K-stability of Log Fano Cone Singularities

by

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Abstract

In this thesis, we define the δ-invariant for log Fano cone singularities, and show that the necessary and sufficient condition for K-semistability is $\delta \geq 1$. This generalizes the result of [Li17] and [Fuj19]. We also prove that on any log Fano cone singularity of dimension n whose δ-invariant is less than $\frac{n+1}{n}$, any valuation computing δ has a finitely generated associated graded ring. This shows a log Fano cone is K-polystable if and only if it is uniformly K-stable. Together with earlier works, this implies the Yau-Tian-Donaldson Conjecture for Fano cone.

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Chapter 1

Introduction

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers. The concept of K-stability was first introduced by Tian and later formulated algebraically by Donaldson, as a criterion to characterise the existence of Kähler-Einstein metrics on Fano manifolds. It was defined by looking at the generalized Futaki invariant of all possible normal $\mathbb{C}^*$-degenerations (called test configurations) of a Fano manifold X. Later, Fujita [Fuj19], Chi Li [Li17] and Blum-Jonsson [BJ17] developed a valuative criterion of K-(semi)stability, namely the δ-invariant. Liu-Xu-Zhuang [LXZ21] proved the higher rank finite generation Conjecture. Together with [BBJ18] and [LTW19], it implies the Yau-Tian-Donaldson Conjecture for general Fano variety.

Log Fano cone singularity A Riemannian manifold is called Sasakian if its Riemannian cone is Kähler. If, in addition, the cone is Ricci-flat, the manifold is called Sasakian-Einstein. Collins and Székelyhidi [CS19] introduced the K-(semi)stability of log Fano cone singularities to characterise the existence of Sasakian-Einstein metric on Fano cones. Later Li-Xu [LX18] gave a purely algebro-geometric definition.

Given a normal affine variety X and a torus $T = (\mathbb{C}^*)^r$ acting on X. We say the action is *good* if it is effective, and there is a unique closed point $x \in X$ lies in the orbit closure of any T- orbit.

We shall call x to be the vertex point of X.

Let $N = \mathrm{Hom}(\mathbb{C}^*, T)$ be the co-weight lattice and $M = N^*$ the weight lattice. We have a weight decomposition $R = \oplus_{\alpha\in\Lambda} R_\alpha$ where $\Lambda = \{\alpha \in M \mid R_\alpha \neq 0\}$. We use $\sigma^\vee \subset M_{\mathbb{Q}}$ to denote the cone generated by Λ over $\mathbb{Q}$. The dual of $\sigma^\vee$ is the *Reeb cone*

$$\mathfrak{t}_{\mathbb{R}}^+ = \{\xi \in N_{\mathbb{R}} \mid \langle \alpha, \xi\rangle > 0 \text{ for any } 0 \neq \alpha \in \Lambda\}.$$

Definition 1.0.1. Let (X, D) be an affine klt pair with a good T-action. For a fixed $\xi_0 \in \mathfrak{t}_{\mathbb{R}}^+$, we call the triple (X, D, ξ_0) a klt singularity with a log Fano cone structure that is polarized by ξ_0.

Following the $\mathbb{Q}$-Fano case, we can also define the notion of (special) test configurations of log Fano cone singularities similarly. Given a test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$, the Futaki invariant is defined as $\mathrm{Fut}\,(\mathcal{X}, \mathcal{D}, \xi_0; \eta) := (D_{-\eta}\widehat{\mathrm{vol}})(\xi_0)$, here $\widehat{\mathrm{vol}}(v)$ is the *normalized volume* of a valuation v (see chapter 2 for more details). Then we can define the K-(semi)stability of log Fano cone singularity similarly.

If ξ_0 is rational, i.e. ξ_0 generates a one dimensional torus, then quotient by $T = \langle\xi_0\rangle$, we get a special test configuration $(\mathcal{Y}, \mathcal{E})$ of the log Fano pair $(Y, E) = ((X, D) - \{x\})/\langle\xi_0\rangle$, and its Futaki invariant is just a rescaling of the Futaki invariant of $(\mathcal{Y}, \mathcal{E})$. Hence the definition here is a generalization of K-stability of $\mathbb{Q}$-Fano variety (i.e. the rank 1 case).

Definition 1.0.2. Let (X, D, ξ_0) be a log Fano cone singularity, the delta invariant (also called stability threshold) is defined as

$$\delta(X, D, \xi_0) = \inf_{v\in \mathrm{Val}_{X,x}^T} \frac{A_{(X,D)}(v)}{S_{(X,D)}(v)}$$

where $\mathrm{Val}_{X,x}^T$ is the set of all T-equivariant valuations centered at x with finite log discrepancy,

$A_{(X,D)}(v)$ is the log discrepancy of v,

$$S_{(X,D)}(v) = \frac{A_{(X,D)}(\mathrm{wt}_{\xi_0})}{\mathrm{vol}(\xi_0)} \int_0^\infty \mathrm{vol}(\mathcal{F}_v R^{(t)}) dt$$

We also define the beta invariant for every valuation $v \in \mathrm{Val}^T_{X,x}$,

$$\beta(v) := \beta_{(X,D,\xi_0)}(v) := A_{(X,D)}(v) - S_{(X,D)}(v)$$

Here $\mathcal{F}_v$ is a filtration on R induced by a T-equivariant valuation v, and $\mathrm{vol}(\mathcal{F}_v R^{(t)})$ comes naturally from the calculation of the volume. See Definition 2.4.3 and Proposition 3.1.3 for details.

Theorem 1.0.3. *Let (X, D, ξ_0) be a log Fano cone singularity, then it is K-semistable if and only if $\delta(X, D, \xi_0) \geq 1$, or equivalently $\beta(v) \geq 0$ for all $v \in \mathrm{Val}^T_{X,x}$.*

When ξ_0 is rational, and $(Y, E) = ((X, D) - \{x\})/\langle \xi_0 \rangle$, the delta invariant we defined here is the same as the delta invariant defined in [Fuj19] and [BJ17]. Hence this is a generalization of the result in the log Fano case.

The idea is to consider two series of valuations on the (speical) test configuration $(\mathcal{X}, \mathcal{D}, \xi_0, \eta)$ of the log Fano cone (X, D, ξ_0). A series of valuations $\mathrm{wt}_{\xi_\epsilon}$ on the central fiber $\mathcal{X}_0$, and a series of valuations w_ϵ on the general fiber which is isomorphic to X. They have the same normalized volume $\widehat{\mathrm{vol}}(\mathrm{wt}_{\xi_\epsilon}) = \widehat{\mathrm{vol}}(w_\epsilon)$.

On the central fiber $\mathcal{X}_0$, we have $\frac{d}{d\epsilon}|_{\epsilon=0}\widehat{\mathrm{vol}}(\mathrm{wt}_{\xi_\epsilon}) = C_1 \cdot \mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0)$. On X we have $\frac{d}{d\epsilon}|_{\epsilon=0}\widehat{\mathrm{vol}}(w_\epsilon) = C_2 \cdot \beta(E)$, where C_1, C_2 are positive constants. The S function in Definition 1.0.2 comes from computing $\frac{d}{dt}|_{\epsilon=0}\widehat{\mathrm{vol}}(w_\epsilon)$. This explains why the delta invariant (or equivalently the beta invariant) could be used as a criterion for K-(semi)stability.

The Delta invariant via filtrations We present another approach to define the delta invariant in Chapter 4. That is to use the Okounkov body. Given a valuation $v \in \mathrm{Val}^T_{X,x}$, it induces a filtration on R by $\mathcal{F}^\lambda R_\alpha = \{f \in R_\alpha \mid v(f) \geq \lambda\}$.

For any *linearly bounded* filtration $\mathcal{F}$ on R (see chapter 2.4), we can define the functions S_m and δ_m by looking at the *jumping numbers* of the filtration. More precisely, the jumping numbers are

$$0 \leq a_{\alpha,1} \leq \cdots \leq a_{\alpha,N_\alpha}$$

defined for $\alpha \in \Lambda$ by

$$a_{\alpha,j} = a_{\alpha,j}(\mathcal{F}) = \inf\{\lambda \in \mathbb{R}_+ \mid \operatorname{codim} \mathcal{F}^\lambda R_\alpha \geq j\}$$

for $1 \leq j \leq N_\alpha$, where $N_\alpha = \dim_{\mathbb{C}} R_\alpha$. We define the rescaled sum of the jumping numbers:

$$S_\alpha(\mathcal{F}) := \frac{1}{\langle \alpha, \xi_0 \rangle N_\alpha} \sum_{j=1}^{N_\alpha} a_{\alpha,j}$$

We say an effective divisor B is an α*-basis type divisor*, if there exists a basis $s_1, \ldots, s_{N_\alpha}$ of R_α, such that

$$B = \frac{\sum_{i=1}^{N_\alpha} \{s_i = 0\}}{\langle \alpha, \xi_0 \rangle N_\alpha}.$$

Then we define

$$\delta_\alpha = \inf\{\operatorname{lct}(X, D; B) \mid B \text{ is an } \alpha\text{-basis type divisor } \}$$

for any $\alpha \in \Lambda$, here $lct(X, D; B)$ is the log-canonical threshold, see [CS08].

For any integer m, we define $R_m := \bigoplus_{m-1 < \langle \alpha, \xi_0 \rangle \leq m} R_\alpha$, so $R = \bigoplus_{m=0}^{+\infty} R_m$. Write $N_\alpha := \dim_{\mathbb{C}} R_\alpha$, and $N_m := \dim_{\mathbb{C}} R_m$

Notice that our definition of R_m is different from the definition in [Wu21]. If ξ_0 is rational and that (X, D) is a cone over (Y, E), then up to rescaling of ξ_0, R_m defined above equals to $H^0(Y, m(-K_Y - E))$. This matches with the definition of R_m in [BJ17] and [LX20].

For R_m we define the jumping numbers, $S_m(\mathcal{F})$, m-basis type divisor and δ_m similarly.

$$S_m(\mathcal{F}) := \frac{1}{mN_m}\sum_{j=1}^{N_m} a_{m,j}$$

for $\alpha \in \Lambda, m \in \mathbb{N}$, and

$$\delta_m = \inf\{\operatorname{lct}(X, D; B) \mid B \text{ is an m-basis type divisor }\}.$$

Finally we have

Theorem 1.0.4. *The limit* $\lim_{m\to\infty}\delta_m$ *exists and equals to* $\delta(X, D, \xi_0)$ *we defined in Definition 1.0.2. Furthermore,*

$$\delta(X, D, \xi_0) = \inf_v \frac{A_{X,D}(v)}{S_{X,D}(v)} = \inf_E \frac{A(\operatorname{ord}_E)}{S(\operatorname{ord}_E)}.$$

where E *runs through all the* T*-invariant prime divisors over* X*.*

Higher Rank Finite Generation Conjecture

Definition 1.0.5. A filtration $\mathcal{F}$ on R gives the associated graded ring $\operatorname{gr}_{\mathcal{F}} R := \oplus_{\alpha\in\Gamma}\oplus_{\lambda\in\mathbb{R}_{\geq 0}} \operatorname{gr}^{\lambda}_{\mathcal{F}} R_\alpha$, where $\operatorname{gr}^{\lambda}_{\mathcal{F}} R_\alpha = \mathcal{F}^{\lambda}R_\alpha / \cup_{\lambda'>\lambda}\mathcal{F}^{\lambda'}R_\alpha$. We use $\mathcal{F}_v$ to denote the filtration induced by a valuation v.

Theorem 1.0.6 (Higher Rank Finite Generation Conjecture)**.** *Let* (X, D, ξ_0) *be a log Fano cone singularity of dimension* n*,* $X = \operatorname{Spec}(R)$*. Assume that* $\delta(X, D, \xi_0) < \frac{n+1}{n}$*. Then for any valuation* v *that computes* $\delta(X, D, \xi_0)$*, the associated graded ring* $\operatorname{gr}_{\mathcal{F}_v} R$ *is finitely generated.*

We follow the idea in [LXZ21]. The key observation is that any valuation v computing $\delta(X, D, \xi_0) < \frac{n+1}{n}$, is an lc place of a $\mathbb{Q}$-complement Γ, and that complement satisfies some further technical conditions (see *special complement* in Definition 6.1.1). Moreover, any divisorial lc place

w of the complement induces a weakly special degeneration. If finite generation holds for a quasi-monomial valuation v, then for any valuation w that lies in the minimal rational affine subspace of the dual complex $\mathcal{DMR}(X, D+\Gamma)$ containing v that are close enough to v, the central fibers of the induced degenerations would be isomorphic to each other, so they are bounded. A key fact is that the inverse is also true.

So we need to show that, given an *monomial lc place* of a special complement, there exists a neighborhood of v in the rational affine subspace of the dual complex, such that the degenerations corresponding to the rational points have bounded central fibers. This is done by giving a positive lower bound of the alpha-invariant.

Notice that this statement does not depend on ξ_0, so we may assume ξ_0 is rational and take the quotient by $\langle \xi_0 \rangle$. Therefore we need to generalize the estimate of the alpha-invariant in [LXZ21] to the toroidal case.

There are several remarkable corollaries of the finite generation result.

Theorem 1.0.7 (Optimal Destabilization Conjecture). *Let (X, D, ξ_0) be a log Fano cone singularity of dimension n. Assume that $\delta(X, D, \xi_0) < \frac{n+1}{n}$. Then $\delta(X, D, \xi_0) \in \mathbb{Q}$ and there exists a divisorial valuation ord_E over X that computes $\delta(X, D, \xi_0)$.*

Theorem 1.0.8 (Yau-Tian-Donaldson Conjecture). *A log Fano cone singularity (X, D, ξ_0) is K-polystable if and only if it is uniformly K-stable. Furthermore, A log Fano cone singularity (X, D, ξ_0) admits a weak Ricci-flat Kähler potential if and only if it is K-polystable.*

Chi Li [Li21] gives a different approach to prove the Yau-Tian-Donaldson Conjecture for Fano cone, using the correspondence with g-weighted stability.

Chapter 2

Preliminaries

2.1 Log Fano Cone Singularity

Assume $X = \mathrm{Spec}_{\mathbb{C}}(R)$ is an affine variety with $\mathbb{Q}$-Gorenstein klt singularities. Denote by T the complex torus $(\mathbb{C}^*)^r$. Assume X admits a good T- action in the following sense.

Definition 2.1.1. Let $X = \mathrm{Spec}(R)$ be a normal affine variety. We say that a T- action is *good* if it is effective and there exists a unique closed point $x \in X$ lies in the closure of any orbit. We call x the vertex of X.

Let $N = \mathrm{Hom}(\mathbb{C}^*, T)$ be the co-weight lattice and $M = N^*$ be the weight lattice. We have the weight decomposition

$$R = \bigoplus_{\alpha \in \Lambda} R_\alpha \text{ where } \Lambda = \{\alpha \in M \mid R_\alpha \neq 0\}$$

The action being good implies $R_0 = \mathbb{C}$, which will always be assumed in the below. An ideal $\mathfrak{a}$ is called homogeneous if $\mathfrak{a} = \oplus_{\alpha\in\Lambda}\mathfrak{a} \cap R_\alpha$. Denote by $\sigma^\vee \subset M_{\mathbb{Q}}$ the cone generated by Λ over $\mathbb{Q}$, which is called the *weight cone* (or the *moment cone*), is the same as the following set

$$\mathfrak{t}_{\mathbb{R}}^{+} = \{\xi \in N_{\mathbb{R}} \mid \langle \alpha, \xi \rangle > 0 \text{ for any } 0 \neq \alpha \in \Lambda\}$$

Definition 2.1.2 (Reeb cone)**.** The above set $\mathfrak{t}_{\mathbb{R}}^{+}$ is called Reeb cone. A vector $\xi \in \mathfrak{t}_{\mathbb{R}}^{+}$ is called a Reeb vector. We define $\mathrm{rank}(\xi)$ to be the dimension of the subtorus T_ξ generated by ξ.

We recall the following structure results for any T-varieties.

Theorem 2.1.3 ([AH06])**.** *Let $X = \mathrm{Spec}(R)$ be a normal affine variety and suppose $T = \mathrm{Spec}(\mathbb{C}([M]))$ acts effectively on X with weight cone $\sigma^\vee \subseteq M_{\mathbb{Q}}$. Then there exists a normal semiprojective variety Y such that $\pi : X \to Y$ is the good quotient under T-action and a polyhedral divisor $\mathfrak{D}$ such that there is an isomorphism of graded algebras:*

$$R \cong H^0(X, \mathcal{O}_X) \cong \bigoplus_{u \in \sigma^\vee \cap M} H^0(Y, \mathcal{O}(\mathfrak{D}(u))) =: R(Y, \mathfrak{D}).$$

In fact, X is equal to $\mathrm{Spec}_{\mathbb{C}}(\oplus_{u\in\sigma^\vee\cap M} H^0(Y, \mathcal{O}(\mathfrak{D}(u))))$.

Theorem 2.1.4 ([LX18])**.** *Assume a T-variety X is determined by the data $(Y, \sigma, \mathfrak{D})$ such that Y is projective, σ is a maximal dimension one cone in $N_{\mathbb{R}}$ and $\mathfrak{D}$ is a polyhedral divisor.*

1. For any T-invariant quasi-monomial valuation v, there exists a quasi-monomial valuation $v^{(0)}$ over Y and $\xi \in M_{\mathbb{R}}$ such that for any $f \cdot \chi^u \in R_u$, we have :

$$v(f \cdot \chi^u) = v^{(0)}(f) + \langle u, \xi \rangle.$$

2. T-invariant divisors on X are either vertical or horizontal. Any horizontal divisor is determined by a divisor Z on Y and a vertex v of $\mathfrak{D}_Z$, and will be denoted by $D_{(Z,v)}$. Any vertical divisor is determined by a ray ρ of σ and will be denoted by E_ρ.

3. Let D be a T-invariant vertical effective $\mathbb{Q}$-divisor. If $K_X + D$ is $\mathbb{Q}$-Cartier, then the log canonical divisor has a representation $K_X + D = \pi^* H + div(\chi^{-u_0})$ where $H = \sum_Z a_Z \cdot Z$ is a principal $\mathbb{Q}$-divisor on Y and $u_0 \in M_{\mathbb{Q}}$. Moreover, the log discrepancy of the horizontal divisor E_ρ is given by:

$$A_{(X,D)}(E_\rho) = \langle u_0, n_\rho \rangle,$$

where n_ρ is the primitive vector along the ray ρ.

Using the above structure theorem, we have the following (see [LX18][Lemma 2.16, Lemma 2.18])

Proposition 2.1.5. *Any Reeb vector ξ gives a quasi-monomial valuation on X*

$$\mathrm{wt}_\xi : f \mapsto \min_{\alpha \in \Lambda} \{ \langle \alpha, \xi \rangle \mid f = \sum_\alpha f_\alpha, f_\alpha \neq 0 \}$$

. The rational rank of wt_ξ is $\mathrm{rank}(\xi)$, the center of wt_ξ is x, and the log discrepancy of wt_ξ is given by $A_{(X,D)}(\mathrm{wt}_\xi) = \langle u_0, \xi \rangle$.

Definition 2.1.6. Using the above notation, for any $\eta \in \mathfrak{t}_{\mathbb{R}}$, we define:

$$A_{(X,D)}(\eta) = \langle u_0, \eta \rangle.$$

Definition 2.1.7 (log Fano cone singularity)**.** Let (X, D) be an affine klt pair with a good torus action, where D is a T-invariant vertical divisor. For a fixed $\xi_0 \in \mathfrak{t}^+_{\mathbb{R}}$, we call the triple (X, D, ξ_0) a klt singularity with a log Fano cone structure that is polarized by ξ_0. We denote T to be the torus generated by ξ_0.

2.2 Valuations and normalized volume

Let X be a normal variety. A *real valuation* of its function field $K(X)$ is a nonconstant valuation map $v : K(X)^\times \to \mathbb{R}$ which is trivial on $\mathbb{C}$.

We say a valuation is centered at a scheme-theoretic point $\xi = c_X(v)$ if $v \geq 0$ on $\mathcal{O}_{X,\xi}$ and $v > 0$ on the maximal ideal $\mathfrak{m}_{X,\xi}$. Let $\mathrm{Val}_{X,x}$ denote all the valuations centered at the closed point $x \in X$. If we have a torus T acting on X, we use Val_X^T to denote all valuations $v \in \mathrm{Val}_X$ that are T-equivariant. For the purpose of this paper, we only care about the valuations that are T-equivariant.

Definition 2.2.1. If $Y \to X$ is a proper birational morphism, with Y normal, and $E \subset Y$ is a prime divisor (called a *prime divisor over* X), then E defines a valuation $\mathrm{ord}_E : \mathbb{C}(X)^* \to \mathbb{Z}$ in Val_X given by order of vanishing at the generic point of E. Any valuation of the form $v = c\,\mathrm{ord}_E$ with $c \in \mathbb{R}_{>0}$ will be called *divisorial*.

Definition 2.2.2 (quasi-monomial valuation)**.** Let $\pi : Y \to X$ be a birational morphism where Y is normal. Let $\eta \in Y$ be a scheme-theoretic point such that Y is regular at η. For a regular system of parameters $(y_1, \dots, y_r)$ of $\mathcal{O}_{Y,\eta}$ and $\alpha \in \mathbb{R}^r_{\geq 0}$, we define a valuation v_α as follows. For $f \in \mathcal{O}_{Y,\eta} - \{0\}$, we may write f in $\widehat{\mathcal{O}_{Y,\eta}} \cong \kappa(\eta)[\![y_1, \dots, y_r]\!]$ as $f = \sum_{\beta \in \mathbb{Z}^r_{\geq 0}} c_\beta y^\beta$, where $c_\beta \in \kappa(\eta)$ and $y^\beta = y_1^{\beta_1} \dots y_r^{\beta_r}$ with $\beta = (\beta_1, \dots, \beta_r)$. We set

$$v_\alpha(f) := \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}.$$

A valuation is called *quasi-monomial* if $v = v_\alpha$ for some $\pi : Y \to X, \eta, (y_1, \dots, y_r)$ and α. It is proven in [ELS03] that a valuation is quasi-monomial if and only if it is an Abhyankar valuation, i.e. v satisfies $\mathrm{trdeg}(v) + \mathrm{rat.rk}(v) = \dim X$ where $\mathrm{trdeg}(v)$ is the transcendental degree of v. From the above defition, we have that for ant $f \in \mathcal{O}_{Y,\eta} - \{0\}$, the function $\alpha \mapsto v_\alpha(f)$ is piecewise rational linear and is concave, i.e.

$$v_{t\alpha_1+(1-t)\alpha_2}(f) \geq t \cdot v_{\alpha_1}(f) + (1-t) \cdot v_{\alpha_2}(f)$$

for any $0 \leq t \leq 1$ and any $\alpha_1, \alpha_2 \in \mathbb{R}^r_{\geq 0}$. In particular, for any non-trivial effective $\mathbb{Q}$- Cartier divisor D (resp. graded sequence $\mathfrak{a}_\bullet$ of ideals) on X, the function $\alpha \mapsto v_\alpha(D)$ (resp. $\alpha \mapsto v_\alpha(\mathfrak{a}_\bullet)$) is piecewise rational linear and concave. If, in addition, $\pi : (Y, E = \sum_{i=1}^{l})E_i) \to X$ is a log smooth model where $(y_i = 0) = E_i$ for $1 \leq i \leq r$ as an irreducible component of E, then we denote the set $\{v_\alpha \mid \alpha \in \mathbb{R}^r_{\geq 0}\}$ by $\mathrm{QM}_\eta(Y, E)$. We also set $\mathrm{QM}(Y, E) := \cup_\eta \mathrm{QM}_\eta(Y, E)$ where η runs through all generic points of $\cap_{i \in I} E_i$ for some non-empty subset $I \subseteq \{1, \dots, ll\}$. Notice that if v is a quasi-monomial valuation and q is its rational rank, then the log resolution $\pi : Y \to X$ can be chosen (by passing to a further blowup) such that $v \in \mathrm{QM}_\eta(Y, E)$ for some codimension q point η.

Given a valuation $v \in \mathrm{Val}_{X,x}$ and any integer m, we define the *associated valuation ideal* $\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\}$.

Definition 2.2.3. Let X be an n-dimensional normal variety. Let $x \in X$ be a closed point. We define the volume of a valuation $v \in \mathrm{Val}_{X,x}$ as

$$\mathrm{vol}_{X.x}(v) = \limsup_{m \to \infty} \frac{l(\mathcal{O}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!}$$

Definition 2.2.4. Let (X, Δ) be a klt log pair. Consider a proper birational morphism from a normal variety $\mu : Y \to X$, and a prime divisor $E \subset Y$. We define the log discrepancy function of valuations $A_{(X,\Delta)}(\mathrm{ord}_E)$ to be:

$$A_{(X,\Delta)}(\mathrm{ord}_E) := 1 + \mathrm{ord}_E(K_Y - \mu^*(K_X + \Delta))$$

The log discrepancy function can be naturally extended to a lower semicontinuous function

$A_{X,\Delta}: \mathrm{Val}_X \to (0, +\infty]$ extending $A_{(X,\Delta)}(\mathrm{ord}_E)$ that is homogeneous of order 1. See [BdFFU15] for details.

We use v_{triv} to denote the trivial valuation, and set

$$\mathrm{Val}_X^\circ := (\mathrm{Val}_X^\circ)^T = \{v \in \mathrm{Val}_X^T \mid A_{X,D}(v) < +\infty \text{ and } v \neq v_{triv}\}.$$

If (X, D) is lc, then $v \in \mathrm{Val}_X$ is an *lc place* of (X, D) if $A_{X,D}(v) = 0$. If (Y, E) is a log smooth model over an lc pair (X, D) satisfying $\mathrm{Supp}(Ex(\pi) + \pi_*^{-1}D) \subseteq E$, then we know that the set of all lc places of (X, D) coincides with $\mathrm{QM}(Y, E')$ where E' is the sum of irreducible components E_i of E satisfying $A_{X,D}(E_i) = 0$. In particular, anuy lc place of (X, D) is a quasi-monomial valuation in $\mathrm{QM}(Y, E)$.

Definition 2.2.5 ([Li18])**.** Let (X, Δ) be an n-dimensional klt log pair. Let $x \in X$ be a closed point. The *normalized volume function of valuations* $\widehat{\mathrm{vol}}_{(X,\Delta),x} : \mathrm{Val}_{X,x} \to (0, +\infty]$ is defined as

$$\widehat{\mathrm{vol}}_{(X,\Delta),x}(v) = \begin{cases} A_{(X,\Delta)}(v)^n \operatorname{vol}_{X,x}(v), & \text{if} A_{(X,\Delta)}(v) < +\infty \\ +\infty, & \text{if} A_{(X,\Delta)}(v) = +\infty \end{cases}$$

Let V be a $\mathbb{Q}$-Fano variety and $X = C(V, -K_V)$ is the affine cone with vertex o. Consider V as the exceptional divisor of the blow up $Bl_oX \to X$, we have the canonical divisorial valuation ord_V on X.

Theorem 2.2.6 ([Li17])**.** *$(V, -K_V)$ is K-semistable if and only if $\widehat{\mathrm{vol}}$ is $\mathbb{C}^*$-equivariantly minimized at ord_V over (X, o).*

2.3 K-semistability of log Fano cone singularity

Following the log Fano case, we can also define the notion of (special) test configuration, Futaki invariant and K-stability for log Fano cone singularities.

Definition 2.3.1. Let (X, D, ξ_0) be a log Fano cone singularity and T be the torus generated by ξ_0. A T-equivariant special test configuration of (X, D, ξ_0) is a quadruple $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with a map $\pi : ((\mathcal{X}, \mathcal{D}) \to \mathbb{A}^1(= \mathbb{C})$ satisfying the following conditions:

(1) π is a flat family of log pairs such that the fibers away from 0 are isomorphic to (X, D) and $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$ is affine, where $\mathcal{R}$ is a finitely generated flat $\mathbb{C}[t]$ algebra. The torus T acts on $\mathcal{X}$, and we write $\mathcal{R} = \oplus_\alpha \mathcal{R}_\alpha$ as the decomposition into weight spaces;

(2) η is an algebraic holomorphic vector field on $\mathcal{X}$ generating a $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{D})$ such that π is $\mathbb{C}^*$-equivariant where $\mathbb{C}^*$ acts on the base $\mathbb{C}$ by multiplication (so that $\pi_*\eta = t\partial_t$ if t is the affine coordinate on $\mathbb{A}^1$) and there is a $\mathbb{C}^*$-equivariant isomorphism $\phi : ((\mathcal{X}, \mathcal{D}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, D) \times \mathbb{C}^*$;

(3) the algebraic holomorphic vector field ξ_0 on $\mathcal{X} \times_{\mathbb{C}} \mathbb{C}^*$ (via the isomorphism ϕ) extends to a holomorphic vector field on $\mathcal{X}$ (still denote by ξ_0) and generates a T-action on $((\mathcal{X}, \mathcal{D})$ that commutes with the $\mathbb{C}^*$- action generated by η and preserves (X_0, D_0);

(4) (X_0, D_0) has klt singularities and $(X_0, D_0, \xi_0|_{X_0})$ is a log Fano cone singularity.

$((\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is a product test configuration if there is a T-equivariant isomorphism $((\mathcal{X}, \mathcal{D}) \cong (X, D) \times \mathbb{C}$ and $\eta = \eta_0 + t\partial_t$ with $\eta_0 \in \mathfrak{t}$.

By abuse of notation, we still denote $\xi_0|_{X_0}$ by ξ_0. For simplicity, we still just say that $((\mathcal{X}, \mathcal{D})$ is a special test configuration if ξ_0, η are clear. We also say $((\mathcal{X}, \mathcal{D}, \xi_0, \eta$ specially degenerates to $(X_0, D_0, \xi_0; \eta)$ (or simply (X_0, D_0)).

Since T-action and $\mathbb{C}^*$-action commute with each other, X_0 has a $T' = (T\times\mathbb{C}^*)$-action generated by $\{\xi_0, \eta\}$. Let $\mathfrak{t}' = \mathrm{Lie}(T')$. For any $\xi \in \mathfrak{t}'^{+}_{\mathbb{R}}$, we have $\mathrm{wt}_\xi \in \mathrm{Val}_{X_0,o'}$ where $o' \in X_0$ is the vertex point of the central fiber X_0. So we can define its volume $\mathrm{vol}(\mathrm{wt}_\xi)$ and normalized volume $\widehat{\mathrm{vol}}(\mathrm{wt}_\xi)$. For simplicity of notations, we will frequently just write ξ in place of wt_ξ.

Remark 2.3.2. The volume $\mathrm{vol}(\xi)$ is given by

$$\mathrm{vol}(\xi) := \mathrm{vol}_X(\mathrm{wt}_\xi) = \lim_{m\to+\infty} \frac{\dim_{\mathbb{C}} R/\mathfrak{a}_m(\mathrm{wt}_\xi)}{m^n/n!}.$$

In [CS19] the volume can be viewed via the index character. Let $X_0 = \mathrm{Spec}(B)$ and $B = \oplus_{\alpha'} B_{\alpha'}$ be the weight decomposition with respect to T'. For any $\xi \in \mathfrak{t}'^{+}_{\mathbb{R}}$, the index character is defined as

$$\Phi(t,\xi) = \sum_{\alpha'} e^{-t\langle \alpha',\xi\rangle} \dim B_{\alpha'}.$$

Then $\Phi(t,\xi)$ has the expansion:

$$\Phi(t,\xi) = \frac{\mathrm{vol}(\xi)}{t^{n+1}} + O(t^{-n}).$$

Definition 2.3.3 ([CS19])**.** Let (X_0, D_0, ξ_0) be a log Fano cone singularity with a good action by $T' \cong (\mathbb{C}^*)^{r+1}$. Denote $\mathrm{vol} = \mathrm{vol}_{(X_0,D_0)}$ on $\mathfrak{t}'^{+}_{\mathbb{R}}$ and $A = A_{(X_0,D_0)}$ on $\mathfrak{t}'_{\mathbb{R}}$. Assume $\xi_0 \in \mathfrak{t}'^{+}_{\mathbb{R}}$. For any $\eta \in \mathfrak{t}'_{\mathbb{R}}$, we define the generalized Futaki invariant to be:

$$\mathrm{Fut}(X_0, D_0, \xi_0; \eta) := (D_{-\eta}\widehat{\mathrm{vol}})(\xi_0) = nA(\xi_0)^{n-1}A(-\eta)\,\mathrm{vol}(\xi_0) + A(\xi)^n \cdot (D_{-\eta}\,\mathrm{vol})(\xi_0).$$

If $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is a special test configuration of (X, D, ξ_0), then the Futaki invariant of $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$, denoted by $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is defined to be $\mathrm{Fut}(X_0, D_0, \xi_0; \eta)$.

Remark 2.3.4. When ξ_0 is rational, i.e. ξ_0 generates a one dimensional torus $T \cong \mathbb{C}^*$, then quotient by T we get a log Fano pair (Y, E). In this case (X, D) is indeed a cone over a log Fano pair. The special test configuration of (X, D, ξ_0) becomes a special test configuration of (Y, E). The Futaki invariant defined in 2.3.3 is a rescaling of $\mathrm{Fut}(Y, E)$ (see [Li17, Lemma 6.20]). This also verifies the definition coincides with the one in [CS19] as any vector could be approximated by rational ones and the Futaki invariant in both definitions are continuous and coincide when ξ_0

is rational.

Definition 2.3.5. Let $(X, D, \xi_0; \eta)$ be a log Fano cone singularity. We say it is K-semistable if for any T-invariant special test configuration $\mathcal{X}$ that degenerates (X, D, ξ_0) to $(X_0, D_0, \xi_0; \eta)$, we have

$$\mathrm{Fut}(X_0, D_0, \xi_0; \eta) \geq 0.$$

We will need the following result later.

Theorem 2.3.6. *[LX18] (X, D, ξ_0) is K-semistable if and only if wt_{ξ_0} is a minimizer of $\widehat{\mathrm{vol}}_{(X,D)}$ in Val_X°.*

2.4 Filtrations

Definition 2.4.1. A filtration $\mathcal{F}$ on $R = \oplus_\alpha R_\alpha$ is a family $\mathcal{F}^\lambda R_\alpha \subseteq R_\alpha$ of $\mathbb{C}$-vector spaces of R_α for $\alpha \in \Lambda$ and $\lambda \in \mathbb{R}^+$, satisfying:

(1) $\mathcal{F}$ is decreasing: $\mathcal{F}^\lambda R_\alpha \subseteq \mathcal{F}^{\lambda'} R_\alpha$ if $\lambda \geq \lambda'$;

(2) $\mathcal{F}$ is left continuous: $\mathcal{F}^\lambda R_\alpha = \cap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_\alpha$ for $\lambda > 0$;

(3) $\mathcal{F}$ is multiplicative: $\mathcal{F}^\lambda R_\alpha \cdot \mathcal{F}^{\lambda'} R_{\alpha'} \subseteq \mathcal{F}^{\lambda+\lambda'} R_{\alpha+\alpha'}$;

(4) $\mathcal{F}$ is $T-$invariant: $\mathcal{F}^\lambda R = \oplus_{\alpha \in \Lambda} \mathcal{F}^\lambda R_\alpha$;

(5) $\mathcal{F}^0 R = R$, and for any $\alpha \in \Lambda$, $\mathcal{F}^\lambda R = 0$ for $\lambda \gg 0$.

Definition 2.4.2. Let $\mathcal{F}$ be a filtration on R. The *associated graded ring* $\mathrm{gr}_{\mathcal{F}} R$ of $\mathcal{F}$ is defined as

$$\mathrm{gr}_{\mathcal{F}} R := \bigoplus_{\alpha \in \Lambda} \bigoplus_{\lambda \in \mathbb{R}_{\geq 0}} \mathrm{gr}_{\mathcal{F}}^{\lambda} R_\alpha$$

where $\mathrm{gr}_{\mathcal{F}}^{\lambda} R_\alpha := \mathcal{F}^\lambda R_\alpha / \cup_{\lambda' > \lambda} \mathcal{F}^{\lambda'} R_\alpha$. We say that $\mathcal{F}$ is *finitely generated* if $\mathrm{gr}_{\mathcal{F}} R$ is finitely generated $\mathbb{C}$- algebra. For a valuation $v \in \mathrm{Val}_X$, we define the *associated graded ring* of v by

$\operatorname{gr}_v R := \operatorname{gr}_{\mathcal{F}_v} R$.

Definition 2.4.3. For any integer m, we define $R_m := \bigoplus_{m-1<\langle\alpha,\xi_0\rangle\leq m} R_\alpha$, so $R = \bigoplus_{m=0}^{+\infty} R_m$. Write $N_\alpha := \dim_{\mathbb{C}} R_\alpha$, and $N_m := \dim_{\mathbb{C}} R_m$ for $m \in \mathbb{N}$ and $M(R) \subset \mathbb{N}$ for the semigroup of $m \in \mathbb{N}$ for which $N_m > 0$. For later convenience, we rescale ξ_0 to make $R_m \neq \emptyset$ for sufficiently large m.

Denote $R^{(t)} = \bigoplus_{k=0}^{+\infty} \mathcal{F}^{kt} R_k$. We define the volume

$$\operatorname{vol}(R^{(t)}) := \limsup_{k\to+\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{kt} R_k}{k^n/n!}$$

Remark 2.4.4. Notice that our definition of R_m is different from the definition in [Wu21]. If ξ_0 is rational, we know (X, D) is a cone over (Y, E), then up to rescaling of ξ_0, R_m defined above equals to $H^0(Y, m(-K_Y - E))$. This matches with the definition of R_m in [BJ17] and [LX20].

We define $R_m^t := \mathcal{F}^{mt} R_m$ for $m \in \mathbb{N}$ and $t \in \mathbb{R}_+$, and set

$$T_\alpha := T_\alpha(\mathcal{F}) := \sup\{t \geq 0 \mid \mathcal{F}^{t\cdot\langle\xi_0,\alpha\rangle} R_\alpha \neq 0\}$$

$$T_m := T_m(\mathcal{F}) := \sup\{t \geq 0 \mid R_m^t = \mathcal{F}^{mt} R_m \neq 0\}.$$

Notice that $\{\mathcal{F}R_m\}_{m\in\mathbb{N}}$ is not a filtration, but we still have

Lemma 2.4.5. *We define the pseudo-effective threshold*

$$T := T(\mathcal{F}) := \sup_m T_m(\mathcal{F}).$$

Then $\lim_{m\to\infty} T_m$ *exists and equals to* T.

Proof. First assume $T < +\infty$, then for any $\epsilon > 0$, we can find some $\alpha_0 \in \Lambda$ such that $T_{\alpha_0} \geq T - \epsilon$.

Let $e_1, e_2, \ldots, e_r$ be a lattice basis for Λ. Suppose $\alpha_0 = \sum_i q_i \cdot e_i$, where $q_i \in \mathbb{N}$. Notice that for any two lattice points $\alpha_1, \alpha_2 \in \Lambda$, we have $\langle\xi_0, \alpha_1\rangle T_{\alpha_1} + \langle\xi_0, \alpha_2\rangle T_{\alpha_2} \leq \langle\xi_0, \alpha_1 + \alpha_2\rangle T_{\alpha_1+\alpha_2}$. So for any $\alpha = n\alpha_0 + \sum_i c_i \cdot e_i \in \Lambda$ where $0 \leq c_i \leq q_i, n \in \mathbb{N}$, we have $\langle \xi_0, \alpha \rangle T_\alpha \geq n\langle \xi_0, \alpha_0 \rangle T_{\alpha_0}$. Since $\langle \xi_0, \alpha \rangle \leq (n+1)\langle \xi_0, \alpha_0 \rangle$, so $T_\alpha \geq \frac{n}{n+1} T_{\alpha_0} \geq \frac{n}{n+1}(T - \epsilon)$. When n is sufficiently large we have $T_\alpha \geq T - 2\,\epsilon$. For sufficiently large m we can always find some α above such that $R_\alpha \subset R_m$, notice that

$$T_m \leq \sup_{m-1 < \langle \xi_0, \alpha \rangle \leq m} T_\alpha \leq \frac{m}{m-1} T_m$$

we have $T_m \geq T - 3\,\epsilon$ when $m \gg 1$. Hence $\lim_{m\to\infty} T_m = T$.

If $T = +\infty$, we just choose any $M > 0$ and find $T_{\alpha_0} > M$. The rest is the same.

□

The filtration is said to be *linearly bounded* if $T(\mathcal{F}) < \infty$. Note that being linearly bounded is not independent of the choice of ξ_0.

Example 2.4.6 (Filtration from test configuration)**.** [Wu21, Prop 3.8] Any test configuration $(\mathcal{X} = \operatorname{Spec} \mathcal{R}, \mathcal{D}, \xi_0; \eta)$ for (X, ξ_0) induces a filtration $\mathcal{F}$ on R defined by

$$\mathcal{F}^\lambda R := \bigoplus_{\alpha \in \Lambda} \{ f \in R_\alpha \mid t^{-\lambda} \bar{f} \in \mathcal{R}_\alpha \}$$

for $\lambda \in \mathbb{Z}_+$, where $\bar{f}$ denotes the pullback of f under the composition $\mathcal{X} \times_{\mathbb{A}^1} (\mathbb{A}^1 - \{0\}) \cong X \times (\mathbb{A}^1 - \{0\}) \to X$, and

$$\mathcal{F}^\lambda R_\alpha := \mathcal{F}^{\lceil \lambda \rceil} R_\alpha$$

for general $\lambda \in \mathbb{R}_+$. This filtration is linearly bounded, and finitely generated as a $\mathbb{Z}$-filtration, i.e. the bi-graded algebra

$$\bigoplus_{\alpha \in \Lambda} \left(\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} \mathcal{F}^\lambda R_\alpha \right)$$

is a finitely generated $\mathbb{C}[t]-$algebra.

Example 2.4.7 (Filtration from valuation)**.** Any valuation $v \in \mathrm{Val}_X$ induces a filtration $\mathcal{F}_v$ on R

as

$$\mathcal{F}_v^{\lambda} R_\alpha := \{s \in R_\alpha \mid v(s) \geq \lambda\}$$

2.5 Complement

Definition 2.5.1. A $\mathbb{Q}$-complement of (X, D) is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} -K_X - D$ such that $(X, D+B)$ is log canonical. A $\mathbb{Q}$-complement B is called an N-complement for $N \in \mathbb{Z}_{>0}$ if $N(K_X + D + B) \sim 0$, and $N(D+B) \geq N\lfloor D \rfloor + \lfloor (N+1)\{D\} \rfloor$ where $\{D\} = D - \lfloor D \rfloor$.

For the purpose of this paper, unless state otherwise, we only discuss T-equivariant complement.

For any $\mathbb{Q}$-complement B of (X, D) we define the *dual complex* of $(X, D+B)$ to be

$$\mathcal{DMR}(X, D+B) := \{v \in \mathrm{Val}_X^{\circ} \mid A_{(X,D+B)}(v) = 0 \text{ and } A_{X,D}(v) = 1\}.$$

In particular, the space of all lc places of $(X, D+B)$ is a cone over $\mathcal{DMR}(X, D+B)$. By abuse of notation, we usually write $v \in \mathcal{DMR}(X, D+B)$ if v is an lc place of $(X, D+B)$.

As in [LXZ21, Lemma 2.28], we have

Lemma 2.5.2. *Assume that v is a divisorial lc place of some $\mathbb{Q}$-complement. Then $\mathrm{gr}_v R$ is finitely generated.*

Chapter 3

A valuative criterion for K-semistability

3.1 A general volume formula

Let (X, D, ξ_0) be a log Fano cone singularity with $T = (\mathbb{C}^*)^r$ action. Let v_1 be a T-invariant valuation centered at the vertex $x \in X$, with $A_{(X,D)}(v_1) < +\infty$. We write $v_0 := \mathrm{wt}_{\xi_0}$ to denote the canonical valuation.

We define the filtration $\mathcal{F}$ on R as follow

$$\mathcal{F}^x R_\alpha = \{f \in R_\alpha \mid v_1(f) \geq x\}.$$

If $A(v_1) < +\infty$, then the filtration is linearly bounded. Indeed by Izumi's theorem, there exists $c_1, c_2 > 0$ such that $c_1 v_0 \leq v \leq c_2 v_0$. If $f \in R_\alpha, v_1(f) \geq x$ then $v_0(f) \geq c_2^{-1} x$ so when $x > c_2 v_0(f)$, we have $\mathcal{F}^x R_\alpha = 0$. So $\mathcal{F}$ is linearly bounded from above. Similarly, if $x < c_1 v_0(f)$ for some $f \in R_\alpha$ then $\mathcal{F}^x R_\alpha = F_\alpha$. So $\mathcal{F}$ is linearly bounded from below.

For later convenience, from now on we will fix the following constant:

$$c_1 := \inf_{\mathfrak{m}} \frac{v_1}{v_0} > 0$$

We still write $\mathcal{F}^x R_k$ to denote $\oplus_{R_\alpha \subset R_m} \mathcal{F}^x R_\alpha$. The filtration $\mathcal{F}$ can help us calculate the volume of v_1 via the following observation.

Lemma 3.1.1. *For any $m \in \mathbb{R}_{>0}$, we have*

$$\sum_{k=0}^{+\infty} \dim_{\mathbb{C}}(R_k/\mathcal{F}^m R_k) = \dim_{\mathbb{C}}(R/\mathfrak{a}_m(v_1)).$$

Notice that because $\mathcal{F}$ is linear bounded, so there are only finitely many nonzero elements in the left hand side.

Proof. For each $\alpha \in \Lambda$, we set $d_\alpha = \dim_{\mathbb{C}}(R_\alpha/\mathcal{F}^m R_\alpha)$. Then we can choose a basis of $R_\alpha/\mathcal{F}^m R_\alpha$:

$$\{[f_i^\alpha]_\alpha \mid f_i^\alpha \in R_\alpha, 1 \le i \le d_\alpha\},$$

here we use $[\cdot]_\alpha$ to denote the quotient class in $R_\alpha/\mathcal{F}^m R_\alpha$. When $\langle \alpha, \xi_0 \rangle > m/c_1$, the set becomes empty. We want to show the set

$$B := \{[f_i^\alpha] \mid 1 \le i \le d_\alpha, 0 \le \langle \alpha, \xi_0 \rangle < m/c_1\}$$

is a basis of $R/\mathfrak{a}_m(v_1)$, here $[\cdot]$ means taking quotient in $R/\mathfrak{a}_m(v_1)$. First we show that the elements in B are linearly independent. Assume we have a nontrivial linear combination of $[f_i^\alpha]$:

$$\sum_{\alpha \in \Lambda} \sum_{i=1}^{d_\alpha} c_i^\alpha [f_i^\alpha] = \left[\sum_{\alpha \in \Lambda} \sum_{i=1}^{d_\alpha} c_i^\alpha f_i^\alpha \right] = [f^{k_1} + \cdots + f^{k_p}] =: [F]$$

where $f^{k_j} \neq 0$ is an element in $R_{k_j} - \mathcal{F}^m R_{k_j}$ and $k_1 < k_2 < \cdots < k_p$. Now $f^{k_1} \notin \mathcal{F}^m R_{k_1}$ and $v_1(F) > k_1$, so that $f^{k_1} + \ldots f^{k_p} \notin \mathfrak{a}_m(v_1)$. Hence $[F] \neq 0 \in R/\mathfrak{a}_m(v_1)$.

Next we show that B indeed spans $R/\mathfrak{a}_m(v_1)$. Suppose on the contrary we have some $\alpha_0 \in \Lambda$ and some element $f \in R_{\alpha_0} - \mathfrak{a}_m(v_1)$ such that $[f] \neq 0 \in R/\mathfrak{a}_m(v_1)$ that cannot be written as a linear combination of $[f_i^\alpha]$. Let us assume $f \in R_k$. We first show that we could find a maximal k such that this thing happens. This is from the fact that the following set

$$\{v_0(g) \mid g \in R - \mathfrak{a}_m(v_1)\}$$

is finite (because v_1 is bounded by v_0).

So we could find some k such that any α_1 such that $\langle \alpha_1, \xi_0 \rangle > k$ and $g \in R_{\alpha_1}$, $[g]$ lies in the span of B.

If $f \in R_k - \mathcal{F}^m R_k$, then since $[f_i^\alpha]_\alpha$ where $k-1 < \langle \alpha, \xi_0 \rangle \leq k$ is a basis of $R_k/\mathcal{F}^m R_k$, we can write f as $\sum_j c_j f_j^\alpha + h_k$ where $h_k \in \mathcal{F}^m R_k$. So there exists some $h \in \mathfrak{a}_m(v_1)$ such that $f = \sum_j c_j f_j^\alpha + h$ and $v_0(h - h_k) > k$. By the maximality of k, $[h - h_k]$ lies in the span of B, so $[f]$ lies in the span of B. This is a contradiction.

If $f \in \mathcal{F}^m R_k \subseteq R_k$. Then by the definition of $\mathcal{F}^m R_k$, we can find some $h \in R$ such that $f + h \in \mathfrak{a}_m(v_1)$ and $v_0(f+h) = v_0(f)$. Since we assumed $[f] \neq 0$ in $R/\mathfrak{a}_m(v_1)$, so $h \neq 0$ and $k' = v_0(h) > v_0(f) = k$. So we know that $[f] = [(f+h) - h] = [-h]$ lies in the span of B. This is still a contradiction. □

Lemma 3.1.2. *Let $\mathcal{F}$ be a linearly bounded filtration on R. For any $u \in \mathbb{R}_+$ and $v > -c_1$, we have*

$$\lim_{p \to +\infty} \frac{n!}{p^n} \sum_{i=0}^{\lfloor up/(v+c_1) \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{up-vi} R_i) = n \int_{c_1}^{+\infty} \mathrm{vol}(R^{(x)}) \frac{u^n dx}{(v+x)^{n+1}}.$$

Proof. Let $\phi(y) = \dim_{\mathbb{C}}(\mathcal{F}^{up-vy} R_{\lfloor y \rfloor})$. Then $\phi(y)$ is an increasing function on $[m, m+1)$ for any $m \in \mathbb{Z}_{\geq 0}$ and $\phi(y) \leq \dim_{\mathbb{C}} R_{\lfloor y \rfloor} \leq C y^{n-1}$. Notice $\mathcal{F}^x$ is decreasing in x, so that $\phi(y) \geq$

$\dim_{\mathbb{C}}(\mathcal{F}^{up-v\lfloor y\rfloor}R_{\lfloor y\rfloor})$. So we have

$$\sum_{i=0}^{\lfloor up/(v+c_1)\rfloor}\dim_{\mathbb{C}}(\mathcal{F}^{up-vi}R_i)\leq\left(\sum_{i=0}^{\lfloor up/(v+c_1)\rfloor-1}\dim_{\mathbb{C}}(\mathcal{F}^{up-vi}R_i)\right)+\dim_{\mathbb{C}}R_{\lfloor up/(v+c_1)\rfloor}$$
$$\leq\left(\int_0^{up/(v+c_1)}\phi(y)dy\right)+O(p^{n-1})$$
$$=\left(p\int_{c_1}^{+\infty}\phi(\frac{up}{v+x})\frac{udx}{(v+x)^2}\right)+O(p^{n-1})$$

Then we have

$$\limsup_{p\to+\infty}\frac{\phi(up/(v+x))}{p^{n-1}/(n-1)!}=\limsup_{p\to+\infty}\frac{\dim_{\mathbb{C}}(\mathcal{F}^{upx/(v+x)}R_{\lfloor up/(v+x)\rfloor})}{p^{n-1}/(n-1)!}$$
$$\leq\limsup_{p\to+\infty}\frac{\dim_{\mathbb{C}}(\mathcal{F}^{upx/(v+x)}R_{\lfloor up/(v+x)\rfloor})}{\lfloor up/(v+x)\rfloor^{n-1}/(n-1)!}\frac{\lfloor up/(v+x)\rfloor^{n-1}}{(up/(v+x))^{n-1}}\frac{u^{n-1}}{(v+x)^{n-1}}$$
$$=\mathrm{vol}(R^{(x)})\frac{u^{n-1}}{(v+x)^{n-1}}.$$

The last equality holds by [BC11]. Now by Fatou's lemma, we have:

$$\limsup_{p\to+\infty}\frac{n!}{p^n}\sum_{i=0}^{\lfloor up/(v+c_1)\rfloor}\dim_{\mathbb{C}}(\mathcal{F}^{up-vi}R_i)\leq n\limsup_{p\to+\infty}\left(\int_{c_1}^{+\infty}\frac{(n-1)!}{p^{n-1}}\phi(\frac{up}{v+x})\frac{udx}{(v+x)^2}+O(p^{-1})\right)$$
$$\leq n\int_{c_1}^{+\infty}\limsup_{p\to+\infty}\frac{\phi(up/(v+x))}{p^{n-1}/(n-1)!}\frac{udx}{(v+x)^2}$$
$$\leq n\int_{c_1}^{+\infty}\mathrm{vol}(R^{(x)})\frac{u^ndx}{(v+x)^{n+1}}.$$

We can prove the other direction similarly. Define $\psi(y)=\dim_{\mathbb{C}}(\mathcal{F}^{up-vy}R_{\lceil y\rceil})$. Then $\phi(y)$ is an increasing function on $(m,m+1]$ for any $m\in\mathbb{Z}_{\geq 0}$ and $\psi(y)\leq\dim_{\mathbb{C}}R_{\lceil y\rceil}\leq Cy^{n-1}$, and that

$\psi(y) \le \dim_{\mathbb{C}}(\mathcal{F}^{up-v\lceil y\rceil} R_{\lceil y\rceil})$. So we have

$$\begin{aligned}
\sum_{i=0}^{\lfloor up/(v+c_1)\rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{up-vi} R_i) &\ge \left(\sum_{i=0}^{\lceil up/(v+c_1)\rceil} \dim_{\mathbb{C}}(\mathcal{F}^{up-vi} R_i)\right) - \dim_{\mathbb{C}} R_{\lceil up/(v+c_1)\rceil} \\
&\ge \left(\int_0^{up/(v+c_1)} \psi(y)dy\right) + O(p^{n-1}) \\
&= \left(p\int_{c_1}^{+\infty} \psi(\frac{up}{v+x})\frac{udx}{(v+x)^2}\right) + O(p^{n-1})
\end{aligned}$$

Then we have

$$\begin{aligned}
\liminf_{p\to+\infty} \frac{\psi(up/(v+x))}{p^{n-1}/(n-1)!} &= \liminf_{p\to+\infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{upx/(v+x)} R_{\lceil up/(v+x)\rceil})}{p^{n-1}/(n-1)!} \\
&\ge \liminf_{p\to+\infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{upx/(v+x)} R_{\lceil up/(v+x)\rceil})}{\lceil up/(v+x)\rceil^{n-1}/(n-1)!} \frac{\lceil up/(v+x)\rceil^{n-1}}{(up/(v+x))^{n-1}} \frac{u^{n-1}}{(v+x)^{n-1}} \\
&= \mathrm{vol}(R^{(x)})\frac{u^{n-1}}{(v+x)^{n-1}}.
\end{aligned}$$

By Fatou's lemma, we get the other direction of the estimate:

$$\begin{aligned}
\liminf_{p\to+\infty} \frac{n!}{p^n} \sum_{i=0}^{\lfloor up/(v+c_1)\rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{up-vi} R_i) &\ge n \liminf_{p\to+\infty} \left(\int_{c_1}^{+\infty} \frac{(n-1)!}{p^{n-1}}\psi(\frac{up}{v+x})\frac{udx}{(v+x)^2} + O(p^{-1})\right) \\
&\ge n\int_{c_1}^{+\infty} \liminf_{p\to+\infty} \frac{\psi(up/(v+x))}{p^{n-1}/(n-1)!}\frac{udx}{(v+x)^2} \\
&\ge n\int_{c_1}^{+\infty} \mathrm{vol}(R^{(x)})\frac{u^n dx}{(v+x)^{n+1}}.
\end{aligned}$$

Therefore we have the identity

$$\lim_{p\to+\infty} \frac{n!}{p^n} \sum_{i=0}^{\lfloor up/(v+c_1)\rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{up-vi} R_i) = n\int_{c_1}^{+\infty} \mathrm{vol}(R^{(x)})\frac{u^n dx}{(v+x)^{n+1}}.$$

∎

Proposition 3.1.3. *Given a valuation $v_1 \in \mathrm{Val}^T_{(X,D),x}$, we have*

$$\mathrm{vol}(v_1) = \frac{1}{c_1^n}\,\mathrm{vol}(\xi_0) - n\int_{c_1}^{+\infty} \mathrm{vol}(R^{(t)})\frac{dt}{t^{n+1}}$$

Proof. By Lemma 3.1.1 we have

$$\sum_{k=0}^{+\infty} \dim_{\mathbb{C}}(R_k/\mathcal{F}^m R_k) = \dim_{\mathbb{C}}(R/\mathfrak{a}_m(v_1)).$$

Now that

$$\begin{aligned} n!\dim_{\mathbb{C}}(R/\mathfrak{a}_m(v_1)) &= n!\sum_{k=0}^{+\infty}\dim_{\mathbb{C}}(R_k/\mathcal{F}^m R_k) \\ &= n!\sum_{k=0}^{\lfloor m/c_1\rfloor}(\dim_{\mathbb{C}} R_k - \dim_{\mathbb{C}}\mathcal{F}^m R_k) \\ &= \frac{m^n}{n!}\mathrm{vol}(\xi_0) - n\int_{c_1}^{+\infty}\mathrm{vol}(R^{(t)})\frac{dt}{t^{n+1}} + O(m^{n-1}) \end{aligned}$$

The last equality uses Lemma 3.1.2. Now

$$\mathrm{vol}(v_1) = \lim_{m\to\infty}\frac{n!}{m^n}\dim_{\mathbb{C}}(R/\mathfrak{a}_m(v_1)) = \frac{1}{c_1^n}\mathrm{vol}(\xi_0) - n\int_{c_1}^{+\infty}\mathrm{vol}(R^{(t)})\frac{dt}{t^{n+1}}$$

. ∎

3.2 Kollár components

Definition 3.2.1. Let $o \in (X, D)$ be a klt singularity. We call a proper birational morphism $\mu : Y \to X$ a Kollár component S, if μ is isomorphic over $X - \{o\}$ and $\mu^{-1}(o)$ is an irreducible divisor S, such that $(Y, S+\mu_*^{-1}D)$ is plt and $(-S)$ is $\mathbb{Q}$-Cartier and ample over X.

If we denote $K_S+\Delta_S=(K_Y+S+\mu_*^{-1}D)|_S$, then (S,Δ_S) is klt log Fano.

Special test configurations from Kollár component Let S be a T-invariant Kollár component over $o\in(X,D)$ and $\pi: Y\to X$ be the plt blow up extracting S and let $K_Y+\pi_*^{-1}D+S|_S=: K_S+\Delta_S$. We can use the deformation to the normal cone construction to get a degeneration of X to an orbifold cone over (S,Δ_S).

Denote the associated ring graded ring of $v_0=\mathrm{ord}_S$ by

$$A=\bigoplus_{k=0}^{+\infty}\mathfrak{a}_k(v_0)/\mathfrak{a}_{k+1}(v_0)=\bigoplus_{k=0}^{+\infty}A_k$$

We have a decomposition

$$\mathfrak{a}_k(v_0)=\bigoplus_{\alpha}\mathfrak{a}_k^{\alpha}(v_0)=\bigoplus_{\alpha}R_\alpha\cap\mathfrak{a}_k(v_0)$$

T acts equivariantly on the extended Rees algebra:

$$\mathcal{R}'=\bigoplus_{k\in\mathbb{Z}}\mathcal{R}'_k:=\bigoplus_{k\in\mathbb{Z}}\mathfrak{a}_k(v_0)t^{-k}\subset R[t,t^{-1}]$$

.

Let $\mathcal{X}=\mathrm{Spec}(\mathcal{R}')$. Then we get a flat family $\pi:\mathcal{X}\to\mathbb{A}^1$ satisfying $X_t=\mathcal{X}\times_{\mathbb{A}^1}\{t\}=X$ and $X_0=\mathcal{X}\times_{\mathbb{A}^1}\{0\}=\mathrm{Spec}(A)$. Let $\mathcal{D}$ be the strict transform of $D\times\mathbb{A}^1$ under the birational morphism $\mathcal{X}\dashrightarrow X\times\mathbb{A}^1$.

Definition 3.2.2. Assume that $o\in(X,D)$ is a klt singularity with a good T-action and S is a T-invariant Kollár component. Let $\mathcal{X}\to\mathbb{A}^1$ be the associated degeneration which degenerates (X,D) to a (X_0,D_0) and admits a $T'=T\times\mathbb{C}^*$-action. For any $f=\sum f_k\in\mathcal{R}'$, $\mathrm{ord}_S(f)=\min\{k\mid f_k\neq 0\}$.

Over X_0, ord_S corresponds to the $\mathbb{C}^*$-action corresponding to the $\mathbb{Z}$- grading, Denote the generating vector by $\xi_S \in \mathfrak{t}'^{+}_{\mathbb{R}}$.

With the above notation, we say that $(\mathcal{X}, \mathcal{D}, \xi_0; \xi_S)$ is the special test configuration associated to the Kollár component S. If ξ_0 and ξ_S are clear, we just use $(\mathcal{X}, \mathcal{D})$ to denote the special test configuration.

Lemma 3.2.3. *[LX18] Let $(\mathcal{X}, \mathcal{D}, \xi_0; \xi_S)$ denote the special test configuration associated to a T-invariant Kollár component S. Let (X_0, D_0) be the corresponding paor on the special fiber. For any $\xi_0 \in \mathfrak{t}'^{+}_{\mathbb{R}}$, let ξ_0 also denote the induced Reeb vector on X_0. Then we have:*

1. $A_{(X,D)}(\mathrm{ord}_S) = A_{(X_0,D_0)}(\mathrm{wt}_{\xi_S})$, $\mathrm{vol}_{(X,D)}(\mathrm{ord}_S) = \mathrm{vol}_{(X_0,D_0)}(\mathrm{wt}_{\xi_S})$
2. $A_{(X,D)}(\mathrm{wt}_{\xi_0}) = A_{(X_0,D_0)}(\mathrm{wt}_{\xi_0})$, $\mathrm{vol}_{(X,D)}(\mathrm{wt}_{\xi_0}) = \mathrm{vol}_{(X_0,D_0)}(\mathrm{wt}_{\xi_0})$

3.3 The valuations w_ϵ and $\mathrm{wt}_{\xi_\epsilon}$

Let $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ be any special test configuration. Let $\xi_\epsilon = \xi_0 - \epsilon\eta \in \mathfrak{t}'^{+}_{\mathbb{R}}$, then wt_ϵ can be considered as a valuation on $\mathcal{X}$. Using the embedding $\mathbb{C}(X) \to \mathbb{C}(\mathcal{X}) = \mathbb{C}(X \times \mathbb{C}^*) = \mathbb{C}(X \times \mathbb{C})$, $\mathrm{wt}_{\xi_\epsilon}$ can be restricted to become a valuation w_ϵ on X. (see [Li17]) Alternatively by equivariantly embedding of $\mathcal{X}$ into $\mathbb{C}^N \times \mathbb{C}$, $\mathrm{wt}_{\xi_\epsilon}$ is induced by a linear holomorphic vector field, still denoted by ξ_ϵ, on $\mathbb{C}^N$. The weight function associated to ξ_ϵ induces a filtration on R whose associated graded ring is equal to the coordinate ring of X_0. By [LX18, Lemma 2.11], this filtration is indeed determined by a valuation w_ϵ on X. As a consequence we have $\mathrm{vol}_{(X,D)}(w_\epsilon) = \mathrm{vol}_{(X_0,D_0)}(\mathrm{wt}_{\xi_\epsilon})$ because w_ϵ and wt_ϵ have the same associated graded ring. On the other hand, we have the following lemma.

Lemma 3.3.1. *Use the above notation, for each fixed ϵ, $A_{(X,D)}(w_\epsilon) = A_{(X_0,D_0)}(\mathrm{wt}_{\xi_\epsilon})$. Therefore $\widehat{\mathrm{vol}}_{(X,D)}(w_\epsilon) = \widehat{\mathrm{vol}}_{(X_0,D_0)}(\mathrm{wt}_{\xi_\epsilon})$. As a consequence:*

$$\mathrm{Fut}(X_0, D_0.\xi_0; \eta) = \frac{d}{d\,\epsilon}\Big|_{\epsilon=0} \widehat{\mathrm{vol}}_{(X_0,D_0)}(\mathrm{wt}_{\xi_\epsilon}) = \frac{d}{d\,\epsilon}\Big|_{\epsilon=0} \widehat{\mathrm{vol}}_{(X,D)}(w_\epsilon)$$

Proof. For sufficiently small $\epsilon > 0$, we choose a sequence of rational vector fields $\xi_{k,\epsilon} \in \mathfrak{t}^+_{\mathbb{Q}}$ approaching ξ_ϵ as $k \to +\infty$. Then the $\mathbb{C}^*$-action generated by $\xi_{k,\epsilon}$ corresponds to a Kollár component $S_{k,\epsilon}$ which is isomorphic to the quotient $X_0/\langle \exp(\mathbb{C} \cdot \xi_{k,\epsilon} \rangle)$. So up to a base change, $(\mathcal{X}, \mathcal{D}, \xi_0; \xi_{k,\epsilon})$ is equivalent to the special test configuration associated to $S_{k,\epsilon}$ and there exists constants $c_{k,\epsilon} > 0$ such that $\mathrm{wt}_{\xi_{k,\epsilon}} |_{\mathbb{C}(X)} = c_{k,\epsilon} \cdot \mathrm{ord}_{S_{k,\epsilon}} \to w_\epsilon$ as $k \to +\infty$. So by Lemma 3.2.3, we know $A_{X,D}(c_{k,\epsilon} \cdot \mathrm{ord}_{S_{k,\epsilon}}) = A_{X_0,D_0}(\mathrm{wt}_{\xi_{S_{k,\epsilon}}})$. By taking a limit $k \to +\infty$, we get $A_{(X,D)}(w_\epsilon) = A_{(X_0,D_0)}(\mathrm{wt}_{\xi_\epsilon})$. $\square$

3.4 Proof of the valuative criterion

First we show that $\beta(v) \geq 0$ for all $v \in \mathrm{Val}^\circ_X$ implies K-semistability of (X, D, ξ_0).

By [LX18, Proposition 3.6], we only need to consider the special test configuration associated to the Kollár components. Let S be a T-invariant Kollár component, and $(\mathcal{X}, \mathcal{D}, \xi_0; \eta = -\xi_S)$ be the corresponding test configuration, we have the valuations w_ϵ as above.

For any $f \in R$, we have

$$w_\epsilon(f) = \min_\alpha \{\langle \alpha, \xi_0 \rangle + \epsilon \, \mathrm{ord}_S(f_\alpha) \mid f = \sum f_\alpha, f_\alpha \neq 0\}$$

so $\mathcal{F}^x_{w_\epsilon} R_\alpha = \mathcal{F}^{(x - \langle \alpha, \xi_0 \rangle)/\epsilon}_{\mathrm{ord}_S} R_\alpha$. Notice that $R_k = \bigoplus_{k-1 < \langle \alpha, \xi_0 \rangle \leq k} R_\alpha$, so

$$\mathcal{F}^{(x-k+1)/\epsilon}_{\mathrm{ord}_S} R_k \subseteq \mathcal{F}^x_{w_\epsilon} R_k \subseteq \mathcal{F}^{(x-k)/\epsilon}_{\mathrm{ord}_S} R_k$$

so $\dim_{\mathbb{C}} \mathcal{F}^{\frac{k(x-1)}{\epsilon} + \frac{1}{\epsilon}}_{\mathrm{ord}_S} R_k \leq \dim_{\mathbb{C}} \mathcal{F}^{kx}_{w_\epsilon} R_k \leq \dim_{\mathbb{C}} \mathcal{F}^{\frac{k(x-1)}{\epsilon}}_{\mathrm{ord}_S} R_k$. Recall that $\mathrm{vol}(\mathcal{F}_v R^{(x)}) = \limsup_{k \to \infty} \dfrac{\dim_{\mathbb{C}} \mathcal{F}^{kx}_v R_k}{k^n/n!}$, so $\mathrm{vol}(\mathcal{F}_{w_\epsilon} R^{(x)}) = \mathrm{vol}(\mathcal{F}_{\mathrm{ord}_S} R^{(\frac{x-1}{\epsilon})})$.

We also have $\inf_{\mathfrak{m}} \dfrac{w_\epsilon}{\mathrm{wt}_{\xi_0}} = 1$, so $c_1 = 1$. By Proposition 3.1.3 we have

$$\begin{aligned}\operatorname{vol}(w_\epsilon) &= \operatorname{vol}(\xi_0) - n\int_1^{+\infty} \operatorname{vol}(\mathcal{F}_{w_\epsilon} R^{(t)}) \frac{dt}{t^{n+1}} \\ &= \operatorname{vol}(\xi_0) - n\int_0^{+\infty} \operatorname{vol}(\mathcal{F}_S R^{(t)}) \frac{\epsilon\, dt}{(1+\epsilon\, t)^{n+1}}\end{aligned}$$

The log discrepancy is given by

$$A_{(X,D)}(w_\epsilon) = A_{(X,D)}(\mathrm{wt}_{\xi_0}) + \epsilon\, A_{(X,D)}(S).$$

So we get the normalized volume of w_ϵ:

$$\widehat{\operatorname{vol}}(w_\epsilon) = (A_X(\xi_0) + \epsilon\, A_X(S))^n (\operatorname{vol}(\xi_0) - n\int_0^{+\infty} \operatorname{vol}(\mathcal{F}_S R^{(t)}) \frac{\epsilon\, dt}{(1+\epsilon\, t)^{n+1}}).$$

The derivative at $\epsilon = 0$ is equal to:

$$\begin{aligned}\frac{d}{d\,\epsilon}\Big|_{\epsilon=0} \widehat{\operatorname{vol}}(w_\epsilon) &= nA_X(\xi_0)^{n-1} A_X(S) \operatorname{vol}(\xi_0) - nA_X(\xi_0)^n \operatorname{vol}(\xi_0) \int_0^{+\infty} \operatorname{vol}(\mathcal{F}_S R^{(t)}) dt \\ &= nA_X(\xi_0)^{n-1} \operatorname{vol}(\xi_0)(A_X(S) - \frac{A_X(\xi_0)}{\operatorname{vol}(\xi_0)} \int_0^{+\infty} \operatorname{vol}(\mathcal{F}_S R^{(t)}) dt) \\ &= nA_X(\xi_0)^{n-1} \operatorname{vol}(\xi_0) \beta(S)\end{aligned}$$

By Lemma 3.3.1, we know the Futaki invariant of $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is precisely $nA_X(\xi_0)^{n-1} \operatorname{vol}(\xi_0)\beta(S)$. So $\beta(S) \geq 0$ implies $\mathrm{Fut}((\mathcal{X}, \mathcal{D}, \xi_0; \eta)) \geq 0$. Hence we proved one side of the criterion.

For the other side, suppose (X, D, ξ_0) is K-semistable, we show that $\beta(\mathrm{ord}_E) \geq 0$ for all T-equivariant divisor E over X.

Given any T-invariant divisor E over X, we can similarly define the valuations w_ϵ to be

$$w_\epsilon(f) = \min_\alpha \{\langle \alpha, \xi_0 \rangle + \epsilon\, \mathrm{ord}_E(f_\alpha) \mid f = \sum f_\alpha, f_\alpha \neq 0\}$$

The above calculation still holds, which gives us $\frac{d}{d\epsilon}\Big|_{\epsilon=0}\widehat{\mathrm{vol}}(w_\epsilon) = nA_X(\xi_0)^{n-1}\,\mathrm{vol}(\xi_0)\beta(E)$. By Theorem 2.3.6, we know $\widehat{\mathrm{vol}}$ reaches its minimum at $\mathrm{wt}_{\xi_0} = w_0$, so $\frac{d}{d\epsilon}\Big|_{\epsilon=0}\widehat{\mathrm{vol}}(w_\epsilon) \geq 0$, hence $\beta(E) \geq 0$.

To prove the other side of the criterion, it suffices to show that $\beta(E) \geq 0$ for all T-equivariant divisor E implies $\beta(v) \geq 0$ for all $v \in \mathrm{Val}_X^\circ$. This is done by Theorem 4.3.5 (notice we do not use any result in this section to prove Theorem 4.3.5 in chapter 4).

Chapter 4

Filtrations and the Delta invariant

4.1 Okounkov bodies

We follow the idea in [Wu21]. Let $\mu : Y \to X$ be a log resolution of X at the vertex x, and set $Y_0 := \mu^{-1}(x) = \sum_{i \in I} b_i E_i$. After possible replacing Y by further blowups at x, one may pick a regular system of parameters $x_1, \ldots, x_n$ for $\mathcal{O}_{Y,y}$ with y the generic point of $\cap_{i=1}^n E_i$ and x_i defining E_i. Then by Cohen structure theorem, $\widehat{\mathcal{O}_{Y,y}} \cong \mathbb{C}[\![x_1, \ldots, x_n]\!]$. This gives us a rank n valuation $v = (v_1, \ldots, v_n) : \mathcal{O}_{Y,y} - \{0\} \to \mathbb{N}^n$ with $v_1 = \mathrm{ord}_{E_1}$ on Y_1,

$$v_i(f) := \mathrm{ord}_{E_i} \left(\frac{f}{\prod_{k<i} x_k^{v_k(f)}} \Big|_{\cap_{j<i} E_j} \right)$$

for $2 \le i \le n$, and $\mathbb{N}^n$ equipped with the lexicographic ordering.

As in [Wu21, Lemma 3.1] we have the following Izumi type estimate.

Lemma 4.1.1. *There is a constant $C > 0$ such that $v_i(f) \le C \,\mathrm{ord}_0(f)$ for all $f \in R$ and $1 \le i \le n$.*

Now for each $m \in \mathbb{N}$, define

$$\Gamma_m := v(R_m) \subseteq \mathbb{N}^n, \ \Gamma := \{(x, m) \mid x \in \Gamma_m, m \in \mathbb{N}_+\}.$$

We denote by $\Sigma(\Gamma) \subseteq \mathbb{R}^{n+1}$ the closed convex cone generated by Γ. We define the convex body of $(X = \operatorname{Spec} R, D, \xi_0)$ by

$$\Delta = \Delta \times \{1\} := \Sigma(\Gamma) \cap (\mathbb{R}^n \times \{1\}).$$

We claim this is indeed a local version of Okounkov body.

Lemma 4.1.2. *Let Γ be as above. Then Γ satisfies the following conditions:*

(1) $\Gamma_0 = \{0\}$,

(2) There exists finitely many $a_i \in \mathbb{N}^n$ such that $(a_i, 1)$ span a subsemigroup $B \subset \mathbb{N}^{n+1}$ containing Γ.

(3) The subgroup generated by Γ in $\mathbb{Z}^{n+1}$ is $\mathbb{Z}^{n+1}$.

Proof. The first condition is a straight forward check. For the second part, we use the Izumi type estimate.

For the second part, by [BFJ14, Prop 4.8], we have some constant $C' > 0$ such that $\operatorname{ord}_0(f) \leq C' v_\xi(f)$. Then we know for all $0 \leq i \leq n$ and $0 \neq f \in R_m$, $v_i(f) \leq CC'$ where C comes from 4.1.1. So that the vectors $(a_1, \ldots, a_n, 1)$ will span a semigroup containing Γ.

For the last part, we write $x_i = f_i/g_i$ with $f_i, g_i \in R$. Then $v(f_i) - v(g_i) = e_i, 1 \leq i \leq n$ where $\{e_i\}$ denotes the standard basis for $\mathbb{Z}^n$. Since $(0, 1) \in \Gamma$, we have that Γ will generate all of $\mathbb{Z}^{n+1}$. $\square$

Now as in [Wu21, Theorem 3.3] we have

Theorem 4.1.3. *For any $m \geq 1$, let $\rho_m := \frac{1}{m^n} \sum_{x \in \Gamma_m} \delta_{m^{-1}x}$ be a positive measure on Δ. Then*

$\lim_{m\to\infty}\rho_m = \rho$ *weakly, where* ρ *denotes the Lebesgue measure on* Δ*. In particular, the limit*

$$\mathrm{vol}(\Delta) = \lim_{m\to\infty} \frac{n!}{m^n} \#\Gamma_m = \lim_{m\to\infty} \frac{n!}{m^n} \dim_{\mathbb{C}} R_m$$

exists and equals $\mathrm{vol}(\xi_0)$.

Follow [BJ17, Lemma 2.2] we have

Lemma 4.1.4. *For every* $\epsilon > 0$*, there exists a* $m_0 = m_0(\epsilon) > 0$ *such that*

$$\int_\Delta g d\rho_m \le \int_\Delta g d\rho + \epsilon$$

for every $m \ge m_0$ *and every concave function* $g : \Delta \to \mathbb{R}$ *satisfying* $0 \le g \le 1$*. Notice that we require the uniformity in* g.

4.2 Concave transform and limit measure

Let Δ be the Okounkov body of (X, D, ξ_0), and $\mathcal{F}$ be a linearly bounded filtration on R. For $t \ge 0$, we define $\Delta^t \subseteq \Delta$ to be the local Okounkov body associated to R^t_m as in [Wu21, Prop 3.10]. More precisely, let $\Gamma^t_m := v(R^t_m)$, $\Gamma^t := \{(x, m) \mid x \in \Gamma_m, m \in \mathbb{N}_+\}$, and $\Delta^t = \Delta^t \times \{1\} = \Sigma(\Gamma^t) \cap (\mathbb{R}^n \times \{1\})$.

Define $G : \Delta \to \mathbb{R}_+$ to be $x \mapsto \sup\{t \in \mathbb{R}_+ \mid x \in \Delta^t\}$. Then G is a concave, upper continuous function taking values in $[0, T(\mathcal{F})]$.

As in [BJ17], we define the limit measure μ of the filtration $\mathcal{F}$ as the pushforward

$$\mu = G_*\rho = -\frac{d}{dt}\,\mathrm{vol}(\Delta^t).$$

Thus μ is a positive measure on $\mathbb{R}_+$ of mass $\mathrm{vol}(\xi_0)$ with support $[0, T(\mathcal{F})]$.

Definition 4.2.1. For a linearly bounded filtration $\mathcal{F}$, we define the *volume* (or the S-invariant) of $\mathcal{F}$ to be

$$S(\mathcal{F}) := \frac{n!}{\mathrm{vol}(\xi_0)} \int_0^{+\infty} \mathrm{vol}(\Delta^t) dt = \frac{n!}{\mathrm{vol}(\xi_0)} \int_0^{+\infty} t d\mu(t) = \frac{1}{\mathrm{vol}(\Delta)} \int_\Delta G d\rho.$$

Jumping Numbers Given a filtration $\mathcal{F}$ on R, consider the *jumping numbers*

$$0 \leq a_{\alpha,1} \leq \cdots \leq a_{\alpha,N_\alpha} = \langle \alpha, \xi_0 \rangle T_\alpha(\mathcal{F})$$

defined for $\alpha \in \Lambda$ by

$$a_{\alpha,j} = a_{\alpha,j}(\mathcal{F}) = \inf\{\lambda \in \mathbb{R}_+ \mid \operatorname{codim} \mathcal{F}^\lambda R_\alpha \geq j\}$$

for $1 \leq j \leq N_\alpha$.

For R_m we also define the jumping numbers

$$0 \leq a_{m,1} \leq \cdots \leq a_{m,N_m} = mT_m(\mathcal{F})$$

for $m \in \mathbb{N}$ by

$$a_{m,j} = a_{m,j}(\mathcal{F}) = \inf\{\lambda \in \mathbb{R}_+ \mid \operatorname{codim} \mathcal{F}^\lambda R_m \geq j\}$$

for $1 \leq j \leq N_m$.

We define the rescaled sum of the jumping numbers:

$$S_\alpha(\mathcal{F}) := \frac{1}{\langle \alpha, \xi_0 \rangle N_\alpha} \sum_{j=1}^{N_\alpha} a_{\alpha,j}, S_m(\mathcal{F}) := \frac{1}{mN_m} \sum_{j=1}^{N_m} a_{m,j}$$

for $\alpha \in \Lambda, m \in \mathbb{N}$.

Define a positive measure $\mu_m = \mu_m^{\mathcal{F}}$ on $\mathbb{R}_+$ by

$$\mu_m = \frac{1}{m^n}\sum_j \delta_{m^{-1}a_{m,j}} = -\frac{1}{m^n}\frac{d}{dt}\dim \mathcal{F}^{mt}R_m.$$

We have the following result as in [BC11, Theorem 1.11], [Wu21, Theorem 3.12]

Theorem 4.2.2. *Let $\mathcal{F}$ be a linearly bounded filtration on R, then we have*

$$\lim_{m\to\infty}\mu_m = \mu$$

in the weak sense of measures on $\mathbb{R}_+$.

Proof. The proof goes along the same lines as in [BC11, Theorem 1.11]. Notice that $\dim_{\mathbb{C}} \mathcal{F}^\lambda R_m = j$ if and only if $a_{m,N_m-j} \le \lambda < a_{m,N_m-j+1}$. So we have

$$\frac{d}{dt}\dim \mathcal{F}^\lambda R_m = -\sum_j \delta_{a_{m,j}}$$

in the sense of distributions. Let $g_m(t) = \frac{1}{m^n}\dim R_m^t$. By 4.1.3 and the Okounkov body construction, we have

$$\lim_{t\to+\infty} g_m(t) = g(t) := \operatorname{vol}\Delta(R_\bullet^t),$$

for $0 \le t < T(\mathcal{F})$. Since g_m are uniformly bounded above, $g_m \to g$ in L^1_{loc} by dominated convergence, and hence $-\mu_m = g_m' \to g' = -\mu$ as distributions. □

Then we can rewrite the $S_m(\mathcal{F})$ as

$$S_m(\mathcal{F}) = \frac{1}{mN_m}\sum_j a_{m,j} = \frac{m^n}{N_m}\int_0^{+\infty} td\mu_m(t).$$

Lemma 4.2.3. *Let $\mathcal{F}$ be a linearly bounded filtration on R, we have*

$$S_m(\mathcal{F}) \leq \frac{m^n}{N_m} \int_{\Delta} G d\rho_m,$$

and furthermore we have

$$S(\mathcal{F}) = \lim_{m \to \infty} S_m(\mathcal{F}).$$

Proof. The limit comes directly from the above theorem. For the inequality, we choose a basis $s_1, s_2, \dots, s_{N_m}$ of R_m such that $v(s_j) = a_{m,j}, 1 \leq j \leq N_m$. Let $r_j := v(s_j)$ where $v = (v_1, \dots, v_n)$ comes from our construction of the Okounkov body in section 2.6. Notice v has trasnscendence degree 0, we have $\Gamma_m = \{r_1, \dots, r_m\}$. So

$$\frac{m^n}{N_m} \int_{\Delta} G d\rho_m = \frac{1}{N_m} \sum_{j=1}^{N_m} G(m^{-1} r_j)$$

and

$$S_m(\mathcal{F}) = \frac{1}{N_m} \sum_{j=1}^{N_m} m^{-1} a_{m,j}.$$

So it suffices to show $G(m^{-1} r_j) \geq m^{-1} a_{m,j}$ for $1 \leq j \; leq N_m$. This is by the definition of G. $\square$

Proposition 4.2.4. *For any $\epsilon > 0$, there exists $m_0 = m_0(\epsilon) > 0$, such that*

$$S_m(\mathcal{F}) \leq (1 + \epsilon) S(\mathcal{F}).$$

Proof. Let $V := \mathrm{vol}(\Delta)$. Take $\epsilon' > 0$ such that $(V^{-1} + \epsilon')(V + (n+1)\,\epsilon') \leq 1 + \epsilon$. Since $0 \leq G \leq T(\mathcal{F})$. By 4.1.4, take $g = G/T(\mathcal{F})$ we could find some m_0 such that

$$\int_{\Delta} G d\rho_m \leq \int_{\Delta} G d\rho + \epsilon'\, T(\mathcal{F}) = V S(\mathcal{F}) + \epsilon'\, T(\mathcal{F}) \leq (V + (n+1)\,\epsilon') S(\mathcal{F})$$

for $m \geq m_0$. By 4.1.3 we could also assume $\frac{m^n}{N_m} \leq V^{-1} + \epsilon'$ for $m \geq m_0$. The above lemma gives us

$$S_m(\mathcal{F}) \leq \frac{m^n}{N_m} \int_{\Delta} G d\rho_m \leq (V^{-1} + \epsilon')(V + (n+1)\,\epsilon')S(\mathcal{F}) \leq (1+\epsilon)S(\mathcal{F})$$

for $m \geq m_0$. □

4.3 The Delta invariant via filtration

Let v be a T-invariant valuation on log Fano cone (X, D, ξ_0). Then we have the filtration $\mathcal{F} = \mathcal{F}_v$ on R by $\mathcal{F}^x R_\alpha = \{f \in R_\alpha \mid v(f) \geq x\}$. We also write $\mathcal{F}^x R_m = \{f \in R_m \mid v(f) \geq x\}$. We write $S_\alpha(v) = S_\alpha(\mathcal{F}_v), S_m(v) = S_m(\mathcal{F}_v)$.

Definition 4.3.1. For any $\alpha \in \Lambda$, we say an effective divisor B is an *α-basis type divisor*, if there exists a basis $s_1, \ldots, s_{N_\alpha}$ of R_α, such that

$$B = \frac{\sum_{i=1}^{N_\alpha} \{s_i = 0\}}{\langle \alpha, \xi_0 \rangle N_\alpha}$$

Similarly for any $m \in \mathbb{N}$, we say an effective divisor B is an *m-basis type divisor*, if there exists a basis $s_1, \ldots, s_{N_m}$ of R_m, such that

$$B = \frac{\sum_{i=1}^{N_m} \{s_i = 0\}}{m N_m}$$

Definition 4.3.2. For any $\alpha \in \Lambda$, we define

$$\delta_\alpha = \inf\{\mathrm{lct}(X, D; B) \mid B \text{ is an } \alpha\text{-basis type divisor }\}$$

For any $m \in \mathbb{N}$, we define

$$\delta_m = \inf\{\mathrm{lct}(X, D; B) \mid B \text{ is an m-basis type divisor } \}$$

here $lct(X, D; B)$ is the log-canonical threshold, see [CS08].

Proposition 4.3.3. *For any $\alpha \in \lambda$, we have*

$$\delta_\alpha = \inf_{v \in \mathrm{Val}_X^\circ} \frac{A_{X,D}(v)}{S_\alpha(v)} = \inf_E \frac{A_{X,D}(\mathrm{ord}_E)}{S_\alpha(\mathrm{ord}_E)}$$

where E runs through all the T-invariant prime divisors over X.

Similarly, for any $m \in M(R)$, we have

$$\delta_m = \inf_{v \in \mathrm{Val}_X^\circ} \frac{A_{X,D}(v)}{S_m(v)} = \inf_E \frac{A_{X,D}(\mathrm{ord}_E)}{S_m(\mathrm{ord}_E)}$$

where E runs through all the T-invariant prime divisors over X.

We need a simple observation.

Lemma 4.3.4. *For any $\alpha \in \Lambda$, and any $v \in \mathrm{Val}_X^\circ$, we have*

$$S_\alpha(v) = \max_{s_j} \frac{1}{\langle \alpha, \xi_0 \rangle N_\alpha} \sum_{j=1}^{N_\alpha} v(s_j),$$

where the maximum is over all bases $s_1, \ldots, s_{N_\alpha}$ of R_α. The similar result holds for R_m.

Proof. For any basis $s_1, \ldots, s_{N_\alpha}$ of R_α, we may assume $v(s_1) \leq \cdots \leq v(s_{N_\alpha})$. Then $v(s_j) \leq a_{alpha,j}$ by the definition of the jumping numbers. Thus $(\langle \alpha, \xi_0 \rangle N_\alpha)^{-1} \sum_j v(s_j) \leq (\langle \alpha, \xi_0 \rangle N_\alpha)^{-1} \sum_j a_{\alpha,j} = S_\alpha(v)$. On the other hand, if we pick basis s_j such that $v(s_j) = a_{\alpha,j}$, then the equality holds. The case of R_m is the same. □

proof of Proposition 4.3.3. Recall that (see [CS08])

$$\operatorname{lct}(X, D; B) = \inf_v \frac{A_{X,D}(v)}{v(B)}.$$

So we have

$$\delta_\alpha = \inf\{\inf_v \frac{A_{X,D}(v)}{v(B)} \mid B \text{ of } \alpha\text{-basis type divisor } \}$$

where the second infimum runs through all divisorial valuations $v \in \operatorname{Val}_X^\circ$. Switching the two infimum and then Lemma 4.3.4 implies the result. □

Theorem 4.3.5. *The limit $\lim_{m\to\infty} \delta_m$ exists and equals to $\delta(X, D, \xi_0)$ we defined in Definition 1.0.2. Furthermore,*

$$\delta(X, D, \xi_0) = \inf_v \frac{A_{X,D}(v)}{S_{X,D}(v)} = \inf_E \frac{A(\operatorname{ord}_E)}{S(\operatorname{ord}_E)}.$$

where E runs through all the T-invariant prime divisors over X.

Proof. Let $\delta := \limsup_m \delta_m$. By Proposition 4.2.4 and Proposition 4.3.3,

$$\limsup_m \delta_m \le \inf_v \frac{A_{X,D}(v)}{S_{X,D}(v)}.$$

On the other hand, for any $\epsilon > 0$, we could find some $m_0 = m_0(\epsilon)$ such that $S_m(v) \le (1+\epsilon)S(v)$ for all $v \in \operatorname{Val}_X^\circ$ and $m \ge m_0$. Therefore

$$\delta = \limsup_m \delta_m = \limsup_m \inf_v \frac{A(v)}{S_m(v)} \ge \frac{1}{1+\epsilon} \inf_v \frac{A(v)}{S(v)}.$$

Hence $\delta = \lim_m \delta_m$. By Lemma 4.2.3 and Proposition 4.3.3, it is straightforward to check that $\delta = \delta(X, D, \xi_0)$. The same argument in the proof of Proposition 4.3.3 shows

$$\inf_v \frac{A_{X,D}(v)}{S_{X,D}(v)} = \inf_E \frac{A(\operatorname{ord}_E)}{S(\operatorname{ord}_E)}$$

. $\square$

Theorem 4.3.5 together with Chapter 3.4 completes the proof of Theorem 1.0.3.

Chapter 5

Valuations computing the stability threshold

In the log Fano case, [BJ17] showed that there exists a valuation computing $\delta(X, D)$ if the ground filed k is uncountable, and in [BLX19] when $\delta(X, D) \leq 1$ for a general ground field, where it is also shown that in this case any minimizer is an lc place of a $\mathbb{Q}$-complement. In [LXZ21] the bound is extended to $\frac{n+1}{n}$. We follow the idea of [LXZ21] here.

Definition 5.0.1. Let $\mathcal{F}$ be a filtration on R. A basis $(s_1, \dots, s_{N_\alpha})$ of R_α is said to be *compatible with* $\mathcal{F}$ if $\mathcal{F}^\lambda R_\alpha$ is spanned by some of the s_i's for every $\lambda \in \mathbb{R}_{\geq 0}$. An α-basis type divisor $B = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} (s_i = 0)$ is said to be *compatible with* $\mathcal{F}$ if $(s_1, \dots, s_{N_\alpha})$ is compatible with $\mathcal{F}$. By abuse of notation, we say that an α-basis type divisor B is compatible with a valuation v if B is compatible with the filtration induced by v on R.

Similarly, a basis $(s_1, \dots, s_{N_m})$ of R_m is said to be *compatible with* $\mathcal{F}$ if $\mathcal{F}^\lambda R_m$ is spanned by some of the s_i's for every $\lambda \in \mathbb{R}_{\geq 0}$. An m-basis type divisor $B = \frac{1}{N_m} \sum_{i=1}^{N_m} (s_i = 0)$ is said to be *compatible with* $\mathcal{F}$ if $(s_1, \dots, s_{N_m})$ is compatible with $\mathcal{F}$.

We recall a useful fact. The proof is the same as in [AZ20]

Lemma 5.0.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations on R. Then for any $\alpha \in \Lambda$ (resp. $m \in \mathbb{Z}_{>0}$), there exists an α-basis type divisor (resp. m-basis type divisor) that is compatible with both $\mathcal{F}$ and $\mathcal{G}$.*

We want to show that when $\delta(X, D, \xi_0) < \frac{n+1}{n}$, the valuation computing δ is an lc place of some $\mathbb{Q}$-complement, and that complement satisfies some further technical properties (which will called *special complement*).

Lemma 5.0.3. *Let (X, D, ξ_0) be a log Fano cone singularity of dimension n and $\delta(X, D, \xi_0) = \delta < \frac{n+1}{n}$. Let v be a T-equivariant valuation computing δ. Let $\alpha \in (0, \min\{\frac{\delta}{n+1}, 1 - \frac{n\delta}{n+1}\}) \cap \mathbb{Q}$. Then for any effective divisor $B \sim_{\mathbb{Q}} -(K_X + D)$, there exists some $\mathbb{Q}$-complement Γ of (X, D) such that $\Gamma \geq \alpha B$ and v is an lc place of $(X, D + \Gamma)$.*

Theorem 5.0.4. *Let (X, D, ξ_0) be a log Fano cone singularity of dimension n, with $\delta(X, D, \xi_0) < \frac{n+1}{n}$. Then,*

(1) there exists a T-equivariant valuation computing $\delta(X, D, \xi_0)$; and

(2) there exists a positive integer N depending only on $\dim(X), \xi_0$ and the coefficients of D such that for any T-equivariant valuation v computing $\delta(X, D, \xi_0)$, there exists an N-complement B of (X, D) which satisfies that v is an lc place of $(X, D + B)$.

Proof. We first show (1). Recall 2.1.4 we know that there exists some $u_0 \in M_{\mathbb{Q}}$ such that $K_X + D = \pi^* H + div(\chi^{-u_0})$ and H is a principal $\mathbb{Q}$-divisor. For any sufficiently divisible $m \in \mathbb{N}$, let $\delta_m := \delta_{mu_0}(X, D, \xi_0)$, and let E_m be a divisor over X such that $\frac{A_{(X,D)}(E_m)}{S_{mu_0}(E_m)} = \delta_m$. Fix a sufficiently large positive integer m_0 and let H_m be a smooth divisor $(f = 0)$ for some $f \in R_{m_0 u_0}$ that does not contain the center of E_m. For any such m, by 5.0.2 we can find some mu_0-basis type divisor B_m which is compatible with both E_m and H_m. We could write $B_m = \Gamma_m + a_m H_m$ where $\mathrm{Supp}(\Gamma_m)$ does not contain H_m. We notice that the coefficient a_m does not depend on the choice of H_m and moreover $\lim_{m\to\infty} a_m = \frac{1}{m_0(n+1)}$. Now we know

$$\mathrm{lct}(X, D; B_m) \leq \frac{A_{(X,D)}(E_m)}{\mathrm{ord}_{E_m}(B_m)} = \frac{A_{X,D}(E_m)}{S_m(E_m)} = \delta_m,$$

where $\operatorname{ord}_{E_m}(D_m) = S_m(E_m)$ comes from the B_m is compatible with E_m. By definition of δ_m we know that $\operatorname{lct}(X, D; B_m) \geq \delta_m$ and we know $\operatorname{lct}(X, D; B_m) = \delta_m$, and that the corresponding log canonical threshold is computed by E_m. Since H_m does not contain the center of E_m we know that $(X, D + \delta_m \Gamma_m)$ is still lc and E_m is an lc place of this pair.

Notice that $\lim_{m\to\infty} \delta_m = \delta(X, D, \xi_0) < \frac{n+1}{n}$. So for sufficiently large m we get

$$\delta_m \Gamma_m = \delta_m(B_m - a_m H_m) \sim_{\mathbb{Q}} -\lambda_m(K_X + D)$$

for $lambda = \delta_m(1 - m_0 a_m) \in (0, 1)$. Thus E_m is an lc place of a $\mathbb{Q}$-complement. The rest of the proof is the same as in [BLX19, Theorem 4.6]: we know that E_m is indeed an lc place of an N-complement for some N that only depends on $\dim(X), \xi_0$ and the coefficients of D. Therefore, after passing to a subsequence, we can find an N-complement B, together with lc places F_m of $(X, D + B)$, such that $\frac{A_{X,D}(E_m)}{S_{X,D}(E_m)} = \frac{A_{X,D}(F_m)}{S_{X,D}(F_m)}$ for all sufficiently divisible $m \in \mathbb{N}_{>0}$. If we take v to be the limit of $(A_{X,D}(F_m))^{-1} \operatorname{ord}_{F_m}$ in $\mathcal{DMR}(X, D + B)$ then v computes $\delta(X, D, \xi_0)$ as

$$\frac{A_{X,D}(v)}{S_{X,D}(v)} = \lim_{m\to\infty} \frac{A_{X,D}(F_m)}{S_{X,D}(F_m)} = \lim_{m\to\infty} \frac{A_{X,D}(E_m)}{S_{X,D}(E_m)} = \lim_{m\to\infty} = \delta(X, D).$$

For (2), we know from 5.0.3 there exists some $\mathbb{Q}$-complement Γ such that v is an lc place of Γ. There exists a log smooth model $(Y, E) \to (X, D+B)$ such that every component $E_i (i = 1, \ldots, M)$ of E is an lc place of $(X, D+\Gamma)$ and every prime divisor on Y with log discrepancy 0 with respect to $(X, D + \Gamma)$ is contained in E. So we know $v \in \operatorname{QM}(Y, E)$. By [BCHM10, Corollary 1.4.3], there exists a $\mathbb{Q}$-factorial birational model $\mu : \tilde{X} \to X$ that extracts exactly the divisors E_i and $Y \dashrightarrow \tilde{X}$ is isomorphic at the generic point of any component of all non-empty intersections of $\cap_{i\in I} E_i$ for $I \subset \{1, \ldots, M\}$. Let $a_i = \operatorname{coeff}_{E_i}(D)$ if E_i is a prime divisor on X, otherwise set $a_i = 0$. Then we can argue as in the proof opf [BLX19, Theorem 3.5]: $(\tilde{X}, \mu_*^{-1} D + \sum_{i=1}^{M} (1 - a_i) E_i)$ has a $\mathbb{Q}$-complement, therefore also has an N-complement, whose pushforward on X gives an

N-complement B of (X, D) that has all E_i as lc places. In particular, it also has v as an lc place. $\square$

Chapter 6

Finite generation

In this chapter we prove the Higer Rank Finite Generation Conjecture for log Fano cone singularities.

Theorem 6.0.1. *Let (X, D, ξ_0) be a log Fano cone singularity of dimension n, $X = \mathrm{Spec}(R)$. Assume that $\delta(X, D, \xi_0) < \frac{n+1}{n}$. Then for any valuation v that computes $\delta(X, D, \xi_0)$, the associated graded ring $\mathrm{gr}_{\mathcal{F}_v} R$ is finitely generated.*

6.1 Special complement

We follow the idea in [LXZ21]. We define the notion of special complement, and show that the existence of a special $\mathbb{Q}$-complement and an lc place v implies the finite generation of the associated graded ring $\mathrm{gr}_v R$.

Definition 6.1.1. Given a a log Fano cone (X, D, ξ_0) with T- action. A (T-equivariant) $\mathbb{Q}$-complement Γ of (X, D, ξ_0) is called *special complement* with respect to a T-equivariant log smooth model $\pi : (Y, E) \to (X, D)$ if $\Gamma_Y = \pi_*^{-1}\Gamma \geq G$ for some effective ample $\mathbb{Q}$-divisor G on Y whose support does not contain any stratum of (Y, E). Here a log smooth model means a log resolution

$\pi : Y \to (X, D)$ and a reduced divisor E on Y such that $E + Ex(\pi) + \pi_*^{-1} D$ has simple normal crossing support. Any valuation $v \in \mathrm{QM}(Y, E) \cap \mathcal{DMR}(X, D + \Gamma)$ is called a *monomial lc place* of the special $\mathbb{Q}$-complement Γ with respect to (Y, E).

Lemma 6.1.2. *Let (X, D, ξ_0) be a log Fano cone singularity of dimension n with $\delta(X, D, \xi_0) < \frac{n+1}{n}$. Let v be a T-invariant valuation computing δ. Then there exists a T-invariant log smooth model $\pi : (Y, E) \to (X, D)$ and a special $\mathbb{Q}$-complement $0 \le \Gamma \sim_{\mathbb{Q}} -(K_X + D)$ with respect to (Y, E), such that $v \in \mathrm{QM}(Y, E) \cap \mathcal{DMR}(X, D + \Gamma)$.*

Proof. Because v is quasi-monomial and T-invariant, we could find a T-invariant log smooth model $\pi : (Y, E) \to (X, D)$ whose exceptional locus supports a π-ample divisor F such that $v \in \mathrm{QM}(Y, E)$. Choose some small $\epsilon > 0$, set $L = -\pi^*(K_X + D) + \epsilon F$ and let G be a general divisor in the $\mathbb{Q}$-linear system $|L|_{\mathbb{Q}}$ whose support does not contain any stratum of (Y, E). Let $B = \pi_* G \sim -(K_X + D)$ and let $\alpha < \min\{\frac{\delta}{n+1}, 1 - \frac{n\delta}{n+1}\}$ be a fixed rational positive number. By 5.0.3, we have some T-invariant complement Γ of (X, D) such that $\Gamma \ge \alpha B$ and v is an lc place of $(X, D + \Gamma)$. Replace G by αG then Γ is indeed a special $\mathbb{Q}$-complement with respect to (Y, E). $\square$

Assume $\mathrm{gr}_v R$ is finitely generated for some v, we define $X_v := \mathrm{Proj}\, \mathrm{gr}_v R$ and D_v is the induced degeneration of D to X_v. More precisely, suppose $D = \sum_{i=1}^{l} a_i D_i$ where D_i is a prime divisor on X and $a_i \in \mathbb{Q}_{\ge 0}$. Let $I_{D_i} \subseteq R$ be the graded ideal of D_i. Let $\mathrm{in}(I_{D_i}) \subseteq \mathrm{gr}_v R$ be the initial ideal of I_{D_i}. Then $D_v := \sum_{i=1}^{l} a_i D_{v,i}$, where $D_{v,i}$ is the divisorial part of the closed subscheme $V(\mathrm{in}(I_{D_i})) \subseteq X_v$. So that $D_{v,i}$ and $V(\mathrm{in}(I_{D_i}))$ coincide away from a codimension 2 subset of X_v.

Theorem 6.1.3. *Let (X, D, ξ_0) be a log Fano cone singularity. Let v be a T-equivariant quasi-monomial valuation on X. The following are equivalent.*

(1) The associated graded ring $\mathrm{gr}_v R$ is finitely generated and the central fiber (X_v, D_v) of the induced degenetation is klt.

(2) The valuation v is a monomial lc place of a special $\mathbb{Q}$-complement Γ with respect to some T-equivariant log smooth model (Y, E).

Theorem 6.1.3 together with Lemma 6.1.2 immediately implies Theorem 6.0.1. The proof of the easier side of Theorem 6.1.3, i.e. (1) $\implies$ (2) is the same as in [LXZ21]. To show the harder side, we need

Theorem 6.1.4. *Let (X, D, ξ_0) be a log Fano cone singularity, and let $0 \le \Gamma \sim_{\mathbb{Q}} -(K_X + D)$ be a T-equivariant $\mathbb{Q}$-complement. Let v_0 be an lc place of $(X, D + \Gamma)$ and let $\Sigma \subseteq \mathcal{DMR}(X, D + \Gamma)$ be the minimal rational PL subspace containing v_0 induced by a fixed log smooth model of (X, D). Then the following are equivalent.*

(1) The associated graded ring $\mathrm{gr}_{v_0} R$ is finitely generated.

(2) There exists an open neighborhood $v_0 \in U \subseteq \Sigma$ such that the set

$$\{(X_v, D_v) \mid v \in U(\mathbb{Q}) := U \cap \Sigma(\mathbb{Q})\}$$

is bounded.

(3) The S-invariant function

$$v \mapsto S_{X,D}(v)$$

is linear on a neighborhood of v_0 in Σ.

6.2 Estimate of alpha invariants

By Lemma 6.1.2 and Theorem 6.1.4, to prove Theorem 6.1.3, we only need to show the boundedness of $\{(X_v, D_v) \mid v \in U(\mathbb{Q})\}$ for some open neighborhood $U \subseteq \Sigma$. This could be proven by showing a lower positive bound of the alpha invariants.

Theorem 6.2.1. *[Jia20] Fix positive integers n, C and three positive numbers V, α_0, δ_0. If we consider the set calP of all n-dimensional log Fano pairs $\{(X, D)\}$ such that $C \cdot D$ is integral, $(-K_X - D)^n = V$ and $\alpha(X, D) \ge \alpha_0$ (resp. $\delta(X, D) \ge \delta_0$), then $\mathcal{P}$ is bounded.*

Notice that this boundedness property is independent from the choice of ξ_0, so we could assume ξ_0 is rational, and take the quotient $((X, D) - \{x\})/\langle \xi_0 \rangle$. We cannot guarantee a simple normal crossing pair, so we need to generalize the calculation in [LXZ21] to the toroidal case.

Theorem 6.2.2. *Let (X, D) be a log Fano pair. Let Γ be a special complement with respect to a resolution $\pi : (Y, E) \to (X, D)$. Let $K \subset \mathcal{DMR}(X, D + \Gamma)$ be a compact subset that is contained in the interior of a simplicial cone in $\mathrm{QM}(Y, E)$. Then there exists some constant $\alpha_0 > 0$ such that for all rational points $v \in K$, the alpha invariant $\alpha(X_v, D_v)$ of the induced degenerations (X_v, D_v) is bounded from below by α_0.*

Lemma 6.2.3. *Let v be a divisorial valuation such that $\mathrm{gr}_v R$ is finitely generated and let $\alpha \in (0, 1)$ be a rational number. Then $\alpha(X_v, D_v) \geq \alpha$ if and only if for all $0 \leq B \sim_{\mathbb{Q}} -(K_X + D)$, there exists some $0 \leq B' \sim_{\mathbb{Q}} -(K_X + D)$ such that $(X, D + \alpha B + (1 - \alpha)B')$ is lc and have v as an lc place.*

We call such B' an (α, v)-*complement* of B.

Proof. We have a $\mathbb{G}_m$-action on (X_v, D_v). Taking the limit under the $\mathbb{G}_m$-action we see that any effective divisor $G \sim_{\mathbb{Q}} -(K_{X_v} + D_v)$ degenerates to some $\mathbb{G}_m$-invariant divisor G_0. Using the semi-continuity of log canonical threshold we have $\mathrm{lct}(X_v, D_v; G) \geq \mathrm{lct}(X_v, D_v; G_0)$, and so $\alpha(X_v, D_v) \geq \alpha$ if and only if $\mathrm{lct}(X_v, D_v; G_0) \geq \alpha$ for all $\mathbb{G}_m$-invariant divisors $G_0 \sim_{\mathbb{Q}} -(K_{X_v} + D_v)$. Any such G_0 is also the specialization of some divisor $0 \leq D \sim_{\mathbb{Q}} -(K_X + D)$ on X, and $\mathrm{lct}(X_v, D_v; G_0) \geq \alpha$ means that v induces a weakly special degeneration of $(X, D + \alpha B)$. By [BLX19, Theorem 3.5], this is equivalent to say, for all sufficiently small $\epsilon \in \mathbb{Q}$, the valuation v is an lc place of a $\mathbb{Q}$-complement of the klt pair $(X, D + (\alpha - \epsilon)B)$, so that B has an $(\alpha - \epsilon, v)$-complement.

It suffices to show this is equivalent to say B has an (α, v)-complement. We could write $v = c \cdot \mathrm{ord}_E$. Because E is an lc place of a $\mathbb{Q}$-complement, by [BCHM10], there exists a birational model $\pi : Y \to X$ that extracts E as the only exceptional divisor, and Y is of Fano type. Moreover, if follows from the existence of $(\alpha-\epsilon, v)$-complement that the pair $(Y, \pi_*^{-1}(D+(\alpha-\epsilon)B)\vee E)$ has a $\mathbb{Q}$-complement for all sufficiently small ϵ. By [HMX14] this implies that $(Y, \pi_*^{-1}(D+\alpha B)\vee E)$ also has a $\mathbb{Q}$-complement, and the pushforward of X is an (α, v)-complement of D. $\square$

Next we want to construct the (α, v)-complements for some uniform constant α. We fix an effective ample $\mathbb{Q}$-divisor G on Y that does not contain any stratum of E and that $\Gamma_Y \geq G$. For any divisorial valuation $v \in \mathcal{DMR}(X, D+\Gamma)\cap \mathrm{QM}(Y,E)$, let $\mu : Z \to Y$ be the corresponding weighted blowup, F the exceptional divisor and (Z, D_Z), (Y, D_Y) to be the crepant pullbacks. Let $D^+ = D_Z \vee 0 \vee F$. Notice that (Z, D^+) is plt. By adjunction we have $K_F + \Phi = (K_Z + D^+)|_F$. Set

$$L := \mu^*\pi^*(K_X + D) - A_{X,D}(F)\cdot F.$$

Since $v = c\cdot \mathrm{ord}_F$ is an lc place of $(X, D+\Gamma)$, and F is not contained in the support of $\mu^*\pi^*\Gamma - A_{X,D}(F)\cdot F \sim_{\mathbb{Q}} L$, so the $\mathbb{Q}$-linear system $|L|_{\mathbb{Q}} \neq \emptyset$ and we define

$$\alpha_v := \mathrm{lct}(F, \Phi; |L_F|_{\mathbb{Q}}).$$

$$\epsilon_v := \sup\{t \geq 0 \mid \mu^*G - tA_{X,D}(F)\cdot F \text{ is nef}\}.$$

We have $\epsilon_v > 0$ because $-F$ is μ-ample, and for any $0 < t < \epsilon_v$ the divisor $\mu^*G - tA_{X,D}(F)F$ is ample.

Lemma 6.2.4. *Given constants $a, b > 0$, there exists some constant $\alpha > 0$ depending only on $a, b, (X, D), \Gamma$ such that $\alpha(X_v, D_v) \geq \alpha$ if $\alpha_v > a$ and $\epsilon_v > b$.*

Proof. According to Lemma 6.2.3, it suffices to find some constant $\alpha > 0$ such that (α, v)-complement exists for any effective divisor $B \sim_{\mathbb{Q}} -(K_X + D)$.

We claim it suffices to find an (α, v)-complement for divisors B such that $v(B) = A_{X,D}(v)$. If so, take a sufficiently small $\epsilon > 0$ such that $G + \epsilon\, \pi^*(K_X + D)$ is ample. Then $T(G; v) \geq \epsilon \cdot T_{X,D}(v)$, and that $T_{X,D}(v) = T(\pi^*\Gamma; v) \geq v(\pi^*\Gamma - G) + T(G; v) \geq v(\Gamma) + \epsilon \cdot T_{X,D}(v) = A_{X,D}()v + \epsilon T_{X,D}(v)$. So $(1-\epsilon)T_{X,D}(v) \geq A_{X,D}(v)$. Notice that by definition of α invariant, we know $\alpha(X, D)T_{X,D}(v) \leq A_{X,D}(v)$.

Since $0 < \alpha \leq \frac{A}{T} \leq 1 - \epsilon$, so we could find some $\lambda \in (0, 1)$ that only depends on ϵ and $\alpha(X, D)$, such that for any $0 \leq p \leq T$, we can find some $0 \leq q < T$ and some $r > \lambda$, such that $rp + (1-r)q = A$. Therefore we could find some constant $0 < \lambda < 1$ that depending only on ϵ and $\alpha(X, D)$ such that for any effective divisor $B \sim_{\mathbb{Q}} -(K_X + D)$, there always exists an effective divisor $B' \sim_{\mathbb{Q}} -(K_X + D)$ and $r \geq \lambda$ such that $rv(B) + (1-r)v(B') = A_{X,D}(v)$.

If an (α, v)-complement exists for $rB + (1-r)B'$, then $(\alpha\lambda, v)$-complement exists for B. Therefore we proved the claim.

Now fix a sufficiently small $t > 0$, and set $s := (1-a)t/(1-t) < b$, then we know that $\mu^*G - sA_{X,D}(F) \cdot F$ is ample. Fix an effective divisor $B \sim_{\mathbb{Q}} -(K_X + D)$ with $v(B) = A_{X,D}(v)$. Let H' be a general member of the $\mathbb{Q}$-linear system $|\mu^*G - sA_{X,D}(F)F|_{\mathbb{Q}}$, and let $H = \mu_*H'$.

We now show that the pair $(Y, D_Y + a\pi^*B + \frac{1-t}{t}H)$ is lc along $\mu(F)$ and has F as its unique lc place.

Notice that $A_{Y,D_Y}(F) - \mathrm{ord}_F(a\pi^*B + \frac{1-t}{t}H) = A_{X,D}(F) - aA_{X,D}(F) - (1-a)A_{X,D}(F) = 0$. Let $B' = \mu^*\pi^*B - \mathrm{ord}_F(B)F = \mu^*\pi^*B - A_{X,D}(F)F \sim_{\mathbb{Q}} L$. Since $(F, \Phi + aB'|_F)$ is kle and so we know (because H is general) $(F, \Phi + aB'|_F + \frac{1-t}{t}H'|_F)$ is also klt. By inversion of adjunction, we know $(Z, D^+ + aB' + \frac{1-t}{t}H')$ is plt along F. Since $D^+ \geq D_Z \vee F$, we deduce that $(Z, D_Z \vee F + aB' + \frac{1-t}{t}H')$ is also plt along F. Then we know

$$K_Z + D_Z \vee F + aB' + \frac{1-t}{t}H' = \mu'(K_Y + D_Y + a\pi^*B + \frac{1-t}{t}H),$$

so that $Y, D_Y + a\pi^*B + \frac{1-t}{t}H$ is lc along $\mu(F)$ and F is the only lc place.

Similarly we know

$$(Y, D_Y + t(a\pi^*B + \frac{1-t}{t}H) + (1-t)(\pi^*\Gamma - G)) = (Y, D_Y + at\pi^*B + (1-t)(\pi^*\Gamma - G + H))$$

is lc along $\mu(F)$ and F is the only lc place of the pair in a neighborhood of $\mu(F)$. So that $\mu(F)$ is a connected component of the non-klt locus of the pair. Since $K_Y + D_Y + at\pi^*B + (1-t)(\pi^*\Gamma - G + H) = \pi^*(K_X + D + atB + (1-t)(\Gamma - \pi_*G + \pi^*H)$, so $(X, D + atB + (1-t)(\pi^*\Gamma = \pi_*G + \pi_*H))$ is lc along $\pi(\mu(F))$. By Kollár-Shokurov connectedness theorem, we know $\pi(\mu(F))$ is a connected component of its non-klt locus.

Similarly we know

$$-(K_X + D + atB + (1-t)(\Gamma - \pi_*G + \pi_*H)) \sim_{\mathbb{Q}} -(1-a)t(K_X + D)$$

is also ample. So we know $(X, D + atB + (1-t)(\Gamma - \pi_*G + \pi_*H))$ is lc everywhere by Kollár-Shokurov connectedness theorem. Notice that $v = c \cdot \mathrm{ord}_F$ is an lc place of $(X, D + atB + (1-t)(\Gamma - \pi_*G + \pi_*H))$, so we could add some effective general divisor $B' \sim_{\mathbb{Q}} -(1-a)t(K_X + D)$ to the pair and so that B has an (at, v)-complement. Here t only depends on a, b.

□

Lemma 6.2.5. *Use the same notation as above. Let $K \subseteq \mathcal{DMR}(X, D + \Gamma)$ be a compact subset contained in the interior of some simplicial cone in* $\mathrm{QM}(Y, E)$. *Then there exists some constants $a > 0$ such that $\alpha_v > a$ for all divisorial valuations $v \in K$.*

Proof. Let $E_i (1 \le i \le r)$ be the irreducible components of E, and that $W = \cap_{i=1}^r E_i$ is the common center of valuations in K on Y. Any divisorial valuation $v \in K$ corresponds to a weighted blowup at W with weight $\mathrm{wt}(E_i) = a_i$ for some $a_i \in \mathbb{N}_{>0}$, and we could assume $\gcd(a_i) = 1$. Notice that K is compact, so we could find some constant $C > 0$ such that $\frac{a_i}{a_j} < C$ for all $1 \le i, j \le r$.

In an open neighborhood of a point $x \in W$, if E_i is given by $(e_i = 0)$, we set $\mathcal{I}_d$ generated by monomials $e_1^{d_1}\dots e_r^{d_r}$ such that $\sum_i a_i d_i \geq d$. Then the weighted blowup is given by $\mathrm{Proj}_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus \mathcal{I}_1 \oplus \dots)$. The exceptional divisor F is a weighted projective space bundle over W with fiber F_0 isomorphic to $(\mathbb{A}^r - \{0\})/\mathbb{G}_m$ with the action $\lambda \cdot (y_1, \dots, y_r) = (\lambda^{a_1} y_1, \dots, \lambda^{a_r} y_r)$. Let $q_i := \gcd(a_1, \dots, \hat{a_i}, \dots, a_r)$, and $q = q_1 \dots q_r$, $a_i' = \frac{a_i q_i}{q}$. Then $F_0 \cong \mathbb{P}(a_1', \dots, a_r')$.

Let $c_i = A_{X,D}(E_i) > 0, b_i = \max\{0, \mathrm{ord}_{E_i}(D_Y)\} < 1$, then we have

$$A_{X,D}(F) = \sum_{i=1}^r a_i A_{X,D}(E_i) = a_1 c_1 + \dots + a_r c_r.$$

Let $L_{F_0} := L|_{F_0} \sim_{\mathbb{Q}} \frac{A_{X,D}(F)}{q} L_0$ where L_0 is the class of $\mathcal{O}(1)$ on $\mathbb{P}(a_1', \dots, a_r')$. We define $\Phi_{F_0} = \Phi|_{F_0} = \sum_{i=1}^r \frac{q_i - 1 + b_i}{q_i}\{x_i = 0\}$, where x_i are the weighted homogeneous coordinates on $\mathbb{P}(a_1', \dots, a_r')$. Let

$$\mathfrak{b}_m := \mu_* \mathcal{O}_Z(-mF)/\mu_* \mathcal{O}_Z(-(m+1)F) \cong \bigoplus \mathcal{O}_W(-\sum_{i=1}^r m_i E_i),$$

where the direct sum runs over all $(m_1, \dots, m_r) \in \mathbb{N}^r$ such that $\sum_{i=1}^r a_i m_i = m$.

For any $m \in \mathbb{N}$, such that mL is Cartier, we have

$$\mu_* \mathcal{O}_F(mL) \cong \mathcal{O}_Y(-m\pi^*(K_X + D)) \otimes \mu_* \mathcal{O}_Z(-m A_{X,D}(F) F)/\mu_* \mathcal{O}_Z(-(m A_{X,D}(F) + 1)F),$$

so

$$\mu_* \mathcal{O}_F(mL) \cong \bigoplus_{\sum_{i=1}^r a_i m_i = m \sum_{i=1}^r a_i c_i} \mathcal{O}_W(-m\pi^*(K_X + D) - (m_1 E_1 + \dots m_r E_r)).$$

Take $C' = \lceil C \sum_{i=1}^r \rceil$, then $\sum_{i=1}^r m_i \leq C' m$. Take a very ample line bundle H_0 such that $H_0 + E_i$ are very ample for all $1 \leq i \leq r$, and $H_0 + \pi^*(K_X + D)$ is ample, then for sufficiently divisible m, we have the inclusion $\mathcal{O}_W(-m\pi^*(K_X + D) - (m_1 E_1 + \dots m_r E_r) \hookrightarrow \mathcal{O}_W((m + \sum_{i=1}^r m_i) H_0) \hookrightarrow \mathcal{O}_W((C' + 1) m H_0)$ for each direct summand in $\mu^* \mathcal{O}_F(mL)$. Therefore for $H = (C' + 1) H_0$, and sufficiently divisible m we have

$$\mu_*\mathcal{O}_F(mL) \hookrightarrow \mathcal{O}_W(mH)^{\oplus N_m}$$

for some $N_m = \mathrm{rank}(\mathfrak{b}_{mA_{X,D}(F)})$.

Notice that F_0 is toric, therefore $\mathrm{lct}(F_0, \Phi_{F_0}; |L_{F_0}|_{\mathbb{Q}})$ is computed by one of torus invariant divisors $\{x_i = 0\}$, so that

$$\mathrm{lct}(F_0, \Phi_{F_0}; |L_{F_0}|_{\mathbb{Q}}) = \frac{\min_{1\le i\le r} a_i(1-b_i)}{\sum_{i=1}^r a_i c_i} \ge a$$

for some constants $a > 0$ depending only on b_i, c_i and C. Set $D_W := (D_Y \vee 0 - \sum_{i=1}^r b_i E_i)|_W$. By Izumi's inequality we have $\mathrm{lct}(W, D_W; |H|_{\mathbb{Q}}) > 0$. So we may assume $lct(W, D_W; |H|_{\mathbb{Q}}) \ge a$ by replacing a by a smaller positive number (notice that a does not depend on (W, D_w)).

Let $0 < t < a$, and $\Phi' \sim_{\mathbb{Q}} L|_F$ be an effective divisor. We claim that $(F, \Phi + t\Phi')$ is lc. Suppose not, then we could find some divisorial valuation v_0 over F such that $A_{F,\Phi+t\Phi'}(v_0) < 0$ and the center of v_0 does not dominate W. Now v_0 restricts to a divisorial valuation w on W.

Consider the birational morphism $g : W_1 \to W$ such that the center of w is a divisor Q on W_1, and let $F_1 = F \times_W W_1$, $\Phi_1 = g^*(\Phi - \mu^* D_W)$, and let P be the preimage of Q in F_1. Notice that $F \to W$ is locally a trivial product $F_0 \times W$, by projection formula, we see

$$H^0(F_1, \mathcal{O}_{F_1}(g^*mL - kP)) = H^0(W_1, g^*\mu_*\mathcal{O}_F(mL) \otimes \mathcal{O}_{W_1}(-kQ)).$$

For sufficiently divisible m we have $\mu_*\mathcal{O}_F(mL) \hookrightarrow \mathcal{O}_W(mH)^{\oplus N_m}$, so that $H^0(F_1, \mathcal{O}_{F_1}(g^*mL - kP))neq0 \implies H^0(W_1, \mathcal{O}_{W_1}(mg^*H - kQ)) \neq 0$ for any $k \in \mathbb{N}$. So $\mathrm{ord}_P(\Phi') \le \sup_{H' \in |H|_{\mathbb{Q}}} \mathrm{ord}_Q(H')$. Notice $\sup_{H' \in |H|_{\mathbb{Q}}} \mathrm{ord}_Q(H') \le \frac{1}{a} A_{W,D_W}(Q) = A_{F,\Phi}(P)$ (because $lct(W, D_W; |H|_{\mathbb{Q}}) \ge a$), so we have $t\,\mathrm{ord}_P(\Phi') < A_{F,\Phi}(P)$ (remember $t < a$). If we write $g^*(K_F + \Phi + t\Phi') = K_{F_1} + \Phi_1 + \lambda P + D$ where $P \not\subset \mathrm{Supp}(D)$ then the coefficient $\lambda \le 1$.

Notice the divisor P is vertical, so over a general fiber of $P \to Q$, we have $D|_{F_0} \sim_{\mathbb{Q}} tg^*\Phi'|_{F_0} \sim_{\mathbb{Q}} tL|_{F_0}$. So that $(P, (\Phi_1+D)|_P)$ is lc along the general fibers of $P \to Q$. So by inversion of adjunction, we know $(F_1, \Phi_1 + \lambda P + Q)$ is also lc along the general fibers of $P \to Q$. So it is lc at the center of v_0. This is a contradiction. So $(F, \Phi + t\Phi')$ is lc and so $\alpha_v \geq a$ as we want.

□

Lemma 6.2.6. *Use the same notation as above. Let $K \subseteq \mathcal{DMR}(X, D+\Gamma)$ be a compact subset contained in the interior of some simplicial cone in* $\mathrm{QM}(Y, E)$*. Then there exists some constants $b > 0$ such that $\epsilon_v > b$ for all divisorial valuations $v \in K$.*

Proof. Follow the notation from above, we set $\mathfrak{a}_m := \mu_*\mathcal{O}_Z(-mA_{X,D}(F)F)$. Remember $a_i/a_j < C$ for all $1 \leq i, j \leq r$, there exists some constant $M \in \mathbb{N}$ such that $\frac{1}{A_{X,D}(F)}\operatorname{ord}_F(f) \geq \frac{1}{M}\mathrm{mult}_W(f)$ for all regular function f around the generic point of W. So that $\mathcal{I}_W^{Mm} \subseteq \mathfrak{a}_m$ for all $m \in \mathbb{N}$.

As in [LXZ21, Claim 4.15], we can find a sequence of ideals $\mathcal{O}_Y \supseteq \mathcal{I}_W \supseteq \cdots \supseteq \mathfrak{a}_m \supseteq \cdots \supseteq \mathcal{I}_W^{Mm}$ on Y such that the quotients of consecutive terms are all isomorphic to $\mathcal{O}_W(-n_1E_1 - \cdots - n_rE_r)$ for some $(n_1 \ldots, n_r) \in \mathbb{N}^r$ with $\sum_{i=1}^r n_i < Mm$.

Now we choose some sufficiently large and divisible integer $m_0, p > 0$ such that:

(1) the line bundles $\frac{p}{M}G$, $(\frac{p}{m}G - E_i)|_W$ are globally generated for all i,

(2) $H^i(W, \mathcal{O}_W(mpGn\sum_i E_i)) = 0$ for all $i, m \in \mathbb{N}_+$ and all $(n_1, \ldots, n_r) \in \mathbb{N}$ with $\sum_i n_i \leq Mm$,

(3) $\mathcal{O}_Y(mpG) \otimes \mathcal{I}_W^{Mm}$ is globally generated and $H^j(Y, \mathcal{O}_Y(mpG) \otimes \mathcal{I}_W^{Mm}) = 0$ for $m \geq m_0$, and $j \in \mathbb{N}_+$.

Let $\mathcal{I}_1 \supseteq \mathcal{I}_2$ be two consecutive terms in the above filtration, then we have the exact sequence

$$0 \to \mathcal{O}_Y(mpG) \otimes \mathcal{I}_2 \to \mathcal{O}_Y(mpG) \otimes \mathcal{I}_1 \to \mathcal{O}_W(mpG|_W) \otimes (\mathcal{I}_1/\mathcal{I}_2) \to 0.$$

Because $(\mathcal{I}_1/\mathcal{I}_2) \cong \mathcal{O}_W(-\sum_i n_iE_i)$ for some $(n_1, \ldots, n_r) \in \mathbb{N}^r$ and $\sum_i n_i \leq Mm$, so we have $H^i(W, \mathcal{O}_W(mpG|_W) \otimes (\mathcal{I}_1/\mathcal{I}_2)) = 0$ for all $i > 0$. So $H^i(Y, \mathcal{O}_Y(mpG|_W) \otimes \mathcal{I}_2) = 0$ for $i > 0$ implies $H^i(Y, \mathcal{O}_Y(mpG|_W) \otimes \mathcal{I}_1) = 0$ for $i > 0$. Now we know $\mathcal{O}_Y(mpG) \otimes \mathcal{I}_2$ is globally generated.

So by induction, we know that $\mathcal{O}_Y(mpG) \otimes \mathfrak{a}_m$ is globally generated for all $m \geq m_0$. So that $p\mu^*G - A_{X,D}(F)F$ is nef. Therefore we know $\epsilon_v > 1/p$, and p does not depend on the valuation v. So we are done. □

Now Theorem 6.2.2 follows from Lemma 6.2.4, Lemma 6.2.5 and Lemma 6.2.6. So we finished the proof of Theorem 6.1.3, and therefore the proof of Theorem 6.0.1.

Chapter 7

Applications

In this chapter we present some applications of the finite generation result.

Theorem 7.0.1 (Optimal Destabilization Conjecture)**.** *Let (X, D, ξ_0) be a log Fano cone singularity of dimension n. Assume that $\delta(X, D, \xi_0) < \frac{n+1}{n}$. Then $\delta(X, D, \xi_0) \in \mathbb{Q}$ and there exists a T-equivariant divisorial valuation ord_E over X that computes $\delta(X, D, \xi_0)$.*

Proof. Let v be a T-equivariant valuation on X that computes $\delta(X, D, \xi_0)$. By Lemma 6.1.2, there exists some T-invariant complement Γ of (X, D) such that $v \in \mathcal{DMR}(X, D + \Gamma)$. Let $\Sigma \subseteq \mathcal{DMR}(X, D + \Gamma)$ be the smallest rational PL subspace containing v. By Theorem 6.1.4, we know the S-invariant function $w \mapsto S_{X,D}(w)$ on Σ is linear in a neighborhood of v. Since v computes $\delta(X, D, \xi_0)$, we know

$$A_{X,D}(v) = \delta(X, D, \xi_0) S_{X,D}(v).$$

Since the log discrepancy function $w \mapsto A_{X,D}(w)$ is linear in a neighborhood of $v \in \Sigma$ and by the definition of δ we know

$$A_{X,D}(w) \geq \delta(X, D, \xi_0) S_{X,D}(w)$$

for all $w \in \Sigma$, we know that

$$A_{X,D}(w) = \delta(X, D, \xi_0) S_{X,D}(w)$$

in a neighborhood $U \subseteq \Sigma$ of v. So we know any divisorial valuation $w \in U(\mathbb{Q})$ also computes $\delta(X, D)$. Since w is a divisorial lc place of a complement, it induces a weakly special test configuration of (X, D). By our calculation of β invariant in section 3, we could see that $\beta(X, D, \xi_0) \in \mathbb{Q}$. Notice that $A_{X,D}(w)$ is rational, so $\delta(X, D, \xi_0) \in \mathbb{Q}$. □

Theorem 7.0.2 (Yau-Tian-Donaldson Conjecture). *A log Fano cone singularity (X, D, ξ_0) is K-polystable if and only if it is uniformly K-stable. Furthermore, A log Fano cone singularity (X, D, ξ_0) admits a weak Ricci-flat Kähler potential if and only if it is K-polystable.*

Proof. Suppose that (X, D, ξ_0) is K-polystable. Let $\mathbb{T} \subseteq \mathrm{Aut}(X, D)$ be a maximal torus (one could assume $T = \mathbb{T}$). We show that $\delta_{\mathbb{T}} > 1$. Suppose not, then we know $\delta_{\mathbb{T}} = 1$ and that $\delta(X, D, \xi_0)$ is computed by some $\mathbb{T}$-invariant quasi-monomial valuation v that is not of the form wt_ξ for any $\xi \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, and v is an lc place of a complement. Let $m \in \mathbb{N}$ be sufficiently divisible and consider the $\mathbb{T}-$invariant linear system

$$\mathcal{M} := \{s \in H^0(-m(K_X + D)) \mid v(s) \geq m \cdot A_{X,D}(v)\}.$$

Let $B_0 \in |\mathcal{M}|$ be a general member and let $B = \frac{1}{m} B_0$. Then $(X, D + \frac{1}{m}\mathcal{M})$ has the same set of lc places as $(X, D + B)$ and so we know $v \in \mathcal{DMR}(X, D + B)$. Notice $\mathbb{T}$ is a connected algebraic group, every lc place of the $\mathbb{T}$-invariant pair $(X, D + \frac{1}{m})$ is automatically $\mathbb{T}$-invariant. So we know $\mathcal{DMR}(X, D + B)$ consists only of $\mathbb{T}$-invariant valuations.

By the same argument as in the proof of Theorem 7.0.1, we see that $\delta(X, D, \xi_0)$ is also computed by some divisorial valuations $w \in \mathcal{DMR}(X, D + B)$ that are sufficiently close to v. Because w is $\mathbb{T}$-invariant, we know w induces a $\mathbb{T}$-equivariant special test configuration $(\mathcal{X}, \mathcal{D})$ of (X, D) with $\mathrm{Fut}(\mathcal{X}, \mathcal{D}) = 0$. Notice $\mathbb{T} \subseteq \mathrm{Aut}(X, D)$ is a maximal torus and $w \neq \mathrm{wt}_\xi$ for any $\xi \in$

$\mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, we know that $(\mathcal{X}, \mathcal{D})$ is not a product test configuration. This contradicts with the assumption that (X, D, ξ_0) is K-polystable. Therefore we show $\delta(X, D, \xi_0) > 1$ and (X, D, ξ_0) is uniformly K-stable. When the ground field is $\mathbb{C}$, the existence of a weak Ricci-flat Kähler potential follows from this equivalence and [HL20]. $\square$

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