



# **MIT Sloan School of Management**

**Working Paper 4254-02  
July 2002**

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# Monotone Equilibrium in Multi-Unit Auctions

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July 16, 2002  
First draft: November 2000

## Abstract

In a large class of multi-unit auctions of identical objects that includes the uniform-price, as-bid (or discriminatory), and Vickrey auctions, a Bayesian Nash equilibrium exists in monotone pure strategies whenever there is a finite price / quantity grid and each bidder's interim expected payoff function satisfies single-crossing in own bid and type. A stronger condition, non-decreasing differences in own bid and type, is satisfied in this class of auctions given (a) independent types and (b) risk-neutral bidders with marginal values that are (c) non-decreasing in own type and have (d) non-increasing differences in own type and others' quantities. A key observation behind this analysis is that each bidder's valuation for what he wins is always modular in own bid in any multi-unit auction in which the allocation is determined by market-clearing. This paper also provides the first proof of pure strategy equilibrium existence in the uniform-price auction when bidders have multi-unit demand and values that are not private.

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\*A previous version of this paper circulated as "Isotone Equilibrium in Multi-Unit Auctions". I thank John McMillan, Robert Wilson, and seminar participants at Carnegie Mellon GSIA, MIT Sloan, NYU Stern, Northwestern, WUSL Olin, U Michigan, U North Carolina, and Yale for their helpful suggestions and critiques. Most especially, I am grateful to Susan Athey for several rounds of detailed comments on an earlier version and Paul Milgrom for suggesting that lattice methods might apply to multi-unit auctions. Any error is, of course, mine alone. This research has been supported by the John Olin Foundation through a grant to the Stanford Institute for Economic Policy Research, as well as by the State Farm Companies Foundation. E-mail: mcadams@mit.edu. Post: MIT Sloan School of Management, E52-448, 50 Memorial Drive, Cambridge, MA 02142

# 1 Introduction

This paper applies the results of McAdams (2002b) to multi-unit auctions in which bidders have multi-unit demand and multi-dimensional types. That paper shows that a pure strategy equilibrium exists in all incomplete information games in which each player has a finite lattice multi-dimensional action space, a multi-dimensional atomless type, and an interim expected payoff function satisfying two non-primitive conditions: (i) single-crossing in own action and type and (ii) quasisupermodularity in own action. Also, this equilibrium is monotone (technically “isotone”): each player’s action is non-decreasing along every dimension of his action space as his type increases along any dimension of his type space. When players have atom types and (i,ii) are satisfied, furthermore, McAdams (2002b) shows that an isotone mixed strategy equilibrium exists: the least upper bound of the actions played with positive probability by a lower type is less than or equal to the greatest lower bound of the actions played by a higher type.

An advantage of this approach is that one may shed all assumptions but those that are needed to verify the relevant ordinal conditions. In particular, I relax several common assumptions:

No assumptions on the relationship between marginal value and quantity. In particular, bidder  $i$ ’s marginal values may be increasing in own quantity, allowing for increasing returns to scale in consumption. (On the other hand, bidders are restricted to submit non-increasing demand schedules in my model.)

No assumptions on the relationship between marginal value and private information, other than that each bidder’s marginal values are non-decreasing in own type. In particular, I allow for all intermediate cases in which values are neither private nor common. Also, bidder  $i$ ’s marginal values need not be monotone in others’ information.

No assumptions on the relationship between marginal value and others’ quantities, other than that each bidder’s marginal value satisfies non-increasing differences in others’ quantities and own type. For example, I allow for stock IPO models in which the value of shares depends both on the underlying equity and the number of shares being sold.

Furthermore, this paper’s approach applies to many commonly studied multi-unit auctions, whereas previous papers in this literature have required dif-

ferent approaches to study different auctions. This paper also is the first to prove existence of a pure strategy equilibrium (monotone or not) in the uniform-price auction when bidders have multi-unit demand and values that are not private. (See the Appendix for a discussion of related literature.) Finally, while other papers have addressed the issue of existence of pure strategy equilibrium in some of the other common auction forms in general settings,<sup>1</sup> these papers do not show that equilibria are monotone in bidders' types. (Existence of monotone pure strategy equilibrium given existence of a pure strategy equilibrium is non-trivial, even when payoffs satisfy the ordinal conditions required by McAdams (2002b). Bidders can have best response strategies that are non-monotone.)

While my analysis is quite general in many respects, I do make certain assumptions that limit its scope. Most importantly, I assume that bidders have stochastically independent types. When bidders have negatively correlated types, it is easy to construct examples in which all equilibria are non-monotone in both single-unit and multi-unit auctions. When bidders have affiliated one-dimensional types, however, McAdams (2002a) proves that all equilibria in the first-price auction are monotone but provides examples in which all equilibria of the uniform-price and as-bid auctions are non-monotone. Thus, monotonicity of equilibrium can be lost even when types are positively correlated given multi-unit demand! The rationing rule that is used to break ties also is important to the analysis. I use a rationing rule that generalizes the coin-flip rule commonly used to study single-unit auctions, but the analysis does not apply when other natural rationing rules are used. For instance, the main technical result that bidders' payoffs are modular in own bid fails given the commonly studied proportional rationing rule; payoffs become submodular in own bid. The issues surrounding how my analysis generalizes beyond the case of independent types and to other rationing rules are worthwhile areas for future research.

On the other hand, the assumption of a finite price-quantity grid is not essential to my results. Kazumori (2002) provides a limiting argument proving that, as this grid becomes arbitrarily fine, a limit of equilibria in finite grid auctions is a monotone pure strategy equilibrium given a continuum grid. Finally, all results extend in a straightforward way to double auction settings with multiple buyers and multiple sellers in which any of them may

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<sup>1</sup>For example, Reny (1999) proves existence of pure strategy equilibrium in a very general model of the as-bid auction.

or may not be strategic agents. (In the one-sided auctions that I explicitly study, there is one non-strategic seller and all buyers are strategic.)

The remainder of the paper is organized as follows: Section 1.1 continues the introduction and defines several terms. Section 2 lays out the model of multi-unit auctions. Section 3 and the Appendix provide the proofs. Section 4, finally, offers concluding remarks.

## 1.1 Definitions

The key technical observation of this paper is that, when the space of permissible bids is endowed with the product order, each bidder's ex post valuation for what he wins is *modular* in own bid. (See Section 2.1.1 for the definition of the product order on bids.)

**Definition.** Let  $(X, \geq, \vee, \wedge)$  be a lattice.  $f : X \rightarrow \Re$  is *modular* in  $x$  iff

$$f(x') + f(x) = f(x' \vee x) + f(x' \wedge x)$$

for all  $x', x \in X$ .

In a finite-dimensional Euclidean setting, modularity with respect to the product order is equivalent to additive separability:  $f(x, t) = \sum_{j=1}^k f^j(x_j, t)$  iff

$$\begin{aligned} f(x^1, t) + f(x^2, t) &= \sum_{j=1}^k (f^j(x_j^1, t) + f^j(x_j^2, t)) \\ &= f(x^1 \vee x^2, t) + f(x^1 \wedge x^2, t) \end{aligned}$$

I will also show that, in many commonly studied auctions, each bidder's ex post payment is modular in own bid. Since modularity is preserved under scaling and addition, these facts imply that each bidder's interim expected surplus is modular, regardless of the structure of uncertainty! (Ex post surplus is the difference between valuation and payment and therefore modular. Interim expected surplus is a weighted sum of ex post surplus across states and therefore modular.) As long as each bidder is an expected surplus maximizer, his interim expected payoff is modular and hence automatically quasisupermodular in own bid.

**Definition (Quasisupermodular).** Let  $(X, \geq, \vee, \wedge)$  be a lattice.  $g : X \rightarrow \mathcal{R}$  is quasisupermodular in  $x$  iff

$$g(x') \geq (>)g(x' \wedge x) \Rightarrow g(x' \vee x) \geq (>)g(x)$$

for all  $x', x \in X$ . (Weak inequality implies weak inequality and strict inequality implies strict inequality.)

Given multi-dimensional atomless types and a finite set of permissible prices and quantities, then, a monotone pure strategy equilibrium exists under any set of additional assumptions that implies single-crossing of interim expected payoff in own bid and type.

**Definition (Single-crossing in  $(x, t)$ ).** Let  $(X, \geq, \vee, \wedge)$  be a lattice and  $(T, \geq)$  a partially ordered set.  $g : X \times T \rightarrow \mathcal{R}$  satisfies *single-crossing in  $(x, t)$*  iff

$$g(x', t) \geq (>)g(x, t) \Rightarrow g(x', t') \geq (>)g(x, t')$$

for all  $x' > x \in X$  and all  $t' > t \in T$ .

The following conditions imply non-decreasing differences in own bid and type, a much stronger property: (a) independent types; (b) risk-neutral bidders; and (c) marginal values non-decreasing in own type with (d) non-increasing differences in own type and others' quantities.

**Definition.** Let  $(X, \geq, \vee, \wedge)$  be a lattice,  $(T, \geq)$ .  $f : X \times T \rightarrow \mathfrak{R}$  has *non-decreasing (or non-increasing) differences in  $(x, t)$*  iff

$$f(x', t') - f(x, t') \geq (\text{or } \leq) f(x', t) - f(x, t)$$

for all  $x' > x \in X$  and all  $t' > t \in T$ .

## 2 Model: Multi-Unit Auctions

I study a class  $\mathcal{A}$  of multi-unit auctions that includes the uniform-price, as-bid (or discriminatory), and Vickrey auctions. A representative auction from this class is denoted by

$$A(n, (\mathbf{p}, \mathbf{q}), S(\cdot), \mathbf{z}(\cdot)) \in \mathcal{A}$$

where  $n$  is the number of bidders,  $(\mathbf{p}, \mathbf{q})$  the grid of permissible prices and quantities,  $S(\cdot)$  the auctioneer's supply correspondence, and  $\mathbf{z}(\cdot)$  the vector of bidder payment functions. I describe these auctions in the rest of this Section: the set of permissible bids in Section 2.1; the rule for allocation of quantity in Sections 2.3 and 2.4; and the rule for payment in Section 2.5. Sections 2.2 and 2.7, finally, lay out assumptions on the structure of payoffs and uncertainty and define the equilibrium concept.

## 2.1 Grids and Bids

There is a finite *grid* of permissible prices  $\mathbf{p}$  and of permissible quantities  $\mathbf{q}$ :

$$\begin{aligned}\mathbf{p} &= \{\emptyset\} \cup \{p^2, \dots, p^{|\mathbf{p}|-1}\} \cup \infty \\ \mathbf{q} &= \{q^1 = 0, q^2, \dots, q^{|\mathbf{q}|}\};\end{aligned}$$

where I denote  $p^1 = \{\emptyset\}$  and  $p^{|\mathbf{p}|} = \infty$ . For notational simplicity, suppose that these grids have constant fineness:

$$\begin{aligned}p^{k+1} - p^k &= \Delta_p \text{ for } k = 2, \dots, |\mathbf{p}| - 2 \\ q^{l+1} - q^l &= \Delta_q \text{ for } l = 1, \dots, |\mathbf{q}| - 1\end{aligned}$$

My “low null price”  $\{\emptyset\}$  and “high null price”  $\infty$  are unconventional. *Interpretation:* If  $q \in D_i(\{\emptyset\})$  (or  $q \in D_i(\infty)$ ), then bidder  $i$  is unwilling to receive more (or less) quantity than  $q$  at any permissible price. Similarly, if  $q \in S(\{\emptyset\})$  (or  $q \in S(\infty)$ ) then the auctioneer is unwilling to supply less (or more) than  $q$ .

A *bid*  $D_i(\cdot) : \mathbf{p} \rightarrow \mathcal{P}(\mathbf{q})$  is a demand correspondence that satisfies the following requirements: (i) well-defined:  $\forall p \in \mathbf{p}, D_i(p) \neq \emptyset$ ; (ii) Inverse well-defined:  $\forall q \in \mathbf{q} \exists p \in \mathbf{p} : q \in D_i(p)$ ; (iii) Order interval-valued:  $\forall p \in \mathbf{p}$ , if  $\{q', q''\} \subset D_i(p)$  and  $q \in \mathbf{q} \cap (q', q'')$ , then  $q \in D_i(p)$ ; (iv) Non-increasing:  $\forall k = 1, \dots, |\mathbf{p}| - 1, \min D_i(p^k) = \max D_i(p^{k+1})$ . These conditions guarantee that a market-clearing allocation exists.

Any demand correspondence  $D_i(\cdot)$  is equivalent to an inverse demand correspondence  $P_i(\cdot)$ :  $p \in P_i(q)$  iff  $q \in D_i(p)$ . I will use both demand and inverse demand correspondence notation for bids, as convenient. For each subset of bidders  $I \in \mathcal{P}(\{1, \dots, n\})$ , define the announced aggregate demand of  $I$  as  $D_I(\cdot) = \sum_{j \in I} D_j(\cdot)$ . That is to say,  $Q \in D_I(p)$  iff there exists  $\{q_j\}_{j \in I}$  such that  $\sum_{j \in I} q_j = Q$  and  $q_j \in D_j(p)$  for all  $j \in I$ . Let  $P_I(\cdot)$  be the corresponding inverse demand correspondence.

### 2.1.1 Lattice Structure

The set of permissible bids,  $\mathcal{D}_i$ , is finite and forms a lattice with respect to the *product order*.  $D^2(\cdot) \geq_P D^1(\cdot)$  in the product order iff, for all  $p \in \mathbf{p}$ ,  $\max D^2(p) \geq \max D^1(p)$  and  $\min D^2(p) \geq \min D^1(p)$ . (For simplicity, I will usually refer to  $\geq_P$  as  $\geq$ .) The meet and join of any two bids  $D^2(\cdot), D^1(\cdot)$  are their lower- and upper envelopes:

$$\begin{aligned} \max D^2 \vee D^1(p) &= \max\{\max D^2(p), \max D^1(p)\} \\ \min D^2 \vee D^1(p) &= \max\{\min D^2(p), \min D^1(p)\} \\ \max D^2 \wedge D^1(p) &= \min\{\max D^2(p), \max D^1(p)\} \\ \min D^2 \wedge D^1(p) &= \min\{\min D^2(p), \min D^1(p)\} \end{aligned}$$

for all  $p \in \mathbf{p}$ .

## 2.2 Types and Valuation

Bidder *types*  $t_i$  are i.i.d. uniformly distributed over  $[0, 1]^{h_i}$ . (Given the assumption of independent atomless types, the assumption that types are uniformly distributed on  $[0, 1]$  is without additional loss of generality.) The *state*  $\mathbf{t}$  bears on each bidder's valuation. Bidder  $i$ 's *valuation* for  $q_i$  shares is

$$\sum_{x \in (0, q_i] \cap \mathbf{q}} v_i(x; q_{-i}, \mathbf{t}) \Delta_q,$$

where  $v_i(q_i; q_{-i}, \mathbf{t})$  is his “*marginal value*” for quantity  $q_i$  when others receive quantities  $q_{-i}$  in state  $\mathbf{t}$ . I assume (only!) that  $v_i$  is (i) piecewise-continuous in  $\mathbf{t}$ , (ii) bounded by  $B$  and  $-B$ , and (iii) non-decreasing in  $t_i$  with (iv) non-increasing differences in  $(t_i, q_{-i})$ .

**Discussion.** Since  $v_i$  may be non-monotone in  $q_i$ , the model applies to procurement settings in which suppliers have increasing returns to scale. A bidder's marginal value may also depend on the allocation to others. For example, in a common value stock IPO model, if the total value of the equity being auctioned may be  $Z(\mathbf{t})$ , the value of each share depends on the total



supply of shares:

$$\sum_{x \in (0, q_i] \cap \mathbf{q}} v_i(x; q_{-i}, \mathbf{t}) \Delta_q = \frac{q_i}{\sum_{j=1}^n q_j} Z(\mathbf{t})$$

$$v_i(x; q_{-i}, \mathbf{t}) \approx \frac{Z(\mathbf{t})}{\sum_{j=1}^n q_j} \left( 1 - \frac{x}{x + \sum_{j \neq i}^n q_j} \right)$$

when  $\Delta_q \approx 0$ . Note that marginal values have non-increasing differences in  $(t_i, q_{-i})$  since any increase in  $Z(\mathbf{t})$  due to more positive information is diluted if others receive more shares.

### 2.3 Supply and Market-Clearing Price

Define  $\mathcal{Q} \equiv \sum_{j=1}^n \mathbf{q}$  to be the grid of all permissible aggregate quantities. The auctioneer prespecifies a *supply correspondence*  $S : \mathbf{p} \rightarrow \mathcal{P}(\mathcal{Q})$ . I require that  $S(\cdot)$  meet requirements that are analogous to those placed on bids: (i) Well-defined  $\forall p \in \mathbf{p}$ ,  $S(p) \neq \emptyset$  (ii) Inverse well-defined  $\forall Q \in \mathcal{Q} \exists p \in \mathbf{p} : Q \in S(p)$  (iii) Order interval-valued  $\forall p \in \mathbf{p}$ , if  $\{Q', Q''\} \subset S(p)$  and  $Q \in \mathcal{Q} \cap (Q', Q'')$ , then  $Q \in S(p)$  (iv) Non-decreasing  $\forall k = 1, \dots, |\mathbf{p}| - 1$ ,  $\max S(p^k) = \min S(p^{k+1})$

The *market-clearing price*  $p^{mc}(\mathbf{D}(\cdot))$  (shorthand  $p^{mc}$ ) is the maximal permissible price that clears the market:

$$p^{mc} \equiv \max \{p \in \mathbf{p} : S(p) \cap D_{1,\dots,n}(p) \neq \emptyset\}$$

The *market-clearing supply*  $S^{mc}(\mathbf{D}(\cdot))$  (shorthand  $S^{mc}$ ), similarly, is the maximal supply that allows the market to clear at that price:

$$S^{mc} \equiv \max (S(p^{mc}) \cap D_{1,\dots,n}(p^{mc}))$$

Given the restrictions on  $\mathbf{D}(\cdot)$  and on  $S(\cdot)$ , such a price (possibly  $\{\emptyset\}$  or  $\infty$ ) always exists. Note that, by definition,  $p^{mc} = \max P_{1,\dots,n}(S^{mc})$ . Another useful alternative characterization (see Lemma 2 in the Appendix) is

$$p^{mc} \equiv \max \{p \in \mathbf{p} : \min S(p) \leq \max D_{1,\dots,n}(p)\}$$

**Discussion.** Supply is unconventional but more general than standard approaches. The two most common modelling approaches are special cases: (i) Fixed supply  $S$  with a minimum price  $p^{min}$  in which the auction is cancelled

if there is not aggregate demand for  $S$  at  $p^{min}$ :  $S(p) = S$  for all  $p > \{\emptyset\}$  and  $S(\{\emptyset\}) = [0, S]$  (ii) Fixed supply  $S$  with a minimum price  $p^{min}$  in which the auctioneer supplies  $\max D_{1,...,n}(p^{min})$  if there is not aggregate demand for  $S$  at  $p^{min}$ :  $S(p) = S$  for all  $p > p^{min}$ ,  $S(p^{min}) = [0, S]$ , and  $S(\{\emptyset\}) = 0$ .

## 2.4 Quantity Rationing

If  $p^{mc} \in \{\{\emptyset\}, \infty\}$ , then the auction is cancelled. Else each bidder receives  $q_i^{mc}(\mathbf{D}(\cdot), \rho)$  (shorthand  $q_i^{mc}$ ), where  $q_i^{mc}$  is a *market-clearing quantity*:  $q_i^{mc} \in D_i(p^{mc})$ . There are several reasonable ways to allocate or “ration” quantity when there are multiple sets of market-clearing quantities. I will use a specific “*randomized rationing rule*” that generalizes the usual randomized tie-breaking in single-unit auctions and proceeds as follows:

- [STEP I]: Each bidder submits a bid  $D_i(\cdot)$  and the auctioneer determines the market-clearing price  $p^{mc}$  and the minimal and maximal quantities  $\underline{q}_i(\mathbf{D}(\cdot))$  (shorthand  $\underline{q}_i$ ) and  $\bar{q}_i(\mathbf{D}(\cdot))$  (shorthand  $\bar{q}_i$ ) that each bidder receives in a market-clearing allocation:

$$\begin{aligned}\underline{q}_i &\equiv \min(D_i(p^{mc}) \cap RS_i(p^{mc})) \\ \bar{q}_i &\equiv \max(D_i(p^{mc}) \cap RS_i(p^{mc}))\end{aligned}$$

where  $RS_i(p) \equiv S(p) - \sum_{j \neq i} D_j(p)$  is the *residual supply* facing bidder  $i$  given others’ bids.

- [STEP II]: The auctioneer orders the bidders into a “rationing ranking” according to a random permutation  $\rho$ .
- [STEP III]: Quantities are then assigned according to  $\rho$ :

- [STEP IIIA]: For  $i = 1, \dots, n$ , set  $q_i^{mc} = \underline{q}_i$ . Set

$$r = S^{mc} - \sum_{j=1}^n \underline{q}_j$$

End if  $r = 0$ ; else set  $j = 1$ .

- [STEP IIIB]: Reset

$$\begin{aligned}q_{\rho^{-1}(j)}^{mc} &= q_{\rho^{-1}(j)}^{mc} + \min\{r, \max \bar{q}_{\rho^{-1}(j)} - \underline{q}_{\rho^{-1}(j)}\} \\ r &= r - \min\{r, \bar{q}_{\rho^{-1}(j)} - \underline{q}_{\rho^{-1}(j)}\}\end{aligned}$$

End if  $r = 0$ ; else reset  $j = j + 1$  and repeat [STEP IIIB].

*Interpretation:* Each bidder receives at least his minimal demanded quantity at  $p^{mc}$ :  $\underline{q}_i \geq \min D_i(p^{mc})$ . Then bidders are served according to  $\rho$ . Beginning with the highest priority bidder,  $\rho^{-1}(1)$ , each bidder receives all of the remaining supply or the maximal quantity that he demanded at  $p^{mc}$ , whichever is less. This algorithm always generates an allocation  $\mathbf{q}^{mc}$  with the property that  $q_i^{mc} \in D_i(p^{mc})$  for each  $i = 1, \dots, n$ . Furthermore, at most one bidder receives a quantity  $q_i^{mc} \in (\underline{q}_i, \bar{q}_i)$ . (This is the final bidder who receives additional quantity.)

## 2.5 Payment

Each bidder  $i$ 's payment function  $z_i$  has form

$$z_i(\mathbf{D}(\cdot), \rho) = \sum_{q \in (0, q_i^{mc}] \cap \mathbf{q}} w_i(\max P_i(q), P_{-i}(\cdot); q) + y_i(p^{mc}; q_i^{mc})$$

There are two notable components to each bidder's payment:

- $\sum_{q \in (0, q_i^{mc}] \cap \mathbf{q}} w_i(\max P_i(q), P_{-i}(\cdot); q)$ : This part of the bidder's payment is simply a sum of marginal payments on the units that he wins. The marginal payment for quantity  $q$  may depend on the price that he bid for that quantity as well the entire profile of others' bids. The as-bid auction and the Vickrey auction provide examples in which payment takes this form.
- $y_i(p^{mc}; q_i^{mc})$ : This part allows for payment to depend on the market-clearing price and the market-clearing allocation. For instance, this allows for an "additional payment specific to the market-clearing quantity  $q_i^{mc}$ ". The uniform-price auction provides an example in which payment takes this form.

Admittedly, the structure of payoffs in my class of auctions  $\mathcal{A}$  is somewhat ad hoc. The main feature of this class that I emphasize is that the three most commonly studied auctions belong to it. My results apply to other sorts of auctions outside of  $\mathcal{A}$  as well in which ex post payments are modular or supermodular in own bid.

### 2.5.1 Example: Uniform-Price Auction

In the uniform-price auction, each bidder pays the market-clearing price on all units that he wins:

$$z_i^U(\mathbf{D}(\cdot); \rho) = p^{mc} q_i^{mc}$$

This payment rule fits within my framework when I set

$$\begin{aligned} w_i(p, P_{-i}(\cdot); q) &= 0 \text{ for all } q > 0 \in \mathfrak{q} \\ y_i(p; q) &= pq \text{ for all } q \in \mathfrak{q} \end{aligned}$$

### 2.5.2 Example: As-Bid Auction

In the as-bid auction, each bidder pays the sum of his marginal bids on the units that he wins:

$$z_i^A(\mathbf{D}(\cdot); \rho) = \sum_{q \in (0, q_i^{mc}] \cap \mathfrak{q}} \max P_i(q) \Delta_q$$

This payment rule fits within my framework when I set

$$\begin{aligned} w_i(p, P_{-i}(\cdot); q) &= p \Delta_q \text{ for all } q > 0 \in \mathfrak{q} \\ y_i(p, P_{-i}(\cdot); q) &= 0 \text{ for all } q \in \mathfrak{q} \end{aligned}$$

### 2.5.3 Example: Vickrey Auction

In the Vickrey auction, each bidder pays the sum of the losing marginal bids of others that would have won if he did not participate in the auction:

$$\begin{aligned} z_i^V(\mathbf{D}(\cdot); \rho) &= \sum_{q \in (0, q_i^{mc}) \cap \mathfrak{q}} \max P_{-i}(S^{mc} - q) \Delta_q \\ &= \sum_{q \in (0, q_i^{mc}] \cap \mathfrak{q}} \min P_{-i}(S^{mc} - q) \Delta_q \end{aligned}$$

where  $P_{-i}(\cdot)$  is the aggregate inverse demand of all other bidders. This payment rule fits within my framework as well when I set

$$\begin{aligned} w_i(p, P_{-i}(\cdot); q) &= \min P_{-i}(q) \Delta_q \text{ for all } q > 0 \in \mathfrak{q} \\ y_i(p, P_{-i}(\cdot); q) &= 0 \end{aligned}$$

## 2.6 Strategies

A *pure strategy* for bidder  $i$  specifies a permissible bid for each type:

$$\begin{aligned} D_i : [0, 1]^{h_i} &\rightarrow \mathcal{D}_i \\ t_i &\mapsto D_i(\cdot; t_i) \end{aligned}$$

As discussed in Section 2.1, any permissible bid may also be represented by a unique inverse demand curve. Let  $P_i(\cdot; t_i)$  be the inverse demand schedule corresponding to  $D_i(\cdot; t_i)$ . Also,  $\mathcal{S}_i$  is the set of bidder  $i$ 's strategies and  $\mathcal{S} \equiv \prod_{j=1}^n \mathcal{S}_j$  and  $\mathcal{S}_{-i} \equiv \prod_{j \neq i} \mathcal{S}_j$  are the sets of strategy profiles of all bidders and of bidders  $-i$  (all others than  $i$ ), respectively.

## 2.7 Payoffs and Equilibrium

When bids  $\mathbf{D}(\cdot)$  are submitted in state  $\mathbf{t}$  and the rationing ranking is  $\rho$ , bidder  $i$ 's *ex post surplus*

$$\begin{aligned} &\Pi_i^{post}(D_i(\cdot), t_i; D_{-i}(\cdot), t_{-i}; \rho) \\ &= \sum_{q \in (0, q_i^{mc}] \cap \mathbf{q}} v_i(q; q_{-i}^{mc}, \mathbf{t}) \Delta_q - z_i(\mathbf{D}(\cdot); \rho) \end{aligned}$$

(where I have suppressed, as usual, the notation indicating the dependence of the allocation on the bids  $\mathbf{D}(\cdot)$  and the rationing ranking  $\rho$ ). Given this specification of ex post payoffs, I define each bidder's *interim expected payoff function*

$$\begin{aligned} &\Pi_i^{int}(D_i(\cdot; t_i), t_i; D_{-i}(\cdot; \cdot)) \\ &= E_{t_{-i}, \rho} [\Pi_i^{post}(D_i(\cdot; t_i), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}; \rho)] \end{aligned}$$

Let  $BR_i(t_i; D_{-i}(\cdot; \cdot)) \equiv \arg \max_{D_i(\cdot) \in \mathcal{D}_i} \Pi_i^{int}(D_i(\cdot), t_i; D_{-i}(\cdot; \cdot))$  be bidder  $i$ 's set of best response bids to the pure strategy profile  $D_{-i}(\cdot; \cdot)$  given type  $t_i$ , and let  $BR_i$  be bidder  $i$ 's *best response correspondence*, mapping profiles of pure strategies into sets of pure strategies:

$$\begin{aligned} BR_i : \mathcal{S}_{-i} &\rightarrow \mathcal{P}(\mathcal{S}_i) \\ D_{-i}(\cdot; \cdot) &\mapsto \{D_i(\cdot; \cdot) : D_i(\cdot; t_i) \in BR_i(t_i; D_{-i}(\cdot; \cdot)) \forall t_i \in T_i\} \end{aligned}$$

A profile  $(D_1^*(\cdot; \cdot), \dots, D_n^*(\cdot; \cdot)) \in \mathcal{S}$  is a *pure strategy equilibrium* iff

$$D_i^*(\cdot; \cdot) \in BR_i(D_{-i}^*(\cdot; \cdot)) \text{ for all } i$$

*Mixed strategy equilibrium* is defined similarly.

A pure strategy  $D_i(\cdot; \cdot)$  is *monotone* iff  $\max D_i(p, t'_i) \geq \max D_i(p, t_i)$  and  $\min D_i(p, t'_i) \geq \min D_i(p, t_i)$  for all  $p \in \mathbf{p}$  whenever  $t'_i > t_i$ .

### 3 Equilibrium in Multi-Unit Auctions

In this section, I prove that an isotone pure strategy equilibrium exists in all auctions  $A(n, (\mathbf{p}, \mathbf{q}), S(\cdot), \mathbf{z}(\cdot)) \in \mathcal{A}$  described in Section 2.

**Theorem 1.** *A monotone pure strategy equilibrium exists in all multi-unit auctions  $A(n, (\mathbf{p}, \mathbf{q}), S(\cdot), \mathbf{z}(\cdot)) \in \mathcal{A}$ .*

*Proof.* In the Appendix, beginning on page 21. □

I discuss the main points of the proof here.

- For all  $t_i$  and all  $D_{-i}(\cdot; \cdot) \in \mathcal{S}_{-i}$ ,  $\Pi_i^{int}(D_i(\cdot), t_i; D_{-i}(\cdot; \cdot))$  is *modular* in  $D_i(\cdot)$ .
- For all  $D_{-i}(\cdot; \cdot) \in \mathcal{S}_{-i}$ ,  $\Pi_i^{int}(D_i(\cdot), t_i; D_{-i}(\cdot; \cdot))$  has *non-decreasing differences* in  $(D_i(\cdot); t_i)$ .

Since modularity implies quasisupermodularity and non-decreasing differences implies single-crossing, the first two points show that the sufficient conditions of McAdams (2002b)'s isotone pure strategy equilibrium existence theorem are satisfied.

Furthermore, if some or all of the bidders have atom types but otherwise all of the assumptions of the model are satisfied, I may still invoke McAdams (2002b) to conclude that a mixed strategy equilibrium exists in which the least upper bound of the bids submitted by type  $t_i$  is less than or equal to the greatest lower bound of the bids submitted by type  $t'_i$  whenever  $t'_i > t_i$ . To save space, the formal statement and proof of this result are omitted.

**Lemma 1.** *Under the assumptions of Section 2,  $\Pi_i^{int}(D_i(\cdot), t_i; D_{-i}(\cdot; \cdot))$  has non-decreasing differences in  $(D_i(\cdot); t_i)$ .*

*Proof.* In Appendix 4 on page 24. □

The intuition for non-decreasing differences is straightforward. No matter what others bid, submitting a higher bid causes one to win (weakly) more quantity, and one's marginal gain from winning more quantity is (weakly) increasing in one's own type. Thus, each bidder's ex post surplus satisfies non-decreasing differences in his own bid and type in every state. Given type independence and risk-neutrality, then, each bidder's interim expected payoff satisfies this non-decreasing differences property.

Proving that payoffs are modular is my central technical contribution, so I label the next result as a theorem.

**Theorem 2.** *Interim expected surplus  $\Pi_i^{int}(D_i(\cdot), t_i, D_{-i}(\cdot, \cdot))$  is modular in  $D_i(\cdot)$  in all auctions  $A(n, (\mathbf{p}, \mathbf{q}), S(\cdot), \mathbf{z}(\cdot)) \in \mathcal{A}$  regardless of the structure of uncertainty when the randomized rationing rule is used.*

*Proof.* Since weighted sums of modular functions are modular, it suffices to show that, for all  $\mathbf{t}$ ,  $D_{-i}(\cdot)$ , and  $\rho$ , ex post surplus

$$\Pi_i^{post}(D_i(\cdot), t_i; D_{-i}(\cdot); t_{-i}, \rho)$$

is modular in  $D_i(\cdot)$ . I break the argument into two parts:

- Valuation of winnings,  $\sum_{q \in (0, q_i^{mc}] \cap \mathbf{q}} v_i(q; q_{-i}^{mc}(\mathbf{D}(\cdot), \rho), \mathbf{t}) \Delta_q$ , is modular in  $D_i(\cdot)$ .
- Payment  $z_i(\mathbf{D}(\cdot), \rho)$  is modular in  $D_i(\cdot)$ .

Modularity of valuation and of payment both arise from the structure imposed on allocations by the market-clearing rule (augmented by the randomized rationing rule). Figure 1 illustrates the basic point. In Figure 1, the join of  $D^1(\cdot)$  and  $D^2(\cdot)$  in the product ordering,  $D^1(\cdot) \vee D^2(\cdot) \equiv D^{1 \vee 2}$ , is traced by unfilled circles. Note that the allocation  $\mathbf{q}^{mc}$  is the same when bidder  $i$  submits bid  $D^2(\cdot)$  or  $D^{1 \vee 2}(\cdot)$  and the same when he submits bid  $D^1(\cdot)$  or  $D^{1 \wedge 2}(\cdot)$ , given that others have submitted the profile of bids  $D_{-i}(\cdot)$ . Since ex post valuation only depends on the allocation and the state, this implies

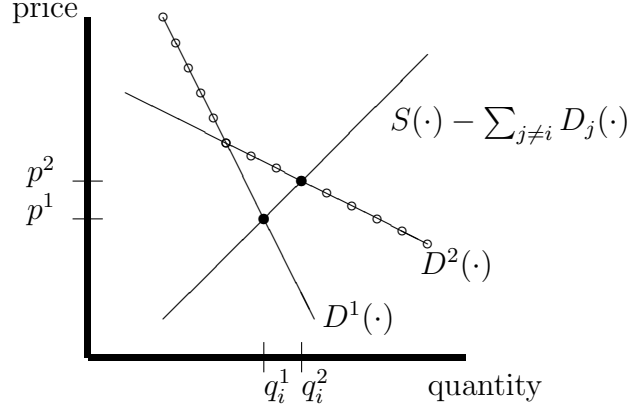


Figure 1: Modularity of ex post valuation given signals  $t_{-i}$

modularity directly:

$$\left\{ \sum_{q \in (0, q_i^1] \cap \mathbf{q}} v_i(q; q_{-i}^1, \mathbf{t}) \Delta_q, \sum_{q \in (0, q_i^2] \cap \mathbf{q}} v_i(q; q_{-i}^2, \mathbf{t}) \Delta_q \right\}$$

$$= \left\{ \sum_{q \in (0, q_i^{1 \vee 2}] \cap \mathbf{q}} v_i(q; q_{-i}^{1 \vee 2}, \mathbf{t}) \Delta_q, \sum_{q \in (0, q_i^{1 \wedge 2}] \cap \mathbf{q}} v_i(q; q_{-i}^{1 \wedge 2}, \mathbf{t}) \Delta_q \right\}$$

implies that

$$\sum_{q \in (0, q_i^1] \cap \mathbf{q}} v_i(q; q_{-i}^1, \mathbf{t}) \Delta_q + \sum_{q \in (0, q_i^2] \cap \mathbf{q}} v_i(q; q_{-i}^2, \mathbf{t}) \Delta_q$$

$$= \sum_{q \in (0, q_i^{1 \vee 2}] \cap \mathbf{q}} v_i(q; q_{-i}^{1 \vee 2}, \mathbf{t}) \Delta_q + \sum_{q \in (0, q_i^{1 \wedge 2}] \cap \mathbf{q}} v_i(q; q_{-i}^{1 \wedge 2}, \mathbf{t}) \Delta_q$$

Of course, Figure 1 is only suggestive since there is a continuum price-quantity grid and no rationing occurs since all bids are strictly downward sloping. A series of Lemmas in the Appendix proves that this and other related phenomena suggested by Figure 1 are in fact general when the randomized rationing rule is used. In the case of constant marginal values up to



a constraint, finally, modularity is preserved when any of the other rationing rules described in the model is used.

Modularity of the payment functions  $\mathbf{z}(\mathbf{D}(\cdot), \rho)$  in  $D_i(\cdot)$  follows for similar reasons. Figures 1, 2, and 3 illustrate, respectively, why payment is modular in the special cases of the uniform-price, as-bid, and Vickrey auctions.

### 3.1 Example: Uniform-Price Auction

In the uniform-price auction, payment is modular since

$$\{(p^1, q^1), (p^2, q^2)\} = \{(p^{1\vee 2}, q^{1\vee 2}), (p^{1\wedge 2}, q^{1\wedge 2})\}$$

(where I now use shorthand  $p^1, \dots, p^{1\wedge 2}$  for the realized market-clearing price given bids  $D_{-i}(\cdot)$  and  $D_i(\cdot) = D^1(\cdot), \dots$ , or  $D^{1\wedge 2}$ ). Graphically, this argument is summarized in Figure 1 by the observation that bidder  $i$ 's residual supply intersects  $D^1(\cdot)$  and  $D^2(\cdot)$  at the same points as it intersects  $D^{1\wedge 2}(\cdot)$  and  $D^{1\vee 2}(\cdot)$ .

### 3.2 Example: As-Bid Auction

Without loss, suppose that  $q^2 \geq q^1$ . In the as-bid auction,

$$\{\max P^1(q), \max P^2(q)\} = \{\max P^{1\vee 2}(q), \max P^{1\wedge 2}(q)\}$$

for all  $q \in (0, q^1] \cap \mathfrak{q}$  and

$$\max P^2(q) = \max P^{1\vee 2}(q)$$

for all  $q \in (q^1, q^2] \cap \mathfrak{q}$ . Thus,

$$\begin{aligned} & z_i^A(D^1(\cdot), D_{-i}(\cdot)) + z_i^A(D^2(\cdot), D_{-i}(\cdot)) \\ &= \sum_{q \in (0, q^1] \cap \mathfrak{q}} (\max P^1(q) + \max P^2(q)) + \sum_{q \in (q^1, q^2] \cap \mathfrak{q}} \max P^2(q) \\ &= \sum_{q \in (0, q^1] \cap \mathfrak{q}} (\max P^{1\vee 2}(q) + \max P^{1\wedge 2}(q)) + \sum_{q \in (q^1, q^2] \cap \mathfrak{q}} \max P^{1\vee 2}(q) \\ &= z_i^A(D^{1\vee 2}(\cdot), D_{-i}(\cdot)) + z_i^A(D^{1\wedge 2}(\cdot), D_{-i}(\cdot)) \end{aligned}$$

Graphically, this argument is summarized in Figure 2 by the observation that

$$\begin{aligned} & z_i^A(D^1(\cdot), D_{-i}(\cdot; t_{-i})) + z_i^A(D^2(\cdot), D_{-i}(\cdot; t_{-i})) \\ &= z_i^A(D^{1\vee 2}(\cdot), D_{-i}(\cdot; t_{-i})) + z_i^A(D^{1\wedge 2}(\cdot), D_{-i}(\cdot; t_{-i})) \\ &= \text{AREA}(A) + 2\text{AREA}(B) + \text{AREA}(C) \end{aligned}$$

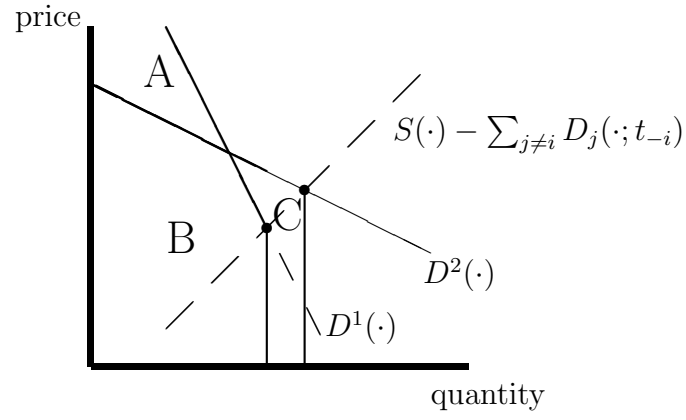


Figure 2: Modularity of ex post payment in as-bid auction

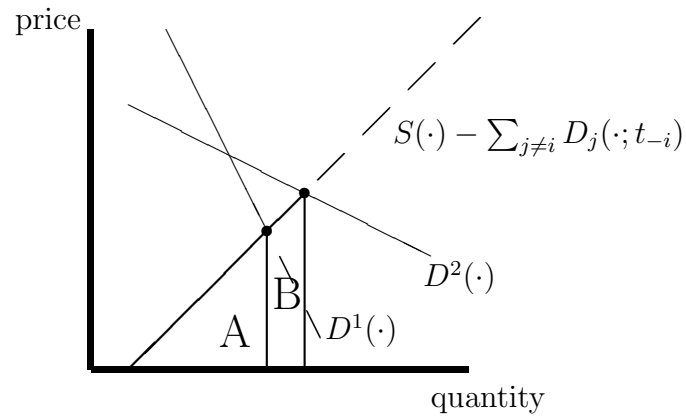


Figure 3: Modularity of ex post payment in Vickrey auction

### 3.3 Example: Vickrey Auction

In the Vickrey auction, the allocation and the bids  $D_{-i}(\cdot)$  determine  $z_i^V$ . Since

$$\{q_{-i}^1, q_{-i}^2\} = \{q_{-i}^{1\vee 2}, q_{-i}^{1\wedge 2}\}$$

and  $D_{-i}(\cdot)$  is held as fixed, then,  $z_i^V$  is modular in  $D_i(\cdot)$ . Graphically, this argument is summarized in Figure 3 by the observation that

$$\begin{aligned} z_i^V(D^1(\cdot), D_{-i}(\cdot; t_{-i})) + z_i^V(D^2(\cdot), D_{-i}(\cdot; t_{-i})) \\ = z_i^V(D^{1\vee 2}(\cdot), D_{-i}(\cdot; t_{-i})) + z_i^V(D^{1\wedge 2}(\cdot), D_{-i}(\cdot; t_{-i})) \\ = 2AREA(A) + AREA(B) \end{aligned}$$

## 4 Concluding Remarks

This paper brings a unified analysis to many multi-unit auctions in which the allocation is determined by market-clearing, including the uniform-price, as-bid, and Vickrey auctions as well as all single-unit auctions in which the high bidder wins.<sup>2</sup> In these auctions, each bidder's expected surplus is modular (or additively separable) in own bid *regardless of the structure of uncertainty*. When bidders are risk-neutral so that expected payoff equals expected surplus, then, expected payoffs are modular in own bid. To verify existence of an monotone pure strategy equilibrium, then, one need only check that expected payoffs satisfy single-crossing in own bid and type when others' follow monotone strategies. But this is exactly the "single-crossing condition" that Athey (2001) requires in the context of single-unit auctions. In short, the observation that payoffs are modular allows us to "reduce" the issue of equilibrium existence in multi-unit auctions to the issue of equilibrium existence in single-unit auctions. Unfortunately, as discussed by McAdams (2002a) and illustrated in the Appendix, payoffs in multi-unit auctions do not satisfy any sort of single-crossing property as they do in single-unit auctions. This makes verifying Athey's single-crossing condition much more challenging in all but the case of independent types covered by this paper.

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<sup>2</sup>Auctions not covered include the endogenous supply auctions studied by Back and Zender (2001) and McAdams (2002c) in which the auctioneer decides how much to supply after receiving the bids.

## Appendix

This paper is the first to prove existence of a pure strategy equilibrium in the uniform-price auction when bidders have multi-unit demand and values that are not private.

Dasgupta and Maskin (1986), Reny (1999), and others do not apply since each bidder's payoff is not quasiconcave, even in Reny (1999)'s weak sense of being "diagonally quasiconcave". (See page 19.)

Vives (1990), Milgrom and Shannon (1994), and others who leverage strategic complementarity do not apply. Each bidder's payoff in the uniform-price auction does not satisfy even the weak single-crossing property in own bid and others' bids. (See page 20.)

Baye, Tian, and Zhou (1993)'s necessary and sufficient conditions for existence of pure strategy equilibrium are extremely difficult to check in multi-unit auction applications.

The purification theorems of Radner and Rosenthal (1985) and Milgrom and Weber (1985) as well as Bresky (2000) and Jackson and Swinkels (2001) require private values.

Jackson, Simon, Swinkels, and Zame (forthcoming) only prove existence of a mixed strategy equilibrium.

**Quasiconcavity:** The following definition is taken from Reny (1999). (Each player has the same action set  $X$ .)

**Definition.** The game  $G = (X, u_i)$  is *diagonally quasiconcave* if  $X$  is convex, and for every player  $i$ , all  $x^1, \dots, x^m \in X$  and all  $\bar{x} \in co\{x^1, \dots, x^m\}$ ,

$$u_i(\bar{x}, \dots, \bar{x}) \geq \min_{1 \leq k \leq m} u_i(\bar{x}, \dots, x^k, \dots, \bar{x})$$

In the uniform-price auction, however, payoffs fail to be diagonally quasiconcave. A simple example makes the point:<sup>3</sup>

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<sup>3</sup>This example (and the next, illustrating failure of strategic complementarity) can be easily modified to apply in settings with a discrete grid of prices and quantities. I suppose a continuum grid only for easy exposition.

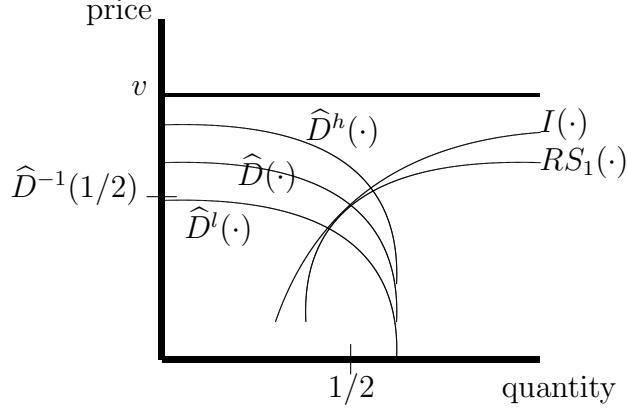


Figure 4: Failure of quasi-concavity of payoffs in the uniform-price auction

**Example 1.** Suppose that there are two bidders and one perfectly divisible good, bidder 1 has constant marginal value  $v$ , and both bidders submit the bid corresponding to a decreasing demand function  $\hat{D}(\cdot) = \hat{D}_1(\cdot) = \hat{D}_2(\cdot)$  such that  $\hat{D}^{-1}(\frac{1}{2}) < v$ . Each bidder then receives quantity  $q^* = \frac{1}{2}$  at price  $\hat{D}^{-1}(\frac{1}{2})$ . Note that bidder 1's payoff is entirely determined by the realized price and his realized quantity, and define 1's isoprofit curve

$$\left\{ (p, I(p)) : I(p)(v - p) = \frac{1}{2}(v - \hat{D}^{-1}(\frac{1}{2})), p \in (-\infty, v] \right\},$$

profit function  $\Pi_1(\hat{D}_1(\cdot), \hat{D}_2(\cdot))$ , and residual supply  $RS_1(\cdot) = 1 - \hat{D}_2(\cdot)$ . As long as  $|\hat{D}'(\frac{1}{2})| = I'(\frac{1}{2})$  and  $|\hat{D}''(\frac{1}{2})| > |I''(\frac{1}{2})|$ , for small enough  $\varepsilon$

$$\Pi_1(\hat{D}^h(\cdot), \hat{D}(\cdot)), \Pi_1(\hat{D}^l(\cdot), \hat{D}(\cdot)) > \Pi_1(\hat{D}(\cdot), \hat{D}(\cdot))$$

where  $\hat{D}^h(q) = \hat{D}(q) + \varepsilon$ ,  $\hat{D}^l(q) = \hat{D}(q) - \varepsilon$ , and  $1/2\hat{D}^h(q) + 1/2\hat{D}^l(q) = \hat{D}(q)$  for all  $q$ . If payoffs were diagonally quasi-concave, however,

$$\min \left\{ \Pi_1(\hat{D}^h(\cdot), \hat{D}(\cdot)), \Pi_1(\hat{D}^l(\cdot), \hat{D}(\cdot)) \right\} \leq \Pi_1(\hat{D}(\cdot), \hat{D}(\cdot)).$$

**Single-Crossing in Own Bid and Others' Bids** Figure 5 illustrates why bids in the uniform-price auction fail to exhibit strategic complementarity in a simple example with two bidders and  $S$  perfectly divisible units.

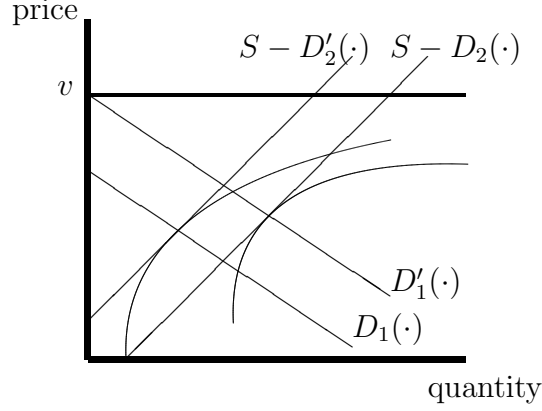


Figure 5: Lack of strategic complementarity in the uniform-price auction

(The observation extends, however, to settings with any number of bidders, indivisible units, and price-elastic supply.)

**Example 2.** Bidder  $i$  has a constant marginal value  $v$  for shares that is common knowledge, i.e.  $v_i(\mathbf{q}; \mathbf{t}) = q_i v$ .  $S - D'_2(\cdot)$ ,  $S - D_2(\cdot)$  are the residual supply curves that would result if bidder 2 submits the bid  $D'_2(\cdot)$  or  $D_2(\cdot)$ . The unlabelled curves in Figure 5, finally, are isoprofit curves of bidder 1. Thus,  $D_1(\cdot)$  is a best response to  $D'_2(\cdot)$  and  $D'_1(\cdot)$  is a best response to  $D_2(\cdot)$ . And although  $D'_2(\cdot) > D_2(\cdot)$  and  $D'_1(\cdot) > D_1(\cdot)$ ,

$$\Pi_1(D_1(\cdot), D_2(\cdot)) < \Pi_1(D'_1(\cdot), D_2(\cdot)), \quad \Pi_1(D_1(\cdot), D'_2(\cdot)) > \Pi_1(D'_1(\cdot), D'_2(\cdot))$$

## Notational Shorthand

In most of the analysis to follow, there is a fixed profile of bids  $D_{-i}(\cdot)$ , a fixed rationing ranking  $\rho$ , and a fixed state  $\mathbf{t}$ . I focus on properties of the realized price, allocation, and payments when bidder  $i$  submits one of two bids  $D^1(\cdot)$  or  $D^2(\cdot)$  or their join  $D^{1 \vee 2}(\cdot) \equiv D^1(\cdot) \vee D^2(\cdot)$  or meet  $D^{1 \wedge 2}(\cdot) \equiv D^1(\cdot) \wedge D^2(\cdot)$ .

I use the following shorthand for price, quantities, and payments:

$$\begin{aligned}
q_j^1 &\equiv q_j^{mc} (D^1(\cdot), D_{-i}(\cdot); \rho), \quad q_j^2 \equiv q_j^{mc} (D^2(\cdot), D_{-i}(\cdot); \rho) \\
q_j^{1\vee 2} &\equiv q_j^{mc} (D^{1\vee 2}(\cdot), D_{-i}(\cdot); \rho), \quad q_j^{1\wedge 2} \equiv q_j^{mc} (D^{1\wedge 2}(\cdot), D_{-i}(\cdot); \rho) \\
p^1 &\equiv p^{mc} (D^1(\cdot), D_{-i}(\cdot)), \quad p^2 \equiv p^{mc} (D^2(\cdot), D_{-i}(\cdot)) \\
p^{1\vee 2} &\equiv p^{mc} (D^{1\vee 2}(\cdot), D_{-i}(\cdot)), \quad p^{1\wedge 2} \equiv p^{mc} (D^{1\wedge 2}(\cdot), D_{-i}(\cdot)) \\
z_j^1 &\equiv z_j (D^1(\cdot), D_{-i}(\cdot); \rho), \quad z_j^2 \equiv z_j (D^2(\cdot), D_{-i}(\cdot); \rho) \\
z_j^{1\vee 2} &\equiv z_j (D^{1\vee 2}(\cdot), D_{-i}(\cdot); \rho), \quad z_j^{1\wedge 2} \equiv z_j (D^{1\wedge 2}(\cdot), D_{-i}(\cdot); \rho)
\end{aligned}$$

When there can be no confusion, furthermore, I will replace  $q_i^1, z_i^1, \dots$  with  $q^1, z^1, \dots$

**Lemma 2.** (*Characterizing Price*) If  $\{D_i(\cdot)\}$  and  $S(\cdot)$  satisfy conditions (i) - (iv) (see pages 6, 8), then  $p^{mc}(\mathbf{D}(\cdot)) = \max\{p \in \mathbf{p} : D_{1,\dots,n}(p) \cap S(p) \neq \emptyset\}$  exists. Furthermore,

$$p^{mc}(\mathbf{D}(\cdot)) = \max\{p \in \mathbf{p} : \max D_{1,\dots,n}(p) \geq \min S(p)\}$$

*Proof.* First, by condition (ii),  $\max \mathbf{q} \in D_{1,\dots,n}(\{\emptyset\})$  and  $0 \in S(\{\emptyset\})$ . Hence,  $\max D_{1,\dots,n}(\{\emptyset\}) > \min S(\{\emptyset\})$  and, since  $\mathbf{p}$  is finite,

$$\tilde{p} \equiv \max\{p \in \mathbf{p} : \max D_{1,\dots,n}(p) \geq \min S(p)\}$$

exists. Clearly,  $D_{1,\dots,n}(p) \cap S(p) = \emptyset$  for all  $p > \tilde{p}$ . Now I need only show that  $D_{1,\dots,n}(\tilde{p}) \cap S(\tilde{p}) \neq \emptyset$ . By condition (iv),

$$\min D_{1,\dots,n}(\tilde{p}) = \max D_{1,\dots,n}(\tilde{p} + \triangle_p) < \min S(\tilde{p} + \triangle_p) = \max S(\tilde{p})$$

By conditions (ii), (iii), and (iv), finally,

$$\begin{aligned}
\min D_{1,\dots,n}(\tilde{p}) &< \max S(\tilde{p}) \quad \text{and} \quad \max D_{1,\dots,n}(\tilde{p}) \geq \min S(\tilde{p}) \\
&\Rightarrow \exists q \in D_{1,\dots,n}(\tilde{p}) \cap S(\tilde{p})
\end{aligned}$$

In particular,  $\tilde{p} = p^{mc}$ . □

Equivalently, for each bidder  $i = 1, \dots, n$ ,  $p^{mc} = p^{mc}(\mathbf{D}(\cdot)) = \max\{p : \max D_i(p) \geq \min RS_i(p)\}$  where  $RS_i(p) = S - \sum_{j \neq i} D_j(p)$  is the residual supply correspondence.

Define bidder  $i$ 's *rationing function*

$$R_i^\rho(p) \equiv S - \sum_{\rho(j)=1}^{\rho(i)-1} \max D_j(p) - \sum_{\rho(j)=\rho(i)+1}^{\rho(j)=n} \min D_j(p)$$

**Lemma 3.** (*Characterizing Quantities*)

$$\begin{aligned} q_i^{mc} &= \min D_i(p^{mc}) \text{ if } R_i^\rho(p^{mc}) \leq \min D_i(p^{mc}) \\ q_i^{mc} &= R_i^\rho(p^{mc}) \text{ if } R_i^\rho(p^{mc}) \in [\min D_i(p^{mc}), \max D_i(p^{mc})] \\ q_i^{mc} &= \max D_i(p^{mc}) \text{ if } R_i^\rho(p^{mc}) \geq \max D_i(p^{mc}) \end{aligned}$$

*Proof.*  $R_i^\rho(p^{mc}) \leq \min D_i(p^{mc})$  iff

$$S^{mc} - \min D_{1,\dots,n}(p^{mc}) \leq \sum_{\rho^{-1}(j)=1}^{\rho^{-1}(i)} \max D_j(p^{mc}) - \min D_j(p^{mc})$$

i.e. if there is no quantity remaining after all others ahead of  $i$  are served in the rationing queue. Similarly,  $R_i^\rho(p^{mc}) \leq \min D_i(p^{mc})$  iff  $i$  can be fully served after all others ahead of  $i$  have been fully served. Finally,  $R_i^\rho(p^{mc}) \in (\min D_i(p^{mc}), \max D_i(p^{mc}))$  iff  $i$  can only be partially served after those ahead of him have been fully served.  $\square$

In more compact notation,

$$q_i^{mc}(\mathbf{D}(\cdot); \rho) = \max\{\min D_i(p^{mc}(\mathbf{D}(\cdot))), \min\{R_i^\rho(\mathbf{D}(\cdot)), \max D_i(p^{mc}(\mathbf{D}(\cdot)))\}\}.$$

Also, by definition,  $R_i^\rho(p^{mc}) \in RS_i(p^{mc})$ .

**Lemma 4.** (*Monotone Prices, Quantities*) Suppose that  $D^2(\cdot) > D^1(\cdot)$ . Then for all  $D_{-i}(\cdot)$ ,  $\rho$

$$p^2 \geq p^1, q_i^2 \geq q_i^1, \text{ and } q_j^2 \leq q_j^1 \text{ for all } j \neq i$$

*Proof.*  $\max D^2(p^1) \geq \max D^1(p^1) \geq \min RS_i(p^1)$ , which implies that  $p^2 \geq p^1$ . Now two cases: (i)  $p^2 > p^1$ :  $q_i^2 \geq \min RS_i(p^2) \geq \max RS_i(p^1) \geq q_i^1$ . (ii)  $p^2 = p^1 = p$ :

$$\begin{aligned} q_i^2 &= \max\{\min D^2(p), \min\{R_i^\rho(p), \max D^2(p)\}\} \\ &\geq \max\{\min D^1(p), \min\{R_i^\rho(p), \max D^1(p)\}\} = q_i^1 \end{aligned}$$

To prove that  $q_j^2 \leq q_j^1$  for  $j \neq i$ , note that  $q_j^{mc}(\mathbf{D}(\cdot))$  is a (weakly) increasing function of  $R_j^\rho(\mathbf{D}(\cdot))$  and that  $R_j^\rho(\mathbf{D}(\cdot))$  is (weakly) decreasing in  $D_i(\cdot)$ .  $\square$



## Proof of Lemma 1

Note that

$$\begin{aligned} & \Pi_i^{int}(D_i(\cdot), t_i; D_{-i}(\cdot; \cdot)) \\ &= E_{t_{-i}, \rho} \left[ \Pi_i^{post} \left( D_i(\cdot; t_i), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}; \rho \right) \middle| t_{-i}, \rho \right] \end{aligned}$$

Since I have assumed independence of types, it suffices to show that  $\Pi_i^{post}(\cdot, \cdot; D_{-i}(\cdot); t_{-i}, \rho)$  has non-decreasing differences for all  $D_{-i}(\cdot)$ , all  $t_{-i}$ , and all  $\rho$ . But since

$$\begin{aligned} & \Pi_i^{post}(D_i(\cdot), t_i; D_{-i}(\cdot), t_{-i}; \rho) \\ &= \sum_{q \in (0, q_i^{mc}(\mathbf{D}(\cdot); \rho)] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc}(\mathbf{D}(\cdot); \rho), \mathbf{t}) \Delta_q - z_i(\mathbf{D}(\cdot); \rho) \end{aligned}$$

this reduces to the requirement that

$$\begin{aligned} & \sum_{q \in (0, q_i^{mc'}] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc'}, t'_i, t_{-i}) \Delta_q - \sum_{q \in (0, q_i^{mc}] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc}, t'_i, t_{-i}) \Delta_q \\ & \geq \sum_{q \in (0, q_i^{mc'}] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc'}, t_i, t_{-i}) \Delta_q - \sum_{q \in (0, q_i^{mc}] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc}, t_i, t_{-i}) \Delta_q \end{aligned}$$

where I use the shorthand notation

$$q_j^{mc'} \equiv q_j^{mc}(D'_i(\cdot), D_{-i}(\cdot); \rho), \quad q_j^{mc} \equiv q_j^{mc}(D_i(\cdot), D_{-i}(\cdot); \rho),$$

(The payment terms cancel out.)

By Lemma 4 in the Appendix,

$$\begin{aligned} & q_i^{mc'} \geq q_i^{mc} \text{ for all } D_{-i}(\cdot) \\ & q_j^{mc'} \leq q_j^{mc} \text{ for all } D_{-i}(\cdot), j \neq i \end{aligned}$$

Thus,

$$\begin{aligned} & \sum_{q \in (0, q_i^{mc'}] \cap \mathfrak{q}} v_i(x; q_{-i}^{mc'}, t'_i, t_{-i}) \Delta_q - \sum_{q \in (0, q_i^{mc}] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc}, t'_i, t_{-i}) \Delta_q \\ & \geq \sum_{q \in (0, q_i^{mc'}] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc'}, t_i, t_{-i}) \Delta_q - \sum_{q \in (0, q_i^{mc}] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc}, t_i, t_{-i}) \Delta_q \end{aligned}$$

follows from the fact that  $v_i$  has non-increasing differences in  $(t_i, q_{-i})$  and

$$\sum_{q \in (q_i^{mc}, q_i^{mc'}] \cap \mathfrak{q}} v_i(x; q_{-i}^{mc'}, t'_i, t_{-i}) \Delta_q \geq \sum_{q \in (q_i^{mc}, q_i^{mc'}] \cap \mathfrak{q}} v_i(x; q_{-i}^{mc'}, t_i, t_{-i}) \Delta_q$$

follows from the fact that  $v_i$  is non-decreasing in  $t_i$ .

## Proof of Theorem 2

Let  $D^2(\cdot)D^1(\cdot) \in \mathcal{D}$  be two permissible bids. By Lemma 4, I may assume without loss that  $D^2(\cdot) \not\leq D^1(\cdot)$  and that either  $p^2 > p^1$  or  $p^2 = p^1$ ,  $q^2 \geq q^1$ .

For a given type  $t_i$  and strategy profile  $D_{-i}(\cdot; \cdot)$ , it suffices to show that  $\Pi_i^{post}(D_i(\cdot), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}, \rho)$  is modular in  $D_i(\cdot)$  for all  $t_{-i}, \rho$ :

$$\begin{aligned} & \Pi_i^{post}(D^1(\cdot), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}, \rho) + \Pi_i^{post}(D^2(\cdot), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}, \rho) \\ &= \Pi_i^{post}(D^{1 \vee 2}(\cdot), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}, \rho) + \Pi_i^{post}(D^{1 \wedge 2}(\cdot), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}, \rho) \end{aligned}$$

Recall that

$$\begin{aligned} \min D^1 \vee D^2(p) &= \max\{\min D^1(p), \min D^2(p)\} \\ \max D^1 \vee D^2(p) &= \max\{\max D^1(p), \max D^2(p)\} \\ \min D^1 \wedge D^2(p) &= \min\{\min D^1(p), \min D^2(p)\} \\ \max D^1 \wedge D^2(p) &= \min\{\max D^1(p), \max D^2(p)\} \end{aligned}$$

and that ex post payoffs are

$$\begin{aligned} & \Pi_i^{post}(D_i(\cdot), t_i; D_{-i}(\cdot; t_{-i}), t_{-i}, \rho) \\ &= \sum_{q \in (0, q_i^{mc}(\mathbf{D}(\cdot), \rho)] \cap \mathfrak{q}} v_i(q; q_{-i}^{mc}(\mathbf{D}(\cdot), \rho), \mathbf{t}) \Delta_q - z_i(\mathbf{D}(\cdot), \rho) \end{aligned}$$

Using my shorthand notation, then, it suffices to show that

$$\begin{aligned} & \sum_{q \in (0, q_i^{1 \wedge 2}] \cap \mathfrak{q}} v_i(q; q_{-i}^{1 \wedge 2}, \mathbf{t}) \Delta_q - \sum_{q \in (0, q_i^1] \cap \mathfrak{q}} v_i(q; q_{-i}^1, \mathbf{t}) \Delta_q \\ &+ \sum_{q \in (0, q_i^{1 \vee 2}] \cap \mathfrak{q}} v_i(q; q_{-i}^{1 \vee 2}, \mathbf{t}) \Delta_q - \sum_{q \in (0, q_i^2] \cap \mathfrak{q}} v_i(q; q_{-i}^2, \mathbf{t}) \Delta_q \\ &= z_i^{1 \wedge 2} - z_i^1 + z_i^{1 \vee 2} - z_i^2 \end{aligned}$$

To complete the proof it therefore suffices to show that

$$\begin{aligned} p^2 &= p^{1\vee 2}, p^1 = p^{1\wedge 2} \\ q_j^2 &= q_j^{1\vee 2}, q_j^1 = q_j^{1\wedge 2} \text{ for all } j = 1, \dots, n \\ z^1 + z^2 &= z^{1\vee 2} + z^{1\wedge 2} \end{aligned}$$

**Lemma 5.**  $p^2 = p^{1\vee 2}$  and  $p^1 = p^{1\wedge 2}$ .

*Proof.* Note that for all  $p > p^2$ ,

$$\max D^1(p), \max D^2(p) < \min RS_i(p) \Rightarrow p^{1\vee 2} \leq p^2$$

Similarly,  $\forall p < p^1$ ,

$$\min D^1(p), \min D^2(p) > \max RS_i(p) \Rightarrow p^{1\wedge 2} \geq p^1$$

On the other hand,  $\max D^2(p^2) \geq \min RS_i(p^2)$  implies that  $\max D^{1\vee 2}(p^2) \geq \min RS_i(p^2)$ , so  $p^{1\vee 2} \geq p^2$ . We conclude that  $p^{1\vee 2} = p^2$ . Similarly,  $\max D^1(p) < \min RS_i(p)$  for all  $p > p^1$  implies that  $\max D^{1\wedge 2}(p) < \min RS_i(p)$  for all  $p > p^1$ , so  $p^{1\wedge 2} \leq p^1$ . I conclude that  $p^{1\wedge 2} = p^1$ .  $\square$

**Lemma 6.** If  $p^1 = p^2$ , then  $q^1 = q^{1\wedge 2}$  and  $q^2 = q^{1\vee 2}$

*Proof.* Let  $p = p^1 = p^2$ . By Lemma 5, I know that  $p = p^{1\vee 2} = p^{1\wedge 2}$  as well. By assumption,  $q^2 \geq q^1$ .  $q^2 = q^1$  exactly when *either* (A)  $R_i^\rho(p) \in D^1(p) \cap D^2(p)$  or (B)  $R_i^\rho(p) \leq \min D^1(p) \cup D^2(p)$  or (C)  $R_i^\rho(p) \geq \max D^1(p) \cup D^2(p)$ . But (A) implies  $R_i^\rho(p) \in D^{1\wedge 2}(p) \cap D^{1\vee 2}(p)$ , (B) implies  $R_i^\rho(p) \leq \min D^{1\wedge 2}(p) \cup D^{1\vee 2}(p)$ , and (C) implies  $R_i^\rho(p) \geq \max D^{1\wedge 2}(p) \cup D^{1\vee 2}(p)$ . So,  $q^2 = q^1$  implies that  $q^2 = q^1 = q^{1\vee 2} = q^{1\wedge 2}$ .

Now suppose that  $q^2 > q^1$ . In this case, it must be that *either*  $\min D^1(p), R_i^\rho(p) < \min D^2(p)$  or  $\max D^1(p) < R_i^\rho(p), \max D^2(p)$ . In either case,  $q^1 = q^{1\wedge 2}$  and  $q^2 = q^{1\vee 2}$ .  $\square$

**Lemma 7.** If  $p^2 > p^1$ , then  $q^2 = q^{1\wedge 2}$  and  $q^1 = q^{1\vee 2}$ .

*Proof.* By Lemma 5, I know that  $p^2 = p^{1\vee 2}$  and  $p^1 = p^{1\wedge 2}$ . Also, given  $R_i^\rho(\cdot)$  and a market-clearing price  $p$ ,  $q_i^{mc}$  is entirely determined by  $\max\{R_i^\rho(p), \min D_i(p)\}$  and  $\min\{R_i^\rho(p), \max D_i(p)\}$ . (See Section ??.) Thus, to prove that  $q^1 = q^{1\wedge 2}$  it suffices to show that

$$\begin{aligned} \min D^1(p^1) &= \min D^{1\wedge 2}(p^1) \text{ or } R_i^\rho(p^1) \geq \min D^1(p^1), \min D^{1\wedge 2}(p^1) \\ &\text{and} \\ \max D^1(p^1) &= \max D^{1\wedge 2}(p^1) \text{ or } R_i^\rho(p^1) \leq \max D^1(p^1), \max D^{1\wedge 2}(p^1) \end{aligned}$$

Similarly, to prove that  $q^2 = q^{1\vee 2}$  it suffices to show that

$$\begin{aligned} \min D^2(p^2) &= \min D^{1\vee 2}(p^2) \text{ or } R_i^p(p^2) \geq \min D^2(p^2), \min D^{1\vee 2}(p^2) \\ &\text{and} \\ \max D^2(p^2) &= \max D^{1\vee 2}(p^2) \text{ or } R_i^p(p^2) \leq \max D^2(p^2), \max D^{1\vee 2}(p^2) \end{aligned}$$

I prove first that

$$\begin{aligned} \min D^1(p^1) &= \min D^{1\wedge 2}(p^1) \\ \max D^2(p^2) &= \max D^{1\vee 2}(p^2) \end{aligned}$$

which is the same thing as to say that

$$\begin{aligned} \min D^1(p^1) &\leq \min D^2(p^1) \\ \max D^2(p^2) &\geq \max D^1(p^2) \end{aligned}$$

By definition of the market-clearing price given bids  $D^1(\cdot)$  and  $D^2(\cdot)$ ,  $p^2 > p^1$  implies that

$$\begin{aligned} \max D^2(p^2) &\geq \min RS_i(p^2) \\ \max D^1(p^1) &\geq \min RS_i(p^1) \\ \max D^1(p^2) &< \min RS_i(p^2) \end{aligned}$$

(The last inequality follows from the fact that  $p^1$  is the highest price such that  $D^1(p) \cap RS_i(p) \neq \emptyset$ .) Furthermore,  $\min RS_i(p^2) \geq \max RS_i(p^1)$  since residual supply must be non-decreasing in  $p$ . Consequently,  $\max D^2(p^2) > \max D^1(p^2)$  which implies that  $\max D^2(p^2) = \max D^{1\vee 2}(p^2)$ . Similarly, since  $\min D^2(p^1) \geq \max D^2(p^2)$  it must be that  $\min D^1(p^1) < \min D^2(p^1)$  which implies that  $\min D^1(p^1) = \min D^{1\wedge 2}(p^1)$ .

Now I will prove that

$$\begin{aligned} \max D^1(p^1) > \max D^{1\wedge 2}(p^1) &\Rightarrow R_i(p^1) \leq \max D^1(p^1), \max D^{1\wedge 2}(p^1) \\ \min D^2(p^2) < \min D^{1\vee 2}(p^2) &\Rightarrow R_i(p^2) \geq \min D^2(p^2), \min D^{1\vee 2}(p^2) \end{aligned}$$

Note the following relationships:

$$\max D^2(p^1) \geq \max D^2(p^2) \geq \min RS_i(p^2) \geq R_i(p^1)$$

(First:  $D^2(\cdot)$  is non-increasing. Second: definition of  $p^2$ . Third:  $\min D_j(p^1) \geq \max D_j(p^2)$  for all  $j$  and by definition

$$R_i(p) \leq \max RS_i(p) = S - \sum_{j \neq i} \max D_j(p).$$

So,

$$\max D^1(p^1) > \max D^{1\wedge 2}(p^1) \Rightarrow \max D^1(p^1) > \max D^2(p^1) \geq R_i(p^1)$$

For the same (dual) reasons,

$$\min D^1(p^2) \leq \min D^1(p^1) \leq \max RS_i(p^1) \leq R_i(p^2)$$

So,

$$\min D^2(p^2) < \min D^{1\vee 2}(p^1) \Rightarrow \min D^2(p^2) < \min D^1(p^2) \leq R_i(p^2)$$

and I am done.  $\square$

**Lemma 8.**  $z^1 + z^2 = z^{1\vee 2} + z^{1\wedge 2}$

*Proof.* Recall that in every auction  $A(n, (\mathbf{p}, \mathbf{q}), S(\cdot), \mathbf{z}(\cdot)) \in \mathcal{A}$ , the payment rule must be of the form

$$z_i(\mathbf{D}(\cdot), \rho) = \sum_{q \in (0, q_i^{mc}] \cap \mathbf{q}} w_i(\max P_i(q), P_{-i}(\cdot); q) + y_i(p^{mc}; q_i^{mc})$$

Since the sum of modular functions are modular, it suffices for us to prove that each additive component of payment is modular in  $D_i(\cdot)$ .

Modularity of  $y_i(p^{mc}; q_i^{mc})$  in  $D_i(\cdot)$  is immediate from the previous results, since  $\{(p^1, q^1), (p^2, q^2)\} = \{(p^{1\wedge 2}, q^{1\wedge 2}), (p^{1\vee 2}, q^{1\vee 2})\}$ . Similarly, modularity of  $I\{q < q_i^{mc}\} w_i(\max P_i(q), P_{-i}(\cdot); q)$  is immediate for each  $q < q^1$  and each  $q > q^2$ . (Recall that, by naming convention, either  $q^1 < q^2$  or  $q^1 = q^2$  and  $p^1 < p^2$ .) Consider then  $q^1 \leq q \leq q^2$ . Modularity of  $I\{q < q_i^{mc}\} w_i(\max P_i(q), P_{-i}(\cdot); q)$  in this case is equivalent to equality between  $w_i(\max P_i^2(q), P_{-i}(\cdot); q)$  and  $w_i(\max P_i^{1\vee 2}(q), P_{-i}(\cdot); q)$ , which itself follows from  $\max P_i^2(q) = \max P_i^{1\vee 2}(q)$ . To prove this, note that  $\max P_i^1(q) < \max P_i^2(q)$ , else  $\max P_i^1(q) > \min P_{-i}(q)$  implying that either  $q^1 > q^2$  or  $q^1 = q^2$  and  $p^1 > p^2$ , a contradiction.  $\square$

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