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### *On Parabolic Restriction of Perverse Sheaves*

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## ON PARABOLIC RESTRICTION OF PERVERSE SHEAVES

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*To Professor Masaki Kashiwara with admiration on the occasion of his 70th birthday*

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## 1. INTRODUCTION

1.1. **Main result.** Let  $G$  be a connected reductive group over a finite field  $\mathbb{F}_q$ . Let  $P \subset G$  be a parabolic, with Levi quotient  $L = P/U$ . One has an adjoint pair of functors of *parabolic induction* and *parabolic restriction*

$$pind_G^P : D(L \backslash L) \rightleftarrows D(G \backslash G) : pres_P^G.$$

Here, by  $D(H \backslash H)$  we mean the equivariant derived category of constructible  $\bar{\mathbb{Q}}_\ell$ -sheaves on  $H$ , equivariant with respect to the adjoint action of  $H$  on itself.

Grothendieck's *sheaf-to-function* correspondence relates these functors to the operators on the space of functions on the corresponding Chevalley groups which send the character of a representation to the character of its parabolic induction and parabolic restriction, respectively.

Recall the subcategory  $D(G \backslash G)^\heartsuit \subset D(G \backslash G)$  of *perverse sheaves*. Lusztig introduced a subcategory  $D(G \backslash G)^{\text{ch}} \subset D(G \backslash G)$ , the subcategory of *character sheaves*, such that  $D(G \backslash G)^{\heartsuit, \text{ch}} := D(G \backslash G)^\heartsuit \cap D(G \backslash G)^{\text{ch}}$  matches tightly under the above-mentioned dictionaries with representations in  $\text{Rep}(G(\mathbb{F}_q))$ . The definition of this subcategory is geometric, it applies to reductive algebraic groups over an arbitrary field.

The functors of parabolic induction and parabolic restriction take character sheaves to character sheaves. One fundamental result of Lusztig is the following:

**Theorem** ([Lu2]). *The functors  $\text{pres}_P^G$  and  $\text{pind}_G^P$ , when restricted to the subcategories of character sheaves, are  $t$ -exact; In other words, they transform perverse character sheaves into perverse character sheaves.*

The main theorem of this note is the following generalization of the above theorem:

**Theorem** (Theorem 5.4). *The functors  $\text{pres}_P^G$  and  $\text{pind}_G^P$  are  $t$ -exact; In other words, they transform perverse sheaves into perverse sheaves.*

Thus, we extend  $t$ -exactness from character sheaves to all adjoint-equivariant sheaves.

Our method works equally well for  $\ell$ -adic sheaves, constructible sheaves in the complex-analytic topology and (not necessarily holonomic)  $D$ -modules (see §2.2 for the list of “sheaf-theoretic settings”).

Our method also works in the close setting of  $G$ -equivariant sheaves on the Lie algebra  $\mathfrak{g}$  (see §5.4). Let us remark that, in the case of  $G$ -equivariant  $D$ -modules on  $\mathfrak{g}$ , the theorem was established earlier in [Gu]. However, in *loc. cit.* the tools of Fourier transform (which is not available in the group case) and singular support (which has been defined for  $\ell$ -adic sheaves only recently [Be], [Sa]) are used. Sam Gunningham has informed us that he also expects his methods to generalize to the group case.

Our method is based on a known expression for the dimension of the intersection of a conjugacy class with a Borel subgroup (see §3, and in particular proposition 3.1), it also relies on Braden's Theorem on the behavior of hyperbolic restriction with respect to Verdier duality [Bra], [DrGa2].

**1.2. A conjecture.** In section 6 we recall the fundamental *Harish-Chandra transform*

$$\text{HC}_* : D(G \backslash G) \rightarrow D(G \backslash (G/U \times G/U)/T)$$

and the *long intertwining transform*

$$R_! : D(G \backslash (G/U \times G/U)/T) \rightarrow D(G \backslash (G/U \times G/U^-)/T).$$

By works [BeFiOs] and [ChYo], the composition  $R_! \circ \text{HC}_*$  is  $t$ -exact when applied to character sheaves. We propose the following conjectural generalization:

**Conjecture** (Conjecture 6.2). *The functor  $R_! \circ \text{HC}_*$  is  $t$ -exact.*

We also explain how the main theorem of this note (in the Borel case) is an evidence towards this conjecture.

**1.3. Two applications.** In the work [GaLo], which deals with perverse sheaves on a torus  $T$ , a  $t$ -exact and conservative Mellin transform

$$\mathcal{M}_* : D(T) \rightarrow D_{coh}^b(\mathcal{C}(T))$$

is introduced, where  $\mathcal{C}(T)$  is, roughly, the space of tame local systems of rank 1 on  $T$ . The Euler characteristic of a perverse sheaf  $\mathcal{F} \in D(T)^\heartsuit$  is interpreted as the generic rank of  $\mathcal{M}_*(\mathcal{F})$ , and hence in particular is non-negative. A closely related fact is that the convolution of a perverse sheaf  $\mathcal{F} \in D(T)^\heartsuit$  with  $\mathcal{L}_\chi \in D(T)^\heartsuit$ , the perverse local system of rank 1 corresponding to  $\chi \in \mathcal{C}(T)$ , is perverse for generic  $\chi$ : in fact  $\mathcal{F} * \mathcal{L}_\chi = V_{\mathcal{F}}(\chi) \otimes \mathcal{L}_\chi$  where the vector space  $V_{\mathcal{F}}(\chi)$  is the fiber of  $\mathcal{M}_*(\mathcal{F})$  at the point  $\chi$ .

We generalize the above two properties replacing the torus by a general reductive group.

**Theorem** (Theorem 7.1). *Let  $\mathcal{F} \in D(G \backslash G)^\heartsuit$ , i.e.  $\mathcal{F}$  is a perverse sheaf on  $G$ , equivariant with respect to the adjoint  $G$ -action. Then the Euler characteristic of  $\mathcal{F}$  is non-negative.*

We prove this theorem by reducing to the torus case, via parabolic restriction. The result is, as far as we know, new in the  $\ell$ -adic setting as well as in the holonomic  $D$ -module setting, and we refer the reader to remark 7.2 for its history and previously known cases.

**Proposition** (Proposition 7.6). *Let  $\mathcal{F} \in D(G \backslash G)^\heartsuit$  be a  $G$ -equivariant perverse sheaf on  $G$  and let  $\mathcal{G} \in D(G \backslash G)^\heartsuit$  be a perverse character sheaf with generic central character (here “generic” depends on  $\mathcal{F}$ ). Then  $\mathcal{F} * \mathcal{G}$  is perverse (where “ $*$ ” denotes convolution on the group).*

This proposition is also proved by reducing to the torus case, via parabolic restriction and induction.

It is tempting to conjecture that Theorem 7.1 and Proposition 7.6 are related to the yet unknown “Mellin transform for reductive groups” which would relate  $D(G \backslash G)^\heartsuit$  to sheaves on a space parametrizing character sheaves, just as their special case established in [GaLo] is related to the functor  $\mathcal{M}_*$ .

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## 2. NOTATIONS AND CONVENTIONS

**2.1. Group-theoretic notations.** We work over an algebraically closed ground field.

We fix a connected reductive algebraic group  $G$ , and a parabolic  $L := \frac{P}{U} \leftarrow^\pi P \rightarrow G$ . Thus, we denote by  $U$  the unipotent radical of  $P$ , by  $L$  the Levi quotient of  $P$ , and by  $\pi : P \rightarrow L$  the quotient map.

For a connected affine algebraic group  $H$ , we always understand  $H$  to act on itself via conjugation so  $H$ -invariant subvarieties and  $H$ -orbits are understood accordingly; as another example,  $H \backslash H$  denotes the quotient stack of  $H$  by the adjoint action of  $H$ . We use the following notations. For  $h \in H$ , we denote

$$\mathcal{O}_h^H := \{k h k^{-1} : k \in H\}.$$

Given an integer  $d \geq 0$ , we denote

$$H^{(d)} := \{h \in H \mid \dim \mathcal{O}_h^H = d\}.$$

If  $H$  acts on a variety  $X$ , we denote

$$X^h := \{x \in X \mid h x = x\}.$$

If  $H$  is reductive, we denote by  $X_H$  the variety of Borels in  $H$ .

**2.2. Sheaf-theoretic notations.** We only consider stacks which are of the form  $H \backslash X$  where  $H$  is an affine algebraic group acting on a variety  $X$ . For a stack  $\mathcal{X}$ , we denote by  $D(\mathcal{X})$  the derived category of “sheaves” on  $\mathcal{X}$ , meaning any one of the following “sheaf-theoretic contexts”<sup>1</sup>:

- (1) Non-holonomic  $D$ -module setting: The ground field is of characteristic zero, and we consider the unbounded derived category of all  $D$ -modules, such as in [DrGa1] etc.
- (2) Holonomic  $D$ -module setting: The ground field is of characteristic zero, and we consider the bounded derived category of holonomic  $D$ -modules.
- (3)  $\ell$ -adic setting: We fix a prime  $\ell$  different from the characteristic of the ground field, and consider the bounded derived category of constructible  $\ell$ -adic sheaves.
- (4) Complex-analytic setting: The ground field is the field of complex numbers, and we consider the bounded derived category of sheaves in the complex-analytic topology, constructible w.r.t. finite algebraic stratifications.

In each setting,  $D(\mathcal{X})$  admits a natural  $t$ -structure (the “perverse” one). We denote by  $D(\mathcal{X})^{\leq 0}, D(\mathcal{X})^{\geq 0}$ , etc. the subcategories of objects concentrated in the specified cohomological degrees.

For a smooth stack  $\mathcal{X}$ , by a *smooth* sheaf in  $D(\mathcal{X})$  we will understand a locally constant sheaf in the sheaf-theoretic contexts (3) and (4), and a coherent  $D$ -module which is locally free as an  $\mathcal{O}$ -module - of finite rank in the sheaf-theoretic context (2) and of perhaps infinite rank in the sheaf-theoretic context (1).

For  $\mathcal{F} \in D(H \backslash X)$ , we denote by  $\underline{\mathcal{F}} \in D(X)$  the corresponding sheaf, i.e. the result of applying to  $\mathcal{F}$  the  $t$ -exact forgetful functor  $p^\circ = p^![-\dim H]$  where  $p : X \rightarrow H \backslash X$ .

We will prefer stating things with the  $(\pi^!, \pi_*)$ -versions (as opposed to  $(\pi^*, \pi_!)$ ) of functors, because they work better in the sheaf-theoretic context (1).

---

<sup>1</sup>Except of section 7, where we do not consider the  $D$ -module setting.

## 3. CONJUGACY CLASSES AND PARABOLICS

In this section, we will provide information on some dimensions involving conjugacy classes, which will be used in the proof of proposition 5.1. We recall that we fix a parabolic  $L \xleftarrow{\pi} P \rightarrow G$ .

**Proposition 3.1.** *Let  $g \in G$ , and let  $B \subset G$  be a Borel subgroup. Then*

$$\dim(\mathcal{O}_g^G \cap B) = \frac{1}{2} \dim \mathcal{O}_g^G.$$

*Proof.* One has

$$\dim \mathcal{O}_g^G + \dim X_G^g = \dim X_G + \dim(\mathcal{O}_g^G \cap B)$$

(because both sides represent the dimension of  $\{(h, x) \in \mathcal{O}_g^G \times X_G \mid hx = x\}$ ), and thus the desired equality is easily seen to be equivalent to the equality

$$\dim Z_G(g) = \text{rank}(G) + 2 \dim X_G^g.$$

For the latter, see [Hu, Chapter 6] and referencenes therein (for attributions, consult *loc. cit.*). In §6.17 of *loc. cit.* the general statement is derived from that for unipotent  $g$ . In §6.8 of *loc. cit.* the statement for unipotent  $g$  is derived from a “density condition”. References for the verification of this condition are given in §6.9 of *loc. cit.* (some low characteristic cases need to be treated independently).  $\square$

**Lemma 3.2** ([Lu1, Proposition 1.2 (a)]). *Let  $g \in G$  and  $\ell \in L$ . Then<sup>2</sup>*

$$\dim(\mathcal{O}_g^G \cap \pi^{-1}(\ell)) \leq \frac{1}{2} (\dim \mathcal{O}_g^G - \dim \mathcal{O}_\ell^L).$$

*Proof.* Let us provide a proof which assumes proposition 3.1; See [Lu1, Proposition 1.2 (a)] for a different proof.

Since

$$\dim(\mathcal{O}_g^G \cap \pi^{-1}(\ell)) = \dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L)) - \dim(\mathcal{O}_\ell^L),$$

the inequality to be established is equivalent to

$$\dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L)) \stackrel{?}{\leq} \frac{1}{2} (\dim(\mathcal{O}_g^G) + \dim(\mathcal{O}_\ell^L)).$$

Let us fix a Borel subgroup  $B \subset L$ . By considering the variety

$$Z_1 = \{(h, x) \in (\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L)) \times X_L \mid \pi(h)x = x\}$$

and its two projections, we see that

$$(3.1) \quad \dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L)) + \dim X_L^\ell = \dim Z_1 = \dim(X_L) + \dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L \cap B)).$$

Similarly, by considering the variety

$$Z_2 = \{(h, x) \in \mathcal{O}_\ell^L \times X_L \mid hx = x\}$$

and its two projections, we see that

$$(3.2) \quad \dim \mathcal{O}_\ell^L + \dim X_L^\ell = \dim Z_2 = \dim X_L + \dim(\mathcal{O}_\ell^L \cap B).$$

We thus have:

$$\begin{aligned} \dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L)) &\stackrel{\text{eq. 3.1}}{=} \dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L \cap B)) + (\dim X_L - \dim X_L^\ell) \stackrel{\text{eq. 3.2}}{=} \\ &= \dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L \cap B)) + (\dim \mathcal{O}_\ell^L - \dim(\mathcal{O}_\ell^L \cap B)) \stackrel{\text{prop. 3.1}}{=} \end{aligned}$$

<sup>2</sup>By convention, the inequality is understood to hold if  $\mathcal{O}_g^G \cap \pi^{-1}(\ell)$  is empty.

$$\begin{aligned}
&= \dim(\mathcal{O}_g^G \cap \pi^{-1}(\mathcal{O}_\ell^L \cap B)) + \frac{1}{2} \dim \mathcal{O}_\ell^L \leq \dim(\mathcal{O}_g^G \cap \pi^{-1}(B)) + \frac{1}{2} \dim \mathcal{O}_\ell^L \stackrel{\text{prop. 3.1}}{=} \\
&= \frac{1}{2} \dim \mathcal{O}_g^G + \frac{1}{2} \dim \mathcal{O}_\ell^L.
\end{aligned}$$

□

**Proposition 3.3.** *Let  $d, e \geq 0$ , and let  $V \subset G^{(d)}$  be a  $G$ -invariant subvariety. Then for every  $\ell \in L^{(e)}$  one has<sup>3</sup>*

$$\dim(V \cap \pi^{-1}(\ell)) \leq \dim V - \dim(V \cap \pi^{-1}(L^{(e)})).$$

*Proof.* For any given  $\ell \in L^{(e)}$ , there are only finitely many  $G$ -orbits  $\mathcal{O} \subset V$  for which  $\mathcal{O} \cap \pi^{-1}(\ell) \neq \emptyset$ . Hence, from lemma 3.2 we deduce

$$(3.3) \quad \dim(V \cap \pi^{-1}(\ell)) \leq \frac{1}{2}(d - e).$$

Thus, in order to establish the desired inequality, it is enough to establish

$$(3.4) \quad \frac{1}{2}(d - e) \stackrel{?}{\leq} \dim V - \dim(V \cap \pi^{-1}(L^{(e)})).$$

Let us denote by  $Z$  the closure inside  $L^{(e)}$  of  $\pi(V \cap \pi^{-1}(L^{(e)}))$ . By inequality 3.3 we have

$$\dim(V \cap \pi^{-1}(L^{(e)})) \leq \dim Z + \frac{1}{2}(d - e),$$

and thus inequality 3.4 will follow if we establish that

$$(3.5) \quad \dim Z - e \stackrel{?}{\leq} \dim V - d.$$

However, this inequality is clear, since the right hand side is equal to the dimension of the closure of  $\chi(V) \subset W \setminus \setminus T$ , where  $\chi : G \rightarrow W \setminus \setminus T$  is the usual "characteristic" map, while the left hand side is equal to the dimension of a subvariety of this closure. □

#### 4. PARABOLIC RESTRICTION AND PARABOLIC INDUCTION

In this section we recall basic facts about parabolic restriction and parabolic induction of adjoint-equivariant sheaves. We recall that we fix a parabolic  $L \xleftarrow{\pi} P \rightarrow G$ .

**4.1. Parabolic restriction.** Recall that the *parabolic restriction* functor

$$\text{pres}_P^G : D(G \setminus G) \rightarrow D(L \setminus L)$$

is given as  $q_* p^!$  where

$$(4.1) \quad \begin{array}{ccc} & P \setminus P & \\ p \swarrow & & \searrow q \\ G \setminus G & & L \setminus L \end{array}$$

and the actions of the groups on themselves are via conjugation.

Let us give two non-equivariant descriptions of parabolic restriction, which we will use later.

<sup>3</sup>By convention, the inequality is understood to hold if  $V \cap \pi^{-1}(\ell)$  is empty.

**Remark 4.1.** Consider the correspondence

$$(4.2) \quad \begin{array}{ccc} & P & \\ \iota \swarrow & & \searrow \pi \\ G & & L \end{array} .$$

Then one has a 2-commutative diagram

$$\begin{array}{ccc} D(G \setminus G) & \xrightarrow{\text{pres}_P^G} & D(L \setminus L) \\ \downarrow & & \downarrow \\ D(G) & \xrightarrow{\pi_* \iota^!} & D(L) \end{array}$$

where the vertical arrows are the  $t$ -exact forgetful functors.

**Remark 4.2.** Consider the correspondence

$$\begin{array}{ccc} & (G \times G/P)_1 & \\ \underline{p} \swarrow & & \searrow \tilde{q} \\ G & & (\frac{G/U \times G/U}{L})_1 \xleftarrow{i} L \end{array}$$

where

$$(G \times G/P)_1 := \{(g, xP) \in G \times G/P \mid x^{-1}gx \in P\},$$

$\frac{G/U \times G/U}{L}$  denotes the quotient by the diagonal right action, and

$$(\frac{G/U \times G/U}{L})_1 := \{(xU, yU) \in \frac{G/U \times G/U}{L} \mid x^{-1}y \in P\}.$$

The maps are

$$\underline{p}(g, xP) = g, \quad \tilde{q}(g, xP) = (xU, gxU), \quad i(\ell) = (U, \ell U).$$

Then one has a 2-commutative diagram

$$\begin{array}{ccc} D(G \setminus G) & \xrightarrow{\text{pres}_P^G} & D(L \setminus L) \\ \downarrow & & \downarrow \\ D(G) & \xrightarrow{i^! \tilde{q}_* \underline{p}^!} & D(L) \end{array}$$

where the vertical arrows are the  $t$ -exact forgetful functors.

**4.2. Parabolic induction.** Recall that  $\text{pres}_P^G$  admits a left adjoint, denoted

$$\text{pind}_G^P : D(L \setminus L) \rightarrow D(G \setminus G),$$

given by  $p_* q^!$  in terms of the correspondence 4.1. Notice that the functor  $\text{pind}_G^P$  is Verdier self-dual (since it is the composition of a pull-back w.r.t. a smooth morphism of relative dimension 0 and a push-forward w.r.t. a proper morphism).



**4.3. Braden’s hyperbolic localization and second adjunction.** Let  $P^- \subset G$  be a parabolic opposite to  $P$ . Notice that the Levi factors of  $P$  and  $P^-$  are then canonically identified (both isomorphic to  $P \cap P^-$ ). Denoting by  $\iota^-, \pi^-$  the arrows as in the diagram 4.2 but with  $P$  replaced by  $P^-$ , Braden’s hyperbolic localization theorem yields (as noted in [DrGa2, §0.2.1]):

**Proposition 4.3.** *One has an isomorphism of functors*

$$\pi_1 \circ \iota^* \cong \pi_*^- \circ (\iota^-)^! : D(G)^{G\text{-mon}} \rightarrow D(L)$$

where  $D(G)^{G\text{-mon}} \subset D(G)$  denotes the full subcategory generated by the image of the forgetful functor  $D(G \setminus G) \rightarrow D(G)$ .

From this proposition one can deduce that the functors  $\text{pres}_P^G$  and  $\text{pres}_{P^-}^G$  are Verdier dual to each other, and from this one obtains:

**Theorem 4.4** (Second adjunction). *The functor  $\text{pres}_{P^-}^G$  is left adjoint to the functor  $\text{pind}_G^P$ .*

## 5. PARABOLIC RESTRICTION AND PARABOLIC INDUCTION ARE $t$ -EXACT

In this section, we will prove the main result of this note, namely that parabolic restriction and parabolic induction of adjoint-equivariant sheaves are  $t$ -exact. We recall that we fix a parabolic  $L \xleftarrow{\pi} P \rightarrow G$ .

**5.1. Left  $t$ -exactness of parabolic restriction.** In this subsection, we establish the following proposition, on the way to proving the main theorem:

**Proposition 5.1.** *The functor  $\text{pres}_P^G$  is left  $t$ -exact.*

*Proof.* It is enough to establish that for any  $0 \neq \mathcal{M} \in D(G \setminus G)^{=0}$  one has  $\text{pres}_P^G \mathcal{M} \in D(\cdot)^{\geq 0}$ , and in the  $D$ -module setting one can assume that  $\mathcal{M}$  is coherent. We can find a non-empty open  $\nu : V \hookrightarrow \text{supp}(\mathcal{M})$  which is connected smooth and  $G$ -invariant, and such that  $\nu^! \mathcal{M}$  is smooth (see §2.2 for what “smooth” means). Moreover, by making  $V$  smaller we can assume that  $V \subset G^{(d)}$  for some  $d \geq 0$ .

By Noetherian induction on the support of  $\mathcal{M}$  we reduce to the case  $\mathcal{M} = \nu_* \mathcal{S}$ , where  $\nu : V \hookrightarrow G$  is as above and  $\mathcal{S} \in D(G \setminus V)^{=0}$  is smooth.

By remark 4.1 it suffices to show that

$$\pi_* \iota^! \underline{\mathcal{S}} \in D(\cdot)^{\geq 0}.$$

Denoting by  $i_e : L^{(e)} \hookrightarrow L$  the inclusion, it is enough to show that

$$i_e^! \pi_* \iota^! \underline{\mathcal{S}} \in D(\cdot)^{\geq 0}$$

for all  $e \geq 0$ . By base change, this is the same as to establish

$$\tilde{\pi}_* \tilde{i}_e^! \underline{\mathcal{S}} \in D(\cdot)^{\geq 0},$$

where

$$\begin{array}{ccc} & V \cap \pi^{-1}(L^{(e)}) & \\ \tilde{i}_e \swarrow & & \searrow \tilde{\pi} \\ V & & L^{(e)} \end{array} .$$

Now, proposition 3.3 states that the dimensions of fibers of  $\tilde{\pi}$  are no bigger than the difference in dimension between the target and source of  $\tilde{i}_e$ . Hence, using lemmas 5.2 and 5.3, the claim follows.  $\square$

We record here the following two lemmas which we have used in the proof above:

**Lemma 5.2.** *Let  $\pi : X \rightarrow Y$  be a morphism of varieties, with  $Y$  connected smooth. Denote  $\dim X \leq e$  and  $\dim Y = d$ . Let  $\mathcal{M} \in D(Y)^{=0}$  be smooth. Then  $\pi^!(\mathcal{M}) \in D(X)^{\geq d-e}$ .*

**Lemma 5.3.** *Let  $\pi : X \rightarrow Y$  be a morphism of varieties, with dimensions of fibers  $\leq f$ . Then  $\pi_*(D(X)^{\geq 0}) \subset D(Y)^{\geq -f}$ .*

**5.2.  $t$ -exactness.** We can now prove the main theorem of this note:

**Theorem 5.4.** *The functors  $pres_P^G$  and  $pind_G^P$  are  $t$ -exact.*

*Proof.* Proposition 5.1 states that  $pres_P^G$  is left  $t$ -exact. Thus,  $pind_G^P$  is right  $t$ -exact. Since  $pind_G^P$  is Verdier self-dual we deduce that  $pind_G^P$  is also left  $t$ -exact - this is clear in the sheaf-theoretic contexts (2), (3), (4), and will be explained in a moment in the sheaf-theoretic context (1). Finally, by theorem 4.4, we deduce, since we just showed that  $pind_G^{P^-}$  is left  $t$ -exact, that the functor  $pres_P^G$  is right  $t$ -exact.

Let us now explain, in the sheaf-theoretic context (1), why the right  $t$ -exactness of  $pind_G^P$  implies its left  $t$ -exactness. It is enough to show that for  $\mathcal{M} \in D(L \setminus L)^{=0}$  which is coherent, one has  $pind_G^P \mathcal{M} \in D(\cdot)^{\geq 0}$ . Since  $pind_G^P \cong \mathbb{D} \circ pind_G^P \circ \mathbb{D}$ , it is enough to show that for every  $i \geq 0$  one has  $\mathbb{D} pind_G^P(H^{-i}(\mathbb{D}\mathcal{M})[i]) \in D(\cdot)^{\geq 0}$ . We will use [Ra] as a reference. The holonomic defect of  $H^{-i}(\mathbb{D}\mathcal{M})$  is  $\leq i$ . Since, by [Ra, Theorem 2.5.1], the standard functors do not increase holonomic defect, the holonomic defect of  $pind_G^P(H^{-i}(\mathbb{D}\mathcal{M})[i])$  is  $\leq i$ . Hence, by [Ra, Proposition 2.6.1] and by the established right  $t$ -exactness of  $pind_G^P$ , we have  $\mathbb{D} pind_G^P(H^{-i}(\mathbb{D}\mathcal{M})[i]) \in D(\cdot)^{\geq 0}$ .  $\square$

**Remark 5.5.** In the sheaf-theoretic contexts (2), (3), (4) we can avoid using  $pind_G^P$  in order to establish the right  $t$ -exactness of  $pres_P^G$  - it simply follows from the left  $t$ -exactness by Verdier duality, using proposition 4.3.

**5.3. Purity and semisimplicity.** Let us notice in passing here that, when we consider the setting of a finite ground field and mixed  $\ell$ -adic sheaves, the functor  $pres_P^G$  preserves complexes of weight  $\geq w$  (as is clear from remark 4.1) and complexes of weight  $\leq w$  (as is clear from proposition 4.3). Thus,  $pres_P^G$  is pure (preserves complexes pure of weight  $w$ ). This allows, in a standard way, to deduce that, now in one of our sheaf-theoretic contexts (3), (4), the functor  $pres_P^G$  preserves semisimplicity of complexes “of geometric origin”. Let us here also notice that  $pind_G^P$  preserves semisimplicity of complexes of geometric origin - this follows from the decomposition theorem.

**5.4. The Lie algebra case.** In the case of  $G$ -equivariant sheaves on the Lie algebra  $\mathfrak{g}$ , one has analogous results.

First of all, Lie algebra versions of all the propositions in section 3 hold. These are stated analogously and proved in the same way, once one has the basic input, which is proposition 3.1. Thus, we want to see that for  $x \in \mathfrak{g}$  and a Borel  $\mathfrak{b} \subset \mathfrak{g}$ , one has

$$\dim(\mathcal{O}_x^{\mathfrak{g}} \cap \mathfrak{b}) = \frac{1}{2} \dim \mathcal{O}_x^{\mathfrak{g}}$$

(where  $\mathcal{O}_x^{\mathfrak{g}} := \{\text{Ad}(g)x : g \in G\}$ ). By arguing in the same manner as in [Hu, §6.17], we reduce to the case when  $x$  is nilpotent. This latter case is handled, for example, in [ChGi, Corollary 3.3.24] (for attributions, consult *loc. cit.*).

Then, everything is defined and proven similarly to the group case. For example, parabolic restriction  $pres_P^G$  is given as  $q_*p^!$  where

$$\begin{array}{ccc} & P \backslash \mathfrak{p} & \\ p \swarrow & & \searrow q \\ G \backslash \mathfrak{g} & & L \backslash \mathfrak{l} \end{array} .$$

We obtain:

**Theorem 5.6.** *The functors*

$$pres_P^G : D(G \backslash \mathfrak{g}) \rightarrow D(L \backslash \mathfrak{l})$$

and

$$pind_G^P : D(L \backslash \mathfrak{l}) \rightarrow D(G \backslash \mathfrak{g})$$

are  $t$ -exact.

Let us notice that in the sheaf-theoretic context of  $D$ -modules, this theorem was proven, by different methods, in [Gu].

## 6. A CONJECTURE ABOUT $t$ -EXACTNESS RELATED TO THE HARISH-CHANDRA TRANSFORM

In this section we provide a conjecture which generalizes theorem 5.4 in the Borel case. We exclude the non-holonomic  $D$ -module setting for simplicity. We fix a pair of opposite Borel subgroups  $B, B^- \subset G$ . We denote by  $U, U^-$  their respective unipotent radicals and  $T := B \cap B^-$ , seen also as the Levi quotient of  $B$  and of  $B^-$ .

**6.1. The Harish-Chandra transform.** Recall the *Harish-Chandra transform*

$$HC_* : D(G \backslash G) \rightarrow D(G \backslash (G/U \times G/U)/T),$$

given by  $q_*p^!$  where

$$\begin{array}{ccc} & G \backslash (G \times G/B) & \\ p \swarrow & & \searrow q \\ G \backslash G & & G \backslash (G/U \times G/U)/T \end{array} .$$

Here  $p(g_1, g_2B) = g_1$  and  $q(g_1, g_2B) = (g_2U, g_1g_2U)$ .

One has a closed embedding

$$i : B \backslash T \cong G \backslash (G/U \times G/U)_1/T \rightarrow G \backslash (G/U \times G/U)/T$$

(here  $B$  acts on  $T$  by projecting onto  $T$  and then acting by conjugation) where

$$(G/U \times G/U)_1 := \{(g_1U, g_2U) \in G/U \times G/U \mid g_1^{-1}g_2 \in B\}$$

and the identification is by  $t \mapsto (U, tU)$ . Let us also denote by

$$\pi : B \backslash T \rightarrow T \backslash T$$

the natural map.

Similarly to remark 4.2, we have

$$(6.1) \quad \pi_* \circ i^! \circ HC_* \cong pres_B^G : D(G \backslash G) \rightarrow D(T \backslash T).$$

6.2. **The long intertwining transform.** Recall the *long intertwining transform*

$$R_! : D(G \backslash (G/U \times G/U)/T) \rightarrow D(G \backslash (G/U \times G/U^-)/T),$$

given by  $s_! r^*$  where

$$\begin{array}{ccc} & G \backslash (G/U \times G/U \times G/U^-)_{w_0, b} / T & \\ & \swarrow \scriptstyle r & \searrow \scriptstyle s \\ G \backslash (G/U \times G/U) / T & & G \backslash (G/U \times G/U^-) / T \end{array}$$

Here

$$(G/U \times G/U \times G/U^-)_{w_0, b} := \{(g_1 U, g_2 U, g_3 U^-) \in G/U \times G/U \times G/U^- \mid g_2^{-1} g_3 \in U U^-\}.$$

The map  $r$  is by projecting to the first and second coordinates, while the map  $s$  is by projecting to the first and third coordinates.

One has an open embedding

$$j : T \backslash T \cong G \backslash (G/U \times G/U^-)_{w_0} / T \rightarrow G \backslash (G/U \times G/U^-) / T$$

where

$$(G/U \times G/U^-)_{w_0} := \{(g_1 U, g_2 U^-) \in G/U \times G/U^- \mid g_1^{-1} g_2 \in B U^-\}$$

and the identification is by  $t \mapsto (U, tU^-)$ .

**Lemma 6.1.** *One has*

$$\pi_* \circ i^! \cong j^! \circ R_! : D(G \backslash (G/U \times G/U)/T) \rightarrow D(T \backslash T).$$

*Proof.* Consider the diagram:

$$\begin{array}{ccccc} & & & T \backslash U^- T & \\ & & & \swarrow \scriptstyle \tilde{j} & \searrow \scriptstyle \tilde{s} \\ & & G \backslash (G/U \times G/U \times G/U^-)_{w_0} / T & & T \backslash T \\ & \swarrow \scriptstyle r & & \searrow \scriptstyle s & \swarrow \scriptstyle j \\ G \backslash (G/U \times G/U) / T & & & G \backslash (G/U \times G/U^-) / T & \end{array}$$

Here  $T$  acts on  $U^- T$  by conjugation. The map  $\tilde{j}$  is given by  $u^- t \mapsto (U, u^- t U, t U^-)$  and the map  $\tilde{s}$  is given by  $u^- t \mapsto t$ . The diamond is Cartesian. Let us also denote by

$$\iota : T \backslash T \rightarrow T \backslash U^- T$$

the inclusion. By the contraction principle, one has  $\tilde{s}_! \cong \iota^!$ .

One now has:

$$j^! \circ R_! = j^! \circ s_! \circ r^* \cong \tilde{s}_! \circ \tilde{j}^! \circ r^* \cong \iota^! \circ \tilde{j}^! \circ r^* \cong (r \circ \tilde{j} \circ \iota)^! [-2 \dim U] = \dots$$

Notice now that  $r \circ \tilde{j} \circ \iota = i \circ \rho$  where  $\rho : T \backslash T \rightarrow B \backslash T$  is the natural map. Also, one has  $\rho^! \cong \pi_* [2 \dim U]$  (because  $\pi$  is a  $U \bullet$ -torsor). We thus further obtain

$$\dots \cong \pi_* \circ i^!$$

as desired.  $\square$

**6.3. The conjecture.** We propose the following conjecture:

**Conjecture 6.2.** *The functor*

$$R_! \circ \mathrm{HC}_* : D(G \backslash G) \rightarrow D(G \backslash (G/U \times G/U^-) / T)$$

*is t-exact.*

**Remark 6.3.** According to [BeFiOs, Corollary 3.4], in the setting of holonomic  $D$ -modules, the conjecture holds when we restrict the domain to that of *character sheaves*. In the  $\ell$ -adic setting, the same assertion can be deduced from [ChYo, Theorem 7.8].

The following proposition is an evidence toward conjecture 6.2.

**Proposition 6.4.** *The functor*

$$j^! \circ R_! \circ \mathrm{HC}_* : D(G \backslash G) \rightarrow D(T \backslash T)$$

*is t-exact.*

*Proof.* This follows by combining lemma 6.1, relation 6.1, and the theorem 5.4.  $\square$

## 7. TWO APPLICATIONS

**7.1. Application to Euler characteristic.** In this subsection, we assume that we are in the holonomic  $D$ -module,  $\ell$ -adic or complex-analytic settings (i.e., we exclude non-holonomic  $D$ -modules).

Recall that for a variety  $X$  and  $\mathcal{G} \in D(X)$ , one defines the *Euler characteristic*  $\mathrm{Eul}(\mathcal{G}) \in \mathbb{Z}$  by:

$$\mathrm{Eul}(\mathcal{G}) := \dim p_* \mathcal{G} = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(p_* \mathcal{G})$$

where  $p : X \rightarrow \bullet$ .

We would like to prove:

**Theorem 7.1.** *Let  $\mathcal{F} \in D(G \backslash G)^{=0}$ . Then  $\mathrm{Eul}(\underline{\mathcal{F}})$  is non-negative.*

**Remarks 7.2.**

- (1) When  $G$  is a torus, the result is proved in [GaLo] in the  $\ell$ -adic setting, in [LoSa] in the holonomic  $D$ -module setting and in [FrKa] in the complex-analytic setting. We will assume the torus case when proving the general case.
- (2) In the complex-analytic setting, the result is proved in [Ki] where the conjectural statement of the result is attributed to M. Kapranov.
- (3) In the  $\ell$ -adic setting or the holonomic  $D$ -module setting (and thus also the complex-analytic setting), assuming in addition that  $\mathcal{F}$  is an intermediate extension from the regular semisimple locus, the result is proved in [Br].
- (4) Thus, we provide a new proof of the result, which holds in the complex-analytic setting as well as in the formerly unestablished  $\ell$ -adic setting.

In order to prove theorem 7.1, let us first establish the following claim:

**Claim 7.3.** *Let  $\mathcal{F} \in D(G \backslash G)$ . Then*

$$\mathrm{Eul}(\underline{\mathrm{pres}}_B^G(\mathcal{F})) = \mathrm{Eul}(\underline{\mathcal{F}})$$

(where  $B \subset G$  is a Borel subgroup).

*Proof.* For the proof, we choose a splitting of  $B \rightarrow T$ , so a maximal torus  $T \subset B$ .

Denote by  $pr : G \times G/B \rightarrow G$  the projection. By lemma 7.4, we have

$$\text{Eul}(pr^! \mathcal{F}) = \text{Eul}(\omega_{G/B}) \cdot \text{Eul}(\mathcal{F}) = |W| \cdot \text{Eul}(\mathcal{F}).$$

With respect to the standard  $G$ -action on  $G \times G/B$  (by  $g(h, xB) = (ghg^{-1}, gxB)$ ), the  $T$ -fixed points are of the form  $(t, wB)$  for  $t \in T, w \in W$ . In particular, they all lie in  $(G \times G/B)_1$  (see remark 4.2 for this notation). Hence, by lemma 7.5, working in the notations of remark 4.2, we have

$$\text{Eul}(pr^! \mathcal{F}) = \text{Eul}(p^! \mathcal{F}) = \text{Eul}(\tilde{q}_* p^! \mathcal{F}).$$

Now, with respect to the diagonal  $G$ -action on  $(\frac{G/U \times G/U}{T})_1$ , the  $T$ -fixed points consist of the disconnected union of copies of  $T$ :  $(wU, wTU)$  for  $w \in W$ . By lemma 7.5 and  $G$ -equivariancy, we obtain:

$$\text{Eul}(\tilde{q}_* p^! \mathcal{F}) = |W| \cdot \text{Eul}(i^! \tilde{q}_* p^! \mathcal{F}) = |W| \cdot \text{Eul}(\underline{\text{pres}}_B^G(\mathcal{F})).$$

□

Now we are ready to prove theorem 7.1.

*Proof (of theorem 7.1).* Let  $\mathcal{F} \in D(G \setminus G)^{=0}$ . By claim 7.3, we have  $\text{Eul}(\mathcal{F}) = \text{Eul}(\underline{\text{pres}}_B^G(\mathcal{F}))$ . Since by theorem 5.4 one has  $\underline{\text{pres}}_B^G(\mathcal{F}) \in D(\ )^{=0}$ , in view of the torus case (remark 7.2.(1)) the theorem follows. □

We record here the following two lemmas which we have used above:

**Lemma 7.4.** *Let  $X, Y$  be varieties, and let  $\mathcal{G} \in D(X)$ . Denote by  $pr : X \times Y \rightarrow X$  the projection. Then*

$$\text{Eul}(pr^! \mathcal{G}) = \text{Eul}(\omega_Y) \cdot \text{Eul}(\mathcal{G}).$$

**Lemma 7.5.** *Let  $X$  be a variety equipped with an action of a torus  $T$ . Let  $\mathcal{F} \in D(T \setminus X)$ . Denoting  $j : X^T \hookrightarrow X$ , we have*

$$\text{Eul}(\mathcal{F}) = \text{Eul}(j^! \mathcal{F}).$$

**7.2. Application to generic perversity.** In this subsection, we restrict ourselves to the  $\ell$ -adic setting. We fix a Borel  $B \subset G$  with Levi  $T$ .

In [GaLo], a  $\bar{\mathbb{Q}}_\ell$ -scheme  $\mathcal{C}(T)$  is defined, whose  $\bar{\mathbb{Q}}_\ell$ -points are the isomorphism classes of tame local systems of rank 1 on  $T$ . Let us denote by  $\mathcal{L}_\chi$  the perverse local system of rank 1 on  $T$  corresponding to  $\chi \in \mathcal{C}(T)(\bar{\mathbb{Q}}_\ell)$ . Recall also the notion of *character sheaves* in  $D(G \setminus G)$  with *central character*  $\chi$  (or  $\mathcal{L}_\chi$ ).

**Proposition 7.6.** *Let  $\mathcal{F} \in D(G \setminus G)^{=0}$ . For a generic  $\chi \in \mathcal{C}(T)(\bar{\mathbb{Q}}_\ell)$  (depending on  $\mathcal{F}$ ), given a character sheaf  $\mathcal{G} \in D(G \setminus G)^{=0}$  with central character  $\chi$ , one has*

$$\mathcal{F} * \mathcal{G} \in D(G \setminus G)^{=0}$$

(where “ $*$ ” denotes the  $!$ -convolution on the group).

*Proof.* *The torus case:* We want to check that for  $\mathcal{F} \in D(T)^{=0}$  and a generic  $\chi \in \mathcal{C}(T)$ , one has  $\mathcal{F} * \mathcal{L}_\chi \in D(T)^{=0}$ . Recall the *Mellin transform*

$$\mathcal{M}_* : D(T) \rightarrow D_{coh}^b(\mathcal{C}(T))$$

from [GaLo]. By [GaLo, Theorem 3.4.7], it is enough to check that  $\mathcal{M}_*(\mathcal{F} * \mathcal{L}_\chi)$  sits in degree 0. By [GaLo, Proposition 3.3.1 (f)], one has

$$\mathcal{M}_*(\mathcal{F} * \mathcal{L}_\chi) \cong \mathcal{M}_*(\mathcal{F}) \otimes \mathcal{M}_*(\mathcal{L}_\chi)$$

(here the “ $\otimes$ ” is in the derived sense). Since  $\mathcal{M}_*(\mathcal{F})$  sits in degree zero and, by [GaLo, Theorem 3.4.3], is generically locally free, and  $\mathcal{M}_*(\mathcal{L}_\chi) \cong (i_\chi)_* \bar{\mathbb{Q}}_\ell$  (where  $i_\chi$  is the inclusion of the point corresponding to  $\chi$ ), we see that indeed that  $\mathcal{M}_*(\mathcal{F}) \otimes \mathcal{M}_*(\mathcal{L}_\chi)$  sits in degree zero for generic  $\chi$ .

*The general case:* We will reduce the general case to the torus case. Recall the morphisms  $i$  and  $\pi$  from §6.1. Also, recall the transform

$$\text{CH} : D(G \backslash (G/U \times G/U)/T) \rightarrow D(G \backslash G)$$

left adjoint to  $\text{HC}_*$ .

Since  $\chi$  is generic (so its stabilizer in  $W$  is trivial),  $\mathcal{G}$  is a direct summand of  $\text{pind}_G^B \mathcal{L}_\chi$ , and hence we may assume  $\mathcal{G} = \text{pind}_G^B \mathcal{L}_\chi$ .

We have:

$$\mathcal{F} * \text{pind}_G^B \mathcal{L}_\chi \cong \text{CH}(\text{HC}_*(\mathcal{F}) * \mathcal{L}_\chi)$$

(where the latter “ $*$ ” denotes  $!$ -convolution w.r.t. the right action of  $T$  on  $G \backslash (G/U \times G/U)/T$  given by  $(xU, yU) * t = (xU, ytU)$ ). Since  $\chi$  is generic (so its stabilizer in  $W$  is trivial), we have

$$\begin{aligned} \text{HC}_*(\mathcal{F}) * \mathcal{L}_\chi &= i_* i^! (\text{HC}_*(\mathcal{F}) * \mathcal{L}_\chi) \cong i_* \pi^* \pi_* i^! (\text{HC}_*(\mathcal{F}) * \mathcal{L}_\chi) \cong \\ &\cong i_* \pi^* ((\pi_* i^! \text{HC}_*(\mathcal{F})) * \mathcal{L}_\chi) \cong i_* \pi^* (\text{pres}_B^G \mathcal{F} * \mathcal{L}_\chi) \end{aligned}$$

and thus

$$\mathcal{F} * \text{pind}_G^B \mathcal{L}_\chi \cong \text{CH}(i_* \pi^* (\text{pres}_B^G \mathcal{F} * \mathcal{L}_\chi)) \cong \text{pind}_G^B (\text{pres}_B^G \mathcal{F} * \mathcal{L}_\chi).$$

Now the claim is clear by the  $t$ -exactness of  $\text{pind}_G^B$  and  $\text{pres}_B^G$  and the torus case above.  $\square$

## REFERENCES

- [Be] A. Beilinson, *Constructible sheaves are holonomic*, *Selecta Math. (N.S.)* **22** (2016), no. 4, 1797-1819.
- [BeFiOs] R. Bezrukavnikov, M. Finkelberg and V. Ostrik, *Character  $D$ -modules via Drinfeld center of Harish-Chandra bimodules*, *Invent. Math.* **188** (2012), 589-620.
- [Bra] T. Braden, *Hyperbolic localization of Intersection Cohomology*, *Transformation Groups* **8** (2003), no. 3, 209-216.
- [Br] A. Braverman, *On Euler characteristic of equivariant sheaves*, *Adv. Math.* **179** (2003), 1-6.
- [ChGi] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser Boston (1997).
- [ChYo] T.H. Chen and A. Yom Din, *A formula for the geometric Jacquet functor and its character sheaf analog*, *GAFA* **27(4)** (2017), 772-797.
- [DrGa1] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, *GAFA* **23** (2013), 149-294.
- [DrGa2] V. Drinfeld and D. Gaitsgory, *On a theorem of Braden*, *Transform. Groups* **19** (2014), 313-358.
- [FrKa] J. Franke and M. Kapranov, *The Gauss map and a noncompact Riemann-Roch formula for constructible sheaves on semiabelian varieties*, *Duke Math. J.* **104** (2000), 171-180.
- [GaLo] O. Gabber and F. Loeser, *Faisceaux pervers  $\ell$ -adiques sur un tore*, *Duke Math. J.* **83** (1996), 501-606.
- [Gu] S. Gunningham, *Generalized Springer Theory for  $D$ -modules on a Reductive Lie Algebra*, to appear in *Selecta Math. (N.S.)*, preprint available at <https://arxiv.org/abs/1510.02452>.

- [Hu] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Math. Surveys and Monographs **43** (1995), Amer. Math. Soc., Providence, R.I..
- [Ki] V. Kiritchenko, *A Gauss-Bonnet theorem for constructible sheaves on reductive groups*, Math. Res. Lett. **9(6)** (2002), 791-800.
- [LoSa] F. Loeser and C. Sabbah, *Caractérisation des  $D$ -modules hypergéométriques irréductibles sur le tore I*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), 735-738; II, **315** (1992), 1263-1264.
- [Lu1] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), 205-272.
- [Lu2] G. Lusztig, *Character sheaves I*, Adv. Math. **56** (1985), 193-237.
- [Ra] S. Raskin, *A generalization of the  $b$ -function lemma*, preprint available at <https://arxiv.org/abs/1611.04940> .
- [Sa] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Invent. Math. **207** (2017), no. 2, 597-695.