Optimal bounds for ancient caloric functions

OPTIMAL BOUNDS FOR ANCIENT CALORIC FUNCTIONS

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Abstract. For any manifold with polynomial volume growth, we show: The dimension of the space of ancient caloric functions with polynomial growth is bounded by the degree of growth times the dimension of harmonic functions with the same growth. As a consequence, we get a sharp bound for the dimension of ancient caloric functions on any space where Yau's 1974 conjecture about polynomial growth harmonic functions holds.

0. Introduction

Given a complete manifold M and a constant d, $\mathcal{H}_d(M)$ is the linear space of harmonic functions of polynomial growth at most d. Namely, $u \in \mathcal{H}_d(M)$ if $\Delta u = 0$ and for some $p \in M$ and a constant C_u depending on u

$$\sup_{B_R(p)} |u| \leq C_u \, (1+R)^d \text{ for all } R\,. \tag{0.1}$$

In 1974, S.T. Yau conjectured that $\mathcal{H}_d(M)$ is finite dimensional for each d when $\text{Ric}_M \geq 0$. The conjecture was settled in [CM2]; see [CM1]–[CM5] for more results.[1] In fact, [CM2]–[CM4] proved finite dimensionality under much weaker assumptions of:

(1) A volume doubling bound.
(2) A scale-invariant Poincaré inequality or meanvalue inequality.

The natural parabolic generalization is a polynomial growth ancient solution of the heat equation. A solution of the heat equation is often called a caloric function. Ancient solutions are ones that are defined for all negative t - these are the solutions that arise in a blow up analysis. Given $d > 0$, $u \in \mathcal{P}_d(M)$ if u is ancient, $\partial_t u = \Delta\, u$ and for some $p \in M$ and a constant C_u

$$\sup_{B_R(p)\times[-R^2,0]} |u| \leq C_u \, (1+R)^d \text{ for all } R\,. \tag{0.2}$$

On $\mathbf{R}^n$, $\mathcal{P}_d$ is the classical space of caloric polynomials that generalize the Hermite polynomials; see [N], [E1], [E2]. More generally, the spaces $\mathcal{P}_d(M)$ play a fundamental role in geometric flows, see [CM6]–[CM8]. They were studied by Calle in her 2006 thesis, [Ca1], [Ca2], in the context of mean curvature flow.

A manifold has polynomial volume growth if there are constants C and d_V so that $\text{Vol}(B_R(p)) \leq C\,(1+R)^{d_V}$ for some $p \in M$, all $R > 0$.[2] Our main result is the following sharp inequality:

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[1] For Yau's 1974 conjecture see: page 117 in [Ya2], problem 48 in [Ya3], Conjecture 2.5 in [Sc], [Ka], [Kz], [DF], Conjecture 1 in [Li1], and problem (1) in [LiTa], amongst others.

[2] A volume doubling space with doubling constant C_D has polynomial volume growth of degree $\log_2 C_D$.

Theorem 0.3. If M has polynomial volume growth and k is a nonnegative integer, then

$$\dim \mathcal{P}_{2k}(M) \leq \sum_{i=0}^{k} \dim \mathcal{H}_{2i}(M)\,. \tag{0.4}$$

The inequality (0.4) is an equality on $\mathbf{R}^n$ (see Corollary 2.18 below). Since $\mathcal{H}_{d_1} \subset \mathcal{H}_{d_2}$ for $d_1 \leq d_2$, Theorem 0.3 implies:

Corollary 0.5. If M has polynomial volume growth, then for all $k \geq 1$

$$\dim \mathcal{P}_{2k}(M) \leq (k+1)\, \dim \mathcal{H}_{2k}(M)\,. \tag{0.6}$$

Combining this with the bound $\dim \mathcal{H}_d(M) \leq C\, d^{n-1}$ when $\text{Ric}_{M^n} \geq 0$ from [CM3] gives:

Corollary 0.7. There exists $C = C(n)$ so that if $\text{Ric}_{M^n} \geq 0$, then for $d \geq 1$

$$\dim\, \mathcal{P}_d(M) \leq C\, d^n\,. \tag{0.8}$$

The exponent n in (0.8) is sharp: There is a constant c depending on n so that for $d \geq 1$

$$c^{-1}\, d^n \leq \dim \mathcal{P}_d(\mathbf{R}^n) \leq c\, d^n\,. \tag{0.9}$$

Recently, Lin and Zhang, [LZ], proved very interesting related results, adapting the methods of [CM2]–[CM4] to get the bound d^{n+1}.

Using parabolic gradient estimates of Li-Yau, [LiY], and Souplet-Zhang, [SoZ], one can show that if $d < 2$ and $\text{Ric} \geq 0$, then $\mathcal{P}_d(M) = \mathcal{H}_d(M)$ consists only of harmonic functions of polynomial growth. In particular, $\mathcal{P}_d(M) = \{\text{Constant functions}\}$ for $d < 1$ and, moreover, $\dim \mathcal{P}_1(M) \leq n+1$, by Li and Tam, [LiTa], with equality if and only if $M = \mathbf{R}^n$ by [ChCM].

The exponent $n-1$ is also sharp in the bound for $\dim \mathcal{H}_d$ when $\text{Ric}_{M^n} \geq 0$. However, as in Weyl's asymptotic formula, the coefficient of d^{n-1} can be related to the volume, [CM3]:

$$\dim \mathcal{H}_d(M) \leq C_n\, \text{V}_M\, d^{n-1} + o(d^{n-1})\,. \tag{0.10}$$

- V_M is the "asymptotic volume ratio" $\lim_{r \to \infty} \text{Vol}(B_r)/r^n$.
- $o(d^{n-1})$ is a function of d with $\lim_{d \to \infty} o(d^{n-1})/d^{n-1} = 0$.

Combining (0.10) with Corollary 0.5 gives $\dim\, \mathcal{P}_d(M) \leq C_n\, \text{V}_M\, d^n + o(d^n)$ when $\text{Ric}_{M^n} \geq 0$.

An interesting feature of these dimension estimates is that they follow from "rough" properties of M and are therefore surprisingly stable under perturbation. For instance, [CM4] proves finite dimensionality of $\mathcal{H}_d$ for manifolds with a volume doubling and a Poincaré inequality, so we also get finite dimensionality for $\mathcal{P}_d$ on these spaces. Unlike a Ricci curvature bound, these properties are stable under bi–Lipschitz transformations (cf. [MS]). Moreover, these properties make sense also for discrete spaces, vastly extending the theory and methods out of the continuous world. Recently Kleiner, [K], (see also Shalom-Tao, [ST], [T1], [T2]) used, in part, this in his new proof of an important and foundational result in geometric group theory, originally due to Gromov, [G]. We expect that the proof of Theorem 0.3 extends to many discrete spaces, allowing a wide range of applications.

1. Ancient solutions of the heat equation

The next lemma gives a reverse Poincaré inequality for the heat equation (cf. [M]).

Lemma 1.1. There is a universal constant c so that if $u_t = \Delta\, u$, then

$$r^2 \int_{B_{\frac{r}{10}} \times [-\frac{r^2}{100},0]} |\nabla u|^2 + r^4 \int_{B_{\frac{r}{10}} \times [-\frac{r^2}{100},0]} u_t^2 \leq c \int_{B_r \times [-r^2,0]} u^2 \, . \tag{1.2}$$

Proof. Let Q_R denote $B_R \times [-R^2, 0]$ and ψ be a cutoff function on M. Since $u_t = \Delta\, u$, integration by parts and the absorbing inequality $4ab \leq a^2 + 4b^2$ give

$$\begin{aligned} \partial_t \int u^2\, \psi^2 &= 2 \int u\, \psi^2\, \Delta u = -2 \int |\nabla u|^2\, \psi^2 - 4 \int u\, \psi\, \langle \nabla \psi, \nabla u \rangle \\ &\leq - \int |\nabla u|^2\, \psi^2 + 4 \int u^2\, |\nabla \psi|^2 \, . \end{aligned} \tag{1.3}$$

Integrating this in time from $-R^2$ to 0 gives

$$\int_{t=0} u^2\, \psi^2 - \int_{t=-R^2} u^2\, \psi^2 \leq \int_{-R^2}^0 \left(- \int |\nabla u|^2\, \psi^2 + 4 \int u^2\, |\nabla \psi|^2 \right) dt \, . \tag{1.4}$$

In particular, we get

$$\int_{-R^2}^0 \int |\nabla u|^2\, \psi^2\, dt \leq \int_{t=-R^2} u^2\, \psi^2 + 4 \int_{-R^2}^0 \int u^2\, |\nabla \psi|^2\, dt \, . \tag{1.5}$$

Let $|\psi| \leq 1$ be one on $B_{R/2}$, have support in B_R, and satisfy $|\nabla \psi| \leq 2/R$, so we get

$$\int_{Q_{R/2}} |\nabla u|^2 \leq \int_{B_R \times \{t=-R^2\}} u^2 + \frac{16}{R^2} \int_{Q_R} u^2 \, . \tag{1.6}$$

Next, we argue similarly to get a bound on u_t^2. Namely, differentiating, then integrating by parts and using that $u_t = \Delta\, u$ gives

$$\begin{aligned} \partial_t \int |\nabla u|^2\, \psi^2 &= 2 \int \langle \nabla u, \nabla u_t \rangle\, \psi^2 = -2 \int u_t^2\, \psi^2 - 4 \int u_t\, \psi\, \langle \nabla u, \nabla \psi \rangle \\ &\leq - \int u_t^2\, \psi^2 + 4 \int |\nabla u|^2\, |\nabla \psi|^2 \, . \end{aligned} \tag{1.7}$$

Integrating (1.7) in time from $-R^2$ to 0 gives

$$\int_{t=0} |\nabla u|^2\, \psi^2 - \int_{t=-R^2} |\nabla u|^2\, \psi^2 \leq \int_{-R^2}^0 \left(- \int u_t^2\, \psi^2 + 4 \int |\nabla u|^2\, |\nabla \psi|^2 \right) dt \, . \tag{1.8}$$

Letting ψ be as above, we conclude that

$$\int_{Q_{R/2}} u_t^2 \leq \frac{16}{R^2} \int_{Q_R} |\nabla u|^2 + \int_{B_R \times \{t=-R^2\}} |\nabla u|^2 \, . \tag{1.9}$$

Next, choose some $r_1 \in [4r/5, r]$ with

$$\int_{B_r \times \{t=-r_1^2\}} u^2 \leq \frac{25}{9\, r^2} \int_{-r^2}^0 \left(\int_{B_r} u^2 \right) dt = \frac{25}{9\, r^2} \int_{Q_r} u^2 \, . \tag{1.10}$$

Applying (1.6) with $R = r_1$ and using the bound (1.10) at r_1 gives

$$\int_{Q_{\frac{2r}{5}}} |\nabla u|^2 \leq \int_{Q_{\frac{r_1}{2}}} |\nabla u|^2 \leq \int_{B_{r_1} \times \{t=-r_1^2\}} u^2 + \frac{16}{r_1^2} \int_{Q_{r_1}} u^2 \leq \frac{20}{r_1^2} \int_{Q_r} u^2 \, . \tag{1.11}$$

For simplicity, c is a constant independent of everything that can change from line to line. It follows from (1.11) that there must exist some $\rho \in [r/5, 2r/5]$ so that

$$\int_{B_{\frac{2r}{5}} \times \{t=-\rho^2\}} |\nabla u|^2 \leq \frac{25}{3r^2} \int_{-\frac{4r^2}{25}}^0 \left(\int_{B_{\frac{2r}{5}}} |\nabla u|^2 \right) dt = \frac{25}{3r^2} \int_{Q_{\frac{2r}{5}}} |\nabla u|^2 \leq \frac{c}{r^4} \int_{Q_r} u^2 \, . \tag{1.12}$$

Now applying (1.9) with $R = \rho$ and using (1.11) and (1.12) gives

$$\int_{Q_{\rho/2}} u_t^2 \leq \frac{16}{\rho^2} \int_{Q_\rho} |\nabla u|^2 + \int_{B_\rho \times \{t=-\rho^2\}} |\nabla u|^2 \leq \frac{c}{r^4} \int_{Q_r} u^2 \, . \tag{1.13}$$

□

Corollary 1.14. If $\text{Vol}(B_R) \leq C\,(1+R)^{d_V}$ and $u \in \mathcal{P}_d(M)$, then $\partial_t^k u \equiv 0$ for $4k > 2d+d_V+2$.

Proof. Since the metric on M is constant in time, $\partial_t - \Delta$ commutes with ∂_t and, thus, $(\partial_t - \Delta)\partial_t^j u = 0$ for every j. Let Q_R denote $B_R \times [-R^2, 0]$. Applying Lemma 1.1 to u on Q_r for some r, then to u_t on $Q_{\frac{r}{10}}$, etc., we get a constant c_k depending just on k so that

$$\int_{Q_{\frac{r}{10^k}}} \left|\partial_t^k u\right|^2 \leq \frac{c_k}{r^{4k}} \int_{Q_r} u^2 \leq \frac{c_k}{r^{4k}}\, r^2 \,\text{Vol}(B_r) \sup_{Q_r} u^2 \leq C\, c_k\, r^{2-4k}\,(1+r)^{2d+d_V} \, . \tag{1.15}$$

Since $4k > 2d + d_V + 2$, the right-hand side goes to zero as $r \to \infty$, giving the corollary. □

We will prove Corollary 0.5 next, though it will eventually be a corollary of Theorem 0.3.

Proof of Corollary 0.5. Choose an integer m with $4m > 2k + d_V + 2$. Corollary 1.14 gives that $\partial_t^m u = 0$ for any $u \in \mathcal{P}_{2k}(M)$. Thus, any $u \in \mathcal{P}_{2k}(M)$ can be written as

$$u = p_0 + t\,p_1 + \cdots + t^{m-1}\,p_{m-1} \, , \tag{1.16}$$

where each p_j is a function on M. Moreover, using the growth bound $u \in \mathcal{P}_{2k}(M)$ for t large and x fixed, we see that $p_j \equiv 0$ for any $j > k$. (See theorem 1.2 in [LZ] for a similar decomposition under more restrictive hypotheses and [KoT] for a splitting result for ancient positive solutions on homogeneous spaces.)

We will show next that the functions p_j grow at most polynomially of degree d. Fix distinct values $-1 < t_1 < t_2 < \cdots < t_k < t_{k+1} = 0$. We claim that the $k+1$-vectors

$$(1, t_i, t_i^2, \ldots, t_i^k) \tag{1.17}$$

are linearly independent in $\mathbf{R}^{k+1}$ for $i = 1, \ldots, k+1$. If this was not the case, then there would be some (non-trivial) $(a_0, \ldots, a_k) \in \mathbf{R}^{k+1}$ that is orthogonal to all of them. But this means that there would be $k+1$ distinct roots to the degree k polynomial

$$a_0 + a_1 t + \cdots + a_k t^k \, , \tag{1.18}$$

which is impossible, and the claim follows. Let $e_j \in \mathbf{R}^{k+1}$ be the standard unit vectors. Since the $(1, t_i, t_i^2, \ldots, t_i^k)$'s span $\mathbf{R}^{k+1}$, we can choose coefficients b_i^j so that for each j

$$e_j = \sum_i b_i^j \,(1, t_i, t_i^2, \ldots, t_i^k) \, . \tag{1.19}$$

It follows from (1.16) and (1.19) that

$$p_j(x) = \sum_i b_i^j u(x, t_i)\,. \tag{1.20}$$

Since $u \in \mathcal{P}_{2k}(M)$, (1.28) implies that each p_j is a linear combination of functions that grow polynomially of degree at most $2k$ and, thus, p_j grows polynomially of degree at most $2k$.

Since u satisfies the heat equation, it follows that $\Delta p_k = 0$ and

$$\Delta p_j = (j+1)\, p_{j+1}\,. \tag{1.21}$$

Thus, we get a linear map $\Psi_0 : \mathcal{P}_{2k}(M) \to \mathcal{H}_{2k}(M)$ given by $\Psi_0(u) = p_k$. Let $\mathcal{K}_0 = \text{Ker}(\Psi_0)$. It follows from this that

$$\dim \mathcal{P}_{2k}(M) \leq \dim \mathcal{K}_0 + \dim \mathcal{H}_{2k}(M)\,. \tag{1.22}$$

If $u \in \mathcal{K}_0$, then $p_k = 0$ and $\Delta p_{k-1} = 0$, so we get a linear map $\Psi_1 : \mathcal{K}_0 \to \mathcal{H}_{2k}(M)$ given by $\Psi_1(u) = p_{k-1}$. Let $\mathcal{K}_1$ be the kernel of Ψ_1 on $\mathcal{K}_0$. It follows as above that

$$\dim \mathcal{K}_0 \leq \dim \mathcal{K}_1 + \dim \mathcal{H}_{2k}(M)\,. \tag{1.23}$$

Repeating this $k+1$ times gives the theorem. □

Lemma 1.24. If $u \in \mathcal{P}_{2k}(M)$ can be written as $u = p_0(x) + t\, p_1(x) + \cdots + t^k\, p_k(x)$, then

$$|p_j(x)| \leq C_j \, \left(1 + |x|^{2(k-j)}\right)\,. \tag{1.25}$$

Proof. By assumption, there is a constant C so that

$$|u(x,t)| \leq C\, \left(1 + |t|^k + |x|^{2k}\right)\,. \tag{1.26}$$

Following the proof of Corollary 0.5, fix $-1 < t_1 < t_2 < \cdots < t_k < t_{k+1} = -\frac{1}{2}$ and coefficients b_i^j so that (1.19) holds for each j. Observe that (1.19) gives for each j

$$\sum_i b_i^j u(x, R^2\, t_i) = \sum_i \sum_\ell b_i^j\, p_\ell(x)\, R^{2j}\, t_i^\ell = \sum_\ell \sum_i b_i^j\, p_\ell(x)\, R^{2j}\, t_i^\ell = R^{2j}\, p_j(x)\,. \tag{1.27}$$

Thus, given $R > 2$ and $x \in B_R$, we get that

$$\begin{aligned} \left|R^{2j}\, p_j(x)\right| &= \left|\sum_i b_i^j u(x, R^2\, t_i)\right| \leq \max_{i,j} |b_i^j| \, \sum_i |u(x, R^2\, t_i)| \\ &\leq \tilde{C}\, \left(1 + |x|^{2k} + \max_i |R^2\, t_i|^k\right) \leq 3\, \tilde{C}\, R^{2k}\,. \end{aligned} \tag{1.28}$$

From this, we conclude that $\sup_{B_R} |p_j| \leq 3\, \tilde{C}\, R^{2k-2j}$. □

Proof. (of Theorem 0.3). Following the proof Corollary 0.5, each $u \in \mathcal{P}_{2k}(M)$, can be expanded as $u = p_0(x) + t\, p_1(x) + \cdots + t^k\, p_k(x)$. By Lemma 1.24, the linear map $\Psi_0 : \mathcal{P}_{2k}(M) \to \mathcal{H}_{2k}(M)$ given by $\Psi_0(u) = p_k$ actually maps into $\mathcal{H}_0(M)$ and, thus,

$$\dim \mathcal{P}_{2k}(M) \leq \dim \mathcal{H}_0(M) + \dim \text{Ker}(\Psi_0)\,. \tag{1.29}$$

Similary, Lemma 1.24 implies that the map Ψ_1 maps the kernel of Ψ_0 to $\mathcal{H}_2(M)$. Applying this repeatedly gives the theorem. □

2. Caloric polynomials

It is a classical fact that $\mathcal{P}_d(\mathbf{R}^n)$ consists of caloric polynomials, i.e., polynomials in x, t that satisfy the heat equation ([E1], [E2], [N]). We compute the dimensions of these spaces.

Given a polynomial in x and t, define its *parabolic degree* by considering t to have degree two. Thus, $x_1^{m_1} x_2^{m_2} t^{m_0}$ has parabolic degree $m_1 + m_2 + 2m_0$. A polynomial in x, t is homogeneous if each monomial has the same parabolic degree. Let A_p^n denote the homogeneous degree p polynomials on $\mathbf{R}^n$. The parabolic homogeneous degree p polynomials $\mathcal{A}_p^n$ are

$$\mathcal{A}_p^n = A_p^n \oplus t\, A_{p-2}^n \oplus t^2\, A_{p-4}^n \oplus \dots \tag{2.1}$$

Lemma 2.2. For each positive integer p, we have $\dim \left(\mathcal{P}_p(\mathbf{R}^n) \cap \mathcal{A}_p^n\right) = \dim A_p^n$ and

$$\dim \mathcal{P}_p(\mathbf{R}^n) = \sum_{j=0}^{p} \dim A_j^n \,. \tag{2.3}$$

Proof. Observe that ∂_t and Δ map $\mathcal{A}_p^n$ to $\mathcal{A}_{p-2}^n$. Moreover, given any $u \in \mathcal{A}_{p-2}^n$, we have

$$(\partial_t - \Delta) \left[t\, u - \frac{1}{2}\, t^2 (\partial_t - \Delta) u + \frac{1}{6}\, t^3 (\partial_t - \Delta)^2 u - \dots \right] = u \,. \tag{2.4}$$

Therefore, the map $(\partial_t - \Delta) : \mathcal{A}_p^n \to \mathcal{A}_{p-2}^n$ is onto. Since the kernel of this map is $\mathcal{P}_p(\mathbf{R}^n) \cap \mathcal{A}_p^n$, we conclude that

$$\dim \left(\mathcal{P}_p(\mathbf{R}^n) \cap \mathcal{A}_p^n\right) = \dim \mathcal{A}_p^n - \dim \mathcal{A}_{p-2}^n = \dim A_p^n \,. \tag{2.5}$$

This gives both claims. □

Lemma 2.6. If $p \geq n$, then

$$\frac{1}{(n-1)!}\, p^{n-1} \leq \dim\, A_p^n \leq \frac{2^{n-1}}{(n-1)!}\, p^{n-1} \,. \tag{2.7}$$

Proof. To get the upper bound, we use that $p \geq n$ to get

$$\dim\, A_p^n = \frac{(p+n-1)!}{p!\,(n-1)!} \leq \frac{(p+n-1)^{n-1}}{(n-1)!} \leq \frac{(2p)^{n-1}}{(n-1)!} = \frac{2^{n-1}}{(n-1)!}\, p^{n-1} \,. \tag{2.8}$$

The lower bound follows similarly since $\frac{(p+n-1)!}{p!\,(n-1)!} \geq \frac{p^{n-1}}{(n-1)!}$. □

The dimension bounds for $\mathcal{P}_d(\mathbf{R}^n)$ in (0.9) follow by combining Lemmas 2.2 and 2.6.

2.1. Harmonic polynomials.

For each j, the Laplacian gives a linear map $\Delta : A_j^n \to A_{j-2}^n$. The kernel $H_j^n \subset A_j^n$ of this map is the linear space of homogeneous harmonic polynomials of degree j on $\mathbf{R}^n$. The next lemma shows that this map is onto:

Lemma 2.9. For each d, the map $\Delta : A_{d+2}^n \to A_d^n$ is onto.

Proof. Take an arbitrary $u \in A_d^n$. For each nonnegative $\ell \leq d/2$, define u_ℓ and v_ℓ by

$$u_\ell = |x|^{2\ell}\, \Delta^\ell u \,, \tag{2.10}$$

$$v_\ell = |x|^2\, u_\ell = |x|^{2\ell+2}\, \Delta^\ell u \,. \tag{2.11}$$

Note that $u_0 = u$. We will use repeatedly that if $v \in A_k^n$, then homogeneity gives

$$\langle x, \nabla v \rangle = k\, v \,. \tag{2.12}$$

Using this and $\Delta\,|x|^2 = 2n$, we get for each ℓ that

$$
\begin{aligned}
\Delta\, v_\ell &= (\ell+1)\,(2n+4\ell)\,|x|^{2\ell}\,\Delta^\ell u + 2\,\langle \nabla |x|^{2(\ell+1)}, \nabla \Delta^\ell u\rangle + |x|^{2(\ell+1)}\,\Delta^{\ell+1} u \\
&= (\ell+1)\,(2n+4\ell)\,|x|^{2\ell}\,\Delta^\ell u + 4\,(\ell+1)\,(d-2\ell)\,|x|^{2\ell}\,\Delta^\ell u + |x|^{2(\ell+1)}\,\Delta^{\ell+1} u \\
&= (\ell+1)\,\left(2n+4d-4\ell\right)\,u_\ell + u_{\ell+1}\,.
\end{aligned}
\tag{2.13}
$$

Thus, if we define positive constants $c_\ell = (\ell+1)\,(2n+4d-4\ell)$, then we have that

$$
\Delta v_\ell = c_\ell\, u_\ell + u_{\ell+1}\,. \tag{2.14}
$$

Let k be the greatest integer less than or equal to $\frac{d}{2}$. Note that $u_{k+1} = v_{k+1} \equiv 0$. It follows from this and (2.14) that

$$
\Delta\,\left(v_k - c_k\, v_{k-1} + c_k\, c_{k-1}\, v_{k-2} - c_k\, c_{k-1}\, c_{k-2}\, v_{k-3} + \dots\right) \tag{2.15}
$$

is a nonzero multiple of $u_0 = u$, giving the lemma. □

Corollary 2.16. For each positive integer k, we have $\dim H^n_k = \dim A^n_k - \dim A^n_{k-2}$ and

$$
\dim\, \mathcal{H}_k(\mathbf{R}^n) = \dim A^n_k + \dim A^n_{k-1}\,. \tag{2.17}
$$

Proof. Note that $\Delta : A^n_j \to A^n_{j-2}$ gives a linear map with kernel equal to H^n_j. The map is onto by Lemma 2.9, giving the first claim. Summing the first claim gives (2.17). □

Corollary 2.18. For each k, (0.4) is an equality on $\mathbf{R}^n$.

Proof. Corollary 2.16 and Lemma 2.2 give

$$
\sum_{j=0}^k \dim \mathcal{H}_{2j}(\mathbf{R}^n) = \sum_{j=0}^k \left(\dim A^n_{2j} + \dim A^n_{2j-1}\right) = \sum_{i=0}^{2k} \dim A^n_i = \dim\, \mathcal{P}_{2k}(\mathbf{R}^n)\,. \tag{2.19}
$$

□

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