# HIERARCHICAL CONTROL AND DECOMPOSITION OF DECENTRALIZED LINEAR STOCHASTIC SYSTEMS

by

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(1974)

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(1976)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

SEPTEMBER 1978

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September, 1978 LIDS-TH-858

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This report is based on the unaltered thesis of Douglas P. Looze, submitted in partial fulfillment of the requirements of the degree of Doctor of Philosophy at the Massachusetts Institute of Technology in September, 1978. The research was conducted at the Decision and Control Sciences group of the M.I.T. Laboratory for Information and Decision Systems (formerly Electronic Systems Laboratory), with partial support extended by ONR contract N00014-76-C-0346, and by DOD under grant ERDA-E(49-18)-2087.

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Submitted to the Department of Electrical Engineering and Computer Science on August 31, 1978 in partial fulfillment of the requirements for the degree of Doctor of Philosophy

## ABSTRACT

The contents of this thesis can be separated into two distinct divisions. The first is concerned with the development of a general theory of decomposition algorithms for optimization problems. The second develops an application of the methodology of the decomposition theory to a decentralized linear stochastic control problem.

An indirect approach to the development of the theory of decomposition of optimization problems is taken. It is assumed that a set of necessary conditions for the optimization can be expressed in the form of a system of nonlinear equations. It is this system of equations which is decomposed. The result is a constructive approach for the decomposition of optimization problems which includes most hierarchical algorithms proposed to date. Because the approach is indirect, the convergence analysis is local in nature. Also, the formulation admits the development and analysis of multiple decompositions (and hence multilevel hierarchies).

The theory of decomposition is applied to a linear stochastic optimal control problem with information flow constraints. The particular problem formulation considered is the interconnected system problem where the

controller is required to be a linear finite dimensional system. The problem is then reformulated as a deterministic minimization and necessary conditions are derived. These conditions are decomposed to take advantage of the interconnected system structure. The convergence properties of the algorithm are examined, and solution algorithms for the decomposed subproblems are proposed. Finally, the algorithm is applied to the linearized model of an inertia wheel attitude control device.

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ACKNOWLEDGEMENTS

It gives me great pleasure to thank Professor Nils Sandell Jr. for his excellent supervision of this thesis. His insight and ideas have proved invaluable.

I also wish to express my appreciation to the readers of this thesis, Professor Michael Athans, Dr. David Castanon, Dr. Paul Houpt and Dr. Alan Laub, for their helpful suggestions and ideas.

I am very greatful to Fifa Monserrate for her wonderful work and long hours spent typing this thesis, and to Art Giordani of the drafting department for his help with the figures.

Finally, I wish to thank my wife Susan. Without her love and support this thesis could never have been finished.

This research was conducted at the Decision and Control Sciences Group, Electronic Systems Laboratory with support provided by the Office of Naval Research. This research was conducted at the M.I.T. Electronic Systems Laboratory with support extended by ONR contract N00014-76-C-0346, and by DOD under grant ERDA-E(49-18)-2087.

## BIOGRAPHICAL NOTE

Douglas P. Looze was born in Green Bay, Wisconsin on November 21, 1951. He attended the Massachusetts Institute of Technology where he received the S.B., S.M., and E.E. degrees in 1974, 1975 and 1976 respectively. He has worked as a research assistant at M.I.T. from 1974-1978 and has held summer positions at the Electronic Systems Laboratory in 1976 and at The Analytical Sciences Corporation in 1977. He is a member of Eta Kappa Nu.

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# 1. INTRODUCTION

## 1.1 General Discussion and Background

In recent years there has developed a considerable literature in what is termed large scale systems theory (see, e.g. Ho and Mitter [1976]; Guardabassi and Locatelli [1976]; Sandell et.al. [1978] gives an extensive literature survey). This theory has touched upon all the traditional areas of system theory. However, in spite of the amount of literature available, the definition of a large scale system is very elusive and the boundary between large scale and non-large scale systems is ill-defined. The classification of a system as large scale is usually determined by the inability of traditional techniques to handle the problems which the system presents, either due to computer time and word length limitations or to the extensive communications required to implement the solution. Both these problems often occur simultaneously in large scale systems.

The general control problem considered in large scale system theory is that of optimal control of a nonlinear stochastic system. An approach often used (Lefkowitz [1966]; Findeisen [1975]; Athans [1971]) is to treat the nonlinearities in a deterministic framework and approximate the stochastic effects as white noise driving a linear system. The resulting control law is computed and implemented in two layers. The upper (economic) layer is a deterministic nonlinear control problem which attempts to optimize the economic objectives of the system. Meanwhile

the lower (stabilizing) layer is a stochastic linear control problem which attempts to maintain the system trajectories near the deterministic economic optimal (and thus, in a sense, optimizes the stochastic stability of the system).

The objective and difficulties associated with these two problems are very different. Due to the equivalence of open and closed loop controls in deterministic optimal control and the difficulty of determining the closed loop solution for nonlinear problems, the economic layer problem is usually solved for the open loop form of the control. The control law may be determined on-line but is implemented only after the solution is obtained. Thus the total amount of computation time is an important consideration; communication is not an issue due to the open loop structure of the solution. In contrast, the stabilizing layer solution must be closed loop. Many communication links are required for systems with a large number of state variables. Computational problems are still important, but are often neglected.

Hierarchical control theory (Mesarovic, et.al. [1970]; Wismer [1971]; Findeisen [1976]; Bailey and Laub [1978]; Smith and Sage [1973]; Singh [1976]; Bernhard [1976]) has been motivated by the success of decomposition techniques in mathematical programming (Lasdon [1970]; Geoffrion [1970]). Since the natural generalization of these techniques is to open loop control of deterministic systems, hierarchical control is mainly concerned with the economic layer problem[1]. The chief

[1]There are exceptions to this and some of the other statements in this paragraph, particularly in the more recent literature (Singh et.al. [1976]; Findeisen and Malinowski [1976]; Chong and Athans [1976]).

characteristic of hierarchical control is the iterative computation of the globally optimal control by coordinated solution of certain subproblems. This computation, which is generally envisioned as taking place off-line, is decentralized in the sense that the subproblems can be solved without knowledge of the global model of the control process. However, the computation does require extensive communication between the subproblems and the coordinating problem. The advantage of the hierarchical control approach is computational. Savings in computer time and space are often apparent in a single processor implementation, but the real advantage of the approach is evident in a multiprocessor environment where the global problem is solved by a set of intercommunicating processors, no one of which is alone powerful enough to solve the problem.

In contrast, decentralized control theory (Witsenhausen [1971]; Ho and Chu [1972]; Ho and Chu [1974]; Sandell and Athans [1978]; Wang and Davison [1973]; Corfmat and Morse [1976]) is exclusively concerned with feedback. Linear problem formulations are most often considered since the theory of nonlinear feedback control is not as well developed as the linear theory. Therefore, the practical application of the theory is primarily to the stabilizing layer control problem. The chief characteristic of decentralized control is that restrictions are placed on the real time information flow between the controllers of the process. However, the control laws that define the controllers mathematically are derived in a completely centralized fashion using full knowledge of

the global system model. Determination of these control laws is usually computationally more difficult than determination of a centralized control law with no restrictions on information flow. The advantage of decentralized control is the reduction (or elimination) of requirements for on-line communication links between the process controllers.

From the above discussion it is apparent that both the hierarchical and decentralized control theories address issues of concern in large scale system theory. Despite this fact, the theories are virtually unrelated at the present time.

## 1.2 Contents and Contributions

The contents of this thesis are logically divided into two parts which have distinct objectives. The first part, contained in Chapter 2, develops a general theory of decomposition for nonlinear equations. The second part contains the remainder of the thesis. By using the theory presented in Chapter 2, it establishes a relationship between the hierarchical and decentralized control theories. Specifically, a hierarchical structure for the computation of a decentralized control law is considered.

One of the main failures of hierarchical control theory to date has been the lack of a general theory of decomposition. As a consequence, many algorithms have been proposed which either are essentially the same as previous algorithms or are formulated only for specific problems.

A recent paper by Cohen [1978] has presented a theory which includes most of the hierarchical algorithms proposed to date. This theory

approaches the decomposition from an optimization viewpoint. As a result, global convergence results are obtained through suitable convexity assumptions on the optimization problem.

Chapter 2 of this thesis presents a parallel development of a similar theory. However, Chapter 2 approaches the decomposition of optimization problems from an indirect viewpoint. It is assumed that necessary conditions for the optimization problem can be stated in the form of a system of nonlinear equations. The hierarchical decomposition theory is then developed through the decomposition of this system of equations. Because the approach is indirect, the convergence results are local in nature. However, the assumptions placed on the optimization problem are less restrictive than those required for a global analysis. When such assumptions are made, the local convergence results also apply globally.

The flavor of the approach of Chapter 2 coincides with the structure and derivation of many of the currently proposed hierarchical algorithms. Most are developed through the use of necessary conditions. It is only after the decomposition is derived that the results are interpreted in the optimization framework. Hence the theory of Chapter 2 applies directly to most hierarchical algorithms.

Another advantage of the formulation of Chapter 2 is that it can be extended to encompass multiple decompositions and the resulting multilevel hierarchies. Because no convexity assumptions are used, convergence conditions can be easily developed from the basic theory.

This extension is begun in Chapter 2, and for the first time a three level hierarchical structure is analyzed.

The second part of this thesis considers a particular linear stochastic control problem. Specifically, the controller structure for the problem is required to be linear with a fixed (but arbitrary) finite dimension. This problem, formulated in Chapter 3, has been considered by many authors in varying contexts (Levine and Athans [1970]; Kosut [1970]; Chong and Athans [1971]; Levine et.al. [1971]; Kwakernaak and Sivan [1972]; Wang [1972]; Galiana et.al. [1973]; Davison et.al. [1973]; Cohen [1977]; Looze et.al. [1978]).

The approach of each of the authors cited in the preceding paragraph is the same. The formulation of the problem allows a deterministic equivalent optimization to be stated. Necessary conditions for this problem can then be derived. Chapter 3 follows the same approach. However, the necessary conditions are stated in a more general form than any others to date.

The necessary conditions for the interconnected subsystem problem are decomposed in Chapter 4 using the theory developed in Chapter 2[1]. The result is a hierarchical structure for the computation of the best linear controller which satisfies the information flow constraints. This algorithm has several important properties. First, the algorithm is shown to converge if the subsystem interactions are sufficiently weak.

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[1] See Sandell [1976]; Looze and Sandell [1977a], [1977b] and [1978]; for earlier but incomplete developments of this idea.

This property is exploited in the development of a practical convergence test. Second, the algorithm produces a stabilizing controller at each iteration. This raises the possibility (discussed briefly in Chapter 4 and more extensively in Chapter 6) of applying the structure in an on-line mode. Finally, it is the first proposed hierarchical algorithm for linear stochastic systems which reduces the computation at each level.

The remainder of the thesis is concerned with the problem of applying the algorithm to a practical problem. Section 4.4 of Chapter 4 attempts to simplify the convergence condition for the iteration. A sufficient condition involving the concept of block diagonal dominance (see Feingold and Varga [1962]) is derived, but still requires considerable computation.

Chapter 5 reviews several solution methods for the infimal and supremal problems which result from the decomposition. Included in the discussion is the possibility of using further decompositions as solution algorithms. This is recommended for the supremal problem. The proposed infimal solution methods also included Newton's algorithm, gradient search algorithms, and an extension of an algorithm used by Levine [1970] and Wang [1972]. The algorithm is applied to the linearized model at an inertia wheel attitude control device.

A summary of the results of this thesis and a discussion of possible research directions is presented in Chapter 6.

## 1.4 Notation

In the following, let $X = X_1 \times \ldots \times X_N$ and Y be Banach spaces and assume $f: X \to Y$. Then define:

$\mathcal{L}(X) \triangleq$ {bounded linear operators $L: X \to X$}

$\mathcal{L}(X,Y) \triangleq$ {bounded linear operators $L: X \to Y$}

$\delta f(x;\Delta x) \triangleq$ Fréchet differential of f at x in the direction $\Delta x$

$\partial f \triangleq$ Fréchet derivative of $f (\partial f \in \mathcal{L}(X,Y))$

$\partial_i f \triangleq$ Fréchet derivative of f with respect to the $i^{\text{th}}$ argument

$f^{(i)} \triangleq i^{\text{th}}$ Fréchet derivative

$o(||h||^n) \triangleq$ any function such that $\lim_{||h|| \to 0} \frac{o(||h||^n)}{||h||^n} = 0$

$\rho\{\cdot\} \triangleq$ spectral radius operator

$\mathbb{R} \triangleq$ set of real numbers

$\mathcal{B} \triangleq$ Borel $\sigma$-algebra of $\mathbb{R}$

$\lambda[t_o,T] \triangleq$ Lebesgue measure on $[t_o,T] \subset \mathbb{R}$

$L_2^n$ ( $\mathbb{R}$, $\mathbb{B}$, $\lambda[t_o,T]$) $\triangleq$ space of all real valued vector functions

$x: [t_o,T] \to \mathbb{R}^n$ such that the Lebesgue integral

$$\int_{t_o}^{T} x'(t)x(t)dt \quad \text{is finite.}$$

$\delta_{ij} \triangleq$ Kronecker delta function

$\delta(t) \triangleq$ Dirac delta function

$E\{\cdot\} \triangleq$ Expectation operator

$A' \triangleq$ transpose of A

$$[A_{ij}] \triangleq \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & \cdots & A_{NN} \end{bmatrix}$$

$$\text{diag}[B_i: i=1,\ldots,N] = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_N \end{bmatrix}$$

# 2. SOLUTION OF NONLINEAR EQUATIONS BY DECOMPOSITION

## 2.1 Introduction

An increasingly common approach to solving optimal control problems utilizes the concepts of decomposition and coordination to develop iterative hierarchical control algorithms (Mesarovic, Macko, and Takahara [1970]; Bailey and Laub [1978]; Smith and Sage [1973]; Singh [1976]; Bernhard [1976]; Lasdon [1970]; Geoffrion [1970]; Singh, Hassan, and Titli [1976]; Findeisen and Malinowski [1976]; Chong and Athans [1976]; Lefkowitz [1966]; Findeisen [1975]). Many of these algorithms exhibit fundamental similarities, but until recently a basic theory of decomposition and coordination has not been available. Now, with the recent paper by Cohen [1978] a general framework for decomposition algorithms is beginning to emerge. The major importance of this type of formulation is that it allows attention to center on classes of algorithms rather than the individual algorithm, thus developing greater insight into the essential similarities and differences betwen algorithms.

This chapter develops an alternate framework for the study of the theory of decomposition and coordination. The general decomposition algorithm is formulated and examined from a numerical analysis viewpoint. The result is a framework which includes a larger class of algorithms for differentiable optimization problems than Cohen's formulation. The approach allows the development

of local convergence results without imposing convexity assumptions on the original optimization problem. Such results have not been obtained to date in the hierarchical control literature.

The precise approach used is to generalize a method used in many hierarchical control algorithms (Singh [1976]). This method involves using iterative algorithms to solve the two point boundary value problem resulting from a deterministic optimal control problem. The particular iterative algorithm used determines the structure of the two level hierarchical controller. This approach extends directly to any optimization problem for which a set of necessary conditions can be expressed as a set of possibly nonlinear equations. The class of decomposition algorithms which results is sufficiently general to include all the open loop hierarchical control algorithms for optimal control problems developed to date. Other common iterative solutions to optimization problems fit the framework also.

Many of the decomposition algorithms governed by the decomposition theory of this chapter can be given hierarchical interpretations. The result is invariably a two level structure. However, the decomposition framework is further developed to allow multiple decompositions of the same problem. Thus multilevel hierarchical structures can be considered. Specific results are derived for three level structures which result from the use of two arbitrary decompositions.

The contents of the chapter are as follows. Section 2.2 provides a brief summary of the needed variational results. Section 2.3 formulates the decomposition framework for nonlinear equations, and relates the framework both to Cohen's work and to the earlier hierarchical formulation of Mesarovic et.al. [1970]. Multiple decompositions of a single problem are discussed in Section 2.4. Section 2.5 summarizes and discusses the results of this chapter.

## 2.2 Derivatives in Abstract Spaces

The concepts of derivatives and differentials in Banach spaces will be needed to relate the nonlinear iterative methods to the linear techniques and for the convergence analysis of the iterations. In the following, let X and Y be normed linear spaces and let f be a possibly nonlinear transformation.

$$f: X \to Y \tag{2.2.1}$$

Definition: (Luenberger [1969]) If for $x \in X$ and each $h \in X$ there exists $\delta f(x;h) \in Y$ which is linear and continuous with respect to h such that

$$\lim_{||h|| \to 0} \frac{||f(x+h)-f(x)-\delta f(x;h)||}{||h||} = 0$$

then f is Fréchet differentiable at x and $\delta f(x;h)$ is the Fréchet differential of f at x with increment h.

The Fréchet differential posesses the following properties.

Lemma 2.1: If the function f of (2.2.1) is Fréchet differentiable at x then it is continuous at x. The Fréchet differential is unique and is given by

$$\delta f(x;h) = \frac{d}{d\alpha} f(x + \alpha h)\Big|_{\alpha=0} \tag{2.2.2}$$

Proof: Luenberger [1969], pp. 176-177.

By definition, if f is Fréchet differentiable at a point x, the differential is of the form

$$\delta f(x;h) = f'(x)h \tag{2.2.3}$$

where

$$f': X \to L(X,Y) \triangleq \{\text{bounded linear operators } L: X \to Y\}$$

The transformation f' is the Fréchet derivative of f and the linear operator f'(x) is the Fréchet derivative of f at x. If the function f is a functional on X (i.e., if Y = R), then f'(x) is often referred to as the gradient of f at x. If f' is continuous on an open set U c X then f is said to be continuously Fréchet differentiable on U.

Since $L(X,Y)$ is a normed linear space, the theory of Fréchet differentials can be applied to the function f'. If the Fréchet derivative of f' exists, it is referred to as the second Fréchet derivative of f, and is denoted f":

$$f'': X \to L(X, L(X,Y)) \tag{2.2.4}$$

If f" is continuous then f is said to be twice continuously Fréchet differentiable. The theory and terminology can be extended to all higher order derivatives in a similar manner. The following notation will be used to denote the $i^{th}$ Fréchet derivative evaluated at $x \in X$:

$$f^{(i)}(x) \in L^{(i)}(X,Y)$$

where $L^{(i)}(X,Y)$ is defined recursively by:

$$L^{(1)}(X,Y) = L(X,Y)$$

$$L^{(i)}(X,Y) = L(X, L^{(i-1)}(X,Y)).$$

Much of the theory of ordinary derivatives extends to Fréchet derivatives. For example, the concept of partial derivatives has a straightforward extension. Let $f:X_1 \times X_2 \times \ldots \times X_n \to Y$ where $X_1,\ldots,X_n$, and Y are normed linear spaces. Then the $i^{th}$ partial Fréchet derivative at $(x_1,\ldots,x_n)$ is defined as the unique map

$$\partial_i f: X_1 \times \ldots \times X_n \to L(X_i,Y)$$

such that for each $h \in X_i$

$$\lim_{||h|| \to 0} \frac{||f(x_1,\ldots,x_i + h,\ldots,x_n) - f(x_1,\ldots,x_n) - \partial_i f(x_1,\ldots,x_n)h||}{||h||} = 0$$

if the limit exists.

Other extensions of the theory of ordinary derivatives which will prove useful are the chain rule for differentiating compositions of functions, the Taylor series expansion of a function, and the implicit function theorem. These are given by the following three theorems.

Theorem 2.1: (Chain Rule) Let X,V,Y be normed linear spaces. Suppose $g: X \to V$ and $h: V \to Y$ are Fréchet differentiable at x and g(x) respectively. Then the composite map $f: X \to Y$ given by

$$f(x) = h(g(x)) \tag{2.2.5}$$

is Fréchet differentiable at x and

$$f'(x) = h'(g(x))g'(x) \tag{2.2.6}$$

Proof: Ortega and Rheinboldt [1970], p. 62.

Theorem 2.2: (Taylor expansion) Let f be n-times Fréchet differentiable at $x^*$. Then there is an open neighborhood $U \subset X$ of $x^*$ such that for each $x \in U$ and $1 \leq m \leq n$

$$f(x) = f(x^*) + \sum_{i=1}^{m} \frac{1}{i!} f^{(i)}(x^*)(x-x^*)^i + o(||x-x^*||^m) \tag{2.2.7}$$

where $f^{(i)}: X \to L^{(i)}(X,Y)$ is the $i^{\text{th}}$ Fréchet derivative and:

$$f^{(i)}(x^*)(x-x^*)^i \triangleq [\ldots[[f^{(i)}(x^*)](\underbrace{x-x^*)]\ldots](x-x^*)}_{i \text{ times}}$$

Proof: Lusternik and Sobolev [1968]

Theorem 2.3: (Implicit Function Theorem) Let X and Y be Banach spaces. Suppose that $f: X \times Y \to X$ is continuous in an open neighborhood $U_o$ of a point $(x_o,y_o)$ for which $f(x_o,y_o)=0$. Assume that $\partial_1 f$ exists in a neighborhood of $(x_o,y_o)$, is continuous at $(x_o,y_o)$ and $\partial_1 f(x_o,y_o)$ is nonsingular. Then there exist open neighborhoods $U_1 \subset X$ of $x_o$ and $U_2 \subset Y$ of $y_o$ such that for each $x \in U_2$ the equation

$$f(x,y) = 0 \tag{2.2.8}$$

has a unique solution

$$x = \hat{f}(y) \tag{2.2.9}$$

and the mapping $\hat{f}: U_2 \to X$ is continuous. Also, if $\partial_2 f$ exists at $(x_o,y_o)$ then $\hat{f}$ is Fréchet differentiable at $y_o$ and

$$\hat{f}'(y_o) = -[\partial_1 f(x_o,y_o)]^{-1} \partial_2 f(x_o,y_o) \tag{2.2.10}$$

Proof: Ortega and Rheinboldt [1970], pp. 128-129.

## 2.3 Decomposition of Nonlinear Equations

Consider the following optimization problem:

$$\min_{u \in U} J(u) \tag{2.3.1}$$

where $U$ is a Banach space and $J$ is twice continuously Frechet differentiable. A necessary condition for $x^*$ to solve (2.3.1) is:

$$J'(u^*) = 0 \tag{2.3.2}$$

Note that $J': U \to L(U)$, and that $L(U)$ is a Banach space. An indirect approach to the solution of (2.3.1) is to solve (2.3.2) for $u^*$.

To insure that $u^*$ is at least locally unique, it will be assumed that $J''(u^*)^{-1}$ exists for all $u^*$ such that $J'(u^*) = 0$. Thus, locally (around $u^*$) $L(U)$ can be identified with $U$ for the purpose of solving (2.3.2). Equation (2.3.2) is then of the form:

$$f(x) = 0, \quad f: X \to X \tag{2.3.3}$$

where X is a Banach space.

Often the nonlinearities of the function f or the dimension of the space $X$ prevent a closed form (or even a finite algorithmic) solution of equation (2.3.3). The usual approach is to try an iterative solution method. Common examples of iterative techniques include Newton's method and (in 1 dimension) the Fibonacci search.

The general approach which will be discussed here is to extend ideas used for the one-point iterative solution of linear equations. Iterative methods for linear equations are well known and commonly used (Varga [1962]; Ortega and Rheinboldt [1970]; Laub [1974]; Athay [1976]). The linear equivalent to (2.3.3) is

$$Ax = b \tag{2.3.4}$$

where $A \in L(X)$

The iteration is determined by splitting (decomposing) the operator A:

$$A = A_o + A_1 \; ; \quad A_o, A_1 \in L(X) \tag{2.3.5}$$

and solving the equation

$$A_o x_{k+1} = b - A_1 x_k \; ; \qquad k=0,1,... \tag{2.3.6}$$

The splitting is chosen such that (2.3.6) is more easily solved than (2.3.4). Two common splittings are demonstrated by the following examples.

Example 2.1: (Jacobi Iteration)

$$A = D + L + U \tag{2.3.7}$$

where D is diagonal, L is strictly lower triangular and U is strictly upper triangular. Then let:

$$A_o = D$$
$$A_1 = L + U \tag{2.3.8}$$

□

Example 2.2: (Gauss-Seidel)

Assume D, L and U are defined as in (2.3.7). Then let:

$$A_o = D + L$$
$$A_1 = U \tag{2.3.9}$$

□

Each of the above iterations can be combined with the concept of under or over-relaxation.

Example 2.3: (Under or over-relaxation)

Choose a splitting as in (2.3.5). Then solve

$$A_o \hat{x}_k = b - A_1 x_k \tag{2.3.10}$$

$$x_{k+1} = x_k + \varepsilon(\hat{x}_k - x_k) \tag{2.3.11}$$

The parameter $\varepsilon$ is called the relaxation parameter. The terms under and over-relaxation correspond to $\varepsilon < 1$ and $\varepsilon > 1$ respectively. In either case, relaxation used with the Gauss-Seidel iteration will be referred to as an SOR iteration (successive over-relaxation). □

The same idea can be extended to the nonlinear equation (2.3.3). The first step is to choose a continuously Fréchet differentiable function $f_o: X \times X \to X$ to create the decomposition:

$$f(x) = f_o(x,y) + f_1(x,y) \tag{2.3.12}$$

Then, the equation

$$f_o(x_{k+1}, x_k) + f_1(x_k,x_k) = 0 \tag{2.3.13}$$

is solved for $x_{k+1}$ at the $k+1^{st}$ iteration. To have the iteration (2.3.13) well defined in a neighborhood of the solution $x^*$, it will be assumed that $\partial_1 f_o^{-1}(x^*,x^*)$ exists.

There are several notable aspects of this formulation. In the past, it was assumed that the cost function J, and hence the necessary conditions for the optimization (represented by f), had to be separable to achieve a decomposition. This conception has persisted even in some of the recent literature (Forestier and Varaiya [1978]). However, the formulation of equations (2.3.12)-(2.3.13) make no such assumption. In fact, the linear decentralized stochastic control problem formulation of Chapter 3 is distinctly nonseparable.

A second observation is that the choice of the function $f_o$ determines the iteration. For this reason, $f_o$ is called the core of the decomposition. The terminology used and the form of this framework is similar to the formulation developed by Cohen [1978]. The formulation of (2.3.12)-(2.3.13) is more general however. Cohen's formulation views the decomposition from an optimization viewpoint,

However,

$$\frac{\partial^2}{\partial y \partial x} K_o(x,y,w,z) = 1 \neq -1 = \frac{\partial^2}{\partial x \partial y} K_o(x,y,w,z)$$

which shows the function K is not analytic. □

Finally, it should be noted that regarding the decomposition framework from a numerical analysis viewpoint allows the introduction of a large number of local convergence and existence results which are not available from an optimization point of view. On the other hand, global convergence results for certain decompositions applied to unconstrained optimization problems are obtained more naturally from an optimization viewpoint. Both are important, but the local analysis has been mostly neglected to date.

Equation (2.3.13) defines an equation of the form

$$g(x_{k+1}, x_k) = 0 \tag{2.3.14}$$

This is the general form for a one-point iteration with the requirement

$$g(x^*, x^*) = f(x^*) = 0$$

(i.e., g and f should solve the same problem). The formulation of (2.3.12)-(2.3.13) is more restrictive. By letting

$$x_{k+1} = x_k = x, \qquad x \in X$$

and hence requires a core $K(u,y)$ to be chosen to split the cost $J$ of (2.3.1) in the same manner that $f_o$ splits $f$ in (2.3.12). For a differentiable problem, this is equivalent to choosing the derivative $\partial K(u,y)$ as a core to decompose the nonlinear equation (2.3.2). However, this restricts the class of cores available for the decomposition to those which are derivatives of a function $K$.

Example 2.4: Let $J(x,y) = x^2 + y^2$. Then $f$ is given by:

$$f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Suppose the core $f_o$ is chosen to be:

$$f_o(x,y;\ w,z) = \begin{bmatrix} x + y \\ y - x \end{bmatrix}$$

It is easy to show there is no core $K_o$ which satisfies

$$J(x,y) = K_o(x,y,w,z) + K_1(x,y,w,z)$$

and achieves the decomposition corresponding to $f_o$. If there were, then

$$\frac{\partial}{\partial x} K_o(x,y,w,z) = x + y$$

$$\frac{\partial}{\partial y} K_o(x,y,w,z) = y - x$$

in (2.3.14) and using (2.3.12), one finds

$$g(x,x) = f_o(x,x) + f_1(x,x) = f(x) \quad \forall\, x \in X \tag{2.3.15}$$

Although (2.3.12)-(2.3.14) do not allow arbitrary one-point iterations, the formulation is general enough to admit most of the common nonlinear algorithms.

Example 2.5: (Newton's method)

Choose the core as:

$$f_o(x,y) = f'(y)x \tag{2.3.16}$$

Then

$$f_1(x,y) = f(x) - f'(y)x \tag{2.3.17}$$

Combining (2.3.16)-(2.3.17) with (2.3.12)-(2.3.13) gives the equation:

$$x_{k+1} = x_k - f'(x_k)^{-1} f(x_k) \tag{2.3.18}$$

□

Example 2.6: (nonlinear Jacobi and Gauss-Seidel)

Let $X = X_1 \times \ldots \times X_N$ and define $f_i: X_1 \times \ldots \times X_n \to X_i$, $(i=1,\ldots,N)$ by $f(x) = (f_1(x_1,\ldots,x_N),\ldots,f_N(x_1,\ldots,x_N))$. Then the nonlinear

Jacobi iteration is formed by choosing

$$f_o(x,y)_i = f_i(y_1,\ldots,y_{i-1}, x_i y_{i-1},\ldots,y_N);\ i=1,\ldots,N$$

This results is the iteration

$$f_i(x_1^k,\ldots,x_{i-1}^k, x_i^{k+1}, x_{i+1}^k,\ldots,x_N^k) = 0; \quad i=1,\ldots,N \qquad (2.3.19)$$

If the core is chosen as:

$$f_o(x,y)_i = f_i(x_1,\ldots,x_i, y_{i+1},\ldots,y_N); \quad i=1,\ldots,N$$

The nonlinear Gauss-Seidel iteration results:

$$f_i(x_1^{k+1},\ldots,x_i^{k+1}, x_{i+1}^k,\ldots,x_N^k) = 0\ ; \quad i=1,\ldots,N \qquad (2.3.20)$$

□

Example 2.7: (Successive under or over-relaxation)

The SOR concept can be extended to nonlinear equations. Equation (2.3.13) is solved for a value $\hat{x}_k$. Then (2.3.11) is used to compute $x_{k+1}$. To see that this fits the formulation (2.3.12)-(2.3.13), substitute for $\hat{x}_k$ in (2.3.13):

$$f_o(\frac{1}{\varepsilon}[x_{k+1} - (1-\varepsilon)x_k], x_k) + f_1(x_k, x_k) = 0 \qquad (2.3.21)$$

This corresponds to choosing the core.

$$\hat{f}_o(x,y) = f_o(\frac{1}{\varepsilon}[x - (1-\varepsilon)y], y)$$

Then:

$$\hat{f}_1(x,y) = f(x) - f_o(\frac{1}{\varepsilon}[x - (1-\varepsilon)y],y]$$

Since:

$$\hat{f}_1(y,y) = f(y) - f_o(y,y) = f_1(y,y)$$

equation (2.3.21) describes the iteration resulting from the core $\hat{f}_o$. □

Example 2.8: (Quasi-linearization) Let the space X be decomposed as in example 2.6. Define the core $f_o$ by:

$$f_o(x,y) = \text{diag}[\partial_i f_i(y): i=1,\ldots,N]x$$
$$\triangleq f_D'(y)x$$

Then

$$f_1(x,y) = f(x) - f_D'(y)x$$

which when combined with (2.3.12)-(2.3.13) defines the iteration:

$$x_{k+1} = x_k - f_D'(x_k)^{-1}f(x_k)$$

Singh and Titli [1975] have applied the method of Example 2.8 to the two point boundary value problem resulting from a deterministic optimal control problem. □

The structure of the computation and information flow in the Gauss-Seidel and Jacobi iterations leads to a two level hierarchical interpretation. In the usual straightforward interpretation, the upper (supremal) level simply transfers the information (most recent solution) as it is needed. This occurs once at the end of each Jacobi iteration but must be done sequentially for the Gauss-Seidel algorithm. The lower (infimal) level problems solve the nonlinear equations $f_i$, either in parallel (Jacobi) or sequentially (Gauss-Seidel).

Certain problems possess a structure which admits a different two level hierarchical interpretation and allows some of the computation to be shifted to the supremal level. Suppose that the Banach space X can be decomposed into two Banach sub-spaces:

$$X = X^S \times X^I \tag{2.3.22}$$

This leads to the definition of $f_S$ and $f_I$ as in Example 2.6. If the space $X^I$ can be further decomposed:

$$X^I = X^I_1 \times \ldots \times X^I_N \tag{2.3.23}$$

and if the resulting subsystems of equations $f_{Ii}$ (i=1,...,N) are such that:

$$f_{Ii}(x_S, x_1, \ldots, x_N) = \overline{f}_{Ii}(x_S, x_i) \tag{2.3.24}$$

then the application of either the Jacobi or Gauss-Seidel algorithm results in a subsystem of equations $f_I$ which decouples into N subproblems $\bar{f}_{Ii}$. These subproblems can be solved in parallel. At the upper level (see Figure 2.1), the system of equations $f_S$ is solved. The N subproblems are solved at the infimal level.

This hierarchical interpretation fits nicely into the framework developed by Mesarovic et.al. [1970]. The coordination principle determined by this decomposition is the interaction prediction principle. The infimal problems solve their subsystem of equations at the current iteration based on the interactions predicted (by the solution of the supremal equations) from the solutions at the last iteration. This principle is almost trivially applicable[1] to the solution of the set of nonlinear equations (2.3.3). When viewed in the context of solving the optimization problem (2.3.1) the principle will not, in general, be applicable since the decomposition is used to solve the necessary (but not sufficient) conditions (2.3.2). The problem is that applicability requires the coordination predicate to be true globally. A natural extension of the existing terminology is the following.

Definition: A coordination principle is locally applicable at $\bar{x}$ if there exists an open neighborhood $U$ containing $\bar{x}$ such that the

---

[1] In the sense of Mesarovic et.al. [1970]. The definition of applicability of a coordination principle states that whenever the coordination predicate is true, the original problem is solved. For the interaction prediction principle, the coordination predicate states that the predicted values of the interaction variables are the solution values.

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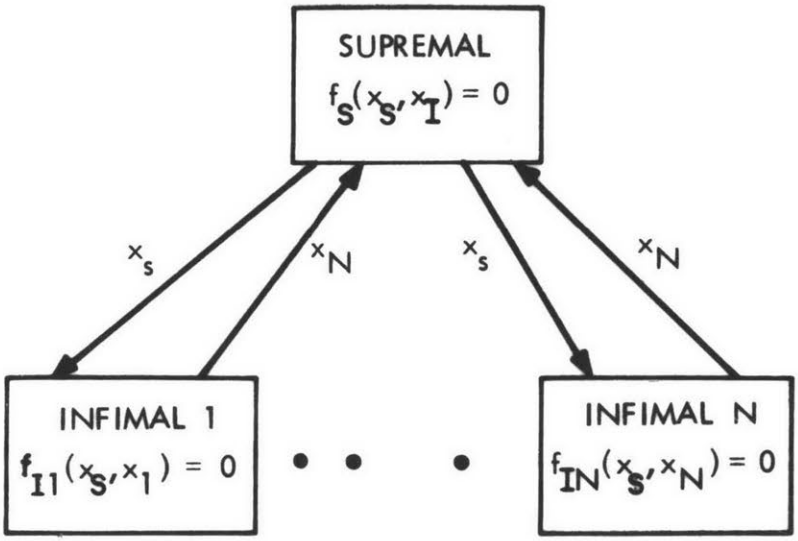


Figure 2.1: Hierarchical Computation Structure

coordination predicate is true for all $x \in U$.

The concept of local coordinability is defined in an analogous manner.

Definition: A problem is locally coordinable at $\bar{x}$ by a given coordination principle if the principle is locally applicable at $\bar{x}$ and there exists a coordination input such that the resulting x solves the overall problem restricted to $U$.

It is easily seen that the optimization problem (2.3.1) is locally coordinable by the coordination principle defined implicitly by the Jacobi or Gauss-Seidel iterations at any local minimum of the original problem.

The type of structure needed for this decomposition is present in the linear stochastic control problem formulation of Chapter 3, and will be exploited by the decomposition presented in Chapter 4. For now, a simple three dimensional problem is developed in Example 2.9 to illustrate the ideas.

Example 2.9: Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$f(x,y,z) = \begin{bmatrix} x - y^2 - z^2 \\ xy^3 - a \\ xz^3 - b \end{bmatrix}$$

Define $X_S = R$ (x is the supremal variable), $X_I = \mathbb{R}^2$ (y and z are the infimal variables) and $f_S$ and $f_I$ by

$$f_S(x,y,z) = x - y^2 - z^2$$

$$f_I(x,y,z) = \begin{bmatrix} xy^3 - a \\ xz^3 - b \end{bmatrix}$$

The Gauss-Seidel iteration will be used. Choose the core to be:

$$f_o(x,y,z;u,v,w) = \begin{bmatrix} x - v^2 - w^2 \\ xy^3 - a \\ xz^3 - b \end{bmatrix}$$

The iteration is:

$$x^{k+1} = (y^k)^2 + (z^k)^2 \tag{2.3.25}$$

$$\left\{ \begin{array}{l} x^{k+1}(y^{k+1})^3 - a = 0 \\ \\ x^{k+1}(z^{k+1})^3 - b = 0 \end{array} \right\} \tag{2.3.26}$$

Equation (2.3.25) is solved at the supremal level while equations (2.3.26) decouple and are solved at the infimal level. □

There are two important considerations which affect the choice of the core. First, equation (2.3.13) should be more easily solved than (2.3.3). This will generally be true for the nonlinear Jacobi

and Gauss-Seidel iterations since the problems are smaller than the original. Newton's method reduces the original problem to the problem of solving a linear set of equations.

The second consideration, of course, is that the sequence $\{x^k\}_{k=0}^{\infty}$ converges to the solution $x^*$ of (2.3.3). The following theorem gives a sufficient condition for the iteration defined by (2.3.14) to converge locally and gives an estimate of the asymptotic rate of convergence.

Theorem 2.4: Let $g: X \times X \to X$ be continuously Frechet differentiable in an open neighborhood $U_o$ of a point $(x^*, x^*) \in X \times X$ for which $g(x^*, x^*) = f(x^*) = 0$. If:

$$\text{(i)} \quad \partial_1 g(x^*, x^*) \text{ is nonsingular}$$

$$\text{(ii)} \quad \gamma = \rho[-\partial_1 g(x^*, x^*)^{-1} \partial_2 g(x^*, x^*)] < 1 \tag{2.3.27}$$

then there exists an open neighborhood $U \subset U_o$ such that for any $x^o \in U$ there is a unique sequence $\{x^k\}_{k=0}^{\infty}$ which satisfies the iteration (2.3.14). Moreover, $\lim_{k\to\infty} x^k = x^*$ and for each $\varepsilon > 0$ there exists an integer $k_o$ such that for all $k > k_o$

$$||x^k - x^*|| \leq (\gamma+\varepsilon)^k \tag{2.3.28}$$

Proof: Ortega and Rheinboldt [1970], pp. 325-326.

The linear operator $-\partial_1 g(x^*,x^*)\ \partial_2 g(x^*,x^*)$ is called the (linearized) iteration operator and the scalar $\gamma$ is the asymptotic convergence rate.

Theorem 2.4 is not in general useful for determining a priori whether the iteration will converge for a given problem since conditions (i) and (ii) make use of the presumably unknown solution. However, this result is used in Chapter 4 to prove a weak coupling condition for the convergence of the decomposition applied to the linear stochastic problem formulation of Chapter 3. One case when condition (2.3.27) can be used directly is if it can be shown to hold for each possible solution $x \in X$. Then local convergence is assured. This is illustrated by the following example.

Example 2.10: Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be defined as in example 2.9 with the same Gauss-Seidel iteration. Then g is defined by:

$$g(x_{k+1}, y_{k+1}, z_{k+1}; x_k, y_k, z_k) = \begin{bmatrix} x_{k+1} - y_k^2 - z_k^2 \\ x_{k+1}\, y_{k+1}^3 - a \\ x_{k+1}\, z_{k+1}^3 - b \end{bmatrix} \qquad (2.3.29)$$

Then, at x:

$$\gamma = \rho\left\{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{y}{3x} & \frac{1}{3xy^2} & 0 \\ -\frac{z}{3x} & 0 & \frac{1}{3xz^2} \end{bmatrix} \begin{bmatrix} 0 & 2y & 2z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

$$= \rho \left\{ \begin{bmatrix} 0 & 2y & 2z \\ 0 & -\frac{y^2}{3x} & -\frac{yz}{3x} \\ 0 & -\frac{yz}{3x} & -\frac{z^2}{3x} \end{bmatrix} \right\} \tag{2.3.30}$$

The characteristic equation of the bracketted matrix is:

$$s^3 + \frac{2}{3}\left(\frac{y^2+z^2}{x}\right)s^2 = 0 \tag{2.3.31}$$

Since $x = y^2 + z^2$ for any possible solution $(x,y,z)$, the spectral radius of the operator evaluated at any point on the submanifold of possible solutions is:

$$\gamma = \frac{2}{3} \tag{2.3.32}$$

□

Condition (2.3.27) can also be used to give insight into the structure which g must posess to generate a convergent iteration. This can be seen more clearly by examining the first order Taylor expansion of g about the solution $x^*$:

$$g(x^k, x^{k-1}) = \partial_1 g(x^*,x^*)\Delta x^k + \partial_2 g(x^*,x^*)\Delta x^{k-1} + o(||x^k - x^*||) \tag{2.3.33}$$

where

$$\Delta x^k \triangleq x^k - x^*$$

If the contribution from the higher order terms were negligible (as would be the case for $x^k \approx x^{k-1} \approx x^*$), equations (2.3.33) and

(2.3.14) could be combined to give an equation which is <u>linear</u> in $x^k$ and $x^{k-1}$:

$$\partial_1 g(x^*,x^*)\ x^k + \partial_2 g(x^*,x^*)\ x^{k-1} = [\partial_1 g(x^*,x^*) + \partial_2 g(x^*,x^*)]x^* \tag{2.3.34}$$

The condition for convergence of the iteration (2.3.34) is identical to (2.3.27).

This analysis also provides additional insight into the relationship between the linear iterative procedure described by equations (2.3.4)-(2.3.6) and the nonlinear decomposition procedure formulated earlier in this section. Equation (2.3.3) can be expanded in a first order Taylor series about $x^*$:

$$0 = f'(x^*)(x-x^*) + o(||x-x^*||) \tag{2.3.35}$$

Again ignoring the higher order terms, (2.3.35) can be rewritten:

$$f'(x^*)\ x = f'(x^*)x^* \tag{2.3.36}$$

The imposed condition (2.3.15) and the properties of partial Fréchet derivatives give

$$f'(x^*) = \partial_1 g(x^*,x^*) + \partial_2 g(x^*,x^*) \tag{2.3.37}$$

Note the correspondence between (2.3.34), (2.3.36)-(2.3.37) and (2.3.4)-(2.3.6). Thus the decomposition procedure (2.3.12)-(2.3.14)

can be viewed as choosing a splitting (as in the linear iteration) of the Fréchet derivative. However, the method is slightly more general since by choosing $f_o$ one determines splittingsof the higher order derivatives also.

## 2.4 Composition of Decompositions

Section 2.3 presented a general formulation for the solution of nonlinear equations by decomposition. One of the objectives which is to be satisfied in the choice of the core of the decomposition is to have equation (2.3.13) be more easily solved than the original equation (2.3.3). However, the equation to be solved (2.3.13) may still be nonlinear (as in the Jacobi and Gauss-Seidel algorithms) or too large to be solved directly. One possible solution is to decompose the iteration equation (2.3.14) to obtain a secondary iteration. An overall iteration defined by the use of a secondary decomposition to solve a set of equations which have resulted from another decomposition will be called a compound iteration. The question of convergence of such a compound iteration will be answered by Theorem 2.5.

Consider the primary one-point iteration defined by:

$$g_p(x_{k+1}, x_k) = 0 \tag{2.4.1}$$

As in Section 2.3, a continuously Frechet differentiable core $g_o$ is chosen to decompose (2.4.1).

$$g_p(x,z) = g_o(x,y,z) + g_1(x,y,z) \qquad \forall\, x,y,z, \in X \tag{2.4.2}$$

Again it is assumed that at the solution $x^*$, $\partial_1 g_o(x^*,x^*,x^*)^{-1}$ exists. The decomposition defines the secondary iteration:

$$g_s(x_{k+1}^{\ell+1}, x_{k+1}^{\ell}, x_k) = g_o(x_{k+1}^{\ell+1}, x_{k+1}^{\ell}, x_k) + g_1(x_{k+1}^{\ell}, x_{k+1}^{\ell}, x_k) = 0 \tag{2.4.3}$$

It will be assumed that the secondary iteration is repeated M times for each value of k. The resulting solution $x_{k+1}^M$ is used to start the secondary iteration at primary iteration k+1. The issue is whether the sequence $\{x_k^M\}_{k=0}^{\infty}$ converges to a value $x^*$ for which $g_p(x^*;x^*) = 0$.

By the implicit function theorem, equation (2.4.3) defines a function $\bar{g}$ in a neighborhood $U_1 \subset X$ of $x^*$:

$$x_{k+1}^{\ell+1} = \bar{g}(x_{k+1}^{\ell}, x_k) \qquad \forall\, x_{k+1}^{\ell}, x_k \in U_1 \subset X \tag{2.4.4}$$

Using $x_k^M$ for $x_k$ as the starting point of (2.4.4), $x_{k+1}^M$ is given by:

$$x_{k+1}^M = \underbrace{\bar{g}(\bar{g}(\ldots\bar{g}}_{M \text{ times}}(x_k^M, x_k^M), x_k^M)\ldots), x_k^M) \tag{2.4.5}$$

Thus $\{x_k^M\}_{k=0}^{\infty}$ converges if and only if (2.4.5) is a convergent iteration. Sufficient conditions for the convergence of $\{x_k^M\}_{k=0}^{\infty}$ are given by the following theorem.

Theorem 2.5: Assume $g_p$, $g_s$ and $\bar{g}$ as defined in (2.4.1)-(2.4.5) are continuously Fréchet differentiable in an open neighborhood $U_o \subset X$ of a point $x^* \in X$ for which $g_p(x^*,x^*) = 0$, and that $\partial_1 g_p(x^*,x^*)$ and $\partial_1 g_s(x^*,x^*,x^*)$ are nonsingular.

Define $\Gamma_s \triangleq - \partial_1 g_s(x^*,x^*,x^*)^{-1} \partial_2 g_s(x^*,x^*,x^*)$

$\Gamma_p \triangleq - \partial_1 g_p(x^*,x^*)^{-1} \partial_2 g_p(x^*,x^*)$

If:

$$\gamma = \rho\{\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p\} < 1 \tag{2.4.6}$$

then there exists an open neighborhood $U \subset X$ of $x^*$ such that for each $x_0^0 \in U$ the sequence $\{x_k^M\}_{k=1}^{\infty}$ defined by (2.4.1)-(2.4.5) converges to $x^*$. In addition, for each $\varepsilon > 0$ there is a positive integer $k_o$ such that

$$||x_k^M - x^*|| < (\gamma+\varepsilon)^k \qquad \forall k \geq k_o \tag{2.4.7}$$

Proof: To simplify notation the arguments of the Fréchet derivatives will not be explicitly specified. All derivatives will be evaluated

at the solution $x^*$.

By Theorem 2.4 and equation (2.4.5), the iteration will converge and satisfy (2.4.7) if

$$\gamma = \rho\{\partial \underbrace{\bar{g}(\bar{g}(\ldots\bar{g}}_{n \text{ times}}(x^*,x^*), x^*)\ldots), x^*)\} < 1 \tag{2.4.8}$$

By direct calculation, condition (2.4.8) becomes:

$$\gamma = \rho\{\partial_1 \bar{g}^M + \sum_{i=0}^{M-1} (\partial_1 \bar{g})^i \partial_2 \bar{g}\} < 1 \tag{2.4.9}$$

From the implicit function theorem, $\partial_1 \bar{g}$ and $\partial_2 \bar{g}$ are

$$\partial_1 \bar{g} = - \partial_1 g_s^{-1} \partial_2 g_s \triangleq \Gamma_s \tag{2.4.10}$$

$$\partial_2 \bar{g} = - \partial_1 g_s^{-1} \partial_3 g_s \tag{2.4.11}$$

Using equation (2.4.3) and the definition of $\Gamma_s$, it is seen that:

$$\partial_1 g_s = \partial_1 g_o \tag{2.4.12}$$

$$\partial_3 g_s = \partial_3 g_o + \partial_3 g_1 \tag{2.4.13}$$

$$\Gamma_s = - \partial_1 g_o^{-1} (\partial_2 g_o + \partial_2 g_1 + \partial_1 g_1) \tag{2.4.14}$$

By equation (2.4.2)

$$\partial_1 g_1 = \partial_1 g_p - \partial_1 g_o \tag{2.4.15}$$

$$\partial_2 g_1 = - \partial_2 g_o \tag{2.4.16}$$

$$\partial_3 g_o + \partial_3 g_1 = \partial_2 g_p \tag{2.4.17}$$

Combining (2.4.12)-(2.4.16) gives

$$\partial_1 g_s^{-1} = (I - \Gamma_s)\partial_1 g_p^{-1} \tag{2.4.18}$$

Substituting (2.4.17)-(2.4.18) into (2.4.11):

$$\begin{aligned} \partial_2 \bar{g} &= - (I - \Gamma_s)\partial_1 g_p^{-1} \partial_2 g_p \\ &= (I - \Gamma_s)\Gamma_p \end{aligned} \tag{2.4.19}$$

Then condition (2.4.9) becomes:

$$\gamma = \rho\{\Gamma_s^M + \sum_{i=0}^{M-1} \Gamma_s^i (I - \Gamma_s)\Gamma_p\} < 1 \tag{2.4.20}$$

This can be rewritten as:

$$\gamma = \rho\{\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p\} < 1 \tag{2.4.21}$$

□

The following example illustrates the concepts involved by analyzing the use of a compound iteration to solve a system of linear equations.

Example 2.11: For the linear problem (Equations (2.3.4)-(2.3.6)), a secondary iteration corresponds to choosing a second splitting to solve (2.3.6):

$$A_o = A_{oo} + A_{o1} \tag{2.4.22}$$

The total iteration (the linear analog of (2.4.4)) becomes:

$$x_{k+1}^{\ell+1} = -A_{oo}^{-1} A_{o1} x_{k+1}^{\ell} - A_{oo}^{-1} A_{o1} x_k + A_{oo}^{-1} b \tag{2.4.23}$$

Using $x_{k+1}^{o} = x_k = x_k^M$ and the discrete variation of constants formula, the linear analog of (2.4.5) is obtained:

$$\begin{aligned} x_{k+1}^{M} &= (-A_{oo}^{-1} A_{o1})^M x_k^M + \sum_{i=0}^{M-1} (-A_{oo}^{-1} A_{01})^i (A_{oo}^{-1} b - A_{oo}^{-1} A_1 x_k^M) \\ &= \left[ \Gamma_s^M - \sum_{i=0}^{M-1} \Gamma_s^i A_{oo}^{-1} A_1 \right] x_k^M + \sum_{i=0}^{M-1} \Gamma_s^i A_{oo}^{-1} b \end{aligned} \tag{2.4.24}$$

Note that $\Gamma_s \triangleq - A_{oo}^{-1} A_{o1}$ is the iteration operator corresponding to the splitting (2.4.22). Now:

$$\begin{aligned} A_{oo}^{-1} A_1 &= A_{oo}^{-1}(A_{oo}+A_{o1})(A_{oo}+A_{o1})^{-1} A_1 \\ &= (I + A_{oo}^{-1} A_{o1}) A_o^{-1} A_1 \\ &= - (I - \Gamma_s)\Gamma_p \end{aligned} \tag{2.4.25}$$

where $\Gamma_p \triangleq - A_o^{-1} A_1$ is the iteration operator corresponding to the

splitting (2.3.5). Using (2.4.25) in (2.4.24):

$$x_{K+1}^{M} = \left[ \Gamma_s^M + \sum_{i=0}^{M-1} \Gamma_s^i (I - \Gamma_s)\Gamma_p \right] x_k^M + \sum_{i=0}^{M-1} \Gamma_s^i A_{oo}^{-1} b$$

$$= \left[ \Gamma_s^M + (I - \Gamma_s^M)\Gamma_p \right] x_k^M + \sum_{i=0}^{M-1} \Gamma_s^i A_{oo}^{-1} b \qquad k=0,1,\ldots, \tag{2.4.26}$$

A necessary (and sufficient) condition for (2.4.26) to converge to the solution of (2.3.4) is

$$\rho\{\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p\} < 1 \tag{2.4.27}$$

which is equivalent to (2.4.6). □

As with Theorem 2.4, condition (2.4.6) of Theorem 2.5 would not often be used to determine convergence in a practical application. However, the theorem does give some insight into the interaction of the primary and secondary iterations. If Newton's method is used as the secondary iteration, the convergence condition simplifies (since $\Gamma_s=0$ for Newton's method; see Ortega and Rheinboldt [1970], p. 311)

$$\rho\{\Gamma_p\} < 1$$

This is just the condition for local convergence of the primary iteration, and is independent of the number of steps in the secondary

iteration.[1] Also, Theorem 2.5 can be used to show that any convergent decomposition can be used as a secondary iteration with an arbitrarily small increase in the overall convergence rate if the number of steps M is sufficiently large.

Theorem 2.6: Assume the conditions of theorem 2.5 hold, and:

$$\rho\{\Gamma_s\} < 1$$

Then for each $\varepsilon > 0$ there is an $M_o$ such that

$$\rho\{\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p\} \leq \rho\{\Gamma_p\} + \varepsilon \quad \forall M \geq M_o \tag{2.4.28}$$

Proof: Since the spectral radius $\rho\{\cdot\}$ is a continuous function of the linear operator, given $\Gamma_p$ and $\varepsilon$ there exists a $\delta$ such that

$||A - \Gamma_p|| < \delta$ implies $|\rho\{A\} - \rho\{\Gamma_p\}| < \varepsilon$.

Then:

$$||\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p - \Gamma_p|| = ||\Gamma_s^M(I - \Gamma_p)||$$

$$\leq ||\Gamma_s^M|| \; ||I - \Gamma_p||$$

Since $\rho\{\Gamma_s\} < 1$, choose $M_o$ such that

---

[1] Of course, the region of convergence and the average (as opposed to asymptotic) rate of convergence may be affected by the number of steps in the secondary iteration.

$$||\Gamma_s^M|| < \frac{\delta}{||I-\Gamma_p||} \qquad \forall M \geq M_o$$

Thus:

$$||\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p - \Gamma_p|| < \delta$$

which gives:

$$|\rho\{\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p\} - \rho\{\Gamma_p\}| < \varepsilon$$

□

The concept of a secondary iteration and the implications of Theorems 2.5 and 2.6 are demonstrated by the following example.

Example 2.12: Consider the primary iteration defined in Example 2.9 (equations (2.3.25)-(2.3.26)). Equation (2.3.26) involves the solution of two decoupled cubic equations. These will be solved by the following iteration:

$$\begin{aligned} x_{k+1}\, y_{k+1}^{\ell}(y_{k+1}^{\ell+1})^2 - a &= 0 \\ x_{k+1}\, z_{k+1}^{\ell}(z_{k+1}^{\ell+1})^2 - b &= 0 \end{aligned} \qquad (2.4.29)$$

Each equation of (2.4.29) can be written in the form:

$$\alpha^{\ell}(\alpha^{\ell+1})^2 - d = 0 \qquad (2.4.30)$$

The iteration operator for (2.4.30) is:

$$\gamma_s = - \frac{1}{2\alpha^{\ell}\alpha^{\ell+1}} \left(\alpha^{\ell+1}\right)^2 \Bigg|_{\alpha^{\ell}=\alpha^{\ell+1}=\alpha^*} = - \frac{1}{2} \tag{2.4.31}$$

Thus the iteration operator for the whole secondary iteration (noting that equation (2.3.25) is solved exactly) is:

$$\Gamma_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \tag{2.4.32}$$

Since $\Gamma_s$ does not depend on the solution, local convergence is assured.

Convergence condition (2.4.6) for the composite iteration with M secondary steps is:

$$\gamma = \rho\{\Gamma_s^M + (I - \Gamma_s^M)\Gamma_p\}$$

$$= \rho\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}^M & 0 \\ 0 & 0 & -\frac{1}{2}^M \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-\left(-\frac{1}{2}\right)^M & 0 \\ 0 & 0 & 1-\left(-\frac{1}{2}\right)^M \end{bmatrix} \begin{bmatrix} 0 & 2y & 2z \\ 0 & -\frac{y^2}{3x} & -\frac{yz}{3x} \\ 0 & -\frac{yz}{3x} & -\frac{z^2}{3x} \end{bmatrix}\right\}$$

$$= \rho \left\{ \left(-\frac{1}{2}\right)^M I + \left[1-\left(-\frac{1}{2}\right)^M\right] \begin{bmatrix} -\frac{y^2}{3x} & -\frac{yz}{3x} \\ -\frac{yz}{3x} & -\frac{z^2}{3x} \end{bmatrix} \right\}^* \tag{2.4.33}$$

Now, if $\lambda$ is an eigenvalue of

$$\begin{bmatrix} -\frac{y^2}{3x} & -\frac{yz}{3x} \\ -\frac{yz}{3x} & -\frac{z^2}{3x} \end{bmatrix} \tag{2.4.34}$$

then $\left(-\frac{1}{2}\right)^M + \left[1-\left(-\frac{1}{2}\right)^M\right]\lambda$ is an eigenvalue of

$$\left(-\frac{1}{2}\right)^M I + \left[1-\left(-\frac{1}{2}\right)^M\right] \begin{bmatrix} -\frac{y^2}{3x} & -\frac{yz}{3x} \\ -\frac{yz}{3x} & -\frac{z^2}{3x} \end{bmatrix} \tag{2.4.35}$$

In Example 2.8, it was seen that the eigenvalues of (2.4.34) at any possible solution value were:

$$\lambda_1 = -\frac{2}{3} \quad , \quad \lambda_2 = 0 \tag{2.4.36}$$

Thus, the convergence condition is:

$$\gamma = \max \left\{ \frac{1}{2}^M \; , \; \left| \left(-\frac{1}{2}\right)^M - \frac{2}{3}\left[1-\left(-\frac{1}{2}\right)^M\right] \right| \right\} < 1 \tag{2.4.37}$$

There are two properties worth noting. First, as M becomes large $\gamma$ approaches $\frac{2}{3}$ (the asymptotic rate of the primary iteration) as predicted by Theorem 2.6. Second, the smallest value of $\gamma$ ($\frac{1}{4}$) occurs for M=2, and this rate is significantly less than the primary rate of convergence. Thus it may be advantageous to use a secondary iteration without carrying the solution to the limit.

The composite iteration with a = -b = 2 was simulated for several different initial guesses, and several values of M. The iteration was continued until $||\underline{x} - \underline{x}^*|| < 10^{-6}$ (where $\underline{x} = (x,y,z)$). The results are contained in Table 2.1. The observed asymptotic rate of convergence agrees in each case with the value predicted by equation (2.4.6). Note that the number of iterations for M=2 is significantly less than for any other value. □

The hierarchical interpretation which was given the primary decomposition in Section 2.3 can be extended to the compound iteration described by (2.4.1)-(2.4.5). In this case the lower level of the original two level hierarchy also becomes a two-level hierarchy. The result (Figure 2.2) is a three level hierarchical structure. This same procedure can be repeated for any problem at any level.

## 2.5 Summary

This chapter has presented a general framework for the solution of nonlinear equations using decomposition algorithms. The class

Table 2.1: Convergence rates for Example 2.10

| Secondary Iterations M | Intial Guess ($y_o = -x_o$) | Iterations (k) | Observed Rate $\|\|\underline{x}-\underline{x}^*\|\|^{1/k}$ | Predicted Rate |
|---|---|---|---|---|
| 1 | 1.0005 | * | * | $\frac{3}{2}$ |
| 2 | $\frac{1}{2}$ | 11 | .259 | $\frac{1}{4}$ |
| | $\frac{3}{2}$ | 10 | .246 | |
| | 50 | 12 | .298 | |
| 3 | $\frac{1}{2}$ | 119 | .890 | $\frac{7}{8}$ |
| | $\frac{3}{2}$ | 103 | .873 | |
| | 50 | 120 | .891 | |
| 4 | $\frac{1}{2}$ | 25 | .571 | $\frac{9}{16}$ |
| | $\frac{3}{2}$ | 24 | .559 | |
| | 50 | 28 | .607 | |
| $\infty$ | $\frac{1}{2}$ | 36 | .674 | $\frac{2}{3}$ |
| | $\frac{3}{2}$ | 34 | .664 | |
| | 50 | 40 | .703 | |

* Did not converge

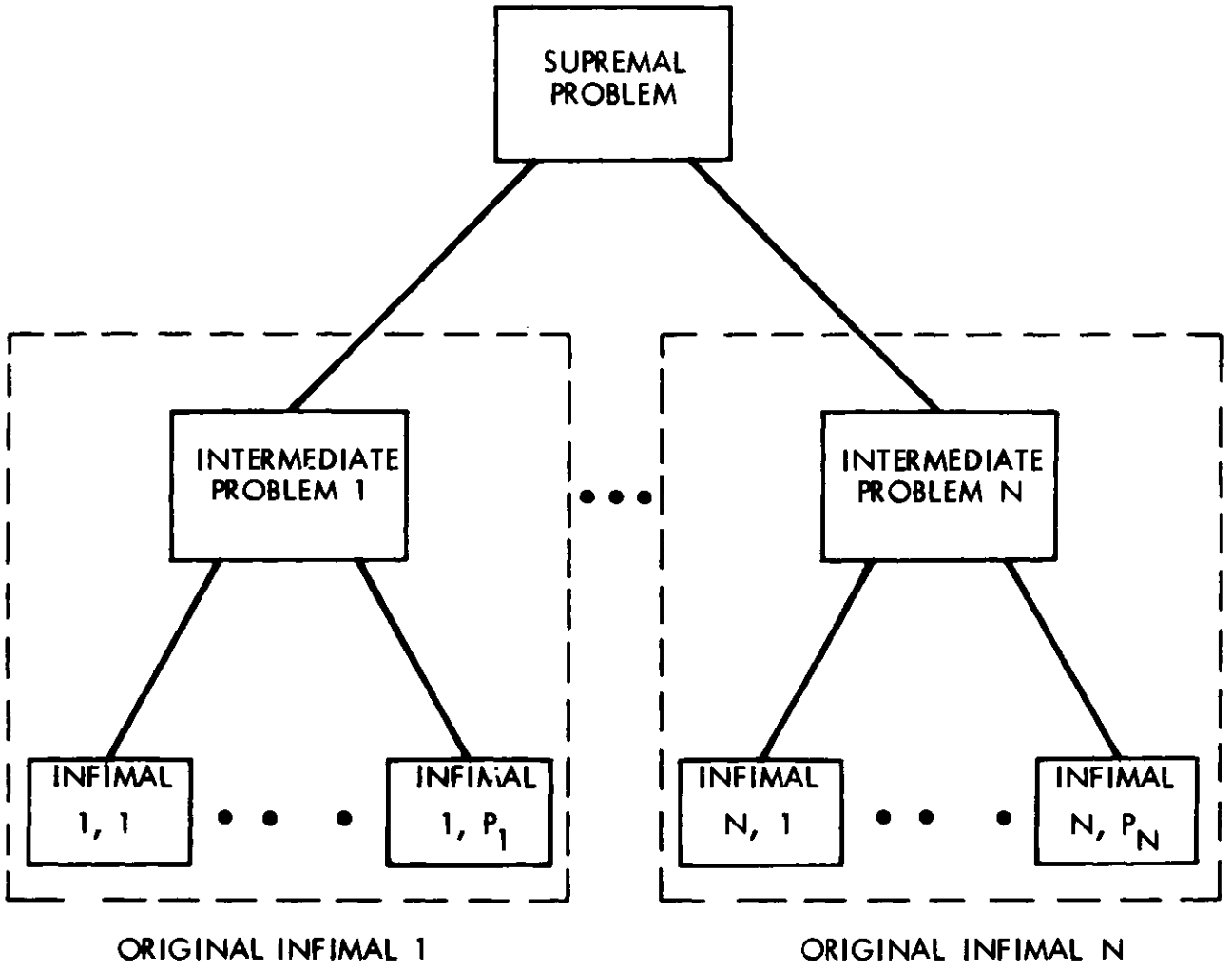


Figure 2.2: Hierarchical Structure of a Compound Decomposition

of algorithms covered by this formulation include many of the classical iterative algorithms (such as Newton's method, and the Jacobi and Gauss-Seidel iterations) as well as many of those algorithms currently used in the hierarchical control literature. The general framework allows the essential similarities and differences of individual algorithms to be clarified. In addition, the formulation provides a constructive approach to designing decomposition algorithms through the ability to specify the decomposition by the choice of the core.

The formulation differs from other decomposition theories in that it approaches the decomposition from a numerical analysis (as opposed to an optimization) point of view. This viewpoint introduces a theory of local convergence to the theory of hierarchical control. The local convergence results provide an important new compliment to the global convergence theory most often found in the hierarchical literature.

Another new development of the formulation of this chapter is the ability to construct and analyze multilevel hierarchies. The formulation of Section 2.4 allows the hierarchical structure of the computation to be tailored to the problem at hand. A local convergence criterion, given in terms of the individual decomposition iteration matrices, was developed for a three-level structure.

The criterion can be used inductively to analyze the local convergence behavior of any multilevel algorithm developed under the framework of Sections 2.3-2.4.

There is a final, important consideration in the development of a particular decomposition algorithm which has been largely neglected. To be successful, the core of a decomposition must be chosen to take advantage of the structure of the problem which is to be solved. There is no widely applicable rule which will result in the choice of a core that simplifies the problem and results in a convergent iteration. Certain classes of problems posess a structure which can be exploited through a corresponding core structure. An example of such a class is developed in the remainder of this thesis. However, insight into a problem remains essential to any practical decomposition.

# 3. THE LINEAR DECENTRALIZED STOCHASTIC CONTROL PROBLEM

## 3.1 Introduction

As discussed in Chapter 1, the linear stochastic control problem formulation is often used as the lower level of a two level control structure in the solution of nonlinear stochastic optimal control problems. If no restrictions on information flow are imposed (i.e., each input can use the entire output history of the system), the (centralized) solution to the linear stochastic control problem is known to be linear and the separation principle applies (Wonham [1968]). Due to the on-line computational and communication requirements of the centralized solution it is often necessary to restrict the amount of information which is available to each input. In general, the optimal control for such problems is no longer linear and the separation principle does not apply. Also, the solution is usually difficult or impossible to compute.

In an attempt to reduce the complexity of the non-classical stochastic control problem, many authors have restricted the class of permissible controls (Sage and Eisenberg [1966]; Levine and Athans [1970]; Kosut [1970]; Chong and Athans [1971]; Levine et.al. [1971]; Wang [1972]; Cohen [1977]; Looze et.al. [1978]). By far the most popular class has been the class of linear controllers with specified dimension which satisfy the information flow

constraints. The reasons are obvious; the class is characterized by a finite number of variables and the solution is easily implemented. A direct result of this approach is that the stochastic optimization problem can be reformulated as a constrained deterministic optimization over the variables which characterize the linear control system. This approach to the linear stochastic control problem is developed in Sections 3.2 and 3.3.

Several authors have derived necessary conditions for special cases of the resulting deterministic optimization. Galiana, et.al. [1973] used a Lagrange multiplier approach, Cohen [1977] used the Pontruyagin minimum principle, and Chong and Athans [1971] used the matrix minimum principle. A variational approach is used in Section 3.4. This approach has two advantages over previous approaches. First, the derivation of the necessary conditions demonstrates explicitly the role of the adjoint equation. The second advantage is that arbitrary parameterizations of the control system fit naturally into the variational framework. The result is a more general set of necessary conditions of which each of the above referencesis a special case.

The remainder of Section 3.4 demonstrates how the general necessary conditions can be used to derive a set of conditions for a specific problem (namely, for a system consisting of interconnected subsystems). Finally, Section 3.5 summarizes the results of this chapter.

## 3.2 Problem Formulation

Consider the linear stochastic system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \xi(t); \qquad x(t_o)=x_o \tag{3.2.1}$$

$$y(t) = C(t)x(t) + \theta(t) \tag{3.2.2}$$

$$x(t),\ \xi(t)\in \mathbb{R}^n;\ y(t),\ \theta(t)\in \mathbb{R}^p;\ u(t)\in \mathbb{R}^m$$

with:

$$\begin{aligned}
&E\{\xi(t)\} = 0\\
&E\{\theta(t)\} = 0\\
&E\{x_o\} = 0\\
&E\{\xi(t)\theta'(\tau)\} = 0\\
&E\{x_o\ \xi'(t)\} = 0\\
&E\{x_o\ \theta'(t)\} = 0\\
&E\{x_o\ x_o'\} = P_o\\
&E\{\xi(t)\xi'(\tau)\} = \Xi(t)\delta(t-\tau)\\
&E\{\theta(t)\theta'(\tau)\} = \Theta(t)\delta(t-\tau)
\end{aligned} \tag{3.2.3}$$

The objective is to choose u(t) to minimize the quadratic cost functional:

$$J = \frac{1}{2} E\{x'(T)K_T x(T) + \int_{t_o}^{T} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt\}$$

$$K_T = K_T' \geq 0;\ Q(t) = Q'(t) \geq 0;\ R(t) = R'(t) > 0 \tag{3.2.4}$$

For the time invariant infinite horizon case, the time averaged quadratic cost

$$J_\infty = \frac{1}{2} \lim_{T\to\infty} E\{\frac{1}{T} \int_o^T [x'(t)Qx(t) + u'(t)Ru(t)]dt\} \qquad (3.2.5)$$

with the same restrictions on Q and R will be used. All time varying matrices are assumed to have elements which are square integrable over the interval $[t_o,T]$ with respect to Lebesgue measure. Equalities (3.2.1)-(3.2.2) are assumed to hold almost everywhere in t with respect to Lebesgue measure.

The notation used in (3.2.1)-(3.2.2) is a formal representation of a stochastic Ito integral. With the assumptions above, the Ito stochastic process x(t) satisfying (3.2.1) exists and is almost surely continuous (Liptser and Shiryayev [1977], Theorem 4.10).

A major objective of this thesis is to derive control laws for systems in which information flow is restricted in various ways. Since the optimal unconstrained control law for such problems is generally nonlinear (Witsenhausen [1968]) and difficult, if not impossible, to compute and implement, a linear finite dimensional structure is imposed:

$$u(t) = - G(t)\hat{x}(t) \qquad (3.2.6)$$

$$\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) + \hat{B}(t)u(t) + H(t)\ [y(t) - \hat{C}(t)\hat{x}(t)];\ \hat{x}(t_o)=0 \qquad (3.2.7)$$

where $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$ $(\hat{n} < \infty)$

Again, the elements of all time varying matrices are assumed to be square integrable with respect to Lebesgue measure on the interval $[t_o, T]$. The matrices $\hat{A}(t)$, $G(t)$ and $H(t)$ are assumed to be parameterized by $\alpha(t) \in \mathbb{R}^s$ $(s < \infty)$, with $\alpha(t)$ also square integrable on the interval $[t_o, T]$. When the infinite horizon, time invariant problem is considered, the parameterization will depend on the time invariant vector $\alpha \in \mathbb{R}^s$. The problem is to choose $\alpha(t) \in \mathbb{R}^s$ $x[t_o, T]$ (or $\alpha \in \mathbb{R}^s$) to minimize the quadratic cost index (3.2.4) (or (3.2.5)).

Several examples will illustrate the generality of the formulation.

Example 3.1: Kalman Filter Based Compensation

$$\hat{n} = n \; ; \; (\hat{A}(t), \hat{B}(t), \hat{C}(t)) = (A(t), B(t), C(t))$$

$$\alpha(t) = (G(t), H(t))$$

□

Example 3.2: N Interconnected Subsystems

$$\dot{x}_i(t) = A_{ii}(t)x_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} A_{ij}(t)x_j(t) + B_i(t)u_i(t) + \xi_i(t); \; x_i(t_o) = x_{io} \tag{3.2.8}$$

$$y_i(t) = C_i(t)x_i(t) + \theta_i(t)$$

$$x_i(t),\ \xi_i(t) \in \mathbb{R}^{n_i}\ ;\ u_i(t) \in \mathbb{R}^{m_i}\ ;\ y_i(t) \in \mathbb{R}^{p_i}$$

$$E\{\xi_i(t)\} = 0;\ E\{\theta_i(t)\} = 0;\ E\{x_{io}\} = 0$$

$$\left.\begin{aligned} &E\{\theta_i(t)\xi_j'(\tau)\} = 0 \quad E\{\theta_i(t)x_{jo}'\} = 0 \quad E\{\xi_i(t)x_{jo}'\} = 0 \\ &E\{\xi_i(t)\xi_j'(\tau)\} = \Xi_i(t)\delta(t-\tau)\delta_{ij} \\ &E\{\theta_i(t)\theta_j'(\tau)\} = \Theta_i(t)\delta(t-\tau)\delta_{ij} \end{aligned}\right\} \quad j=1,\ldots,N$$

$$E\{x_{io}\ x_{jo}'\} = P_{io}\ \delta_{ij} \qquad i=1,\ldots,N \qquad (3.2.9)$$

$$J = \sum_{i=1}^{N} E\left\{ \int_{t_o}^{T} [x_i'(t)Q_i(t)x_i(t) + u_i'(t)R_i(t)u_i(t)]dt + x_i'(T)K_{T_i}x_i(T) \right\} \qquad (3.2.10)$$

If the time invariant, infinite horizon problem is being considered the cost functional is:

$$J_\infty = \sum_{i=1}^{N} \lim_{T\to\infty} \frac{1}{T} E\left\{ \int_{t_o}^{T} [x_i'(t)Q_i x_i(t) + u_i'(t)R_i u_i(t)]dt \right\} \qquad (3.2.11)$$

The system matrices can be put in the form of (3.2.1)-(3.2.5) by defining:

$$A(t) = [A_{ij}(t)]$$

$$B(t) = \text{diag}[B_i(t);\ i=1,\ldots,N] \quad C(t) = \text{diag}[C_i(t):\ i=1,\ldots,N]$$

$$Q(t) = \text{diag}[Q_i(t):\ i=1,\ldots,N] \quad R(t) = \text{diag}[R_i(t):\ i=1,\ldots,N]$$

$$\Xi(t) = \text{diag}[\Xi_i(t):\ i=1,\ldots,N] \quad \Theta(t) = \text{diag}[\Theta_i(t):\ i=1,\ldots,N]$$

$$K_T = \text{diag}[K_{Ti} : i=1,\ldots,N]$$

$$n = \sum_{i=1}^{N} n_i \ ; \quad p = \sum_{i=1}^{N} p_i \ ; \quad m = \sum_{i=1}^{N} m_i$$

To achieve the form of equations (3.2.6)-(3.2.7), define the controller matrices:

$$\hat{A}(t) = [\hat{A}_{ij}(t)] \qquad \hat{A}_{ij}(t) \in \mathbb{R}^{\hat{n}_i \times \hat{n}_j}$$

$$\hat{B}(t) = \text{diag}[\hat{B}_i(t):\ i=1,\ldots,N] \quad \hat{B}_i \in \mathbb{R}^{\hat{n}_i \times m_i}$$

$$\hat{C}(t) = \text{diag}[\hat{C}_i(t):\ i=1,\ldots,N] \quad \hat{C}_i \in \mathbb{R}^{p_i \times \hat{n}_i}$$

$$G(t) = [G_{ij}(t)] \qquad G_{ij}(t) \in \mathbb{R}^{m_i \times \hat{n}_j}$$

$$H(t) = [H_{ij}(t)] \qquad H_{ij}(t) \in \mathbb{R}^{\hat{n}_i \times p_j}$$

$$\hat{n} = \sum_{j=1}^{N} \hat{n}_i$$

□

Example 3.3: Completely Decentralized Dynamic Feedback

(Figure 3.1)

$A(t)$, $B(t)$, $C(t)$, $Q(t)$, $R(t)$, $K_T$, $\Xi(t)$ and $\Theta(t)$ as in Example 3.2.

$$\hat{n}_i = n_i \qquad i=1,\ldots,N$$

$$\hat{A}(t) = \text{diag}[A_{ii}(t): \ i=1,\ldots,N]$$

$$\hat{B}(t) = B(t)$$

$$\hat{C}(t) = C(t)$$

$$G(t) = \text{diag}[G_i(t): \ i=1,\ldots,N]$$

$$H(t) = \text{diag}[H_i(t): \ i=1,\ldots,N]$$

$$\alpha(t) = (G(t), H(t))$$

□

These examples will be discussed further in the sequel. Henceforth, it will be assumed that the system (3.2.1)-(3.2.5) has the structure of Example 3.2. Examples 3.1 and 3.3 will then be handled by specialization.

## 3.3 Reformulation as a Functional Minimization

Equations (3.2.1)-(3.2.7) can be rewritten in closed loop form for the time varying problem as:

$$\min_{\alpha(\cdot)\in L_2^s(\mathcal{R},\mathcal{B},\lambda[t_o,\ T])} J(\alpha(\cdot)) \tag{3.3.1}$$

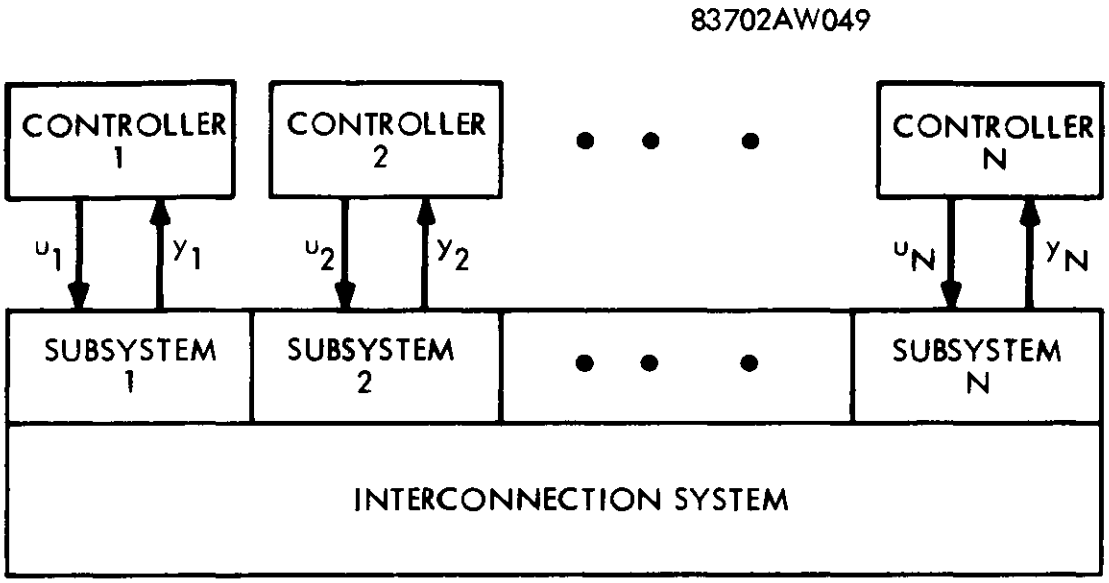


Figure 3.1: Decentralized Control of an Interconnected System

subject to

$$\dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{\xi}(t);\ \tilde{x}(t_o) = \tilde{x}_o \tag{3.3.2}$$

For the time invariant problem, the equivalent problem is:

$$\min_{\alpha \in R^s} \quad J_\infty(\alpha) \tag{3.3.3}$$

subject to:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{\xi}(t);\quad \tilde{x}(t_o) = \tilde{x}_o \tag{3.3.4}$$

The quantities in the above equations are defined as:

$$J(\alpha(\cdot)) = \frac{1}{2} E\{\tilde{x}'(T)\tilde{K}_T\tilde{x}(T) + \int_{t_o}^{T} \tilde{x}'(t)\tilde{Q}(t)\tilde{x}(t)dt\} \tag{3.3.5}$$

$$J_\infty(\alpha) = \frac{1}{2} \operatorname*{Lim}_{T\to\infty} E\{\frac{1}{T} \int_o^T \tilde{x}'(t)\tilde{Q}\tilde{x}(t)dt\} \tag{3.3.6}$$

$$\tilde{x}_i(t) = \begin{bmatrix} x_i(t) \\ \hat{x}_i(t) \end{bmatrix} \qquad \tilde{x}(t) = [\tilde{x}_i(t)]$$

$$\tilde{A}_{ij}(t) = \begin{bmatrix} A_{ij}(t) & -B_iG_{ij}(t) \\ H_{ij}C_j(t) & \hat{A}_{ij}(t) - \hat{B}_i(t)G_{ij}(t) - H_{ij}(t)\hat{C}_j(t) \end{bmatrix}$$

$$\tilde{A}(t) = [\tilde{A}_{ij}(t)]$$

$$\tilde{Q}_{ij}(t) = \begin{bmatrix} Q_i(t)\delta_{ij} & 0 \\ 0 & \sum_{k=1}^{N} G'_{ki}(t) R_k(t) G_{kj}(t) \end{bmatrix}$$

$$\tilde{Q}(t) = [\tilde{Q}_{ij}(t)]$$

$$\tilde{\xi}_i(t) = \begin{bmatrix} \xi_i(t) \\ \sum_{k=1}^{N} H_{ik}(t)\theta_k(t) \end{bmatrix} \qquad \tilde{\xi}(t) = [\tilde{\xi}_i(t)]$$

$$E\{\tilde{\xi}_i(t)\tilde{\xi}'_j(\tau)\} \triangleq \tilde{\Xi}_{ij}(t)\delta(t-\tau) = \begin{bmatrix} \Xi_i(t)\ \delta_{ij} & 0 \\ 0 & \sum_{k=1}^{N} H_{ik}(t)\Theta_k(t)H'_{jk}(t) \end{bmatrix} \delta(t-\tau)$$

$$\tilde{\Xi}(t) = [\tilde{\Xi}_{ij}(t)]$$

$$\tilde{x}_{io} = \begin{bmatrix} x_{io} \\ 0 \end{bmatrix} \qquad \tilde{x}_o = [\tilde{x}_{io}]$$

$$\tilde{K}_{Ti} = \begin{bmatrix} K_{Ti} & 0 \\ 0 & 0 \end{bmatrix} \qquad \tilde{K}_T = \text{diag}[\tilde{K}_{Ti};\ i=1,\ldots,N]$$

Since x(t) has a finite second moment, the Fubini Theorem for stochastic integrals (Liptser and Shiryayev [1977], Theorem 1.9) allows the expectation and integration operators to be interchanged. The following two series of equalities, for (3.3.5) and (3.3.6) respectively, follow from trace identities and the linearity of the trace, expectation and integration operators:

$$J(\alpha(\cdot)) = \frac{1}{2} E\{\tilde{x}'(T)\tilde{K}_T\tilde{x}(T)\} + \frac{1}{2}\int_{t_o}^{T} E\{\tilde{x}'(t)\tilde{Q}(t)\tilde{x}(t)\}dt$$

$$= \frac{1}{2} E\{tr[\tilde{K}_T\tilde{x}(T)\tilde{x}'(T)]\} + \frac{1}{2}\int_{t_o}^{T} E\{tr[\tilde{Q}(t)\tilde{x}(t)\tilde{x}'(t)]\}dt$$

$$= \frac{1}{2} tr\{\tilde{K}_T E[\tilde{x}(T)\tilde{x}'(T)] + \int_{t_o}^{T} \tilde{Q}(t)E[\tilde{x}(t)\tilde{x}'(t)]dt\} \qquad (3.3.7)$$

$$J_\infty(\alpha) = \frac{1}{2}\lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} E\{\tilde{x}'(t)\tilde{Q}\ \tilde{x}(t)\}dt$$

$$= \frac{1}{2}\lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} E\{tr[\tilde{Q}\ \tilde{x}(t)\tilde{x}'(t)]\}dt$$

$$= \frac{1}{2} tr\{\tilde{Q}\lim_{T\to\infty} \frac{1}{T}\int_{o}^{T} E[\tilde{x}(t)\tilde{x}'(t)]dt]\} \qquad (3.3.8)$$

Now define:

$$\tilde{P}(t) \triangleq E\{\tilde{x}(t)\tilde{x}'(t)\} \qquad (3.3.9)$$

Then:

$$\dot{\tilde{P}}(t) = \tilde{A}(t)\tilde{P}(t) + \tilde{P}(t)\tilde{A}'(t) + \tilde{\Xi}(t); \quad \tilde{P}(t_o) = \tilde{P}_o \tag{3.3.10}$$

where

$$\tilde{P}_o = E\{\tilde{x}_o\tilde{x}_o'\}$$

If the closed loop system is time invariant with an infinite horizon cost, $\tilde{P}(t)$ approaches a constant matrix $\tilde{P}$ as t increases. For all $\alpha$ such that $\tilde{A}$ is stable, the matrix $\tilde{P}$ satisfies:

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{\Xi} = 0 \tag{3.3.11}$$

Also:

$$\lim_{T\to\infty} \frac{1}{T} \int_o^T \tilde{P}(t)dt = \tilde{P} \tag{3.3.12}$$

Equations (3.3.1) and (3.3.2) or (3.3.1) and (3.3.3) can now be expressed as equivalent functional or static minimization problems respectively:

Time Varying Functional Minimization

$$\min_{\alpha(t)\in L_2^s(\mathbb{R},\mathbb{B},\lambda)} \left\{ J(\alpha(\cdot)) = \frac{1}{2}\, tr[\tilde{K}_T\tilde{P}(T) + \int_{t_o}^{T} \tilde{Q}(t)\tilde{P}(t)dt] \right\} \tag{3.3.13}$$

subject to

$$\dot{\tilde{P}}(t) = \tilde{A}(t)\tilde{P}(t) + \tilde{P}(t)\tilde{A}'(t) + \tilde{\Xi}(t); \quad \tilde{P}(t_o) = \tilde{P}_o \tag{3.3.14}$$

Time Invariant, Infinite Horizon Static Minimization

$$\min_{\alpha \in A} \{J_\infty(\alpha) = \frac{1}{2} \operatorname{tr}[\tilde{\tilde{Q}}\tilde{\tilde{P}}]\} \tag{3.3.15}$$

subject to

$$0 = \tilde{\tilde{A}}\tilde{\tilde{P}} + \tilde{\tilde{P}}\tilde{\tilde{A}}' + \tilde{\tilde{\Xi}} \tag{3.3.16}$$

where $A \triangleq \{\alpha: \tilde{A} \text{ is asymptotically stable}\}$ .

Note that in the two above formulations the explicit dependence of $\tilde{A}(t)$, $\tilde{Q}(t)$ and $\tilde{\Xi}(t)$ on $\alpha(t)$ has been suppressed to simplify the notation. This dependence will be noted only when it is necessary to clarify the discussion.

The solutions of the minimization problems presented by equations (3.3.13)-(3.3.14) and (3.3.15)-(3.3.16) depend critically on the parameterization of the matrices $\hat{A}(t; \alpha(\cdot))$, $G(t; \alpha(\cdot))$ and $H(t; \alpha(\cdot))$. The parameterization for equations (3.3.15)-(3.3.16) must be general enough that the set $A$ is non-empty. In either problem, any parameterization will generally result in several local minima[1]. Overparameterization of the control system matrices can compound this problem, resulting in an infinity of solutions. A solution

[1] In fact the set $A$ in the static minimization problem may be disconnected, having at least one minimum in each disjoint region.

$\alpha^*(t)$ of (3.3.13)-(3.3.14) or (3.3.15)-(3.3.16)) is locally unique if and only if there is an open neighborhood U of $\alpha^*(t)$ such that

$$J(\alpha(\cdot)) > J(\alpha^*(\cdot)) \qquad \forall \alpha(t) \in U \tag{3.3.17}$$

$$J_\infty(\alpha) > J_\infty(\alpha^*) \qquad \forall \alpha \in U \tag{3.3.18}$$

respectively. Conditions (3.3.17) and (3.3.18) will hold if the problem is not overparameterized.

An obvious necessary condition for (3.3.16) or (3.3.17) to hold is that the controller impulse response matrix

$$H(t, \tau; \alpha(\cdot)) = G(t; \alpha(\cdot))\Phi_{A_f}(t, \tau; \alpha(\cdot))H(\tau; \alpha(\cdot)) \tag{3.3.19}$$

where

$$\frac{d}{dt}\Phi_{A_f}(t, \tau; \alpha(\cdot)) = [\hat{A}(t;\alpha(\cdot))-\hat{B}(t)G(t;\alpha(\cdot))-H(t;\alpha(\cdot)\hat{C}(t)]\Phi_{A_f}(t,\tau,\alpha(\cdot))$$

be uniquely determined in a neighborhood of $\alpha^*$. For the time invariant infinite horizon problem (3.3.15)-(3.3.16), equation (3.3.19) can be transformed to the frequency domain transfer function:

$$F(\delta,\alpha) = G(\alpha)[\delta I-\hat{A}(\alpha)-\hat{B}G(\alpha)-H(\alpha)\hat{C}]H(\alpha) \tag{3.3.20}$$

Glover and Willems [1974] give several tests which can be performed to determine if (3.3.20) is unique in some neighborhood of $\alpha^*$.

Essentially, the implicit function theorem is used to determine a matrix which has full rank if and only if the problem is not overparameterized.

The decomposition algorithm of Chapter 4 will require that the solutions to (3.3.13)-(3.3.14) or (3.3.15)-(3.3.16) be locally unique if convergence is to be guaranteed. The necessary condition that the transfer function (3.3.19) be unique gives a weaker but more easily determined necessary condition. The frequency domain transfer function (3.3.20) implies an even easier test. Since $\bar{F}(\delta,\alpha)$ is determined by at most $2\hat{n}mp$ independent parameters, it is necessary that:

$$s < 2\hat{n}mp \tag{3.3.21}$$

For the remainder of this thesis only parameterizations of G(t) and H(t) will be considered. It will be assumed that $\hat{A}(t)$ is chosen from other considerations. This assures that condition (3.3.21) is satisfied for the time invariant infinite horizon problem (3.3.15)-(3.3.16) since the maximum number of free parameters in G and H is:

$$s_{max} = \hat{n}(m+p) \tag{3.3.22}$$

Inequality (3.3.21) is satisfied since

$$\begin{aligned} 2\hat{n}mp &= \hat{n}\, m(p+p) \\ &\geq \hat{n}\, m(p+1) \\ &= \hat{n}\, mp + \hat{n}m \\ &\geq \hat{n}(p+m) \end{aligned} \tag{3.3.23}$$

The two inequalities in (3.3.23) are tight if and only if p=m=1, i.e., in the single input-single output case.

## 3.3 Optimality Conditions

There are numerous authors who have treated special cases or closely related versions of the problem formulated in the previous section (Sage and Eisenberg [1966]; Levine and Athans [1970]; Kosut [1970]; Levine, Johnson and Athans [1971]; Chong and Athans [1971]; Wang [1972]; Kwakernaak and Sivan [1972]; Davison, Rau and Palmay [1973]; Galiana, et.al. [1973]; Cohen [1977]; Looze, Houpt, Sandell and Athans [1978]); note that the basic problem formulation even predates the state space era (Newton, Gould and Kaisen [1975]). The necessary conditions stated in this section are more general than any to date, and the derivation involving explicit use of variational ideas has some claims to novelty, but the results obtained should not be surprising to those familiar with the cited literature.

Define the following Hilbert spaces:

$$X \triangleq L_2^{nxn}(\mathbb{R},\mathcal{B},\lambda[t_o,T]); \quad \langle A,B\rangle_X \triangleq \operatorname{tr}\int_{t_o}^{T} A'(t)B(t)dt \qquad (3.4.1)$$

$$Y \triangleq \mathbb{R}^{nxn}; \quad \langle A,B\rangle_Y \triangleq \operatorname{tr} A'B \qquad (3.4.2)$$

Since both X and Y are Hilbert spaces, the dual spaces $X^*$ and $Y^*$ can be identified with X and Y respectively. Given a linear operator $F: U \to V$ with U and V Hilbert spaces, the adjoint operator

$F^*: V \to U$ (more precisely $F^*: V^* \to U^*$) is defined by:

$$\langle v, Fu\rangle_V = \langle F^* v, u\rangle_U \qquad \forall\, v \in V,\ u \in U \tag{3.4.3}$$

The following linear operators will be needed:[1]

$$F_{\underset{A}{\sim}}: X \to X: \tilde{\Xi}(\cdot) \to \int_{t_o}^{t} \Phi_{\underset{A}{\sim}}(t,\sigma)\tilde{\Xi}(\sigma)\Phi_{\underset{A}{\sim}}(t,\sigma)\,d\sigma \tag{3.4.4}$$

$$F^T_{\underset{A}{\sim}}: X \to Y: \tilde{\Xi}(\cdot) \to \int_{t_o}^{T} \Phi_{\underset{A}{\sim}}(T,\sigma)\tilde{\Xi}(\sigma)\Phi'_{\underset{A}{\sim}}(T,\sigma)\,d\sigma \tag{3.4.5}$$

$$H_{\underset{A}{\sim}}: Y \to X: \tilde{P}_o \to \Phi_{\underset{A}{\sim}}(t, t_o)\tilde{P}_o\,\Phi'_{\underset{A}{\sim}}(t,t_o) \tag{3.4.6}$$

$$H^T_{\underset{A}{\sim}}: Y \to Y: \tilde{P}_o \to \Phi_{\underset{A}{\sim}}(T,t_o)\tilde{P}_o\,\Phi'_{\underset{A}{\sim}}(T,t_o) \tag{3.4.7}$$

$$L_{\underset{\tilde{A}}{\sim}}: Y \to Y: \tilde{P} \to \tilde{A}\tilde{P} + \tilde{P}\tilde{A}' \tag{3.4.8}$$

The matrix $\Phi_{\underset{A}{\sim}}(t,\sigma)$ is the transition matrix of the closed loop system, and satisfies

$$\frac{d}{dt}\Phi_{\underset{A}{\sim}}(t,\sigma) = \tilde{A}(t;\alpha(\cdot))\Phi_{\underset{A}{\sim}}(t,\sigma);\ \Phi_{\underset{A}{\sim}}(\sigma,\sigma) = I \tag{3.4.9}$$

---

[1] Again, the explicit dependence of matrices on t and $\alpha(\cdot)$ is noted only where it is needed for clarity.

The adjoints of the operators defined in (3.4.4)-(3.4.8) are given by (Appendix A, Lemma A.1):

$$F^{*}_{\underset{\sim}{A}}(\tilde{Q}) = \int_{t}^{T} \Phi'_{\underset{\sim}{A}}(\sigma,t)\tilde{Q}(\sigma)\Phi_{\underset{\sim}{A}}(\sigma,t)\,d\sigma \tag{3.4.10}$$

$$F^{T^{*}}_{\underset{\sim}{A}}(\tilde{K}_{T}) = \Phi'_{\underset{\sim}{A}}(T,t)\tilde{K}_{T}\Phi_{\underset{\sim}{A}}(T,t) \tag{3.4.11}$$

$$H^{*}_{\underset{\sim}{A}}(\tilde{Q}) = \int_{t_o}^{T} \Phi'_{\underset{\sim}{A}}(t,t_o)\tilde{Q}(t)\Phi_{\underset{\sim}{A}}(t,t_o)\,dt \tag{3.4.12}$$

$$H^{T^{*}}_{\underset{\sim}{A}}(\tilde{K}_{T}) = \Phi'_{\underset{\sim}{A}}(T,t_o)\tilde{K}_{T}\Phi_{\underset{\sim}{A}}(T,t_o) \tag{3.4.13}$$

$$L^{*}_{\underset{\sim}{A}}(\tilde{K}) = \tilde{A}'\tilde{K} + \tilde{K}\tilde{A} \tag{3.4.14}$$

The Fréchet differential of $J(\alpha)$ at $\alpha$ in the direction $\Delta\alpha$ will be denoted by (see Section 2.2):

$$\delta J(\alpha;\Delta\alpha) = \frac{d}{d\varepsilon} J(\alpha+\varepsilon\Delta\alpha)\Big|_{\varepsilon=0} \tag{3.4.15}$$

Similarly, let

$$\delta\tilde{A}(t;\alpha;\Delta\alpha) = \frac{d}{d\varepsilon} \tilde{A}(t;\alpha+\varepsilon\Delta\alpha)\Big|_{\varepsilon=0} \tag{3.4.16}$$

$$\delta\tilde{Q}(t;\alpha;\Delta\alpha) = \frac{d}{d\varepsilon} \tilde{Q}(t;\alpha+\varepsilon\Delta\alpha)\Big|_{\varepsilon=0} \tag{3.4.17}$$

$$\delta\tilde{\Xi}(t;\alpha;\Delta\alpha) = \frac{d}{d\varepsilon}\,\tilde{\Xi}(t;\alpha+\varepsilon\Delta\alpha)\Big|_{\varepsilon=0} \tag{3.4.18}$$

$$\delta J_{\infty}(\alpha;\Delta\alpha) = \frac{d}{d\varepsilon}\,J_{\infty}(\alpha+\varepsilon\Delta\alpha)\Big|_{\varepsilon=0} \tag{3.4.19}$$

The following theorem characterizes the Fréchet differential of (3.3.13)-(3.3.14).

Theorem 3.1: Let $J(\alpha)$ be defined by (3.3.13)-(3.3.14). Then the Fréchet differential of $J$ at $\alpha$ in the direction $\Delta\alpha$ is given by:

$$\delta J(\alpha;\Delta\alpha) = \mathrm{tr}\Big\{\int_{t_o}^{\dot{T}} [\tilde{P}(t)\tilde{K}(t)\delta\tilde{A}(t;\alpha;\Delta\alpha) + \frac{1}{2}\,\tilde{P}(t)\delta\tilde{Q}(t;\alpha;\Delta\alpha) + \frac{1}{2}\,\tilde{K}(t)\delta\tilde{\Xi}(t;\alpha;\Delta\alpha)]dt\Big\} \tag{3.4.20}$$

where:

$$\dot{\tilde{K}}(t) = -\tilde{A}'(t)\tilde{K}(t) - \tilde{K}(t)\tilde{A}(t) - \tilde{Q}(t);\quad \tilde{K}(T) = \tilde{K}_T \tag{3.4.21}$$

$$\dot{\tilde{P}}(t) = \tilde{A}(t)\tilde{P}(t) + \tilde{P}(t)\tilde{A}'(t) + \tilde{\Xi}(t);\ \tilde{P}(t_o) = \tilde{P}_o \tag{3.4.22}$$

Proof: From equation (3.3.14)

$$\begin{aligned} \tilde{P}(t) &= \underset{A}{H}(t,P_o) + \underset{A}{F}(\tilde{\Xi}) \\ \tilde{P}(T) &= \underset{A}{H}^{T}(P_o) + \underset{A}{F}^{T}(\tilde{\Xi}) \end{aligned} \tag{3.4.23}$$

Using (3.4.23) and the definitions of the inner products on X and Y in (3.4.1)-(3.4.2):

$$J(\alpha) = \frac{1}{2}\Big\{\langle \tilde{Q}, H_{\underset{\sim}{A}}(\tilde{P}_o)\rangle_X + \langle \tilde{Q}, F_{\underset{\sim}{A}}(\tilde{\Xi})\rangle_X + \langle \tilde{K}_T, H^T_{\underset{\sim}{A}}(\tilde{P}_o)\rangle_Y + \langle \tilde{K}_T, F^T_{\underset{\sim}{A}}(\tilde{\Xi})\rangle_Y \tag{3.4.24}$$

The Frechet differential (3.4.15) is:

$$\begin{aligned}\delta J(\alpha;\Delta\alpha) = \frac{1}{2}\frac{d}{d\varepsilon}\{&\langle \tilde{Q}(\alpha+\varepsilon\Delta\alpha), H_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o)\rangle_X \\ &+ \langle \tilde{Q}(\alpha+\varepsilon\Delta\alpha), F_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha+\varepsilon\Delta\alpha))\rangle_X \\ &+ \langle \tilde{K}_T, H^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o)\rangle_Y \\ &+ \langle \tilde{K}_T, F^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha+\varepsilon\Delta\alpha))\rangle_Y\Big\}\Big|_{\varepsilon=0}\end{aligned} \tag{3.4.25}$$

The computation will proceed term by term. The first term is:

$$\begin{aligned}&\frac{d}{d\varepsilon}\langle \tilde{Q}(\alpha+\varepsilon\Delta\alpha), H_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o)\rangle_X\Big|_{\varepsilon=0} \\ &= \langle \delta\tilde{Q}(\alpha;\Delta\alpha), H_{\underset{\sim}{A}}(P_o)\rangle_X + \langle \tilde{Q}, \frac{d}{d\varepsilon}H_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(P_o)\rangle_X\Big|_{\varepsilon=0}\end{aligned} \tag{3.4.26}$$

The second term of (3.4.25) is:

$$\begin{aligned}&\frac{d}{d\varepsilon}\langle \tilde{Q}(\alpha+\varepsilon\Delta\alpha), F_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha+\varepsilon\Delta\alpha))\rangle_X\Big|_{\varepsilon=0} \\ &= \langle \delta Q(\alpha;\Delta\alpha), F_{\underset{\sim}{A}}(\tilde{\Xi})\rangle_X + \langle \tilde{Q}, \frac{d}{d\varepsilon}F_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha))\Big|_{\varepsilon=0}\rangle_X\end{aligned} \tag{3.4.27}$$

Continuation

$$+ \langle \tilde{Q},\ F_{\underset{\sim}{A}}(\delta\tilde{\Xi}(\alpha;\Delta\alpha)) \rangle_X \tag{3.4.27}$$

$$= \langle \delta\tilde{Q}(\alpha;\Delta\alpha),\ F_{\underset{\sim}{A}}(\tilde{\Xi}) \rangle_X + \langle F^*_{\underset{\sim}{A}}(\tilde{Q}),\ \delta\tilde{\Xi}(\alpha;\Delta\alpha) \rangle_X$$

$$+ \langle \tilde{Q},\ \frac{d}{d\varepsilon} F_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi})\Big|_{\varepsilon=0} \rangle_X$$

The third term of (3.4.25) is:

$$\frac{d}{d\varepsilon} \langle \tilde{K}_T,\ H^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o) \rangle_Y\Big|_{\varepsilon=0} = \langle \tilde{K}_T,\ \frac{d}{d\varepsilon} H^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o)\Big|_{\varepsilon=0} \rangle_Y \tag{3.4.28}$$

The last term of (3.4.25) is:

$$\frac{d}{d\varepsilon} \langle \tilde{K}_T,\ F^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha+\varepsilon\Delta\alpha)) \rangle_Y\Big|_{\varepsilon=0}$$

$$= \langle \tilde{K}_T,\ F^T_{\underset{\sim}{A}}(\delta\tilde{\Xi}(\alpha;\Delta\alpha)) \rangle_Y + \langle \tilde{K}_T,\ \frac{d}{d\varepsilon} F^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha))\Big|_{\varepsilon=0} \rangle_Y$$

$$= \langle F^{T^*}_{\underset{\sim}{A}}(K_T),\ \delta\tilde{\Xi}(\alpha;\Delta\alpha) \rangle_X + \langle \tilde{K}_T,\ \frac{d}{d\varepsilon} F^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi})\Big|_{\varepsilon=0} \rangle_Y \tag{3.4.29}$$

Combining (3.4.26)-(3.4.29) in (3.4.25) gives:

$$\delta J(\alpha;\Delta\alpha) = \frac{1}{2}\Big\{ \langle \delta\tilde{Q}(\alpha;\Delta\alpha),\ H_{\underset{\sim}{A}}(\tilde{P}_o) + F_{\underset{\sim}{A}}(\tilde{\Xi}) \rangle_X \tag{3.4.30}$$

$$+ \langle F^{T^*}_{\underset{\sim}{A}}(\tilde{K}_T) + F^*_{\underset{\sim}{A}}(\tilde{Q}),\ \delta\tilde{\Xi}(\alpha;\Delta\alpha) \rangle_X$$

$$+ \langle \tilde{Q},\ \frac{d}{d\varepsilon}\left[H_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o) + F_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha))\right]_{\varepsilon=0} \rangle_X$$

$$+ \langle \tilde{K}_T,\ \frac{d}{d\varepsilon}\left[H^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o) + F^T_{\underset{\sim}{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha))\right]_{\varepsilon=0} \rangle_Y \Big\}$$

Using equations (3.4.4)-(3.4.7) and defining:

$$\delta\underset{\sim}{\Phi}_{A}(t,\sigma) = \frac{d}{d\varepsilon}\,\underset{\sim}{\Phi}_{A(\alpha+\varepsilon\Delta\alpha)}(t,\sigma)\Big|_{\varepsilon=0} \tag{3.4.31}$$

gives:

$$\frac{d}{d\varepsilon}\,(\underset{\sim}{H}_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o) + \underset{\sim}{F}_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi}(\alpha)) \tag{3.4.32}$$

$$= \delta\underset{\sim}{\Phi}_{A}(t,t_o)\tilde{P}_o\underset{\sim}{\Phi}'_{A}(t,t_o) + \underset{\sim}{\Phi}_{\tilde{A}}(t,t_o)\tilde{P}_o\delta\underset{\sim}{\Phi}'_{A}(t,t_o)$$

$$+ \int_{t_o}^{T}\delta\underset{\sim}{\Phi}_{A}(t,\sigma)\tilde{\Xi}(\sigma)\underset{\sim}{\Phi}'_{\tilde{A}}(t,\sigma)\,d\sigma + \int_{t_o}^{T}\underset{\sim}{\Phi}_{\tilde{A}}(t,\sigma)\tilde{\Xi}(\sigma)\delta\underset{\sim}{\Phi}'_{A}(t,\sigma)\,d\sigma$$

$$= S(t) + S'(t)$$

where

$$S(t) \triangleq \delta\underset{\sim}{\Phi}_{A}(t,t_o)\tilde{P}_o\underset{\sim}{\Phi}'_{A}(t,t_o) + \int_{t_o}^{T}\delta\underset{\sim}{\Phi}_{A}(t,\sigma)\tilde{\Xi}(\sigma)\underset{\sim}{\Phi}'_{A}(t,\sigma)\,d\sigma$$

Since $\underset{\sim}{\Phi}_{A}(t,\sigma)$ is defined by (3.4.9), it is given by

$$\frac{d}{dt}\,\delta\underset{\sim}{\Phi}_{A}(t,\sigma) = \tilde{A}(t)\delta\underset{\sim}{\Phi}_{A}(t,\sigma) + \delta\tilde{A}(t;\sigma;\Delta\alpha)\underset{\sim}{\Phi}_{A}(t,\sigma);\delta\underset{\sim}{\Phi}_{A}(\sigma,\sigma) = 0 \tag{3.4.33}$$

Using the variation of constants formula, $\delta\underset{\sim}{\Phi}_{A}(t,\sigma)$ is

$$\delta\underset{\sim}{\Phi}_{A}(t,\sigma) = \int_{\sigma}^{t}\underset{\sim}{\Phi}_{A}(t,\tau)\delta\tilde{A}(\tau;\alpha;\Delta\alpha)\underset{\sim}{\Phi}_{A}(\tau,\sigma)\,d\tau \tag{3.4.34}$$

Now combine (3.4.34) with the definition of S(t):

$$S(t) = \int_{t_o}^{t} \underset{\sim}{\Phi}_{\tilde{A}}(t,\tau)\delta\tilde{A}(\tau;\alpha;\Delta\alpha)\underset{\sim}{\Phi}_{A}(\tau,t_o)\tilde{P}_o\underset{\sim}{\Phi}'_{\tilde{A}}(t,t_o)d\tau$$

$$+ \int_{t_o}^{t}\int_{\sigma}^{t} \underset{\sim}{\Phi}_{\tilde{A}}(t,\tau)\delta\tilde{A}(\tau;\alpha;\Delta\alpha)\underset{\sim}{\Phi}_{A}(\tau,\sigma)\tilde{\Xi}(\sigma)\underset{\sim}{\Phi}'_{A}(t,\sigma)d\tau d\sigma \qquad (3.4.35)$$

Using the composition rule and interchanging the integrations in the second term gives:

$$S(t) = \int_{t_o}^{t} \underset{\sim}{\Phi}_{A}(t,\tau)\left[\delta\tilde{A}(\tau;\alpha;\Delta\alpha)\underset{\sim}{\Phi}_{\tilde{A}}(\tau,t_o)\tilde{P}_o\underset{\sim}{\Phi}'_{A}(\tau,t_o)\right]\underset{\sim}{\Phi}'_{\tilde{A}}(t,\tau)d\tau$$

$$+ \int_{t_o}^{t}\int_{t_o}^{\tau} \underset{\sim}{\Phi}_{A}(t,\tau)\delta\tilde{A}(\tau;\alpha;\Delta\alpha)\underset{\sim}{\Phi}_{\tilde{A}}(\tau,\sigma)\tilde{\Xi}(\sigma)\underset{\sim}{\Phi}'_{\tilde{A}}(\tau,\sigma)\underset{\sim}{\Phi}'_{A}(t,\tau)d\sigma d\tau$$

$$= \underset{\sim}{F}_{A}(\delta\tilde{A}(\alpha;\Delta\alpha)\underset{\sim}{H}_{A}(\tilde{P}_o))$$

$$+ \int_{t_o}^{t} \underset{\sim}{\Phi}_{\tilde{A}}(t,\tau)\left[\delta\tilde{A}(\tau;\alpha;\Delta\alpha)\int_{t_o}^{\tau}\underset{\sim}{\Phi}_{\tilde{A}}(\tau,\sigma)\tilde{\Xi}(\sigma)\underset{\sim}{\Phi}'_{A}(\tau,\sigma)d\sigma\right]\underset{\sim}{\Phi}'_{A}(t,\tau)d\tau$$

$$= \underset{\sim}{F}_{A}(\delta\tilde{A}(\alpha;\Delta\alpha)\left[\underset{\sim}{H}_{A}(\tilde{P}_o) + \underset{\sim}{F}_{A}(\tilde{\Xi})\right]) \qquad (3.4.36)$$

Thus (3.4.32) is:

$$\frac{d}{d\varepsilon}\left(H_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o) + F_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi})\right)$$

$$= F_{\tilde{A}}(\tilde{P}(t)\delta A'(t;\alpha;\Delta\alpha) + \delta\tilde{A}(t;\alpha;\Delta\alpha)\tilde{P}(t)) \tag{3.4.37}$$

Similarly

$$\frac{d}{d\varepsilon}\left(H^T_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{P}_o) + F^T_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)}(\tilde{\Xi})\right) \tag{3.4.38}$$

$$= S(T) + S'(T)$$

$$= F^T_{\tilde{A}}(\tilde{P}(t)\delta\tilde{A}'(t;\alpha;\Delta\alpha) + \delta\tilde{A}(t;\alpha;\Delta\alpha)\tilde{P}(t))$$

Substituting (3.4.37)-(3.4.38) in (3.4.30) gives:

$$\begin{aligned}
\delta J(\alpha;\Delta\alpha) = \frac{1}{2}\Big\{ & <\delta\tilde{Q}(\alpha;\Delta\alpha),\ \tilde{P}>_X \\
& + <\tilde{K},\ \delta\tilde{\Xi}(\alpha;\Delta\alpha)>_X \\
& + <\tilde{Q},\ F_{\tilde{A}}(\tilde{P}\delta\tilde{A}'(\alpha;\Delta\alpha) + \delta\tilde{A}(\alpha;\Delta\alpha)\tilde{P})>_X \\
& + <\tilde{K}_T,\ F^T_{\tilde{A}}(\tilde{P}\delta\tilde{A}'(\alpha;\Delta\alpha) + \delta\tilde{A}(\alpha;\Delta\alpha)\tilde{P})>_Y \Big\} \\
= \frac{1}{2}\Big\{ & <\delta\tilde{Q}(\alpha;\Delta\alpha),\tilde{P}>_X + <\tilde{K},\delta\tilde{\Xi}(\alpha;\Delta\alpha)>_X \\
& + <F^*_{\tilde{A}}(\tilde{Q}) + F^{T^*}_{\tilde{A}}(\tilde{K}_T),\ \tilde{P}\delta\tilde{A}'(\alpha;\Delta\alpha) + \delta\tilde{A}(\alpha;\Delta\alpha)\tilde{P}>_X \Big\} \\
= \frac{1}{2}\{ & <\delta\tilde{Q}(\alpha;\Delta\alpha),\tilde{P}>_X + <\tilde{K},\ \delta\tilde{\Xi}(\alpha;\Delta\alpha)>_X \\
& + <\tilde{K},\ \tilde{P}\delta\tilde{A}'(\alpha;\Delta\alpha) + \delta\tilde{A}(\alpha;\Delta\alpha)\tilde{P}>_X\}
\end{aligned} \tag{3.4.39}$$

where $\tilde{K}(t)$ is defined by:

$$\tilde{K}(t) = \underset{\sim}{F}^{*}_{A}(\tilde{Q}) + \underset{\sim}{F}^{T^{*}}_{A}(\tilde{K}_{T}) \tag{3.4.40}$$

Thus K(t) satisfies (3.4.21) and P(t) satisfies (3.4.22). Using the definition of $\langle \cdot,\cdot \rangle_X$ in (3.4.39) gives:

$$\delta J(\alpha;\Delta\alpha) = \frac{1}{2}\,\mathrm{tr} \int_{t_o}^{T} [\delta\tilde{Q}'(t;\alpha;\Delta\alpha)\tilde{P}(t) + \tilde{K}(t)\delta\tilde{\Xi}(t;\alpha;\Delta\alpha) \tag{3.4.41}$$

$$+ \tilde{K}(t)\tilde{P}(t)\delta\tilde{A}'(t;\alpha;\Delta\alpha) + \tilde{K}(t)\delta\tilde{A}(t;\alpha;\Delta\alpha)\tilde{P}(t)]dt$$

$$= \mathrm{tr} \int_{t_o}^{T} [\tilde{P}(t)\tilde{K}(t)\delta\tilde{A}(t;\alpha;\Delta\alpha) + \frac{1}{2}\tilde{P}(t)\delta\tilde{Q}(t;\alpha;\Delta\alpha)$$

$$+ \frac{1}{2}\tilde{K}(t)\delta\tilde{\Xi}(t;\alpha;\Delta\alpha)]dt$$

□

The above proof is rather complex due to the time varying structure of the problem. A simpler approach can be used to prove the corresponding theorem for the time invariant problem.

Theorem 3.2: Let $J_\infty(\alpha)$ be defined by (3.3.15)-(3.3.16). Then the Fréchet differential of $J_\infty$ at $\alpha$ in the direction $\Delta\alpha$ is given by:

$$\delta J_\infty(\alpha;\Delta\alpha) = \mathrm{tr}\{\tilde{P}\tilde{K}\delta\tilde{A}(\alpha;\Delta\alpha) + \frac{1}{2}\tilde{P}\delta\tilde{Q}(\alpha;\Delta\alpha) \tag{3.4.42}$$

$$+ \frac{1}{2}\tilde{K}\delta\tilde{\Xi}(\alpha;\Delta\alpha)\}$$

where:

$$\tilde{A}'\tilde{K} + \tilde{K}\tilde{A} + \tilde{Q} = 0 \tag{3.4.43}$$

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{\Xi} = 0 \tag{3.4.44}$$

Proof: From the definition of $L_{\underset{A}{\sim}}$, $\tilde{P}$ is found by:

$$\tilde{P} = - L_{\underset{A}{\sim}}^{-1} (\tilde{\Xi}) \tag{3.4.45}$$

Since $\tilde{Q}$ is symmetric, $J_\infty(\alpha)$ (3.3.15) is given by:

$$J_\infty(\alpha) = - \frac{1}{2} \langle \tilde{Q}, L_{\underset{A}{\sim}}^{-1} (\tilde{\Xi}) \rangle_Y$$

The Frechet differential of $J_\infty(\alpha)$ is:

$$\begin{aligned} \delta J_\infty(\alpha;\Delta\alpha) &= - \frac{1}{2} \langle \delta\tilde{Q}(\alpha;\Delta\alpha), L_{\underset{A}{\sim}}^{-1}(\tilde{\Xi}) \rangle_Y \\ &\quad - \frac{1}{2} \langle \tilde{Q}, \frac{d}{d\varepsilon} L_{\underset{A(\alpha+\varepsilon\Delta\alpha)}{\sim}}^{-1} (\tilde{\Xi}(\alpha+\varepsilon\Delta\alpha)) \Big|_{\varepsilon=0} \rangle_Y \\ &= \frac{1}{2} \langle \delta\tilde{Q}(\alpha;\Delta\alpha), \tilde{P} \rangle_Y \\ &\quad - \frac{1}{2} \langle \tilde{Q}, \frac{d}{d\varepsilon} L_{\underset{A(\alpha+\varepsilon\Delta\alpha)}{\sim}}^{-1} (\tilde{\Xi}(\alpha)) \Big|_{\varepsilon=0} \rangle_Y \\ &\quad - \frac{1}{2} \langle \tilde{Q}, L_{\underset{A}{\sim}}^{-1}(\delta\tilde{\Xi}(\alpha;\Delta\alpha)) \rangle_Y \end{aligned} \tag{3.4.46}$$

The derivative in the second term above can be computed using the definition of $L_{\underset{A}{\sim}}$ and $L_{\underset{A}{\sim}}^{-1}$ in the following manner:

$$0 = \frac{d}{d\varepsilon}\tilde{\Xi}(\alpha) = \frac{d}{d\varepsilon} L_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)} L^{-1}_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)} (\tilde{\Xi}(\alpha))\Big|_{\varepsilon=0}$$

Thus

$$0 = L_{\delta\tilde{A}(\alpha;\Delta\alpha)} \circ L^{-1}_{\tilde{A}(\alpha)} (\tilde{\Xi}(\alpha)) + L_{\tilde{A}(\alpha)} \circ \left[\frac{d}{d\varepsilon} L^{-1}_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)} (\tilde{\Xi}(\alpha))\right]_{\varepsilon=0} \tag{3.4.47}$$

Using (3.4.45) and solving equation (3.4.47) for the bracketted term gives:

$$\frac{d}{d\varepsilon} L^{-1}_{\tilde{A}(\alpha+\varepsilon\Delta\alpha)} (\tilde{\Xi}(\alpha))\Big|_{\varepsilon=0} = L^{-1}_{\tilde{A}} \left[L_{\delta\tilde{A}(\alpha;\Delta\alpha)} (\tilde{P})\right] \tag{3.4.48}$$

Using (3.4.48) and the definition of the adjoint operator, (3.4.46) becomes:

$$\begin{aligned}\delta J(\alpha;\Delta\alpha) &= \frac{1}{2} \langle \delta\tilde{Q}(\alpha;\Delta\alpha), \tilde{P}\rangle_Y \\ &\quad - \frac{1}{2} \langle L^{*-1}_{\tilde{A}} (\tilde{Q}), L_{\delta\tilde{A}(\alpha;\Delta\alpha)} (\tilde{P}) + \delta\tilde{\Xi}(\alpha;\Delta\alpha)\rangle_Y\end{aligned} \tag{3.4.49}$$

Then, by defining

$$\tilde{K} = - L^{*-1}_{\tilde{A}} (\tilde{Q})$$

and using the definition of the Lyapunov operator and the inner product on Y equation (3.4.46) is equivalent to

$$\delta J(\alpha;\Delta\alpha) = \mathrm{tr}\{\frac{1}{2}\delta\tilde{Q}(\alpha;\Delta\alpha)\tilde{P} + \frac{1}{2}\tilde{K}\delta\tilde{\Xi}(\alpha;\Delta\alpha) \tag{3.4.50}$$

$$+ \frac{1}{2}[\tilde{K}\delta\tilde{A}(\alpha;\Delta\alpha)\tilde{P} + \tilde{K}\tilde{P}\delta\tilde{A}'(\alpha;\Delta\alpha)]\}$$

$$+ \mathrm{tr}\{\delta\tilde{A}(\alpha;\Delta\alpha)\tilde{P}\tilde{K} + \frac{1}{2}\tilde{P}\delta\tilde{Q}(\alpha;\Delta\alpha) + \frac{1}{2}\tilde{K}\delta\Xi(\alpha;\Delta\alpha)\} \tag{3.4.50}$$

By (3.4.14), K solves (3.4.43).

□

Theorems 3.1 and 3.2 can be used directly in a gradient search (see Section 2.2)to solve either of the two minimization problems of Section 3.3 for arbitrary parameterizations.

With the interconnected system structure assumed in Section 3.2 (Example 3.2), these conditions can be developed further.

Theorem 3.3: a) Assume that $G^*_{ij}(t)$ and $H^*_{ij}(t)$ $(i=1,\ldots,N;j=1,\ldots,N)$ are optimal for the functional minimization problem (3.3.13)-(3.3.14). Then $G^*_{ij}(t)$ and $H^*_{ij}(t)$ satisfy:

$$\mathrm{tr}\int_{t_o}^{T}\{\Delta G'_{ij}(t)\{-B'_i(t)\,(\tilde{K}^*(t)\tilde{P}^*(t))_{2i-1,2j} - \hat{B}'_i(t)\,(\tilde{K}^*(t)\tilde{P}^*(t))_{2i,2j}$$

$$+ \sum_{\ell=1}^{N} R_i(t)G^*_{i\ell}(t)\tilde{P}^*_{2\ell,2j}(t)\}\}dt = 0 \tag{3.4.51}$$

$$\mathrm{tr}\int_{t_o}^{T}\{\{(\tilde{K}^*(t)\tilde{P}^*(t))_{2i,2j-1}C'_j(t) - (\tilde{K}^*(t)\tilde{P}^*(t))_{2i,2j}\hat{C}'_j(t)$$

$$+ \sum_{\ell=1}^{N}\tilde{K}^*_{2i,2\ell}(t)H^*_{\ell j}(t)\Theta_j(t)\}\Delta\tilde{H}'_{ij}(t)\}dt = 0 \tag{3.4.52}$$

$$\dot{\tilde{K}}^*(t) = -\tilde{A}^{*'}(t)\,\tilde{K}^*(t) - \tilde{K}^*(t)\tilde{A}^*(t) - \tilde{Q}^*(t);\ \tilde{K}^*(T) = \tilde{K}_T \tag{3.4.53}$$

$$\dot{\tilde{P}}^*(t) = \tilde{A}^*(t)\tilde{P}^*(t) + \tilde{P}^*(t)\tilde{A}^{*\prime}(t) + \tilde{\Xi}^*(t);\ \tilde{P}^*(t_o) = \tilde{P}_o \tag{3.4.54}$$

b) Assume $G^*_{ij}$ and $H^*_{ij}$ $(i=1,\ldots,N;\ j=1,\ldots,N)$ are optimal for the static minimization problem (3.3.15)-(3.3.16). Then $G^*_{ij}$ and $H^*_{ij}$ satisfy:

$$\mathrm{tr}\{\Delta G'_{ij}\{-B_i'(\tilde{K}^*\tilde{P}^*)_{2i-1,2j} - \hat{B}_i'(\tilde{K}^*\tilde{P}^*)_{2i,2j} \tag{3.4.55}$$

$$+ \sum_{\ell=1}^{N} R_i G^*_{i\ell}\tilde{P}^*_{2\ell,2j}\}\} = 0$$

$$\mathrm{tr}\{\{(\tilde{K}^*\tilde{P}^*)_{2i,2j-1}C_j' - (\tilde{K}^*\tilde{P}^*)_{2i,2j}\hat{C}_j' \tag{3.4.56}$$

$$+ \sum_{\ell=1}^{N}\tilde{K}^*_{2i,2\ell}H^*_{\ell j}\Theta_j\}\Delta H'_{ij}\} = 0$$

$$(\tilde{A}'\tilde{K} + \tilde{K}\tilde{A} + \tilde{Q})^* = 0 \tag{3.4.57}$$

$$(\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{\Xi})^* = 0 \tag{3.4.58}$$

Here the subscripts denote the block of the indicated matrix, partitioned conformally with the closed loop system matrices defined in (3.3.1)-(3.3.2).

Proof:

The proof involves a simple algebraic manipulation of the equations from Theorems 3.1 and 3.2, and is identical for parts a and b. The steps will be shown for part b.

Assume $\tilde{P}$ and $\tilde{K}$ are given by (3.4.57)-(3.4.58)

Define

$$\widetilde{PK}_{ij} \triangleq \begin{bmatrix} (\tilde{\tilde{PK}})_{2i-1,2j-1} & (\tilde{\tilde{PK}})_{2i-1,2j} \\ (\tilde{\tilde{PK}})_{2i,2j-1} & (\tilde{\tilde{PK}})_{2i,2j} \end{bmatrix}$$

using the notation as in Theorem 3.2. Then,

$$\delta\tilde{A}(\alpha;\Delta\alpha)_{ij} = \begin{bmatrix} 0 & -B_i \; G_{ij} \\ \Delta H_{ij} \; C_j & -\hat{B}_i \; \Delta G_{ij} - \Delta H_{ij} \; \hat{C}_j \end{bmatrix}$$

The first term of the right hand side of (3.4.42) is:

$$tr\{\delta\tilde{A}(\alpha;\Delta\alpha)\tilde{\tilde{PK}}\} = tr \sum_{i=1}^{N} \sum_{j=1}^{N} \delta\tilde{A}(\alpha;\Delta\alpha)_{ij} \; \widetilde{PK}_{ji}$$

since the trace of a block matrix is the sum of the traces of the diagonal blocks. Then:

$$tr\{\delta\tilde{A}(\alpha;\Delta\alpha)_{ij} \; \widetilde{PK}_{ji}\} = tr\{-B_i \; \Delta G_{ij} (\tilde{\tilde{PK}})_{2j,2i-1} \; -\hat{B}_i \; \Delta G_{ij} (\tilde{\tilde{PK}})_{2j,2i}$$

$$+ \Delta H_{ij} \; C_j (\tilde{\tilde{PK}})_{2j-1,2i} - \Delta H_{ij} \; \hat{C}_j (\tilde{\tilde{PK}})_{2j,2i}\}$$

Thus:

$$\mathrm{tr}\{\delta\tilde{A}(\alpha;\Delta\alpha)\tilde{\tilde{PK}}\} = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\{-[(\tilde{\tilde{PK}})_{2j,2i-1}B_i + (\tilde{\tilde{PK}})_{2j,2i}\hat{B}_i]\Delta G_{ij}$$

$$+ \Delta H_{ij}[C_j(\tilde{\tilde{PK}})_{2j-1,2i} - \hat{C}_j(\tilde{\tilde{PK}})_{2j,2i}]\}$$

where the linearity of the trace and the following property have been used:

$$\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\}, \quad A \in \mathbb{R}^{n \times m}, \quad B \in \mathbb{R}^{m \times n}$$

Next, using $\mathrm{tr}\{A\} = \mathrm{tr}\{A'\}$:

$$\mathrm{tr}\{\delta\tilde{A}(\alpha;\Delta\alpha)\tilde{\tilde{PK}}\} = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\{-\Delta G_{ij}'[B_i'(\tilde{\tilde{KP}})_{2i-1,2j} + \hat{B}_i'(\tilde{\tilde{KP}})_{2i,2j}]$$

$$+ [(\tilde{\tilde{KP}})_{2i,2j-1}C_j' - (\tilde{\tilde{KP}})_{2i,2j}\hat{C}_j']\Delta H_{ij}\}$$

Similarly,
$$\delta\tilde{\Xi}(\alpha;\Delta\alpha)_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{\ell=1}^{N}\{\Delta H_{i\ell}\Theta_\ell H_{j\ell}' + H_{i\ell}\Theta_\ell \Delta H_{j\ell}'\} \end{bmatrix} \tag{3.4.59}$$

$$\delta\tilde{Q}(\alpha;\Delta\alpha)_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{\ell=1}^{N}\{\Delta G_{\ell i}' R_\ell G_{\ell j} + G_{\ell i}' R_\ell \Delta G_{\ell j}\} \end{bmatrix}$$

Then:

$$\frac{1}{2}\,\mathrm{tr}\{\tilde{P}\delta\tilde{Q}(\alpha;\Delta\alpha)\} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\{\tilde{P}_{ij}\,\delta\tilde{Q}(\alpha;\Delta\alpha)_{ji}\}$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\{\tilde{P}_{2i,2j}\sum_{\ell=1}^{N}[\Delta G'_{\ell j}R_\ell G_{\ell i} + G'_{\ell j}R_\ell \Delta G_{\ell i}]\}$$

$$= \frac{1}{2}\,\mathrm{tr}\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}[\tilde{P}_{2i,2j}\Delta G'_{\ell j}R_\ell G_{\ell i} + \tilde{P}_{2i,2j}G'_{\ell j}R_\ell \Delta G_{\ell i}]\right\}$$

$$= \frac{1}{2}\,\mathrm{tr}\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}[\Delta G'_{\ell j}R_\ell G_{\ell i}\tilde{P}_{2i,2j} + \Delta G'_{\ell i}R_\ell G_{\ell j}\tilde{P}'_{2i,2j}]\right\}$$

$$= \frac{1}{2}\,\mathrm{tr}\left\{\sum_{\ell=1}^{N}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\Delta G'_{\ell j}R_\ell G_{\ell i}\tilde{P}_{2i,2j} + \sum_{j=1}^{N}\sum_{i=1}^{N}\Delta G'_{\ell i}R_\ell G_{\ell j}\tilde{P}_{2j,2i}\right]\right\} \tag{3.4.60}$$

The first equality follows from the property of the trace of a block matrix; the second by multiplying the bracketted term from the previous equation; the third by rearranging terms; the fourth from the trace identities used previously; and the fifth by rearranging terms and the fact

$$\tilde{P}'_{2i,2j} = \tilde{P}_{2j,2i}$$

Switching i and j in the indexing of the second term of (3.4.60) shows the two terms are equal. Hence:

$$\frac{1}{2}\,\mathrm{tr}\{\tilde{P}\delta\tilde{Q}(\alpha;\Delta\alpha)\} = \sum_{\ell=1}^{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\{\Delta G'_{\ell j}R_\ell G_{\ell i}\tilde{P}_{2i,2j}\}$$

$$= \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\{\Delta G'_{ij}R_i G_{i\ell}\tilde{P}_{2\ell,2j}\}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\{\Delta G'_{ij}\sum_{\ell=1}^{N}R_i G_{i\ell}\tilde{P}_{2\ell,2j}\} \qquad (3.4.61)$$

The second equality follows by interchanging the i and $\ell$ indexing.

In a completely analogous manner:

$$\frac{1}{2}\,\mathrm{tr}\{\tilde{K}\delta\tilde{\Xi}(\alpha;\Delta\alpha)\} = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\left\{\left[\sum_{\ell=1}^{N}\tilde{K}_{2i,2\ell}H_{\ell j}\Theta_j\right]\Delta H'_{ij}\right\} \qquad (3.4.62)$$

Combining (3.4.59), (3.4.61) and (3.4.62), and using

$$\delta J(\alpha;\Delta\alpha) = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\Big\{\Delta G'_{ij}\{-B'_i(\tilde{\tilde{K}}\tilde{\tilde{P}})_{2i-1,2j} - \hat{B}'_i(\tilde{\tilde{K}}\tilde{\tilde{P}})_{2i,2j} \qquad (3.4.63)$$

$$+ \sum_{\ell=1}^{N}R_i G_{i\ell}\tilde{P}_{2\ell,2j}\Big\}$$

$$+ \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\left\{(\tilde{\tilde{K}}\tilde{\tilde{P}})_{2i,2j-1}C'_j - (\tilde{\tilde{K}}\tilde{\tilde{P}})_{2i,2j}\hat{C}'_j + \sum_{\ell=1}^{N}\tilde{K}_{2i,2\ell}H_{\ell j}\Theta_j\}\Delta H'_{ij}\right\}$$

A necessary condition for $\alpha^* = (G_{ij}{}^*, H_{ij}{}^*: i,j=1,\ldots,N)$ to be optimal is:

$$\delta J(\alpha^*;\Delta\alpha) = 0 \qquad (3.4.64)$$

If (3.4.55)-(3.4.56) are satisfied, then so is (3.4.64). Conversely, if (3.4.64) is true then

$$\Delta H_{pq} = 0 \qquad \forall p,q = 1,\ldots,N$$

$$\Delta G_{pq} = 0 \qquad \forall p,q = 1,\ldots,N;\ p\neq i,\ q\neq j$$

results in (3.4.55). Similarly

$$\Delta H_{pq} = 0 \qquad \forall p,q = 1,\ldots,N;\ p\neq i,\ q\neq j$$

$$\Delta G_{pq} = 0 \qquad \forall p,q = 1,\ldots,N$$

results in (3.4.56). Thus, (3.4.55)-(3.4.56) with $\tilde{K}$ and $\tilde{P}$ given by (3.4.57)-(3.4.58) are equivalent to (3.4.64).

□

Theorems 3.1 and 3.2, and the approach of Theorem 3.3 can be used to derive necessary conditions for many closely related problems. A special case is the classical situation of Example 3.1. Then the above theorem applies with N=1 and $\Delta G$ and $\Delta H$ arbitrary. Equations (3.4.51)-(3.4.52) are equivalent to:

$$-B'(t)[(K(t)P(t))_{12} + (K(t)P(t))_{22}] + R(t)G(t)P_{22}(t) = 0 \qquad (3.4.65)$$

$$[(K(t)P(t))_{21} - (K(t)P(t))_{22}]C'(t) + K_{22}(t)H(t)\Theta(t) = 0 \qquad (3.4.66)$$

With a bit of manipulation, equations (3.4.53)-(3.4.54) and (3.4.65)-(3.4.66) can be used to show that the equations for the filter and control gains decouple (a version of the separation principle) and to obtain the control and filter Riccati equations. This calculation is performed in Appendix B. The insight provided by these manipulations will be useful in the solution of the subproblems formed by the decomposition algorithm, and will be discussed further in Chapter 5.

The similarity between equations (3.4.65)-(3.4.66), (3.4.51)-(3.4.52) and (3.4.55)-(3.4.56) will be exploited through the following notation in the remainder of this thesis.

$$\mathcal{D}_c(G;P,K,B,\hat{B},R) = -B'(KP)_{12} - \hat{B}'(KP)_{22} + RGP_{22} \tag{3.3.67}$$

$$\mathcal{D}_f(H;P,K,C,\hat{C},\Theta) = (KP)_{21}\, C' - (KP)_{22}\, \hat{C}' + K_{22}H\,\Theta \tag{3.3.68}$$

In the general case when some of the elements of G and H have been fixed a priori (e.g., Example 3.3) the corresponding elements of $\Delta G$ and $\Delta H$ are 0. Then equations (3.4.51)-(3.4.54) or (3.4.55)-(3.4.58) will result in a system of $2n^2 + s$ coupled nonlinear equations in $2n^2 + s$ unknowns. Of course, as was noted earlier, the equations may be dependent if the parameterization is too general.

There are two particular examples when G and H are constrained which will be used extensively in Chapters 4 and 5. The first requires that G and H be identically 0. The minimization problem is of course

trivial, but the decomposition approach of Chapter 4 will result in an iterative solution method for the solution of the Lyapunov equations (3.4.53)-(3.4.54) or (3.4.57)-(3.4.58). The result for the time invariant equations is similar to a technique studied by Athay [1976] and Lehtomaki [1978].

The second example corresponds to Example 3.3. In this case it is required that $G_{ij}$ and $H_{ij}$ for $i \neq j$ be identically 0 with $G_{ii}$ and $H_{ii}$ unconstrained. Equations (3.4.51)-(3.4.52) become:

$$-B_i'(t)\ (K^*(t)P^*(t))_{2i-1,2i} - \hat{B}_i'(t)(K^*(t)P^*(t))_{2i,2i} + R_i(t)G^*_{ii}(t)P^*_{2i,2i}(t) = 0 \qquad (3.4.69)$$

$$(K^*(t)P^*(t))_{2i,2i-1}\ C_i' - (K^*(t)P^*(t))_{2i,2i}\ \hat{C}_i'(t) + K^*_{2i,2i}(t)H^*_{ii}(t)\Theta_i(t) = 0 \qquad (3.4.70)$$

The time invariant equations (3.4.55)-(3.4.56) are also equivalent in this case to (3.4.69)-(3.4.70) without the time dependence.

## 3.5 Summary and Discussion

This chapter has considered a suboptimal approach to solving nonclassical linear stochastic optimal control problems. The class of admissible controls was restricted to those controls which can be generated as the output of a finite specified dimensional linear system which uses as input the output of the system to be controlled.

Information flow restrictions are handled through constraints on the variables which characterize the controller.

Several practical advantages of this formulation were discussed earlier (see Chapter 1 and Section 3.1). One of the most important advantages is that the nonclassical stochastic optimization can be reformulated as a deterministic nonlinear optimization. Then a set of necessary conditions for the solution of the stochastic optimization problem can be derived.

Two points about the results of Section 3.4 should be noted. First, Theorems 3.1 and 3.2 apply to any problem which can be put in the form of the optimizations in Section 3.3. In addition to the linear stochastic problem, other examples which produce optimizations of this form are the output feedback problem, the model reduction problem and the reduced order observer problem (see, for example, Galiana et.al. [1973]). The applicability of Theorems 3.1 and 3.2 are a direct result of the generality of their derivation.

The other point is that the conditions presented in Section 3.4 may have many solutions. This problem arises because the condtions are satisfied for any stationary points of the optimization. Since the nonclassical stochastic optimization may have several local minima there will not, in general, be a unique solution to the necessary conditions.

# 4. DECOMPOSITION OF THE LINEAR STOCHASTIC CONTROL PROBLEM

## 4.1 Introduction

There are several approaches to solving the linear stochastic control problem formulated in Chapter 3. The derivatives presented in Theorems 3.1 and 3.2 can be used to solve the minimization problems of Section 3.3 ((3.3.13)-(3.3.14) or (3.3.15)-(3.3.16)) directly (Kwakernaak and Sivan [1972]; Davison, Rau and Palmay [1973]; Looze, Houpt, Sandell and Athans [1978]). Newton's method or any quasi-Newton method (see Dennis and More' [1977]) can be used to solve the nonlinear equations which result from the necessary conditions presented in Theorem 3.3. Levine et.al. [1971] and Wang [1972] used iterative methods to solve similar sets of equations resulting from output feedback problems and deterministic decentralized control problems respectively. Both methods fit the decomposition framework developed in Chapter 2. However, none of the above methods utilizes the structure of the interconnected system problem.

In many problems the interconnected system problem possesses a physical weak coupling; i.e. the interactions between subsystems are much less important than the self-dynamics of the subsystems. In the linear stochastic control problem formulation of Chapter 3, the weak coupling is manifested in the off diagonal blocks of the system matrix. If each of the off diagonal blocks are zero, the optimization problems and their necessary conditions decouple into N independent optimization

problems whose solutions are simply the centralized linear-quadratic-Gaussian subproblem solutions. Since the subsystem interactions were assumed to be relatively insignificant, it seems reasonable that the solution to the overall problem should be near the solution of the decoupled problem.

The above structure is similar to the structure exploited in Example 2.7. The decomposition developed in the discussion preceding Example 2.7 will be used in Section 4.2 to exploit the weak coupling structure. The remainder of this chapter will discuss the convergence of the resulting iteration. Although the convergence condition of Theorem 2.4 cannot be practically evaluated in most cases, it will be used in Section 4.3 to derive a weak coupling convergence result. Section 4.4 develops related conditions which can be used as guidelines to determine whether the iteration will converge. Section 4.5 discusses the results of Chapter 4.

## 4.2 The Gauss-Seidel Decomposition

The necesarry conditions given by Theorem 3.3 result in a system of nonlinear equations of the general form:

$$f(\beta) = 0 \qquad \beta \in \mathcal{B} \tag{4.2.1}$$

where $\mathcal{B}$ is a Banach space. The exact form of f in (4.2.1) will depend on the particular parameterization used. For the purposes of this chapter, it will be assumed that the individual elements of each parameterized block of G and H are parameters. The parameterized

blocks of G and H will be specified by the index sets $I_G$ and $I_H$.

$$I_G \triangleq \{(i,j): G_{ij} \text{ is parameterized}\} \tag{4.2.2}$$

$$I_H \triangleq \{(i,j): H_{ij} \text{ is parameterized}\} \tag{4.2.3}$$

A one-point stationary iterative method will be used to solve equation (4.2.1). Such methods were discussed in Chapter 2. The particular iteration to be used will be chosen to take advantage of the weakly coupled interconnected system structure assumed in Chapter 3. The key observation which affects the choice of the iteration is that if the systems are not coupled ($A_{ij}=0$ for $i\neq j$) then equations (3.4.51)-(3.4.54) and (3.4.55)-(3.4.58) decouple into N independent systems of nonlinear equations which correspond to the centralized necessary conditions of the subsystems. The resulting solution G, H, P, and K are block diagonal. If the systems are coupled, but the off diagonal blocks are fixed and the off diagonal equations (i.e. the equations resulting from the off diagonal blocks of the Lyapunov equations and the Fréchet derivatives $\left\{\frac{\partial J}{dG_{ij}}, \frac{\partial J}{\partial H_{ij}} : i\neq j\right\}$ are not enforced, then the problem again decouples into independent systems of nonlinear equations.

The structure of the necessary conditions described above is exactly that specified by (2.3.22)-(2.3.24). To exploit this structure, the core will be chosen to implement the Gauss-Seidel algorithm (note that the Jacobi algorithm could also be used). The remaining step

is to specify the decomposition of the space $\mathcal{B}$ as in (2.3.22). This decomposition involves choosing a partial basis for the space, and corresponds to assigning the parameters and variables to either the upper or lower level of the hierarchical structure which results.

Using the structure of the necessary conditions as a guideline, the decomposition that will be used is:

$$\mathcal{B} = \mathcal{B}^S \times \mathcal{B}^I \tag{4.2.4}$$

$$\mathcal{B}^S \triangleq \{\tilde{K}_{ij}, \tilde{P}_{ij}, G_{k\ell}, H_{qr}: (i\neq j;\ i=1,\ldots,N;\ j=1,\ldots,N), \tag{4.2.5}$$

$$(k\neq\ell;\ (k,\ell)\in I_G),\ (q\neq r;\ (q,r)\in I_H\}$$

$$\mathcal{B}^I \triangleq \{\tilde{K}_{ii}, \tilde{P}_{ii}, G_{jj}, H_{kk}: i=1,\ldots,N;\ (j,j)\in I_G;\ (k,k)\in I_H\} \tag{4.2.6}$$

$$f_S \triangleq \{\dot{\tilde{K}}_{ij} = -(\tilde{A}'\tilde{K} + \tilde{K}\tilde{A} + \tilde{Q})_{ij};\ \dot{\tilde{P}}_{ij} = (\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{\Xi})_{ij}\ ; \tag{4.2.7}$$

$$\frac{\partial J}{\partial G_{k\ell}} = 0;\ \ \frac{\partial J}{\partial H_{qr}} = 0:\ (i\neq j;\ i=1,\ldots,N;\ j=1,\ldots,N),$$

$$(k\neq\ell;(k,\ell)\in I_G),\ (q\neq r;\ (q,r)\in I_H)\}$$

$$f_I \triangleq \{\dot{\tilde{K}}_{ii} = -(\tilde{A}'\tilde{K} + \tilde{K}\tilde{A} + \tilde{Q})_{ii}\ ;\ \dot{\tilde{P}}_{ii} = (\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{\Xi})_{ii}\ ; \tag{4.2.8}$$

$$\frac{\partial J}{\partial G_{jj}} = 0;\ \ \frac{\partial J}{\partial H_{kk}} = 0:\ i=1,\ldots,N;\ (j,j)\in I_G\ ;$$

$$(k,k)\in I_H\}$$

Above $\tilde{K}_{ij}$ and $\tilde{P}_{ij}$ are the blocks of $\tilde{K}$ and $\tilde{P}$ partitioned conformally

with $\tilde{A}_{ij}$, and the subscripts of the Lyapunov equations indicate the blocks of the equations corresponding to $\tilde{A}_{ij}$. In the time invariant case, the assignment of variables is analogous, with the time derivatives replaced by the zero matrix.

The resulting iteration is:

$$f_S(\beta_S^{k+1};\ \beta_I^k)=0 \qquad \beta_S^{k+1} \in \mathcal{B}^S;\ \beta_I^k \in \mathcal{B}^I \tag{4.2.9}$$

$$f_I(\beta_I^{k+1};\ \beta_S^{k+1})=0 \qquad \beta_I^{k+1} \in \mathcal{B}^I \tag{4.2.10}$$

With the decomposition given by (4.2.4)-(4.2.8) the solution of $f_I$ (equation (4.2.10)) for $\beta_I^{k+1}$ decouples into N independent smaller problems. Define

$$\bar{K}_{ij}^k = \begin{cases} \tilde{K}_{ij}^k & i \neq j \\ \tilde{K}_{ii}^{k-1} & i=j \end{cases} \tag{4.2.11}$$

$$\bar{K}^k = [\bar{K}_{ij}^k] \tag{4.2.12}$$

i.e., $\bar{K}^k$ is the matrix with diagonal blocks equal to the most recently obtained subsystem matrices $\tilde{K}_{ii}$ and the off diagonal supremal matrices $\tilde{K}_{ij}$ which are to be computed. In a similar manner define $\bar{P}^k$, $\bar{G}^k$, $\bar{H}^k$, $\bar{A}^k$, $\bar{Q}^k$, and $\bar{\Xi}^k$.

The decomposed problem (4.2.9)-(4.2.10) can then be written as a two level hierarchy in terms of the original system matrices and parameters. To simplify notation, the time dependence notations will

be dropped, and only the time varying case will be considered. The same idea can be applied to the time invariant problem. The resulting equations are identical to the time varying case with the time derivatives replace by zero matrices.

Supremal Problem

$$-B_i'(\bar{K}^k\bar{P}^k)_{2i-1,2j} - \hat{B}_i'(\bar{K}^k\bar{P}^k)_{2i,2j} + R_i \sum_{\ell=1}^{N} \bar{G}^k_{i\ell}\,\bar{P}^k_{2\ell,2j} = 0; \qquad i=j,\ (i,j)\in I_G \tag{4.2.13}$$

$$(\bar{K}^k\bar{P}^k)_{2i,2j-1}C_j' - (\bar{K}^k\bar{P}^k)_{2i,2j}\hat{C}_j' + \sum_{\ell=1}^{N} \bar{K}^k_{2i,2\ell}\,\bar{H}^k_{\ell,j}\,\Theta_j = 0 \qquad i\neq j,\ (i,j)\in I_H \tag{4.2.14}$$

$$\dot{\bar{P}}^k_{ij} = [\bar{A}^k\bar{P}^k + \bar{P}^k(\bar{A}^k)' + \bar{\Xi}^k]_{i,j};\ \bar{P}^k_{ij}(t_o) = \tilde{P}_{oij} \tag{4.2.15}$$

$$i\neq j;\ i=1,\ldots,N;\ j=1,\ldots,N$$

$$\dot{\bar{K}}^k_{ij} = -[(\bar{A}^k)'\bar{K}^k + \bar{K}^k\bar{A}^k + \bar{Q}^k]_{i,j};\ \bar{K}^k_{ij}(T) = \tilde{K}_{Tij} \tag{4.2.16}$$

Infimal Problems (i=1,...,N)

$$\mathcal{D}_c(G^k_{ii};\ \tilde{P}^k_{ii},\ \tilde{K}^k_{ii},\ B_i,\ \hat{B}_i,\ R_i) + S^k_i = 0; \quad (i,i)\in I_G \tag{4.2.17}$$

$$\mathcal{D}_f(H^k_{ii};\ \tilde{P}^k_{ii},\ \tilde{K}^k_{ii},\ C_i,\ \hat{C}_i,\ \Theta_i) + T^k_i = 0; \quad (i,i)\in I_H \tag{4.2.18}$$

$$\dot{\tilde{P}}^k_{ii} = \tilde{A}^k_{ii}\,\tilde{P}^k_{ii} + \tilde{P}^k_{ii}\,(\tilde{A}^k_{ii})' + \tilde{\Xi}^k_{ii} + D^k_i;\ \tilde{P}^k_{ii}(t_o) = \tilde{P}_{oii} \tag{4.2.19}$$

$$\dot{\tilde{K}}^k_{ii} = -(\tilde{A}^k_{ii})'\,\tilde{K}^k_{ii} - \tilde{K}^k_{ii}\,\tilde{A}^k_{ii} - \tilde{Q}^k_{ii} + E^k_i;\ \tilde{K}^k_{ii}(T) = \tilde{K}_{Tii} \tag{4.2.20}$$

where:

$$S_i^k \triangleq \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{D}_c(G_{ij}^k;\ \tilde{P}_{ji}^k,\ \tilde{K}_{ij}^k,\ B_i,\ \hat{B}_i,\ R_i) \tag{4.2.21}$$

$$T_i^k \triangleq \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathcal{D}_f(H_{ij}^k;\ \tilde{P}_{ji}^k,\ \tilde{K}_{ij}^k,\ C_i,\ \hat{C}_i,\ \Theta_i) \tag{4.2.22}$$

$$D_i^k \triangleq \sum_{\substack{j=1 \\ j\neq i}}^{N} [\tilde{A}_{ij}^k\ \tilde{P}_{ji}^k + \tilde{P}_{ij}^k\ (\tilde{A}_{ij}^k)'] \tag{4.2.23}$$

$$E_i^k \triangleq \sum_{\substack{j=1 \\ j\neq i}}^{N} [(\tilde{A}_{ji}^k)'\ \tilde{K}_{ji}^k + \tilde{K}_{ij}^k\ \tilde{A}_{ji}^k] \tag{4.2.24}$$

There are several properties of the above decomposition ((4.2.13)-(4.2.24)) which should be noted. In general both the supremal and infimal problems are nonlinear. However, in the completely decentralized problem equations (4.2.13)-(4.2.14) are not present. In this particular case the supremal problem solves the <u>linear</u> matrix equations (4.2.15)-(4.2.16). Equations (4.2.15)-(4.2.16) are defined by the restriction of the linear Lyapunov operator $L_{\bar{A}}$ to the subspace of symmetric matrices with 0 along the block diagonal. Let the space X be defined as in (3.4.1) and define:

$$X_{ij} \triangleq L_2^{(n_i+\hat{n}_i)x(n_j+\hat{n}_j)}(\mathbb{R},\mathcal{B},\lambda[t_o,T]);\ \langle x,y\rangle_{X_{ij}} \triangleq \operatorname{tr}\int_{t_o}^{T} x'(t)y(t)dt \tag{2.4.25}$$

$$\mathcal{P} \triangleq \{\{P_{ij}: i=1,\ldots,N;\ j=1,\ldots,N;\ i\neq j\}: P_{ij} \in X_{ij}\} \tag{4.2.26}$$

$$\langle P,Q\rangle_{\mathcal{P}} \triangleq \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \langle P_{ij}, Q_{ij}\rangle_{X_{ij}}$$

$$\pi: \mathcal{P} \to X: \{P_{ij}: i=1,\ldots,N;\ j=1,\ldots,N;\ i\neq j\} \to P_o \tag{4.2.27}$$

$$\pi^* \triangleq \text{adjoint of } \pi \tag{4.2.28}$$

$$P_o \triangleq \begin{bmatrix} 0 & P_{12} & \cdots & P_{1N} \\ P_{21} & 0 & \cdots & P_{2N} \\ \vdots & \vdots & & \vdots \\ P_{N1} & P_{N2} & \cdots & 0 \end{bmatrix} \tag{4.2.29}$$

Then the restricted Lyapunov operators in (4.2.15) and (4.2.16) are $\pi^* \circ L_{\bar{A}} \circ \pi$ and $\pi^* \circ L_{\bar{A}}^{-1} \circ \pi$ respectively. Similar definitions of $\pi$ and $\pi^*$ can be made for the time invariant case to result in the same form for the operators. Since $\pi$ is a projection, the operators in (4.2.15)-(4.2.16) are nonsingular if $\bar{A}$ is stable.

The infimal problems are nonlinear for any non-trivial parameterization. However, the structure of the nonlinear equations (4.2.17)-(4.2.20) is similar to the centralized necessary conditions (3.4.65)-(3.4.66) and (3.4.53)-(3.4.54). In fact, the centralized necessary conditions are a special case of (4.2.17)-(4.2.20) with N=1 and $S_i^k$, $T_i^k$, $D_i^k$ and $E_i^k$ being zero matrices.

The hierarchical structure of equations (4.2.13)-(4.2.24) involves a sequential information flow and computation. This pattern is a direct result of the Gauss-Seidel algorithm used to decompose the necessary conditions. As an alternative, the Jacobi algorithm can be used to introduce a parallel computation structure. The only modification to equations (4.2.13)-(4.2.24) is to replace $S_i^k$, $T_i^k$, $D_i^k$ and $E_i^k$ in equations (4.2.17)-(4.2.20) with $S_i^{k-1}$, $T_i^{k-1}$, $D_i^{k-1}$ and $E_i^{k-1}$. The Gauss-Seidel algorithm generally has better convergence characteristics than the Jacobi algorithm. The region of convergence is usually larger, and the rate of convergence is usually faster for the Gauss-Seidel iteration. However, if processing capability is available to solve the supremal and infimal problems in parallel, the total amount of computation time required may be less for the Jacobi iteration. The choice of algorithms must depend on the particular problem being solved.

## 4.3 Convergence for Weakly Coupled Systems

The iteration described by the decomposition of Section 4.2 can be written in the form of equation (2.3.14):

$$g(\beta^{k+1};\ \beta^k) = 0 \qquad (\beta^{k+1},\ \beta^k) \in B \times B \tag{4.3.1}$$

by defining (see (4.2.9)-(4.2.10))

$$g(\beta^{k+1};\ \beta^k) = \begin{bmatrix} f_S(\beta_S^{k+1};\ \beta_I^k) \\ f_I(\beta_I^{k+1};\ \beta_S^{k+1}) \end{bmatrix} \tag{4.3.2}$$

The iteration resulting from (4.3.1) will exhibit local convergence if the conditions of Theorem 2.4 are satisfied. The function g defined by (4.3.1)-(4.3.2) is Fréchet differentiable, and the calculation of the linear operators $\partial_1 g(\beta_S,\beta_I)$ and $\partial_2 g(\beta_S,\beta_I)$ from (4.2.9)-(4.2.10) and (4.2.13)-(4.2.24) is straightforward. The operators consist of terms involving the closed loop matrices $\tilde{A}_{ij}$ and the matrices $\tilde{K}$ and $\tilde{P}$. However, the evaluation of conditions (i) and (ii) of Theorem 2.4 is not practical for two reasons. First, as discussed in Section 2.3, the derivatives are evaluated at the presumably unknown solution. Secondly, the operator $\partial_1 g(\beta_S,\beta_I)$ must be inverted and the spectral radius of the linear operator $\partial_1 g^{-1}\ \partial_2 g$ must be evaluated. For the time varying problem the linear operators are infinite dimensional. Even in the time invariant case the operator $\partial_1 g^{-1}\ \partial_2 g$ can be represented as a $(2n^2 + s)x(2n^2 + s)$ matrix. The amount of computation required to check these conditions is often prohibitive.

However, Theorem 2.4 can be used to identify a class of problems for which the decomposition of the preceding section will converge. Given an arbitrary matrix $A = [A_{ij}]$ with $A_{ij} \in L_2^{n_i x n_j}(\mathbb{R},\mathcal{B},\lambda[t_o,T])$ (or $A_{ij} \in \mathbb{R}^{n_i x n_j}$) define:

$$A_D \triangleq \text{diag}[A_{ii}\ ;\ i=1,\ldots,N] \tag{4.3.3}$$

$$A_o \triangleq A - A_o \tag{4.3.4}$$

$$A_o \triangleq \{A_o\colon A_o \text{ as defined in (4.3.3)-(4.3.4)}\} \tag{4.3.5}$$

Note that $A_o$ is a sub-space of linear operators over the space $L_2^n(\mathbb{R},\mathcal{B},\lambda[t_o,T])$ or $\mathbb{R}^n$. Given any norm on either of the latter spaces, $A_o$ will assume the corresponding induced norm. Also define:

$(\beta_{So},\beta_{Io}) = (0,\beta_{Io}) \triangleq$ The optimal centralized solutions for the subsystems when $A_o = 0$. (4.3.6)

$(\beta_S^k,\beta_I^k)_{k=o}^{\infty} \triangleq$ The sequence generated by (4.3.1) . (4.3.7)

Using definitions (4.3.3)-(4.3.7) the following theorem can be stated.

Theorem 4.1: Assume there exist open neighborhoods $U_o$ of $0 \in A_o$ and $V_o$ of $(\beta_{So},\beta_{Io}) \in \mathcal{B}^S \times \mathcal{B}^I$ such that $A_o \in U_o$ and $(\beta_S,\beta_I) \in V_o$ implies $\partial g$ is nonsingular. Then there exist open neighborhoods U of $0 \in A_o$ and $V \subset \mathcal{B}^S \times \mathcal{B}^I$ such that

(i) $(\beta_S^o,\beta_I^o) \in V$

(ii) $A_o \in U$

implies

$$\lim_{k\to\infty} (\beta_S^k,\beta_I^k) = (\beta_S^*,\beta_I^*) \tag{4.3.8}$$

where $(\beta_S^*,\beta_I^*)$ is a local minimum of (3.2.13)-(3.2.14) (or(3.2.15)-(3.2.16)).

Proof: By direct calculation , the linear operators $\partial_1 g$ and $\partial_2 g$ are jointly continuous in $(\beta_S, \beta_I)$ and $A_o$ . Since $\partial g$ is nonsingular for $A_o \in U_o$ and $(\beta_S, \beta_I) \in V_o$, the implicit function theorem (Theorem 2.3) implies the variables $(\beta_S, \beta_I)$ are continuous functions of $A_o$. Hence $\partial_1 g$ and $\partial_2 g$ also depend continuously on $A_o$.

For any norm on $\mathcal{B}$ the induced norm on the space of linear operators $L(\mathcal{B})$ satisfies

$$\rho[\partial_1 g^{-1} \partial_2 g] \leq ||\partial_1 g^{-1} \partial_2 g|| \leq ||\partial_1 g^{-1}|| \; ||\partial_2 g|| \tag{4.3.9}$$

Again by direct calculation and use of the fact that $(\beta_S^*, \beta_I^*) = (\beta_{So}, \beta_{Io}) = (0, \beta_{Io})$ for $A_o = 0$:

$$\partial_2 g(\beta^*, \beta^*) = 0 \tag{4.3.10}$$

Since $\partial g(\beta^*, \beta^*)$ is nonsingular, (4.3.10) implies:

$$||\partial_1 g^{-1}(\beta^*, \beta^*)|| = M_o < \infty \tag{4.3.11}$$

Because $\partial_1 g$ and $\partial_2 g$ depend continuously on $A_o$, there exists an open neighborhood $\hat{U}$ of $0 \in A_o$ such that for all $A_o \in \hat{U}$:

$$||\partial_1 g^{-1}(\beta^*, \beta^*)|| < M_o + 1 \tag{4.3.12}$$

$$||\partial_2 g(\beta^*, \beta^*)|| < \frac{1}{M_o + 1} \tag{4.3.13}$$

where $\beta^*$ satisfies

$$g(\beta^*,\beta^*) = 0$$

for the given $A_o$. Combining (4.3.9) and (4.3.12)-(4.3.13) gives:

$$\rho[\partial_1 g^{-1}(\beta^*,\beta^*)\partial_2 g(\beta^*,\beta^*)] < 1 \qquad (4.3.14)$$

for all $A_o \in U$. Since (4.3.12) implies $\partial_1 g^{-1}(\beta^*,\beta^*)$ exists, by Theorem 2.4 there exists a neighborhood $V \subset B^S \times B^I$ (depending on $A_o$) such that the sequence $(\beta_S^k, \beta_I^k)_{k=0}^{\infty}$ converges to $(\beta_S^*, \beta_I^*)$. Since the function generated by the implicit function theorem for the dependence of g on $A_o$ is unique and continuous in an open neighborhood $U_1$ of $0 \in A_o$, the Frechet derivative $\partial g$ is nonsingular and $(\beta_{So}, \beta_{Io})$ is a minimum for $A_o = 0$, the limit of the sequence (4.3.8) is a local minimum of the corresponding optimization (3.3.13)-(3.3.14) or (3.3.15)-(3.3.16).

□

Theorem 4.1 serves to reinforce the intuition and insight which led to the decomposition of Section 4.2. For an arbitrary set of subsystems, the theorem says that sufficiently weak coupling (in terms of the magnitude of the coupling) will result in a convergent algorithm. Since it was the ultimate weak coupling situation (no interactions between subsystems) which inspired the decomposition, the result is reassuring.

Theorem 4.1 also gives insight into the importance of the parameterization of the original problem. The requirement that $\partial g$ be nonsingular in the appropriate regions is equivalent to requiring that the problem not be overparameterized. If the problem is over-parameterized, then conditions (i) and (ii) of Theorem 2.4 can not be satisfied simultaneously, as is demonstrated by the following argument. By the property of Fréchet derivatives,

$$\partial g = \partial_1 g + \partial_2 g$$

If $\partial g$ is singular, there is an element $x$ such that

$$(\partial g)\, x = (\partial_1 g)x + (\partial_2 g)x = 0 \tag{4.3.15}$$

Assuming $\partial_1 g$ is nonsingular, a brief manipulation of (4.3.15) results in the equation:

$$(I - \partial_1 g^{-1} \partial_2 g)x = 0 \tag{4.3.16}$$

Hence $[\partial_1 g^{-1} \partial_2 g]$ has an eigenvalue with modulus unity and does not satisfy condition (ii) of Theorem 2.4. Thus the choice of a parameterization which results in a locally unique solution to the minimization is crucial.

Finally, Theorem 2.4 can be used to demonstrate another property of the iteration defined by (4.2.13)-(4.2.24). It should be noted that if the convergence criterion of Theorem 2.4 is satisfied there will

be an integer $k_o$ such that the sequence $\{(\beta_S^k, \beta_I^k)\}_{k=k_o}^{\infty}$ converges monotonely to $(\beta_S^*, \beta_I^*)$. In the time invariant case, there exists another integer M such that each of the iterates $(\beta_S^k, \beta_I^k)$ for $k > M$ stabilize the system. Thus if the initial value $(\beta_S^o, \beta_I^o)$ is sufficiently close to $(\beta_S^*, \beta_I^*)$ and the iteration converges, each of the iterates $(\beta_S^k, \beta_I^k)$ will be stabilizing. For both the time varying and time invariant cases, a sufficiently close initial value will lead to a monotonely decreasing cost.

This suggests that the iteration prescribed by (4.2.13)-(4.2.24) can be applied in an on-line mode. The solutions $G^k$ and $H^k$ obtained at the end of the $k^{th}$ iteration can be applied until the $(k+1)^{st}$ iteration is completed. For an initial guess sufficiently close to the optimum and a convergent iteration, the system will be stable at all times and the value of the cost functional will decrease at each iteration.

## 4.4 Practical Convergence Tests

As mentioned in the previous section and in Section 2.3, the local convergence condition is not often useful for predicting the success or failure of a particular algorithm. One obvious problem is that the condition must be evaluated at the presumably unknown solution. The amount of computation involved in inverting $\partial_1 g$ and computing the spectral radius of $\partial_1 g^{-1} \partial_2 g$ can also be prohibitive. This section attempts to deal with these problems by finding simpler, more practical

tests which can be used to determine whether the iteration will converge.

As will be seen, the problem is a difficult one, and no complete solution is found. By restricting attention to the time invariant situation several partial results and guidelines are developed which may prove useful for certain problems. These results depend on properties of the individual problems for success. Hence, a major factor in the choice of which set of tests to use should be insight into the structure of the problem being considered.

The first step is to simplify the convergence condition. The iteration of concern is described by the function g given by (4.3.2) and repeated here:

$$g(\beta^{k+1}, \beta^{k}) = \begin{bmatrix} f_S(\beta_S^{k+1}, \beta_I^{k}) \\ f_I(\beta_I^{k+1}, \beta_S^{k+1}) \end{bmatrix} = 0 \qquad (4.4.1)$$

Here $f_S$, $f_I$, $\beta_S$ and $\beta_I$ are as defined in (4.2.5)-(4.2.24) and $\beta = (\beta_S, \beta_I)$. The local convergence condition for (4.4.1) is (Theorem 2.4)

$$\rho\{\partial_1 g^{-1}(\beta^*, \beta^*)\ \partial_2 g(\beta^*, \beta^*)\} < 1 \qquad (4.4.2)$$

From (4.4.1), $\partial_1 g$ and $\partial_2 g$ are given by:

$$\partial_1 g(\beta^*,\beta^*)\Delta\beta = \begin{bmatrix} \partial_1 f_S(\beta_S^*,\beta_I^*)\Delta\beta_S \\ \partial_1 f_I(\beta_I^*,\beta_S^*)\Delta\beta_I + \partial_2 f_I(\beta_I^*,\beta_S^*)\Delta\beta_S \end{bmatrix} \tag{4.4.3}$$

$$\partial_2 g(\beta^*,\beta^*)\Delta\beta = \begin{bmatrix} \partial_2 f_S(\beta_S^*,\beta_I^*)\Delta\beta_I \\ 0 \end{bmatrix} \tag{4.4.4}$$

Equations (4.4.3)-(4.4.4) and the subsequent development can be simplified by using the following matrix like notation for the partitioned linear operators of (4.4.3)-(4.4.4):

$$\partial_1 g(\beta^*,\beta^*)\Delta\beta = \begin{bmatrix} \partial_1 f_S(\beta_S^*,\beta_I^*) & 0 \\ \partial_2 f_I(\beta_I^*,\beta_S^*) & \partial_1 f_I(\beta_I^*,\beta_S^*) \end{bmatrix} \begin{bmatrix} \Delta\beta_S \\ \Delta\beta_I \end{bmatrix} \tag{4.4.5}$$

$$\partial_2 g(\beta^*,\beta^*)\Delta\beta = \begin{bmatrix} 0 & \partial_2 f_S(\beta_S^*,\beta_I^*) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\beta_S \\ \Delta\beta_2 \end{bmatrix} \tag{4.4.6}$$

Note that the blocks of the operators are themselves linear operators, and not matrices. However, the composition of partitioned operators follows the same notational rules as does multiplication of matrices.

Using this notation and equation (4.4.5), $\partial_1 g^{-1}$ can be computed:

$$\partial_1 g^{-1}(\beta^*,\beta^*) = \left[\begin{array}{c:c} \partial_1 f_S^{-1}(\beta_S^*\ \beta_I^*) & 0 \\ \hdashline -\partial_1 f_I^{-1}(\beta_I^*,\beta_S^*)\partial_2 f_I(\beta_I^*,\beta_S^*)\partial_1 f_S^{-1}(\beta_S^*,\beta_I^*) & \partial_1 f_I^{-1}(\beta_I^*,\beta_S^*) \end{array}\right] \tag{4.4.7}$$

Composing (4.4.7) with (4.4.6) in condition (4.4.2) gives:

$$\rho\{\partial_1 g^{-1}(\beta^*,\beta^*)\partial_2 g(\beta^*,\beta^*)\} \tag{4.4.8}$$

$$= \rho\left\{\begin{bmatrix} 0 & \partial_1 f_S^{-1}(\beta_S^*,\beta_I^*)\partial_2 f_S(\beta_S^*,\beta_I^*) \\ 0 & -\partial_I f_I^{-1}(\beta_I^*,\beta_S^*)\partial_2 f_I(\beta_I^*,\beta_S^*)\partial_1 f_S^{-1}(\beta_S^*,\beta_I^*)\partial_2 f_S(\beta_S^*,\beta_I^*) \end{bmatrix}\right\}$$

Thus:

$$\rho\{\partial_1 g^{-1}(\beta^*,\beta^*)\partial_2 g(\beta^*,\beta^*)\} = \rho\{\partial_1 f_I^{-1}(\beta_I^*,\beta_S^*)\partial_2 f_I(\beta_I^*,\beta_S^*)\partial_1 f_S^{-1}(\beta_S^*,\beta_I^*)\partial_2 f_S(\beta_S^*,\beta_I^*)\} \tag{4.4.9}$$

Condition (4.4.9) is certainly less costly (in terms of computation involved) to evaluate than (4.4.2). However, the linear operators $\partial_1 f_I$ and $\partial_1 f_S$ must still be inverted. The computation involved in performing these inversions or in computing the indicated spectral radius may still be too great. In addition, the linear operators still must be evaluated at the solution.

The latter problem will be considered first. In general, the only way to avoid this problem is to show that condition (4.4.9) holds for each possible solution. Usually this approach will not be possible. However, the continuity of the Fréchet derivatives and the assumed weak coupling between subsystems can be used for many problems to develop a good approximation to (4.4.9).

The basic idea is as follows. Since the linear operators in (4.4.9) are continuous functions of $\beta$ and the spectral radius is also

continuous, if condition (4.4.9) holds for some $\hat{\beta}$ sufficiently close to $\beta^*$, then it will also hold at $\beta^*$. By using certain properties of the structure of the problem it is often possible to determine a good approximation to $\beta^*$ relatively easily.

There are two possible choices of $\hat{\beta}$ which will be discussed here. Since the problem is assumed to be weakly coupled, one would expect the centralized solutions of the subsystem problems would be close to the overall solution. Indeed, the proof of Theorem 4.1 demonstrated this fact. The second possibility is to use the open loop system as choice for $\hat{\beta}$ (i.e., use G=0, H=0). This should work well whenever the controls are heavily penalized and the observation noise covariance is large. Either of the above two choices for $\hat{\beta}$ will be satisfactory for some systems and unsatisfactory for others. The important idea is the concept of choosing a good approximation to $\beta^*$ based on insight into the structure of the systems.

Even when a good approximation to $\beta^*$ is found, condition (4.4.9) may be too difficult to evaluate practically. The condition can be simplified further at the expense of weakening its sufficiency. For any norm on the space of linear operators $L(\mathcal{B},\mathcal{B})$ which is subordinate to a norm on $\mathcal{B}$, the spectral radius satisfies:

$$\rho(A) \leq ||A|| \qquad A \in L(\mathcal{B},\mathcal{B}) \tag{4.4.10}$$

Using (4.4.10) with condition (4.4.9) gives the following sufficient condition for local convergence:

$$||\partial_1 f_I^{-1}(\beta_I^*, \beta_S^*)||\ ||\partial_2 f_I(\beta_I^*, \beta_S^*)||\ ||\partial_1 f_S^{-1}(\beta_S^*, \beta_I^*)||\ ||\partial_2 f_S(\beta_S^*, \beta_I^*)|| < 1 \tag{4.4.11}$$

Weakening the condition still further, the iteration defined by (4.4.1) will converge locally if:

$$\begin{aligned} &||\partial_1 f_I^{-1}(\beta_I^*, \beta_S^*)||\ ||\partial_2 f_I(\beta_I^*, \beta_S^*)|| < 1 \\ &||\partial_1 f_S^{-1}(\beta_S^*, \beta_I^*)||\ ||\partial_2 f_S(\beta_S^*, \beta_I^*)|| < 1 \end{aligned} \tag{4.4.12}$$

If relations (4.4.12) are satisfied, then the total derivative $\partial g$ is an example of the class of strict block diagonally dominant linear operators (see Feingold and Varga [1962]):

Definition: Let $A = [A_{ij}]$ where $A \in L(X)$, $X = X_1 \times \ldots \times X_N$ is a product of N Banach spaces and $A_{ij} \in L(X_j, X_i)$ for $i,j=1,\ldots,N$. Then A is strictly block diagonally dominant if

$$||A_{ii}^{-1}|| \sum_{\substack{j=1 \\ j \neq i}}^{N} ||A_{ij}|| < 1 \tag{4.4.13}$$

where the norms on the indicated operators are induced by the Banach space norms.

The concept of a strictly diagonally dominant matrix (i.e. each $X_i = \mathbb{R}$ in the above definition) has been shown to be a sufficient condition for convergence of the Jacobi and Gauss-Seidel iterations

(see Varga [1962]). Similarly, the strict block diagonal dominance condition for linear operators can be shown to be sufficient for the convergence of the corresponding block Jacobi and block Gauss-Seidel iterations.

Theorem 4.2: Let $A \in \mathcal{L}(X)$ be a strictly block diagonally dominant linear operator. Then both the corresponding block Jacobi and block Gauss-Seidel iterations converge.

Proof: Let the linear operators D, E and F be given by (using the partitioned operator notation):

$$D = \text{diag}[A_{ii}: i=1,\ldots,N] \tag{4.4.14}$$

$$E = [-A_{ij}: \ j=1,\ldots,N-1; \ i=j+1,\ldots,N] \tag{4.4.15}$$

$$F = [-A_{ij}; \ i=1,\ldots,N-1; \ j=i+1,\ldots,N] \tag{4.4.16}$$

i.e., D is block diagonal, E is strictly block lower triangular and F is strictly block upper triangular. Then:

$$A = D - E - F \tag{4.4.17}$$

The block Jacobi iteration is defined by the splitting (see Example 2.1):

$$A_o = D \tag{4.4.18}$$

$$A_1 = - E - F \tag{4.4.19}$$

The corresponding convergence condition is:

$$\rho\{D^{-1}(E+F)\} < 1 \tag{4.4.20}$$

Let:

$$B \triangleq D^{-1}(E+F); \qquad B \in L(X) \tag{4.4.21}$$

Then

$$B_{ij} = \begin{cases} -A_{ii}^{-1} A_{ij} & i \neq j \\ 0 & i = j \end{cases}; \qquad B_{ij} \in L(X_j, X_i) \tag{4.4.22}$$

Similarly, the block Gauss-Seidel iteration is defined by the splitting (Example 2.2):

$$A_o = D - E \tag{4.4.23}$$

$$A_1 = - F \tag{4.4.24}$$

The corresponding convergence condition is:

$$\rho\{(D-E)^{-1}F\} < 1 \tag{4.4.25}$$

Let:

$$C \triangleq (D-E)^{-1} F \tag{4.4.26}$$

$$= (I-D^{-1}E)^{-1}(D^{-1}F); \; C \in L(X)$$

Also define:

$$L = D^{-1} E \tag{4.4.27}$$

$$U = D^{-1} F \tag{4.4.28}$$

Then

$$L_{ij} = \begin{cases} -A_{ii}^{-1} A_{ij} & j=1,\ldots,N-1;\ i=j+1,\ldots,N \\ 0 & j=1,\ldots,N;\ i=1,\ldots,j \end{cases} \tag{4.4.29}$$

$$U_{ij} = \begin{cases} -A_{ii}^{-1} A_{ij} & i=1,\ldots,N-1;\ j=i+1,\ldots,N \\ 0 & i=1,\ldots,N;\ j=1,\ldots,i \end{cases} \tag{4.4.30}$$

In terms of L and U, C is given by:

$$C = (I-L)^{-1} U \tag{4.4.31}$$

For the remainder of this proof, the norm on X will be taken as

$$||x|| \triangleq \max_i ||x_i||;\quad x = [x_1,\ldots,x_N] \in X,\ x_i \in X_i \tag{4.4.32}$$

where the norm on the right hand side of (4.4.32) is taken as the norm on $X_i$. The corresponding norm induced on $L(X)$ is

$$||A|| \triangleq \max_i \sum_{j=1}^{N} ||A_{ij}|| ; \tag{4.4.33}$$

$$A = [A_{ij}] \in L(X),\ A_{ij} \in L(X_j, X_i)$$

where again the norm on the right hand side of (4.4.33) is the norm on $L(X_j, X_i)$ induced by the norms on $X_j$ and $X_i$.

By (4.4.10) and using (4.4.33) and (4.4.22):

$$\rho\{B\} \leq ||B|| \tag{4.4.34}$$

$$= \max_i \sum_{j=1}^{N} ||B_{ij}||$$

$$= \max_i \sum_{\substack{j=1 \\ j\neq i}}^{N} ||A_{ii}^{-1} A_{ij}||$$

$$\leq \max_i ||A_{ii}^{-1}|| \sum_{\substack{j=1 \\ j\neq i}}^{N} ||A_{ij}||$$

If the strict block diagonal dominance condition (4.4.13) holds then

$$\rho\{B\} \leq \max_i ||A_{ii}^{-1}|| \sum_{\substack{j=1 \\ j\neq i}}^{N} ||A_{ij}|| < 1 \tag{4.4.35}$$

Thus the block Jacobi iteration converges.

Similarly, by (4.4.10)

$$\rho\{C\} \leq ||C|| = \max_i \sum_{j=1}^{N} ||C_{ij}|| \tag{4.4.36}$$

Since L is strictly block lower triangular,

$$(I-L)^{-1} = \sum_{m=0}^{N-1} L^m \tag{4.4.37}$$

Thus, C is given by:

$$C = \sum_{m=0}^{N-1} L^m U \tag{4.4.38}$$

Define $C^M$ by:

$$C^M \triangleq \sum_{m=0}^{M} L^m U$$

Given $C^M$, $C^{M+1}$ can be found by the recursion relation:

$$C^{M+1} = L\, C^M + U \tag{4.4.39}$$

Using the fact that L is strictly block lower triangular, $C^{M+1}_{ij}$ can be written:

$$C^{M+1}_{ij} = \sum_{k=1}^{i-1} L_{ik}\, C^M_{kj} + U_{ij} \tag{4.4.40}$$

Now, (4.4.36) is equivalent to

$$\rho\{C\} \leq ||C^{N-1}|| = \max_i \sum_{j=1}^{N} ||C^{N-1}_{ij}|| \tag{4.4.41}$$

The proof proceeds by induction on the exponent M; i.e., it will be proven (assuming strict block diagonal dominance) that

$$||C^M|| < 1 \tag{4.4.42}$$

for all $M \geq 0$. For M=0,

$$C^0_{ij} = U_{ij} \qquad i=1,\ldots,N;\ j=1,\ldots,N \tag{4.4.43}$$

Then, assuming (4.4.13) holds

$$||C^{o}|| = \max_{i} \sum_{j=i+1}^{N} ||U_{ij}|| \tag{4.4.44}$$

$$= \max_{i} ||A_{ii}^{-1}|| \sum_{j=i+1}^{N} ||A_{ij}||$$

$$\leq 1$$

For the induction step, assume

$$||C^{M}|| = \max_{i} \sum_{j=1}^{N} ||C_{ij}^{M}|| < 1 \tag{4.4.45}$$

By (4.4.40)

$$||C^{M+1}|| = \max_{i} \left\{ \sum_{j=1}^{N} ||C_{ij}^{M+1}|| \right\} \tag{4.4.46}$$

$$= \max_{i} \left\{ \sum_{j=1}^{N} || \sum_{k=1}^{i-1} L_{ik} C_{kj}^{M} + U_{ij} || \right\}$$

$$\leq \max_{i} \left\{ \sum_{j=1}^{N} \sum_{k=1}^{i-1} ||L_{ik}|| \, ||C_{kj}^{M}|| + ||U_{ij}|| \right\}$$

$$= \max_{i} \left\{ \sum_{k=1}^{i-1} ||L_{ik}|| \left[ \sum_{j=1}^{N} ||C_{kj}^{M}|| \right] + \sum_{j=1}^{N} ||U_{ij}|| \right\}$$

By (4.4.45), the bracketted term above is less than unity. Also,

$U_{ij}=0$ for $j \leq i$. Relation (4.4.47) becomes

$$||C^{M+1}|| \leq \max_i \left\{ \sum_{k=1}^{i-1} ||L_{ik}|| + \sum_{j=i+1}^{N} ||U_{ij}|| \right\} \tag{4.4.47}$$

$$= \max_i \left\{ \sum_{k=1}^{i-1} ||A_{ii}^{-1} A_{ik}|| + \sum_{j=i+1}^{N} ||A_{ii}^{-1} A_{ij}|| \right\}$$

$$\leq \max_i ||A_{ii}^{-1}|| \sum_{\substack{j=1 \\ j \neq i}}^{N} ||A_{ij}||$$

Again assuming strict block diagonal dominance, relation (4.4.47) becomes

$$||C^{M+1}|| < 1 \tag{4.4.48}$$

This concludes the induction step.

By the above induction, $||C^M|| < 1$ for each $M > 0$. In particular, it holds for M=N-1 in (4.4.41). Thus

$$\rho\{C\} \leq 1 \tag{4.4.49}$$

and hence the Gauss-Seidel iteration is convergent. □

Theorem 4.2 also applies to the nonlinear Jacobi and Gauss-Seidel iterations. The proof of Theorem 4.2 involved bounding the spectral radius of the corresponding linear iteration operators. As shown at the end of Section 2.3 ((2.3.34)-(2.3.37)) the decompositions which result in the nonlinear Jacobi and Gauss-Seidel iterations correspond to splittings of the derivative of the original equation. The splittings are identical to the splittings of the linear Jacobi and Gauss-Seidel

algorithms. The convergence condition is given in terms of the splittings of the derivative, and again is identical to the linear case. Thus, using the notation from Examples 2.6 and 2.7, Theorem 4.2 can be applied by defining:

$$A = \partial f(x) \tag{4.4.50}$$

$$A_{ij} = \partial_i f_j(x_1,\ldots,x_N) \tag{4.4.51}$$

When applied to the decomposition of Section 4.2 (using the notation of equation (4.4.1)), the strict block diagonal dominance condition (4.4.13) becomes

$$||\partial_1 f_S^{-1}(\beta_S^*,\beta_I^*)||\ ||\partial_2 f_S(\beta_S^*\ \beta_I^*)|| < 1 \tag{4.4.52}$$

$$\max_{1\le i\le N} ||\partial_1 f_{Ii}(\beta_{Ii}^*,\beta_S^*)||\ ||\partial_2 f_{Ii}(\beta_{Ii}^*,\beta_S^*)|| < 1$$

By Theorem 4.2, relations (4.4.52) are sufficient conditions for the local convergence of the decomposition of Section 4.2. Note that the form of (4.4.52) is similar to that of (4.4.12). In fact, (4.4.52) could have been derived from (4.4.12) by using the norm on X (see (4.4.32)) that was used in the proof of Theorem 4.2. Also, relations (4.4.52) and the strict block diagonal dominance condition from which they are derived can be interpreted in the context of the problem formulation of Chapter 3 as a weak coupling condition for the interconnected system structure.

Although the strict block diagonal dominance condition for the decomposition of Section 4.2 (condition (4.4.52)) is easier to evaluate than the spectral condition (4.4.2) or its simplification (4.4.9), there is still a need to compute the norms of the inverses of several linear operators. The difficulty involved in explicitly performing the inversion of these linear operators can be demonstrated by considering the time invariant, completely decentralized problem. The partial derivatives involved in (4.4.52) have a less complicated structure for this problem, but are still extremely complex:

$$[\partial_1 f_S(\beta_S,\beta_I)\Delta\beta_S]_{ij} = \begin{bmatrix} \sum_{\substack{\ell=1 \\ \ell\neq i,j}}^{N} (\bar{A}_{i\ell}\Delta\bar{P}_{\ell j} + \Delta\bar{P}_{i\ell}\bar{A}'_{j\ell}) + \bar{A}_{ii}\Delta\bar{P}_{ij} + \Delta\bar{P}_{ij}\bar{A}'_{ij} \\ \sum_{\substack{\ell=1 \\ \ell\neq i,j}}^{N} (\bar{A}'_{\ell i}\Delta\bar{K}_{\ell j} + \Delta\bar{K}_{i\ell}\bar{A}_{\ell j}) + \bar{A}'_{ii}\Delta\bar{K}_{ij} + \Delta\bar{K}_{ij}\bar{A}_{jj} \end{bmatrix} \tag{4.4.53}$$

$$[\partial_2 f_S(\beta_S,\beta_I)\Delta\beta_I]_{ij} = \begin{bmatrix} \bar{A}_{ij}\Delta\bar{P}_{jj} + \Delta\bar{P}_{ii}\bar{A}'_{ji} \\ \bar{A}'_{ji}\Delta\bar{K}_{jj} + \Delta\bar{K}_{ii}\bar{A}_{ij} \end{bmatrix}_{i,j=1,\ldots,N;\ i\neq j} \tag{4.4.54}$$

$$[\partial_1 f_I(\beta_I,\beta_S)\Delta\beta_I]_{ii} = \begin{bmatrix} -B'_i\,(\bar{K}_{ii}\Delta\bar{P}_{ii} + \Delta\bar{K}_{ii}\bar{P}_{ii})_{12} - \hat{B}'_i\,(\bar{K}_{ii}\Delta\bar{P}_{ii} + \Delta\bar{K}_{ii}\bar{P}_{ii})_{22} \\ \qquad + R_i(\Delta G_i P_{2i,2i} + G_i\Delta P_{2i,2i}) \\ (\bar{K}_{ii}\Delta\bar{P}_{ii} + \Delta\bar{K}_{ii}\bar{P}_{ii})_{21}C'_i - (\bar{K}_{ii}\Delta P_{ii} + \Delta K_{ii}P_{ii})_{22}\,\hat{C}'_i \\ \qquad + (\Delta K_{2i,2i}H_i + K_{2i,2i}\Delta H_i)\Theta_i \\ (\bar{A}_{ii}\Delta\bar{P}_{ii} + \Delta\bar{P}_{ii}\bar{A}'_{ii}) + \Delta\bar{A}_{ii}\bar{P}_{ii} + \bar{P}_{ii}\Delta\bar{A}_{ii} + \Delta\bar{\Xi}_{ii} \\ (\bar{A}'_{ii}\Delta\bar{K}_{ii} + \Delta\bar{K}_{ii}\bar{A}_{ii}) + \Delta\bar{A}'_{ii}\bar{K}_{ii} + \bar{K}_{ii}\Delta\bar{A}_{ii} + \Delta\bar{Q}_{ii} \end{bmatrix} \tag{4.4.55}$$

$$[\partial_2 f_I(\beta_I,\beta_S)\Delta\beta_S]_{ii}=\begin{bmatrix} -\sum_{\substack{j=1\\ j\neq i}}^{N} B_i' (\Delta\bar{K}_{ij}\bar{P}_{ji} + \bar{K}_{ij}\Delta\bar{P}_{ji})_{12} + \hat{B}_i'(\Delta\bar{K}_{ij}\bar{P}_{ji} + \bar{K}_{ij}\Delta\bar{P}_{ji})_{22} \\ \sum_{\substack{j=1\\ j\neq i}}^{N} (\Delta\bar{K}_{ij}\bar{P}_{ji} + \bar{K}_{ij}\Delta\bar{P}_{ji})_{21} C_i' - (\Delta\bar{K}_{ij}\bar{P}_{ji} + \bar{K}_{ij}\Delta\bar{P}_{ji})_{22} \hat{C}_i' \\ \sum_{\substack{j=1\\ j\neq i}}^{N} [\bar{A}_{ij}\Delta\bar{P}_{ji} + \Delta\bar{P}_{ij}\bar{A}_{ij}'] \\ \sum_{\substack{j=1\\ j\neq i}}^{N} [\bar{A}_{ji}'\Delta\bar{K}_{ji} + \Delta\bar{K}_{ij}\bar{A}_{ji}] \end{bmatrix} \tag{4.4.56}$$

It is obvious that (4.4.53) and (4.4.55) are difficult to invert in the present form. Kronecker products (see Bellman [1970] for a detailed discussion) can be used to place (4.4.53) and (4.4.55) in matrix form. The dimensions of the matrices which result from (4.4.53) and (4.4.55) are $\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}(n_i+\hat{n}_i)(n_j+\hat{n}_j)$ and $\sum_{i=1}^{n}[2(n_i+\hat{n}_i)^2+\hat{n}_i(m_i+p_i)]$ respectively. Thus the dimensions of these matrices can be large even for relatively small problems and the problem of calculating the norm of the inverse is still difficult.

Another approach which can be used to circumvent this difficulty is to bound or estimate the norms of the inverse of the operators in (4.4.53) and (4.4.55) in terms of the operators without explicitly inverting them. This approach essentially requires the determination

of the condition numbers of the operators (see, for example, Wilkinson [1965]) without inverting the operators. This is also a difficult problem. There exist methods to calculate (such as singular value decomposition; see Golub and Reinsch [1971]) or estimate (Cline et.al. [1977]) these norms when the operator is in matrix form. Again, the dimensions of the matrices could prohibit the use of such methods.

The problem with the approaches described above is that the operators to be inverted do not have a simple form, and can be expressed as matrices only at the expense of increasing the dimension of the problem considerably. Hence an exact solution may not be possible in many problems. If such is the case, a desirable approach would be to develop a guideline which, while not sufficient to ensure convergence, would give insight to the nature of the problem and would also be practical to compute.

One such guideline is to check the strict block diagonal dominance conditions (4.4.52) for the Lyapunov operator corresponding to the closed loop system matrix $\bar{A}$. There are several reasons why one might expect the test using the Lyapunov operator to provide a good indication of convergence of the overall problem. First, the operators in (4.4.53)-(4.4.54) are exactly those which occur in the decomposition of the Lyapunov operator. Also, the subsystem Lyapunov operators occur in the operator in (4.4.55). Second, when no perturbations are allowed in the control and filter gain matrices (i.e. $\Delta G$ and $\Delta H$ are required to be zero matrices), the decomposition developed in Section 4.2 becomes an iterative

method for the solution of the Lyapunov equations (3.3.57)-(3.3.58). Finally, from the usual system theoretic interpretations of the Lyapunov operator in the context of covariance and cost-to-go equations, one would expect that the weak coupling in the system should also be manifested in the Lyapunov operator. Conversely, if the Lyapunov operator is weakly coupled, the overall system is most likely weakly coupled also. These considerations are definitely ad hoc in nature; however, the similarity between the Lyapunov operator and (4.4.53)-(4.4.56), and the usual interpretations of the Lyapunov operator in system theory lend support to this approach.

It is still necessary to invert several linear operators to test the Lyapunov operator for block diagonal dominance. Define the operator $S_{AB}: \mathbb{R}^{mxn} \to \mathbb{R}^{mxn}$ for $A \in \mathbb{R}^{mxm}$ and $B \in \mathbb{R}^{nxn}$ by

$$S_{AB}(X) = AX + XB' \quad ; \qquad X \in R^{mxn} \tag{4.4.57}$$

Also, let $L_A = S_{AA}$. Then the operators which must be inverted are $L_{\tilde{A}_{ii}}$ and the projected Lyapunov operator $\pi^* \circ L_{\tilde{A}} \circ \pi$ (where $\pi$ is as defined in (4.2.25)-(4.2.29)). Inverting these operators is not much easier than inverting the operators in (4.4.53) and (4.4.55).

There are several bounds on the norm of the inverse of the Lyapunov operator $L_A$, but these are either very conservative or applicable only to particular forms of the matrix A (see Athay [1976]). The following theorem provides a tight bound which is useful for a large class of

matrices for the matrix norm induced by the Euclidean vector norm.

Theorem 4.3: Let $\sigma_{max}(A) = \sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A) = \sigma_{min}(A) \geq 0$ for $A \in R^{nxn}$ denote the singular values (see Golub and Reinsh [1971]) of the matrix A. Also let $\{\lambda_i(A): i=1,\ldots,n\}$ denote the eigenvalues of the matrix A.

For an arbitrary square matrix A, let let $A_s = \frac{1}{2}(A+A')$ denote the symmetric part of A and let $A_a = \frac{1}{2}(A-A')$ denote the antisymmetric part of A. Then the singular values of $S_{AB}$ are bounded by:

$$\sigma_{max}(S_{AB}) \leq \sigma_{max}(A) + \sigma_{max}(B) \tag{4.4.58}$$

$$\sigma^2_{min}(S_{AB}) \geq \sigma^2_{min}(A) + \sigma^2_{min}(B) + \min_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \{\lambda_i(A_s)\lambda_j(B_s)\} - \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \{\lambda_i(A_a)\lambda_j(B_a)\} \tag{4.4.59}$$

The proof of theorem 4.3 will require the following lemma.

Lemma 4.1: Let $A, B \in R^{nxn}$ be symmetric matrices, and let $\lambda_{max}(A) = \lambda_1(A) \geq \ldots \geq \lambda_n(A) = \lambda_{min}(A)$ denote the eigenvalues of A. Similarly, let $\lambda_{max}(B) = \lambda_1(B) \geq \ldots \geq \lambda_n(B) = \lambda_{min}(B)$ denote the eigenvalues of B. Then

$$\lambda_{max}(A+B) \leq \lambda_{max}(A) + \lambda_{max}(B) \tag{4.4.60}$$

$$\lambda_{min}(A+B) \geq \lambda_{min}(A) + \lambda_{min}(B) \tag{4.4.61}$$

Proof (of Lemma 4.1): Let $||\cdot||$ denote the Euclidean norm on $\mathbb{R}^n$. Since A+B is symmetric,

$$\begin{aligned}\lambda_{max}(A+B) &= \max_{||x||=1} x'(A+B)x \\ &\leq \max_{||x||=1} x'Ax + \max_{||x||=1} x'Bx \\ &= \lambda_{max}(A) + \lambda_{max}(B)\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_{min}(A+B) &= \min_{||x||=1} x'(A+B)x \\ &\geq \min_{||x||=1} x'Ax + \min_{||x||=1} x'Bx \\ &= \lambda_{min}(A) + \lambda_{min}(B)\end{aligned}$$

Also needed for the proof of Theorem 4.3 is the notation and properties of the Kronecker product. Bellman [1970] (Chapter 2) provides a detailed discussion of this subject.

Proof (of Theorem 4.3): Without loss of generality, it is assumed that $S_{AB}$ is represented in matrix form:

$$S_{AB} = A \otimes I + I \otimes B \tag{4.4.62}$$

The square of the singular values of $S_{AB}$ are the eigenvalues of:

$$S'_{AB} S_{AB} = (A \otimes I + I \otimes B)'(A \otimes I + I \otimes B) \qquad (4.4.63)$$

Using properties of the Kronecker product, equation (4.4.63) simplifies to:

$$S'_{AB} S_{AB} = (A'A \otimes I + I \otimes B'B)+(A' \otimes B + A \otimes B') \qquad (4.4.64)$$

The first term on the right of (4.4.64) is a symmetric matrix with eigenvalues

$$\sigma_i^2(A) + \sigma_j^2(B);\ \{i=1,\ldots,m\};\ \{j=1,\ldots,n\}\ .$$

The second term on the right is twice the symmetric part of $A' \otimes B$. The largest singular value (also the largest eigenvalue) of $(A' \otimes B + A \otimes B')$ is bound (using Lemma 4.1) by:

$$\sigma_{max}(A' \otimes B + A \otimes B') \leq 2\,\sigma_{max}(A' \otimes B) = 2[\sigma_{max}(A)\sigma_{max}(B)] \qquad (4.4.65)$$

The final equality follows from the series of equalities (Bellman [1970]):

$$\begin{aligned}\sigma_1^2(A' \otimes B) &= \sigma_1[(A' \otimes B)'(A \otimes B)] \\ &= \sigma_1[A\,A' \otimes B'B] \\ &= \sigma_1^2(A)\ \sigma_1^2(B)\end{aligned}$$

Since both terms on the right of (4.4.64) are symmetric,

$$\sigma_1^2(S_{AB}) = \sigma_1(S_{AB}'S_{AB}) \leq \sigma_1(A'A \otimes I + I \otimes B'B) + \sigma_1(A' \otimes B + A' \otimes B')$$

$$\leq \sigma_1^2(A) + \sigma_1^2(B) + 2\sigma_1(A)\sigma_1(B)$$

$$= [\sigma_1(A) + \sigma_1(B)]^2 \qquad (4.4.66)$$

which is identical to (4.4.58).

Returning to (4.4.64), the minimum singular value of $(A' \otimes B + A \otimes B')$ is:

$$\sigma_{min}[(A' \otimes B + A \otimes B')] = \min_{||x||=1} x' (A' \otimes B + A \otimes B')x$$

$$= 2 \min_{||x||=1} x' (A' \otimes B)x \qquad (4.4.67)$$

where $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^{mxn}$. Let $A_s$ and $A_a$ denote the symmetric and anti symmetric parts of A. Similarly define $B_s$ and $B_a$. Then for any x,

$$x'(A' \otimes B)x = x'[(A_s' + A_a') \otimes (B_s + B_a)]x \qquad (4.4.68)$$

$$= x'[(A_s - A_a) \times (B_s + B_a)]x$$

$$= x'[A_s \otimes B_s - A_a \otimes B_s - A_a \otimes B_a + A_s \otimes B_a]x$$

Since

$$(A_a \otimes B_s)' = - A_a \otimes B_s \qquad (4.4.69)$$

$$(A_s \otimes B_a)' = - A_s \otimes B_a \qquad (4.4.70)$$

(i.e. the matrices $A_a \otimes B_s$ and $A_s \otimes B_a$ are antisymmetric) equation (4.4.68) becomes

$$x'[A' \otimes B]x = x'[A_s \otimes B_s - A_a \otimes B_a]x$$

Substituting (4.4.71) in (4.4.67) gives

$$\sigma_{min}[(A' \otimes B + A' \otimes B')] = 2 \min_{||x||=1} x'[A_s \otimes B_s - A_a \otimes B_a]x$$

$$\geq 2 \{ \min_{||x||=1} x'(A_s \otimes B_s)x - \max_{||x||=1} x'(A_a \otimes B_a)x \} \tag{4.4.72}$$

Since $(A_s \otimes B_s)$ and $(A_a \otimes B_a)$ are symmetric, the minimum and maximum in (4.4.72) are the smallest and largest eigenvalues, respectively, of the corresponding matrices. Thus, the quantities on the right of (4.4.72) are:

$$\min_{||x||=1} x'(A_s \otimes B_s)x = \min_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \{\lambda_i(A_s)\lambda_j(B_s)\} \tag{4.4.73}$$

$$\max_{||x||=1} x'(A_a \otimes B_a)x = \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \{\lambda_i(A_a)\lambda_j(B_a)\} \tag{4.4.74}$$

Now, the minimum singular value of $S_{AB}$ is bounded by:

$$\sigma_n^2(S_{AB}) = \sigma_n(S_{AB}' S_{AB}) \tag{4.4.75}$$

$$\geq \sigma_n(A'A \otimes I + I \otimes B'B) + \sigma_n(A' \otimes B + A' \otimes B')$$

Combining (4.4.72)-(4.4.74) and substituting the result in (4.4.75) gives (4.4.59). □

The importance of Theorem 4.3 lies in the fact that the Euclidean induced norms on $S_{AB}$ and $S_{AB}^{-1}$ are given by

$$||S_{AB}|| = \sigma_1(S_{AB}) \tag{4.4.76}$$

$$||S_{AB}^{-1}|| = \frac{1}{\sigma_n(S_{AB})} \tag{4.4.77}$$

Thus the norms in (4.4.76) and (4.4.77) are bounded by:

$$||S_{AB}|| \leq \sigma_1(A) + \sigma_2(B) \tag{4.4.78}$$

$$||S_{AB}^{-1}|| \leq \left[\sigma_m^2(A) + \sigma_n^2(B) + \min_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \lambda_i(A_s)\lambda_j(B_s) - \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \lambda_i(A_a)\lambda_j(B_a)\right]^{-1/2} \tag{4.4.79}$$

These bounds can be evaluated through operations on the original A and B matrices rather than the Kronecker expansion $S_{AB}$.

At this point it should be noted that the bound (4.4.59) may not give any useful information (i.e., the right hand side of (4.4.59) may be negative). It is easily seen that the last term

$$- \max_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \lambda_i(A_a)\lambda_j(B_a)$$

is always non-positive since the eigenvalues of an antisymmetric matrix are purely imaginary and occur in complex conjugate pairs. Thus if the antisymmetric part of A and B are non-zero, the bound could be negative

as the following example shows.

Example 4.1: Consider $L_A = S_{AA}$ with

$$A = \begin{bmatrix} -1 & \alpha \\ 0 & -1 \end{bmatrix}$$

The singular values of A are

$$\sigma_1^2 = 1 + \frac{\alpha^2}{2} + \frac{\alpha}{2}\sqrt{4 + \alpha^2}$$

$$\sigma_2^2 = 1 + \frac{\alpha^2}{2} - \frac{\alpha}{2}\sqrt{4 + \alpha^2}$$

The symmetric and antisymmetric part of A are

$$A_s = \begin{bmatrix} -1 & \frac{\alpha}{2} \\ \frac{\alpha}{2} & -1 \end{bmatrix} \qquad A_a = \begin{bmatrix} 0 & \frac{\alpha}{2} \\ -\frac{\alpha}{2} & 0 \end{bmatrix}$$

The eigenvalues of the above two matrices are:

$$\lambda_1(A_s) = -1 + \frac{\alpha}{2}; \lambda_2(A_s) = -1 - \frac{\alpha}{2}$$

$$\lambda_1(A_a) = j\frac{\alpha}{2}; \quad \lambda_2(A_a) = -j\frac{\alpha}{2}$$

where j denotes $\sqrt{-1}$ in the above. Using the above values, the bounds in (4.4.58)-(4.4.59) are:

$$\sigma_1^2(L_A) \leq 2\left[1 + \frac{\alpha^2}{2} + \frac{\alpha}{2}\sqrt{4+\alpha^2}\right]^{1/2} \tag{4.4.80}$$

$$\sigma_4(L_A) \leq \left[3 + \frac{\alpha^2}{2} - \alpha\sqrt{4+\alpha^2}\right]^{1/2} \tag{4.4.81}$$

The right hand side of (4.4.81) is negative whenever

$$\alpha > \left[4\sqrt{7} - 2\right]^{1/2} \approx 2.93 \tag{4.4.82}$$

However, taking $\alpha=3$ the singular values of $L_A$ are found to be:

$$\sigma_1(L_A) = \left[16\sqrt{252}\right]^{1/2} \approx 5.65$$

$$\sigma_2(L_A) = 4$$

$$\sigma_3(L_A) = 2$$

$$\sigma_4(L_A) = \left[16 - \sqrt{252}\right]^{1/2} \approx .354$$

□

The above example demonstrated a case in which the bound (4.4.59) was not useful. However, there are many cases in which the bound will be accurate. For example, the bound is exact for stable symmetric matrices since

$$A_a = 0 \qquad B_a = 0$$

and

$$\min_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \lambda_i(A_s)\lambda_j(B_s) = \sigma_{min}(A)\sigma_{min}(B)$$

Thus one would expect the bound to be good for matrices which are almost symmetric. There are other cases in which $A_a$ and $B_a$ are not small but the bound in (4.4.59) is still tight.

Example 4.2: Again consider $L_A = S_{AA}$ with

$$A = \begin{bmatrix} -\alpha & \omega \\ -\omega & -\alpha \end{bmatrix} \qquad \begin{matrix} \alpha > 0 \\ \omega \geq 0 \end{matrix}$$

The singular values of A are

$$\sigma_1(A) = \sigma_2(A) = \sqrt{\alpha^2 + \omega^2}$$

The symmetric and antisymmetric parts of A are

$$A_s = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix} \qquad A_a = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

The eigenvalues of the above two matrices are:

$$\lambda_1(A_s) = \lambda_2(A_s) = -\alpha$$

$$\lambda_1(A_a) = j\omega \qquad \lambda_2(A_a) = -j\omega$$

Using the above values, the bounds in (4.4.58)-(4.4.59) becomes

$$\sigma_1(L_A) \leq 2\sqrt{\alpha^2 + \omega^2}$$

$$\sigma_{min}(L_A) \geq 2\alpha$$

The singular values of $L_A$ are:

$$\sigma_1(L_A) = 2\sqrt{\alpha^2 + \omega^2}$$

$$\sigma_2(L_A) = 2\sqrt{\alpha^2 + \omega^2}$$

$$\sigma_3(L_A) = 2\alpha$$

$$\sigma_4(L_A) = 2\alpha$$

Thus the bounds are tight. □

To conclude this section, it is useful to summarize the results of this section and outline the procedure one would use to test for convergence of the iteration defined in Section 4.2. Equations (4.4.9), (4.4.11), (4.4.12) and (4.4.52) are a series of sufficient conditions which become successively weaker but also successively easier to evaluate. To evaluate these conditions exactly the solution is required. However, the continuity of the derivatives implies that approximations to the solution can be used to evaluate the conditions.

There is obviously a tradeoff between computational complexity and the strength of the test for (4.4.9), (4.4.11)-(4.4.12) and (4.4.52). Also, there may be problems for which even the simplest test (4.4.52) is too complex to evaluate. For such problems the same decomposition applied to the corresponding Lyapunov equation may provide a good indication of the convergence properties of the original decomposition. The tests (4.4.9), (4.4.11), (4.4.12) and (4.4.52) applied to the Lyapunov equation decomposition are more easily evaluated. Also, the bounds given in Theorem 4.3 can be used to simplify the tests further.

## 4.5 Summary

Using the decomposition framework of Chapter 2, Section 4.2 developed a two level hierarchical computation structure corresponding to the nonlinear Gauss-Seidel iteration for the solution of the linear stochastic control problem formulated in Chapter 3. Several properties of the iteration were discussed in Section 4.3. The local convergence of the iteration for sufficiently weakly coupled systems was demonstrated. Section 4.3 also showed that the value of the cost decreased at each iteration if the starting point of the iteration is sufficiently close to the solution and if the iteration converges.

Section 4.4 discussed practical a priori tests for convergence of the decomposition procedure of Section 4.2. Several simplifications of the convergence condition of Theorem 2.4 were developed. For situation when the simplified tests can not be used, a guideline based on the analysis of an iterative solution of a Lyapunov equation was presented. In the process of developing the simplified tests and guidelines, two new results were developed. The first showed that if the Fréchet derivative of the decomposed function were strictly block diagonally dominant then the corresponding Gauss-Seidel and Jacobi iterations are locally convergent. The second result provided upper and lower bounds for the singular values of the Sylvester operator (4.4.57). In addition to providing bounds for the convergence conditions of this chapter, the latter result can also be used to bound the condition number of the Sylvester operator.

Finally, it should be noted that the concluding remarks of Section 2.5 apply to this chapter also. Insight into the structure of the system which is being decomposed is the most important ingrediant in the choice of the decomposition and the design of a convergence test.

# 5. SOLUTION METHODS FOR THE DECOMPOSED PROBLEM

## 5.1 Introduction

The purpose of decomposing the linear stochastic optimal control problem (formulated in Chapter 3) was to reduce the computational burden associated with computing the best linear controller that satisfies the information flow constraints of the problem. The reduction is accomplished by replacing the original problem by a group of smaller subproblems which are repeatedly solved. To achieve the purpose of the decomposition, efficient solution methods to the subproblems must be available.

The supremal and infimal subproblems developed in Chapter 4 are still nonlinear, and the supremal can be relatively large. However, as will be seen in this chapter, there is additional structure present in both types of subproblems. This structure will be exploited to develop secondary decompositions which will lead to efficient solution methods.

The supremal problem of Chapter 4 will be studied in Section 5.2. A decomposition algorithm which further exploits the weakly coupled subsystem structure is developed. A second method which is only applicable to the completely decentralized problem is also developed. Section 5.3 extends several previously developed algorithms to the infimal problem of Chapter 4. Both Sections 5.2 and 5.3 consider the convergence characteristics of the proposed algorithms. Section 5.4 presents an application of the algorithms of Chapter 4 and Sections

5.2 - 5.3. Finally, Section 5.5 discusses the algorithms in the context of the overall problem solution.

## 5.2 The Supremal Problem

The decomposition algorithm presented in Chapter 4 was developed to take advantage of the assumed weak interactions between subsystems. The same line of thinking leads to a similar decomposition algorithm for the solution of the supremal problem. To motivate the decomposition of Chapter 4, the completely decoupled interconnected system was examined as the limiting case of the general weakly coupled system. For this case, the solution to the problem was simply to take the centralized optimal control problem solutions of the subsystems as the diagonal blocks of G, H, $\tilde{P}$ and $\tilde{K}$ and the zero matrix as the off diagonal blocks. The equations for the off diagonal blocks were still present. However, the solutions were the null solutions because they driving terms were linear functions of the subsystem coupling matrices and hence were zero. These off diagonal equation blocks constitute the supremal problem of Chapter 4. Hence the same logic which led to the decomposition of the original problem can be used to motivate a similar decomposition of the supremal problem.

As in Chapter 4, a Gauss-Seidel iteration (see Example 2.6) will be used. Again, the corresponding Jacobi algorithm requires only minor modifications to the following discussion. At each iteration (i.e., for fixed $\beta_1$), the supremal problem (4.2.13)-(4.2.16) is of the form:

$$h(\beta_S) = 0 \qquad \beta_S \in \mathcal{B}^S \tag{5.2.1}$$

where $h: \mathcal{B}^S \to \mathcal{B}^S$. To exploit the weakly coupled structure of the problem as outlined in the preceding paragraph, define the decomposition of $\mathcal{B}$.

$$\mathcal{B}^S = \mathcal{B}_{12} \times \mathcal{B}_{23} \times \ldots \times \mathcal{B}_{N,1} \times \mathcal{B}_{13} \times \ldots \times \mathcal{B}_{N-2,N} \times \mathcal{B}_{14} \times \ldots \times \mathcal{B}_{1N} \tag{5.2.2}$$

$$\mathcal{B}_{ij} \triangleq \{\tilde{K}_{ij}, \tilde{P}_{ij}, G_{ij}, G_{ji}, H_{ij}, H_{ji}\} \quad i=1,\ldots,N-1;\ j=2,\ldots,N;\ i\neq j \tag{5.2.3}$$

$$h_{ij} \triangleq \{\dot{\tilde{K}}_{ij} = -[\tilde{A}'\tilde{K} + \tilde{K}\tilde{A} + \tilde{Q}]_{ij}\ ; \tilde{K}_{ij}(t) = (\tilde{K}_T)_{ij}\ ; \tag{5.2.4}$$

$$\dot{\tilde{P}}_{ij} = [\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{\Xi}]_{ij}\ ;\ \tilde{P}_{ij}(t_o) = (\tilde{P}_o)_{ij}\ ;$$

$$\frac{\partial J}{\partial G_{ij}} = 0;\ \frac{\partial J}{\partial G_{ji}} = 0;\ \frac{\partial J}{\partial H_{ij}} = 0;\ \frac{\partial J}{\partial H_{ji}} = 0\}$$

$$i=1,\ldots,N-1;\ j=2,\ldots,N;\ i\neq j$$

In the time invariant case, the assignment of variables and equations is analogous with time derivatives replaced by the zero matrix.

When the Gauss-Seidel algorithm is used with more than two subproblems, the order in which the subproblems are solved can affect the rate of convergence (see Fox [1964]). The ordering implied by the decomposition (5.2.2) can be expected to produce good results when the forward and backward coupling between subproblems are of the same order of magnitude (see Lehtomaki [1978]). The resulting iteration is:

$$h_{ij}(\beta_{12}^{k+1},\ldots,\beta_{i-1,j-1}^{k+1},\beta_{i,j}^{k+1},\beta_{i+1,j+1}^{k},\ldots,\beta_{iN}^{k}) = 0 \qquad (5.2.5)$$

$$\ell=1,\ldots,N-1;\ i=1,\ldots,N-\ell;\ j=i+\ell$$

Note that the above ordering amounts to solving for the first super diagonal blocks first, the second super diagonal blocks second, etc. This ordering is illustrated in Figure 5.1.

Given an (i,j) pair, for fixed $\beta_I$ (i.e. fixed $\{\tilde{K}_{ii}, \tilde{P}_{ii}, G_{ii}, H_{ii}: i=1,\ldots,N\}$) define the following notation (similar to that in (4.2.11)-(4.2.12)).

$$\bar{\bar{K}}_{\ell m}^{k} = \begin{cases} \tilde{K}_{\ell m}^{k} & |\ell-m| < |i-j| \text{ and } n\neq\ell \\ \tilde{K}_{\ell m}^{k} & |\ell-m| = |i-j|,\ m\neq\ell, \text{ and } k \leq i \\ \tilde{K}_{\ell\ell} & \ell=m \\ \tilde{K}_{\ell m}^{k-1} & \text{otherwise} \end{cases} \qquad (5.2.6)$$

$$\bar{\bar{K}}^{k} = \left[\bar{\bar{K}}_{\ell m}^{k}\right] \qquad (5.2.7)$$

i.e., for the $(i,j)^{th}$ problem $\bar{\bar{K}}^{k}$ is the matrix with blocks equal to the most recent solutions of a 1 the other problems. In a similar manner, define for the $(i,j)^{th}$ problem $\bar{\bar{P}}^{k}$, $\bar{\bar{G}}^{k}$, $\bar{\bar{H}}^{k}$, $\bar{\bar{A}}^{k}$, $\bar{\bar{Q}}^{k}$ and $\bar{\bar{\Xi}}^{k}$. It should be noted that the above definition of the bar notation depends on the indices i and j, but this dependence is not explicit in the notation.

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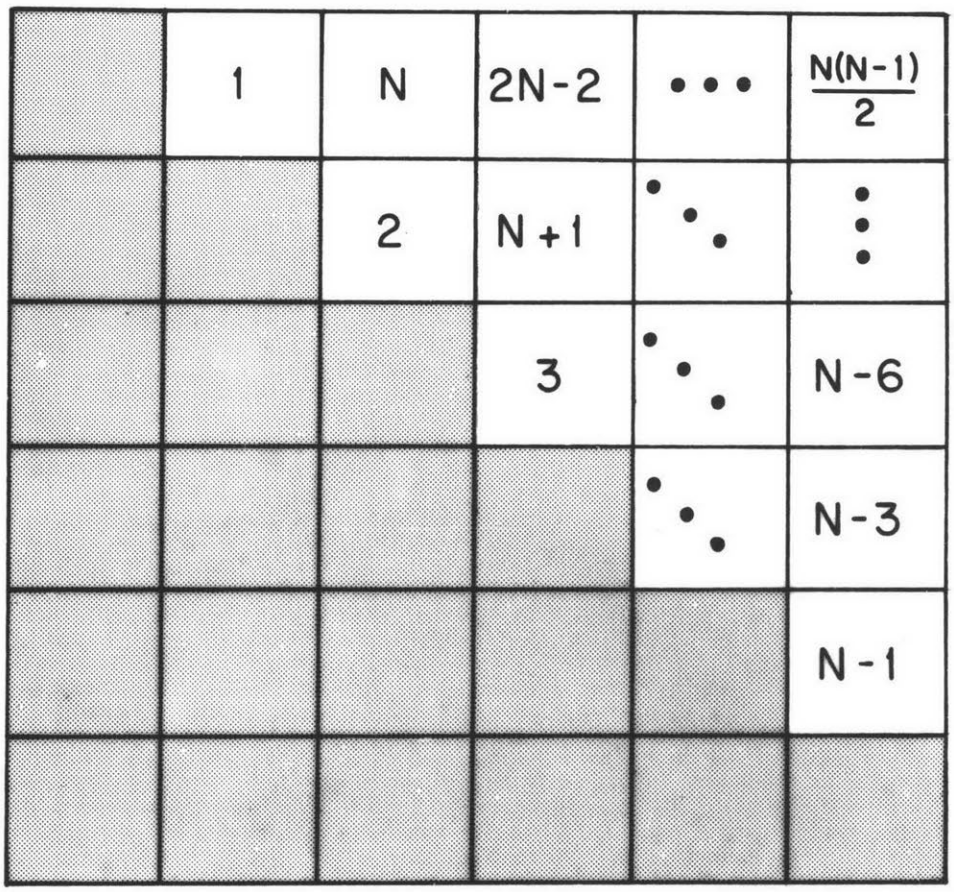


Figure 5.1: Order of Solution of Subproblems for the Gauss-Seidel Algorithm

Using the bar notation of (5.2.6)-(5.2.7), equation (5.2.5) can be rewritten in terms of the system matrices as follows.

Supremal Subproblem (i,j)

$$- B_i'[(\bar{K}_{ij}^k \bar{P}_{jj}^k)_{12} + (\bar{K}_{ii}^k \bar{P}_{ij}^k)_{12}] - \hat{B}_i'[(\bar{K}_{ij}^k \bar{P}_{jj}^k)_{22} + (\bar{K}_{ii}^k \bar{P}_{ij}^k)_{22}]$$
$$+ R_i \bar{G}_{ij}^k (\bar{P}_{jj}^k)_{22} + R_i \bar{G}_{ii} (\bar{P}_{ij}^k)_{22} + S_{ij}^k = 0 \tag{5.2.8}$$

$$[(\bar{K}_{ij}^k \bar{P}_{jj}^k)_{21} + (\bar{K}_{ii}^k \bar{P}_{ij}^k)_{21}] C_j - [(\bar{K}_{ij}^k \bar{P}_{jj}^k)_{22} + (\bar{K}_{ii}^k \bar{P}_{ij}^k)_{22}]\hat{C}_j' \tag{5.2.9}$$
$$+ (\bar{K}_{ii}^k)_{22} \bar{H}_{ij}^k \Theta_j + (\bar{K}_{ij}^k)_{22} \bar{H}_{jj}^k \Theta_j + T_{ij}^k = 0$$

$$-B_j'[(\bar{K}_{ji}^k \bar{P}_{ii}^k)_{12} + (\bar{K}_{jj}^k \bar{P}_{ji}^k)_{12}] - \hat{B}_j'[(\bar{K}_{ji}^k \bar{P}_{ii}^k)_{22} + (\bar{K}_{jj}^k \bar{P}_{ji}^k)_{22}] \tag{5.2.10}$$
$$+ R_j \bar{G}_{ji}^k (\bar{P}_{ii}^k)_{22} + R_j \bar{G}_{jj}^k (\bar{P}_{ji}^k)_{22} + S_{ji}^k = 0$$

$$[(\bar{K}_{ji}^k \bar{P}_{ii}^k)_{21} + (\bar{K}_{jj}^k \bar{P}_{ji}^k)_{21}] C_i' - [(\bar{K}_{ji}^k \bar{P}_{ii}^k)_{22} + (\bar{K}_{jj}^k \bar{P}_{ji}^k)_{22}]\hat{C}_i \tag{5.2.11}$$
$$+ (\bar{K}_{jj}^k)_{22} \bar{H}_{ji}^k \Theta_i + (\bar{K}_{ji}^k)_{22} \bar{H}_{jj}^k \Theta_j + T_{ji}^k = 0$$

$$\bar{A}_{ij}^k \bar{P}_{ij}^k + \bar{P}_{ij}^k (\bar{A}_{jj}^k)' + \bar{A}_{ij}^k \bar{P}_{jj}^k + \bar{P}_{ii}^k (\bar{A}_{ji}^k) + \bar{\Xi}_{ij}^k + D_{ij}^k = 0 \tag{5.2.12}$$

$$(\bar{A}_{ii}^k)' \bar{K}_{ij}^k + \bar{K}_{ij}^k \bar{A}_{jj}^k + (\bar{A}_{ji}^k)' \bar{K}_{jj}^k + \bar{K}_{ii}^k \bar{A}_{ij}^k + \bar{Q}_{ij}^k + E_{ij}^k = 0 \tag{5.2.13}$$

where

$$S_{ij}^{k} = \sum_{\substack{\ell=1 \\ \ell\neq i,j}}^{N} \mathcal{D}_C(\bar{G}_{i\ell}^{k};\ \bar{P}_{\ell j}^{k},\ \bar{K}_{i\ell}^{k},\ B_i,\ \hat{B}_i,\ R_i) \tag{5.2.14}$$

$$T_{ij}^{k} = \sum_{\substack{\ell=1 \\ \ell\neq i,j}}^{N} \mathcal{D}_f(\bar{H}_{\ell j}^{k};\ \bar{P}_{\ell j}^{k},\ \bar{K}_{i\ell}^{k},\ C_j,\ \hat{C}_j,\ \Theta_j) \tag{5.2.15}$$

$$D_{ij}^{k} = \sum_{\substack{\ell=1 \\ \ell\neq i,j}}^{N} \left[\ \bar{A}_{i\ell}^{k}\ \bar{P}_{\ell j}^{k} + \bar{P}_{i\ell}^{k}(\bar{A}_{j\ell}^{k})'\right] \tag{5.2.16}$$

$$E_{ij}^{k} = \sum_{\substack{\ell=1 \\ \ell\neq i,j}}^{N} (\bar{A}_{\ell i}^{k})'\ \bar{K}_{\ell j}^{k} + \bar{K}_{i\ell}^{k}\ \bar{A}_{\ell j}^{k} \tag{5.2.17}$$

Although the above problem appears to be extremely complex upon first examination, it is linear in the variables of the problem ($G_{ij}$, $G_{ji}$, $H_{ij}$, $H_{ji}$, $\tilde{P}_{ij}$ and $\tilde{K}_{ij}$). Also, in the completely decentralized problem ($G_{ij} = 0$, $H_{ij} = 0$ for $i,j=1,\ldots,N$ and $i\neq j$) the problem reduces to the solution of two Sylvester equations. Such equations can be solved efficiently (see Bartels and Stewart [1972]).

A second property of (5.2.8)-(5.2.17) in the completely decentralized problem should also be noted. If the original system dynamics matrix is block tri-diagonal the N-M problems for which $|i-j| = M$ are not coupled. Thus these problems can be solved in parallel. This property also suggest the possibility of interleaving the Jacobi algorithm with

the Gauss-Seidel algorithm. The original supremal problem is decomposed using the Gauss-Seidel algorithm into N-1 problems consisting of all the equations and variables on the same super block diagonal; i.e. all (i,j) blocks of equations and matrices for which $\{|i-j| = M;\ M=1,\ldots,N-1\}$ would be solved at the same time. Then each of the resulting N-1 problems is once more decomposed using the Jacobi algorithm along the same lines as the decomposition in (5.2.2)-(5.2.4). The only change to equations (5.2.8)-(5.2.17) is the iteration indexing for several of the matrices. By redefining the bar notation for the $(i,j)^{th}$ problem as

$$\bar{K}^k_{\ell m} = \begin{cases} \tilde{K}^k_{\ell m} & |\ell-m| < |i-j| \quad \ell \neq m \\ \tilde{K}^k_{\ell m} & \ell=i, \quad m=j \\ K_{\ell\ell} & \ell=m \\ \tilde{K}^{k-1}_{\ell m} & \text{otherwise} \end{cases} \tag{5.2.18}$$

equations (5.2.8)-(5.2.17) apply directly to the new decomposition.

The latter decomposition algorithm for the solution of the supremal problem would be expected to work well when the interactions between subsystems decrease as the difference between their indexing increases. Also, this algorithm possesses a multilevel hierarchical interpretation (see Figure 5.2). Each group of problems on the same super block diagonal is viewed as a level in the hierarchy. The $M^{th}$ level consists of N-M decision units which have two functions. The

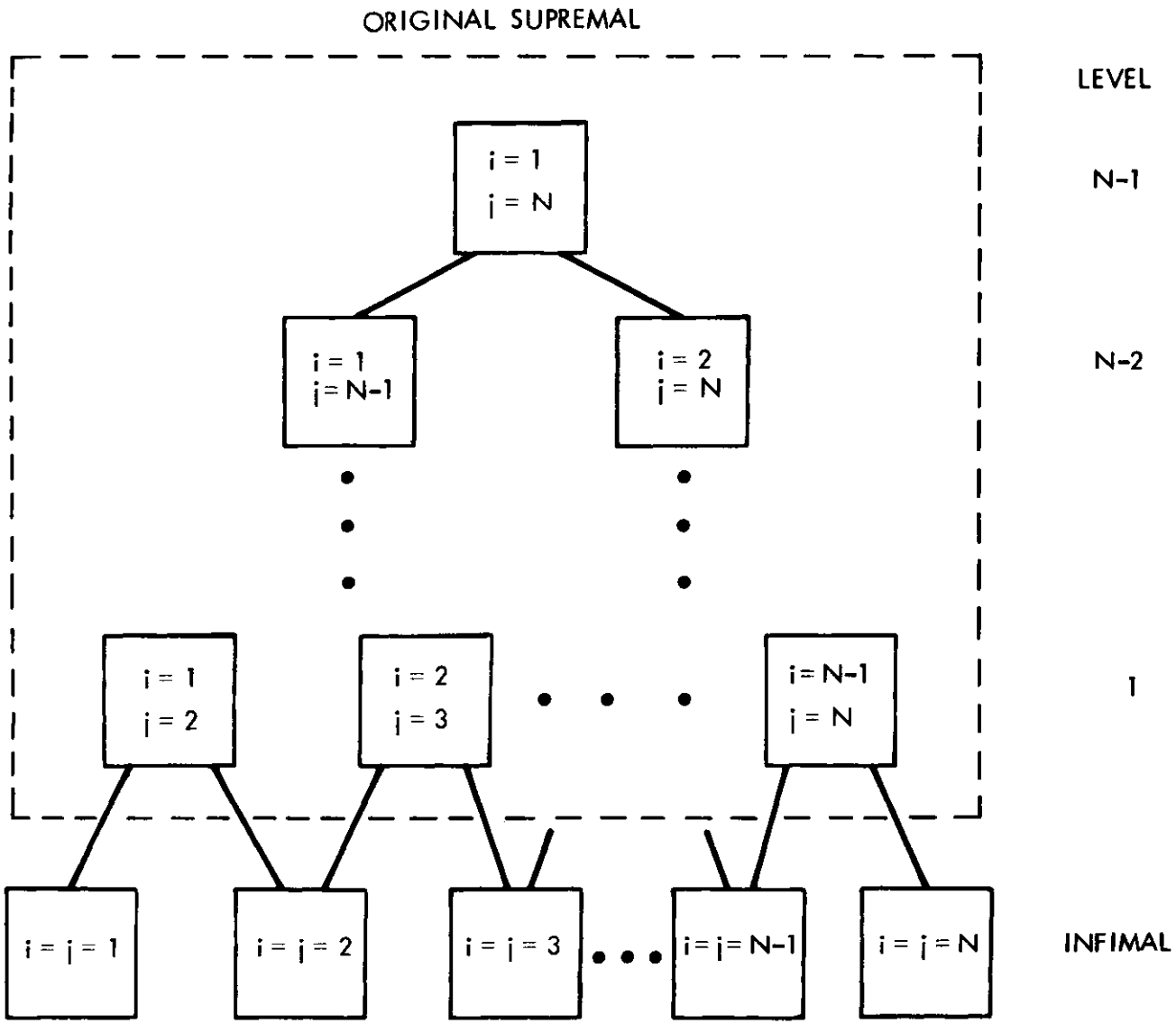


Figure 5.2: Multilevel Hierarchical Interpretation of the Interleaved Gauss-Seidel-Jacobi Supremal Decomposition

first is to solve the corresponding (i,j) problem (equations (5.2.8)-(5.2.18)). The second task is to relay information concerning other problem solutions to the levels above and below.

The convergence properties of the above two decompositions can be analyzed independently from the original primary decomposition of Chapter 4. This analysis can then be combined with the results of Section 2.4 to analyze the total iteration. As pointed out in Section 4.3 and 4.4, the exact analysis of convergence for the type of problems being considered is difficult. However, the supremal decompositions of this section can be shown to converge for sufficiently weakly coupled systems in a manner directly analogous to the development of Section 4.3.

Recall the following definition for an arbitrary matrix $A = [A_{ij}]$ with $A_{ij} \in L_2^{n_i x n_j}(\mathbb{R},\mathcal{B},\lambda[t_o,T])$ (or $A_{ij} \in \mathbb{R}^{n_i x n_j}$):

$$A_D \triangleq \text{diag}[A_{ii}: i=1,\ldots,N] \tag{5.2.19}$$

$$A_o \triangleq A - A_D \tag{5.2.20}$$

$$\mathcal{A}_o \triangleq \{A_o: A_o \text{ as defined in (5.2.19)-(5.2.20)}\} \tag{5.2.21}$$

Again, $\mathcal{A}_o$ is a sub-space of linear operators over the space $L_2^n(\mathbb{R},\mathcal{B},\lambda[t_o,T])$ or $\mathbb{R}^n$. Given any norm on either of the latter spaces, $\mathcal{A}_o$ will assume the corresponding induced norm. Note that (5.2.8)-(5.2.18) define an iteration of the form:

$$g(\beta_S^{k+1},\beta_S^k) = 0 \tag{5.2.22}$$

For $A_o = 0$, it is easily verified that $\beta_S^* = 0$ and $\partial_2 g(0,0) = 0$. Let $\{\beta_S^k\}_{k=0}^{\infty}$ denote the sequence generated by (5.2.22).

Using the above definitions, the following theorem demonstrates the importance of a weakly coupled subsystem structure to the supremal decomposition algorithms of this section.

Theorem 5.1: Assume there exist open neighborhoods $U_o$ of $0 \in A_o$ and $V_o$ of $0 \in \mathcal{B}^S$ such that $A_o \in U_o$ and $\beta_S \in V_o$ implies $\partial g$ is nonsingular. Then there exist open neighborhoods U of $0 \in A_o$ and $V \subset \mathcal{B}^S$ such that

(i) $\beta_S^o \in V$

(ii) $A_o \in U$

implies

$$\lim_{k\to\infty} (\beta_S^k) = \beta_S^* \tag{5.2.23}$$

where $\beta_S^*$ is a solution of (5.2.1).

Proof: By direct calculation, the linear operators $\partial_1 g$ and $\partial_2 g$ are jointly continuous in $\beta_S$ and $A_o$. Since $\partial g$ is nonsingular for $A_o \in U_o$ and $\beta_S \in V_o$, the implicit function theorem (Theorem 2.3) implies that $\beta_S$ is a continuous function of $A_o$. Hence $\partial_1 g$ and $\partial_2 g$ also depend continuously on $A_o$.

For any norm on $\mathcal{B}_S$, the induced norm on the space of linear operators $L(\mathcal{B}_S)$ satisfies

$$\rho[\partial_1 g^{-1}\, \partial_2 g] \leq ||\partial_1 g^{-1}\, \partial_2 g|| \leq ||\partial_1 g^{-1}||\, ||\partial_2 g|| \tag{5.2.24}$$

As noted previously for $A_o = 0$ the solution to equation (5.2.1) is $\beta_S^* = 0$. Also,

$$\partial_2 g(\beta_S^*, \beta_S^*) = 0 \tag{5.2.25}$$

Since $\partial g(\beta_S^*, \beta_S^*)$ is nonsingular, (5.2.25) implies:

$$||\partial_1 g^{-1}(\beta_S^*, \beta_S^*)|| = M_o < \infty \tag{5.2.26}$$

Because $\partial_1 g$ and $\partial_2 g$ vary continuously with $A_o$, there exists an open neighborhood $\hat{U}$ of $0 \in A_o$ such that for all $A_o \in \hat{U}$

$$||\partial_1 g^{-1}(\beta_S^*, \beta_S^*)|| \leq M_o + 1 \tag{5.2.27}$$

$$||\partial_2 g(\beta_S^*, \beta_S^*)|| \leq \frac{1}{M_o + 1} \tag{5.2.28}$$

where $\beta^*$ satisfies

$$g(\beta_S^*, \beta_S^*) = 0$$

for the given $A_o$. Combining (5.2.24) and (5.2.27)-(5.2.28) gives

$$\rho[\partial_1 g^{-1}(\beta_S^*,\beta_S^*)\partial_2 g(\beta_S^*,\beta_S^*)] < 1 \qquad (5.2.29)$$

for each $A_o \in U$. Since (5.2.27) implies $\partial_1 g^{-1}(\beta_S^*,\beta_S^*)$ exists, by Theorem 2.4 there exists a neighborhood $V \subset \mathcal{B}$ (depending on $A_o$) such that the sequence $\{\beta_S^k\}_{k=0}^{\infty}$ converges to $\beta_S^*$. The function generated by the implicit function theorem for the dependence of g on $A_o$ is unique and continuous in an open neighborhood $U_1$ of $0 \in A_o$. Also, the Frechet derivative $\partial g$ is nonsingular and $\beta_S = 0$ solves (5.2.1) for $A_o = 0$. Hence the limit of the sequence (5.2.23) solves (5.2.1) for the corresponding $A_o$.

□

When any of the supremal decompositions of this section are composed with the original decomposition of Chapter 4, the same proof that was used in Theorems 4.1 and 5.1 can be used to prove an analogous theorem for the overall iteration. However, Theorem 2.5 can be used along with Theorems 4.1 and 5.1 to simplify the proof.

Theorem 5.2: Consider the sequence $\{\beta_S^{M,k}, \beta_I^k\}$ resulting from using $M \geq 1$ steps of (5.2.8)-(5.2.17) to solve the supremal subproblem of (4.2.13)-(4.2.24). Assuming the conditions of Theorems 4.1 and 5.1, there exist open neighborhoods U of $0 \in A_o$ and $V \subset \mathcal{B}^S \times \mathcal{B}^I$

such that

$$\text{(i)} \quad (\beta_S^{0,0},\ \beta_I) \in V$$

$$\text{(ii)} \quad A_o \in U$$

implies

$$\lim_{k\to\infty} (\beta_S^{M,k},\beta_I^k) = (\beta_S^*,\beta_I^*) \tag{5.2.30}$$

where $(\beta_S^*,\beta_I^*)$ is a local minimum of the functional minimization (3.2.13)-(3.2.14) (or the static minimization (3.2.15)-(3.2.16)).

Proof: Denote by $\Gamma_s$ the iteration operator resulting from (5.2.8)-(5.2.17) and the exact solution of the infimal problems, and by $\Gamma_p$ the iteration matrix corresponding to (4.2.13)-(4.2.24). In the proofs of Theorems 4.1 and 5.1, the quantities $||\Gamma_s||$ and $||\Gamma_p||$ are bounded can be bounded by:

$$\begin{aligned} ||\Gamma_s|| &< \tfrac{1}{3} \\ ||\Gamma_p|| &< \tfrac{1}{3} \end{aligned} \tag{5.2.31}$$

simply by choosing $\hat{U}$ such that the right hand sides of (4.3.13) and (5.2.28) is $\frac{1}{3(M_o+1)}$. Then, by Theorem 2.5, a sufficient condition for local convergence to the solution of the necessary conditions resulting from the functional or static minimizations is

$$\rho\{\Gamma_s^M + (I-\Gamma_s^M)\Gamma_p\} < 1 \tag{5.2.32}$$

Using the properties of induced norms, the left hand side of (5.2.33) is bounded by:

$$\rho\{\Gamma_s^M + (I-\Gamma_s^M)\Gamma_p\} \leq ||\Gamma_p||^M + (1 + ||\Gamma_s||^M)||\Gamma_p|| \tag{5.2.33}$$

Combining (5.2.31) and (5.2.33) shows that (5.2.32) is satisfied. Finally, the uniqueness of the functions generated in the proofs of Theorems 4.1 and 5.1 implies that the limit (5.2.30) locally solves the minimization. □

Theorems 4.1 and 5.1-5.2 are all of the same genre; a decomposition based on logic induced by a weak coupling assumption was postulated and the corresponding theorem demonstrated that the logic was not flawed. Although the conclusions of the theorems are assuring, they do not give any practical tests for verifying convergence for particular problems.

If bounds on the norms of the iteration operators for the supremal decomposition of this section and the primary decomposition of Chapter 4 can be computed, or if the operators can be computed exactly then Theorem 2.5 can be applied to provide a convergence test. However, when only one step of the supremal iteration is used during each step of the primary iteration a more efficient analysis is possible. In this scheme the resulting overall iteration is simply a Gauss-Seidel

algorithm corresponding to the following decomposition (see Example 2.6):

$$\mathcal{B} = \mathcal{B}_{12} \times \mathcal{B}_{23} \times \ldots \times \mathcal{B}_{2N} \times \mathcal{B}_{1N} \times \mathcal{B}_{11} \times \ldots \times \mathcal{B}_{NN} \tag{5.2.34}$$

using the order implied. Thus the convergence analysis for the overall system can be simplified to the convergence analysis of the common Gauss-Seidel iteration.

By Theorem 4.2, a sufficient condition for the convergence of a Gauss-Seidel iteration is the strict block diagonal dominance condition (4.4.13). For the iteration defined by the decomposition (5.2.34) with $\mathcal{B}_{ij}$ defined as in (5.2.3) and (4.2.7) and $h_{ij}$ defined as in (5.2.4) and (4.2.8) ($\mathcal{B}_{ii} \triangleq \mathcal{B}_i^I$ and $h_{ii} \triangleq f_{Ii}$), the block diagonal dominance condition is:

$$\left\|(\partial_{ij} h_{ij})^{-1}\right\| \sum_{\ell=1}^{N} \sum_{\substack{m=1 \\ (\ell,m)\neq(i,j)}}^{N} \left\|\partial_{\ell m} h_{ij}\right\| < 1 \qquad \begin{matrix} i=1,\ldots,N; \\ j=i,\ldots,N \end{matrix} \tag{5.2.35}$$

where $\partial_{\ell m} h_{ij} \triangleq$ partial Fréchet derivative of $h_{ij}$ with respect to $\beta_{\ell m}$, evaluated at the solution. If the Lyapunov test guideline which was discussed in Section 4.4 is used, then condition (5.2.35) becomes

$$||S_{A_{ii}A_{jj}}^{-1}||\left[\sum_{\substack{\ell=1\\ \ell\neq i,j}}^{N}||A_{i\ell}|| + \sum_{\substack{\ell=1\\ \ell\neq j}}^{N}||A'_{j\ell}||\right] < 1 \tag{5.2.36}$$

Note that this test assumes even more significance in the time invariant completely decentralized problem since each of the conditions which are to be tested in (5.2.36) are exactly those which are to be tested in (5.2.35) except for the case i=j. Thus one might expect that the test (5.3.36) would give a reasonable indication of the convergence properties of the overall iteration.

For the time invariant completely decentralized problem another solution method is possible. In this case, the supremal problem reduces to the solution of the two linear equations

$$\pi^* \circ L_{\tilde{\mathbf{A}}} \circ \pi\,(\tilde{P}_S) = \tilde{\Xi}_S \qquad \tilde{P}_S,\ \tilde{\Xi}_S \in \mathcal{P} \tag{5.2.37}$$

$$\pi^* \circ L_{\tilde{A}'} \circ \pi\,(\tilde{K}_S) = \tilde{Q}_S \qquad \tilde{K}_S,\ \tilde{Q}_S \in \mathcal{P} \tag{5.2.38}$$

where, recalling the notation of (4.2.25)-(4.2.29), $\pi$ and $\mathcal{P}$ are defined by:

$$\mathcal{P} \triangleq \left\{ P_{ij}:\ i=1,\ldots,N;\ j=1,\ldots,N;\ i\neq j\}:\ P_{ij} \in \mathbb{R}^{(n_i+\hat{n}_i)\times(n_j+\hat{n}_j)} \right\} \tag{5.2.39}$$

$$\pi:\ \mathcal{P} \to \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})} :\{P_{ij}:\ i=1,\ldots,N;\ j=1,\ldots,N;\ i\neq j\} \to P_o \tag{5.2.40}$$

$$P_o \triangleq \begin{bmatrix} 0 & P_{12} & \cdots & P_{1N} \\ P_{21} & 0 & \cdots & P_{2N} \\ \vdots & \vdots & & \vdots \\ P_{N1} & P_{N2} & & 0 \end{bmatrix} \tag{5.2.41}$$

$$\pi^* \triangleq \text{adjoint of } \pi \tag{5.2.42}$$

To find an explicit representation for $\pi$, it is necessary to represent the Lyapunov operator as an $(n+\hat{n})^2 \times (n+\hat{n})^2$ matrix through the use of Kronecker product notation. Thus, it will temporarily be assumed that $L_{\tilde{A}}$ is represented as an $N^2 \times N^2$ block matrix $K_{\tilde{A}}$:

$$(K_{\tilde{A}})_{(i-1)N+k,(j-1)N+\ell} = [\tilde{A}_{ij} \otimes I]\delta_{k\ell} + [I \otimes \tilde{A}_{k\ell}]\delta_{ij} \; ; \qquad i,jk,\ \ell=1,\ldots,N \tag{5.2.43}$$

$$K_{\tilde{A}} = [(K_{\tilde{A}})_{ij} : i=1,\ldots,N^2;\ j=1,\ldots,N^2] \tag{5.2.44}$$

To represent the equation

$$L_{\tilde{A}}(\tilde{P}) = \tilde{\Xi} \tag{5.2.45}$$

using the $K_{\tilde{A}}$ notation, the matrices $\tilde{P}$ and $\tilde{\Xi}$ must be represented as vectors $\tilde{P}_v$ and $\tilde{\Xi}_v$. The vector representation which corresponds to the representation (5.2.43)-(5.2.44) is:

$$\tilde{P}_v = \begin{bmatrix} (\tilde{P}_1)_v \\ \vdots \\ (\tilde{P}_N)_v \end{bmatrix} ; \quad (\tilde{P}_i)_v = \begin{bmatrix} (\tilde{P}_{i1})_v \\ \vdots \\ (\tilde{P}_{iN})_v \end{bmatrix}$$

$$(\tilde{P}_{ij})_v = \begin{bmatrix} \tilde{P}^{ij}_1 \\ \vdots \\ \tilde{P}^{ij}_{(n+\hat{n})} \end{bmatrix}; \quad \tilde{P}^{ij}_k = \begin{bmatrix} \tilde{P}^{ij}_{k1} \\ \vdots \\ \tilde{P}^{ij}_{k(n+\hat{n})} \end{bmatrix} \qquad (5.2.46)$$

In words, the vector representation $\tilde{P}_v$ of the matrix $\tilde{\tilde{P}}$ consists of a particular ordering of the elements of the matrix $\tilde{P}$. This ordering is determined by first ordering the blocks row-wise and then ordering the elements of the blocks row-wise. The same ordering applies for the vector $\tilde{\Xi}_v$, and is demonstrated by the following example.

Example 5.1: Let N=2, and let

$A = [A_{ij}] \in \mathbb{R}^{4x4}$ where $A_{ij} \in \mathbb{R}^{2x2}$ for each i and j. The matrix $K_A$ is given by:

$$K_A = \begin{bmatrix} A_{11} \otimes I + I \otimes A_{11} & I \otimes A_{12} & A_{12} \otimes I & 0 \\ I \otimes A_{21} & A_{11} \otimes I + I \otimes A_{22} & 0 & A_{12} \otimes I \\ A_{21} \otimes I & 0 & A_{22} \otimes I + I \otimes A_{11} & I \otimes A_{12} \\ 0 & A_{21} \otimes I & I \otimes A_{21} & A_{22} \otimes I + I \otimes A_{22} \end{bmatrix} \qquad (5.2.47)$$

Also, $P_v$ is given by:

$$P_v' = [P_{11}P_{12}P_{21}P_{22}\,\vdots\,P_{12}P_{14}P_{23}P_{24}\,\vdots\,P_{31}P_{32}P_{41}P_{42}\,\vdots\,P_{33}P_{34}P_{43}P_{44}]' \qquad (5.2.48)$$

The vector $\Xi_v$ is represented similarly. Then the equation

$$L_A(P) = \Xi$$

is represented by

$$K_A\, P_v = \Xi_v$$

□

Finally, it is assumed that elements of $\mathcal{P}$ are represented similarly. That is, if $\tilde{P}_s \in \mathcal{P}$ then $\tilde{P}_s$ is represented as the vector:

$$\tilde{P}_s = \begin{bmatrix} \tilde{P}_1^s \\ \vdots \\ \tilde{P}_N^s \end{bmatrix} \quad ; \quad \tilde{P}_i^s = \begin{bmatrix} \tilde{P}_{i1} \\ \vdots \\ \tilde{P}_{i,i-1} \\ \tilde{P}_{i,i+1} \\ \vdots \\ \tilde{P}_{i,N} \end{bmatrix} \qquad (5.2.49)$$

with $\tilde{P}_{ij}$ defined as in (5.2.46).

With the representations (5.2.43)-(5.2.46) and (5.2.49), the restricted Lyapunov equations (5.2.37)-(5.2.38) are:

$$\pi' K_{\tilde{A}} \pi \tilde{P}_s = \tilde{\Xi}_s \tag{5.2.50}$$

$$\pi' K_{\tilde{A}'} \pi \tilde{K}_s = \tilde{Q}_s \tag{5.2.51}$$

where $\pi \in \mathbb{R}^{(n+\hat{n})^2 \times [(n+\hat{n})^2 - \sum_{i=1}^{N} (n_i+\hat{n}_i)^2]}$ is given by:

$$\pi_i = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{R}^{(n_i+\hat{n}_i)(n+\hat{n}) \times (n_i+\hat{n}_i)(n+\hat{n}-n_i-\hat{n}_i)}$$

$$\bar{\pi} = \text{diag}[\pi_i : i=1,\ldots,N-1] \tag{5.2.52}$$

$$\pi = \begin{bmatrix} \bar{\pi} \\ 0 \end{bmatrix}$$

Thus, the supremal problem is equivalent to solving two systems of $(N-1)(n+\hat{n})$ linear equations. The coefficient matrices $\pi' K_{\tilde{A}} \pi$ and $\pi' K_{\tilde{A}'} \pi$ can be found without matrix multiplications. The projection $\pi$ simply serves to eliminate the rows and columns of $K_{\tilde{A}}$ which lie in the blocks $(K_{\tilde{A}})_{ij}$ for which $i=1 \bmod N$ and $j=1 \bmod N$ respectively. Hence $\pi' K_{\tilde{A}} \pi$ and $\pi' K_{\tilde{A}'} \pi$ can be formed from $\tilde{A}$ using (5.2.43)-(5.2.44) and the preceding comment.

A problem which plagues this approach is the same problem which always occurs when Kronecker products are used to solve Lyapunov equations. The number of multiplications required to solve the

problem using this method is on the order of $(n+\hat{n})^6$. Other methods which exploit the structure of the Lyapunov operator to reduce the amount of computation could be used if a representation of $\pi$ could be found which corresponds to the usual Lyapunov operator representation. Unfortunately, such a representation has not been discovered. Thus, the above method will be impractical for most applications.

## 5.3 Infimal Problem Solution

This section discusses and compares four possible solution methods for the infimal problem which results from the decomposition of Chapter 4. Three of the methods - Newton's algorithm, the gradient search algorithm, and the gain approximation algorithm - are applicable to the general infimal problem. These are examined in Subsections (5.3.1)-(5.3.3) respectively. The fourth method requires the filter dimension $\hat{n}$ to be the same as the system dimension n. This method, discussed in Subsection (5.3.4), is a decomposition algorithm which requires only the solution of Riccati and Lyapunov equations which are of the order of system dimension n.

To simplify the notation, the subscript notation which differentiates the infimal problems will be dropped. In addition, only the time invariant infinite horizon problem will be considered although each of the four methods generalizes to the more general time varying finite horizon problem. The general form of the infimal problem is:

$$- B'(KP)_{12} - \hat{B}'(KP)_{22} + RGP_{22} + S = 0 \tag{5.3.1}$$

$$(KP)_{21} C' - (KP)_{22} \hat{C}' + K_{22} H \Theta + T = 0 \tag{5.3.2}$$

$$\tilde{A}P + P\tilde{A}' + \tilde{\Xi} + D = 0 \tag{5.3.3}$$

$$\tilde{A}'K + K\tilde{A} + \tilde{Q} + E = 0 \tag{5.3.4}$$

where

$$\tilde{A} = \begin{bmatrix} A & -BG \\ HC & \hat{A}-\hat{B}G-H\hat{C} \end{bmatrix}$$

$$\tilde{\Xi} = \begin{bmatrix} \Xi & 0 \\ 0 & H\Theta H' \end{bmatrix}$$

$$Q = \begin{bmatrix} Q & 0 \\ 0 & G'RG \end{bmatrix}$$

The matrices to be solved for are G, H, K and P, and the Hilbert space of these variables will be denoted by $\mathcal{B}^I$ as in Chapter 4.

5.3.1. Newton's Algorithm

Newton's algorithm is a well known and popular iterative method for solving systems of nonlinear equations (see, for example, Dennis and Moré [ 1977] and Example 2.5). The popularity stems from two desirable properties:

i) The equations to be solved at each iteration are linear.

ii) The algorithm exhibits local superlinear convergence.

Newton's iteration is defined by the decomposition given in equations (2.3.16)-(2.3.17). For the system of equations

$$f(\beta) = 0 \qquad f: \mathcal{B} \to \mathcal{B} \tag{5.3.5}$$

the decomposition (2.3.16)-(2.3.17) results in the iteration equation

$$f'(\beta_k)[\beta_{k+1} - \beta_k] = - f(\beta_k) \tag{5.3.6}$$

Equation (5.3.6) is a system of linear equations in $(\beta_{k+1} - \beta_k)$.

To apply Newton's method to the infimal problem it is necessary to compute the Fréchet derivative of equations (5.3.1)-(5.3.4) with respect to G, H, P and K. The Fréchet differential is:

$$f'(\beta)\Delta\beta = \begin{bmatrix} R\Delta GP_{22} + RG\Delta P_{22} - B'\,(\Delta KP+K\Delta P)_{21} - \hat{B}'(\Delta KP+K\Delta P)_{22} \\ K_{22}\Delta H\Theta + \Delta K_{22}H\Theta + (\Delta KP+K\Delta P)_{12}\,C' - (\Delta KP+K\Delta P)\hat{C}' \\ \tilde{A}\Delta P + \Delta P\tilde{A}' + \Delta\tilde{A}P + P\Delta\tilde{A}' + \Delta\tilde{\Xi} \\ \tilde{A}'\Delta K + \Delta K\tilde{A} + \Delta\tilde{A}'K + K\Delta\tilde{A} + \Delta\tilde{Q} \end{bmatrix} \tag{5.3.7}$$

where

$$\Delta\tilde{A} = \begin{bmatrix} 0 & -B\Delta G \\ \Delta HC & -\hat{B}\Delta G - \Delta H\hat{C} \end{bmatrix}$$

$$\Delta\tilde{\Xi} = \begin{bmatrix} 0 & 0 \\ 0 & \Delta H\Theta H' + H\Theta\Delta H' \end{bmatrix}$$

$$\Delta\tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & \Delta G'RG + G'R\Delta G \end{bmatrix}$$

$$\beta = (G,H,P,K) \in \mathcal{B}^{I}$$

By defining

$$\Delta\beta = \beta_{k+1} - \beta_k$$

and evaluating (5.3.7) and (5.3.1)-(5.3.4) at $\beta_k$, these equations can be used with (5.3.6) to solve the infimal problem. The resulting system of linear equations is complex. However, the system can be put in standard matrix-vector form through the use of Kronecker products. The result is a system of $2(n+\hat{n})^2 + \hat{n}(m+p)$ equations in the same number of unknowns. It is easy to see that even for small subproblems the equations which result from Newton's method can have a large dimension.

5.3.2 Gradient Search

The original motivation for studying decomposition algorithms for systems of nonlinear equations developed from an attempt to solve optimization problems indirectly by solving the resulting necessary conditions. Given a set of nonlinear equations such as (5.3.1)-(5.3.4), a natural question to ask is whether such equations could have originated from an optimization problem. The answer to this question involves anti-differentiating equations (5.3.1)-(5.3.4); i.e., a function must be found whose derivative is the left hand side of (5.3.1)-(5.3.4).

The strong similarity between (5.3.1)-(5.3.4) and the centralized necessary conditions (3.4.55)-(3.4.56) and (3.4.65)-(3.4.66) leads to a natural choice for a function in the anti-differentiation process. Let $\hat{J}: \mathbb{B}^I \to R$ be given by:

$$\hat{J}(\beta) = \frac{1}{2} \text{tr} \{(\tilde{Q}+E)P\} + \text{tr } G'S + \text{tr } TH' \tag{5.3.8}$$

where P is given by:

$$\tilde{A}P + P\tilde{A}' + \tilde{\Xi} + D = 0 \tag{5.3.9}$$

The following theorem shows that a solution $\beta^*$ to equations (5.3.1)-(5.3.4) is a stationary point for $\hat{J}(\beta)$.

Theorem 5.3: The gradient $\hat{J}'(\beta)$ is given by the left hand side of (5.3.1)-(5.3.2) with P and K given by (5.3.3)-(5.3.4).

Proof: Since $\hat{J}(\beta)$ separates into three additive terms, the gradient $\hat{J}'(\beta)$ can be computed by adding the gradients of the three terms. The differential of the second and third terms are

$$\partial\{\text{tr } G'S + \text{tr } TH'\} = \text{tr}\{\Delta G'S + T\Delta H'\} \tag{5.3.10}$$

Using Theorems 3.1 and 3.3 and the fact that D and E are constant matrices, the differential of the first term is:

$$\begin{aligned}\frac{1}{2}\,\partial\text{tr}\{(\tilde{Q}+E)P\} = \text{tr}\{&\Delta G'[RGP_{22} - B'(KP)_{12} - \hat{B}'(KP)_{22}] \\ &+ [K_{22}H\Theta + (KP)_{21}C' - (KP)_{22}\hat{C}']\Delta H'\}\end{aligned} \tag{5.3.11}$$

where K is given by (5.3.4). Combining (5.3.10) and (5.3.11) and the fact that $\Delta G$ and $\Delta H$ are arbitrary proves the theorem. □

Since the gradient $\hat{J}(\beta)$ vanishes at the solution to (5.3.1)-(5.3.4), it would be nice to be able to formulate the following optimization problem.

$$\min \hat{J}(\beta) \tag{5.3.12}$$

subject to

$$\tilde{A}P + P\tilde{A}' + \tilde{\Xi} + D = 0 \tag{5.3.13}$$

This optimization problem may not be well posed. The difficulty occurs because either $(\tilde{Q}+E)$ or P may not be positive definite (note

that since the terms tr G'S and tr TH' are linear in $\beta$ they do not affect the well posedness of the problem). However, the problem will be well posed for sufficiently small D and E.

Define the following quantities:

$$\alpha = (\alpha_D, \alpha_E) \in \mathbb{R}^+ \times \mathbb{R}^+ \tag{5.3.14}$$

$$||\alpha|| \triangleq \alpha_D + \alpha_E \tag{5.3.15}$$

$$\tilde{D} = \frac{1}{\alpha_D} D \tag{5.3.16}$$

$$\tilde{E} = \frac{1}{\alpha_E} E \tag{5.3.17}$$

Then let $\hat{J}_\alpha$ be defined by

$$\hat{J}_\alpha(\beta) = \frac{1}{2} \operatorname{tr}(\tilde{Q}+\alpha_E\tilde{E})P_\alpha + \operatorname{tr}\{G'S + TH'\} \tag{5.3.18}$$

where

$$\tilde{A}P_\alpha + P_\alpha\tilde{A}' + \tilde{\Xi} + \alpha_D \tilde{D} = 0 \tag{5.3.19}$$

Note that for $\alpha = (||D||, ||E||)$,

$$\hat{J}_\alpha(\beta) = \hat{J}(\beta)$$

Theorem 5.4: Consider the family of minimization problems parameterized by $\alpha$:

$$\min_{\beta \in \mathcal{B}^I} \hat{J}_\alpha(\beta) \tag{5.3.20}$$

If for $\alpha=(0,0)$ the above problem possesses a solution $\beta^*$ then there exists on $\varepsilon_o > 0$ and an open neighborhood $U \subset \mathcal{B}^I$ such that for each $\alpha$: $||\alpha|| < \varepsilon_o$ the following minimization problem is well posed:

$$\min_{\beta \in U} \hat{J}_\alpha(\beta) \tag{5.3.21}$$

Proof: Since $\beta^*$ minimizes $\hat{J}_o(\beta)$, the gradient and second Frechet derivative of $\hat{J}_o$ satisfy

$$\hat{J}_o'(\beta^*) = 0 \tag{5.3.22}$$

$$\hat{J}_o''(\beta^*) > 0 \tag{5.3.23}$$

Because $\hat{J}_o''$ is continuous, there exist an open neighborhood $U \subset \mathcal{B}^I$ of $\beta_o^*$ such that

$$\hat{J}_o''(\beta) > 0 \qquad \forall \beta \in U \tag{5.3.24}$$

Also, it is easily seen that both $\hat{J}_\alpha'$ and $\hat{J}_\alpha''$ are continuous in $\alpha$. Thus, by the implicit function theorem and equation (5.3.22) there exists an open neighborhood $N \subset \mathbb{R}^+ \times \mathbb{R}^+$ of 0 and a continuous function $\beta_\bullet^*: N \to \mathcal{B}^I$ such that

$$\hat{J}_\alpha'(\beta_\alpha^*) = 0 \tag{5.3.25}$$

Since $\beta_\bullet^*$ is continuous, $\beta_o^* = \beta^*$ and thus there exists $\varepsilon_1 > 0$ such that if $\alpha_1 \in \mathbb{R}^+ \times \mathbb{R}^+$ with $||\alpha_1|| < \varepsilon_1$

$$\beta_{\alpha_1}^* \in U \tag{5.3.26}$$

Similarly, because $J_\alpha''$ is continuous in $\alpha$ there exists $\varepsilon_2 > 0$ such that if $\alpha_2 \in \mathbb{R}^+ \times \mathbb{R}^+$

$$J_{\alpha_2}''(\beta) > 0 \qquad \forall \beta \in U \tag{5.3.27}$$

Let

$$\varepsilon_o = \min(\varepsilon_1, \varepsilon_2) \tag{5.3.28}$$

Then for all $\alpha$ such that $||\alpha|| < \varepsilon_o$, there exists a $\beta_\alpha^* \in U$ which satisfies (5.3.25) and (5.3.27), and hence solves (5.3.21). □

The above theorem implies that the infimal problems will be well posed minimization problems for sufficiently small D and E if the minimization problem (5.3.20) with $\alpha=(0,0)$ has a solution. For $\alpha=(0,0)$, the problem (5.3.20) is simply the deterministic equivalent of a centralized stochastic optimal control problem with the linear modification terms tr[G'S] and tr[TH'] added to the cost. However, these terms do not affect the convexity properties (local or global) of the optimization problem. Hence, the problem (5.3.20) will be well

posed if the centralized stochastic optimization problem corresponding to the decoupled subsystem is well posed. Necessary and sufficient conditions for the latter are that the pairs (A,B) and $(A, \sqrt{\Xi})$[1] be stabilizable and that the pairs $(A, \sqrt{Q})$ and (A,C) be detectable (Wonham [1978b]). Since the matrices D and E for the $i^{th}$ subsystem will be small if the system is weakly coupled and the detectability and stabilizability assumptions are standard, the minimization (5.3.12)-(5.3.13) will be well posed for sufficiently weakly coupled systems.

When the minimization problem (5.3.12)-(5.3.13) is well posed, any gradient search algorithm (see, for example, Rosenbloom [1956]; Hestenes [1956]; Kelley [1962]; Fletcher and Powell [1963]; Fletcher and Reeves [1964]; or Wolfe [1976]) can be used. The gradient of (5.3.12)-(5.3.13) with respect to G and H is given by the left hand side of (5.3.1)-(5.3.2) with P and K given by (5.3.3)-(5.3.4). Then the solution to (5.3.12)-(5.3.13) is also the desired solution to (5.3.1)-(5.3.4).

### 5.3.1 Gain Approximation Algorithm

This algorithm involves a straightforward decomposition of the problem defined by (5.3.1)-(5.3.4). The decomposition corresponds to the successive over-relaxation (SOR) algorithm for the Gauss-Seidel

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[1] Given a symmetric positive semidefinite matrix $\Xi$, the matrix $\sqrt{\Xi}$ is defined as the unique symmetric positive semidefinite matrix W such that $W^2 = \Xi$.

iteration (see Examples 2.6-2.7) with the relaxation parameter $\varepsilon$ varying with both the iteration index and with the subproblem being solved. However, the resulting iteration also corresponds to a downhill search algorithm for the minimization problem (5.3.12)-(5.3.13). This algorithm (with $\varepsilon \equiv 1$), has been used by Levine and Athans [1970] to find the optimal constant feedback gains for the output feedback problem, and by Wang [1972] to solve for the best deterministic decentralized linear constant feedback law. The algorithm is generalized in this subsection to solve equations (5.3.1)-(5.3.4).

Define the following decomposition of (5.3.1)-(5.3.4) (as in Example 2.6):

$$\mathcal{B}^I \triangleq \mathcal{B}^I_G \times \mathcal{B}^I_H \times \mathcal{B}^I_K \times \mathcal{B}^I_P \tag{5.3.29}$$

$$\mathcal{B}^I_G \triangleq \{G \in \mathbb{R}^{mxn}\} \tag{5.3.30}$$

$$\mathcal{B}^I_H \triangleq \{H \in \mathbb{R}^{nxp}\} \tag{5.2.31}$$

$$\mathcal{B}^I_K \triangleq \{K \in \mathbb{R}^{(n+\hat{n})x(n+\hat{n})}\} \tag{5.3.32}$$

$$\mathcal{B}^I_P \triangleq \{P \in \mathbb{R}^{(n+\hat{n})x(n+\hat{n})}\} \tag{5.3.33}$$

The subsets of equations $f_G$, $f_H$, $f_K$ and $f_P$ are taken as (5.3.1), (5.3.2), (5.3.4) and (5.3.3) respectively. The resulting iteration (including the relaxation parameter $\varepsilon_k$) is:

$$\hat{G}_k = R^{-1}\{B'(K_k P_k)_{12} + \hat{B}'(K_k P_k)_{22} - S\}(P_k)_{22}^{-1} \tag{5.3.34}$$

$$\hat{H}_k = (K_k)_{22}^{-1} \{(K_k P_k)_{22} \hat{C}' - (K_k P_k)_{21} C' - T\} \Theta^{-1} \tag{5.3.35}$$

$$G_{k+1} = G_k + \varepsilon_k [\hat{G}_k - G_k] \tag{5.3.36}$$

$$H_{k+1} = H_k + \varepsilon_k [\hat{H}_k - H_k] \tag{5.3.37}$$

$$\tilde{A}'_{k+1} K_{k+1} + K_{k+1} \tilde{A}_{k+1} + \tilde{Q}_{k+1} + E = 0 \tag{5.3.38}$$

$$\tilde{A}_{k+1} P_{k+1} + P_{k+1} \tilde{A}'_{k+1} + \tilde{\Xi}_{k+1} + D = 0 \tag{5.3.39}$$

The following theorem shows that $\varepsilon_k$ can be chosen at each iteration such that $0 < \varepsilon_k \leq 1$ and the cost function (5.3.8) is reduced at each iteration.

Theorem: Let $\hat{J}(\beta)$ be given by (5.3.8) and let the sequence $\{\beta_k\}_{k=0}^{\infty}$ be generated by (5.3.34)-(5.3.39). Then at iteration k+1 the relaxation parameter $\varepsilon_k$ can be chosen such that $0 < \varepsilon \leq 1$ and

$$\hat{J}(\beta_{k+1}) < \hat{J}(\beta_k) \tag{5.3.40}$$

if $\beta_{k+1} \neq \beta_k$.

Proof: For any matrix F, define $\Delta F_k$ and $\Delta \hat{F}_k$ by

$$\Delta F_k = F_{k+1} - F_k \tag{5.3.41}$$

$$\Delta \hat{F}_k = \hat{F}_k - F_k \tag{5.3.42}$$

The difference in cost from iteration k to k+1 is given by:

$$\Delta\hat{J}(\beta_k) = \hat{J}(\beta_{k+1}) - \hat{J}(\beta_k) \tag{5.3.43}$$

$$= \frac{1}{2}\,\mathrm{tr}[(\tilde{Q}_{k+1}+E)P_{k+1}] - \frac{1}{2}\,\mathrm{tr}[(\tilde{Q}_k+E)P_k] + \mathrm{tr}[\Delta G_k'S + T\Delta H_k']$$

$$= \frac{1}{2}\,\mathrm{tr}[\Delta\tilde{Q}_k P_{k+1}] + \frac{1}{2}\,\mathrm{tr}[(\tilde{Q}_k + E)\Delta P_k] + \mathrm{tr}[\Delta G_k'S + T\Delta H_k']$$

The first term of (5.3.43) is given by:

$$\mathrm{tr}[\Delta\tilde{Q}_k P_{k+1}] = \mathrm{tr}\left\{\begin{bmatrix} 0 & 0 \\ 0 & (G_k+\varepsilon_k\Delta\hat{G}_k)'R(G_k+\varepsilon_k\Delta\hat{G}_k) - G_k'RG_k \end{bmatrix} P_k\right\} + \mathrm{tr}\Delta\tilde{Q}_k\Delta\tilde{P}_k \tag{5.3.44}$$

$$= 2\varepsilon_k\,\mathrm{tr}\{\Delta\hat{G}_k'\,R\,G_k[(P_k)_{22} + (\Delta P_k)_{22}]\} + o(\varepsilon_k)$$

Now, $\Delta P_k$ can be found by rewriting (5.3.38):

$$0 = \tilde{A}_k P_{k+1} + P_{k+1}\tilde{A}_k' + \Delta\tilde{A}_k P_{k+1} + P_{k+1}\Delta\tilde{A}_k' + \tilde{\Xi}_k \tag{5.3.45}$$

The following equation results from replacing k+1 by k in (5.3.38) and subtracting the result from (5.3.45):

$$0 = \tilde{A}_k\Delta P_k + \Delta P_k\tilde{A}_k' + \Delta\tilde{A}_k P_{k+1} + P_{k+1}\Delta\tilde{A}_k' + \Delta\tilde{\Xi}_k \tag{5.3.46}$$

Thus, using the definition of $L_{\tilde{A}_k}$ (equation(3.4.8 )), $\Delta P_k$ is given by:

$$\Delta P_k = -L_{\tilde{A}_k}^{-1}[\Delta\tilde{A}_k P_{k+1} + P_{k+1}\Delta\tilde{A}_k' + \Delta\tilde{\Xi}_k] \tag{5.3.47}$$

$$= -\varepsilon_k L_{\tilde{A}_k}^{-1}[S]$$

where

$$S = \begin{bmatrix} 0 & -B\Delta\hat{G}_k \\ \Delta\hat{H}_k C & -\hat{B}\Delta\hat{G}_k - \Delta\hat{H}_k\hat{C} \end{bmatrix} P_{k+1} + P_{k+1} \begin{bmatrix} 0 & -B\Delta\hat{G}_k \\ \Delta\hat{H}_k C - \hat{B}\Delta\hat{G}_k - \Delta\hat{H}_k\hat{C} \end{bmatrix}'$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & \Delta\hat{H}_k\Theta H_k' + H_k\Theta\Delta\hat{H}_k' + \varepsilon_k\Delta\hat{H}_k\Theta\Delta\hat{H}_k' \end{bmatrix} \tag{5.3.48}$$

Substituting (5.3.47) in (5.3.44) gives

$$\mathrm{tr}[\Delta\tilde{Q}_k P_{k+1}] = 2\varepsilon_k \mathrm{tr}\{\Delta\hat{G}_k' R\, G_k(P_k)_{22}\} + o(\varepsilon_k) \tag{5.3.49}$$

Now consider the second term of (5.3.43). Using equation (5.3.47) and the definition of the adjoint gives:

$$\begin{aligned} \mathrm{tr}(\tilde{Q}_k+E)\Delta P_k &= -\varepsilon_k \mathrm{tr}[(\tilde{Q}_k+E)L_{\tilde{A}_k}^{-1}(S)] \\ &= -\varepsilon_k \mathrm{tr}[L_{\tilde{A}_k}^{*-1}(\tilde{Q}_k+E)S] \\ &= \varepsilon_k \mathrm{tr}[K_k S] \end{aligned} \tag{5.3.50}$$

From (5.3.47) it is obvious that

$$P_{k+1} = P_k + o(\varepsilon_k) \tag{5.3.51}$$

Now, using (5.3.48) and (5.3.51) in (5.3.50) and the properties of the trace operator gives

$$\mathrm{tr}(\tilde{Q}_k+E)\Delta P_k = 2\varepsilon_k \mathrm{tr}\{\Delta\hat{G}_k'[-B'(K_kP_k)_{12} - \hat{B}'(K_kP_k)_{22}] \tag{5.3.52}$$

$$+ [(K_kP_k)_{21} C' - (K_kP_k)_{22} \hat{C}' + (K_k)_{22} H_k\Theta]\Delta H_k'\}$$

$$+ o(\varepsilon_k)$$

Combining (5.3.49), (5.3.52) and (5.3.43) gives the following equation for $\Delta\hat{J}(\beta_k)$:

$$\Delta\hat{J}(\beta_k) = \varepsilon_k \mathrm{tr}\{\Delta\hat{G}_k'[-B'(K_kP_k)_{12} - \hat{B}'(K_kP_k)_{22} + RG_k(P_k)_{22} + S] \tag{5.3.53}$$

$$+ [(K_kP_k)_{21} C' - (K_kP_k)_{22} \hat{C}' + (K_k)_{22} H_k\Theta + T]\Delta H_k'\}$$

$$+ o(\varepsilon_k)$$

Using the definition of $\hat{G}_k$ and $\hat{H}_k$ , (5.3.53) can be rewritten as:

$$\Delta\hat{J}(\beta_k) = -\varepsilon_k \mathrm{tr}\{\Delta\hat{G}_k' R \hat{G}_k(P_k)_{22} + (K_k)_{22} \hat{H}_k\Theta \Delta\hat{H}_k \tag{5.3.54}$$

$$- \Delta\hat{G}_k' R G_k(P_k)_{22} - (K_k)_{22} H_k\Theta \Delta\hat{H}_k'\} + o(\varepsilon_k)$$

Hence, equation (5.3.54) becomes

$$\Delta\hat{J}(\beta_k) = -\varepsilon_k \mathrm{tr}\{\Delta\hat{G}_k' R \Delta\hat{G}_k (P_k)_{22} + (K_k)_{22}\Delta\hat{H}_k \Theta \Delta\hat{H}_k'\} + o(\varepsilon_k) \tag{5.3.55}$$

Since the bracketted term is a positive definite quadratic form, it is possible to choose $\varepsilon_k$ such that $0 < \varepsilon_k \leq 1$ and

$$J(\beta_{k+1}) - J(\beta_k) < 0 \tag{5.3.56}$$

if $\Delta\hat{G}_k$ or $\Delta\hat{H}_k$ are non-zero. But, if $\Delta\hat{G}_k$ or $\Delta\hat{H}_k$ are non-zero then $\beta_{k+1}$ and $\beta_k$ are not equal. If $\beta_{k+1}$ and $\beta_k$ are not equal then $\Delta\hat{G}_k$ or $\Delta\hat{H}_k$ are non-zero. □

The above theorem implies that the direction $[\hat{G}_k - G_k]$ determined by the gain approximation algorithm at the $k^{th}$ iteration is a downhill direction for the minimization problem (5.2.12)-(5.2.13). Thus this algorithm is in fact a descent method. However, it will in general be more effective computationally to use a gradient or accelerated gradient method. The computation for the gradient algorithms and the iteration (5.3.34)-(5.3.39) will be approximately the same at each step, but the gradient direction will generally be superior. Also, acceleration algorithms are available for the gradient method. Hence the gradient would be expected to converge more rapidly.

An exception to the preceding remarks is the following situation. If it could be determined a priori that the cost would decrease at each iteration for some fixed $\varepsilon$, then the relaxation parameter could be fixed at this value. In general, both the gain approximation algorithm and gradient algorithms require several function evaluations at each iteration to determine a relaxation (stepsize) parameter value which decreases the cost. The cost evaluation requires the solution of equation (5.3.9) and the evaluation of (5.3.8). Hence a significant reduction in the amount of computation per iteration would occur if the cost were only evaluated once.

In general, it is not possible to determine such a value for the relaxation parameter. However, for the output feedback problem Levine [1969] was able to show that the cost decreased at each iteration for $\varepsilon_k=1$. Thus, it may be advantageous to use the gain approximation algorithm with $\varepsilon_k=1$ rather than a gradient algorithm for the output feedback problem.

### 5.3.4 Decomposition to Riccati and Linear Matrix Equations

This subsection develops an algorithm which requires the subsystem filters have the same dynamics as the subsystem. It is motivated in part by the derivation of the centralized gains in Appendix B and in part by a previous decomposition developed by Sandell [1976]. A major advantage of this decomposition is that only Riccati and linear matrix equations of the subsystem dimension need be solved at each iteration. However, unlike the previous decompositions considered in this thesis, local convergence cannot be demonstrated with only the weak coupling assumption.

For the remainder of this section it will be assumed that in equations (5.3.1)-(5.3.4):

$$\hat{n} = n$$
$$\hat{A} = A;\ \hat{B} = B;\ \hat{C} = C \tag{5.3.57}$$

With these assumptions, the original equations can be transformed as follows. Define

$$W \triangleq \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \tag{5.3.58}$$

Then $W^{-1}$ is given by

$$W^{-1} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \tag{5.3.59}$$

By premultiplying equations (5.3.3)-(5.3.4) by W and $(W^{-1})'$ respectively and postmultiplying by $W'$ and $W^{-1}$ respectively, these equations can be written as:

$$\bar{A}\,\bar{P} + \bar{P}\,\bar{A}' + \bar{\Xi} + \bar{D} = 0 \tag{5.3.60}$$

$$\bar{A}'\bar{K} + \bar{K}\,\bar{A} + \bar{Q} + \bar{E} = 0 \tag{5.3.61}$$

where

$$\bar{A} \triangleq \tilde{W}\tilde{A}W^{-1} = \begin{bmatrix} A - HC & 0 \\ HC & A-BG \end{bmatrix}$$

$$\bar{\Xi} \triangleq \tilde{W}\tilde{\Xi}W' = \begin{bmatrix} \Xi + H\Theta H' & -H\Theta H' \\ -H\Theta H' & H\Theta H' \end{bmatrix}$$

$$\bar{Q} \triangleq (W^{-1})'\,\tilde{Q}\,W^{-1} = \begin{bmatrix} Q & Q \\ Q & Q + G'RG \end{bmatrix}$$

$$\bar{D} \triangleq WDW'$$

$$\bar{E} \triangleq (W^{-1})'\,E\,W^{-1}$$

$$\bar{P} \triangleq WPW' = \begin{bmatrix} P_{11}+P_{22} - P_{12} - P_{21} & P_{12} - P_{22} \\ P_{21} - P_{22} & P_{22} \end{bmatrix}$$

$$\bar{K} = (W^{-1})' \, K W^{-1} = \begin{bmatrix} K_{11} & K_{11} + K_{12} \\ K_{11} + K_{21} & K_{11} + K_{22} + K_{12} + K_{21} \end{bmatrix}$$

Using the definitions of $\bar{K}$ and $\bar{P}$, equations (5.3.1)-(5.3.2) becomes:

$$R\,G\,\bar{P}_{22} - B'(\bar{K}\,\bar{P})_{22} + S = 0 \tag{5.3.62}$$

$$(\bar{K}_{11} + \bar{K}_{22} - \bar{K}_{21} - \bar{K}_{12})H\Theta + [(\bar{K}\,\bar{P})_{21} - (\bar{K}\,\bar{P})_{11}]C' + T = 0 \tag{5.3.63}$$

Equations (5.3.60)-(5.3.61) can be partitioned as:

$$(A-HC)\bar{P}_{11} + \bar{P}_{11}(A-HC)' + \Xi + H\Theta H' + \bar{D}_{11} = 0 \tag{5.3.64}$$

$$(A-HC)\bar{P}_{12} + \bar{P}_{12}(A-BG)' + \bar{P}_{11}C'H' - H\Theta H' + \bar{D}_{12} = 0 \tag{5.3.65}$$

$$(A-BG)\bar{P}_{22} + \bar{P}_{22}(A-BG)' + HCP_{12} + P_{21}C'H' + H\Theta H' + \bar{D}_{22} = 0 \tag{5.3.66}$$

$$(A-HC)'\bar{K}_{11} + \bar{K}_{11}(A-HC) + C'H'\bar{K}_{21} + \bar{K}_{12}HC + Q + \bar{E}_{11} = 0 \tag{5.3.67}$$

$$(A-HC)'\bar{K}_{12} + \bar{K}_{12}(A-BG) + C'H'\bar{K}_{22} + Q + \bar{E}_{12} = 0 \tag{5.3.68}$$

$$(A-BG)'\bar{K}_{22} + \bar{K}_{22}(A-BG) + Q + G'RG + \bar{E}_{22} = 0 \tag{5.3.69}$$

Note that equations (5.3.62)-(5.3.69) are identical to the time invariant equivalent of equations (B.9)-(B.10) and (B.16)-(B.21) of Appendix B, when D, E, S and T are zero. In this case, Appendix B showed that

$$\bar{P}_{12} = 0 \tag{5.3.70}$$

$$\bar{K}_{22} - \bar{K}_{12} = 0 \tag{5.3.71}$$

When D, E, S and T are sufficiently small, $\bar{P}_{12}$ and $(\bar{K}_{22} - \bar{K}_{12})$ will be small also. Thus a decomposition guided by the centralized solution derivation may be practical for some problems.

First consider equations (5.3.62)-(5.3.63). Expanding (5.3.62) and solving for G gives:

$$G = R^{-1}B'\bar{K}_{22} + R^{-1}B'\bar{K}_{21}\bar{P}_{12}\bar{\bar{P}}_{22}^{-1} - R^{-1}S\,\bar{\bar{P}}_{22}^{-1} \tag{5.3.72}$$

Performing the same operations on (5.3.63) results in the equation:

$$H = \bar{P}_{11}C'\Theta^{-1} + (\bar{K}_{11} + \bar{K}_{22} - \bar{K}_{21} - \bar{K}_{12})^{-1}[\bar{K}_{12} - \bar{K}_{22})(\bar{P}_{11} + \bar{P}_{21})C' + T]\Theta^{-1} \tag{5.3.73}$$

When (5.3.70)-(5.3.71) hold (and S and T are zero), the above two equations for G and H are just those for the centralized feedback and filter gains. It will be assumed that at the $\ell^{\underline{th}}$ iteration the terms

$$\hat{S} = R^{-1}[B'\bar{K}_{21}\bar{P}_{12} - S]\bar{\bar{P}}_{22}^{-1} \tag{5.3.74}$$

$$\hat{T} = (\bar{K}_{11}+\bar{K}_{22} - \bar{K}_{21} - \bar{K}_{12})^{-1}[(\bar{K}_{12}-\bar{K}_{22})(\bar{P}_{11}+\bar{P}_{12})C' + T]\Theta^{-1} \quad (5.3.75)$$

will be evaluated using the values computed at the $(\ell-1)^{th}$ iteration. Thus, $C^{\ell}$ and $H^{\ell}$ are given by the equations

$$G^{\ell} = R^{-1}B'\bar{K}_{22}^{\ell} + \hat{\Lambda}^{\ell-1} \quad (5.3.76)$$

$$H = \bar{P}_{11}^{\ell} C' \Theta^{-1} + \hat{T}^{\ell-1} \quad (5.3.77)$$

The equations for $\bar{K}_{22}^{\ell}$ and $\bar{P}_{11}^{\ell}$ can be found by substituting (5.3.76)-(5.3.77) into (5.3.64)-(5.3.69):

$$(A-\hat{T}^{\ell-1}C)\bar{P}_{11}^{\ell} + \bar{P}_{11}^{\ell}(A-\hat{T}^{\ell-1}C)' - \bar{P}_{11}^{\ell} C' \Theta^{-1}C \bar{P}_{11}^{\ell} + \Xi + \hat{D}_{11}^{\ell-1} = 0 \quad (5.3.78)$$

$$(A-B\hat{\Lambda}^{\ell-1})'\bar{K}_{22}^{\ell} + \bar{K}_{22}^{\ell}(A-B\hat{\Lambda}^{\ell-1}) - \bar{K}_{22}^{\ell}BR^{-1}B'\bar{K}_{22}^{\ell} + Q + \hat{E}_{22}^{\ell-1} = 0 \quad (5.3.79)$$

where

$$\hat{D}_{11}^{\ell-1} = D_{11} + (\hat{T}^{\ell-1})\Theta(\hat{T}^{\ell-1})'$$

$$\hat{E}_{22}^{\ell-1} = \bar{E}_{22} + (\hat{S}^{\ell-1})' R(\hat{S}^{\ell-1})$$

Equations (5.3.78)-(5.3.79) are Riccati equations of dimension n, and can be solved for $\bar{P}_{11}^{\ell}$ and $\bar{K}_{22}^{\ell}$ using only values computed at the previous iteration. Then $G^{\ell}$ and $H^{\ell}$ can be computed using equations (5.3.76)-(5.3.77). Finally, these values can be used to solve, in

order, the linear equations (5.3.65), (5.3.66), (5.3.68) and (5.3.67).

Using the scheme described in the preceding paragraph, the iteration becomes:

$$(A-\hat{T}^{\ell-1}C)\bar{P}_{11}^{\ell} + \bar{P}_{11}^{\ell}(A-\hat{T}^{\ell-1}C)' - \bar{P}_{11}^{\ell}C'\Theta^{-1}C\bar{P}_{11}^{\ell} + \Xi + \hat{D}_{11}^{\ell-1} = 0 \qquad (5.3.80)$$

$$(A-B\hat{S}^{\ell-1})'\bar{K}_{22}^{\ell} + \bar{K}_{22}^{\ell}(A-B\hat{S}^{\ell-1}) - \bar{K}_{22}^{\ell}BR^{-1}B'\bar{K}_{22}^{\ell} + Q + \hat{E}_{22}^{\ell-1} = 0 \qquad (5.3.81)$$

$$G^{\ell} = R^{-1}B'\bar{K}_{22}^{\ell} + \hat{S}^{\ell-1} \qquad (5.3.82)$$

$$H^{\ell} = \bar{P}_{11}^{\ell}C'\Theta^{-1} + \hat{T}^{\ell-1} \qquad (5.3.83)$$

$$(A-H^{\ell}C)\bar{P}_{12}^{\ell} + \bar{P}_{12}^{\ell}(A-BG^{\ell})' + \bar{P}_{11}^{\ell}C'(H^{\ell})' - H^{\ell}\Theta(H^{\ell})' + \bar{D}_{12} = 0 \qquad (5.3.84)$$

$$(A-BG^{\ell})\bar{P}_{22}^{\ell} + \bar{P}_{22}^{\ell}(A-BG^{\ell})' + H^{\ell}C\bar{P}_{12}^{\ell} + \bar{P}_{21}^{\ell}C'(H^{\ell})' + H^{\ell}\Theta(H^{\ell})' + \bar{D}_{22} = 0 \qquad (5.3.85)$$

$$(A-H^{\ell}C)'\bar{K}_{12}^{\ell} + \bar{K}_{12}^{\ell}(A-BG^{\ell}) + C'(H^{\ell})'\bar{K}_{22}^{\ell} + Q + \bar{E}_{12} = 0 \qquad (5.3.86)$$

$$(A-H^{\ell}C)'\bar{K}_{22}^{\ell} + \bar{K}_{22}^{\ell}(A-H^{\ell}C) + C'(H^{\ell})'\bar{K}_{21}^{\ell} + \bar{K}_{12}^{\ell}H^{\ell}C + Q + \bar{E}_{11} = 0 \qquad (5.3.87)$$

The iteration defined by (5.3.80)-(5.3.87) requires the solution of two Riccati equations, two Lyapunov equations and two Sylvester equations.

Unlike previous decompositions which have been considered, the iteration defined by (5.3.80)-(5.3.87) cannot be guaranteed to

converge for sufficiently weakly coupled systems. The problem is that the Fréchet differentials of (5.3.80)-(5.3.87) with respect to variables evaluated at the $(\ell-1)^{\underline{th}}$ iteration do not vanish even in the centralized problem (i.e. when S, T, D and E are zero). To see this, consider the Fréchet differential (denoted by $\Delta$) of (5.3.81) with respect to $\bar{P}_{12}^{\ell-1}$ evaluated at the solution to the centralized problem:

$$\Delta = - \bar{P}_{22}^{-1} \bar{K}_{12} \Delta\bar{P}_{21} B R^{-1} B' \bar{K}_{22} - \bar{K}_{22} B R^{-1} B' \Delta\bar{P}_{12} \bar{K}_{21} \bar{P}_{22}^{-1} \tag{5.3.88}$$

This term is generally nonzero.

## 5.4 Example

This section applies the decomposition of Chapter 4 to an example used by Wang [1972]. The example consists of the linearized dynamics of an inertia wheel spacecraft attitude control device. This device can be regarded as an interconnected system consisting of three subsystems, one corresponding to each of the three axes.

The state variables for each subsystem are the roll, pitch and yaw angles and their time rates of change. The control inputs are the torques produced by the motors oriented along the three body axes, and the outputs are the observed angles. Thus each of the subsystems can be represented by a single input, single output double integrator plant:

$$\dot{x}_i = A_{ii} x_i + B_i u + \sum_{\substack{j=1 \\ j\neq i}}^{3} A_{ij} x_j + \xi_i \tag{5.4.1}$$

$$\dot{y}_i = C_i x_i + \theta_i \qquad (5.4.2)$$

where

$$A_{ii} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad i=1,2,3$$

$$A_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{\bar{I}\Omega_k}{I_i} \end{bmatrix} \qquad i=1,2,3;\quad j=1,2,3;\quad j\neq i;\ k\neq i;\ k\neq j$$

$$B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad i=1,2,3$$

$$C_i = [1 \quad 0]$$

$$E\{\xi_i(t)\xi_j'(\tau)\} = \Xi_i\, \delta(t-\tau)\delta_{ij}$$

$$E\{\theta_i(t)\theta_j'(\tau)\} = \Theta_i\, \delta(t-\tau)\delta_{ij}$$

The constant parameters $I_i$ and $\Omega_i$ are the moments of inertia of the spacecraft about the $i^{\text{th}}$ body axis and the angular velocity of the $i^{\text{th}}$ inertia wheel respectively. The parameter $\bar{I}$ is the moment of inertia of the inertia wheels about their axes of rotation. The values of these parameters are taken as:

$$
\begin{aligned}
I_1 &= 37.5 \\
I_2 &= 40.0 \\
I_3 &= 42.5 \\
\bar{I} &= 0.00084 \\
\Omega_1 &= \Omega_2 = \Omega_3 \triangleq \Omega
\end{aligned}
\tag{5.4.3}
$$

The value of $\Omega$ will be varied to control the strength of the coupling between the subsystems.

The spectral densities of the white noise sources $\xi_i$ and $\theta_i$ will be taken as:

$$
\Xi_i = \begin{bmatrix} 0 & 0 \\ 0 & .1 \end{bmatrix} \tag{5.4.4}
$$

$$
\Theta_i = [.01] \tag{5.4.5}
$$

Finally, the cost functional which is to be minimized is the infinite horizon time averaged cost:

$$
J = \frac{1}{2} \lim_{t_f \to \infty} E\left\{ \frac{1}{t_f} \int_0^{t_f} \sum_{i=1}^{3} [x_i' Q_i x_i + u_i^2] dt \right\}
$$

where:

$$
Q_1 = \begin{bmatrix} 1.0 & 0 \\ 0 & 5.0 \end{bmatrix}
$$

$$Q_2 = \begin{bmatrix} 1.5 & 0 \\ 0 & 3.0 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.5 \end{bmatrix}$$

Using the technique of Example 3.2, equations (5.4.1)-(5.4.6) can be expressed as the formulation of Chapter 3 requires. It was assumed that a completely decentralized control system was desired (see Example 3.3). The problem was then solved using the necessary conditions of Chapter 3 and the decomposition of Chapter 4. The supremal problem of Chapter 4 was solved using the decomposition of Section 5.2 and the infimal problems were solved using a gradient search technique as described in Subsection (5.3.2).

The strength of the subsystem interactions were controled by varying the parameter $\Omega$. Five cases were investigated, with $\Omega$ given by the following values:

Case 1: $\Omega = 1.137$

Case 2: $\Omega = 11.37$

Case 3: $\Omega = 113.7$

Case 4: $\Omega = 300.0$

Case 5: $\Omega = 1137.0$

The initial guess for each case was the decoupled centralized solution.

The results were disappointing. Convergence occurred for the first three cases. Cases 1 and 2 required only one complete iteration to converge to the desired tolerance (a relative change in the element of the gain matrices of less than $10^{-6}$). Case 3 required seven iterations. However, the difference in the cost between the centralized decoupled controller and the optimal decentralized gain was zero to machine accuracy.

Two problems occurred as the subsystem interactions increased. First, case 4 diverged even though the subproblems could be solved accurately. This problem can probably be overcome even for considerably stronger interactions by using a relaxation (SOR) algorithm (see Lehtomaki [1978] for several encouraging applications of SOR decompositions to Lyapunov equations). The second problem occurred in case 5. The subproblems become ill posed as minimization problems. To correct this difficulty another solution method for the infimal problems must be found.

## 5.5 Summary and Discussion

This chapter has presented several solution methods for the supremal and infimal problems. Section 5.2 developed a class of decompositions which can be used to solve the supremal problem. This class allows the computations to be structured to take advantage of the particular problem

being solved. All the algorithms contained in the class possess two important features:

i) The subproblems are linear

ii) The resulting iteration converges for sufficiently weakly coupled systems.

All of the four methods discussed in Section 5.3 suffer some drawback. Newton's method requires the solution of a system of linear equations which can have prohibitively large dimensions even for relatively small subproblems. Descent methods require the subproblems to be well posed. This may not occur even for relatively weakly coupled systems as was demonstrated by the example of Section 5.4. Subsection (5.3.4) developed a decomposition which resulted in Riccati and linear matrix equations at each iteration. However, a specific system structure is required and convergence cannot generally be guaranteed even for weakly coupled systems.

The example of Section 5.4 showed the importance of further developing a good infimal solution algorithm. Is also demonstrated the need to develop and use a successive over-relaxation algorithm.

# 6. CONCLUSION

## 6.1 Summary

The contents of this thesis can be separated into two distinct divisions. The first, contained in Chapter 2, is concerned with the development of a general theory of decomposition algorithms for optimization problems. The remainder of the thesis develops an application of the methodology of Chapter 2 to a decentralized linear stochastic optimal control problem.

In spite of the large amount of literature which has become available in recent years, there is a lack of a unifying theory of hierarchical control and coordination. The intent of the development of Chapter 2 is directed at this need. The decomposition formulation provides a means of classifying and analyzing groups of decomposition algorithms which are fundamentally the same but differ only in the partition of the variable space. Thus the need for a separate analysis for each application is avoided and the basic features of the individual algorithms are exposed.

Chapter 2 approaches the theory of decomposition of optimization problems indirectly. It is assumed that a set of necessary conditions for the optimization can be expressed in the form of a system of nonlinear equations. This system of equations is then decomposed using an extension of the splitting methods for solving linear equations.

There are several important and desirable features of the theory which results from the approach described in the preceding paragraph. First, the theory is sufficiently general that it includes all of the hierarchical algorithms proposed to date and many of the classical one point iterative algorithms. In fact, it can be demonstrated that several of the hierarchical algorithms are simply special cases of the classical algorithms.

The second major feature is the ability to specify individual decompositions through the choice of the core function. This is important for two reasons. First, it admits a constructive approach to designing decomposition algorithms. The consistency conditions which are required to ensure that the decomposition solves the original problem are imbedded in the formulation. At the same time, all the properties of the decomposition can be related directly to the core. Hence, the technical considerations can be suppressed and the more important properties can be examined easily through the use of this decomposition approach.

A third feature of the theory of Chapter 2 is the convergence analysis. Because few assumptions are placed on the original optimization problem, the convergence results are local in nature and have not been considered to date in the hierarchical literature.

Finally, the decomposition formulation allows multiple decompositions of the same problem to be analyzed in terms of the original decompositions. When such decompositions are viewed in the context of

hierarchical theory, the result is a multilevel structure. For the first time, such hierarchies with more than two levels have been developed and explicitly analyzed.

The second part of this thesis applies the theory of Chapter 2 to a linear stochastic optimal control problem. The particular control problem considered requires the controller to be a linear finite dimensional system. This problem is then reformulated as a deterministic minimization and necessary conditions are derived. These equations are then decomposed for the interconnected system problem formulation using the Gauss-Seidel decomposition algorithm.

The decomposition results in a two level structure for the computation of the best linear controller which satisfies the problem constraints. The computational burden is reduced at both the supremal and infimal level problems. The supremal problem can be further decomposed and results in a multilevel hierarchy. When this multiple decomposition is used, all the subproblems are linear except for the original infimal problems. The latter problems are similar in structure to the necessary conditions which result from centralized optimal control problems.

The convergence of the resulting algorithm was analyzed using the local convergence results of Chapter 2. Although no definitive results or tests were derived, several guidelines and conservative sufficient tests were proposed. In addition, the algorithm was shown to

converge if the subsystem interactions were sufficiently weak.

The algorithm was applied to the linearized model of an inertia wheel attitude control device. The results were disappointing for this problem, but demonstrated that the algorithm suffers several problems which need more attention. First, in any application the algorithm will probably have to use a relaxation technique. Second, the infimal problems in the example were solved as minimization problems. However, these became ill posed even when the subsystem interactions were weak. Thus another solution method must be used.

## 6.2 Further Research

The theory begun in Chapter 2 provides just the basic fundamentals needed to unify the area of decomposition and coordination. There are many directions future research in this area could take. First, the theory needs further development and study. Included under this direction would be the development of the relationships between the local and global convergence theories, and the relationships between properties of the decompositions and their respective cores. A second direction would be the study of individual classes of algorithms. Also, the theory of multiple decompositions has just begun in this thesis.

The application of the Gauss-Seidel decomposition to linear stochastic optimal control problems demonstrated several difficulties. Most of the research directions are technical in nature. Based on the results of the example, a relaxation algorithm must be used. Also,

new solution methods for the infimal problems need to be developed. Although the results of the example were disappointing, this approach should still be examined further.

## APPENDIX A: ADJOINTS OF SEVERAL LINEAR OPERATORS

Given a linear operator $L: X \to Y$ where X and Y are Hilbert spaces with inner products $\langle \cdot,\cdot \rangle_X$ and $\langle \cdot,\cdot \rangle_Y$, the adjoint of L, if it exists, is denoted by $L^*: Y \to X$ and defined by

$$\langle y, L x \rangle_Y = \langle L^* y, x \rangle_X \qquad \forall x \in X,\ y \in Y \tag{A.1}$$

Let X and Y be defined as the following Hilbert spaces.

$$X \triangleq L_2^{nxn}(\mathbb{R},\mathcal{B},\lambda[t_o,T]);\ \langle x,y \rangle_X = \operatorname{tr} \int_{t_o}^{T} x'(t)y(t) \qquad \forall x,y \in X \tag{A.2}$$

$$Y \triangleq \mathbb{R}^{nxn};\quad \langle x,y \rangle_Y = \operatorname{tr} x'y \qquad \forall x,y \in Y \tag{A.3}$$

Consider the following linear operators:

$$F_A: X \to X:\ \Xi(\cdot) \to \int_{t_o}^{t} \Phi_A(t,\sigma)\Xi(\sigma)\Phi_A'(t,\sigma)\,d\sigma \tag{A.4}$$

$$F_A^T: X \to Y:\ \Xi(\cdot) \to \int_{t_o}^{T} \Phi_A(T,\sigma)\Xi(\sigma)\Phi_A'(T,\sigma)\,d\sigma \tag{A.5}$$

$$H_A: \quad Y \to X:\ P \to \Phi_A(t,t_o)P\,\Phi_A'(t,t_o) \tag{A.6}$$

$$H_A^T: \quad Y \to Y:\ P \to \Phi_A(T,t_o)P\,\Phi_A'(T,t_o) \tag{A.7}$$

$$L_A: \quad Y \to Y:\ P \to AP + PA' \tag{A.8}$$

where $\Phi_A(t,\sigma)$ is the transition matrix defined by:

$$\frac{d}{dt}\ \Phi_A(t,\sigma) = A(t)\Phi(t,\sigma);\ \Phi(\sigma,\sigma) = I \tag{A.9}$$

The adjoints of (A.4)-(A.8) are given by the following lemma.

Lemma A.1: Let X and Y be Hilbert spaces defined as in (A.2)-(A.3). Then the adjoints of the linear operators in (A.4)-(A.8) are given by:

$$F_A^*:\ X \to X:\ Q(\cdot) \to \int_t^T \Phi_A'(\sigma,t)Q(\sigma)\Phi_A(\sigma,t)d\sigma \tag{A.10}$$

$$F_A^{T*}:\ Y \to X:\ K \to \Phi_A'(T,t)K\ \Phi_A(T,t) \tag{A.11}$$

$$H_A^*:\ X \to Y:\ Q(\cdot) \to \int_{t_o}^T \Phi_A'(\sigma,t_o)Q(\sigma)\Phi_A(\sigma,t_o)d\sigma \tag{A.12}$$

$$H_A^{T*}:\ Y \to Y:\ K \to \Phi_A'(T,t_o)K\ \Phi_A(T,t_o) \tag{A.13}$$

$$L_A^*:\ Y \to Y:\ K \to A'K + KA \tag{A.14}$$

Proof: The proof involves standard manipulations using the definitions of inner products on X and Y and the definition of the adjoint (A.1).

a) First, consider $L = F_A$. For arbitrary $\Xi(\cdot)$ and $Q(\cdot)$ in X, the inner product is (using (A.4)).

$$\langle Q(\cdot),\ F_A(\Xi(\cdot))\rangle_X \tag{A.15}$$

$$= \text{tr} \int_{t_o}^T \int_{t_o}^t Q'(t)\Phi_A(t,\sigma)\Xi(\sigma)\Phi_A'(t,\sigma)d\sigma dt$$

Interchanging the integrals in (A.15) gives

$$\langle Q(\cdot), F_A(\Xi(\cdot))\rangle_X = \text{tr} \int_{t_o}^{T} \int_{\sigma}^{T} Q'(t)\Phi_A(t,\sigma)\Xi(\sigma)\Phi_A'(t,\sigma)\,dt\,d\sigma \qquad (A.16)$$

Since

$$\text{tr } AB = \text{tr } BA \qquad \forall\, A \in \mathbb{R}^{mxn} \quad B \in \mathbb{R}^{nxm} \qquad (A.17)$$

equation (A.16) becomes

$$\langle Q(\cdot), F_A(\Xi(\cdot))\rangle_X = \text{tr} \int_{t_o}^{T} \left[ \int_{\sigma}^{T} \Phi_A'(t,\sigma)Q'(t)\Phi_A(t,\sigma)\,dt \right] \Xi(\sigma)\,d\sigma \qquad (A.18)$$

Now, switching the roles of t and $\sigma$ in (A.18) gives

$$\begin{aligned} \langle Q(\cdot), F_A(\Xi(\cdot))\rangle_X &= \text{tr} \int_{t_o}^{T} \left[ \int_{t}^{T} \Phi_A'(\sigma,t)Q(\sigma)\Phi_A(\sigma,t)\,d\sigma \right]' \Xi(t)\,dt \\ &= \langle F_A^*(Q(\cdot)),\ \Xi(\cdot)\rangle_X \end{aligned} \qquad (A.19)$$

where $F_A^*$ has been defined as (A.10).

b) Let $L = F_A^T$ in (A.1). Then, using (A.5) the inner product on Y is:

$$\langle K, F_A^T(\Xi(\cdot))\rangle_Y = \text{tr } K' \int_{t_o}^{T} \Phi_A(T,t)\Xi(t)\Phi_A'(T,t)\,dt \qquad (A.20)$$

Using (A.17), (A.20) becomes

$$\begin{aligned} \langle K, F_A^T(\Xi(\cdot))\rangle_Y &= \text{tr} \int_{t_o}^{T} \left[ \Phi_A'(T,t)K\,\Phi_A(T,t) \right]' \Xi(t)\,dt \\ &= \langle F_A^{T^*}(K),\ \Xi(\cdot)\rangle_X \end{aligned} \qquad (A.21)$$

where $F_A^{T^*}$ has been defined as (A.11).

c) Let $L = H_A$ in (A.1). Then, using (A.6) the inner product on X is:

$$\langle Q(\cdot),\ H_A(P)\rangle_X = \mathrm{tr} \int_{t_o}^{T} Q'(t)\Phi_A(t,t_o)P\ \Phi_A'(t,t_o)dt \tag{A.22}$$

Again using (A.17), (A.22) becomes

$$\begin{aligned}\langle Q(\cdot),\ H_A(P)\rangle_X &= \mathrm{tr}\left[\int_{t_o}^{T} \Phi_A'(t,t_o)Q(t)\Phi_A(t,t_o)dt\right]'P \\ &= \langle H_A^*(Q(\cdot)),\ P\rangle_Y\end{aligned} \tag{A.23}$$

where $H_A^*$ has been defined as (A.12).

d) Let $L = H_A^T$ in (A.1). Then, using (A.7) the inner product on Y is:

$$\langle K, H_A^T(P)\rangle_Y = \mathrm{tr}\ K'\ \Phi_A(T,t_o)P\ \Phi_A'(T,t_o) \tag{A.24}$$

Using (A.17), (A.24) becomes

$$\begin{aligned}\langle K,\ H_A^T(P)\rangle_Y &= \mathrm{tr}\left[\Phi_A'(T,t_o)K\ \Phi_A(T,t_o)\right]'P \\ &= \langle H_A^{T^*}(K),\ P\rangle_Y\end{aligned} \tag{A.25}$$

where $H_A^{T^*}$ has been defined as (A.13).

e) Finally, let $L = L_A$ in (A.1). Then, using (A.8) the inner product on Y is:

$$\langle K, L_A(P)\rangle_Y = \text{tr } K'(AP + PA') \tag{A.26}$$

Using (A.17), (A.26) becomes

$$\begin{aligned} \langle K, L_A(P)\rangle_Y &= \text{tr } (A'K + KA)P \\ &= \langle L_A^*(K), P\rangle_Y \end{aligned} \tag{A.27}$$

where $L_A^*$ has been defined as (A.14).

□

## APPENDIX B: A SOLUTION TO THE CENTRALIZED NECESSARY CONDITIONS

Consider the problem formulation in (3.2.1)-(3.2.7) with N=1, $\hat{A}(t) = A(t)$, $\hat{B}(t) = B(t)$ and $\hat{C}(t) = C(t)$. Let the state of the closed loop system be denoted by $\tilde{x}_B(t)$ where:

$$\tilde{x}_B(t) \triangleq \begin{bmatrix} x(t) - \hat{x}(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix} \tag{B.1}$$

Then $\tilde{x}_B(t)$ is given by the differential equation

$$\dot{\tilde{x}}_B(t) = \tilde{A}(t)\tilde{x}_B(t) + \tilde{\xi}(t); \quad \tilde{x}_B(t_o) = \tilde{x}_{oB} \tag{B.2}$$

where

$$\tilde{A}(t) = \begin{bmatrix} A(t) - H(t)C(t) & 0 \\ H(t)C(t) & A(t) - B(t)G(t) \end{bmatrix} \tag{B.3}$$

$$E\{\tilde{\xi}(t)\tilde{\xi}'(t)\} \triangleq \Xi(t) = \begin{bmatrix} \Xi + H\Theta H' & - H\Theta H' \\ - H\Theta H' & H\Theta H' \end{bmatrix} \tag{B.4}$$

$$E\{\tilde{x}_{oB}\tilde{x}_{oB}'\} \triangleq P_{oB} = \begin{bmatrix} P_o & 0 \\ 0 & 0 \end{bmatrix}$$

In terms of $\tilde{x}(t)$, the cost (3.2.4) is given by:

$$J = \text{tr } E\left\{ \int_{t_o}^{T} \tilde{x}'(t)\tilde{Q}(t)\tilde{x}(t)dt + \tilde{x}'(T)K_{TB}\,\tilde{x}(T) \right\} \tag{B.5}$$

where

$$\tilde{Q}(t) = \begin{bmatrix} Q & Q \\ Q & Q + G'RG \end{bmatrix}$$

$$K_{TB} = \begin{bmatrix} K_T & K_T \\ K_T & K_T \end{bmatrix}$$

The optimization problem corresponding to (B.1)-(B.5) can be placed in the form of (3.3.13)-(3.3.14):

$$\min_{\substack{G(\cdot)\in L_2^{mxn}(\mathbb{R},\mathbb{B},\lambda) \\ H(\cdot)\in L_2^{nxp}(\mathbb{R},\mathbb{B},\lambda)}} J(G(\cdot),H(\cdot)) \tag{B.6}$$

subject to

$$\frac{d}{dt}P(t) = \tilde{A}(t)P(t) + P(t)\tilde{A}'(t) + \tilde{\Xi}(t);\ P(t_o) = P_{oB} \tag{B.7}$$

where

$$J(G(\cdot),H(\cdot)) = \text{tr} \int_{t_o}^{T} \tilde{Q}(t)P(t)dt \tag{B.8}$$

Theorem 3.1 can be used to derive necessary conditions for (B.6)-(B.8). Suppressing the time dependence of the matrices involved, the necessary conditions are:

$$[-(K_{11}P_{11} + K_{12}P_{12}') + (K_{12}'P_{11} + K_{22}P_{12}')]C' \tag{B.9}$$
$$+ (K_{11} + K_{22} - K_{12} - K_{12}')H\Theta = 0$$

$$- B'(K_{12}'P_{12} + K_{22}P_{22}) + RGP_{22} = 0 \tag{B.10}$$

$$\dot{P} = \tilde{A}P + P\tilde{A}' + \tilde{\Xi}; \quad P(t_o) = \tilde{P}_{oB} \tag{B.11}$$

$$\dot{K} = -\tilde{A}K - K\tilde{A} - \tilde{Q}; \quad K(T) = K_{TB} \tag{B.12}$$

Note that conditions (B.9)-(B.12) are equivalent to (3.4.65)-(3.4.66) and (3.4.53)-(3.4.54) since the state $\tilde{x}_B(t)$ in this appendix is related to $\tilde{x}(t)$ of Chapter 3 by the transformation

$$\tilde{x}_B(t) = W\tilde{x}(t) \tag{B.13}$$

where

$$W = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} .$$

Thus K(t) and P(t) in this appendix are related to $\tilde{K}(t)$ and $\tilde{P}(t)$ in Chapter 3 by:

$$K(t) = W'\tilde{K}(t)W \tag{B.14}$$

$$P(t) = W\tilde{P}(t)W' \tag{B.15}$$

Equations (B.11) and (B.12) can be written in partitioned form as:

$$\dot{P}_{11} = (A-HC)P_{11} + P_{11}(A-HC)' + \Xi + H\Theta H';\ P_{11}(t_o) = P_o \tag{B.16}$$

$$\dot{P}_{12} = (A-HC)P_{12} + P_{12}(A-BG)' + P_{11}C'H' - H\Theta H';\ P_{12}(t_o) = 0 \tag{B.17}$$

$$\dot{P}_{22} = (A-BG)P_{22} + P_{22}(A-BG)' + HCP_{12} + P_{12}'C'H' + H\Theta H';\ P_{22}(t_o) = 0 \tag{B.18}$$

$$\dot{K}_{11} = -(A-HC)'\ K_{11} - K_{11}(A-HC) - C'H'K_{12}' - K_{12}HC - Q;\ K_{11}(T) = K_T \tag{B.19}$$

$$\dot{K}_{12} = -(A-HC)'\ K_{12} - K_{12}(A-BG) - C'H'K_{22} - Q;\ K_{12}(T) = K_T \tag{B.20}$$

$$\dot{K}_{22} = -(A-BG)'\ K_{22} - K_{22}(A-BG) - Q - G'RG;\ K_{22}(T) = K_T \tag{B.21}$$

To demonstrate that the classical linear quadratic Gaussian (LQG) solution satisfies the necessary conditions (B.9)-(B.10) and (B.16)-(B.21), it will be assumed that H(t) is given by:

$$H = \sum C'\ \Theta^{-1} \tag{B.22}$$

where $\sum \triangleq E(x-\hat{x})(x-\hat{x})' = P_{11}$

Assuming (B.22), the remainder of the classical LQG solution will be derived and (B.22) will be shown to be a consistent assumption. Note that the choice of the form of H could be motivated by (B.17) and physical considerations. One might suspect that the covariance of the

estimate error and the estimate ($P_{12}$) should be zero for all $t \in [t_o,T]$. From (B.17), this could occur only if:

$$[P_{11}C' - H\Theta]H' = 0 \qquad \forall\, t \in [t_o,T] \tag{B.23}$$

The assumption (B.22) of the form of H ensures that (B.23) will hold.

Assuming (B.22), equation (B.17) becomes:

$$\dot{P}_{12} = (A-HC)P_{12} + P_{12}(A-BG)' \; ; \quad P_{12}(t_o) = 0 \tag{B.24}$$

Thus

$$P_{12}(t) = 0 \qquad \forall\, t \in [t_o,T] \tag{B.25}$$

Using (B.22) and (B.25) in (B.16) gives the Riccati equation for the Kalman filter:

$$\dot{P}_{11} = AP_{11} + P_{11}A' + \Xi - P_{11}C'\Theta CP_{11}; \quad P_{11}(t_o) = P_o \tag{B.26}$$

Also, (B.18) becomes

$$\dot{P}_{22} = (A-BG)P_{22} + P_{22}(A-BG)' + H\Theta H' \; ; \quad P_{22}(t_o) = 0 \tag{B.27}$$

Note that $P_{22}$, the covariance of the estimate, is determined by the closed loop deterministic system dynamics driven by the white noise process $H\theta$.

Now, using (B.22) and (B.25) in (B.10) gives:

$$-B' K_{22}P_{22} + RGP_{22} = 0 \tag{B.28}$$

Or, equivalently

$$G = R^{-1}B'K_{22} \tag{B.29}$$

Then equation (B.21) becomes the control Riccati equation:

$$\dot{K}_{22} = -A'K_{22} - K_{22}A - Q + K_{22}\,BRB'K_{22};\quad K_{22}(T) = K_T \tag{B.30}$$

Now, subtracting equation (B.20) from (B.21) gives:

$$\frac{d}{dt}(K_{22} - K_{12}) = -(A-HC)'(K_{22} - K_{12}) - (K_{22} - K_{12})(A-BG);$$

$$K_{22}(T) - K_{12}(T) = 0 \tag{B.31}$$

Hence

$$K_{22} = K_{12} \qquad \forall\, t \in [t_o, T] \tag{B.32}$$

Finally, using (B.25) in (B.9) gives

$$-(K_{11} - K_{12}')P_{11}C' + [(K_{11} - K_{12}') + (K_{22} - K_{12})]H\Theta = 0 \tag{B.33}$$

Substituting (B.32) in (B.33) gives

$$-(K_{11} - K_{12}')\,(P_{11}C' - H\Theta) = 0 \tag{B.34}$$

Thus (B.22) satisfies (B.34) and hence, as expected, the classical LQG solution given by (B.22), (B.26), (B.29) and (B.30) satisfies the necessary conditions (B.9)-(B.12).

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