomorphic maps $\pi_i = \pi \circ \iota_{t_i} : (K, \iota_{t_i}^\star g_{t_i}, \iota_{t_i}^\star J) \to X_0 \subset \mathbb{C}P^N$. Furthermore, if we regard these maps as harmonic maps, then we have

$$|d\pi_i|^2_{\omega_{\varphi_t}, \omega_{FS}} = \mathrm{Tr}_{\omega_{\varphi_t}} \pi_i^\star \omega_{FS} \leqslant C$$

hence by the regularity theory of harmonic maps ([49]), we have uniform C^∞ estimates on the maps $\|\pi_i^l\|_{C^{k,\alpha}}(K) \leqslant C_K$, for some constant C_K independent of i, which allows us to extract a limit of the maps $\pi_i : K \to \mathbb{C}P^N$ to a map $\pi_\infty : K \to \mathbb{C}P^N$ and since the convergence of the maps are smooth, and the convergence of the metrics $\iota_{ti}^\star g_{t_i} \to g_\infty$ and the complex structures $\iota_{t_i}^\star J \to J_\infty$ are all smooth, it follows that the holomorphicity of the maps π_i passes to the limit, and hence the map π_∞ is holomorphic. □

Proposition 5.6. *In fact we have* $\mathcal{R} = \iota_\infty(X_0^{reg})$*.*

Proof. The proof is the same as in [48, Lemma 2.2]. □

5.2. **Identification of X_0 with the geometry of singular Calabi-Yau.** In this section, we identify the geometry of the singular Calabi-Yau current $X_\infty = \overline{(X_0^{reg}, g_\infty)}$ with the variety X_0 itself. This result is the analogue of the result in [51], where the similar thing was shown in the compact case, our proof follows the approach in [51], adapted to the non-compact case. The idea is based on ideas developed in [23] together with a new gradient estimate for the potential φ_t with respect to the Calabi-Yau metrics ω_{φ_t}.

5.2.1. *A gradient bound for φ_0.* The goal of this section is to prove the following estimate

Proposition 5.7. *The following bound hold*

$$\sup_{\pi^{-1}(X_0^{reg})} |\nabla_{\omega_{\varphi_0}} \varphi_0| \leqslant C$$

Proposition 5.8. *If we set let $v_t = \varphi_t - t\dot{\varphi}_t$, then we have a uniform estimate*

$$\sup_X |v_t| \leqslant C$$

Proof. Recall from the construction of $\hat{\omega}_t$ (Proposition 3.2) that

$$\begin{aligned}\hat{\omega}_t &= \omega_t + i\partial\bar{\partial}u_t \\ &= (1-t)\omega_0 + t\omega_1 + i\partial\bar{\partial}u_t\end{aligned}$$

where $\omega_0 = \pi^\star \omega_{X_0}$ and ω_{X_0} is a Kähler metric on X_0. So we have

$$\begin{aligned}\Delta_{\varphi_t}\varphi_t &= n - \mathrm{Tr}_{\varphi_t}\hat{\omega}_t \\ &= n - (1-t)\mathrm{Tr}_{\varphi_t}\omega_0 - t\mathrm{Tr}_{\varphi_t}\omega_1 - \Delta_{\varphi_t}u_t.\end{aligned}$$

Recall that by the construction of $\hat{\omega}_t$, Proposition 3.1, we have

$$\log \frac{(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n}{\hat{\omega}_t^n} = f_t \in C^\infty_{-\gamma-2}.$$

for some $0 < \gamma < 2n-2$. Differentiating the equation, we have

$$\Delta_{\varphi_t}\dot{\varphi}_t = \dot{f}_t - \mathrm{Tr}_{\varphi_t}\frac{\partial}{\partial t}\hat{\omega}_t + \mathrm{Tr}_{\hat{\omega}_t}\frac{\partial}{\partial t}\hat{\omega}_t \in C^\infty_{-\gamma-2}(X) \tag{5.3}$$

so we have $\dot{\varphi} \in C^\infty_{-\gamma}(X)$ for $t > 0$.

If we differentiate the equation $(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = i^{n^2}\Omega \wedge \bar{\Omega}$ with respect to t, we obtain another expression for $\Delta_{\varphi_t}\dot{\varphi}_t$

$$\Delta_{\varphi_t}\dot{\varphi}_t = -\Delta_{\varphi_t}\dot{u}_t + \mathrm{Tr}_{\varphi_t}(\omega_0 - \omega_1) \tag{5.4}$$

The equations (5.3) and (5.4) imply that v_t satisfy the two equations

$$\Delta_{\varphi_t}v_t = n - \mathrm{Tr}_{\varphi_t}\omega_0 - \Delta_{\varphi_t}(u_t - t\dot{u}_t) \tag{5.5}$$

and

$$\Delta_{\varphi_t}v_t = n - \mathrm{Tr}_{\varphi_t}\hat{\omega}_t - t(\dot{f}_t - \mathrm{Tr}_{\varphi_t}\frac{\partial}{\partial t}\hat{\omega}_t + \mathrm{Tr}_{\hat{\omega}_t}\frac{\partial}{\partial t}\hat{\omega}_t) \tag{5.6}$$

From the first equation and Proposition 5.1, we see that $|\Delta_{\varphi_t}\dot{\varphi}_t| \leqslant C$ uniformly in t. From the second equation we see that $|\Delta_{\varphi_t}v_t| \leqslant Cr^{-\gamma-2}$ away from a compact set K, so we have a uniform bound $|\Delta_{\varphi_t}v_t|_{L^p} \leqslant C$ for $p > \frac{2n}{\gamma+2}$. Since $v_t \in C^\infty_{-\gamma}$, we can do integrate by parts to get

$$\begin{aligned}-\int_X |v_t|^{p-2}v_t i\partial\bar{\partial}v_t \wedge \omega_{\varphi_t}^{n-1} &= \lim_{R\to\infty}(p-1)\int_{B_R}|v_t|^{p-2}i\partial v_t \wedge \bar{\partial}v_t \wedge \omega_{\varphi_t}^{n-1} \\ &\quad - \lim_{R\to\infty}\left(\int_{\partial B_R}|v_t|^{p-2}v_t i\bar{\partial}v_t \wedge \omega_{\varphi_t}^{n-1}\right) \\ &= \frac{4(p-1)}{p^2}\int_X i\partial|v_t|^{\frac{p}{2}} \wedge \bar{\partial}|v_t|^{\frac{p}{2}} \wedge \omega_{\varphi_t}^{n-1}\end{aligned}$$

the boundary term goes away when $p > \frac{2n-2}{\gamma}$ since $|\nabla^k v_t| = O(r^{-\gamma-k})$. Hence we get

$$\int_X |\partial |v_t|^{\frac{p}{2}}|^2 \omega_{\varphi_t}^n = -\frac{np^2}{4(p-1)} \int_X |v_t|^{p-2} v_t (\Delta_{\varphi_t} v_t) \omega_{\varphi_t}^n$$

combined with the Sobolev inequality, one gets

$$\left(\int_X |v_t|^{p\frac{n}{n-1}} i^{n^2} \Omega \wedge \bar{\Omega} \right)^{\frac{n-1}{n}} \leqslant C \frac{np^2}{p-1} \int_X |v_t|^{p-1} |\Delta_{\varphi_t} v_t| i^{n^2} \Omega \wedge \bar{\Omega}$$

applying Holder, we get

$$\|v_t\|_{L^{p\frac{n}{n-1}}} \leqslant C \frac{np^2}{p-1} \|\Delta_{\varphi_t} v_t\|_{L^{\frac{np}{n+p+1}}}$$

hence for $p > \frac{2n}{\gamma}$, we have

$$\|v_t\|_{L^p} \leqslant C_p \tag{5.7}$$

where C_p depends on $\|\Delta_{\varphi_t} v_t\|_{L^{\frac{np}{n+p}}}$. and also for $p > \frac{2n-2}{\gamma}$

$$\|v_t\|^p_{p\frac{n}{n-1}} \leqslant C \frac{np^2}{p-1} \|v_t\|^{p-1}_{L^p} \|\Delta_{\varphi_t} v_t\|_{L^p}$$

we can then apply Moser iteration to this to get the estimate

$$\|v_t\|_{L^\infty} \leqslant B_p \|v_t\|_{L^p} \leqslant B_p C_p$$

where C_p is the constant from (5.7) and B_p depends only on the L^p norm of $\|\Delta_{\varphi_t} v_t\|_{L^p}$. □

Corollary 5.9. *For any compact set $K \subset\subset \pi^{-1}(X_0^{reg})$, we have an estimate*

$$|v_t|_{C^{k,\alpha}(K)} \leqslant C(K, k, \alpha)$$

uniformly in t as $t \to 0$.

Proof. This follows from the equation (5.5) and the fact that ω_{φ_t} and the right hand side of the equation is uniformly bounded in $C^\infty_{loc}(\pi^{-1}(X_0^{reg}))$. □

Proposition 5.10. *We also have the following local uniform gradient estimate for v_t.*

$$\sup_K |\nabla_t v_t| \leqslant C_K$$

for any $K \subset\subset X$.

Proof. By the Bochner formula, we have

$$\begin{aligned}
\Delta_{\varphi_t} |\nabla v_t|^2_{g_{\varphi_t}} &= |\nabla\nabla v_t|^2_{g_{\varphi_t}} + |\partial\bar{\partial} v_t|^2_{g_{\varphi_t}} - 2\mathrm{Re}(\nabla v_t \cdot \nabla \mathrm{Tr}_{\varphi_t} \omega_0) - 2\mathrm{Re}(\nabla v_t \cdot \nabla \Delta_{\varphi_t}(u_t - t\dot{u}_t)) \\
&\geqslant -2|\nabla v_t|^2_{g_{\varphi_t}} - |\nabla \mathrm{Tr}_{\omega_{\varphi_t}} \omega_0|^2_{g_{\varphi_t}} - |\nabla \Delta_{\varphi_t}(u_t - t\dot{u}_t)|^2_{g_{\varphi_t}}
\end{aligned}$$

we also have from (5.2),

$$\Delta_{\omega_{\varphi_t}} \mathrm{Tr}_{\omega_{\varphi_t}} \omega_0 \geqslant -C + c_0 |\nabla \mathrm{Tr}_{\omega_{\varphi_t}} \omega_0|^2$$

If we set $H_t = |\nabla v_t|^2_{g_{\varphi_t}} + A\mathrm{Tr}_{\omega_{\varphi_t}} \omega_0$, then $H_t \geqslant 0$ and satisfies

$$\Delta_{\varphi_t} H_t \geqslant -H_t - C$$

We can apply Moser iteration to this, since ω_{φ_t} has uniform Ricci bounds and volume lower bound, this then gives the estimate

$$\|H_t\|_{L^\infty(B_{g_\infty,R}(p))} \leqslant C \|H_t\|_{L^2(B_{g_\infty,2R}(p))} \tag{5.8}$$

for R sufficiently large. Note that for R sufficiently large ω_{φ_t} converge uniformly in C^∞ to g_∞ on the region $B_{g_\infty,2R}(p) \setminus B_{g_\infty,R}(p)$, hence we can also choose cutoff functions with uniformly controlled gradients and standard Moser iteration gives the inequality. Now it suffices to show that $\|H_t\|_{L^2(B_{g_\infty,2R})}$ is bounded.

$$\begin{aligned}\int_{B_{2R}} |H_t|^2 &\leqslant \|H_t\|_{L^\infty(B_{2R})}\int_{B_{2R}} |H_t| \\ &\leqslant C\|H_t\|_{L^2(B_{4R})}\|H_t\|_{L^1(B_{2R})} \\ &\leqslant C(\|H_t\|_{L^2(B_{2R})}+\|H_t\|_{L^2(B_{4R}\backslash B_{2R})})\|H_t\|_{L^1(B_{2R})}\end{aligned}$$

if R is sufficiently large, then $B_{4R}\backslash B_{2R}$ doesn't contain any of $\pi^{-1}(X_0^{sing})$, hence $\|H_t\|_{L^2(B_{4R}\backslash B_{2R})}$ is uniformly bounded in t on $B_{4R}\setminus B_{2R}$ by the Corollary above. So we have

$$\|H_t\|^2_{L^2(B_{2R})}\leqslant C(\|H_t\|_{L^2(B_{2R})}+C)\|H_t\|_{L^1(B_{2R})}$$

hence either $\|H_t\|_{L^2(B_{2R})}$ is bounded by 1 and we are done, or we get the bound

$$\|H_t\|_{L^2(B_{2R})}\leqslant C\|H_t\|_{L^1(B_{2R})} \tag{5.9}$$

so it suffices to prove an L^1 bound for H_t on compact sets.

Choose cutoff function η such that $\eta = 1$ on B_{2R} for all t, then

$$\begin{aligned}\int_{B_{2R}} H_t &\leqslant \int_X \eta^2 H_t \\ &\leqslant \int_X \eta^2|\nabla v_t|^2\omega^n_{\varphi_t} + C \\ &\leqslant -\int_X \eta^2 v_t(\Delta_{\varphi_t} v_t) + 2\int_X \eta|\nabla\eta||\tilde{v}_t||\nabla\tilde{v}_t|+C\end{aligned}$$

and so we have

$$\int_{B_{2R}} H_t \leqslant C\int_X (\eta^2 + |\nabla\eta|^2)v_t^2 \leqslant C$$

which gives us the L^1 bound, combined with (5.9) and (5.8), we get

$$\|H_t\|_{L^\infty(B_R)}\leqslant C$$

as desired. □

Proposition 5.11. *For any $x \in \iota_\infty(X_0^{reg})$, we have a bound*

$$|\dot{\varphi}_t(x)|\leqslant C$$

for some constant C potentially depending on the point x.

Proof. Fix $x \in \iota_\infty(X_0^{reg})$, then fix a ball $B_{g_\infty,\varepsilon}(x) \subset \iota_\infty(X_0^{reg})$ on which the metrics g_{φ_t} converge smoothly to g_∞, also fix a set $K \subset\subset X$ containing all of $\pi_\infty^{-1}(X_0^{sing})$ and also $B_{g_\infty,\varepsilon}(x)$. Then by the Green's formula representation formula, we have

$$\begin{aligned}\dot{\varphi}_t(x) &= -\int_X \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y) \\ &= -\int_{B_{g_\infty,\varepsilon}(x)} \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y) - \int_{K\backslash B_{g_\infty,\varepsilon}(x)} \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y) \\ &\quad -\int_{X\backslash K} \Delta_{\varphi_t}\dot{\varphi}_t(y)G_t(x,y)\omega^n_{\varphi_t}(y)\end{aligned}$$

where $G_t(x,y)$ is the positive decaying Green's function on (X,ω_{φ_t}). By the estimates for Green's function [44, p.190], [36, 37], the Green's functions $G_t(x,y)$ satisfy the uniform estimates

$$C^{-1}d_t(x,y)^{2-2n} \leqslant G_t(x,y) \leqslant Cd_t(x,y)^{2-2n}$$

where d_t is the distance function induced by g_{φ_t}. And since $\Delta_{\varphi_t}\dot{\varphi}_t = -\Delta_{\varphi_t}\dot{u}_t + \mathrm{Tr}_{\varphi_t}(\omega_0 - \omega_1)$, this implies $|\Delta_{\varphi_t}\dot{\varphi}_t| \leqslant |\Delta_{\varphi_t}\dot{u}_t| + \mathrm{Tr}_{\varphi_t}(\omega_0 + \omega_1) \leqslant C + \mathrm{Tr}_{\varphi_t}\omega_1$ and we have

$$|\dot{\varphi}_t(x)| \leqslant \int_{B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_t(x,y)^{2-2n}\omega^n_{\varphi_t}(y) + \int_{K\setminus B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_t(x,y)^{2-2n}\omega^n_{\varphi_t}(y)$$

$$+ \int_{X\setminus K} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_t(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \tag{5.10}$$

and we analyze the three terms in the above formula seperately. For the first term, we note that $\Delta_{\varphi_t}\dot{\varphi}_t$ is uniformly bounded on $B_{g_\infty,\varepsilon}(x)$, so

$$\int_{B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_{\varphi_t}(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \leqslant C\int_{B_{g_\infty,\varepsilon}(x)} d_t(x,y)^{2-2n} \leqslant C$$

For the second term, observe that on $K \setminus B_{g_\infty,\varepsilon}(x)$, $d_t(x,y)^{2-2n}$ is bounded by $C\varepsilon^{2-2n}$, so

$$\int_{K\setminus B_{g_\infty,\varepsilon}(x)} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_{\varphi_t}(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \leqslant C\left(1 + \int_{K\setminus B_{g_\infty,\varepsilon}(x)} \mathrm{Tr}_{\varphi_t}\omega_1\right)$$

hence it suffices to bound the integral of $\mathrm{Tr}_{\varphi_t}\omega_1$, to do this, we integrate by parts

$$\begin{aligned}\int_K \omega_1 \wedge \omega^{n-1}_{\varphi_t} = \int_K \omega_1 \wedge (\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^{n-1} &= \int_K \omega_1 \wedge \hat{\omega}_t^{n-1} \\ &+ \int_{\partial K} \partial\varphi_t \wedge \omega_1 \wedge \left(\sum_{l=0}^{n-2}\binom{n-1}{l}\hat{\omega}_t^l \wedge (i\partial\bar{\partial}\varphi_t)^{n-2-l}\right) \\ &\leqslant C\end{aligned}$$

because φ_t and its derivatives are all bounded on the boundary of K.
The last term in (5.10) is bounded because $|\Delta_{\varphi_t}\dot{\varphi}_t| \leqslant C d_t(x,y)^{-2-\beta}$ on $X \setminus K$, so we have

$$\int_{X\setminus K} |\Delta_{\varphi_t}\dot{\varphi}_t|(y) d_{\varphi_t}(x,y)^{2-2n}\omega^n_{\varphi_t}(y) \leqslant C\int_{X\setminus K} d_t(x,y)^{-2n-\beta} \leqslant C$$

and we get our result. ☐

proof of Proposition 5.7. Note that we already know $|\nabla_{g_\infty}\varphi_0|$ is bounded and decaying at infinity, so it suffices to prove that it's bounded near $\pi_\infty^{-1}(X_0^{sing})$. Fix a compact set K containing $\pi_\infty^{-1}(X_0^{sing})$, then by Proposition 5.10, $|\nabla v_t| \leqslant C$, but on X_0^{reg}, v_t converges to φ_0 smoothly on compact sets, hence we get our result. ☐

The main goal of this gradient bound is to show the following.

Proposition 5.12. *For any holomorphic section $s \in H^0(X_0, L^k)$ satisfies*

$$\sup_K |s|_{h^k_\infty} \leqslant C$$

and

$$\sup_K |\nabla s|_{h^k_\infty, k\omega_{\varphi_0}} \leqslant C$$

Proof. Locally we can write $h_\infty = e^{-\varphi_0}\hat{h}_0$ where $-i\partial\bar{\partial}\log\hat{h}_0 = \hat{\omega}_0$, since $\hat{\omega}_0 = \pi^\star\omega_{FS}$ on K, we have simply $h_\infty = e^{-\varphi_0}h_{FS}$, and by the C^0 bound for φ_0, it follows that $|s|_{h^k_\infty} \leqslant C|s|_{h^k_{FS}} \leqslant C$. To see the bound for the gradient, we note

$$|\nabla_{h^k_\infty}s|^2_{h^k_\infty,k\omega_{\varphi_0}} = |\nabla_{h^k_{FS}}s + k(\partial\varphi_0)s|_{h^k_\infty,k\omega_{\varphi_0}} \leqslant |\nabla_{h^k_{FS}}s|_{h^k_{FS},k\omega_{\varphi_0}} + k|\nabla\varphi_0|_{k\omega_{\varphi_0}}|s|_{h^k_\infty}$$

and by the gradient estimate (5.7) $|\nabla\varphi_0|_{k\omega_{\varphi_0}}\leqslant C$, so the second term is bounded, and by the estimate (5.1), we have $|\nabla_{h^k_{FS}}s|_{h^k_{FS},k\omega_{\varphi_0}}\leqslant C|\nabla_{h^k_{FS}}s|_{h^k_{FS},k\omega_{FS}}\leqslant C$ and we get the bound that we wanted. □

We will need the boundedness of $|s|_{h^k_\infty}$ and $|\nabla s|_{h^k_\infty,k\omega_{\varphi_0}}$ to make the Moser iteration argument work with cutoff functions in the next section.

5.2.2. *L^2 estimates on X_0.* The argument of this section follows in the same way as in [51], with minor modifications.

We first quote a proposition stating the existence of good cutoff functions on X_∞ from [23].

Lemma 5.13. [23, Proposition 3.5] *There exist cutoff functions ρ_ε on X_∞ satisfying the following*

(1) $0\leqslant\rho_\varepsilon\leqslant 1$
(2) *$supp(\rho_\varepsilon)\subset\subset\mathcal{R}=X_0^{reg}$*
(3) *For any compact set $K\subset\subset\mathcal{R}$, there exist $\varepsilon_K>0$ such that for all $\varepsilon<\varepsilon_K$, we have $\rho_\varepsilon=1$ on K.*
(4) *$\int_X|\nabla\rho_\varepsilon|^2\to 0$ as $\varepsilon\to 0$.*

We recall the following version of Hormander's L^2 estimates for the $\bar\partial$ equation.

Theorem 5.14. [20, Cor 5.3] *Let (M,ω) be a Kähler manifold. Assume M is weakly pseudoconvex. Let (L,h) be a Hermitian line bundle with curvature with (possibly) singular Hermitian metric h, and suppose*

$$-i\partial\bar\partial\log h+Ric(\omega)\geqslant\gamma(x)\omega$$

then for any $\beta\in\bigwedge^{0,1}\otimes L$, with $\bar\partial\beta=0$, there exist a section $s\in L$ satisfying $\bar\partial s=\beta$ with

$$\int_M|s|_h^2\omega^n\leqslant\int_M\frac{1}{\gamma}|\beta|^2_{h,\omega}\omega^n,$$

provided the integral on the RHS is finite.

Now we will prove a version of the above theorem on X equipped with a singular metric ω_{φ_0} that we constructed. First we fix a Hermitian metric h_0 on L such that $-i\partial\bar\partial\log h_0=\hat\omega_0$, which is possibly by the $\partial\bar\partial$-Lemma.

Theorem 5.15. *Let $h_\infty=e^{-\varphi_0}h_0$, so $-i\partial\bar\partial\log h_\infty=\omega_{\varphi_0}$, and $K\subset\subset X$ a compact subset with pseudoconvex boundary. Then for any $\beta\in\bigwedge^{0,1}\otimes L^k$, with compact support and $supp(\beta)\subset X_0^{reg}\cap K$ and $\bar\partial\beta=0$, there exist a section $u\in H^0(L^k)$ satisfying $\bar\partial u=\beta$ with*

$$\int_K|u|^2_{h^k_\infty}\omega^n_{\varphi_0}\leqslant\int_K|\beta|^2_{h^k_\infty,k\omega_{\varphi_0}}\omega^n_{\varphi_0}$$

Proof. By Assumption 1, we know that $\hat\omega_0+i\partial\bar\partial\psi_\varepsilon\geqslant\varepsilon\omega$, which implies $\hat\omega_0+ti\partial\bar\partial\psi_\varepsilon\geqslant(1-t)\hat\omega_0+t\varepsilon\omega$. By the discussion in the previous sections, we can solve

$$\omega^n_{\varphi_t}=\big((1-t)\hat\omega_0+t\varepsilon\omega+i\partial\bar\partial\tilde\varphi_t\big)^n=i^{n^2}\Omega\wedge\bar\Omega$$

with $\tilde\varphi_t$ is bounded on any compact set $K\subset\subset X$, uniformly as $t\to 0$ and $\tilde\varphi_t\to\varphi_0$ in $L^\infty_{loc}(X)$ and in $C^\infty_{loc}(X_0^{reg})$. We pick a metric h_0 on L such that $-i\partial\bar\partial\log h_0=\hat\omega_0$, then if we set $\tilde h_t=e^{-t\psi_\varepsilon-\tilde\varphi_t}h_0$, it satisfies

$$-i\partial\bar\partial\log\tilde h_t^k=k(\hat\omega_0+ti\partial\bar\partial\psi_\varepsilon+i\partial\bar\partial\tilde\varphi_t)\geqslant k\omega_{\varphi_t}$$

By the previous lemma, we can always solve $\bar\partial u_t=\beta$, satisfying the estimate

$$\int_K|u_t|^2_{\tilde h_t^k}\omega^n_{\varphi_t}\leqslant\int_K|\beta|^2_{\tilde h_t^k,k\omega_{\varphi_t}}\omega^n_{\varphi_t}=\int_K e^{-tk\psi_\varepsilon-k\tilde\varphi_t}|\beta|^2_{h_0^k,k\omega_{\varphi_t}}\omega^n_{\varphi_t}$$

Since β is compactly supported on X_0^{reg}, $\omega_{\varphi_t} \to \omega_{\varphi_0}$ on the support of β, and $e^{-tk\psi_\varepsilon} \to 1$ in L^1_{loc}, so we have

$$\lim_{t\to 0}\int_K |\beta|^2_{\tilde{h}_t^k, k\omega_{\varphi_t}}\omega^n_{\varphi_t} = \int_K e^{-k\varphi_0}|\beta|^2_{h_0^k, k\omega_{\varphi_0}}\omega^n_{\varphi_0}$$

and since $e^{-tk\psi_\varepsilon - k\tilde{\varphi}_t}$ is bounded from below on any compact set K, it follows that

$$\int_K |u_t|^2_{h_0^k} i^{n^2}\Omega\wedge\bar{\Omega} \leqslant C\int_K e^{-tk\psi_\varepsilon - k\tilde{\varphi}_t}|u_t|^2_{h_0^k} i^{n^2}\Omega\wedge\bar{\Omega} = C\int_K |u_t|^2_{\tilde{h}_t^k}\omega^n_{\varphi_t} \leqslant C$$

hence there exist a weakly convergent subsequence $u_t \rightharpoonup u$ in $L^2(K, h_0^k)$ and the equation $\bar{\partial}u_t = \beta$ carries through the limit in the weak convergence, so we have $\bar{\partial}u = \beta$. Since the sections $u_t - u$ are holomorphic and weakly converge to 0, it follows that the convergence is smooth it happens strongly, hence we have

$$\int_K e^{-k\varphi_0}|u|^2_{h_0^k} i^{n^2}\Omega\wedge\bar{\Omega} \leqslant \int_K e^{-k\varphi_0}|\beta|^2_{h_0^k,\omega_{\varphi_0}} i^{n^2}\Omega\wedge\bar{\Omega}$$

□

Proposition 5.16. *The following Sobolev inequality hold for $f \in L^\infty \cap H^1(X_0^{reg}, \omega_\infty)$*

$$\left(\int_{X_0^{reg}} |f|^{2\frac{n}{n-1}}\omega^n_\infty\right)^{\frac{n-1}{n}} \leqslant C\int_{X_0^{reg}} |\nabla f|^2_{g_\infty}\omega^n_\infty$$

Proof. Without loss of generality, we can assume $f \geqslant 0$. If f is supported in X_0^{reg}, this follows from [19]. For $f \in L^\infty$, we can define $f_\varepsilon = f\rho_\varepsilon$, f_ε is supported in X_0^{reg}, then we clearly have $\|f_\varepsilon\|_{L^2} \to \|f\|_{L^2}$, and we also have

$$\int_X |\nabla f_\varepsilon|^2 = \int_X \rho_\varepsilon^2|\nabla f|^2 + \int_X f^2|\nabla\rho_\varepsilon|^2 + 2\int_X f\rho_\varepsilon\langle\nabla f, \nabla\rho_\varepsilon\rangle$$

the second and third term goes to 0 as $\varepsilon \to 0$ because $\int_X |\nabla\rho_\varepsilon|^2 \to 0$, and this gives what we wanted. □

Lemma 5.17. *Suppose $u \geqslant 0$ is a bounded function on X_0^{reg} that satisfy*

$$\Delta_{\omega_\infty} u \geqslant -Au$$

then for $R \geqslant 1$ sufficiently large (so that $X_0^{sing} \subset B_R(p)$), we have the estimate

$$\|u\|_{L^\infty(B_R(p))} \leqslant C(A + CR^{-2})^{\frac{n}{2}}\|u\|_{L^2(B_{2R}(p))}$$

Proof. Using the Sobolev inequality above and the cutoff function, we can do Moser iteration on (X_0^{reg}, g_∞)

$$\begin{aligned}
A\int_X \eta^2\rho_\varepsilon^2 u^{p+1}\omega^n_\infty &\geqslant \int_X \eta^2\rho_\varepsilon^2 u^p(-\Delta u)\omega^n_\infty \\
&= \frac{4p}{(p+1)^2}\int_X \eta^2\rho_\varepsilon^2|\nabla u^{\frac{p+1}{2}}|^2\omega^n_\infty + 2\int_X \rho_\varepsilon^2\eta(\nabla\eta\cdot\nabla u)u^p\omega^n_\infty \\
&\quad + \frac{4}{(p+1)}\int_X \eta^2\rho_\varepsilon(\nabla\rho_\varepsilon\cdot\nabla u^{\frac{p+1}{2}})u^{\frac{p+1}{2}}\omega^n_\infty \\
&\geqslant \frac{4p}{(p+1)^2}\int_X \eta^2\rho_\varepsilon^2|\nabla u^{\frac{p+1}{2}}|^2\omega^n_\infty + 2\int_X \rho_\varepsilon^2\eta(\nabla\eta\cdot\nabla u)u^p\omega^n_\infty \\
&\quad - \frac{4}{(p+1)}\left(\int_X \eta^2\rho_\varepsilon^2|\nabla u^{\frac{p+1}{2}}|^2\omega^n_\infty\right)^{\frac{1}{2}}\left(\int_X \eta^2|\nabla\rho_\varepsilon|^2 u^{p+1}\omega^n_\infty\right)^{\frac{1}{2}}
\end{aligned}$$

when u is bounded, we can take a limit as ε goes to 0 and the last term will disappear, so we have

$$\begin{aligned} A\int_X \eta^2 u^{p+1}\omega_\infty^n &\geqslant \frac{4p}{(p+1)^2}\int_X \eta^2|\nabla u^{\frac{p+1}{2}}|^2\omega_\infty^n + \frac{4}{p+1}\int_X \eta(\nabla\eta\cdot\nabla u^{\frac{p+1}{2}})u^{\frac{p+1}{2}}\omega_\infty^n \\ &\geqslant \frac{3p}{(p+1)^2}\int_X \eta^2|\nabla u^{\frac{p+1}{2}}|^2\omega_\infty^n - \frac{16}{p}\int_X |\nabla\eta|^2 u^{p+1}\omega_\infty^n \end{aligned}$$

which implies

$$\int_X |\nabla \eta u^{\frac{p+1}{2}}|^2\omega_\infty^n \leqslant \frac{(p+1)^2}{p}\int_X (A\eta^2 + \frac{17}{p}|\nabla\eta|^2)u^{p+1}\omega_\infty^n$$

then by the Sobolev inquality from Proposition 5.16, we have for any $p>0$,

$$\left(\int_X |\eta u|^{(p+1)\frac{n}{n-1}}\omega_\infty^n\right)^{\frac{n-1}{n}} \leqslant \frac{C(p+1)^2}{p}\int_X (A\eta^2 + \frac{17}{p}|\nabla\eta|^2)u^{p+1}\omega_\infty^n$$

by carefully choosing cutoff functions $0\leqslant \eta_k\leqslant 1$ such that $\mathrm{supp}(\eta_k)\subset B_{(1+2^{-k})R}$, $\eta_k=1$ on $B_{(1+2^{-k-1})R}$ and $|\nabla\eta_k|\leqslant CR^{-1}2^k$, and set $p_k = 2(\frac{n}{n-1})^k$, then for $k=0,1,2,\dots$ we have

$$\|u\|^{p_k}_{L^{p_{k+1}}(B_{(1+2^{-k-1})R})}\leqslant C(Ap_k + CR^{-2}4^k)\|u\|^{p_k}_{L^{p_k}(B_{(1+2^{-k})R})}$$

iterating gives

$$\sup_{B_R} u \leqslant C_{sob}^{\frac{n}{2}}C(2A+CR^{-2})^{\frac{n}{2}}\|u\|_{L^2(B_{2R})}$$

□

We now prove L^2 estimates for holomorphic sections of L^k.

Proposition 5.18. *If s is a holomorphic section of (L^k, h_∞^k), then the following estimates hold on (X_0^{reg}, kg_∞) for R large enough so that $B_R(p)$ contains all of X_0^{sing},*

$$\sup_{B_R(p)}|s|_{h_\infty^k}\leqslant C\|s\|_{L^2_{h_\infty^k, kg_\infty}(B_{2R}(p))}$$

$$\sup_{B_R(p)}|\nabla s|_{h_\infty^k, kg_\infty}\leqslant C\|s\|_{L^2_{h_\infty^k, kg_\infty}(B_{2R}(p))}$$

Proof. For a holomorphic section s, we have $\nabla_{\bar{j}}s=0$, so $g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i s = -ns$. It follows then from standard calculations that

$$\Delta|s|\geqslant -n|s|$$

and

$$\Delta|\nabla s|\geqslant -(n+2)|\nabla s|$$

so now we can apply Lemma 5.17 with $u=|s|$ and $u=|\nabla s|$ to get

$$\|s\|_{L^\infty(B_R)}\leqslant C\|s\|_{L^2(B_{2R})}$$

and

$$\|\nabla s\|_{L^\infty(B_R)}\leqslant C\|\nabla s\|_{L^2(B_{2R})} \tag{5.11}$$

and it suffices to show that $\|\nabla s\|_{L^2(B_{2R})}\leqslant C\|s\|_{L^2(B_{3R})}$. We use integration by parts

$$\begin{aligned}\int_X \eta^2\rho_\varepsilon^2|\nabla s|^2 &= \int_X \eta^2\rho_\varepsilon^2 h g_\infty^{i\bar{j}}\nabla_i s\nabla_{\bar{j}}\bar{s}\omega_\infty^n \\ &= -\int_X \eta^2\rho_\varepsilon^2 h g_\infty^{i\bar{j}}\nabla_{\bar{j}}\nabla_i s\bar{s}\omega_\infty^n - 2\int_X \nabla_{\bar{j}}(\eta^2\rho_\varepsilon^2) h g_\infty^{i\bar{j}}\nabla_i s\bar{s}\omega_\infty^n \\ &\leqslant n\int_X \eta^2\rho_\varepsilon^2|s|^2 + 2\int_X \eta\rho_\varepsilon(\rho_\varepsilon|\nabla\eta| + \eta|\nabla\rho_\varepsilon|)|s||\nabla s| \\ &\leqslant C\int_X(\eta^2+|\nabla\eta|^2)\rho_\varepsilon^2|s|^2 + \varepsilon\int_X \eta^2\rho_\varepsilon^2|\nabla s|^2 + C\int_X \eta^2|\nabla\rho_\varepsilon|^2|s|^2\end{aligned}$$

taking ε to 0 gives

$$\int_X \eta^2|\nabla s|^2 \leqslant C\int_X(\eta^2+|\nabla\eta|^2)|s|^2$$

by choosing $0\leqslant\eta\leqslant 1$ so that $\mathrm{supp}(\eta)\subset B_{4R}$ and $\eta=1$ on B_{2R}, this gives $\|\nabla s\|_{L^2(B_{2R})}\leqslant \|s\|_{L^2(B_{4R})}$ Combined with estimate (5.11), this gives the desired estimates. □

Corollary 5.19. *For any holomorphic sections $s_0, s_1 \in H^0(L^k|_K)$ on K, the function $|s_i|_{h_\infty^k}$ extends as a lipshitz function on K and this function vanishes precisely on the set $\pi_\infty^{-1}(\{s_i=0\})$. Also, $\frac{s_0}{s_1}$ extends as a locally Lipshitz function defined on the set $\{|s_1|_{h_\infty^k}>0\}$.*

Proof. This follows immediately from Kato's inequality

$$|\nabla|s|_{h_\infty^k}|_{g_\infty}\leqslant|\nabla s|_{h_\infty^k,kg_\infty}\leqslant C$$

$$\left|\nabla\frac{s_0}{s_1}\right|_{g_\infty}\leqslant\frac{|s_1|_{h_\infty^k}|\nabla s_0|_{h_\infty^k,kg_\infty}+|s_0|_{h_\infty^k}|\nabla s_1|_{h_\infty^k,kg_\infty}}{|s_1|_{h_\infty^k}^2}\leqslant\frac{C}{|s_1|_{h_\infty^k}^2}$$

and the fact that $K=\overline{(K\cap X_0^{reg},g_\infty)}$. □

In this section we prove that the map $\pi_\infty : X_\infty \to X_0$ is injective, hence it is an isomorphism.

Proposition 5.20. *For any $p,q\in X_\infty$ with $p\neq q$ there exist an $k=k(p,q)>0$ and $s_p,s_q\in H^0(L^k)$ such that*

$$|s_p(p)|_{h_\infty^k}, |s_q(q)|_{h_\infty^k}\geqslant\frac{2}{5}$$

and

$$|s_p(q)|_{h_\infty^k}, |s_q(p)|_{h_\infty^k}\leqslant\frac{1}{3}$$

Proof. This follows from the same argument as Proposition 3.9 in [51]. □

Proposition 5.21. *The map $\pi_\infty : X_\infty\to X_0$ is an homeomorphism.*

Proof. It's clear that the map is surjective and restricts to a homeomorphism on $X_0^{reg}\subset X_\infty$, it suffices to show that is seperates points near X_0^{sing}. Given $p,q\in K$, suppose for a contradiction that $\pi_\infty(p)=\pi_\infty(q)$, then for any $k>0$, and any two sections $s_0,s_1\in H^0(K\cap X_0^{reg},L^k)$, by the normality of X_0, we know that these two sections extend over the singular set to two sections of $s_0',s_1'\in H^0(\pi_\infty(K),L^k)$, hence we must have $\frac{s_0(p)}{s_1(p)}=\frac{s_0(q)}{s_1(q)}$. But if $d_{X_\infty}(p,q)>0$, then by the previous lemma, there exist $k>0$ and we can construct sections $s_p,s_q\in H^0(K\cap X_0^{reg},L^k)$ such that $|s_p|_{h_\infty^k}(p),|s_q|_{h_\infty^k}(q)\geqslant\frac{2}{5}>\frac{1}{3}\geqslant|s_p|_{h_\infty^k}(q),|s_q|_{h_\infty^k}(p)$ which contradicts $\frac{s_p(p)}{s_q(p)}=\frac{s_p(q)}{s_q(q)}$.

Observe that the singular set $\mathcal{S} \subset X_\infty$ is closed and of finite diameter, from which we can see that $\pi_\infty : X_\infty \to X_0$ is a proper map, hence closed, and this implies π_∞^{-1} is also continuous. Thus π_∞ is a homeomorphism. □

proof of Theorem 1.2. This is just a combination of Proposition 5.1, Proposition 5.5 and Proposition 5.21. □

6. Examples and Applications

In this section we apply Theorems 1.1 and Theorem 1.2 to study certain explicit examples of crepant resolutions.

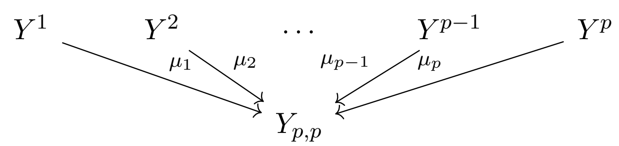
6.1. **Small Resolutions of Brieskorn-Pham cones.** Consider the quasi-homogeneous affine varieties

$$Y_{p,q} = \{xy + z^p - w^q = 0\} \subset \mathbb{C}^4,$$

where we assume that $p \leqslant q$. These singularities, which are compound du Val of type cA_p are Gorenstein and log-terminal and by the main result of [12], $Y_{p,q}$ admits a conical Calabi-Yau metric if and only if $q < 2p$. Let r denote the radial function of the Calabi-Yau cone metric. The Euler vector field $r\frac{\partial}{\partial r}$ associated with the cone structure is given by the real part of the holomorphic vector Reeb field ξ acting on the coordinates (x, y, z, w) with weights

$$\frac{3}{2(p+q)}(pq, pq, 2q, 2p);$$

in particular, the $Y_{p,q}$ are quasi-regular Calabi-Yau cones. A result of Katz [33] says that the $Y_{p,q}$ admits a small (and hence crepant) resolution $\mu : Y \to Y_{p,p}$ if and only if $p = q$. In fact, $Y_{p,p}$ admits p inequivalent small resolutions



with each pair Y^i, Y^j related by a flop; the $\frac{p(p-1)}{2}$ flops are in correspondence with the reflections in the Weyl group of the A_{p-1} Dynkin diagram [42]. When $p = 2$, this recovers the Atiyah flop [1]. The exceptional locus of each contraction μ_j is a chain of $p-1$ rational curves with normal bundle $(-1,-1)$ intersecting transversally. Explicitly, let $\zeta = e^{\frac{2\pi\sqrt{-1}}{p}}$, and write

$$z^p - w^p = \prod_{j=0}^{p-1}(z - \zeta^j w).$$

Fix $1 \leqslant \ell \leqslant p-1$ and consider the rational map $\nu_\ell : Y_{p,p} \to \mathbb{P}^1$ defined by

$$\nu_\ell(x, y, z, w) = ([x : \prod_{j=0}^{\ell}(z - \zeta^j w)]) \in \mathbb{P}^1. \tag{6.1}$$

Then a small resolution $\mu : Y \to Y_{p,p}$ (say Y^1 for concreteness) is obtained by taking the closure of the graph of

$$\nu_1 \times \cdots \times \nu_{p-1} : Y_{p,p} \to \mathbb{P}^1_{(1)} \times \cdots \times \mathbb{P}^1_{(p-1)}.$$

There are also corresponding partial resolutions $\overline{Y}$ by projecting out some collection of the ν_j. Fix $1 \leqslant i \leqslant p$, and let $\overline{Y}$ be any partial resolution whose contraction $\bar{\pi} : \overline{Y} \to Y_{p,p}$ factors through ν_i. Clearly these resolutions are obtained by repeatedly blowing-up along the lines $x = z - \zeta^j w = 0$. There is a divisor E_i defined by $-E_i = \nu_i^{-1}(p)$ for a generic point $p \in \mathbb{P}^1$, and these divisors satisfy $\mathcal{O}_{\overline{Y}}(-E_i)|_{\mathrm{Exc}(\nu_i)} = \mathcal{O}_{\mathbb{P}^1}(1)$, and $\mathcal{O}_{\overline{Y}}(E_i)$ is trivial on any other component of $\mathrm{Exc}(\bar{\pi})$. Furthermore, if $\overline{Y}$ is obtained from $\nu_{i_1} \times \cdots \times \nu_{i_k}$ then $\bigotimes_{j=1}^k \mathcal{O}_{\overline{Y}}(-E_{i_j})^{\otimes \ell_j}$ is ample for any $\ell_j \in \mathbb{Z}_{>0}$. These statements follows straightforwardly from the corresponding statements for the blow-ups of the ambient $\mathbb{C}^4$.

Let us fix a small resolution $\mu : Y \to Y_{p,p}$. By Hartog's theorem the holomorphic Reeb vector field extends over $\mathrm{Exc}(\mu)$ and generates a holomorphic retraction onto $\mathrm{Exc}(\mu)$. Thus we have

$$H^{1,1}(Y, \mathbb{R}) = \bigoplus_{i=1}^{p-1} H^{1,1}(\mathbb{P}^1_{(i)}, \mathbb{R}) = \bigoplus_{i=1}^{p-1} \mathbb{R} \cdot [E_i]$$

By the above discussion, the classes $\sum_{i=1}^{p-1}(-t_i)[E_i]$ are Kähler on Y, provided $t_i > 0$ for all i, and semi-positive for $t_i \geqslant 0$. Each of these cohomology classes is 2-almost compactly supported. Fix a class $[\alpha_0] = \sum_{i=1}^{p-1}(-t_i)[E_i]$ where $t_i \geqslant 0$, and at least one $t_j = 0$ and let $[\omega] \in H^{1,1}(Y, \mathbb{R})$ be any Kähler class. Let $[\omega_t] = (1-t)[\alpha_0] + t[\omega]$ be a linear family of Kähler classes. Then by [28] (see also [15]) there is an asymptotically conical Calabi-Yau metric $\omega_{t,CY}$ in $[\omega_t]$ for all $t > 0$.

Since the cone at infinity is quasi-regular we can apply Lemma 3.3 to conclude that there is a Kähler current in $[\alpha_0]$ which is smooth on the complement of

$$V := \left\{ \mathbb{P}^1_j \subset Y : \int_{\mathbb{P}^1_{(j)}} \alpha_0 = 0 \right\}$$

Let $\overline{Y}$ be the partial resolution obtained by contracting V, and let $\hat{\pi} : Y \to \overline{Y}$ be the contraction map. If $[\alpha_0] \in H^{1,1}(Y, \mathbb{Q})$ then, by the preceding discussion, after rescaling we can assume that $[\alpha_0] = \pi^* c_1(L)$ for some ample line bundle $L \to \overline{Y}$. Applying Theorem 1.1 and Theorem 1.2 we obtain

Proposition 6.1. *In the above situation we have*

(1) $\overline{Y}_{reg}$ *admits a smooth Ricci-flat metric* $\bar{\omega}$*, asymptotic to the Calabi-Yau metric on* $Y_{p,p}$ *at infinity, and with* $\overline{(\overline{Y}_{reg}, \bar{\omega})}$ *homeomorphic to* $\overline{Y}$.

(2) *As* $t \to 0$ $(Y, \omega_{t,CY})$ *converges in the Gromov-Hausdorff sense to* $\overline{(\overline{Y}_{reg}, \bar{\omega})}$.

(3) *In particular, if we take* $[\alpha_0] = 0$*, the flops of the* $Y_{p,p}$ *are continuous in the Gromov-Hausdorff sense.*

Proof. The only point which is not an immediate consequence of Theorems 1.1 and 1.2 is the third point. However, by the uniqueness part of Theorem 1.1, the limiting limiting Calabi-Yau metric $\bar{\omega}$ on $Y_{p,p}$ is isometric to the conical Calabi-Yau metric from [12]. Alternatively, this can be seen as follows. Let ω_c denote the Calabi-Yau metric on $Y_{p,p}$. Clearly $t\omega_{1,CY}$ is a Calabi-Yau metric in $t[\omega]$ asymptotic to $t\omega_c$. Let $\hat{\xi}$ denote the extension of the holomorphic Reeb vector field on Y, and, for $\lambda \in \mathbb{C}$ let $\varphi_\lambda : Y \to Y$ denote the λ-flow of $\hat{\xi}$. Then

$$\left(\varphi_{\frac{1}{\sqrt{t}}}\right)^* t\omega_{1,CY}$$

is Calabi-Yau, asymptotic to ω_c, and lies in the cohomology class $t[\omega]$ and hence is equal to $\omega_{t,CY}$ by the uniqueness results of [15]. From this description, and the convergence result of Theorem 1.1 it follows that $\omega_{t,CY}$ converges to $\mu_i^* \omega_c$ on compact sets of $Y \backslash \mathrm{Exc}(\mu_i)$. □

It's not hard to check that if a partial resolution $\overline{Y}$ is obtained by blowing up $0 < k < p-1$ lines $x = z - \zeta^j w = 0$, then $\overline{Y}$ has an isolated singularity biholomorphic to a neighborhood of the singular point in $Y_{p-k,p-k}$. More precisely, suppose for simplicity that $\overline{Y}$ is obtained by blowing-up the lines $x = z - \zeta^j w = 0$ for $0 \leqslant j \leqslant k < p-1$. Then $\overline{Y}$ has an isolated singularity biholomorphic to

$$\widetilde{Y}_{p-k,p-k} := \{xy = \prod_{j=k}^{p-1}(z - \zeta^j w)\} \subset \mathbb{C}^4$$

which is deformation equivalent to $Y_{p-k,p-k}$ and admits a conical Calabi-Yau metric by argument of [12]. The link of this singularity is topologically $(p-k-1)\#(S^2 \times S^3)$ and it comes equipped with a Sasaki-Einstein metric. It was shown in [12] that the volume of these Sasaki-Einstein metrics is given by

$$\frac{2(2(p-k))^3}{27(p-k)^4} = \frac{16}{27(p-k)}$$

Thus $\overline{Y}$ yields a cobordism between $(p-k-1)\#(S^2 \times S^3)$ and $\#(p-1)(S^2 \times S^3)$. It is natural to expect that that the metric $\bar{\omega}$ on $\overline{Y}$, close to the singular point, is close to the conical Calabi-Yau metric on $\tilde{Y}_{p-k,p-k}$. At the very least, we expect

Conjecture 2. *Let $(\overline{Y}, d)$ denote the metric space obtained as the completion of $(\overline{Y}_{reg}, \bar{\omega})$. Then the tangent cone to $(\overline{Y}, d)$ at the singular point is isometric to $\widetilde{Y}_{p-k,p-k}$ equipped with its conical Calabi-Yau metric.*

Let $y \in \overline{Y}$ denote the singular point, and consider the function

$$\mathbb{R}_{>0} \ni r \mapsto v(r) := \frac{\mathrm{Vol}_{\bar{\omega}}(B_{\bar{\omega}}(y,r))}{r^6}$$

Since $(\overline{Y}, \overline{\omega})$ is Calabi-Yau, $v(r)$ is monotone decreasing by the Bishop-Gromov comparison theorem. Furthermore, assuming Conjecture 2, since $\bar{\omega}$ is asymptotic to the conical Calabi-Yau metric on $Y_{p,p}$ we have

$$\frac{16}{27(p-k)} = \lim_{r\to 0} v(r) \geqslant \lim_{r\to+\infty} v(r) = \frac{16}{27p}.$$

Note that the equality case of Bishop-Gromov already shows that if $k = 0$, then the metric is conical.

While deducing $k \geqslant 0$ in this way is not particularly interesting, this discussion holds for any asymptotically conical Calabi-Yau variety with or without singularities (indeed, a smooth, asymptotically conical Calabi-Yau variety is naturally a cobordism between the standard Sasaki-Einstein structure on the sphere and the link of the cone at infinity). Suppose $(\overline{Y}, \bar{\omega})$ is a asymptotically conical Calabi-Yau variety with asymptotic cone C_∞, and with a singular point y. Assume that a neighborhood of y is biholomorphic to a neighborhood of an isolated singular point in some quasi-homogeneous affine variety C_0 admitting a conical Calabi-Yau metric. Assuming that $\bar{\omega}$ is close to the Calabi-Yau metric on C_0 near the singularity at y, the volume ratio of geodesic balls centered at y will decrease (by Bishop-Gromov) from the volume ratio of the cone $v(C_0)$ to the volume of ratio of the cone at infinity, $v(C_\infty)$. Since these volume ratios are algebraic invariants of the singularities C_0, C_∞, this situation is obstructed in general; for example one cannot take $C_0 = Y_{p,p}$ and $C_\infty = Y_{p-k,p-k}$.

It is tempting to speculate that the volume function on Sasaki-Einstein structures could give rise to a sort of Morse function on the space of Sasaki-Einstein manifolds. For two Sasaki-Einstein manifolds S_0, S_∞ with corresponding cones C_0, C_∞ a Calabi-Yau space $(\overline{Y}, \bar{\omega})$ with an isolated singularity C_0 and cone C_∞ at infinity could be regarded as a kind of flow line of the Morse function between S_0 and S_∞. We will give further examples of this discussion below.

6.2. **Examples from Fano manifolds.** Let us next indicate how to construct examples starting from Fano manifolds with a different singular structure than the previous examples. Suppose X is a Fano manifold of dimension n. Let $\tilde{X} = \mathrm{Bl}_p X$ be the blow up of X at a point and let $\tilde{E} \subset \tilde{X}$ be the exceptional divisor. Assume in addition that that $\tilde{X}$ is Fano and $-K_{\tilde{X}}$ is base-point free. Assume that $\tilde{X}$ has a Kähler-Einstein metric, or more generally that the affine cone over $\tilde{X}$, Spec $\bigoplus_{m\geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$, admits a conical Calabi-Yau metric. This holds, for example, whenever $\tilde{X}$ is toric, by [26]. It is not difficult to generate examples satisfying these assumptions. For example

- Let $X = \mathbb{P}^n$, with p a torus invariant point. Then $\tilde{X} = \mathrm{Bl_p}\mathbb{P}^\mathrm{n}$ is Fano and $-K_{\tilde{X}}$ is base point free. These manifolds do not admit Kähler-Einstein metrics, as can be seen from Matsushima's obstruction. However, they are toric, and so the theorem of Futaki-Ono-Wang implies the existence of a Calabi-Yau cone metric on the affine cone $C :=$ Spec $\bigoplus_{m\geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$. Note that the conical Calabi-Yau structure on C need not be quasi-regular, as happens for example when $n = 2$ [27, 26, 40].
- Let X be a del Pezzo surface with $K_X^2 \geqslant 3$, and p chosen sufficiently generic so that $\tilde{X} = \mathrm{Bl}_p X$ is Fano. The global generation of $-K_{\tilde{X}}$ follows from Reider's Theorem [46]. Furthermore, a theorems of Tian-Yau [55] and Tian [52] say that X admits a Kähler-Einstein metric if $K_X^2 < 8$. If, however, $K_X^2 = 8, 9$ then X does not admit a Kähler-Einstein metric by Matsushima's obstruction [43]. On the other hand, in these latter examples, if p is chosen so that $\tilde{X}$ is toric, then the affine cone Spec $\bigoplus_{m\geqslant 0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$ admits a conical Calabi-Yau metric thanks to results of Futaki-Ono-Wang [26] (See also [12]). In these examples the Calabi-Yau cone structure is not quasi-regular [40, 26].

Let $Y = K_{\tilde{X}}$ be the total space of the canonical bundle, and let $p : Y \to \tilde{X}$ be the projection. The pull-back p^* identifies $H^{1,1}(Y, \mathbb{R}) = H^{1,1}(\tilde{X}, \mathbb{R})$, and Y admits an asymptotically conical Calabi-Yau metric in any Kähler class in $H^{1,1}(Y, \mathbb{R})$ [15]. Suppose $[\alpha] \in H^{1,1}(X, \mathbb{R})$ is a Kähler class, so that $p^*[\pi^*\alpha] \in H^{1,1}(Y, \mathbb{R})$ is a nef class on Y admitting a semi-positive representative. By regarding the exceptional divisor of the blow-up $\pi : \tilde{X} \to X$ as a subvariety of the zero section in Y, we get a natural codimension 2 subvariety, $E \subset Y$ (explicitly $E = p^{-1}(\tilde{E}) \cap \{$ zero section $\}$). Our goal is to show that if $[\omega_t] = (1-t)[p^*\pi^*\alpha] + t[\omega] \in H^{1,1}(Y, \mathbb{R})$ and $\omega_{t,CY}$ are conical Calabi-Yau metrics in $[\omega_t]$ then, as $t \to 0$, $(Y, \omega_{t,CY})$ Gromov-Hausdorff converges to a variety Z with an isolated, Gorenstein, log-terminal singularity which is obtained from Y by contracting E to a point. As a first step, we need to verify that Assumption 1 holds, since the failure of the cone at infinity to be quasi-regular means that Lemma 3.3 does not apply in general.

Lemma 6.2. *The cohomology class $p^*[\pi^*\alpha]$ contains a Kähler current which is smooth outside of E.*

Proof. It is a standard fact that we can choose a hermitian metric $h_{\tilde{E}}$ on $\mathcal{O}_{\tilde{X}}(\tilde{E})$ such that

$$\pi^*\alpha + \varepsilon\sqrt{-1}\partial\overline{\partial}\log h_{\tilde{E}} > \omega_{\tilde{X}} \tag{6.2}$$

for some $\varepsilon > 0$ and $\omega_{\tilde{X}}$ a Kähler form on $\tilde{X}$. Let $s_{\tilde{E}}$ denote the defining section of $\tilde{E} \subset \tilde{X}$. After scaling we may assume that $|s_{\tilde{E}}|^2_{h_{\tilde{E}}} < 1$. The current $\tilde{T} := \pi^*\alpha + \varepsilon\sqrt{-1}\partial\overline{\partial}\log|s_{\tilde{E}}|^2_{h_{\tilde{E}}}$ is a Kähler current on $\tilde{X}$ which is singular along $\tilde{E} \subset \tilde{X}$. Let $h_{\tilde{X}}$ be a negatively curved metric on $K_{\tilde{X}}$, and let s denote a coordinate on the fibers of $K_{\tilde{X}}$. We claim that

$$T = p^*\pi^*\alpha + \sqrt{-1}\partial\overline{\partial}(|s|^2_h + \varepsilon\log(p^*|s_{\tilde{E}}|^2_{h_{\tilde{E}}} + |s|^2_{h_{\tilde{X}}})) \tag{6.3}$$

is a Kähler current. This can be verified by a straightforward calculation, which we leave to the reader.

□

The next step is to show that there is a space Z, and a map $\Phi : Y \to Z$ which is an isomorphism outside E and contracts E to a point, which is an isolated, Gorenstein log-terminal singularity in Z. Let us begin with a local description of this map and the resulting singularity. Note that the normal bundle of $E \subset Y$ is given by

$$N_{E/Y} = \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1))$$

which follows from $K_{\tilde{X}} = \pi^* K_X + (n-1)\tilde{E}$. There is a contraction map

$$\nu : \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1)) \to C$$

contracting the zero section of $N_{E/Y}$ to a point. Explicitly, this map is given by [41, Page 314]

$$\begin{aligned} N_{E/Y} &= \operatorname{Spec} \bigoplus_{m \geqslant 0} \operatorname{Sym}^m \left(\mathcal{O}_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}((n-1))\right) \\ &\to \operatorname{Spec} \bigoplus_{m \geqslant 0} H^0\left(\mathbb{P}^{n-1}, \operatorname{Sym}^m \left(\mathcal{O}_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}((n-1))\right)\right) = C_0 \end{aligned}$$

Since

$$H^0\left(\mathbb{P}^{n-1}, \operatorname{Sym}^m \left(\mathcal{O}_{\mathbb{P}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}((n-1))\right)\right) = H^0(\mathbb{P}(N_{E/Y}), \mathcal{O}_{\mathbb{P}(N_{E/Y})}(m))$$

we see that C_0 is the affine cone over $\mathbb{P}(N_{E/Y})$ obtained by blowing down the zero section of $\mathcal{O}_{\mathbb{P}(N_{E/Y})}(-1)$. We claim that $\mathbb{P}(N_{E/Y})$ is Fano. In general, the canonical bundle of a projective bundle $\pi : \mathbb{P}(V) \to X$, where V has rank r is given by

$$K_{\mathbb{P}(V)} = \mathcal{O}_{\mathbb{P}(V)}(-r-1) \otimes \pi^*(\det V^*) \otimes \pi^* K_X.$$

Applying this formula in the current scenario yields

$$K_{\mathbb{P}(N_{E/Y})} = \mathcal{O}_{\mathbb{P}(N_{E/Y})}(-3).$$

Since $N_{E/Y}$ is a direct sum of negative line bundles, $\mathcal{O}_{\mathbb{P}(N_{E/Y})}(3)$ is ample. It follows from this that C_0 has an isolated Gorenstein, log-terminal singularity and $K_{C_0} \sim \mathcal{O}_{C_0}$ is trivial. Finally, since $N_{E/Y} \to \mathbb{P}^{n-1}$ is a direct sum of line bundles, $\mathbb{P}(N_{E/Y})$ is toric. Therefore the result of Futaki-Ono-Wang [26] (see also [12]) says that C_0 admits a conical Calabi-Yau metric for some choice of Reeb vector field.

Next we will globalize this construction using the input of an ample line bundle L on X. First note that a section $f \in H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$ naturally induces a holomorphic function $f \in H^0(Y, \mathcal{O}_Y)$ vanishing to order m on $\tilde{X} = \{\text{ zero section }\} \subset Y$. Let $f_1 \ldots, f_M$ be generators of the coordinate ring $\bigoplus_{m>0} H^0(\tilde{X}, -K_{\tilde{X}}^{\otimes m})$. Since $-K_{\tilde{X}}$ is ample and globally generated, the holomorphic functions $f_1 \ldots, f_M$ separate points and tangent vectors on $Y \backslash \tilde{X}$, and generate the normal bundle to $\tilde{X}$ in Y. Let L be a very ample line bundle on X, and let $\{s_0, \ldots, s_N\}$ be a basis of $H^0(X, L)$. Fix coordinates $(z_1, \ldots, z_n)$ on X centered at p. Up to making a linear change of coordinates we can assume that $s_0(p) \neq 0$, and near p we have

$$\frac{s_i(z)}{s_0(z)} = z_i + O(z^2) \quad 1 \leqslant i \leqslant n, \qquad \frac{s_j(z)}{s_0(p)} = O(z^2) \quad n \leqslant j \leqslant N$$

By inspection the sections $\{p^*\pi^* s_i\}_{0 \leqslant i \leqslant N}$ separate points and tangents in $\tilde{X} \backslash \tilde{E}$ and generate the normal bundle to $\tilde{E}$ in $\tilde{X}$. Now consider the map $\Phi : Y \to \mathbb{P}^N \times \mathbb{P}^M$ defined by

$$\Phi(z) := ([p^*\pi^* s_0(z) : \cdots : p^*\pi^* s_N(z)], [1 : f_1(z) : \cdots : f_M(z)]) \in \mathbb{P}^N \times \mathbb{P}^M \tag{6.4}$$

By the preceeding discussion this map is an isomorphism on $Y\backslash E$, and $\Phi(E) = [1:0:\cdots:0]\times[1:0:\cdots:0]$. Since the differential $d\Phi$ is an isomorphism on $N_{E/Y}$, the germ of Φ agrees with the contraction ν on $N_{E/Y}$. Note also that $\Phi|_{\tilde{X}} = \pi$ (composed with the imbedding X into projective space by sections of L). Let $Z = \Phi(Y)$. From the local description above Z has an isolated Gorenstein, log-terminal singularity, and $K_Z = \mathcal{O}_Z$. The map $\Phi : Y \to Z$ is therefore a small, and hence crepant, resolution of Z. It follows from the construction that we can describe Z has the relative spectrum

$$Z = \underline{\mathrm{Spec}}\,(K_X \otimes \mathfrak{m}_p) \to X$$

where $\mathfrak{m}_p$ is the ideal sheaf of $p \in X$. In order to apply Theorems 1.1 and 1.2 it suffices to show

Lemma 6.3. *In the above setting, there is an ample line bundle L' on Z such that $p^*c_1(\pi^*L) = \Phi^*c_1(L')$.*

Proof. Since Z is normal and Φ is projective with connected fibers we have $\Phi_*\mathcal{O}_Y = \mathcal{O}_Z$, and $f_1, \ldots, f_M$ extend over the singular point to global sections of $\mathcal{O}_Z$. Furthermore, there is a natural projection

$$\hat{p} : Z \to X$$

obtained by projecting from Z onto the $\mathbb{P}^N$ factor in (6.4) and we have $\pi\circ p = \Phi\circ p = \hat{p}\circ\Phi$. Thus

$$[p^*\pi^*s_0 : \cdots : p^*\pi^*s_N] = [\hat{p}^*s_0 : \cdots : \hat{p}^*s_N].$$

Combining this observation with the Segre embedding $\mathbb{P}^N \times \mathbb{P}^M \hookrightarrow \mathbb{P}^{(N+1)(M+1)-1}$ it follows that $L' := \hat{p}^*L$ is ample on Z. Since

$$p^*c_1(\pi^*L) = \Phi^*c_1(\hat{p}^*L)$$

the lemma follows. □

We can now conclude

Corollary 6.4. *With notation as above, consider the family of Kähler classes $[\omega_t] = (1-t)p^*c_1(\pi^*L) + t[\omega] \in H^{1,1}(Y, \mathbb{R})$ for $t > 0$. Let $\omega_{t,CY}$ be the asymptotically conical Kähler metrics in $[\omega_t]$. Then there is a incomplete, asymptotically conical Calabi-Yau metric $\overline{\omega}$ on Z_{reg} such that $\overline{(Z_{reg}, \bar{\omega})} = (Z, d)$ and*

$$(Y, \omega_{t,CY}) \to_{GH} (Z, d).$$

Proof. Combine Lemmas 6.2 6.3 with Theorems 1.1 and 1.2. □

It is again natural to conjecture

Conjecture 3. *Let (Z, d) be the metric space structure on Z induced from Y by Theorem 1.2. Then the tangent cone to (Z, d) at the singular point $z \in Z$ is isometric to the blow down of the zero section in $\mathcal{O}_{\mathbb{P}(V)}(-1)$ where*

$$V := \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1))$$

equipped with its conical Calabi-Yau metric.

Assuming this conjecture, the space Z can be viewed as a kind of cobordism between Sasaki-Einstein manifolds, and the speculative discussion from Section 6.1 can be applied in the same way.

References

[1] Atiyah, M. F. *On analytic surfaces with double points*, Proc. Roy. Soc. London Ser. A **247** (1958), 237–244.

[2] Bando, S., Kobayashi, R. *Ricci-flat Kähler metrics on affine algebraic manifolds*, Geometry and analysis on manifolds (Katata/Kyoto, 1987), 20–31, Lecture Notes in Math., 1339, Springer, Berlin, 1988.

[3] Bando, S., Kobayashi, R. *Ricci-flat Kähler metrics on affine algebraic manifolds II*, Math. Ann. 287 (1990), no. 1, 175–180.

[4] Bedford, E., Taylor, B. A. *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. **37** (1976), no. 1, 1–44.

[5] Calabi E. *Métriques kählériennes et fibrés holomorphes*, Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 2, 269-294.

[6] Cheeger, J., Colding, T. H. *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) 144 (1996), no. 1, 189-237.

[7] Cheeger, J., Colding, T. H. *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. 46 (1997), no. 3, 406-480.

[8] Cheeger, J., Colding, T. H. *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom. 54 (2000), no. 1, 13-35.

[9] Cheeger, J., Colding, T. H. *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom. 54 (2000), no. 1, 37-74.

[10] Cheeger, J., Colding, T. H., Tian, G. *On the singularities of spaces with bounded Ricci curvature*, Geom. Funct. Anal. 12 (2002), no. 5, 873-914.

[11] Cheng, S.Y. Yau, S.T. *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. 33 (1980), no. 4, 507–544.

[12] Collins, T. C., Székelyhidi, G. *Sasaki-Einstein metrics and K-stability*, Geom. Topol. **23** (2019), no. 3, 1339-1413.

[13] Collins, T. C., Tosatti, V. *Kähler currents and null loci*, Invent. Math. 202 (2015), no. 3, 1167-1198.

[14] Conlon, R. *On the construction of asymptotically conical Calabi-Yau manifolds*, Ph.D. thesis, Imperial College London, 2011.

[15] Conlon, R., Hein, H. *Asymptotically conical Calabi-Yau manifolds, I*, Duke Math. J. 162 (2013), no. 15, 2855-2902.

[16] Conlon, R., Hein, H. *Asymptotically conical Calabi-Yau metrics on quasi-projective varieties*, Geom. Funct. Anal. 25 (2015), no. 2, 517-552.

[17] Conlon, R., Hein, H. *Asymptotically conical Calabi-Yau manifolds III*, arXiv.1405.7140

[18] Corti, A., Haskins, M., Nordström, J., Pacini, T. *Asymptotically cylindrical Calabi-Yau* 3*-folds from weak Fano* 3*-folds*, Geom. Topol. **17** (2013), no. 4, 1955-2059

[19] Croke, C. *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 4, 419-435.

[20] Demailly, J.-P. *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, 1. International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

[21] Demailly, J.-P., Păun, M. *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), no. 3, 1247–1274.

[22] Donaldson, S. Kronheimer, P. *The geometry of four-manifolds*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990. x+440 pp.

[23] Donaldson, S., Sun, S. *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math. 213 (2014), no. 1, 63-106.

[24] Eyssidieux, P., Guedj, V., Zeriahi, A. *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. 22 (2009), no. 3, 607-639.

[25] Faulk, M. *Asymptotically conical Calabi-Yau orbifolds, I*, arxiv.1809.01556

[26] Futaki, A., Ono, H., Wang, G. *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds.*, J. Differential Geom. **83** (2009), no. 3, 585–635.

[27] Gauntlett, J., Martelli, D., Sparks, J., Waldram, D., *Sasaki-Einstein metrics on* $S^2 \times S^3$, Adv. Theor. Math. Phys. **8** (2004), no. 4, 711–734

[28] Goto, R. *Calabi-Yau structures and Einstein-Sasakian structures on crepant resolutions of isolated singularities*, J. Math. Soc. Japan 64 (2012), no. 3, 1005-1052.

[29] Grauert, H. *Aber Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. 146 (1962), 331-368.

[30] Grauert, H. Remmert, R. *Plurisubharmonische Funktionen in komplexen Raumen (German)*, Math. Z. 65 (1956), 175-194.
[31] Haskins, M., Hein, H,-J., Nordström, J. *Asymptotically cylindrical Calabi-Yau manifolds*, J. Differential Geom. **101** (2015), no. 2, 213–265.
[32] Joyce, D. *Compact manifolds with special holonomy*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. xii+436 pp.
[33] Katz, S. *Small resolutions of Gorenstein threefold singularities*, Algebraic geometry: Sundance 1988, 61–70, Contemp. Math., **116**, Amer. Math. Soc., Providence, RI, 1991.
[34] Kolodziej, S. *The complex Monge-Ampere equation*, Acta Math. 180 (1998), no. 1, 69-117.
[35] Li, C. *On sharp rates and analytic compactifications of asymptotically conical Kähler metrics*, arxiv.1405.2433
[36] Li, P. Tam, L.F. *Green's functions, harmonic functions, and volume comparison*, J. Differential Geom. 41 (1995), no. 2, 277-318.
[37] Li, P., Yau, S.-T. *On the parabolic kernel of the Schrodinger operator*, Acta Math. 156 (1986), no. 3-4, 153-201.
[38] Lockhart, R., McOwen, R. *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12 (1985), no. 3, 409 - 447.
[39] Marshall, S. *Deformations of special Lagrangian submanifolds*, Ph.D. thesis, University of Oxford, 2002.
[40] Martelli, D., Sparks, J., Yau, S.-T. *Sasaki-Einstein manifolds and volume minimization*, Comm. Math. Phys. **280** (2008), no. 3, 611–673.
[41] Matsuki, K. *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.
[42] Matsuki, K. *Weyl groups and birational transformations among minimal models*, Mem. Amer. Math. Soc. **116** (1995), no. 557
[43] Matsushima, Y. *Sur la structure du groupe d'homéomorphismes analytiques d'ube certain variíété Kählerienne*, Nagoya Math. J. **11** (1957), 145–150.
[44] Mok, N. Siu, Y.-T., Yau, S.-T. *The Poincaré-Lelong equation on complete Kähler manifolds*, Compositio Math. 44 (1981), no. 1-3, 183-218.
[45] Phong, D. H., Sesum, N., Sturm, J. *Multiplier ideal sheaves and the Kähler-Ricci flow*, Comm. Anal. Geom. 15 (2007), no. 3, 613-632.
[46] Reider, I. *Vector bundles of rank* 2 *and linear systems on algebraic surfaces*, Ann. of Math. (2) **127** (1988), no. 2, 309–316.
[47] Rong, X., Zhang, Y. *Continuity of extremal transitions and flops for Calabi-Yau manifolds. Appendix B by Mark Gross*, J. Differential Geom. 89 (2011), no. 2, 233-269.
[48] Rong, X., Zhang, Y. *Degenerations of Ricci-flat Calabi-Yau manifolds*, Commun. Contemp. Math. 15 (2013), no. 4, 1250057, 8 pp.
[49] Schoen, R. *Analytic aspects of the harmonic map problem*, Math. Sci. Res. Inst. Publ. 2 (1984) 321-358.
[50] Sherman, M. ,Weinkove, B. *Local Calabi and curvature estimates for the Chern-Ricci flow*, New York J. Math. 19 (2013), 565-582.
[51] Song, J. *Riemannian geometry of Kähler-Einstein currents*, arXiv.1404.0445
[52] Tian, G. *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), no. 1, 101–172.
[53] Tian, G., Yau, S.-T. *Complete Kähler manifolds with zero Ricci curvature*, J. Amer. Math. Soc. 3 (1990), no. 3, 579-609.
[54] Tian, G., Yau, S.-T. *Complete Kähler manifolds with zero Ricci curvature II*, Invent. Math. 106 (1991), no. 1, 27-60.
[55] Tian, G., Yau, S.-T. *Kähler-Einstein metrics on complex surfaces with* $C_1 > 0$, Comm. Math. Phys. **112** (1987), no. 1, 175–203.
[56] Tosatti, V. *Calabi-Yau manifolds and their degenerations*, Ann. N.Y. Acad. Sci. **1260** (2012), 8–13
[57] Tosatti, V. *Degenerations of Calabi-Yau metrics*, in *Geometry and Physics in Cracow*, Acta Phys. Polon. B Proc. Suppl. **4** (2011), no. 3, 495–505.
[58] Tosatti, V. *Collapsing Calabi-Yau manifolds*, arXiv:2003.00673
[59] Tosatti, V. *Limits of Calabi-Yau metrics when the Kähler class degenerates*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 4, 755-776.
[60] Tsuji, H. *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), no. 1, 123-133.
[61] van Coevering, C. *Ricci-flat Kähler metrics on crepant resolutions of Kähler cones*, Math. Ann. 347 (2010), no. 3, 581-611.

[62] van Coevering, C. *A Construction of Complete Ricci-flat Kähler Manifolds*, arXiv:0803.0112
[63] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation. I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.
[64] Yau, S.-T. *Survey on partial differential equations in differential geometry*, Seminar on differential geometry, p. 3–71, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

Email address: tristanc@mit.edu

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139

Email address: bguo@rutgers.edu

Department of Mathematics and Computer Science, Rutgers University– Newark, 101 Warren Street, Newark, NJ 07102

Email address: tong@math.columbia.edu

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027