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THE DELIGNE CATEGORY $\text{REP}(\text{GL}_t)$*

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Citation: UTIRALOVA, ALEXANDRA. 2022. "HARISH-CHANDRA BIMODULES IN THE DELIGNE CATEGORY $\text{REP}(\text{GL}_t)$."

As Published: <https://doi.org/10.1007/s00031-021-09689-2>

Publisher: Springer US

Persistent URL: <https://hdl.handle.net/1721.1/146366>

Version: Author's final manuscript: final author's manuscript post peer review, without publisher's formatting or copy editing

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HARISH-CHANDRA BIMODULES IN THE DELIGNE CATEGORY $\text{Rep}(\text{GL}_t)$

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Abstract. In this paper, we study the category of Harish-Chandra bimodules $HC_{\chi,\psi}$ in the Deligne category $\text{Rep}(\text{GL}_t)$. In particular, we answer Question 3.25 posed in Pavel Etingof's paper [7] and determine for which central characters χ and ψ this category is not zero.

1. Introduction

Representation theory in complex rank first started as an example in the paper by Deligne and Milne [5], where the category $\text{Rep}(\text{GL}_t)$ for t not necessarily integer was introduced. It was further developed in later papers by Deligne, where he introduced the categories $\text{Rep}(\text{O}_t)$, $\text{Rep}(\text{Sp}_{2t})$ and $\text{Rep}(S_t)$, interpolating the categories of representations of the groups O_n , Sp_{2n} and S_n correspondingly, and also suggested the ultraproduct realization of these categories [2], [3].

The goal of this paper is to study the categories $HC_{\chi,\psi}$ of Harish-Chandra bimodules for the Lie algebra \mathfrak{gl}_t in the Deligne category $\text{Rep}(\text{GL}_t)$ where t is a generic complex number, which interpolate the categories of Harish-Chandra bimodules for $\mathfrak{gl}_n(\mathbb{C})$ with fixed central characters to noninteger values of n . Namely, we determine for which values of central characters this category is not zero. This answers Question 3.25 in [7].

Harish-Chandra bimodules for GL_t were defined in [7] as follows. A $(\mathfrak{gl}_t, \mathfrak{gl}_t)$ -bimodule $M \in \text{Ind}(\text{Rep}(\text{GL}_t))$ is a Harish-Chandra bimodule if it is finitely generated (i.e., it is a quotient of $U(\mathfrak{gl}_t) \otimes U(\mathfrak{gl}_t) \otimes X$ for some $X \in \text{Rep}(\text{GL}_t)$), the action of the diagonal copy of $\mathfrak{gl}_t \subset \mathfrak{gl}_t \oplus \mathfrak{gl}_t$ is natural, and both copies of $Z(U(\mathfrak{gl}_t))$ act locally finitely on M . The category $HC_{\chi,\psi}$ is then the category of Harish-Chandra bimodules on which the left copy of the center $Z(U(\mathfrak{gl}_t))$ acts by a central character χ and the right one acts by ψ .

Our main result is Theorem 3.19, which shows that $HC_{\chi,\psi}$ is nonzero if and only if $\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{c_i u}$ (central characters for \mathfrak{gl}_t and the generating function notation for them are defined in 2.7).

Acknowledgements. I would like to thank Pavel Etingof for suggesting this problem to me and for all the valuable discussions we had about it.

DOI: 10.1007/s00031

Received February 5, 2020. Accepted August 29, 2021.

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2. Preliminaries

2.1. Notations

From now on, let $\mathbb{k} := \overline{\mathbb{Q}}$, let \mathfrak{h} , \mathfrak{b} and \mathfrak{n}_- denote the standard Cartan subalgebra, Borel subalgebra of upper-triangular matrices, and the subalgebra of strictly lower-triangular matrices in $\mathfrak{gl}_n(\mathbb{k})$; correspondingly, for any $\lambda \in \mathfrak{h}^*$, let \mathbb{k}_λ be the extension to \mathfrak{b} of the one-dimensional \mathfrak{h} -module corresponding to λ . Let $W \simeq S_n$ be the Weyl group of $\mathfrak{gl}_n(\mathbb{k})$, let Λ^+ be the lattice of integral dominant weights, and ρ be the half sum of all positive roots of $\mathfrak{gl}_n(\mathbb{k})$. We denote by $V^{(n)}$ the n -dimensional defining representation of GL_n . By Δ we will always mean the coproduct map.

2.2. Basic results and definitions

Definition 2.1. The Deligne category $\mathrm{Rep}(\mathrm{GL}_t)$ is the Karoubian envelope (formally adjoining images of idempotents) of the rigid symmetric \mathbb{C} -linear tensor category generated by a single object V of dimension $t \in \mathbb{C}$, such that $\mathrm{End}(V^{\otimes m}) = \mathbb{C}[S_m]$ for all $m \geq 1$.

Let us denote the object $V^{\otimes r} \otimes (V^*)^{\otimes s} \in \mathrm{Rep}(\mathrm{GL}_t)$ by $[r, s]$. The following theorem is copied from [7, Thm. 2.9].

Theorem 2.2 ([3]). *The category $\mathrm{Rep}(\mathrm{GL}_t)$ has the following universal property: if \mathcal{D} is a rigid tensor category then isomorphism classes of (possibly nonfaithful) symmetric tensor functors $\mathrm{Rep}(\mathrm{GL}_t) \rightarrow \mathcal{D}$ are in bijection with isomorphism classes of objects $X \in \mathcal{D}$ of dimension t , via $F \mapsto F([1, 0])$.*

It turns out that for noninteger values of t the categories $\mathrm{Rep}(\mathrm{GL}_t)$ are abelian and semisimple (see for example [8, Subsect. 9.12]).

We will now state the result showing that $\mathcal{C} := \mathrm{Rep}(\mathrm{GL}_t)$ can be constructed as a subcategory in the ultraproduct of the categories \mathcal{C}_n , where \mathcal{C}_n is the category of finite-dimensional representations of GL_n . For more details on the ultrafilters and ultraproducts, see [10] or [12]. A very detailed and nice explanation of the following construction can be found in [11]. The original statement for transcendental t is due to Pierre Deligne [3] (but is left without proof). And the similar statement for all values of t (requiring passing to positive characteristics) was proved in [9] by Nate Harman.

Let \mathcal{F} be a nonprincipal ultrafilter on \mathbb{N} . We will fix some isomorphism of fields $\prod_{\mathcal{F}} \mathbb{k} \simeq \mathbb{C}$.

Theorem 2.3 ([3], [9]). *\mathcal{C} is equivalent to the full subcategory $\tilde{\mathcal{C}}$ in $\prod_{\mathcal{F}} \mathcal{C}_n$ generated by $\tilde{V} := \prod_{\mathcal{F}} V^{(n)}$ under the operations of taking duals, tensor products, direct sums, and direct summands if t is the image of $\prod_{\mathcal{F}} n$ under the isomorphism $\prod_{\mathcal{F}} \mathbb{k} \simeq \mathbb{C}$.*

The proof of this statement is almost identical to the proof of [9, Thm. 1.1] or [10, Thm. 1.4.1].

Proof. Clearly, the categorical dimension of \tilde{V} is t . Therefore, by Theorem 2.2, we get a symmetric tensor functor $F : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ with $F([1, 0]) = \tilde{V}$. Now, the category $\mathrm{Rep}(\mathrm{GL}_t)$ is generated by $[1, 0]$ under the operations of taking duals,

tensor products, direct sums, and direct summands. Thus, F is essentially surjective. It is left to show that it is fully faithful. Since both \mathcal{C} and $\tilde{\mathcal{C}}$ are Karoubian envelopes of additive categories generated by $[r, s]$ and $\tilde{V}^{\otimes r} \otimes (\tilde{V}^*)^{\otimes s}$ correspondingly, it is enough to check that F induces an isomorphism of algebras

$$\text{End}_{\mathcal{C}}([r, s]) \rightarrow \text{End}_{\tilde{\mathcal{C}}}(\tilde{V}^{\otimes r} \otimes (\tilde{V}^*)^{\otimes s}).$$

But this is an easy consequence of the Schur-Weyl duality, since both algebras are isomorphic to the walled Brauer algebra $B_{r,s}(t)$ (it is ensured for the ultraproduct, since it holds for $\text{End}_{\text{GL}_n}((V^{(n)})^{\otimes r} \otimes ((V^{(n)})^*)^{\otimes s})$ for all $n > r + s$). \square

Remark 2.4. Clearly, one can only obtain transcendental numbers t as the image of $\prod_{\mathcal{F}} n$. To get this construction for algebraic t , one would need to consider the representations of GL_n over some fields of positive characteristic (see [9]).

However, applying automorphisms of \mathbb{C} over \mathbb{k} , one can show that any transcendental t can be obtained in this manner.

So, from now on we assume for simplicity that t is nonalgebraic. We expect, however, that similar results hold for all noninteger values of t .

Let us denote by $\text{Ind}(\mathcal{C})$ the category of ind-objects (i.e., filtered colimits of regular objects) of \mathcal{C} .

Definition 2.5. Let $\mathfrak{g} = \mathfrak{gl}_t = V \otimes V^*$. It is a Lie algebra in \mathcal{C} . Let us denote the commutator map $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ by c . We define its universal enveloping algebra $U(\mathfrak{g}) \in \text{Ind}(\mathcal{C})$ as a quotient of the tensor algebra $T(\mathfrak{g}) := \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$ by the ideal generated by the image of the map

$$\begin{aligned} r : \mathfrak{g} \otimes \mathfrak{g} &\rightarrow \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \subset T(\mathfrak{g}), \\ r &= c \oplus (\sigma_{\mathfrak{g}} - \text{id}_{\mathfrak{g}}), \end{aligned}$$

where $\sigma_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the permutation of tensor factors.

For any object $X = \prod_{\mathcal{F}} X^{(n)}$ of \mathcal{C} we can define an action map

$$a_X : \mathfrak{g} \otimes X \rightarrow X$$

as the ultraproduct $\prod_{\mathcal{F}} a_{X^{(n)}}$, where $a_{X^{(n)}} : \mathfrak{gl}_n(\mathbb{k}) \otimes X^{(n)} \rightarrow X^{(n)}$ is the natural action of $\mathfrak{gl}_n(\mathbb{k})$ on $X^{(n)}$. Clearly, a_X is a Lie algebra action; i.e.,

$$(a_X \oplus (a_X \circ (\text{id}_{\mathfrak{g}} \otimes a_X))) \circ (r \otimes \text{id}_X) = 0$$

as a map from $\mathfrak{g} \otimes \mathfrak{g} \otimes X$ to X (where r is the map $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g})$ defined in 2.5). That is, a_X induces the action of $U(\mathfrak{g})$ on X . We will refer to this action as the *natural* action of \mathfrak{g} (or $U(\mathfrak{g})$).

Clearly, the natural action of \mathfrak{g} on \mathcal{C} extends to $\text{Ind}(\mathcal{C})$.

We define the Poincaré–Birkhoff–Witt (or PBW) filtration F on $U(\mathfrak{g})$ as the image of $T^i(\mathfrak{g}) := \mathfrak{g}^{\otimes i}$ under the quotient map $T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$. We have

$$F_i U(\mathfrak{g}) := \prod_{\mathcal{F}} F_i U(\mathfrak{gl}_n(\mathbb{k})),$$

where we abuse the notation and denote by F the PBW-filtration on $U(\mathfrak{gl}_n(\mathbb{k}))$ as well.

The center of $U(\mathfrak{g})$

$$Z(U(\mathfrak{g})) = U(\mathfrak{g})^{\text{GL}_t} = \text{Hom}(\mathbb{1}, U(\mathfrak{g})) = \bigcup_i \text{Hom}(\mathbb{1}, F_i U(\mathfrak{g}))$$

is a filtered algebra in the category of \mathbb{C} -vector spaces. We have

$$F_i Z(U(\mathfrak{g})) := Z(U(\mathfrak{g})) \cap F_i U(\mathfrak{g}) = \prod_{\mathcal{F}} (Z(U(\mathfrak{gl}_n(\mathbb{k}))) \cap F_i U(\mathfrak{gl}_n(\mathbb{k}))).$$

The Harish-Chandra isomorphism tells us that $Z(U(\mathfrak{gl}_n(\mathbb{k})))$ is isomorphic to the algebra of symmetric polynomials $\mathbb{k}[\mathfrak{h}^*]^W$ with filtration given by the degree. Given a central element C we recover the corresponding symmetric polynomial p by looking at the action of C on the Verma module $M_\lambda := U(\mathfrak{gl}_n(\mathbb{k})) \otimes_{U(\mathfrak{b})} \mathbb{k}_{\lambda-\rho}$ with the highest weight $\lambda - \rho \in \mathfrak{h}^*$. We have $p(\lambda) = C|_{M_\lambda}$.

Remark 2.6. The shift of the highest weight in the definition of M_λ is needed to replace the dot-action of W on \mathfrak{h}^* with the usual action.

Symmetric polynomials in n variables are freely generated (as an algebra) by the first n power sums polynomials $p_k := \sum_i x_i^k$. For any k and any n we define the element $C_k \in Z(U(\mathfrak{gl}_n(\mathbb{k})))$ to be the image of p_k under the isomorphism $\mathbb{k}[x_1, \dots, x_n]^{S_n} \rightarrow Z(U(\mathfrak{gl}_n(\mathbb{k})))$: i.e., C_k is the central element, which acts on each M_λ via the constant $\sum_i \lambda_i^k$.

We get that $Z(U(\mathfrak{gl}_n(\mathbb{k}))) \simeq \mathbb{k}[C_1, \dots, C_n]$ with $\deg C_k = k$. And therefore,

$$\begin{aligned} Z(U(\mathfrak{g})) &\simeq \mathbb{C}[C_1, C_2, \dots], \\ C_i &:= \prod_{\mathcal{F}} C_i \text{ (by the abuse of notation).} \end{aligned}$$

Definition 2.7. A central character is an algebra homomorphism

$$\psi : Z(U(\mathfrak{g})) \rightarrow \mathbb{C}.$$

By the result above, ψ is completely determined by the numbers $\psi_k = \psi(C_k)$. For convenience, let us adopt the (exponential) generating function notation

$$\psi(u) = \frac{1}{(e^u - 1)} \sum \frac{1}{k!} \psi_k u^k \in \mathbb{C}((u)),$$

where we put $\psi_0 = 1$.

Remark 2.8. We divide the generating function by $(e^u - 1)$ only for the reason that it yields better looking formulas, and we treat the factor $1/(e^u - 1)$ formally.

For each $\mathfrak{gl}_n(\mathbb{k})$, let us fix a central character $\psi^{(n)} : Z(U(\mathfrak{gl}_n(\mathbb{k}))) \rightarrow \mathbb{k}$. It is determined by n numbers $\psi_k^{(n)} = \psi^{(n)}(C_k)$ with $1 \leq k \leq n$. However, since

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the elements C_k are defined for all k , we can define the (exponential) generating function

$$\psi^{(n)}(u) = \frac{1}{(e^u - 1)} \sum \frac{1}{k!} \psi_k^{(n)} u^k \in \mathbb{k}((u)).$$

Due to the algebraic independence of $\{C_k\}_{k \leq n}$ in $Z(U(\mathfrak{gl}_n(\mathbb{k})))$ we can choose $\psi_k^{(n)}$ to be an arbitrary number for $n > k$. Thus, any central character ψ of \mathfrak{g} can be obtained as an ultraproduct of central characters $\psi^{(n)}$ of $\mathfrak{gl}_n(\mathbb{k})$ (i.e., $\psi_k = \prod_{\mathcal{F}} \psi_k^{(n)}$). We write $\psi = \prod_{\mathcal{F}} \psi^{(n)}$.

3. The category $HC_{\chi, \psi}$

3.1. Definitions and notations

We will henceforth consider objects with the action of $\mathfrak{g} \oplus \mathfrak{g}$. Let us denote by \mathfrak{g}_l , \mathfrak{g}_r and \mathfrak{g}_d the left, right and diagonal copy of \mathfrak{g} inside $\mathfrak{g} \oplus \mathfrak{g}$ (with the diagonal copy being the image of the map $\text{id}_{\mathfrak{g}} \oplus \text{id}_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$). Moreover, we will freely switch between the left action of $\mathfrak{g} \oplus \mathfrak{g}$ and the double action of \mathfrak{g} both on the left and the right, the first given by the action of \mathfrak{g}_l and the second by minus the action of \mathfrak{g}_r .

Clearly, any left \mathfrak{g} -module M in $\text{Ind}(\mathcal{C})$ with the action map $f : \mathfrak{g} \otimes M \rightarrow M$ has a unique action of $\mathfrak{g} \oplus \mathfrak{g}$ s.t. \mathfrak{g}_l acts by the original action and the diagonal copy \mathfrak{g}_d acts naturally: i.e., via the map a_M . Indeed, one defines the new action map $(\mathfrak{g} \oplus \mathfrak{g}) \otimes M \rightarrow M$ as $f \oplus (a_M - f)$.

Definition 3.1. A Harish-Chandra bimodule for GL_t is a $\mathfrak{g} \oplus \mathfrak{g}$ -module $M \in \text{Ind}(\mathcal{C})$, such that

- (1) M is finitely generated: i.e., it is a quotient of $(U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes X$ for some $X \in \mathcal{C}$;
- (2) \mathfrak{g}_d acts naturally (i.e., via a_M) on M ;
- (3) the center $Z(U(\mathfrak{g}) \otimes U(\mathfrak{g}))$ acts locally finitely on M : i.e., $\text{Ann}_{Z(U(\mathfrak{g}) \otimes U(\mathfrak{g}))} M$ is an ideal of finite codimension.

Any maximal ideal in $Z(U(\mathfrak{g}) \otimes U(\mathfrak{g}))$ (i.e., ideal of codimension 1) corresponds to a pair of central characters $\chi, \psi : Z(U(\mathfrak{g})) \rightarrow \mathbb{C}$. Denote by $HC_{\chi, \psi}$ the category of Harish-Chandra bimodules on which $Z(U(\mathfrak{g}_l))$ acts via χ and $Z(U(\mathfrak{g}_r))$ acts via ψ .

Example. Let $U_{\chi} := U(\mathfrak{g})/(z - \chi(z))$, where z runs over all elements in the center. Clearly, $U_{\chi} \in HC_{\chi, \chi}$.

Note. For each i we have $F_i U_{\chi} = \prod_{\mathcal{F}} F_i U(\mathfrak{gl}_n(\mathbb{k})) / (z - \chi^{(n)}(z))$, where $\chi = \prod_{\mathcal{F}} \chi^{(n)}$.

We will abuse the notation and denote by U_{χ} the quotient $U(\mathfrak{gl}_n(\mathbb{k})) / (z - \chi(z))$ too, when it is clear from the context which one we refer to.

Clearly, every irreducible Harish-Chandra bimodule must lie in one of the categories $HC_{\chi, \psi}$, so they are interesting to study.

Remark 3.2. Note that for regular Harish-Chandra bimodules (for GL_n), condition (2) of Definition 3.1 translates to the fact that one can integrate the action of the diagonal copy $(\mathfrak{gl}_n)_d$ on M to the action of GL_n ; and condition (3) is equivalent to M having a *finite K -type*, which means that for any simple GL_n representation L the multiplicity space $\mathrm{Hom}_{\mathrm{GL}_n}(L, M)$ is finite-dimensional (see [1, Prop. 5.3]). However, in $\mathrm{Rep}(\mathrm{GL}_t)$ the latter is no longer true. Harish-Chandra bimodules of finite K -type are studied to some extent in [13]. One can also find there some nontrivial examples of irreducible Harish-Chandra bimodules.

For any object $X \in \mathcal{C}$, the tensor product $U_\psi \otimes X$ is naturally a left \mathfrak{g} -module with \mathfrak{g} acting on U_ψ by multiplication on the left and on X naturally (so it is easy to deduce that the right action of \mathfrak{g} is by (minus) multiplication on the right and affects only the U_ψ part of the tensor product). Let $N(\chi, \psi, X) := (U_\psi \otimes X)_\chi$ denote the quotient $(U_\psi \otimes X)/(z - \chi(z))(U_\psi \otimes X)$, where we factor out by the action of the left copy of $Z(U(\mathfrak{g}))$. Clearly, $N(\chi, \psi, X) \in HC_{\chi, \psi}$.

3.2. Bimodules $N(\chi, \psi, X)$

Lemma 3.3 (see [7, Sect. 3.6]). *The category $HC_{\chi, \psi}$ is nonzero iff $N(\chi, \psi, X) \neq 0$ for some $X \in \mathcal{C}$.*

Proof. For any $X \in \mathcal{C}$ we have that as $(\mathfrak{g}, \mathfrak{g})$ -bimodules

$$N(\chi, \psi, X) := U_\chi \otimes_{U(\mathfrak{gl})} (U_\psi \otimes X) \simeq (U_\chi \otimes U_\psi^{\mathrm{op}}) \otimes_{U(\mathfrak{g}_d)} X,$$

where \mathfrak{g}_d acts on the right on $U_\chi \otimes U_\psi^{\mathrm{op}}$.

Now suppose $M \in HC_{\chi, \psi}$, then M is finitely generated: i.e., it is generated by some subobject X (that lies in \mathcal{C}). There is a natural morphism $N(\chi, \psi, X) \rightarrow M$ whose image is the subbimodule of M generated by X (since X generates M , it is surjective). Thus, if M is nonzero then so is $N(\chi, \psi, X)$. \square

Corollary 3.4. *The category $HC_{\chi, \psi}$ is nonzero if and only if $N(\chi, \psi, [r, s]) \neq 0$ for some $[r, s]$.*

So, we want to understand for which χ, ψ the bimodule $(U_\psi \otimes [r, s])_\chi$ is not zero. For this purpose, let us first look at $(U_\psi \otimes V)_\chi$ and understand for which χ and ψ it is not zero.

3.3. The basic case for $\mathfrak{gl}_n(\mathbb{k})$

Let us look at the finite-dimensional case, namely $\mathfrak{gl}_n(\mathbb{k})$. We have a homomorphism $U_\psi \rightarrow \mathrm{End}_{\mathbb{k}}(M_\lambda)$ for some $\lambda = (\lambda_1, \dots, \lambda_n)$ (determined up to permutation of λ_i) which is injective by Duflo's theorem (which states that the annihilator of a Verma module is the ideal generated by the kernel of the corresponding central character, see [6, Thm. 8.4.3]). So there is an injective morphism of $\mathfrak{gl}_n(\mathbb{k})$ -bimodules

$$U_\psi \otimes V^{(n)} \hookrightarrow \mathrm{Hom}_{\mathbb{k}}(M_\lambda, M_\lambda \otimes V^{(n)}),$$

where the right action of $\mathfrak{gl}_n(\mathbb{k})$ is on the source and the left action is on the target. Thus, it is enough to consider the action of the center on $M_\lambda \otimes V^{(n)}$ (instead of considering the left action on $U_\psi \otimes V^{(n)}$). For generic (without nontrivial

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stabilizers in W) weight λ we have $M_\lambda \otimes V^{(n)} = \bigoplus_{i=1}^n M_{\lambda+e_i}$, where $\lambda + e_i = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_n)$. When restricted to the direct summand $M_{\lambda+e_l}$, each C_k acts by $\sum_{i=1}^n \lambda_i^k + (\lambda_l + 1)^k - \lambda_l^k$.

Let $\Omega := \frac{1}{2}(\Delta(C_2) - C_2 \otimes 1 - 1 \otimes 1)$. Then

$$(2\Omega + 1)|_{M_{\lambda+e_l}} = 2\lambda_l + 1$$

and thus,

$$(\Delta(C_k) - C_k \otimes 1)|_{U_\psi \otimes V^{(n)}} = (\Omega|_{U_\psi \otimes V^{(n)}} + 1)^k - \Omega^k|_{U_\psi \otimes V^{(n)}}.$$

Let us denote. Then, we have proved the following lemma:

Lemma 3.5. *For generic central character ψ (i.e., such that U_ψ acts on M_λ with generic λ), we have:*

$$\Delta(C_k)|_{U_\psi \otimes V^{(n)}} - C_k \otimes 1|_{U_\psi \otimes V^{(n)}} = P_k(\Omega|_{U_\psi \otimes V^{(n)}}).$$

Now let us identify the \mathbb{k} -algebra $\text{End}_{\mathbb{k}}(M_\lambda)$ with $A := \text{End}_{\mathbb{k}}(U(\mathfrak{n}_-))$ via the natural isomorphism of vector spaces M_λ and $U(\mathfrak{n}_-)$. The representations $U(\mathfrak{gl}_n(\mathbb{k})) \rightarrow \text{End}_{\mathbb{k}}(M_\lambda)$ thus give us a family of algebra homomorphisms to A depending on λ polynomially. That is to say, we have a map of algebras

$$\varphi : U(\mathfrak{gl}_n(\mathbb{k})) \rightarrow A \otimes \mathbb{k}[x_1, \dots, x_n],$$

which composed with the quotient map $\mathbb{k}[x_1, \dots, x_n] \rightarrow \mathbb{k}$ sending x_i to λ_i , and the natural isomorphism between A and $\text{End}_{\mathbb{k}}(M_\lambda)$, gives us precisely the representation of $\mathfrak{gl}_n(\mathbb{k})$ on M_λ . Let us denote by \mathfrak{m}_λ the maximal ideal generated by $x_i - \lambda_i, 1 \leq i \leq n$.

The family of representations of $\mathfrak{gl}_n(\mathbb{k}) \oplus \mathfrak{gl}_n(\mathbb{k})$ on $M_\lambda \otimes V^{(n)}$ for varying λ (with the left copy acting on M_λ and the right copy acting on $V^{(n)}$) produces a map $\varphi \otimes \rho_{V^{(n)}} : U(\mathfrak{gl}_n(\mathbb{k})) \otimes U(\mathfrak{gl}_n(\mathbb{k})) \rightarrow A \otimes \mathbb{k}[x_1, \dots, x_n] \otimes \text{End}_{\mathbb{k}}(V^{(n)})$. We have proved above that the image of $\Delta(C_k) - C_k \otimes 1 - P_k(\Omega)$ under $\varphi \otimes \rho_{V^{(n)}}$ has to lie in $A \otimes (\bigcap_{\lambda \text{ generic}} \mathfrak{m}_\lambda) \otimes \text{End}_{\mathbb{k}}(V^{(n)})$. Since the set of $\lambda \in \mathbb{k}^n$ such that some $\lambda_i = \lambda_j$ for $i \neq j$ (i.e., λ has a nontrivial stabilizer in W and hence is nongeneric by our definition) is closed in \mathbb{k}^n , the intersection $\bigcap_{\lambda \text{ generic}} \mathfrak{m}_\lambda$ is zero. Thus, $\Delta(C_k) - C_k \otimes 1 - P_k(\Omega)$ acts as zero on $M_\lambda \otimes V^{(n)}$ for any (not necessary generic) λ .

Corollary 3.6. *For any central character ψ of $U(\mathfrak{gl}_n(\mathbb{k}))$*

$$\Delta(C_k)|_{U_\psi \otimes V^{(n)}} - C_k \otimes 1|_{U_\psi \otimes V^{(n)}} = P_k(\Omega|_{U_\psi \otimes V^{(n)}}).$$

Remark 3.7. We have

$$\frac{1}{(e^u - 1)} \sum \frac{1}{k!} P_k(b) u^k = e^{bu}.$$

Recall that Ω was defined as $\frac{1}{2}(\Delta(C_2) - C_2 \otimes 1 - 1 \otimes 1)$. The element $C_k \in Z(U((\mathfrak{gl}_n)_r))$ acts on $U_\psi \otimes V^{(n)}$ as $C_k \otimes 1$, and $C_k \in Z(U((\mathfrak{gl}_n)_l))$ acts on $U_\psi \otimes V^{(n)}$ as $\Delta(C_k)$. Thus, Ω acts on $(U_\psi \otimes V^{(n)})_\chi$ as a constant $b = \frac{1}{2}(\chi_2 - \psi_2 - 1)$. So, by Corollary 3.6, $(U_\psi \otimes V^{(n)})_\chi \neq 0$ only when $\chi(u) - \psi(u) = e^{bu}$.

Thus, for each $\mathfrak{gl}_n(\mathbb{k})$ we must have $\chi^{(n)}(u) - \psi^{(n)}(u) = e^{b_n u}$ for some $b_n \in \mathbb{k}$. So, in the case of \mathfrak{gl}_t , $\chi(u) - \psi(u) = e^{bu}$ for $b = \prod_{\mathcal{F}} b_n$.

3.4. The basic case for \mathfrak{gl}_t

We have just proved the following statement.

Lemma 3.8. *For any central characters ψ, χ of $U(\mathfrak{g})$, the bimodule $(U_\psi \otimes V)_\chi$ is nonzero only if*

$$\chi(u) - \psi(u) = e^{bu}$$

for some $b \in \mathbb{C}$.

Lemma 3.9. *For any finitely generated $(\mathfrak{g}, \mathfrak{g})$ -bimodule M in $\text{Ind}(\mathcal{C})$*

$$(\Delta(C_k) - C_k \otimes 1)|_{M \otimes V} = P_k(\Omega|_{M \otimes V}).$$

Proof. We have proved this statement for $M = U_\psi$. It is easy to see that the same proof works for the module $M = U_\psi \otimes U(\mathfrak{g}) \otimes X$, where $X \in \mathcal{C}$ and $U(\mathfrak{gl}_t)$ acts only on the U_ψ part of the tensor product. X is represented by some sequence of GL_n -modules (X_1, X_2, \dots) , so we can pass to a finite dimensional case. There, $U_\psi \otimes U(\mathfrak{gl}_n(\mathbb{k})) \otimes X_n \otimes V \hookrightarrow \bigoplus_{l=1}^n \text{Hom}_{\mathbb{k}}(M_\lambda, M_{\lambda+e_l}) \otimes U(\mathfrak{gl}_n(\mathbb{k})) \otimes X_n$ and $\mathfrak{gl}_n(\mathbb{k})_l$ acts only on the Hom-part of the tensor product, moreover only on the target (i.e., on $M_{\lambda+e_l}$), and we can repeat the proof from above. Thus, we have obtained the following:

$$(\Delta(C_k) - C_k \otimes 1)|_{(U_\psi \otimes U(\mathfrak{g})^{\text{op}} \otimes X) \otimes V} = P_k(\Omega|_{(U_\psi \otimes U(\mathfrak{g})^{\text{op}} \otimes X) \otimes V}).$$

For any $n \in \mathbb{Z}_{>0}$, the universal enveloping algebra $U(\mathfrak{gl}_n(\mathbb{k}))$ embeds into $\bigoplus_{\chi^{(n)}} U_{\chi^{(n)}}$, where the sum runs over all central characters $\chi^{(n)} : Z(U(\mathfrak{gl}_n(\mathbb{k}))) \rightarrow \mathbb{k}$. Therefore, the same holds for $U(\mathfrak{g})$. Hence, the statement of the lemma holds for $M = U(\mathfrak{g}) \otimes U(\mathfrak{g})^{\text{op}} \otimes X$, where $X \in \mathcal{C}$.

Finally, by definition, any finitely generated $U(\mathfrak{g})$ -module is a quotient of $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes X$ with $X \in \mathcal{C}$ and $U(\mathfrak{gl}_t)$ acting only on the leftmost part of this tensor product. This ends the proof. \square

Remark 3.10. A similar reasoning shows that for any finitely generated $U(\mathfrak{g})$ -module M in $\text{Ind}(\mathcal{C})$,

$$(\Delta(C_k) - C_k \otimes 1)|_{M \otimes V^*} = \overline{P}_k(\Omega|_{M \otimes V^*}),$$

where $\overline{P}_k(c) = (c-1)^k - c^k$.

Note. It is easy to see that

$$\frac{1}{(e^u - 1)} \sum \overline{P}_k(c) u^k = -e^{(c-1)u}.$$

3.5. The general case

Now we are ready to consider the action of the central elements on $U_\psi \otimes [r, s]$. Note that the central element C of $Z(U(\mathfrak{g}_r))$ acts on this module as $(C \otimes 1)_{U_\psi, [r, s]}$, and if it is considered as an element of $Z(U(\mathfrak{gl}_t))$ then it acts as $\Delta(C)_{U_\psi, [r, s]}$, meaning that the first tensor factor acts on U_ψ and the second acts on $[r, s]$.

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Let us adopt some convenient notation. Let A be any element in $Z(U(\mathfrak{g}) \otimes U(\mathfrak{g}))$. Denote by A_j its image under the following map $\tau_j : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g})^{\otimes(m+1)}$:

$$\tau_j = (\Delta^{j-1} \otimes \text{id}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-j}.$$

We also denote by A_j the corresponding operator acting on $X_0 \otimes X_1 \otimes \cdots \otimes X_m$, where each X_i is a $U(\mathfrak{g})$ -module.

Let us view $U_\psi \otimes [r, s]$ as the string of tensor factors

$$U_\psi \otimes V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*.$$

Then

$$\begin{aligned} & (\Delta(C_k) - C_k \otimes 1)_{U_\psi, [r, s]} \\ &= \Delta^{r+s}(C_k) - (C_k \otimes 1)_1 \\ &= (\Delta^{r+s-1} \otimes \text{id})(\Delta(C_k) - C_k \otimes 1) + \Delta^{r+s-1}(C_k) \otimes 1 - (C_k \otimes 1)_1 \\ &= (\Delta(C_k) - C_k \otimes 1)_{r+s} + \Delta^{r+s-1}(C_k) \otimes 1 - (C_k \otimes 1)_1 \\ &= \dots \\ &= (\Delta(C_k) - C_k \otimes 1)_{r+s} + (\Delta(C_k) - C_k \otimes 1)_{r+s-1} + \cdots \\ &\quad + (\Delta(C_k) - C_k \otimes 1)_1. \end{aligned}$$

By Lemma 3.9 and Remark 3.10, this is equal to

$$\bar{P}_k(\Omega_{r+s}) + \cdots + \bar{P}_k(\Omega_{r+1}) + P_k(\Omega_r) + \cdots P_k(\Omega_1).$$

Theorem 3.11. $(U_\psi \otimes [r, s])_\chi \neq 0$ only if there exist numbers $b_1, \dots, b_r, c_1, \dots, c_s \in \mathbb{C}$ such that

$$\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{(c_i-1)u}.$$

Proof. Let us consider $\text{End}_{\mathbb{K}}((U_\psi \otimes [r, s])_\chi)$. It is a nonzero algebra and there is a homomorphism from $\mathbb{C}[\Omega_1, \dots, \Omega_{r+s}]$ to it. Its image is a nonzero finitely generated commutative algebra where

$$\bar{P}_k(\Omega_{r+s}) + \cdots + \bar{P}_k(\Omega_{r+1}) + P_k(\Omega_r) + \cdots P_k(\Omega_1) = \chi_k - \psi_k.$$

Thus, by the Nullstellensatz, there exists a maximal ideal in this algebra: i.e., numbers $b_1, \dots, b_r, c_1, \dots, c_s$, such that

$$\bar{P}_k(c_s) + \cdots + \bar{P}_k(c_1) + P_k(b_r) + \cdots P_k(b_1) = \chi_k - \psi_k.$$

This ends the proof. \square

Corollary 3.12. If $HC_{\chi, \psi} \neq 0$, then there exist numbers $b_1, \dots, b_r, c_1, \dots, c_s \in \mathbb{C}$ such that

$$\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{(c_i-1)u}.$$

3.6. The reverse direction: constructing a nonzero object in $HC_{\chi,\psi}$

Now we want to prove the converse: i.e., if such numbers exist, then the category is nonzero. The remainder of this paper will be devoted to proving the following theorem.

Theorem 3.13. *Suppose there exist numbers $b_1, \dots, b_r, c_1, \dots, c_s \in \mathbb{C}$ such that*

$$\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{(c_i - 1)u}.$$

Then $HC_{\chi,\psi} \neq 0$.

To prove this, we are going to show that $(U_\psi \otimes X)_\chi$ is nonzero for some $X \in \mathcal{C}$.

Lemma 3.14. *Let $b_i = \prod_{\mathcal{F}} b_i^{(n)}$, for $1 \leq i \leq r$ and $c_i = \prod_{\mathcal{F}} c_i^{(n)}$, for $1 \leq i \leq s$, with $b_i^{(n)}, c_i^{(n)} \in \mathbb{k}$.*

Then for any $\psi(u)$, there exists a presentation of it as an ultraproduct of $\psi^{(n)}(u)$ —central characters of $\mathfrak{gl}_n(\mathbb{k})$ — so that $\psi^{(n)}(u) = 0$ for $n \leq r + s$ and when $n > r + s$ then $\mathfrak{gl}_n(\mathbb{k})$ acts with central character $\psi^{(n)}$ on some $M_{\mu^{(n)}}$, where

$$\begin{aligned} \mu_i^{(n)} &= b_i^{(n)}, & 1 \leq i \leq r, \\ \mu_{n-j+1}^{(n)} &= c_j^{(n)}, & 1 \leq j \leq s. \end{aligned}$$

Proof. Let us take an arbitrary presentation of the numbers ψ_k as an ultraproduct: $\psi_k = \prod_{\mathcal{F}} \phi_k^{(n)}$.

We will prove that for a fixed k we can change finitely many of the numbers $\phi_k^{(n)}$, namely, those with $n < k + s + r$, so that the resulting central characters satisfy the condition above.

For a fixed n , consider the following equations on $m := n - r - s$ variables x_i for $k = 1, \dots, m$:

$$(b_1^{(n)})^k + \dots + (b_r^{(n)})^k + (c_1^{(n)})^k + \dots + (c_s^{(n)})^k + x_1^k + \dots + x_m^k = \phi_k^{(n)}. \quad (3.14.1)$$

They are equations on the first m power sums of x_i . The ring of symmetric polynomials in m variables is freely generated by the first m power sums, so these equations determine a point in the maximal spectrum of $\mathbb{k}[x_1, \dots, x_m]^{S_n}$. The inclusion $\mathbb{k}[x_1, \dots, x_m]^{S_n} \hookrightarrow \mathbb{k}[x_1, \dots, x_m]$ induces a surjective map on the maximal spectra (the quotient map by the S_n -action): thus, there exists a solution $x_i = a_i^{(n)}$ for $1 \leq i \leq m$.

Put

$$\begin{aligned} \mu_1^{(n)} &= b_1^{(n)}, \dots, \mu_r^{(n)} = b_r^{(n)}, \\ \mu_{r+1}^{(n)} &= a_1^{(n)}, \dots, \mu_{n-s}^{(n)} = a_{n-r-s}^{(n)}, \\ \mu_{n-s+1}^{(n)} &= c_s^{(n)}, \dots, \mu_n^{(n)} = c_1^{(n)}. \end{aligned}$$

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Consider the central character $\psi^{(n)}$ corresponding to the $\mu^{(n)}$ above. Then for $1 \leq k \leq n - r - s$ we have

$$\psi_k^{(n)} := \sum_{i=1}^n (\mu_i^{(n)})^k = \sum_{i=1}^r (b_i^{(n)})^k + \sum_{i=1}^m (a_i^{(n)})^k + \sum_{i=1}^s (c_i^{(n)})^k = \phi_k^{(n)} \text{ (by 3.14.1).}$$

Putting $\psi^{(n)} = 0$ for $n \leq r + s$, we see that for a fixed k we have $\psi_k^{(n)} = \phi_k^{(n)}$ for all $n \geq k + r + s$ and hence, $\prod_{\mathcal{F}} \psi_k^{(n)} = \prod_{\mathcal{F}} \phi_k^{(n)} = \psi_k$. \square

Lemma 3.15. *Let $n > r + s$ and*

$$\lambda = \mu + e_1 + \cdots + e_r - e_{n-s+1} - \cdots - e_n.$$

Suppose $Z(U(\mathfrak{gl}_n(\mathbb{k})))$ acts on M_λ with character χ and on M_μ with character ψ . Then

$$\chi(u) - \psi(u) = e^{\mu_1 u} + \cdots + e^{\mu_r u} - e^{(\mu_{n-s+1}-1)u} - \cdots - e^{(\mu_n-1)u}.$$

Proof. The element C_k acts on $M_{\mu+e_l}$ as $\sum_j \mu_j^k + P_k(\mu_l)$ and on $M_{\mu-e_l}$ as $\sum_j \mu_j^k + \bar{P}_k(\mu_l)$. Thus,

$$C_k|_{M_{\mu+e_l}} - C_k|_{M_\mu} = P_k(\mu_l)$$

and

$$C_k|_{M_{\mu-e_l}} - C_k|_{M_\mu} = \bar{P}_k(\mu_l).$$

We put $\lambda^{[0]} := \lambda$, $\lambda^{[i]} := \lambda^{[i-1]} - e_i$, $1 \leq i \leq r$ and $\mu^{[i]} := \mu^{[i-1]} - e_{n-i+1}$, $1 \leq i \leq s$ with $\mu^{[0]} := \mu$ (thus $\mu^{[s]} = \lambda^{[r]}$):

$$\begin{aligned} \mu^{[0]} &= (\mu_1, \dots, \mu_n), \\ \mu^{[1]} &= (\mu_1, \dots, \mu_{n-1}, \mu_n - 1), \\ \mu^{[2]} &= (\mu_1, \dots, \mu_{n-2}, \mu_{n-1} - 1, \mu_n - 1), \\ &\dots\dots\dots \\ \mu^{[s]} &= \lambda^{[r]} = (\mu_1, \dots, \mu_{n-s}, \mu_{n-s+1} - 1, \dots, \mu_n - 1), \\ \lambda^{[r-1]} &= (\mu_1, \dots, \mu_{r-1}, \mu_r + 1, \mu_{r+1}, \dots, \mu_{n-s}, \mu_{n-s+1} - 1, \dots, \mu_n - 1), \\ &\dots\dots\dots \\ \lambda^{[1]} &= (\mu_1, \mu_2 + 1, \dots, \mu_r + 1, \mu_{r+1}, \dots, \mu_{n-s}, \mu_{n-s+1} - 1, \dots, \mu_n - 1), \\ \lambda^{[0]} &= (\mu_1 + 1, \dots, \mu_r + 1, \mu_{r+1}, \dots, \mu_{n-s}, \mu_{n-s+1} - 1, \dots, \mu_n - 1). \end{aligned}$$

Now let $\chi^{[i]}$ be the central character corresponding to $\lambda^{[i]}$ and $\psi^{[i]}$ be the central character corresponding to $\mu^{[i]}$. Thus, $\chi_k^{[i]} - \chi_k^{[i+1]} = P_k(\lambda_{i+1}^{[i+1]}) = P_k(\mu_{i+1})$ and $\psi^{[i+1]} - \psi^{[i]} = \bar{P}_k(\mu_{n-i}^{[i]}) = \bar{P}_k(\mu_{n-i})$. Summing up these equations for all χ 's and ψ 's we obtain

$$\chi_k - \psi_k = P_k(\mu_1) + \cdots + P_k(\mu_r) + \bar{P}_k(\mu_{n-s+1}) + \cdots + \bar{P}_k(\mu_n),$$

which leads to the desired relation for generating functions $\chi(u)$ and $\psi(u)$. \square

Corollary 3.16 (of Lemmas 3.14 and 3.15)). *Suppose*

$$\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{(c_i-1)u}$$

for some $b_i \in \mathbb{C}, 1 \leq i \leq r, c_j \in \mathbb{C}, 1 \leq j \leq s$. Then there exist presentations of χ and ψ as ultraproducts of $\chi^{(n)}$ and $\psi^{(n)}$ correspondingly, so that for any $n > r + s$ the algebra $U(\mathfrak{gl}_n(\mathbb{k}))$ acts on some $M_{\lambda^{(n)}}$ with central character $\chi^{(n)}$ and if $\mu^{(n)} = \lambda^{(n)} - e_1 - \dots - e_r + e_{n-s+1} + \dots + e_n$, then $U(\mathfrak{gl}_n(\mathbb{k}))$ acts on $M_{\mu^{(n)}}$ with central character $\psi^{(n)}$.

Proof. We apply Lemma 3.14 to $\psi(u), b_i, c_j$ to obtain central characters $\psi^{(n)}$ and weights $\mu^{(n)}$, satisfying the conditions of the lemma. Then for $n > r + s$, we put $\lambda^{(n)} = \mu^{(n)} + e_1 + \dots + e_r - e_{n-s+1} - \dots - e_n$ and denote by $\chi^{(n)}$ the central character corresponding to $\lambda^{(n)}$.

We use Lemma 3.15 for $\lambda^{(n)}$ and $\mu^{(n)}$ to see that

$$\chi^{(n)}(u) - \psi^{(n)}(u) = e^{b_1^{(n)}u} + \dots + e^{b_r^{(n)}u} - e^{(c_1^{(n)}-1)u} - \dots - e^{(c_s^{(n)}-1)u}.$$

We put $\chi^{(n)} = 0$ for $n \leq r + s$. If now $\tilde{\chi}(u) = \prod_{\mathcal{F}} \chi^{(n)}(u)$, then

$$\tilde{\chi}(u) - \psi(u) = e^{b_1 u} + \dots + e^{b_r u} - e^{(c_1-1)u} - \dots - e^{(c_s-1)u}$$

and thus $\tilde{\chi}(u) = \chi(u)$ and we are done. \square

Lemma 3.17. *Let λ, μ , be such that $\lambda - \mu \in \Lambda^+$ and suppose X is a finite-dimensional $\mathfrak{gl}_n(\mathbb{k})$ -module, with maximal weight $\lambda - \mu$. Let χ be the central character corresponding to λ and ψ be the central character corresponding to μ . Then we have*

$$(U_\psi \otimes X)_\chi \neq 0.$$

Moreover, if $F_0 U_\psi$ is the zeroth filtered component that is the span of 1, the quotient $(F_0 U_\psi \otimes X)_\chi$ is nonzero.

Proof. Consider a natural surjective map of $U(\mathfrak{gl}_n(\mathbb{k}))$ -modules

$$U_\psi \otimes X \rightarrow M_\mu \otimes X \rightarrow 0,$$

which sends $u \otimes x$ to $uv_\mu \otimes x$, where v_μ is the highest weight vector in M_μ .

Taking tensor product with U_χ over $U(\mathfrak{gl}_n(\mathbb{k}))$ for some $\chi: Z(U(\mathfrak{gl}_n(\mathbb{k}))) \rightarrow \mathbb{k}$ is right exact, thus we have

$$(U_\psi \otimes X)_\chi \rightarrow (M_\mu \otimes X)_\chi \rightarrow 0.$$

So it is left to prove that $(M_\mu \otimes X)_\chi \neq 0$ and that the image of $v_\mu \otimes X$ is nonzero.

We note that λ is maximal among weights of $M_\mu \otimes X$. Using the standard argument—i.e., taking the sum of all submodules of $M_\mu \otimes X$ (that are naturally objects of the category \mathcal{O}) that do not contain weight λ —one can show that

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$M_\mu \otimes X$ has a simple quotient-module L with highest weight λ . And if $x_{\lambda-\mu}$ is a vector of maximal weight in X , then the highest weight vector of L is the image of $v_\mu \otimes x_{\lambda-\mu}$.

Since L is invariant under taking tensor product with U_χ over $U(\mathfrak{gl}_n(\mathbb{k}))$, there is a surjective map

$$(M_\mu \otimes X)_\chi \rightarrow L \rightarrow 0.$$

And thus $(M_\mu \otimes X)_\chi$ and consequently, $(U_\psi \otimes X)_\chi$ is nonzero. Moreover, the image of $1 \otimes x_{\lambda-\mu}$ is nonzero in L . Thus, $(F_0 U_\psi \otimes X)_\chi$ is nonzero. \square

Theorem 3.18. *Suppose there exist numbers $b_1, \dots, b_r, c_1, \dots, c_s \in \mathbb{C}$ such that*

$$\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{(c_i-1)u}.$$

Then $(U_\psi \otimes S^r V \otimes S^s V^)_\chi \neq 0$.*

Proof. By Corollary 3.16, there exist weights $\lambda^{(n)}$ and $\mu^{(n)}$ of $\mathfrak{gl}_n(\mathbb{k})$ with

$$\mu^{(n)} = \lambda^{(n)} - e_1 - \dots - e_r + e_{n-s+1} + \dots + e_n,$$

and central characters $\chi^{(n)}, \psi^{(n)}$ corresponding to these weights, such that

$$\chi = \prod_{\mathcal{F}} \chi^{(n)}, \psi = \prod_{\mathcal{F}} \psi^{(n)}.$$

Now, for every $n > r + s$ the module $S^r V^{(n)} \otimes S^s (V^{(n)})^*$ has maximal weight

$$\lambda^{(n)} - \mu^{(n)} = e_1 + \dots + e_r - e_{n-s+1} - \dots - e_n.$$

Thus by Lemma 3.17,

$$(U_{\psi^{(n)}} \otimes S^r V^{(n)} \otimes S^s (V^{(n)})^*)_{\chi^{(n)}} \neq 0,$$

when $n > s + r$. Moreover,

$$(F_0 U_{\psi^{(n)}} \otimes S^r V^{(n)} \otimes S^s (V^{(n)})^*)_{\chi^{(n)}} \neq 0.$$

We have

$$(F_0 U_\psi \otimes S^r V \otimes S^s V^*)_\chi = \prod_{\mathcal{F}} (F_0 U_{\psi^{(n)}} \otimes S^r V^{(n)} \otimes S^s (V^{(n)})^*)_{\chi^{(n)}} \neq 0.$$

And therefore,

$$(U_\psi \otimes S^r V \otimes S^s V^*)_\chi = \text{colim}_i \prod_{\mathcal{F}} (F_i U_{\psi^{(n)}} \otimes S^r V^{(n)} \otimes S^s (V^{(n)})^*)_{\chi^{(n)}} \neq 0. \quad \square$$

We have now constructed a nonzero object in the category $HC_{\chi, \psi}$ with

$$\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{j=1}^s e^{(c_j-1)u}$$

for any complex numbers $b_i, 1 \leq i \leq r, c_j, 1 \leq j \leq s$, and thus we have proved Theorem 3.13.

3.7. Summary

The following theorem summarizes Corollary 3.12 and Theorem 3.13 and is the main result of this paper.

Theorem 3.19. *The category $HC_{\chi,\psi}$ is nonzero if and only if there exist complex numbers $b_1, \dots, b_r, c_1, \dots, c_s$ such that*

$$\chi(u) - \psi(u) = \sum_{i=1}^r e^{b_i u} - \sum_{i=1}^s e^{c_i u}.$$

Note. We replaced $c_i - 1$ with c_i (as compared to Theorem 3.13) to simplify the formula.

Examples. Theorem 3.18 provides us with an example of a nonzero bimodule in $HC_{\chi,\psi}$ if χ, ψ satisfy the conditions of Theorem 3.19. We have

$$0 \neq (U_\psi \otimes S^r V \otimes S^s V^*)_\chi \in HC_{\chi,\psi}.$$

Other examples can be found in [13], where we describe a construction of a family of irreducible Harish-Chandra bimodules. These bimodules are a generalization of finite-dimensional bimodules in the classical case and have finite K-type (see Remark 3.2). The corresponding central characters are computed in Section 5 of [13].

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