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When Efficiency meets Equity in Congestion Pricing and Revenue Refunding Schemes

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Abstract

Congestion pricing has long been hailed as a means to mitigate traffic congestion; however, its practical adoption has been limited due to the resulting social inequity issue, e.g., low-income users are priced out of certain roads. This issue has spurred interest in the design of equitable mechanisms that aim to refund the collected toll revenues as lump-sum transfers to users. Although revenue refunding has been extensively studied for over three decades, there has been no thorough characterization of how such schemes can be designed to simultaneously achieve system efficiency and equity objectives.

In this work, we bridge this gap through the study of *congestion pricing and revenue refunding* (CPRR) schemes in non-atomic congestion games. We first develop CPRR schemes, which in comparison to the untolled case, simultaneously (i) increase system efficiency and (ii) decrease wealth inequality, while being (iii) *user-favorable*: irrespective of their initial wealth or values-of-time (which may differ across users) users would experience a lower travel cost after the implementation of the proposed scheme. We then characterize the set of optimal user-favorable CPRR schemes that simultaneously maximize system efficiency and minimize wealth inequality. These results assume a well-studied behavior model of users minimizing a linear function of their travel times and tolls, without considering refunds. We also study a more complex behavior model wherein users are influenced by and react to the amount of refund that they receive. Although, in general, the two models can result in different outcomes in terms of system efficiency and wealth inequality, we establish that those outcomes coincide when the aforementioned optimal CPRR scheme is implemented. Overall, our work demonstrates that through appropriate refunding policies we can achieve system efficiency while reducing wealth inequality.

1 Introduction

The study of road congestion pricing is central to transportation economics and traces back to 1920 with the seminal work of Pigou [1]. Since then, the marginal cost pricing of roads, where users pay for the externalities they impose on others, has been widely accepted as a mechanism to alleviate traffic congestion. In particular, congestion pricing can be used to steer users away from the user equilibrium (UE) traffic pattern, which forms when users selfishly minimize their own travel times [2], towards the system optimal (SO) traffic pattern [3]. Despite the system-wide benefits of congestion pricing, its practical adoption has been limited [4]. A primary driving force behind the public opposition to congestion pricing has been the resultant inequity, e.g., high income users are likely to get the most benefit with shorter travel times while low income users suffer exceedingly large travel times since they avoid the high toll roads. Several empirical works have noted the regressive nature of congestion pricing [5, 6] and a recent theoretical work [7] has characterized the influence of road tolls on the Gini coefficient, a measure of wealth inequality. Most notably, the latter paper [7] developed an *Inequity Theorem* for users travelling between the same origin-destination (O-D) pair, and proved that any form of road tolls increases wealth inequality. These rigorous critiques are mirrored by opinions expressed in the popular press that congestion fees amount to “a tax on the working class [8].”

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The lack of support for congestion pricing due to its social inequity issues [9, 10] has led to a growing interest in the design of congestion-pricing schemes that are more equitable [11]. One approach that has been proposed recently to alleviate the inequity issues of congestion pricing is direct revenue redistribution, i.e., refunding the toll revenues to users in the form of lump-sum transfers. We note that the idea of revenue refunding is analogous to that of feebates¹, where refunds are used as a means to induce desirable behavior in society. Our work is centered on the design of congestion pricing and revenue refunding (CPRR) schemes that improve system performance, reduce wealth inequality, and benefit every user irrespective of their wealth or value-of-time. We view our work as paving the way for the design of practical, sustainable, and publicly acceptable congestion pricing schemes.

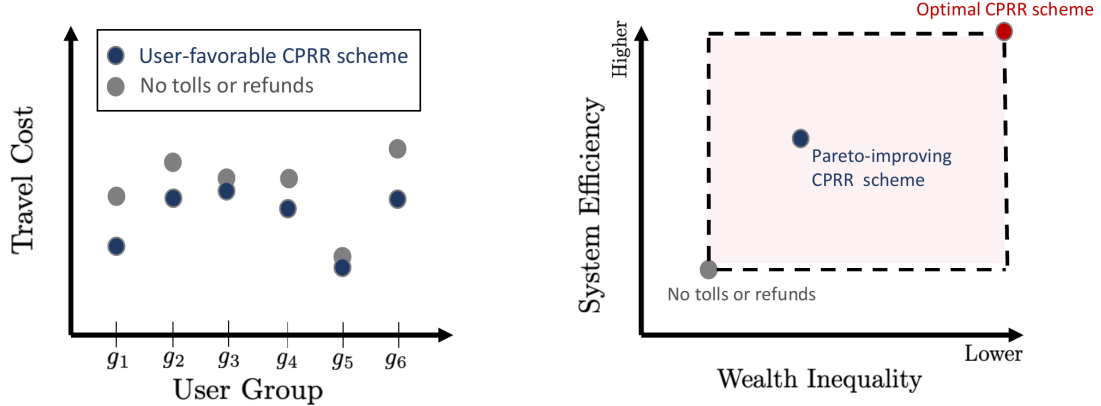


Figure 1: We develop CPRR schemes that improve both system efficiency and wealth inequality, while being *favorable* to all users. On the left we illustrate the concept of a user-favorable CPRR scheme, where every user group, irrespective of their initial income, has a lower travel cost after the implementation of the scheme compared with the untolled setting. Note that users in a given group have the same value-of-time, income, and O-D pair. On the right we illustrate the concept of *Pareto-improving* user-favorable CPRR schemes, which increase system efficiency, and decrease wealth inequality, compared with the untolled case (the pink region represents the set of all Pareto-improving outcomes of user-favorable schemes). We also illustrate an *optimal* user-favorable CPRR scheme, which simultaneously achieves the highest system efficiency and the lowest wealth inequality.

Contributions In this work, we present the first study of the wealth-inequality effects of CPRR schemes in non-atomic congestion games, with a specific focus on devising CPRR schemes that simultaneously reduce the total system cost, i.e., the sum of the travel times on all edges of the network weighted by the corresponding values-of-time, without increasing the level of wealth inequality. We consider the setting of heterogeneous users, with differing values-of-time and income, who seek to minimize their individual travel cost in the system. As in previous work [7], we incorporate the income elasticity of travel time, i.e., the lost income due to a loss of time, to reason about the income distribution of users before and after the imposition of a CPRR scheme.

To capture the behavior of selfish users, we study the effect of the Nash equilibria induced by CPRR schemes on the level of wealth inequality in society for non-atomic congestion games. We consider two notions of equilibrium formation: (i) exogenous equilibrium, wherein users minimize a linear function of their travel time and tolls, without considering refunds, as in [12], and (ii) endogenous equilibrium, an equilibrium notion we introduce, wherein users additionally consider refunds in their travel cost minimization. For these two notions of equilibria, our contributions are three-fold:

1. We develop CPRR schemes that improve both system efficiency and wealth inequality, while being *favorable* to all users. Under the exogenous equilibrium model, we establish the existence of a CPRR

¹<https://theicct.org/spotlight/feebate-systems>

scheme that, compared with the untolled outcome, (i) is user-favorable, i.e., every user group, irrespective of their initial wealth, has a lower travel cost after the implementation of the scheme, (ii) lowers total system cost, and (iii) decreases the wealth inequality (see Figure 1). We call such CPRR schemes *Pareto improving*. For the special case when all travel demand is between a single O-D pair and each user’s value-of-time is proportional to their income, we further show that the same CPRR scheme reduces wealth inequality relative to the ex-ante income distribution, i.e., the income profiles of users prior to the making their trips. In particular, our results show that it is possible to reverse the wealth-inequality effects of congestion pricing that were established in the *Inequity Theorem* [7] through appropriate revenue refunding schemes.

2. *We characterize the set of optimal CPRR schemes that are favorable to all users.* In particular, we establish in the exogenous equilibrium setting that the optimal CPRR schemes are those that simultaneously minimize total system cost and level of wealth inequality among all CPRR schemes that are favorable to any user (see Figure 1). To establish this claim, we (i) develop an explicit characterization of the income distribution of users after the completion of their trip, and (ii) prove a monotonic relationship between the minimum achievable level of wealth-inequality and the total system cost. Furthermore, when the discrete Gini coefficient is used as a measure of wealth inequality, we show that a simple max-min allocation of the refunds among user groups with different levels of income will be optimal.
3. *Finally, we show that even when users endogenize the effect of refunds on their travel decisions, the resultant equilibrium remains the same under any, aforementioned, optimal revenue refunding scheme.* We first present an example to illustrate that the endogenous equilibrium need not always coincide with the exogenous equilibrium. This is because, under the endogenous equilibrium model, users may have an incentive to deviate to increase their refund amount, and thereby decrease their travel costs. While the two equilibrium notions do not agree in general, we then show that any exogenous equilibrium induced by an optimal CPRR scheme is also an endogenous equilibrium.

Our work demonstrates that if we utilize the collected toll revenues to devise appropriate refunding policies then we can achieve system efficiency whilst also progressing towards reduced inequality. Furthermore, in doing so, we ensure that our designed schemes are publicly acceptable since we guarantee that each user is at least as well off as before the introduction of the CPRR scheme. As a result, we view our work as a significant step in shifting the discussion around congestion pricing from one that has focused on the societal inequity impacts of road tolls to one that centers around *how* to best preserve equity through the distribution of toll revenues.

Organization This paper is organized as follows. Section 2 reviews related literature. We then present a model of traffic flow as well as metrics to evaluate the inequality of the wealth distribution and the efficiency of a traffic assignment in Section 3. We then prove the existence of Pareto improving and optimal CPRR schemes for the exogenous setting in Sections 4, and 5, respectively. Next, we show that any exogenous equilibria of an optimal CPRR scheme is also an endogenous equilibria (Section 6). Finally, we present a discussion of how our work fits into the broader conversation around equitable transportation in Section 7. We conclude the paper and provide directions for future work in Section 8.

2 Related Work

The design of mechanisms that satisfy both system efficiency and user fairness desiderata has been a centerpiece of algorithm design for a range of applications including resource allocation, classification tasks for machine learning algorithms and fair traffic routing. For instance, in a seminal work, Bertsimas et al. [13] quantified the loss in efficiency in resource allocation settings when the allocation outcomes are required to satisfy certain fairness criteria. In another landmark work, Dwork et al. [14] studied group-based fairness notions in machine learning classification tasks to prevent discrimination against individuals belonging to disadvantaged groups. In the context of traffic routing, Jahn et al. [15] introduced a fairness-constrained traffic-assignment problem to achieve a balance between the total travel time of a traffic assignment and the

level of fairness, i.e., the maximum discrepancy between the travel times of users travelling between the same O-D pair [16], that it provides. Subsequent work on fair traffic routing has focused on developing algorithms to solve the fairness constrained traffic assignment problem [17, 18, 19], whilst obtaining methods to price roads to enforce the fairness constrained flows in practice [20].

Resolving the trade-off between efficiency and fairness is particularly important for allocation mechanisms involving monetary transfers given the welfare impacts of such mechanisms on low-income groups. Although achieving system efficiency involves allocating goods to users with the highest willingness to pay, in many settings, e.g., cancer treatment, the needs of users are not well expressed by their willingness to pay [21]. Since Weitzman’s seminal work [21] on accounting for agent’s needs in allocation decisions, there has been a rich line of work on taking into account redistributive considerations in the allocation of scarce resources to users. For instance, Besley and Coate [22] analyzed the free provision of a low-quality public good to low-income users by taxing individuals that consume the same good of a higher quality in the private market. More recently, Condorelli [23] studied the allocation of identical objects to agents with the objective of maximizing agent’s values that may be different from their willingness to pay. This analysis was then extended to the allocation of heterogeneous objects to a continuum of agents [24].

In the context of congestion pricing, revenue redistribution has long been considered as a means to alleviate the inequity issues of congestion pricing [25]. Several revenue redistribution strategies have been proposed in the literature, such as the lump-sum transfer of toll revenues to users [26]. In the setting of Vickrey’s bottleneck congestion model [27]—a benchmark representation of peak-period traffic congestion on a single lane—Arnott et al. [28] investigated how a uniform lump-sum payment of toll revenues can be used to make heterogeneous users better off than prior to the implementation of the tolls and refunds. Daganzo [29] developed a novel strategy in the bottleneck congestion model, wherein only a fraction of the users are tolled while the remaining users are exempt from tolls. To extend the application of revenue redistribution schemes to a two parallel-routes setting, Adler and Cetin [30] designed a mechanism wherein the revenue collected from users on the more desirable route was directly transferred to users travelling on the less desirable route. In more general networks with a single O-D pair, Eliasson [5] established the existence of a tolling mechanism with uniform revenue refunds that reduced the travel cost for each user while also decreasing the total system travel time as compared to before the tolling reform. The extension of this result to general road networks with a multiple O-D pair travel demand and heterogeneous users was investigated by Guo and Yang [12]. Our work builds on the latter result [12] by characterizing the influence of congestion pricing and revenue refunding schemes on the level of wealth inequality in society, while designing schemes that simultaneously reduce total system cost and do not increase the level of wealth inequality relative to that under the untolled user equilibrium outcome.

3 Preliminaries

In this section, we introduce basic definitions and concepts on traffic flow, congestion pricing and revenue refunding (CPRR) schemes, and efficiency and wealth-inequality metrics through which we evaluate the quality of CPRR schemes.

3.1 Elements of Traffic Flow

We model the road network as a directed graph $G = (V, E)$, with the vertex and edge sets V and E , respectively. Each edge $e \in E$ has a flow-dependent travel-time function $t_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which maps x_e , the traffic flow rate on edge e , to the travel time $t_e(x_e)$. As is standard in the literature, we assume that the function t_e , for each $e \in E$, is differentiable, convex and monotonically increasing.

Users make trips in the transportation network and belong to a discrete set of user groups based on their (i) value-of-time, (ii) income, and (iii) O-D pair. Let \mathcal{G} denote the set of all user groups, and let $v_g > 0$, $q_g > 0$ and $w_g = (s_g, t_g)$ denote the value-of-time, income and O-D pair represented by an origin s_g and destination t_g , respectively, for each user in group $g \in \mathcal{G}$. The total travel demand d_g of user group g represents the amount of flow to be routed on a set of directed paths \mathcal{P}_g , which is the set of all simple paths connecting O-D pair w_g .

A path flow pattern $\mathbf{f} = \{f_{P,g} : g \in \mathcal{G}, P \in \mathcal{P}_g\}$ specifies for each user group g , the amount of flow $f_{P,g} \geq 0$ routed on a path $P \in \mathcal{P}_g$, where $f_{P,g} \geq 0$. In particular, a flow \mathbf{f} must satisfy the user demand,

i.e.,

$$\sum_{P \in \mathcal{P}_g} f_{P,g} = d_g, \quad \forall g \in \mathcal{G}.$$

We denote the set of all non-negative flows that satisfy this constraint as Ω .

The corresponding edge flows associated with a path flow $\mathbf{f} = \{f_{P,g} : g \in \mathcal{G}, P \in \mathcal{P}_g\}$ is represented as

$$\sum_{P \in \mathcal{P}_g : e \in P} f_{P,g} = x_e^g, \quad \forall e \in E, \quad (1)$$

$$\sum_{g \in \mathcal{G}} x_e^g = x_e, \quad \forall e \in E, \quad (2)$$

where $e \in P$ denotes whether edge e is in path P , while x_e^g represents the flow of users in group g on edge e . For conciseness, we denote $\mathbf{x} = \{x_e\}_{e \in E}$ as the vector of edge flows and $\mathbf{x}^g = \{x_e^g\}_{e \in E}$ denote the vector of edge flows for user group g .

3.2 Congestion Pricing and Revenue Refunding Schemes

A congestion pricing and revenue refunding (CPRR) scheme is defined by a tuple $(\boldsymbol{\tau}, \mathbf{r})$, where (i) $\boldsymbol{\tau} = \{\tau_e : e \in E\}$ is a vector of edge prices (or tolls), and (ii) $\mathbf{r} = \{r_g : g \in \mathcal{G}\}$ is a vector of group-specific revenue refunds, where each user in group g receives a lump-sum transfer of r_g . In other words, everybody pays the same toll for using an edge independent of their group, and all users with the same income, value-of-time and O-D pair get the same refund, irrespective of the actual route. Under the CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ and a vector of edge flows \mathbf{x} , the total value of tolls collected is given by $\Pi = \sum_{e \in E} \tau_e x_e$. In this work we consider CPRR schemes such that the total sum of the revenue refunds equals the total sum of the revenue collected from the edge tolls, i.e., $\sum_{g \in \mathcal{G}} r_g d_g = \Pi$. In addition, we consider revenue refunding schemes that depend only on the groups \mathcal{G} and the total revenue Π induced by a flow \mathbf{f} , but not on the specific paths that the users take under \mathbf{f} . We leave the study of more complex refund schemes for future work (see Section 8).

The total travel cost incurred by the user consists of two components. The first one is a linear function of their travel time and tolls, which is a commonly-used modelling approach [31, 32]. In addition, we have a component which reflects the refund received. The overall model that we use here, which is formally defined below, has been previously considered in the literature [12].

Definition 1 (User Travel Cost). Consider a CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ and a flow pattern \mathbf{f} with edge flow \mathbf{x} , and suppose that a user belongs to a group $g \in \mathcal{G}$. Then, the total cost incurred by a user when traversing a path $P \in \mathcal{P}_g$ such that $f_{P,g} > 0$ is given by

$$\mu_P^g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r}) = \sum_{e \in P} (v_g t_e(x_e) + \tau_e) - r_g. \quad (3)$$

With slight abuse of notation, we will use $\mu_P^g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{0})$ to denote a path cost that does not include refund, and $\mu_P^g(\mathbf{f}, \mathbf{0}, \mathbf{0})$ to denote a path cost that does not account for tolls or refunds, where $\mathbf{0}$ is a vector of zeros. Throughout this paper we will consider in many cases *equilibrium* flow patterns which emerge from different user behavior models that we define formally in the following sections. Relevant to the discussion here is that equilibrium flows equalize the user travel cost of all the users of a given group. That is, if \mathbf{f} is an equilibrium for a CPRR scheme then $\mu_P^g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r}) = \mu_Q^g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r})$ for any group $g \in \mathcal{G}$ and any two paths $P, Q \in \mathcal{P}_g$ such that $f_{P,g}, f_{Q,g} > 0$. In such a case we drop the path dependence in the notation and use $\mu^g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r})$ to denote the travel cost of any user within the group g .

3.3 System Efficiency and Wealth Inequality Metrics

We evaluate the quality of a CPRR scheme using two metrics: (i) system efficiency, which is measured through the total system cost, and (ii) wealth inequality.

Total System Cost: We measure the efficiency of the system through the total system cost, which, for any feasible path flow \mathbf{f} with corresponding edge flows \mathbf{x} and group specific edge flows \mathbf{x}^g , is the sum of travel times weighted by the users' values-of-time across all edges of the network [12, 31, 32], i.e.,

$$C(\mathbf{f}) := \sum_{e \in E} \sum_{g \in \mathcal{G}} v_g x_e^g t_e(x_e).$$

We denote by $C^* := \min_{\mathbf{f} \in \Omega} C(\mathbf{f})$ the widely studied cost-based system optimum.

Wealth Inequality: We measure the impact of a CPRR scheme on wealth inequality in the following manner. For a profile of incomes $\mathbf{q} = \{q_g : g \in \mathcal{G}\}$, we let a function $W : \mathbb{R}_{\geq 0}^{|\mathcal{G}|} \rightarrow \mathbb{R}_{\geq 0}$ measure the level of wealth inequality of society. We say that an income distribution $\tilde{\mathbf{q}}$ has a lower level of wealth inequality than \mathbf{q} if and only if $W(\tilde{\mathbf{q}}) \leq W(\mathbf{q})$.

In this work, we assume that the wealth-inequality measure $W(\cdot)$ satisfies the following properties:

1. **Scale Independence:** The wealth-inequality measure remains unchanged after rescaling incomes by the same positive constant, i.e., $W(\lambda \mathbf{q}) = W(\mathbf{q})$ for any $\lambda > 0$.
2. **Regressive Taxes Increase Inequality:** The wealth-inequality measure increases if the incomes of users are scaled by constants that increase as the income increases. That is, for two income profiles \mathbf{q} and $\tilde{\mathbf{q}}$ with $\tilde{q}_g = \delta_g q_g$, where $0 < \delta_g \leq \delta_{g'}$ if $q_g \leq q_{g'}$ for any two groups g, g' , then $W(\tilde{\mathbf{q}}) \geq W(\mathbf{q})$.
3. **Progressive Taxes Decrease Inequality:** The wealth-inequality measure decreases if the incomes of users are scaled by constants that decrease as the income increases. That is, for two income profiles \mathbf{q} and $\tilde{\mathbf{q}}$ with $\tilde{q}_g = \delta_g q_g$, where $0 < \delta_g \leq \delta_{g'}$ if $q_{g'} \leq q_g$ for any two groups g, g' , then $W(\tilde{\mathbf{q}}) \leq W(\mathbf{q})$.

We note that the above properties are fairly natural [7, 33] and hold for commonly used wealth-inequality measures, such as the discrete Gini coefficient, which we elucidate in detail in Section 5.1. Furthermore, we note that the above properties jointly imply the following important property of the wealth-inequality measure W :

- **Constant Income Transfer Property:** If the initial income distribution is \mathbf{q} and each user is transferred a non-negative (non-positive) amount of money λ ($-\lambda$) where $0 \leq \lambda < \min_{g \in \mathcal{G}} q_g$, then the wealth inequality cannot increase (decrease). That is, $W(\mathbf{q} + \lambda \mathbf{1}) \leq W(\mathbf{q})$ and $W(\mathbf{q} - \lambda \mathbf{1}) \geq W(\mathbf{q})$, where $\mathbf{1}$ is a vector of ones.

We defer a proof of how the constant income transfer property follows from the regressive and progressive tax properties to Appendix 9.1.

When using the wealth inequality measure W , we are interested in understanding the influence of a flow \mathbf{f} for a given CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ on the income distribution of users. To this end, we define the income profile of users before making their trip as the *ex-ante income distribution* $\mathbf{q}^0 > \mathbf{0}$ and that after making their trip as the *ex-post income distribution*, which is defined as follows.

Definition 2 (Ex-Post Income Distribution). For a given CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ and an equilibrium flow \mathbf{f} , the induced ex-post income distribution of users is denoted by $\mathbf{q}(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r})$ and defined as follows. For a given group g , we have that

$$q_g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r}) := q_g^0 - \beta \mu^g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r}),$$

where \mathbf{q}^0 is the ex-ante income distribution and β is a small constant representing the relative importance of the congestion game to an individual's well-being [7].

Since the trip made by users is one among a suite of factors influencing the income of users, we assume that the constant β is small enough so that the ex-post income of all users is strictly positive.

To conclude this section, we note that in this paper we consider time-invariant travel demand that is fixed for all user groups and assume fractional flows, both of which are standard assumptions in the traffic routing literature [34], as well as in game theory in the context of non-atomic congestion games [16]. Furthermore, we assume that the different attributes (i.e., the income, value-of-time and O-D pair) of the user groups are known, and can be used in the design of CPRR schemes. We assume in this work, similar to much of the prior literature in traffic routing with heterogeneous groups of users, that the values-of-time, travel demand, and incomes of different groups of users are known [7, 31, 32].

4 Pareto Improving CPRR Schemes

The social inequity issue surrounding the regressive nature of congestion pricing has been documented in several empirical and theoretical works, while also having spurred political opposition to its implementation in practice. In this section, we show that if the tolls collected from congestion pricing are refunded to users in an appropriate way then the wealth inequality effects of congestion pricing can be reversed. Throughout this section and the next we assume that user behavior is characterized through the *exogenous equilibrium* model wherein users minimize a linear function of their travel time and tolls, without considering refunds.

After formally defining exogenous equilibrium below, we develop a CPRR scheme that simultaneously decreases the total system cost of all users while not increasing the level of wealth inequality relative to the untolled outcome, a property which we refer to as *Pareto improving*. Moreover, when designing the scheme, we ensure that it is politically acceptable for implementation by guaranteeing that each individual user is at least as well off in terms of the travel cost μ^g under the CPRR scheme than that without the implementation of congestion pricing or refunds.

Next, we consider the important special case of travel demand when users travel between the same O-D pair, and have values-of-time that are in proportion to their income. In this setting, we establish the existence of a Pareto improving CPRR scheme that results in an ex-post income distribution that has a lower wealth inequality as compared to that of the ex-ante income distribution. Note that this result is stronger than the more general case with multiple O-D pairs considered above, since the wealth-inequality measure of the ex-ante income distribution is lower than that of the ex-post income distribution for the untolled case.

4.1 Exogenous Equilibrium

To capture the strategic behavior of users, we present below the standard model of Nash equilibrium with heterogeneous users, which we call exogenous equilibrium. The exogenous setting is commonly studied in the context of non-atomic congestion games without [31, 32] or with refunds [12]. As the name suggests, in exogenous equilibrium revenue refunds are assumed to be *exogenous* and do not influence the behavior and route choice of users in the transportation network. That is, users minimize a linear function of their travel time and tolls, without considering refunds.

We note that such a model of user behavior can be quite realistic in certain settings, especially since accounting for refunds when making route choices may often be too complex and involve quite sophisticated decision making on the part of users. Furthermore, for users to reason about how their path choice will influence their refund, they must know the refunding policy, which may typically not be known in practice, thereby making the notion of an exogenous equilibrium more appropriate in such settings. We do consider the more sophisticated *endogenous* setting in Section 6.

The following definition formalizes the notion of an exogenous equilibrium, which only depends on the congestion pricing component τ of a CPRR scheme (τ, r) .

Definition 3 (Exogenous Equilibrium). For a given congestion-pricing scheme τ , a path flow pattern \mathbf{f} is an exogenous equilibrium if for each group $g \in \mathcal{G}$ it holds that $f_{P,g} > 0$ for some path $P \in \mathcal{P}_g$ if and only if

$$\mu_P^g(\mathbf{f}, \tau, \mathbf{0}) \leq \mu_Q^g(\mathbf{f}, \tau, \mathbf{0}), \quad \forall Q \in \mathcal{P}_g.$$

In such a case, we say in short that \mathbf{f} is an exogenous τ -equilibrium.

A key property of any exogenous τ -equilibrium \mathbf{f} is that all users within a given group $g \in \mathcal{G}$ incur the same travel cost without refunds, irrespective of the path on which they travel. Hence, we drop the path dependence in the notation and denote the user travel cost without refunds for any user in group g at flow \mathbf{f} as $\mu^g(\mathbf{f}, \tau, \mathbf{0})$. Additionally, since the refund r_g is the same for all users, the travel cost with refunds is denoted as $\mu^g(\mathbf{f}, \tau, \mathbf{r})$.

Another useful property of exogenous equilibrium is that for a given congestion-pricing scheme τ , the resulting total system cost, user travel cost, and ex-post income distribution are invariant under the different τ -equilibria (see Problem (4) and Appendix 9.2 for a discussion). That is for any two τ -equilibria \mathbf{f}, \mathbf{f}' it holds that $C(\mathbf{f}) = C(\mathbf{f}')$, $\mu^g(\mathbf{f}, \tau, \mathbf{0}) = \mu^g(\mathbf{f}', \tau, \mathbf{0})$, and $q(\mathbf{f}, \tau, \mathbf{r}) = q(\mathbf{f}', \tau, \mathbf{r})$. Thus, we will use the simplified notation $C_\tau := C(\mathbf{f})$, $\mu^g(\tau, \mathbf{r}) := \mu^g(\mathbf{f}, \tau, \mathbf{r})$, and $q(\tau, \mathbf{r}) := q(\mathbf{f}, \tau, \mathbf{r})$ for some exogenous τ -equilibrium \mathbf{f} , when considering the exogenous equilibrium model. In this context, note that C_0 corresponds

to the untolled total system cost, and this quantity is identical for both the exogenous and endogenous equilibrium (we consider the latter in Section 6).

4.2 User-Favorable Pareto Improving CPRR Schemes

To ensure that the CPRR schemes we develop are politically acceptable, we restrict to the class of schemes that result in equilibrium outcomes wherein each individual user is at least as well off as compared to that under the untolled user equilibrium outcome, a property we refer to as user-favorable (see Figure 1). We note that the definition below readily extends to the setting of endogenous equilibrium as well.

Definition 4 (User-Favorable CPRR Schemes). A CPRR scheme (τ, \mathbf{r}) is user-favorable if for any (exogenous) τ -equilibrium the travel cost of any user group g does not increase with respect to any untolled $\mathbf{0}$ -equilibrium \mathbf{f}^0 , i.e., $\mu^g(\tau, \mathbf{r}) \leq \mu^g(\mathbf{0}, \mathbf{0})$.

We now present the main result of this section. In particular, we establish that any pricing scheme τ that improves the system efficiency compared to the untolled case, can be paired with a revenue refunding scheme \mathbf{r} such that wealth inequality relative to the ex-post income distribution under the untolled setting is not increased, i.e., the CPRR scheme (τ, \mathbf{r}) is Pareto improving (see Figure 1) and user-favorable. Note that designing CPRR schemes that achieve a lower level of wealth inequality and total system cost as compared to that of the untolled user equilibrium outcome is desirable since this implies that the CPRR scheme improves upon both the system efficiency and fairness metrics relative to the status-quo traffic equilibrium pattern. We discuss other useful aspects of this result in Section 7.

Proposition 1 (Existence of Pareto Improving CPRR Scheme). *Let τ be a congestion-pricing scheme such that $C_\tau \leq C_0$, where C_0 is the untolled total system cost. Then there exists a refund scheme \mathbf{r} such that (τ, \mathbf{r}) is user-favorable and does not increase wealth inequality, i.e., $W(\mathbf{q}(\tau, \mathbf{r})) \leq W(\mathbf{q}(\mathbf{0}, \mathbf{0}))$. That is, the scheme (τ, \mathbf{r}) is Pareto improving.*

Note that Proposition 1 relies on the key observation that an exogenous equilibrium is completely defined through the road tolls τ , and is thus oblivious of the refund \mathbf{r} . Furthermore, we remark that both Definition 4 and Proposition 1 can readily be extended to incorporate the notions of user-favorable and Pareto improving CPRR schemes relative to any status-quo traffic equilibrium pattern beyond the untolled user equilibrium. For simplicity, we prove those properties relative to the untolled setting. We now prove Proposition 1 by leveraging a class of user-favorable CPRR schemes that were developed recently [12, Theorem 1].

Lemma 1 (Existence of user-favorable CPRR Scheme [12]). *Let τ be a congestion pricing scheme such that $C_\tau \leq C_0$. Then, for any $\alpha_g \geq 0$ with $\sum_{g \in \mathcal{G}} \alpha_g = 1$, the CPRR scheme (τ, \mathbf{r}) with refunds given by*

$$r_g = \mu^g(\tau, \mathbf{0}) - \mu^g(\mathbf{0}, \mathbf{0}) + \frac{\alpha_g}{d_g}(C_0 - C_\tau),$$

for each group g , is user-favorable.

The above lemma states that as long as the edge tolls τ reduce the total system cost there exists a method to refund revenues that makes every user at least as well off as compared to that under the untolled case. We note that the above lemma was previously proven for the case of a strict inequality where $\alpha_g > 0$ and $C_\tau < C_0$. Since we consider the weak inequality $\alpha_g \geq 0$ and $C_\tau \leq C_0$, which suffices to show that the CPRR scheme is user-favorable, for completeness we provide a proof of Lemma 1 in Appendix 9.3. We now leverage Lemma 1 to complete the proof of Proposition 1.

Proof of Proposition 1. For the collected toll revenues, we construct a special case of the revenue refunding scheme from Lemma 1. In particular, consider the refunding scheme where $\alpha_g = \frac{d_g}{\sum_{g \in \mathcal{G}} d_g}$, which gives the refund

$$r_g = \mu^g(\tau, \mathbf{0}) - \mu^g(\mathbf{0}, \mathbf{0}) + \frac{1}{\sum_{g \in \mathcal{G}} d_g}(C_0 - C_\tau)$$

to each user in group g . We now show that under this revenue refunding scheme, the ex-post income distribution $\hat{\mathbf{q}} = \mathbf{q}(\boldsymbol{\tau}, \mathbf{r})$ has a lower wealth inequality measure relative to the untolled user equilibrium ex-post income distribution $\tilde{\mathbf{q}} = \mathbf{q}(\mathbf{0}, \mathbf{0})$. That is, we show that $W(\hat{\mathbf{q}}) \leq W(\tilde{\mathbf{q}})$.

To see this, we begin by considering the ex-ante income distribution \mathbf{q}^0 . Under the untolled user equilibrium, users in group g incur a travel cost $\mu^g(\mathbf{0}, \mathbf{0})$, and thus the ex-post income distribution of users in group g is given by $\tilde{q}_g = q_g^0 - \beta \mu^g(\mathbf{0}, \mathbf{0})$, where β is the scaling factor as in Definition 2. On the other hand, under the CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$, the ex-post income distribution of users in group g is given by

$$\begin{aligned} \hat{q}_g &= q_g^0 - \beta (\mu^g(\boldsymbol{\tau}, \mathbf{0}) - r_g) \\ &= q_g^0 - \beta \left(\mu^g(\boldsymbol{\tau}, \mathbf{0}) - \left[\mu^g(\boldsymbol{\tau}, \mathbf{0}) - \mu^g(\mathbf{0}, \mathbf{0}) + \frac{1}{\sum_{g \in \mathcal{G}} d_g} (C_0 - C_{\boldsymbol{\tau}}) \right] \right) \\ &= q_g^0 - \beta \left(\mu^g(\mathbf{0}, \mathbf{0}) - \frac{1}{\sum_{g \in \mathcal{G}} d_g} (C_0 - C_{\boldsymbol{\tau}}) \right) \\ &= \tilde{q}_g + \beta \frac{1}{\sum_{g \in \mathcal{G}} d_g} (C_0 - C_{\boldsymbol{\tau}}), \end{aligned}$$

where we used that $\tilde{q}_g = q_g^0 - \beta \mu^g(\mathbf{0}, \mathbf{0})$ to derive the last equality. Since the above relation is true for all groups g , we observe that $\hat{\mathbf{q}} = \tilde{\mathbf{q}} + \lambda \mathbf{1}$, where $\lambda = \frac{\beta}{\sum_{g \in \mathcal{G}} d_g} (C_0 - C_{\boldsymbol{\tau}}) \geq 0$. Finally, the result that $W(\hat{\mathbf{q}}) \leq W(\tilde{\mathbf{q}})$ follows by the constant income transfer property (Section 3), establishing our claim. \square

Proposition 1 establishes the existence of a user-favorable CPRR scheme that simultaneously decreases the total system cost and reduces the wealth inequality relative to that of the untolled outcome. We now present an important consequence of this result for the setting when all users travel between the same O-D pair and have values-of-time that are proportional to their incomes. In this setting, we establish the existence of a revenue refunding scheme that decreases the wealth inequality relative to the ex-ante income distribution, which is a stronger result than Proposition 1.

Corollary 1 (CPRR Decreases Wealth Inequality for Single O-D Pair). *Consider the setting where the O-D pairs are identical across all user groups, i.e., $w_g = w_{g'}$ for every $g, g' \in \mathcal{G}$, and assume that users have values-of-time that are proportional to their incomes, i.e., $v_g = \omega q_g^0$ for some $\omega > 0$ for each group g . Let $\boldsymbol{\tau}$ be road tolls such that $C_{\boldsymbol{\tau}} \leq C_0$. Then, for a small enough value of β (Definition 2), there exists a revenue refunding scheme \mathbf{r} such that the CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ is user-favorable and $W(\mathbf{q}(\boldsymbol{\tau}, \mathbf{r})) \leq W(\mathbf{q}^0)$, where \mathbf{q}^0 is the ex-ante income distribution.*

Proof. Consider the same user-favorable CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ as in the proof of Proposition 1. We now show that the wealth inequality of the ex-post income distribution resulting from $(\boldsymbol{\tau}, \mathbf{r})$ is lower than the wealth inequality of the ex-ante income distribution, i.e., $W(\hat{\mathbf{q}}) \leq W(\mathbf{q})$, where $\hat{\mathbf{q}} = \mathbf{q}(\boldsymbol{\tau}, \mathbf{r})$. To see this, we first show that $W(\mathbf{q}(\mathbf{0}, \mathbf{0})) = W(\mathbf{q}^0)$, i.e., the wealth inequality measure of the ex-ante income distribution is exactly equal to that of the untolled ex-post income distribution $\tilde{\mathbf{q}} = \mathbf{q}(\mathbf{0}, \mathbf{0})$. The proof of this result lies in the key observation that for any $\mathbf{0}$ -equilibrium flow \mathbf{f}^0 all users incur the same travel time, denoted as γ , since they travel between the same O-D pair. This observation leads to a travel cost of $\mu^g(\mathbf{0}, \mathbf{0}) = \omega q_g^0 \gamma$ for each group g . Then, for the untolled setting, the ex-post income distribution of users in group g is given by

$$\tilde{q}_g = q_g^0 - \beta \mu^g(\mathbf{0}, \mathbf{0}) = q_g^0 - \beta \omega q_g^0 \gamma = q_g^0 (1 - \beta \omega \gamma).$$

From the above, it follows that $\tilde{\mathbf{q}} = \lambda_1 \mathbf{q}^0$ for $\lambda_1 = 1 - \beta \omega \gamma$. Thus, for β small enough it holds that $\lambda_1 > 0$. Under this assumption, due to the scale-independence property (Section 3) of the wealth-inequality measure it then follows that $W(\tilde{\mathbf{q}}) = W(\mathbf{q}^0)$. Finally, since $W(\hat{\mathbf{q}}) \leq W(\tilde{\mathbf{q}})$ by the proof of Proposition 1 it follows that $W(\hat{\mathbf{q}}) \leq W(\tilde{\mathbf{q}}) = W(\mathbf{q}^0)$, thus proving our claim. \square

The above result shows that in some scenarios, any form of road tolls that decreases the total system cost $C_{\boldsymbol{\tau}}$ relative to C_0 , coupled with the appropriate revenue refunding policy, will not increase the level of wealth inequality in comparison to the ex-ante distribution \mathbf{q}^0 . This result indicates that the appropriate

refunding can reverse the negative consequences of tolls on wealth inequality, as was established in the “Inequity Theorem” [7]. In particular, the “Inequity Theorem” asserts that for the setting considered in Corollary 1 any form of road tolls increase the level of wealth inequality compared with the ex-ante income distribution \mathbf{q}^0 in the absence of a refund intervention.

A main ingredient in Corollary 1 is the fact that the wealth inequality measure of the ex-ante income distribution \mathbf{q}^0 is exactly equal to that of the ex-post income distribution under the untolled user equilibrium. This result holds when users travel between the same O-D pair and have values-of-time that scale proportionally with their incomes. However, it does not hold in general for users travelling between different O-D pairs, since in such a case users may incur different travel times at the untolled user equilibrium. For the multiple O-D pair setting, we show in Proposition 2 that there are travel demand instances when no CPRR scheme can reduce income inequality relative to that of the ex-ante income distribution.

Proposition 2 (Increase in Income Inequality for Multiple O-D Pairs). *There exists a two O-D pair setting such that for any user-favorable CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ it holds that $W(\mathbf{q}(\boldsymbol{\tau}, \mathbf{r})) \geq W(\mathbf{q}^0)$.*

For a proof of Proposition 2, see Appendix 9.4. Given that there may be multiple O-D pair instances when it may not be possible to achieve a lower wealth inequality measure relative to the ex-ante income distribution, for the rest of this paper we devise CPRR schemes that reduce the wealth inequality measure relative to the ex-post income distribution under the untolled user equilibrium outcome rather than relative to the ex-ante income distribution. Note that doing so is reasonable, since we look to design CPRR schemes that improve on the status quo traffic pattern, which is typically described by the untolled user equilibrium setting.

5 Optimal CPRR Schemes

In the previous section, we established the existence of a user-favorable CPRR scheme that simultaneously reduces total system cost without increasing wealth inequality relative to untolled outcome. In this section, we prove the existence of optimal CPRR schemes that achieve total system cost and wealth inequality that cannot be improved by any other user-favorable CPRR scheme. In particular, we establish that the optimal CPRR schemes are those that induce exogenous equilibrium flows with the minimum total system cost while also resulting in ex-post income distributions with the lowest level of wealth inequality among the class of all user-favorable CPRR schemes (see Figure 1). We then provide a specific example to show how the optimal revenue refunding scheme can be derived for the discrete Gini coefficient, a commonly used wealth inequality measure.

We first present the main result of this section, which characterizes the set of optimal CPRR schemes.

Theorem 1 (Optimal CPRR Scheme). *There exists a user-favorable CPRR scheme $(\boldsymbol{\tau}^*, \mathbf{r}^*)$ such that for any user-favorable CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ it holds that $C_{\boldsymbol{\tau}^*} \leq C_{\boldsymbol{\tau}}$ and $W(\mathbf{q}(\boldsymbol{\tau}^*, \mathbf{r}^*)) \leq W(\mathbf{q}(\boldsymbol{\tau}, \mathbf{r}))$.*

The proof of this theorem is constructive, in the sense that it provides a recipe for computing the optimal CPRR scheme $(\boldsymbol{\tau}^*, \mathbf{r}^*)$. The proof relies on two intermediate results that are of independent interest. First, the ex-post income distribution of any user-favorable CPRR scheme is the same as the ex-post income distribution under the untolled user equilibrium outcome plus some nonnegative transfer value, which may vary across groups. We summarize this observation in the following lemma.

Lemma 2 (Ex-post Income Distribution). *Let $\boldsymbol{\tau}$ be road tolls such that $C_{\boldsymbol{\tau}} \leq C_0$. Then, under any set of refunds \mathbf{r} such that the CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ is user-favorable, the ex-post income of any user belonging to group g is*

$$q_g(\boldsymbol{\tau}, \mathbf{r}) = q_g(\mathbf{0}, \mathbf{0}) + \beta c_g,$$

where the transfer value c_g is non-negative and satisfies the relation $\sum_{g \in \mathcal{G}} c_g d_g = C_0 - C_{\boldsymbol{\tau}}$.

Proof. Denote the ex-post income of group g as $\hat{q}_g = q_g(\boldsymbol{\tau}, \mathbf{r})$. We now prove the ex-post income relation using the definition of a user-favorable CPRR scheme. In particular, for any user-favorable CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$ the user travel cost does not increase from the untolled case, i.e., $\mu^g(\boldsymbol{\tau}, \mathbf{r}) \leq \mu^g(\mathbf{0}, \mathbf{0})$. As it holds that

$\mu^g(\boldsymbol{\tau}, \mathbf{r}) = \mu^g(\boldsymbol{\tau}, \mathbf{0}) - r_g$, we observe that for some $c_g \geq 0$ the following relation must hold for each user in group g : $\mu^g(\boldsymbol{\tau}, \mathbf{0}) - r_g + c_g = \mu^g(\mathbf{0}, \mathbf{0})$. Then, for an ex-ante income distribution \mathbf{q}^0 , the ex-post income of each user belonging to group g is given by

$$\hat{q}_g = q_g^0 - \beta(\mu^g(\boldsymbol{\tau}, \mathbf{0}) - r_g) = q_g^0 - \beta\mu^g(\mathbf{0}, \mathbf{0}) + \beta c_g = q_g(\mathbf{0}, \mathbf{0}) + \beta c_g,$$

where the second equality follows since $\mu^g(\boldsymbol{\tau}, \mathbf{0}) - r_g = \mu^g(\mathbf{0}, \mathbf{0}) - c_g$ and the last equality follows from the observation that the ex-post income of users in group g for the untolled setting is given by $q_g(\mathbf{0}, \mathbf{0}) = q_g^0 - \beta\mu^g(\mathbf{0}, \mathbf{0})$.

Next, to show that $\sum_{g \in \mathcal{G}} c_g d_g = C_0 - C_\tau$ we characterize the quantities C_0 and C_τ . In particular, observe that by definition $C_0 = C(\mathbf{f}^0)$ and $C_\tau = C(\mathbf{f})$, where \mathbf{f}^0 is the untolled $\mathbf{0}$ -equilibrium and \mathbf{f} is an exogenous $\boldsymbol{\tau}$ -equilibrium. Now, note that both flows \mathbf{f}^0 and \mathbf{f} can be expressed in closed form. In particular, for a given congestion-pricing scheme $\boldsymbol{\tau}'$ the exogenous $\boldsymbol{\tau}'$ -equilibrium $\mathbf{h}(\boldsymbol{\tau}')$ can be written as

$$\mathbf{h}(\boldsymbol{\tau}') = \arg \min_{\mathbf{h}' \in \Omega} \sum_{e \in E} \int_0^{x(\mathbf{h}')_e} t_e(\omega) d\omega + \sum_{e \in E} \sum_{g \in \mathcal{G}} \frac{1}{v_g} x(\mathbf{h}')_e^g \tau_e, \quad (4)$$

where $\mathbf{x}(\mathbf{f}')$ denotes the edge representation of a path flow \mathbf{f}' . We note that this program corresponds to the *multi-class user-equilibrium optimization problem* [35].

Given this representation of the flow $\mathbf{h}(\boldsymbol{\tau}')$, we derive the following relation that relates the total system cost $C_{\boldsymbol{\tau}'}$ to the amount of collected revenues, by analyzing the KKT conditions of this minimization problem. In particular, it holds that

$$C_{\boldsymbol{\tau}'} = \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}', \mathbf{0}) d_g - \sum_{e \in E} \tau'_e x(\mathbf{h}(\boldsymbol{\tau}'))_e. \quad (5)$$

Note that the edge flow $\mathbf{x}(\mathbf{h}(\boldsymbol{\tau}'))$ is unique by the strict convexity of the travel-time function. We defer the proof of Equation (5) to Appendix 9.2.

We now leverage Equation (5) to obtain that $C_\tau = \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}, \mathbf{0}) d_g - \sum_{e \in E} \tau_e x(\mathbf{f})_e$, where $\mathbf{x}(\mathbf{f}) = \mathbf{x}(\mathbf{h}(\boldsymbol{\tau}))$. Furthermore, from Equation (5) for the untolled setting, we obtain that $C_0 = \sum_{g \in \mathcal{G}} \mu^g(\mathbf{0}, \mathbf{0}) d_g$. Finally, using these two relations and leveraging the fact that $c_g = \mu^g(\mathbf{0}, \mathbf{0}) - \mu^g(\boldsymbol{\tau}, \mathbf{0}) + r_g$ we get

$$\begin{aligned} \sum_{g \in \mathcal{G}} c_g d_g &= \sum_{g \in \mathcal{G}} (\mu^g(\mathbf{0}, \mathbf{0}) - \mu^g(\boldsymbol{\tau}, \mathbf{0}) + r_g) d_g, \\ &= \sum_{g \in \mathcal{G}} \mu^g(\mathbf{0}, \mathbf{0}) d_g - \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}, \mathbf{0}) d_g + \sum_{g \in \mathcal{G}} r_g d_g, \\ &= C_0 - \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}, \mathbf{0}) d_g + \sum_{e \in E} \tau_e x(\mathbf{f})_e, \\ &= C_0 - C_\tau. \end{aligned}$$

Here we used the properties $C_0 = \sum_{g \in \mathcal{G}} \mu^g(\mathbf{0}, \mathbf{0}) d_g$ and $\sum_{g \in \mathcal{G}} r_g d_g = \sum_{e \in E} \tau_e x(\mathbf{f})_e$, to derive the third equality, and that $C_\tau = \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}, \mathbf{0}) d_g - \sum_{e \in E} \tau_e x(\mathbf{f})_e$ to derive the last equality. This proves our claim. \square

The above lemma highlights that under any user-favorable CPRR scheme $(\boldsymbol{\tau}, \mathbf{r})$, each user's income is at least the ex-post income of the user under the untolled case.

The second result required to prove Theorem 1 relies on the observation that there is a monotonic relationship between the minimum achievable wealth-inequality measure and the total system cost.

Lemma 3 (Monotonicity of Refunds). *Suppose that there are two congestion-pricing schemes $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$ with total system costs satisfying $C_{\boldsymbol{\tau}_A} \leq C_{\boldsymbol{\tau}_B} \leq C_0$. Then there exists a revenue refunding scheme \mathbf{r}_A such that $(\boldsymbol{\tau}_A, \mathbf{r}_A)$ is user-favorable and achieves a lower wealth inequality measure than any user-favorable CPRR scheme $(\boldsymbol{\tau}_B, \mathbf{r}_B)$ for any revenue refunds \mathbf{r}_B , i.e., $W(\mathbf{q}(\boldsymbol{\tau}_A, \mathbf{r}_A)) \leq W(\mathbf{q}(\boldsymbol{\tau}_B, \mathbf{r}_B))$.*

Proof. We prove this claim by constructing for each revenue refunding scheme \mathbf{r}_B under the tolling scheme τ_B , a revenue refunding scheme \mathbf{r}_A under the tolling scheme τ_A that achieves a lower wealth inequality measure. To this end, we first introduce some notation. Let c_g^A and c_g^B be non-negative transfers for each group g as in Lemma 2, where $\sum_{g \in \mathcal{G}} c_g^A d_g = C_0 - C_{\tau_A}$ and $\sum_{g \in \mathcal{G}} c_g^B d_g = C_0 - C_{\tau_B}$ must hold for the feasibility of the scheme.

Then, by Lemma 2 we have that the ex-post income of users in group g can be expressed as: $q_g(\tau_A, \mathbf{r}_A) = q_g(\mathbf{0}, \mathbf{0}) + \beta c_g^A$ and $q_g(\tau_B, \mathbf{r}_B) = q_g(\mathbf{0}, \mathbf{0}) + \beta c_g^B$. Let $c_g^A = c_g^B + \frac{1}{\sum_{g \in \mathcal{G}} d_g} (C_{\tau_B} - C_{\tau_A})$. We now show that the refunding \mathbf{r}_A is feasible.

$$\begin{aligned} \sum_{g \in \mathcal{G}} c_g^A d_g &= \sum_{g \in \mathcal{G}} \left(c_g^B d_g + \frac{d_g}{\sum_{g \in \mathcal{G}} d_g} (C_{\tau_B} - C_{\tau_A}) \right) \\ &= \sum_{g \in \mathcal{G}} c_g^B d_g + C_{\tau_B} - C_{\tau_A} \\ &= C_0 - C_{\tau_B} + C_{\tau_B} - C_{\tau_A} \\ &= C_0 - C_{\tau_A}, \end{aligned}$$

Here we leveraged the fact that $\sum_{g \in \mathcal{G}} c_g^B d_g = C_0 - C_{\tau_B}$.

Under the above defined non-negative transfer c_g^A , we observe that the ex-post income distribution under the CPRR scheme (τ_A, \mathbf{r}_A) is the same as the ex-post income distribution under the CPRR scheme (τ_B, \mathbf{r}_B) plus a constant positive transfer, which is equal for all users. That is, we have $\mathbf{q}(\tau_A, \mathbf{r}_A) = \mathbf{q}(\tau_B, \mathbf{r}_B) + \lambda \mathbf{1}$ for $\lambda = \frac{\beta}{\sum_{g \in \mathcal{G}} d_g} (C_{\tau_B} - C_{\tau_A}) \geq 0$. Finally, by the constant income transfer property (Section 3) it follows that $W(\mathbf{q}(\tau_A, \mathbf{r}_A)) \leq W(\mathbf{q}(\tau_B, \mathbf{r}_B))$. \square

The above result establishes a very natural property of any user-favorable revenue-refunding policy for which the total refund remaining after satisfying the user-favorable condition is $C_0 - C_{\tau}$. In particular, a smaller total system cost yields a larger amount of remaining refund $C_0 - C_{\tau}$, which, in turn, results in a greater degree of freedom in distributing these refunds to achieve an overall lower level of wealth inequality.

Finally, Theorem 1 follows directly by the monotonicity relation established in Lemma 3, and prescribes a two-step procedure to find an optimal CPRR scheme that is also user-favorable. In particular, choose a congestion pricing scheme τ^* such that the total travel cost is minimized, i.e., $C_{\tau^*} = C^*$. Next, select the revenue refunding scheme \mathbf{r}^* to be such that the expression $W(\mathbf{q}(\tau^*, \mathbf{r}^*))$ is minimized and (τ^*, \mathbf{r}^*) is user-favorable through an appropriate selection of transfers c_g . Now, let (τ, \mathbf{r}) be some user-favorable CPRR scheme. By definition of τ^* , it holds that $C_{\tau^*} \leq C_{\tau}$. Moreover, Lemma 3 ensures that $W(\mathbf{q}(\tau^*, \mathbf{r}^*)) \leq W(\mathbf{q}(\tau, \mathbf{r}))$ is satisfied.

5.1 Optimal CPRR Scheme for the Discrete Gini Coefficient

We now consider a specific type of wealth inequality measure W called the *discrete Gini coefficient*, which is a commonly used wealth inequality measure when users have finitely many incomes levels [11]. By exploiting the structure of this measure, we provide a concrete recipe for computing the optimal CPRR scheme (τ^*, \mathbf{r}^*) . We note that the problem of computing tolls τ^* that minimize the total system cost has been widely studied in the literature [35, 36]. Thus, we assume for the remainder of this section that the optimal congestion-pricing policy τ^* is given, and focus our attention on deriving the optimal revenue refunding policy \mathbf{r}^* given τ^* .

We first present the discrete Gini coefficient measure and show that it is a valid wealth-inequality measure, i.e., it satisfies the scale independence, regressive and progressive tax properties.

Definition 5 (Discrete Gini Coefficient). For an income distribution \mathbf{q} , and the vector $\mathbf{d} = \{d_g : g \in G\}$ denoting the number of users belonging to each group, the discrete Gini coefficient W is given by

$$W(\mathbf{q}) = \frac{1}{2 \left(\sum_{g \in \mathcal{G}} d_g \right)^2} \sum_{g_1, g_2 \in \mathcal{G}} d_{g_1} d_{g_2} |q_{g_1} - q_{g_2}|,$$

where $\Delta(\mathbf{q}) = \frac{\sum_{g \in \mathcal{G}} q_g d_g}{\sum_{g \in \mathcal{G}} d_g}$ is the mean income for the income distribution \mathbf{q} (see, e.g., [11]).

A few comments about the discrete Gini coefficient as a wealth inequality measure are in order. First, the discrete Gini coefficient is zero if all users have the same income, i.e., there is perfect equality in society. Next, due to the absolute value of the difference between user incomes in the numerator, the discrete Gini coefficient is larger if the dispersion of incomes between different user groups is greater. Finally, note here that we do not write the discrete Gini coefficient measure as a function of the vector of demands $\mathbf{d} = \{d_g : g \in \mathcal{G}\}$ since we assume that user demands are fixed in this work.

Remark 1. The discrete Gini coefficient measure satisfies the scale independence, regressive and progressive tax properties required for it to be a valid wealth inequality measure W . To see this, note that scaling all incomes by a positive constant $\lambda > 0$ scales both the numerator and denominator by a factor of λ , which results in scale independence. The proof of the discrete Gini coefficient satisfying the regressive tax property is presented in Appendix 9.5. The discrete Gini coefficient satisfies the progressive tax property following a similar argument. Thus, it is a valid wealth-inequality measure.

For the discrete Gini coefficient, we now present a mathematical program for computing the revenue refunding policy \mathbf{r}^* . To this end, we first observe that by Lemma 2 for any user-favorable CPRR scheme $(\boldsymbol{\tau}^*, \mathbf{r}^*)$ each user's ex-post income is given by $q_g(\boldsymbol{\tau}^*, \mathbf{r}^*) = q_g(\mathbf{0}, \mathbf{0}) + c_g$ for some $c_g \geq 0$, where $\sum_{g \in \mathcal{G}} c_g d_g = C_0 - C_{\boldsymbol{\tau}^*}$. Thus, the choice of the optimal revenue refunds \mathbf{r}^* can be reduced to computing the optimal transfers c_g . In particular, we formulate the computation of the optimal transfers c_g to minimize the discrete Gini coefficient through the following optimization problem:

$$\min_{c_g} \quad W(\mathbf{q}(\mathbf{0}, \mathbf{0}) + \mathbf{c}) = \frac{1}{2 \left(\sum_{g \in \mathcal{G}} d_g \right)^2} \sum_{g_1, g_2 \in \mathcal{G}} d_{g_1} d_{g_2} |\tilde{q}_{g_1} + c_{g_1} - \tilde{q}_{g_2} - c_{g_2}|, \quad (6a)$$

$$\text{s.t.} \quad \sum_{g \in \mathcal{G}} c_g d_g = C_0 - C_{\boldsymbol{\tau}^*}, \quad (6b)$$

$$c_g \geq 0, \quad \forall g \in \mathcal{G}, \quad (6c)$$

where we denote $\tilde{q}_g = q_g(\mathbf{0}, \mathbf{0})$ for conciseness, $\mathbf{c} = \{c_g : g \in \mathcal{G}\}$, and $\mathbf{q}(\mathbf{0}, \mathbf{0}) + \mathbf{c}$ represents the income distribution of users after receiving the revenue refunds. Furthermore, noting that $\Delta(\mathbf{q}(\mathbf{0}, \mathbf{0}) + \mathbf{c}) = \frac{C_0 - C_{\boldsymbol{\tau}^*} + \sum_{g \in \mathcal{G}} q_g(\mathbf{0}, \mathbf{0}) d_g}{\sum_{g \in \mathcal{G}} d_g}$ is a fixed quantity, the above problem can be solved via a linear program (see [37, Chapter 6]). The optimal revenue refunding policy \mathbf{r}^* corresponding to the above linear program results in a natural max-min outcome. In particular, Algorithm 1 describes the revenue refunding process, wherein users in the lowest income groups, denoted by \mathcal{G}_{\min} , are provided refunds until their incomes equal that of the second lowest income groups. This process is repeated until all the revenue refunds are completely exhausted and is depicted in Figure 2.

The following proposition establishes that the optimal vector of non-negative transfers corresponding to the solution to the linear Program (6a)-(6c) is equal to the vector of transfers \mathbf{c} computed through Algorithm 1.

Proposition 3 (Optimal Revenue Refunding Scheme). *The vector of transfers $\mathbf{c} = \{c_g : g \in \mathcal{G}\}$ output by Algorithm 1 is equal to the optimal vector of transfers computed through the solution to the linear Program (6a)-(6c).*

For a proof of Proposition 3, see Appendix 9.6. We note here that the greedy process of revenue refunding elucidated in Proposition 3 is reminiscent of Rawl's difference principle of giving the greatest benefit to the most disadvantaged groups of society [38].

6 Endogenous Impact of Revenue Refunding

Thus far, we have considered the setting when all users minimize a linear function of their travel times and tolls without considering refunds. In this section we consider the setting of the *endogenous* equilibrium,

Algorithm 1: Max-Min Revenue Refunding

Input : Untolled exogenous equilibrium ex-post income distribution: $\tilde{q} := q(\mathbf{0}, \mathbf{0})$, Total system cost C_0 for the untolled setting, Total system cost C_τ under the tolls τ .

Output: Nonnegative transfers $c = \{c_g : g \in \mathcal{G}\}$

$Z \leftarrow C_0 - C_\tau$ /* Total refund assigned */

$c \leftarrow \mathbf{0}$ /* initialize transfers */

while $Z > 0$ **do**

$q_{\min} \leftarrow \min_{g \in \mathcal{G}} \tilde{q}_g$ /* Lowest income */

$\mathcal{G}_{\min} \leftarrow \{g \in \mathcal{G} \mid \tilde{q}_g = q_{\min}\}$ /* Lowest income groups */

$q_{\text{next}} \leftarrow \min_{g \in \mathcal{G} \setminus \mathcal{G}_{\min}} \tilde{q}_g$ /* Second-lowest income */

$Y \leftarrow \min \left\{ q_{\text{next}} - q_{\min}, \frac{Z}{\sum_{g \in \mathcal{G}_{\min}} d_g} \right\}$ /* Additional income per person */

$\tilde{q}_g \leftarrow \tilde{q}_g + Y, \quad \forall g \in \mathcal{G}_{\min}$ /* Update income */

$c_g \leftarrow c_g + Y, \quad \forall g \in \mathcal{G}_{\min}$ /* Update transfer */

$Z \leftarrow Z - Y \sum_{g \in \mathcal{G}_{\min}} d_g$ /* Update remaining refund */

end

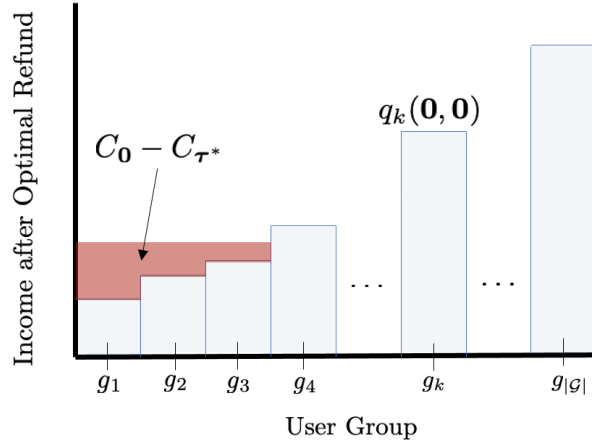


Figure 2: The optimal solution of Algorithm 1 is analogous to a max-min allocation. The height of the grey bars represents the ex-post income of users under the untolled outcome, while the red region denotes the amount of transfer c_g that is given to the different user groups. The total sum of the transfers given to users adds up exactly to $C_0 - C_{\tau^*}$ under the set of tolls τ^* .

wherein users minimize a linear function of not only their travel times and tolls but also refunds. Under this new model of user behavior, we first show that, in general, endogenous equilibria do not coincide with exogenous equilibria. Despite this result, we obtain that any exogenous equilibrium induced by an optimal CPRR scheme is also an endogenous equilibrium.

We begin by introducing the notion of an endogenous equilibrium. Since we are in the setting of a non-atomic congestion game, wherein users are infinitesimal, a unilateral deviation by any user will not influence their overall refunds since the flow of users in the network remains unchanged and the tolls are fixed. However, if each user group is treated as a coalition then a change in the strategy of the entire group, i.e., the flow sent on each feasible path, will likely result in a change in the overall network flow as well as the revenues obtained by each user in the group. Thus, we define an endogenous equilibrium with respect to the strategy of each user group rather than each infinitesimal user.

Definition 6 (Endogenous Equilibrium). Let (τ, \mathbf{r}) be a CPRR scheme, and let \mathbf{f} be a flow pattern. Then \mathbf{f} is an endogenous (τ, \mathbf{r}) -equilibrium if for each group $g \in \mathcal{G}$, every path $P \in \mathcal{P}_g$ such that $f_{P,g} > 0$, and

any flow pattern \mathbf{f}' such that

$$\mathbf{f}'_{P',g'} = \mathbf{f}_{P',g'}, \forall g' \in \mathcal{G} \setminus \{g\}, P' \in \mathcal{P}_{g'},$$

it holds that

$$\mu_P^g(\mathbf{f}, \boldsymbol{\tau}, \mathbf{r}(\mathbf{f}, \boldsymbol{\tau})) \leq \mu_P^g(\mathbf{f}', \boldsymbol{\tau}, \mathbf{r}(\mathbf{f}', \boldsymbol{\tau})), \quad \forall Q \in \mathcal{P}_g.$$

Here \mathbf{f}' denotes a flow that results from \mathbf{f} where exactly one group changes its path assignment.

A few comments about the above definition are in order. First, it is clear that the above definition of endogenous equilibrium is a stronger notion than the standard Nash equilibrium considered in non-atomic congestion games. This is because every endogenous equilibrium is a Nash equilibrium when users minimize their travel costs including refunds but not every Nash equilibrium is necessarily an endogenous equilibrium.

Next, we restrict the set of possible coalitions to those corresponding to strategies for a given user group. This is often reasonable, since users belonging to similar income levels that make similar trips, i.e., travel between the same O-D pair, are more likely to be socially connected with each other and share travel information as compared to users across groups. As a result, we do not consider the setting of equilibrium formation that is robust to any arbitrary set of coalitions [39], and defer this as an interesting direction for future research.

Furthermore, we can view the endogenous equilibrium as a non-atomic analogue of the atomic equilibrium setting, wherein each group g controls a flow of d_g . While in the atomic setting, each group can only send their flow on one path, since this setting is non-atomic the flows can be dispersed across multiple paths whose travel costs are equal.

6.1 Endogenous Equilibria Differ from Exogenous Equilibria

We show that, in general, the endogenous and exogenous equilibria are not the same. To this end, we first recall that an exogenous equilibrium only depends on the tolling scheme $\boldsymbol{\tau}$ and is completely independent of the refunds \mathbf{r} . On the other hand, since users take into account revenue refunds in the case of the endogenous equilibrium, each user must know the refunding policy \mathbf{r} to reason about their strategies in the congestion game. In particular, each user (and coalition of users within a group) must be able to reason about how a change in their strategy, i.e., the path(s) on which they travel, will change the total amount of refund they receive, and in effect their travel cost. Thus, for this section, we restrict to a specific class of revenue refunds that satisfy the revenue-refunding policy of Algorithm 1. That is, users are given refunds through a process that results in an outcome analogous to a max-min allocation. We now construct a counterexample to show that an exogenous $\boldsymbol{\tau}$ -equilibrium flow may no longer be an equilibrium when users take into account refunds in their travel cost minimization.

Proposition 4 (Non-Equivalence of Equilibria). *There exists a setting with (i) a two-edge parallel network, (ii) three income classes, and (iii) tolls $\boldsymbol{\tau}$, such that the induced exogenous $\boldsymbol{\tau}$ -equilibrium is not an endogenous $(\boldsymbol{\tau}, \mathbf{r})$ -equilibrium, where \mathbf{r} results from the max-min revenue refunding policy (Algorithm 1).*

Sketch of Proof. We formally define the instance that is described in Figure 3. Consider a two edge parallel network, having one origin and one destination, with travel time functions $t_1(x_1) = 2x_1$ and $t_2(x_2) = 4 + x_2$. Consider three user classes H, M, L representing high, medium, and low incomes, respectively. Let the demands of the three classes be $d_H = 2, d_M = 1, d_L = 5$, where the incomes are $q_H = 2q_M, q_M$ and q_L , where $q_M(1 - 0.008) = q_L(1 - 0.010) + \frac{0.014q_M}{5}$, and the relative importance of the congestion game is given by a factor $\beta = 1$. Further, let the values-of-time of the users be scaled proportions of their income by a factor of 0.001, i.e., $v_H = 0.002q_M, v_M = 0.001q_M, v_L = 0.001q_L$.

We define the following congestion pricing: $\tau_1 = 0.008q_M$, and $\tau_2 = 0$, i.e., edge 2 is untolled. Given this pricing $\boldsymbol{\tau}$, we define the refunding policy \mathbf{r} , which is known to all users, and is derived from Algorithm 1. That is, we first provide enough refunds to ensure that all groups exactly meet their untolled user equilibrium costs, and then give any remaining refunds to the lowest income group (until their income equals that of the second lowest income group and so on).

Using the above scenario, we first derive the untolled user equilibrium solution, which allows to characterize the refunding scheme \mathbf{r} . Then we compute an exogenous equilibrium and finally show that this is not an endogenous equilibrium by describing a deviation which improves the travel costs of users in group M . \square

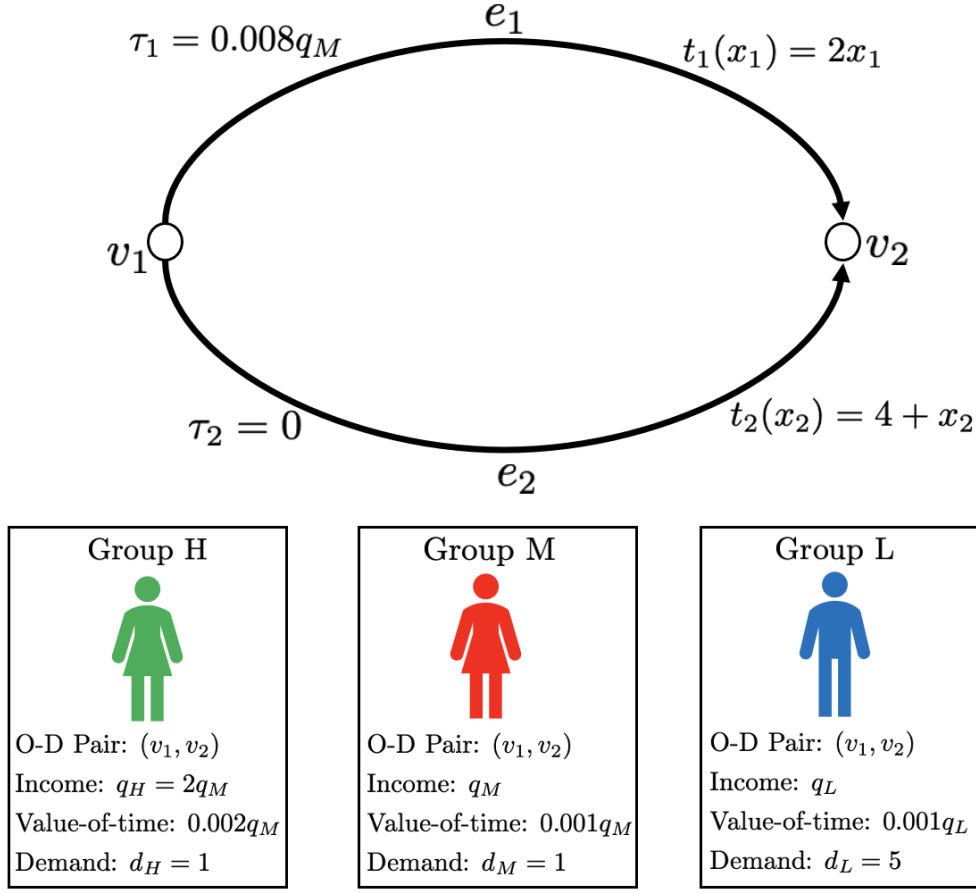


Figure 3: A two-edge parallel network and three user-group instance for which the exogenous τ -equilibrium is not an endogenous (τ, \mathbf{r}) -equilibrium, under the revenue refunding scheme \mathbf{r} resulting from Algorithm 1. For $\tau = (0.008q_M, 0)$, only users in the high income group H use edge e_1 at the exogenous τ -equilibrium. However, this is not an endogenous (τ, \mathbf{r}) -equilibrium since all the users in group M can deviate to use edge e_1 , which will result in a strictly lower travel cost for all users in that group.

For a complete proof of Proposition 4, see Appendix 9.7. The above proposition is quite natural, since low-income users may take routes that were previously unaffordable when taking into account revenue refunds in their route selection process.

6.2 Endogenous Equilibria Coincide with Exogenous Equilibria at the Optimal Solution

While Proposition 4 indicates that, in general, the exogenous and endogenous equilibria do not coincide, we now establish that any exogenous equilibrium induced by an optimal user-favorable CPRR scheme (τ^*, \mathbf{r}^*) , where the refund satisfies a mild condition, is also an endogenous equilibrium. In particular, we have the following lemma:

Lemma 4 (Optimal CPRR Scheme under Endogenous Equilibria). *Let (τ^*, \mathbf{r}^*) be an optimal user-favorable CPRR scheme under the exogenous equilibrium model and let \mathbf{f}^* be its exogenous equilibrium. In addition, let \mathbf{f}^0 be a $\mathbf{0}$ -equilibrium. Further, suppose that the refunding scheme \mathbf{r}^* is defined as $r_g^* := \mu^g(\tau^*, \mathbf{0}) - \mu^g(\mathbf{0}, \mathbf{0}) + c_g$, where the non-negative transfer $c_g(\tau^*)$ is monotonically non-increasing in C_{τ^*} for each group g . Then \mathbf{f}^* is also an endogenous (τ^*, \mathbf{r}^*) -equilibrium.*

Proof. As in the analysis of Lemma 2, for any user-favorable CPRR scheme (τ^*, r^*) it holds that the travel cost to users in group g under the exogenous τ^* -equilibrium f^* is given by $r_g^* = \mu^g(\tau^*, \mathbf{0}) - \mu^g(\mathbf{0}, \mathbf{0}) + c_g$, where $c_g \geq 0$ and $\sum_{g \in \mathcal{G}} c_g d_g = C_{\mathbf{0}} - C_{\tau^*}$.

We now consider the emerging behavior of users for the endogenous setting. Since $\mu^g(\mathbf{0}, \mathbf{0})$ is a fixed quantity representing the travel cost at the untolled $\mathbf{0}$ -equilibrium f^0 , we have that the best response of any coalition within a group g under the endogenous equilibrium, when minimizing each user's individual travel cost $\mu^g(\mathbf{0}, \mathbf{0}) - c_g$, is to maximize c_g .

Next, since for each user group g , c_g is monotonically non-decreasing in $C_{\mathbf{0}} - C_{\tau^*}$, we have that c_g is maximized for each user group g when $C_{\mathbf{0}} - C_{\tau^*}$ is maximized. Since $C_{\mathbf{0}}$ is fixed, we have that $C_{\mathbf{0}} - C_{\tau^*}$ is maximized for any flow f^* with the minimum total system cost. This implies that each user's non-negative transfer c_g is maximized for any flow f^* with the minimum total system cost. Thus, any exogenous τ^* -equilibrium flow f^* that achieves the minimum total system cost is also an endogenous equilibrium, since a deviation by any coalition of users in group g can never result in a higher non-negative transfer c_g than that at the minimum total system cost solution. \square

Lemma 4 establishes that any exogenous equilibrium flow arising from an optimal CPRR scheme is also an endogenous equilibrium. We note that the condition that the non-negative transfer c_g for any group g is monotonically non-increasing in C_{τ} is not demanding. For instance, the optimal refunding scheme, i.e., the one minimizing wealth inequality, corresponding to the discrete Gini coefficient respects this monotonicity relation, as observed in Proposition 3.

7 Discussion

A core tenet of sustainable transportation entails achieving a balance between *economic*, *equity* and *environmental* goals [40]. The results demonstrated in this paper challenge the traditional notion that these goals are in tension with each other by making progress towards achieving each of these sustainable transportation goals simultaneously. In particular, our work directly addresses the economic and equity goals through the development of CPRR schemes that both minimize the total system cost and reverse the wealth inequality effects of congestion pricing. Furthermore, the schemes we develop achieve another economic goal—all users are left at least as well off under the CPRR schemes as compared to that prior to any implementation of congestion pricing or refunds. This property suggests that users would *favor* this pricing and refunding scheme. Finally, as the environmental impact of a scheme is often proportional to the total travel time of all users, we note that the total system cost objective, which we seek to minimize within optimal CPRR schemes (Theorem 1), can be treated as an imperfect proxy for the total environmental pollution in the system. Environmental goals can be more directly incorporated within a CPRR scheme through appropriate congestion pricing schemes, e.g., aiming to minimize air pollution, while potentially improving total system cost and wealth inequality (Proposition 1).

Our work demonstrated that if we look at congestion pricing from the lens of refunding the collected tolls then we can not only achieve system efficiency but also reduce wealth inequality. As a result, we view our work as a significant step in shifting the discussion around congestion pricing from one that has focused on the societal inequity impacts of road tolls to one that centers around how to best distribute the revenues collected to different sections of society. While refunding toll revenues is not a novel idea and has been proposed as early as in [25], our work provided a thorough characterization of how such schemes can be designed to simultaneously achieve system efficiency and equity objectives. Furthermore, in doing so, we ensured that all users are at least as well off as compared to before the introduction of the CPRR scheme, thereby making the scheme publicly acceptable to all groups of society.

We believe that the results of our work pave the way for the design of sustainable, publicly-acceptable congestion-pricing schemes, but significant practical challenges remain. For instance, we assume centralized knowledge of the values-of-time of each user group. In practice these may not be known, and could confound successful implementation of an optimal CPRR scheme. It is also important to note the degree to which the CPRR scheme is successful relies on the full implementation of the tolls *and* refunds. If policymakers implement the congestion pricing scheme but fail to deliver refunds, low-income users of the system will be made worse off, facing higher costs, worse travel times, or both. Underprivileged residents would have legitimate claims that the system was not working, undermining public trust in the system. Thus the

onus is on policy makers to manage the entire life cycle of the CPRR scheme and ensure its successful and sustainable implementation. The difference between an equitable, optimal congestion pricing scheme and one that disproportionately burdens the poor depends significantly on how the toll revenue is spent.

8 Conclusion

In this paper, we studied and designed user-favorable congestion pricing and revenue refunding (CPRR) schemes that mitigate the regressive wealth inequality effects of congestion pricing. We considered two models of user behaviour: (i) users minimizing a linear function of their travel times and tolls, without considering refunds, and (ii) users explicitly taking into account refunds in their travel costs. Using the first model, we developed CPRR schemes that improved both system efficiency and wealth inequality, while being favorable for all users, as compared to the untolled outcome. We then characterized the set of optimal CPRR schemes. Finally, we showed that even when users endogenize the effect of refunds on their travel decisions, the resultant equilibrium remains the same under any optimal revenue-refunding scheme.

There are several interesting directions for further research. The first would be to relax some of the commonly-used assumptions in transportation research and game theory, to improve the applicability to practice. One example is to consider nonlinear travel cost functions. In addition, we currently assume time-invariant travel demand and traffic flows, which motivates the possible generalization to dynamic settings, e.g., through the incorporation of the cell transmission model [41]. We have also assumed that the only decisions made by users are route choices, whereas in reality there are other options, such as changing departure time or travel mode. A possible way to overcome this limitation is by incorporating elastic-demand models into our traffic-assignment formulations [34, 42].

It would also be interesting to extend these results to the setting of anonymous revenue refunding schemes that do not rely on any knowledge of each individual user’s value-of-time. Furthermore, it would be valuable to explore more complex behavior such as inter-group coalitions [39]. Finally, it would be worthwhile to investigate refunding mechanisms wherein some portion of the collected revenues is not directly refunded to users as lump-sum transfers, but instead used to cover operational costs or to improve transportation infrastructure.

References

- [1] A. Pigou. *Wealth and Welfare*. London, Macmillan and Co., 1912.
- [2] T. Roughgarden and É. Tardos. How bad is selfish routing? *J. ACM*, 49(2):236–259, 2002.
- [3] Y. Sheffi. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Prentice-Hall, 1985.
- [4] Kenneth A. Small and Jia Yan. The value of “value pricing” of roads: Second-best pricing and product differentiation. *Journal of Urban Economics*, 49(2):310 – 336, 2001.
- [5] Jonas Eliasson. Road pricing with limited information and heterogeneous users: A successful case. *The annals of regional science*, 35(4):595–604, 2001.
- [6] Michael Manville and Emily Goldman. Would congestion pricing harm the poor? do free roads help the poor? *Journal of Planning Education and Research*, 38(3):329–344, 2018.
- [7] Kurtulus Gemici, Elias Koutsoupas, Barnabé Monnot, Christos H. Papadimitriou, and Georgios Piliouras. Wealth Inequality and the Price of Anarchy. In *International Symposium on Theoretical Aspects of Computer Science*, volume 126 of *Leibniz International Proceedings in Informatics*, pages 31:1–31:16. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019. ISBN 978-3-95977-100-9.
- [8] Azi Paybarah. Congestion pricing: Mass transit savior or tax on the working class? *New York Times*, March 2019.
- [9] S. Jaensirisak, M. Wardman, and A. D. May. Explaining variations in public acceptability of road pricing schemes. *Journal of Transport Economics and Policy*, 39(2):127–153, 2005.

- [10] Martin Wachs. Then and now: The evolution of congestion pricing in transportation and where we stand today. In *Transportation Research Board Conference Proceedings*, 2005.
- [11] Di Wu, Yafeng Yin, Siriphong Lawphongpanich, and Hai Yang. Design of more equitable congestion pricing and tradable credit schemes for multimodal transportation networks. *Transportation Research Part B: Methodological*, 46(9):1273–1287, 2012.
- [12] Xiaolei Guo and Hai Yang. Pareto-improving congestion pricing and revenue refunding with multiple user classes. *Transportation Research Part B: Methodological*, 44(8):972–982, 2010.
- [13] D. Bertsimas, Vivek Farias, and N. Trichakis. The price of fairness. *Operations Research*, 59(1):17–31, 2011.
- [14] Cynthia Dwork, Moritz Hardt, Toniann Pitassi, Omer Reingold, and Richard Zemel. Fairness through awareness. In *Innovations in Theoretical Computer Science Conference*, page 214–226, New York, NY, USA, 2012. Association for Computing Machinery.
- [15] O. Jahn, R. Möhring, A. Schulz, and N. Stier-Moses. System-optimal routing of traffic flows with user constraints in networks with congestion. *Operations Research*, 53(4):600–616, 2005.
- [16] T. Roughgarden. How unfair is optimal routing? In *Symposium on Discrete Algorithms*, pages 203–204. ACM/SIAM, 2002.
- [17] E. Angelelli, I. Arsik, V. Morandi, M. Savelsbergh, and M. Speranza. Proactive route guidance to avoid congestion. *Transportation Research Part B: Methodological*, 94:1–21, 2016.
- [18] E. Angelelli, V. Morandi, and M. Speranza. Congestion avoiding heuristic path generation for the proactive route guidance. *Computers & Operations Research*, 99:234–248, 2018.
- [19] E. Angelelli, V. Morandi, M. Savelsbergh, and M. Speranza. System optimal routing of traffic flows with user constraints using linear programming. *European Journal of Operational Research*, 2020.
- [20] Devansh Jalota, Kiril Solovey, Stephen Zoepf, and Marco Pavone. Balancing fairness and efficiency in traffic routing via interpolated traffic assignment. *CoRR*, abs/2104.00098, 2021.
- [21] Martin L. Weitzman. Is the price system or rationing more efficient in getting a commodity to those who need it most? *Bell Journal of Economics*, 8(2):517–524, 1977.
- [22] Timothy Besley and Stephen Coate. Public provision of private goods and the redistribution of income. *The American Economic Review*, 81(4):979–984, 1991.
- [23] Daniele Condorelli. Market and non-market mechanisms for the optimal allocation of scarce resources. *Games and Economic Behavior*, 82:582–591, 2013.
- [24] Mohammad Akbarpour, Piotr Dworczak, and Scott Duke Kominers. Redistributive allocation mechanisms. GRAPE Working Papers 40, GRAPE Group for Research in Applied Economics, 2020. URL <https://ideas.repec.org/p/fme/wpaper/40.html>.
- [25] Kenneth A Small. Using the revenues from congestion pricing. *Transportation*, 19(4):359–381, 1992.
- [26] Phil B Goodwin. The rule of three: a possible solution to the political problem of competing objectives for road pricing. *Traffic engineering & control*, 30(10):495–497, 1989.
- [27] William S Vickrey. Congestion theory and transport investment. *The American Economic Review*, 59(2):251–260, 1969.
- [28] Richard Arnott, André de Palma, and Robin Lindsey. The welfare effects of congestion tolls with heterogeneous commuters. *Journal of Transport Economics and Policy*, 28(2):139–161, 1994.
- [29] Carlos F. Daganzo. A pareto optimum congestion reduction scheme. *Transportation Research Part B: Methodological*, 29(2):139–154, 1995.
- [30] Jeffrey L. Adler and Mecit Cetin. A direct redistribution model of congestion pricing. *Transportation Research Part B: Methodological*, 35(5):447–460, 2001.

- [31] Richard Cole, Yevgeniy Dodis, and Tim Roughgarden. Pricing network edges for heterogeneous selfish users. In *Symposium on Theory of Computing*, page 521–530. Association for Computing Machinery, 2003.
- [32] L. Fleischer, K. Jain, and M. Mahdian. Tolls for heterogeneous selfish users in multicommodity networks and generalized congestion games. In *Symposium on Foundations of Computer Science*, pages 277–285. IEEE, 2004.
- [33] Era Dabla-Norris, Kalpana Kochhar, Nujin Suphaphiphat, Frantisek Ricka, and Evridiki Tsounta. *Causes and consequences of income inequality: A global perspective*. International Monetary Fund, 2015.
- [34] Michael Patriksson. *The Traffic Assignment Problem: Models and Methods*. Dover Publications, 2015.
- [35] Hai Yang and Hai-Jun Huang. The multi-class, multi-criteria traffic network equilibrium and systems optimum problem. *Transportation Research Part B: Methodological*, 38(1):1–15, 2004.
- [36] H. Yang and H. J. Huang. *Mathematical and Economic Theory of Road Pricing*. Emerald Publishing, 2005.
- [37] Johannes Bisschop. *AIMMS optimization modeling*. lulu.com, 2016.
- [38] John Rawls. *A theory of justice*. Harvard university press, 2020.
- [39] Nir Andelman, Michal Feldman, and Yishay Mansour. Strong price of anarchy. *Games and Economic Behavior*, 65(2):289–317, 2009.
- [40] Ralph P. Hall and Joseph M. Sussman. Promoting the concept of sustainable transportation within the federal system - the need to reinvent the U.S. URL <http://dspace.mit.edu/handle/1721.1/102801>.
- [41] Carlos F Daganzo. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. *Transportation Research Part B: Methodological*, 28(4):269–287, 1994.
- [42] Nathan H Gartner. Optimal traffic assignment with elastic demands: A review part I. analysis framework. *Transportation Science*, 14(2):174–191, 1980.

9 Appendix

9.1 Constant Income Transfer Property

In this section, we show that the constant income transfer property follows directly from the regressive and progressive tax properties of the wealth inequality measure W , as claimed in Section 3. In particular, we show that if the initial income distribution is \mathbf{q} and each person is transferred a non-positive amount of money $-\lambda$, where $0 \leq \lambda < \min_{g \in \mathcal{G}} q_g$, then the wealth inequality cannot decrease, i.e., $W(\mathbf{q} - \lambda \mathbf{1}) \geq W(\mathbf{q})$.

We note that at the new income distribution $\bar{\mathbf{q}} = \mathbf{q} - \lambda \mathbf{1}$, each user in group g has the following income:

$$\bar{q}_g = q_g - \lambda = q_g \left(1 - \frac{\lambda}{q_g}\right).$$

Note that if $q_g \leq q_{g'}$ for any two groups g, g' , then $1 - \frac{\lambda}{q_g} \leq 1 - \frac{\lambda}{q_{g'}}$. Thus, by the regressive tax property, we observe that $W(\mathbf{q} - \lambda \mathbf{1}) \geq W(\mathbf{q})$. We finally note that the claim that $W(\mathbf{q} + \lambda \mathbf{1}) \leq W(\mathbf{q})$ for any $0 \leq \lambda < \min_{g \in \mathcal{G}} q_g$ follows by a similar analysis wherein we use the progressive tax property. This proves our claim that the wealth inequality measure W satisfies the constant income transfer property.

9.2 Proof of Equation 5

In this section, we use the first order necessary and sufficient KKT conditions of the well studied multi-class user equilibrium optimization problem [35]

$$\mathbf{f} = \arg \min_{\mathbf{f}' \in \Omega} \sum_{e \in E} \int_0^{x'_e} t_e(\omega) d\omega + \sum_{e \in E} \sum_{g \in \mathcal{G}} \frac{1}{v_g} x'_e{}^g \tau_e,$$

to prove that the following holds:

$$C_{\boldsymbol{\tau}} = \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}, \mathbf{0}) d_g - \sum_{e \in E} \tau_e x_e. \quad (7)$$

Here $\boldsymbol{\tau}$ is congestion-pricing scheme and \mathbf{f} is an exogenous $\boldsymbol{\tau}$ -equilibrium with edge flow representation \mathbf{x} . Note that the edge flows \mathbf{x} are unique by the strict convexity of the travel time function.

The following exogenous-equilibrium conditions follow directly from the KKT conditions of the above optimization problem:

$$\begin{aligned} \sum_{e \in P} (v_g t_e(x_e) + \tau_e) &= \mu^g(\boldsymbol{\tau}, \mathbf{0}), \quad \text{if } f_{P,g} > 0, P \in \mathcal{P}_g, g \in \mathcal{G}, \\ \sum_{e \in P} (v_g t_e(x_e) + \tau_e) &\geq \mu^g(\boldsymbol{\tau}, \mathbf{0}), \quad \text{if } f_{P,g} = 0, P \in \mathcal{P}_g, g \in \mathcal{G}. \end{aligned}$$

From the above equilibrium conditions and the fact that the sum of the path flows for any group adds up to d_g , i.e., $\sum_{P \in \mathcal{P}_g} f_{P,g} = d_g$, we obtain that:

$$\begin{aligned} \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}, \mathbf{0}) d_g &= \sum_{g \in \mathcal{G}} \sum_{P \in \mathcal{P}_g} f_{P,g} \mu^g(\boldsymbol{\tau}, \mathbf{0}), \\ &= \sum_{g \in \mathcal{G}} \sum_{P \in \mathcal{P}_g} f_{P,g} \sum_{e \in P} (v_g t_e(x_e) + \tau_e), \\ &= \sum_{g \in \mathcal{G}} \sum_{P \in \mathcal{P}_g} f_{P,g} \sum_{e \in E} (v_g t_e(x_e) + \tau_e) \delta_{e,P}, \\ &= \sum_{e \in E} \sum_{g \in \mathcal{G}} \sum_{P \in \mathcal{P}_g} f_{P,g} (v_g t_e(x_e) + \tau_e) \delta_{e,P}, \\ &= \sum_{e \in E} \sum_{g \in \mathcal{G}} \sum_{P \in \mathcal{P}_g: e \in P} f_{P,g} (v_g t_e(x_e) + \tau_e), \\ &= \sum_{e \in E} \sum_{g \in \mathcal{G}} (v_g t_e(x_e) + \tau_e) \sum_{P \in \mathcal{P}_g: e \in P} f_{P,g}, \\ &= \sum_{e \in E} \sum_{g \in \mathcal{G}} x_e^g (v_g t_e(x_e) + \tau_e), \\ &= \sum_{e \in E} \sum_{g \in \mathcal{G}} x_e^g v_g t_e(x_e) + \sum_{e \in E} x_e \tau_e \end{aligned}$$

where $\delta_{e,P} = 1$ if edge $e \in P$ and otherwise it is 0. Note that the above analysis implies Equation (5) since $C_{\boldsymbol{\tau}} = \sum_{e \in E} \sum_{g \in \mathcal{G}} x_e^g v_g t_e(x_e) = \sum_{g \in \mathcal{G}} \mu^g(\boldsymbol{\tau}, \mathbf{0}) d_g - \sum_{e \in E} x_e \tau_e$. This proves our claim.

Remark 2. We note that since the total tolls collected and user travel costs $\mu^g(\boldsymbol{\tau}, \mathbf{0})$ are unique at any equilibrium flow [12], the total travel cost $C_{\boldsymbol{\tau}}$ is also unique for any equilibrium induced by the edge tolls $\boldsymbol{\tau}$. Furthermore, the ex-post income of each user group g is also the same under any equilibrium induced by the edge tolls $\boldsymbol{\tau}$ since the user travel cost $\mu^g(\boldsymbol{\tau}, \mathbf{0})$ is unique at any equilibrium flow [12].

9.3 Proof of Lemma 1

Lemma 1 is a slight variation on a previous result [12]. We provide a proof for this modified lemma for the sake of completeness.

To prove the lemma, we first show that the total sum of the revenue refunds $\sum_{g \in \mathcal{G}} r_g d_g$ is equal to the total revenue collected $\sum_{e \in E} \tau_e x_e$, where x_e is the flow on edge e corresponding to the edge decomposition

of any exogenous τ -equilibrium flow \mathbf{f} . Indeed, it follows that

$$\begin{aligned}
\sum_{g \in \mathcal{G}} r_g d_g &= \sum_{g \in \mathcal{G}} \left[\mu^g(\tau, \mathbf{0}) - \mu^g(\mathbf{0}, \mathbf{0}) + \frac{\alpha_g}{d_g} (C_{\mathbf{0}} - C_{\tau}) \right] d_g \\
&= \sum_{g \in \mathcal{G}} \mu^g(\tau, \mathbf{0}) d_g - \sum_{g \in \mathcal{G}} \mu^g(\mathbf{0}, \mathbf{0}) d_g + \sum_{g \in \mathcal{G}} \alpha_g (C_{\mathbf{0}} - C_{\tau}) \\
&= C_{\tau} + \sum_{e \in E} \tau_e x_e - C_{\mathbf{0}} + C_{\mathbf{0}} - C_{\tau} \\
&= \sum_{e \in E} \tau_e x_e,
\end{aligned}$$

where we have used Equation (5) to obtain that $C_{\tau} = \sum_{g \in \mathcal{G}} \mu^g(\mathbf{f}, \tau, \mathbf{0}) d_g - \sum_{e \in E} \tau_e x_e$. Furthermore, we leveraged Equation (5) for the untolled outcome to obtain that $C_{\mathbf{0}} = \sum_{g \in \mathcal{G}} \mu^g(\mathbf{0}, \mathbf{0}) d_g$.

Next, we show that this CPRR scheme is user-favorable, which follows from the following chain of inequalities:

$$\begin{aligned}
\mu^g(\tau, \mathbf{r}) &= \mu^g(\tau, \mathbf{0}) - r_g \\
&= \mu^g(\tau, \mathbf{0}) - \left(\mu^g(\tau, \mathbf{0}) - \mu^g(\mathbf{0}, \mathbf{0}) + \frac{\alpha_g}{d_g} (C_{\mathbf{0}} - C_{\tau}) \right) \\
&= \mu^g(\mathbf{0}, \mathbf{0}) - \alpha_g (C_{\mathbf{0}} - C_{\tau}) \\
&\leq \mu^g(\mathbf{0}, \mathbf{0}).
\end{aligned}$$

□

9.4 Proof of Proposition 2

We show that there exists a two O-D pair setting such that for any user-favorable CPRR scheme (τ, \mathbf{r}) it holds that $W(\mathbf{q}(\tau, \mathbf{r})) \geq W(\mathbf{q}^0)$.

We begin by formally defining the instance depicted in Figure 4. Consider a graph with four nodes, v_1, v_2, v_3, v_4 and three edges $e_1 = (v_1, v_2)$, $e_2 = (v_3, v_4)$ and $e_3 = (v_3, v_4)$, where there are two possible ways to get from v_3 to v_4 . We define the travel time on edge e_1 as $t_1(x_1) = \frac{x_1}{2}$, that on edge e_2 as $t_2(x_2) = x_2$ and that on edge e_3 as $t_3(x_3) = 1$. Further consider two user types, one with a high income q_H and value-of-time ωq_H that make trips between O-D pair $w_H = (v_1, v_2)$, and the other with a low income q_L and value-of-time ωq_L that make trips between O-D pair $w_L = (v_3, v_4)$. Let the demand of the high income users be $d_H = 1$ and that of the low income users be $d_L = 1$. Then at the untolled user equilibrium outcome it follows that all high income users traverse their only edge e_1 , while all the low income users traverse the edge e_2 . At this equilibrium flow, the cost to the high income users is $\omega q_H \frac{1}{2}$, since the travel time of the edge e_1 is $\frac{1}{2}$, and that to the low income users is ωq_L , since the travel time on edge e_2 is one.

Next, we note that under any CPRR scheme (τ, \mathbf{r}) users in the high income group will continue to use edge e_1 since this is the only available edge on which they can travel. Thus, for this scheme to be user-favorable it must be that any tolls collected from the high income users is directly refunded back within the groups. To see this, if there were tolls collected from high-income users that were given to low income users then some high income users would incur strictly higher costs than at the untolled $\mathbf{0}$ -equilibrium outcome. We similarly observe that all collected refunds from the low income groups must be completely refunded to users within the low income group to ensure that the CPRR scheme is user-favorable. Note that the above argument stems from the fact that the travel paths of the two user groups are completely disjoint, and so any CPRR scheme (τ, \mathbf{r}) must refund all the collected revenues from each user group directly back to that user group to ensure that the scheme is user-favorable.

Thus, we have for any user-favorable CPRR scheme (τ, \mathbf{r}) that all the users incur the same costs as that under the $\mathbf{0}$ -equilibrium outcome. Now, under the untolled user equilibrium, we observe that the ex-post income of the high income group is $q_H = q_H - \beta \frac{\omega q_H}{2} = q_H(1 - \beta \frac{\omega}{2})$ and the ex-post income of the low income group is $q_L = q_L - \beta \omega q_L = q_L(1 - \beta \omega)$. The above analysis implies that the untolled user equilibrium outcome

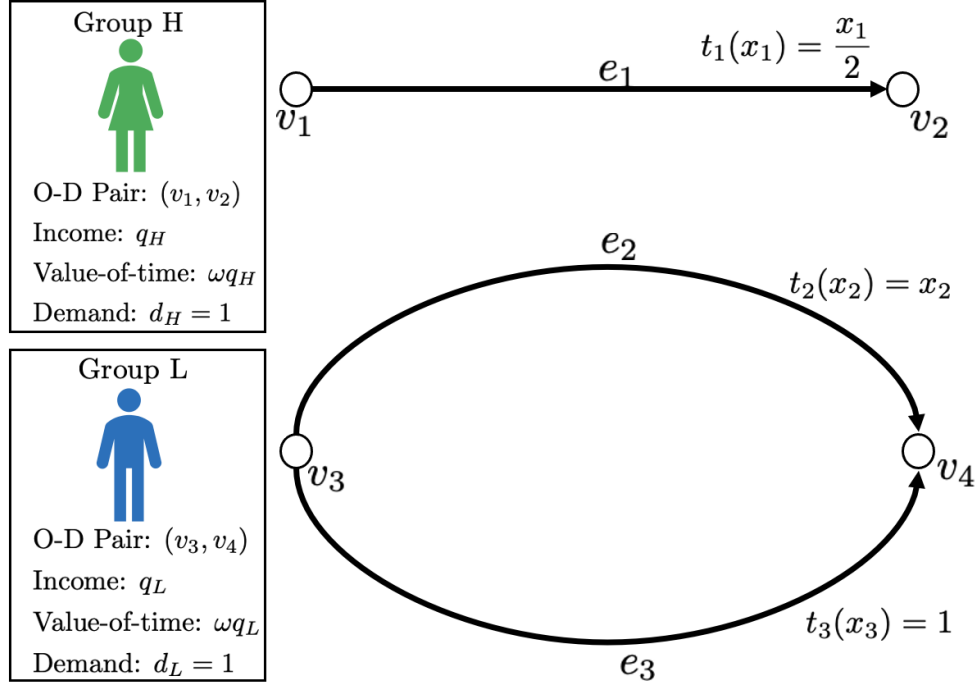


Figure 4: A two O-D pair and two user group instance for which the wealth inequality of the ex-post income distribution under any CPRR scheme is at least the wealth inequality of the ex-ante income distribution. Under the above defined attributes for the different user groups, the user group H with a higher income and value-of-time incurs a strictly lower cost as a proportion of their income as compared to user group L , indicating the regressive nature of any valid CPRR scheme.

results in a regressive tax, i.e., lower income users are charged a greater fraction of their incomes than higher income users. Since the function W satisfies the property that regressive taxes increase inequality, we have that the wealth inequality of the ex-post income distribution is greater than that of the ex-ante income distribution. \square

9.5 Discrete Gini Coefficient Satisfies the Regressive Income Tax Property

We show that for any two income profiles \mathbf{q} and $\tilde{\mathbf{q}}$ with $\tilde{q}_g = \delta_g q_g$, where $0 < \delta_g \leq \delta_{g'}$ if $q_g \leq q_{g'}$ for any two groups g, g' , then $W(\tilde{\mathbf{q}}) \geq W(\mathbf{q})$ for the discrete Gini coefficient wealth inequality measure.

To prove this, we first note by the scale independence property that, we can restrict our attention to scaling factors δ_g for each g that leave the mean income of all users unchanged, i.e., $\Delta(\tilde{\mathbf{q}}) = \Delta(\mathbf{q})$. This is because, if the mean income of the new income profile $\tilde{\mathbf{q}}$ is different than that of \mathbf{q} then we can multiply the new incomes with a scaling constant $\lambda > 0$ to ensure that the mean income of $\lambda\tilde{\mathbf{q}}$ is exactly that of \mathbf{q} . Then, by scale independence we have that $W(\lambda\tilde{\mathbf{q}}) = W(\tilde{\mathbf{q}})$, and so, without loss of generality, we focus on the set of scaling factors δ_g for each g , such that $\Delta(\tilde{\mathbf{q}}) = \Delta(\mathbf{q})$.

Next, we note the following:

$$\begin{aligned}
W(\mathbf{q}) &= \frac{1}{2 \left(\sum_{g \in \mathcal{G}} d_g \right)^2 \Delta(\mathbf{q})} \sum_{g_1, g_2 \in \mathcal{G}} d_{g_1} d_{g_2} |q_{g_1} - q_{g_2}| \\
&= \frac{1}{2 \left(\sum_{g \in \mathcal{G}} d_g \right)^2 \Delta(\tilde{\mathbf{q}})} \sum_{g_1, g_2 \in \mathcal{G}} d_{g_1} d_{g_2} |q_{g_1} - q_{g_2}| \\
&\leq \frac{1}{2 \left(\sum_{g \in \mathcal{G}} d_g \right)^2 \Delta(\tilde{\mathbf{q}})} \sum_{g_1, g_2 \in \mathcal{G}} d_{g_1} d_{g_2} |\delta_{g_1} q_{g_1} - \delta_{g_2} q_{g_2}| \\
&= \frac{1}{2 \left(\sum_{g \in \mathcal{G}} d_g \right)^2 \Delta(\tilde{\mathbf{q}})} \sum_{g_1, g_2 \in \mathcal{G}} d_{g_1} d_{g_2} |\tilde{q}_{g_1} - \tilde{q}_{g_2}| \\
&= W(\tilde{\mathbf{q}}),
\end{aligned}$$

where the second equality follows since $\Delta(\tilde{\mathbf{q}}) = \Delta(\mathbf{q})$, and the third inequality follows since $|q_{g_1} - q_{g_2}| \leq |\delta_{g_1} q_{g_1} - \delta_{g_2} q_{g_2}|$ for any g_1, g_2 as $0 < \delta_g \leq \delta_{g'}$ if $q_g \leq q_{g'}$ for any two groups g, g' .

Thus, we have shown that the discrete Gini coefficient wealth inequality measure satisfies the regressive tax property.

9.6 Proof of Proposition 3

We first note that since $\frac{1}{2 \left(\sum_{g \in \mathcal{G}} d_g \right)^2 \Delta(\mathbf{q}(\mathbf{0}, \mathbf{0}) + \mathbf{c})}$ is a constant the optimal solution of the linear Program (6a)-(6c) is equal to the optimal solution of the following linear program

$$\min_{c_g} \sum_{g_1, g_2 \in \mathcal{G}} d_{g_1} d_{g_2} |\tilde{q}_{g_1} + c_{g_1} - \tilde{q}_{g_2} - c_{g_2}|, \quad (8a)$$

$$\text{s.t.} \quad \sum_{g \in \mathcal{G}} c_g d_g = C_0 - C_\tau, \quad (8b)$$

$$c_g \geq 0, \quad \forall g \in \mathcal{G}, \quad (8c)$$

where $\tilde{q}_g = q_g(\mathbf{0}, \mathbf{0})$ for conciseness.

We now compute the optimal solution of the linear Program (8a)-(8c) by deriving the first order conditions of the optimization problem. To this end, let λ be the dual variable of the Constraint (8b) and l_g be the dual variable of the Constraint (8c) for each group g . Then, we have the following first order derivative condition of the Lagrangian of Problem (8a)-(8c):

$$2 \sum_{g_2: q_{g_2} + r_{g_2} < q_g + r_g} d_{g_2} d_g - 2 \sum_{g_2: q_{g_2} + r_{g_2} > q_g + r_g} d_{g_2} d_g - \lambda d_g - l_g = 0, \quad \forall g \in \mathcal{G}.$$

Since $l_g \geq 0$ for each group g , the above equation implies for each group g that:

$$\begin{aligned}
2 \sum_{g_2: q_{g_2} + r_{g_2} < q_g + r_g} d_{g_2} - 2 \sum_{g_2: q_{g_2} + r_{g_2} > q_g + r_g} d_{g_2} &\geq \lambda, \quad \forall g \in \mathcal{G}, \\
2 \sum_{g_2: q_{g_2} + r_{g_2} < q_g + r_g} d_{g_2} - 2 \sum_{g_2: q_{g_2} + r_{g_2} > q_g + r_g} d_{g_2} &= \lambda, \quad \forall g \in \mathcal{G}, \text{ s.t. } c_g > 0.
\end{aligned}$$

We thus observe that the income group(s) that receive strictly positive transfers $c_g > 0$ are those for whom the above equation is met with an equality. Since λ is a fixed quantity, it follows that the above equation is met with equality for groups g with the minimum value of the following term:

$$2 \sum_{g_2: q_{g_2} + r_{g_2} < q_g + r_g} d_{g_2} - 2 \sum_{g_2: q_{g_2} + r_{g_2} > q_g + r_g} d_{g_2}$$

Note that this term is the smallest only for the lowest income groups after their revenue refunds. That is, all groups that receive a refund that results in a strictly positive transfer $c_g > 0$ have exactly the same income.

The above observation implies that one way to achieve the optimal solution of the linear Program (6a)-(6c) is to provide positive transfers to the lowest income users until their incomes equalize with the second lowest income group. Then both these groups can be given revenue until their income rises to the third lowest income group and so on, as described in Algorithm 1. This process can be repeated until the total pool of transfers $C_0 - C_\tau$ is completely exhausted. \square

9.7 Proof of Proposition 4

We now complete the proof of the counterexample presented in Section 6.1. In particular, using the scenario described in the sketch-of-proof, we first derive the untolled user equilibrium solution, which allows to characterize the refunding scheme \mathbf{r} . Then we compute an exogenous equilibrium and finally show that this is not an endogenous equilibrium by describing a deviation which improves the travel costs of users in group M .

Untolled User Equilibrium. Under this setting it is easy to see that at the user equilibrium the flow on the two edges are $x_1^{UE} = x_2^{UE} = 4$ giving a travel time of 8 to all users. Now, under the UE solution, the costs to the three groups are:

$$\begin{aligned}\text{Group H User Travel Cost} &= 8(0.002q_M) = 0.016q_M, \\ \text{Group M User Travel Cost} &= 0.008q_M, \\ \text{Group L User Travel Cost} &= 0.008q_L.\end{aligned}$$

Exogenous Equilibrium Under Tolls. Now consider a setting with tolls where $\tau_1 = 0.008q_M$. In this setting, the traffic equilibrium (when refunds are exogenous) is given by $x_1^{\tau_1} = 2$ and $x_2^{\tau_1} = 6$, since only users in group H are willing to travel on edge 1 at this toll. This leads to a travel time of $t_1(x_1^{\tau_1}) = 4$ on edge 1, and a travel time of $t_2(x_2^{\tau_1}) = 10$ on edge 2. Note that under this equilibrium, the costs to the different groups are:

$$\begin{aligned}\text{Group H User Travel Cost without Refunds} &= 4(0.002q_M) + 0.008q_M = 0.016q_M, \\ \text{Group M User Travel Cost without Refunds} &= 10(0.001q_M) = 0.010q_M, \\ \text{Group L User Travel Cost without Refunds} &= 0.010q_L.\end{aligned}$$

Note that the total toll revenues collected are $x_1^{\tau_1}\tau_1 = 2(0.008q_M) = 0.016q_M$. Based on the refunding policy, we must first ensure that we give enough refunds so that each user incurs the same cost as that under the untolled user equilibrium. Then any remaining refunds are given to users based on the policy given in Algorithm 1. Following this procedure of giving refunds, we obtain the following set of refunds for each user group:

$$\begin{aligned}\text{Total Refund given to Group H} &= 0, \\ \text{Total Refund given to Group M} &= 0.002q_M, \\ \text{Total Refund given to Group L} &= 0.014q_M.\end{aligned}$$

Note here that users in group H are given no refund since their cost under the tolled exogenous equilibrium is the same as that under the untolled user equilibrium. Group M is given a refund that is exactly equal to the difference travel cost of the user group under the tolled exogenous equilibrium and the untolled user equilibrium, i.e., $0.010q_M - 0.008q_M = 0.002q_M$. Finally, the remaining refund of $0.016q_M - 0.002q_M = 0.014q_M$ is given to users in group L . This is because each user in group L receives a refund of $\frac{0.014q_M}{5}$ and at this amount of refund, users in group M and L have exactly the same income since $q_M(1 - 0.008) = q_L(1 - 0.010) + \frac{0.014q_M}{5}$.

Profitable Deviation under Endogenous Equilibrium. Now, we claim that users in group M have a profitable deviation under the above refunding policy. In particular, we can consider the deviation where all of the users in group M deviate to link 1. In this case, we have the edge flows $\tilde{x}_1^{\tau_1} = 3$ and $\tilde{x}_2^{\tau_1} = 5$, and so the travel time on edge 1 is $t_1(x_1^{\tau_1}) = 6$ and that on edge 2 is $t_2(x_2^{\tau_1}) = 9$. This leads to the following total cost to the different groups without refund:

$$\begin{aligned} \text{Group H User Travel Cost without Refunds} &= 6(0.002q_M) + 0.008q_M = 0.020q_M, \\ \text{Group M User Travel Cost without Refunds} &= 6(0.001q_M) + 0.008q_M = 0.014q_M, \\ \text{Group L User Travel Cost without Refunds} &= 9(0.001q_L) = 0.009q_L. \end{aligned}$$

Then we have that the total tolls collected are $\tilde{x}_1^{\tau_1}\tau_1 = 3(0.008q_M) = 0.024q_M$. Now, we must transfer the amount $0.004q^M$ to group H to equalize their cost to the untolled UE cost. For group M , we must transfer $0.006q^M$ to equalize their cost to the untolled UE cost. This leaves a total refund of $0.014q^M$. Now, we will distribute this remaining income such that the incomes of users in group M and L are equal. Note that in doing so we will also have ensured that users in group L are at least as well off as under the untolled user equilibrium outcome. In particular, we look to find the value y that satisfies the following equation:

$$\frac{y}{5}q_M + q_L(1 - 0.009) = (0.014 - y)q_M + q_M(1 - 0.008).$$

Here y is some constant. We can solve for y using the above relation that $q_M(1 - 0.008) = q_L(1 - 0.010) + \frac{0.014q_M}{5}$, which gives us that $0.014 - y > 0$, implying that users in group M get strictly lower costs than the untolled user equilibrium outcome. Since the cost of user group M was exactly their untolled user equilibrium cost, we observe here that user group M has a profitable deviation implying that the exogenous equilibrium is not an equilibrium when users take revenue refunds into account in their travel decisions. \square