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VIANNEY PERCHET, Crest, ENSAE & Criteo AI LAB, France
PHILIPPE RIGOLLET, Massachusetts Institute of Technology, U.S.A
THIBAUT LE GOUIC, Aix Marseille University, Centrale Marseille, CNRS, I2M, France

A century ago, Emile Borel published his seminal paper on *the theory of play and integral equations with skew-symmetric kernels* [1]. Borel describes what is now called the Blotto game: a resource-allocation game in which two players compete for over n different battlefields by simultaneously allocating resources to each battlefield. The following two additional characteristics are perhaps the most salient features of the Blotto game:

(1) *Winner-takes-all*: For each battlefield, the player allocating the most resources to a given battlefield wins the battlefield.
(2) *Fixed budget*: each player is subject to a fixed—and deterministic—budget that mixed strategies should satisfy almost surely.

Despite its century-long existence, Nash equilibria for the Blotto game are only known under various restrictions on the main parameters of the problem: the budget of each player and the value given to each battlefield. Moreover, previous solutions for two-player games have consisted in constructing explicit solutions. Because of the budget constraints, these strategies can be decomposed into two parts: *marginal distributions* that indicate which (random) strategy to play on each battlefield and a *coupling* that correlates the marginal strategies in such a way to ensure that the budget constraint is satisfied almost surely.

The first part may be studied independently of the second by considering what is known as the (General) *Lotto* game. In this game, the budget constraint needs only be enforced in *expectation* with respect to the randomization of the mixed strategies. While this setup lacks a defining characteristic of the Blotto game (fixed budget), it has the advantage of lending itself to more amenable computations. Indeed, unlike the Blotto game, a complete solution to the Lotto game was recently proposed in [2] where the authors describe an explicit Nash equilibrium in the most general case: asymmetric budget, asymmetric and heterogeneous values.

In light of this progress, a natural question is whether the marginal solutions discovered in [2] could be *coupled* in such a way that the budget constraint is satisfied almost surely. We provide a positive answer to this question by appealing to an existing result from the theory of *joint mixability* [5]. Mixability asks the following question: Can n random variables $X_1, \ldots, X_n$ with prescribed marginal distributions $X_i \sim P_i$, be coupled in such a way that $\mathrm{var}(X_1 + \cdots + X_n) = 0$. Joint mixability is precisely the step required to go from a Lotto solution to a Blotto one by coupling the marginals of the Lotto solution in such a way that the budget constraint is satisfied.

In this paper, we exploit a simple connection between joint mixability and the theory of *multi-marginal couplings*. We propose an algorithmic solution to the Blotto problem by efficiently constructing a coupling that satisfies the budget constraint almost surely and can be easily sampled from. Our construction relies on three key steps: first, we reduce the problem to a small number of marginals to bypass the inherent NP-hardness of multi-marginal problems, second, we discretize the marginals and finally, we employ a multi-marginal version of the Sinkhorn algorithm [3, 4] to construct a coupling of the discretized marginals. This procedure outputs a coupling with continuous marginals that are close to the ones prescribed by the Lotto solutions of [2] and

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from which it is straightforward to sample. Furthermore, we quantify the combined effect of discretization error and of the Sinkhorn algorithm on the value of the game, effectively leading to an *approximate* Nash equilibrium and even to an approximately optimal solution in the case of symmetric values.

For symmetric battlefield values and asymmetric budget, the Blotto game is constant-sum so optimal solutions exist, and our algorithm samples from an ϵ-optimal solution in time $\tilde{O}(n^2 + \varepsilon^{-4})$, independently of budgets and battlefield values. In the case of asymmetric values where optimal solutions need not exist but Nash equilibria do, our algorithm samples from an ε-Nash equilibrium with similar complexity but where implicit constants depend on various parameters of the game such as battlefield values.

The full paper is available at https://arxiv.org/abs/2202.07318

CCS Concepts: • **Theory of computation** → **Exact and approximate computation of equilibria**; *Solution concepts in game theory*; • **Mathematics of computing** → *Probabilistic algorithms.*

Additional Key Words and Phrases: Blotto, Equilibria, Sampling, Mixability, Sinkhorn

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