Combinatorial Analogs of Brouwer's Fixed Point Theorem on a Bounded Polyhedron

by

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Abstract

In this paper, we present a combinatorial theorem on a bounded polyhedron for an unrestricted integer labelling of a triangulation of the polyhedron, which can be interpreted as an extension of the Generalized Sperner lemma. When the labelling function is dual-proper, this theorem specializes to a second theorem on the polyhedron, that is an extension of Scarf's dual Sperner lemma. These results are shown to be analogs of Brouwer's fixed point theorem on a polyhedron, and are shown to generalize two combinatorial theorems on the simplotope as well.

The paper contains two other results of interest. We present a projective transformation lemma that shows that if $\chi = \{x \in R^n | Ax \leq e\}$ is a bounded polyhedron, then $\chi' = \{x \in R^n | (A - eoy)x \leq e\}$ is combinatorially equivalent to χ if and only if y is an element of the interior of the polar of χ. Secondly, the appendix contains a pseudomanifold construction for a polyhedron and its dual that may be of interest to researchers in triangulations based on primal and dual polyhedra.

Key words: polyhedron, triangulation, pseudomanifold, fixed-point, integer label, simplex.

1. Introduction

In an article published in 1928, Emanuel Sperner demonstrated a purely combinatorial lemma on the n-simplex that implied the fixed-point theorem of Brouwer for continuous functions. The connection between combinatorial theorems and topological theorems was further investigated by Tucker [24], who developed a combinatorial lemma that implied the antipodal point theorems of Borsuk and Ulam, and of Lusternik and Schnirelman [19]. Kuhn [15] and Fan [5] later examined combinatorial results on the n-cube that imply Brouwer's fixed point theorem.

With the development of fixed-point computation algorithms stemming from Scarf's seminal work [21], there has been a resurgence of research in combinatorial analogs of Brouwer's theorem. Such analogs of Brouwer's theorem on the simplex include Scarf's "dual" Sperner lemma [22], the Generalized Sperner lemma [10], and of course, the original Sperner lemma [23]. Analogs of Brouwer's theorem on the cube include a pair of dual lemmas presented in [6], one of which is analogous to the constructive algorithm in van der Laan and Talman [17]. Recently, these combinatorial results have been extended to simplotopes (see Freund [7] and van der Laan, Talman, and Van der Heyden [18]), for which the simplex and cubical theorems are special cases.

In this paper, we present a combinatorial theorem on a bounded polyhedron for an unrestricted labelling of a triangulation of the polyhedron, which can be interpreted as an extension of the Generalized Sperner lemma. This theorem is the main theorem of section 3, theorem 1. When the labelling function is dual-proper.

theorem 1 specializes to a second combinatorial theorem on the polyhedron, that is an extension of Scarf's dual Sperner lemma. These results are shown in section 3, and their relationship to results on the simplex and simplotope are also shown. Section 4 contains a combinatorial proof of theorem 1, and hence of theorem 2.

In section 5, we address the issue of an extension of Sperner's lemma to a bounded polyhedron. We present such an extension as theorem 5 of the section. However, the proof of theorem 5 is based on Brouwer's theorem; it is an open question whether a purely combinatorial proof of theorem 5 can be demonstrated.

The paper contains two other results of interest. In section 3, we present a projective transformation lemma, that shows that if $\chi = \{x \in R^n | Ax \leq e\}$ is a bounded polyhedron, then $\chi' = \{x' \in R^n | (A - e \circ y)x' \leq e\}$ is combinatorially equivalent to χ if and only if y is an element of the interior of the polar of χ. This lemma is used in the proof of theorem 1, but it may also have applications elsewhere. Secondly, the appendix contains a pseudomanifold construction for a polyhedron and its dual that may be of interest to researchers in triangulations based on primal and dual polyhedra.

2. Notation

Let R^n denote real n-dimensional space, and define e to be the vector of 1's, namely $e = (1, \ldots, 1)$. Let $x \cdot y$ and $x \circ y$ denote inner and outer product, respectively. Let ϕ denote the empty set, and let $|S|$ denote the cardinality of a set S. For two sets S, T, let $S \backslash T = \{x | x \in S, x \notin T\}$, and let $S \Delta T = \{x | x \in S \cup T, x \notin S \cap T\}$. If $x \in S$, we denote $S \backslash \{x\}$ by $S \backslash x$ to ease the notational burden. Let $v^0, \ldots, v^m$ be vectors in R^n. If the matrix

$$\begin{bmatrix} v^0 & \ldots & v^m \\ 1 & \ldots & 1 \end{bmatrix}$$

has rank $(m+1)$, then the convex hull of $v^0, \ldots, v^m$, denoted $\langle v^0, \ldots, v^m \rangle$, is said to be a <u>real m-dimensional simplex</u>, or more simply an m-simplex. If $\sigma = \langle v^0, \ldots, v^m \rangle$ is an m-simplex and $\{v^{j_0}, \ldots, v^{j_k}\}$ is a nonempty subset of $\{v^0, \ldots, v^m\}$, then $\tau = \langle v^{j_0}, \ldots, v^{j_k} \rangle$ is a <u>k-face</u> or <u>face</u> of σ.

Let χ be a <u>cell</u> in R^n, i.e. a nonempty bounded polyhedron in R^n. Let T be a finite collection of m-simplices σ together with all of their faces. T is a finite <u>triangulation</u> of χ if

i) $\bigcup_{\sigma \in T} \sigma = \chi$,

ii) $\sigma, \tau \in T$ imply $\sigma \cap \tau \in T$, and

iii) If σ is an $(m-1)$-simplex of T, σ is a face of at most two m-simplices of T.

An <u>abstract complex</u> consists of a set of vertices K^0 and a set of finite nonempty subsets of K^0, denoted K, such that

i) $v \in K^0$ implies $\{v\} \in K$, and

ii) $\phi \neq x \subset y \in K$ implies $x \in K$.

An element x of K is called in abstract simplex, or more simply a simplex. If $x \in K$ and $|x| = n + 1$, then x is called an n-simplex, where $|\cdot|$ denotes cardinality. Technically, an abstract complex is defined by the pair (K^0, K). However, since the set K^0 is implied by K, it is convenient to denote the complex by K alone. An abstract complex K is said to be finite if K^0 is finite.

An n-dimensional pseudomanifold, or more simply an n-pseudomanifold, where $n \geq 1$, is a complex K such that

i) $x \in K$ implies there exists $y \in K$ with $|y| = n + 1$ and $x \subset y$, and

ii) if $x \in K$ and $|x| = n$, then there are at most two n-simplices of K that contain x.

Let K be an n-pseudomanifold, where $n \geq 1$. The boundary of K, denoted ∂K, is defined to be the set of simplices $x \in K$ such that x is contained in an (n-1)-simplex $y \in K$, and y is a subset of exactly one n-simplex of K.

Let χ be an m-cell in R^n, and let T be a finite triangulation of χ. For each nonempty face τ of each m-simplex σ of T, define $\bar{\tau} = \{v|v \text{ is a vertex of } \tau\}$. Then the collection $K = \{\bar{\tau}|\tau$ is a nonempty face of a simplex of T$\}$ is an m-pseudomanifold, and is called the m-pseudomanifold corresponding to T.

If A and b are a matrix and a vector, let A_i and b_i denote the i^{th} row and component of A and b respectively, and let A_β and b_β denote the submatrix and subvector of A and b corresponding to the rows and components of A and b indexed by β, respectively.

A vector x is lexicographically greater than or equal to y, written $x \succeq y$, if $x = y$ or the first nonzero component of $x-y$ is positive. A *matrix* A is lexicographically greater than or equal to a matrix B, written $A \succeq B$, if $(A-B)_i \succeq 0$ for every row i of $(A-B)$.

3. The Main Theorem

Consider a bounded polyhedron χ of the form $\chi = \{x \in R^n | Ax \leq b\}$, where A and b are a given (mxn)-matrix and m-vector, respectively. Let T be a finite triangulation of χ, let K^o denote the set of vertices of T, and let K be the pseudomanifold corresponding to T. Let $M = \{1, \ldots, m\}$ be the set of constraint row indices, and let $L(\cdot):K^o \rightarrow M$ be a labelling function that assigns a constraint row index i to each vertex v of K^o. Our interest lies in ascertaining the combinatorial implications of such a labelling function, under boundary conditions or not, in the spirit of and as a generalization of other combinatorial theorems on the simplex, cube, and simplotope [5,6,7,9,15,17,18,22,24]. Toward this goal, we will make the following assumptions on χ, some of which will be relaxed later on:

A1 (Bounded). χ is bounded, i.e., there is a vector $\lambda \geq 0$, $\lambda \neq 0$, such that $\lambda A = 0$.

A2 (Solid). χ has an interior, i.e., there exists $x^o \in \chi$ such that $Ax^o < b$.

A3 (Nonredundant). There is no redundant constraint governing χ, i.e. there is no $i \in M$ and $\lambda \geq 0$, with $\lambda_i = 0$ such that $\lambda A = A_i$ and $\lambda \cdot b \leq b_i$. If χ is solid, this means that for every $i \in M$, there exists $x \in \chi$ such that $A_i x = b_i$ and $A_k x < b_k$ for every $k \in M \setminus i$.

A4 (Centered). χ contains the origin in its relative interior, i.e. $b \geq 0$.

A5 (Centered and Scaled). χ contains the origin in its relative interior, and the rows of A have been scaled so that each b_i equals 0 or 1.

Assume for the remainder of this section that χ is bounded, solid, nonredundant, and centered and scaled. Then, in particular, b=e. Let $\chi^o = \{y \in R^n | y = \lambda A, \lambda \geq 0 \; \lambda \cdot b = 1\}$. Then χ^o is bounded, solid, and centered. Furthermore, χ^o can alternately be described as $\chi^o = \{y \in R^n | y \cdot x \leq 1$ for all $x \in \chi\}$, whereby χ^o is seen to be the polar of χ (see [20]). χ^o is also a combinatorial dual of χ, i.e., there is a one-to-one inclusion reversing mapping from the k-faces of χ to the (n-k-1)-faces of χ^o, see [12].

Because χ is nonredundant, each row of A is an extreme point of χ^o. Furthermore, every point $y \in \chi^o$ can be expressed as a convex combination of (n+1) extreme points of χ^o, i.e., (n+1) rows of A. A point $y \in \chi^o$ is called a regular point of χ^o if y cannot be expressed as a convex combination of n or fewer rows of χ^o. Because χ is bounded, χ^o is solid, and so almost every point in χ^o is a regular point of χ^o, i.e., the set of points in χ^o that are not regular are a set of measure zero, and χ^o has positive measure. Figure 1 illustrates the above remarks. In the figure, y^1 is a regular point, and y^3 is not a regular point. The circled numbers on the boundary of χ in the figure indicate the row constraint index for the facets indicated.

For a subset $\alpha \subset M$, define $S_\alpha = \{y \in R^n | y = \lambda_\alpha A_\alpha, \; \lambda_\alpha \geq 0, \; \lambda_\alpha \cdot b_\alpha = 1\}$, i.e., S_α is the convex hull of the rows of A indexed over α. We have $S_\alpha = \chi^o$ for $\alpha = M$, and $S_\alpha \subset \chi^o$ for all $\alpha \subset M$. For every $y \in \chi^o$ define $G_y = \{\alpha \subset M | y \in S_\alpha\}$. Then G_y consists of the row index sets of vertices of cells S_α that contain the point y. Referring to figure 1 again, we see that G_{y^1} consists of the four sets {1,3,4}, {1,3,5}, {1,2,4}, and {1,2,5}, plus all other subsets of M that contain one

$$\chi = \{x \in R^2 \mid Ax \leq b\}, \text{ where } A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ -1 & 2 \\ -1 & -2 \\ 3 & -2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

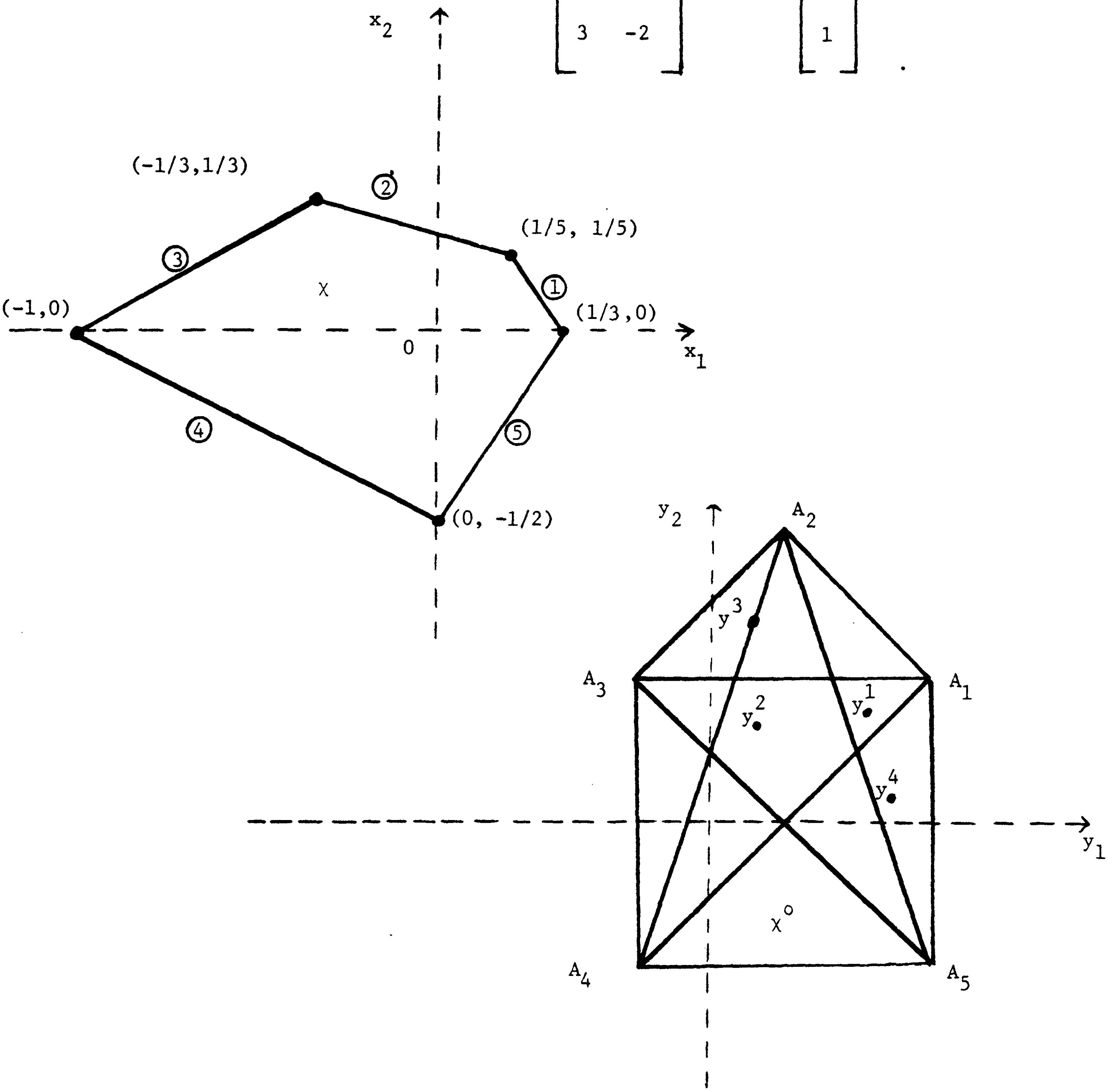

Figure 1

of these four sets. Likewise, the minimal members of $G_y{}^4$ are $\{1,2,5\}$, $\{1,3,5\}$, and $\{1,4,5\}$. Regarding $G_y{}^3$, the minimal members of $G_y{}^3$ are $\{1,2,3\}$, $\{2,4\}$. amd $\{2,3,5\}$.

Now let T be a finite triangulation of χ, let K be the pseudomanifold corresponding to T, and let $L(\cdot):K^o \to M$ be a labelling function from K^o, the set of vertices of K, to M, the set of constraint row indices of χ. For a simplex $\sigma \in K$, let $L(\sigma) = \{i \in M | i = L(v) \text{ for some } v \in K\}$. For a given subset S of χ, define $C(S) = \{i \in M | A_i x = b_i \text{ for all } x \in S\}$. For a point $x \in \chi$, define $C(x) = C(\{x\})$. The mapping $C(\cdot)$ identifies the "carrier" hyperplanes of the set S or point x.

With the above notation in hand, we can state our main theorem:

<u>Theorem 1</u>. Let χ be a polyhedron that is bounded, solid, nonredundant, and centered and scaled. Let T be a finite triangulation of χ, let K be the pseudomanifold corresponding to T, and let $L(\cdot):K^o \to M$ be a labelling function. Then

(i) for any regular point $y \in \chi^o$, there are an odd number of simplices $\sigma \in K$ such that $(L(\sigma) \cup C(\sigma)) \in G_y$, and hence at least one.

(ii) for any point $y \in \text{int } \chi^o$, there is at least one simplex $\sigma \in K$ such that $(L(\sigma) \cup C(\sigma)) \in G_y$.

To illustrate the theorem, let us continue with the example of figure 1. Figure 2 shows a triangulation T of χ and a labelling of K^o. Regarding y^1, a regular point of χ^o, there are five simplices σ of K for which $(L(\sigma) \cup C(\sigma)) \in G_y{}^1 = \{\{1,3,4\}, \{1,3,5\}, \{1,2,4\}, \{1,2,5\}\}$, namely $\{t\}$, $\{w,v\}$, $\{a\}$, $\{f,g,k\}$, and $\{p,q,u\}$. Note that $L(\{w,v\}) = \{1,3\}$, $C(\{w,v\}) = \{5\}$, and hence

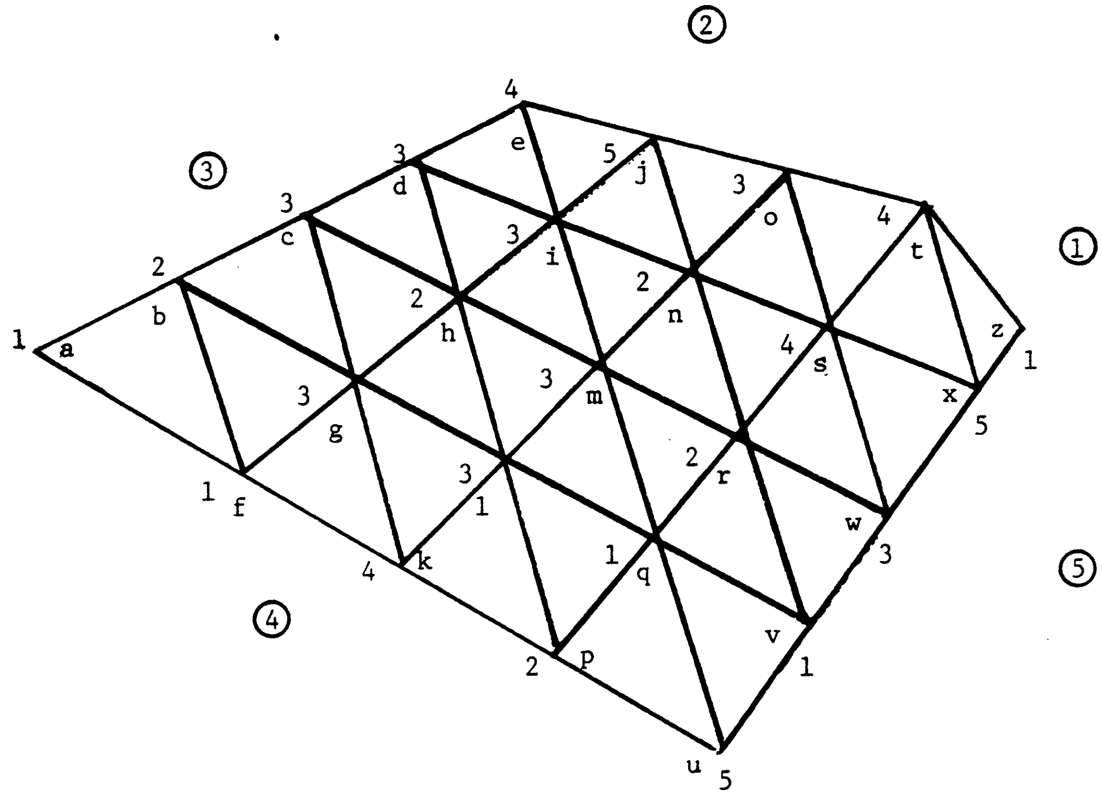

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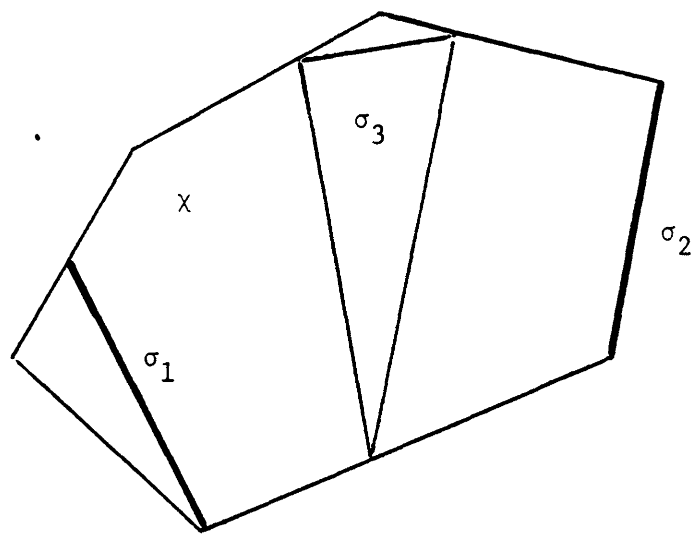
Figure 2

$(L(\{w,v\}) \cup C(\{u,v\})) = \{1,3,5\} \in G_y{}^1$. Regarding y^4, there are three simplices $\sigma \in K$ for which $(L(\sigma) \cup C(\sigma)) \in G_y{}^4 = \{\{1,2,4\}, \{1,3,5\}, \{1,4,5\}\}$, namely $\{p,q,u\}$, $\{w,v\}$, and $\{x,t,z\}$. In the case of the pentagon χ in figure 1, theorem 1 actually makes eleven assertions about the oddness of certain instances of labels, one assertion for each of the eleven regions composing χ^o.

The assertions of theorem 1 do not depend on any special restrictions of the labelling $L(\cdot)$ on the boundary of χ. If we restrict the labelling $L(\cdot)$ on the boundary of χ, we can obtain a stronger form of theorem 1. A labelling $L(\cdot):K^o \rightarrow M$ is called <u>dual proper</u> if $L(v) \in C(v)$ for all $v \in \partial\chi$, $v \in K^o$. If $L(\cdot)$ is dual-proper, $L(v)$ must index a binding constraint at v if v lies on the boundary of χ. This restriction was first introduced by Scarf [22] for the simplex. The denotation here is consistent with the notion of a dual proper labelling as used in [7]. A triangulation T of χ is said to be <u>bridgeless</u> if for each $\sigma \in T$, the intersection of all faces of χ that meet σ is nonempty. This concept is illustrated in Figure 3, for n=2. In the figure, each of the simplices σ_1, σ_2, and σ_3 fails the intersection property. Essentially, if T is bridgeless, then no simplex σ of T meets too many faces of χ that are disparate.

If $L(\cdot)$ is dual-proper and T is bridgeless, we have the following stronger version of theorem 1:



Cases where the intersection of the faces that meet σ are empty

Figure 3

Theorem 2. Let χ be a polyhedron that is bounded, solid, nonredundant, and centered and scaled. Let T be a finite triangulation of χ and let K be the pseudomanifold corresponding to T. Let $L(\cdot): K^o \to M$ be a labelling function on K^o. If $L(\cdot)$ is dual-proper and T is bridgeless, then:

(i) for any regular point $y \in \chi^o$, there are an odd number of simplices $\sigma \in K$ such that $L(\sigma) \in G_y$, and hence at least one.

(ii) for any point $y \in \text{int } \chi^o$, there is at least one simplex $\sigma \in K$ such that $L(\sigma) \in G_y$.

Theorem 2 can be deduced from theorem 1 as follows:

Proof of Theorem 2: Assuming theorem 1 is true, it suffices to show that for each $y \in \text{int } \chi^o$, that if $(L(\sigma) \cup C(\sigma)) \in G_y$, then $C(\sigma) = \phi$. Suppose not. Then there exists $\bar{\sigma} \in K$ such that $(L(\bar{\sigma}) \cup C(\bar{\sigma})) \in G_y$ and $C(\bar{\sigma}) \neq \phi$. Because $C(\bar{\sigma}) \neq \phi$, $\bar{\sigma} \in \partial\chi$, whereby each vertex v of $\bar{\sigma}$ must satisfy $L(v) \in C(v)$. If $L(v)=i$, then v, and hence $\bar{\sigma}$, meets the facet F_i defined by $F_i = \{x \in \chi \mid A_i x = b_i\}$. Therefore $\bar{\sigma}$ meets every facet F_i for $i \in L(\bar{\sigma})$. Furthermore, $\bar{\sigma}$ meets every facet F_i for $i \in C(\bar{\sigma})$. Denoting $\alpha = (L(\bar{\sigma}) \cup C(\bar{\sigma}))$, we have $\bar{\sigma}$ meets F_i for every $i \in \alpha$. Thus $\bigcap_{i \in \alpha} F_i \neq \phi$, because T is bridgeless. Let $\bar{x} \in \bigcap_{i \in \alpha} F_i$, i.e. $A_\alpha \bar{x} = b_\alpha$. Since $\alpha \in G_y$, there exists $\lambda_\alpha \geq 0$ for which $b_\alpha \cdot \lambda_\alpha = 1$ and $y = \lambda_\alpha A_\alpha$. However, $y \cdot \bar{x} = \lambda_\alpha A_\alpha \bar{x} = \lambda_\alpha b_\alpha = 1$. But since $y \in \text{int } \chi^o$, there exists $\theta > 0$ such that $(y + \theta y) \in \chi^o$. Thus $(1+\theta)y \in \chi^o$ and $(1+\theta)y \cdot \bar{x} = 1+\theta > 1$. However, for any $y \in \chi^o$, $x \in \chi$, $y \cdot x \leq 1$, contradicting $(1+\theta)y \cdot \bar{x} > 1$. Thus $C(\bar{\sigma}) = \phi$, and the theorem is proved. [X]

Theorems 1 and 2 (without the oddness assertion) are equivalent to the fixed point theorem of L.E.J. Brouwer [2], stated below:

Brouwer's theorem on a bounded polyhedron. Let χ be a nonempty bounded polyhedron, and let $f(\cdot)$: $\chi \to \chi$ be a continuous function. Then there exists a fixed point of $f(\cdot)$, i.e. a point $x^* \in \chi$ such that $f(x^*) = x^*$.

In order to demonstrate the equivalence of theorems 1 and 2 to Brouwer's theorem, we will use the following lemma, which relates the equilivance of polyhedral representations under projective transformation.

Projective Transformation Lemma. Let $\chi = \{x \in R^n | Ax \leq b\}$ be a polyhedron that is bounded, solid, and centered and scaled, and let $\chi^o = \{y \in R^n | y = \lambda A, \lambda \geq 0, b \cdot \lambda = 1\}$. For any given $y \in$ int χ^o, the set $\chi' = \{x' \in R^n | (A - e \circ y)x' \leq b\}$ is combinatorially equivalent to χ, and $\chi'^o = \chi^o - y$. The mapping $g(x) = x/(1 - y \cdot x)$ maps faces of χ onto the faces of χ' and is inclusion preserving. Furthermore, T is a triangulation of χ if and only if T' is a triangulation of χ', where T' is the collection of simplices $\sigma' = g(\sigma)$ for every $\sigma \in T$.

PROOF: Since $y \in$ int χ^o, $y \cdot x < 1$ for all $x \in \chi$. Consider the mapping $g(\cdot)$: $\chi \to \chi'$, given by $g(x) = x/(1 - y \cdot x)$. It is easy to verify that $g(\cdot)$ maps χ onto χ' continuously, that $x \in \chi$ satisfies $A_i x = b_i$ if and only if $(A - e \circ y)_i g(x) = b_i$, and $g^{-1}(\cdot)$ is given by $g^{-1}(x') = x'/(1 + y \cdot x')$. Thus χ and χ' are combinatorially equivalent. That T' is a triangulation of χ' follows from the fact that $g(\cdot)$ maps affine sets to affine sets and convex sets to convex sets. The mappings $g(\cdot)$ and $g^{-1}(\cdot)$ are, of course, projective transformations. [X]

Proof of theorem 1 (without the oddness assertion) from Brouwer's theorem:

Let χ, T, $L(\cdot)$ and K be given as in theorem 1. Let $y \in \operatorname{int} \chi^o$ be given, define χ' and T' as in the projective transformation lemma, let K' be the pseudomanifold corresponding to T', and define $L'(v') = L(g^{-1}(v'))$ for $v' \in K^{o\prime}$. For each $v' \in K'^{o}$, define $h(v') = A_{L'(v')} - y$, and extend $h(\cdot)$ in a PL manner over all of χ'. Define $f(x') = \arg\min_{z' \in \chi'} \|z' - x' + h(x')\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm.

Because $h(\cdot)$ is continuous, $f(\cdot)$ is continuous and so contains a fixed point $\bar{x}'$. Let $\bar{\sigma}'$ be the smallest simplex σ' in T' that contains $\bar{x}'$, and let $\gamma = L(\bar{\sigma}')$, $\beta = C(\bar{\sigma}')$, and $\alpha = \gamma \cup \beta$. Then the Karush-Kuhn-Tucker conditions state that $\bar{x}' - \bar{x}' + h(\bar{x}') = -\bar{\lambda}_\beta (A - eoy)_\beta$, for some $\lambda_\beta \geq 0$. Furthermore, $h(\bar{x}') = \bar{\lambda}_\gamma (A - eoy)_\gamma$ for some particular $\bar{\lambda}_\gamma \geq 0$, $\bar{\lambda}_\gamma \cdot e_\gamma = 1$. Therefore, $\bar{\lambda}_\beta (A-eoy)_\beta + \bar{\lambda}_\gamma (A-eoy)_\gamma = 0$, whereby $\bar{\lambda}_\alpha (A-eoy)_\alpha = 0$ has a nonnegative and nonzero solution. Upon rescaling the multipliers $\bar{\lambda}_\alpha$ so that they sum to unity, we have $\bar{\lambda}_\alpha A = y$, $\bar{\lambda}_\alpha \geq 0$, $\bar{\lambda}_\alpha \cdot e_\alpha = 1$. Thus $\alpha \in G_y$ and $(L(\bar{\sigma}') \cup C(\bar{\sigma}')) = \alpha$, whereby the simplex $\bar{\sigma} \in T$ defined by $\bar{\sigma} = g^{-1}(\bar{\sigma}')$ has $(L(\bar{\sigma}) \cup C(\bar{\sigma})) = \alpha \in G_y$, proving the result. [X]

The construction of the function $f(\cdot)$ was introduced by Eaves [3] to convert the stationary point problem of $h(\cdot)$ to a fixed point problem on $f(\cdot)$.

Proof of Brouwer's theorem from Theorem 1: Let χ be a polyhedron that is bounded, solid, nonredundant, and centered and scaled, and let $f(\cdot)$: $\chi \to \chi$ be a continuous function. Let T be a finite triangulation of χ and K be the pseudomanifold corresponding to T. Let $L(\cdot)$ be a labelling function on K^{o} defined so that L(v) equals any index i for which $A_i(v-f(v)) \geq 0$. Because χ is bounded, such an index must exist. Let y be a given regular point of χ^{o} . Then there exists a simplex $\sigma \in K$ such that $(L(\sigma) \cup C(\sigma)) \in G_y$. Taking a limit of sequences of such σ as the mesh of T goes to zero, we conclude that there exists $\bar{x} \in \chi$, $\alpha \in G_y$, and $\beta \subseteq \alpha$ such that $A_\beta(\bar{x}-f(\bar{x})) \geq 0$, and $(\beta \cup C(\bar{x})) \supseteq \alpha$. Let $\gamma = C(\bar{x})$. Because $A_\gamma(\bar{x}-f(\bar{x})) \geq 0$, $A_\alpha(\bar{x}-f(\bar{x})) \geq 0$. However, since $\alpha \in G_y$, there exists $\lambda_\alpha \geq 0$ with the property that $e \cdot \lambda_\alpha = 1$ and $\lambda_\alpha A_\alpha = y$. Thus $y \cdot (\bar{x}-f(\bar{x})) = \lambda_\alpha A_\alpha(\bar{x}-f(\bar{x})) \geq 0$. However, because y is regular $G_y = G_z$ for all z sufficiently close to y. Thus $z \cdot (\bar{x}-f(\bar{x})) \geq 0$ for all z sufficiently close to y, whereby $\bar{x}-f(\bar{x}) = 0$, proving Brouwer's theorem. [X]

The equivalence of Brouwer's theorem and theorem 2 (without the oddness assertion) can be accomplished in a manner that parallels the above arguments.

Relation of Theorems 1 and 2 to combinatorial results on the simplex and simplotope.

We show how theorems 1 and 2 specialize to known results on the simplex and the simplotope. The three major combinatorial results on the simplex, namely Sperner's lemma [23], Scarf's dual Sperner lemma [22], and the Generalized Sperner lemma [10], all assert the existence of an odd number of simplices with certain label configurations. However, when these three results are extended to the cube and simplotope, the oddness assertion disappears, and the dimension of the specially labelled simplices of interest is reduced (see [7] and [18]). The inability to assert that there are an odd number of specially labelled simplices stems from the constructive proofs of these simplotope theorems. Herein, by casting the simplex and simplotope theorems as instances of theorems 1 and 2 for particular values of $\bar{y} \in \chi^o$, we will see that the oddness assertion holds on the simplex precisely because $\bar{y}$ is a regular point in χ^o, and the oddness assertion on the simplotope (and hence the cube) does not hold, precisely because $\bar{y}$ is not a regular point in χ^o.

Let $S^n = \{x \in R^n | \ x \leq e, \ -e \cdot x \leq 1\}$. Then S^n is an n-dimensional simplex. By defining

$$A^n = \begin{bmatrix} I \\ -e^T \end{bmatrix} \text{ and } b = \begin{bmatrix} e \\ 1 \end{bmatrix}$$

we can write S^n as $S^n = \{x \in R^n | A^n x \leq b\}$. Let T be a triangulation of S^n, K the pseudomanifold corresponding to T, and $L(\cdot): K^o \rightarrow M$, where $M = \{1, \ldots, m\} = \{1, \ldots, n+1\}$, because $m = n+1$. For $\chi = S^n$, the set $\chi^o = \{y | y = \lambda A^n, \ \lambda \geq 0, \ e \cdot \lambda = 1\}$ is an n-simplex that contains the origin, and any $y \in$ int χ^o is a regular point in χ^o. In particular, $\bar{y} = 0$ is a regular point in χ^o, and $G_{\bar{y}} = \{M\} = \{\{1, \ldots, n+1\}\}$. Because S^n is bounded, solid, nonredundant, and centered and scaled, we can apply theorem 1, and assert that there are an odd number of simplices $\sigma \in K$ with the property that $(L(\sigma) \cup C(\sigma)) \in G_{\bar{y}}$, i.e., $L(\sigma) \cup C(\sigma) = \{1, \ldots, n+1\}$. This is precisely the Generalized Sperner lemma [10], and is seen to follow as a specific instance of theorem 1.

Now suppose that the labelling $L(\cdot)$ is dual-proper, i.e. for each $v \in \partial S^n$, $L(v) = i$ must be chosen so that $A_i v = b_i$. Furthermore, suppose that no simplex of T meets every facet F_i of S^n, where $F_i = \{x \in S^n | A_i x = b_i\}$, $i=1, \ldots, n+1$. Then it can be shown that for any simplex $\bar{\sigma}$ of T, the intersection of all faces of S^n that meet $\bar{\sigma}$ is nonempty, i.e. T is bridgeless, whereby the conditions of theorem 2 are satisfied. Thus there exists an odd number of simplices $\sigma \in K$ such that $L(\sigma) \in G_{\bar{y}}$, i.e. $L(\sigma) = \{1, \ldots, n+1\}$. This latter result is precisely Scarf's dual Sperner lemma [22], and it is seen to follow as a specific instance of theorem 2.

We now turn our attention to theorems on the simplotope. A simplotope S is defined to be the cross-product of n simplices, $S = S^{m_1} \times \ldots \times S^{m_p}$, where, for simplicity, we will assume that each $m_j \geq 1$, $j=1, \ldots, p$. Any point $x \in S$ is a vector in R^N, where $N = \sum_{j=1}^{p} m_j$, and x can be written as $x = (x^1; \ldots; x^p)$, where each each $x^j \in R^{m_j}$, $j=1, \ldots, p$, and x is the concatenization of the n vectors x^j, $j=1, \ldots, p$. Defining A^n as above, let us define A as the $(N+p)x(N)$ matrix:

$$A = \begin{bmatrix} A^{m_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & A^{m_p} \end{bmatrix},$$

where A^{m_j} is as described previously.

Then S can be described as $S = \{x \in R^n \mid Ax \geq b\}$ where $b \in R^{N+p}$ and $b=e$. Define $M = \{(j,k) \mid j=1, \ldots, p,\ k=1, \ldots, m_j+1\}$. The rows of A can be indexed by the ordered pairs $(j,k) \in M$ where row (j,k) of A is in fact row number $(\sum_{i=1}^{j-1} (m_i+1) + k)$ of A. Likewise, a vector $\lambda \in R^{N+p}$ will be indexed by the ordered pairs $(j,k) \in M$. Let T be a triangulation of S, let K be the pseudomanifold corresponding to T, and let $L(\cdot):K^o \rightarrow M$ be a labelling function. For $\chi = S$, χ is bounded, solid, nonredundant, and centered and scaled, and so the conditions of theorem 1 are met. We have $\chi^o = \{y \in R^n \mid y=\lambda A,\ e\cdot\lambda=1,$ $\lambda \geq 0\}$ and $\bar{y}=0 \in \chi^o$. However, $\bar{y}=0$ is not a regular point of $\underline{Y}$. To see this, pick any one index j from among $j \in \{1, \ldots, p\}$.

Then set

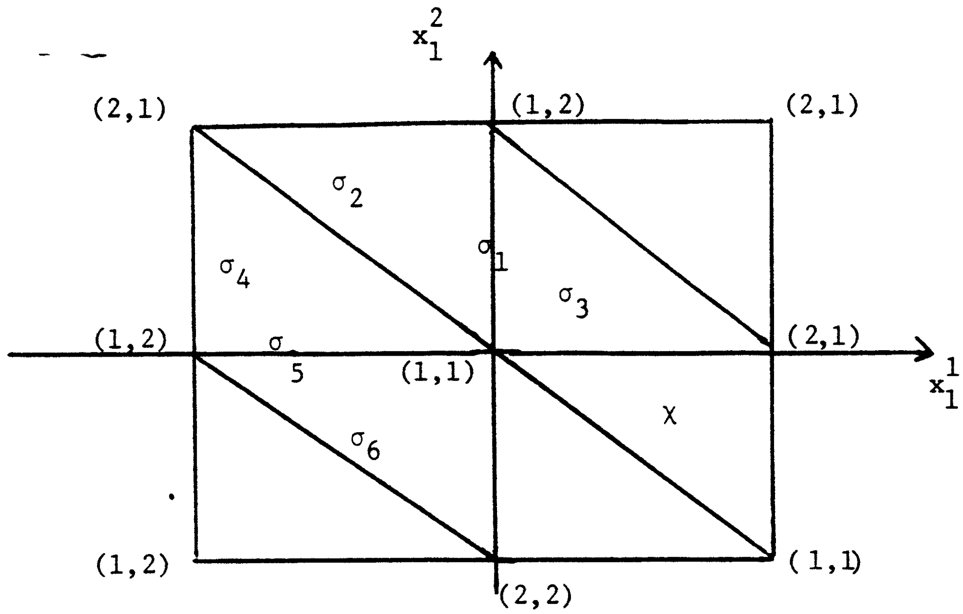
$$\bar{\lambda}_{(i,k)} = \begin{cases} 0 & \text{if } i \neq j \\ 1/(m_j+1) & \text{if } i = j, \end{cases}$$

for each $(i,k) \in M$, and note that $\bar{\lambda} \geq 0$, $e\cdot\bar{\lambda}=1$, and $\bar{\lambda}A=0=\bar{y}$. If we define $\alpha_j=\{(j,1), \ldots, (j,m_j+1)\}$, we see that $0 \in S_{\alpha_j}$, but $|\alpha_j| = m_j+1 < N+1$, so long as $n > 1$. Thus $\bar{y}=0$ is not a regular point of χ^o. Thus, by theorem 1, we can only assert that there exists at least one simplex σ of K such that $(L(\sigma) \cup C(\sigma)) \in G_{\bar{y}}$. However $G_{\bar{y}} = \{\alpha \subset M \mid \bar{y} \in S_\alpha\} = \{\alpha \subset M \mid \alpha = \alpha_j$ for some $j \in \{1, \ldots, p\}\}$. Thus there exists a simplex σ of K such that $(L(\sigma) \cup C(\sigma)) \supseteq \{(j,1), \ldots, (j,m_j+1)\}$ for some $j \in \{1, \ldots, p\}$. This is precisely theorem 1 of [7] or lemma 3.2 of [18].

Figure 4 illustrates the theorem for $m_1=m_2=1$, and $n=2$. With $\bar{y}=0$, $G_{\bar{y}} = \{\{(1,1), (1,2), (2,1)\}, \{(1,1), (1,2), (2,2)\}, \{(2,1), (2,2), (1,1)\}, \{(2,1), (2,2), (1,2)\}$. There are six simplices of S with $(L(\sigma) \cup C(\sigma)) \in G_y$, namely $\sigma_1, \ldots, \sigma_6$ in the figure. The set χ^o is the convex hull of the points (1,0), (-1,0) (0,1) and (0,-1), the diamond shown in the figure. As the figure shows, $\bar{y}=0$ is not a regular point.

Suppose now that the labelling $L(\cdot):K^o \to M$ is dual proper, i.e. for each $v \in \partial S$, $L(v)$ must be chosen so that $A_i x = b_i$. Furthermore, suppose that no simplex $\sigma \in K$ meets each facet $F_{(j,k)} = \{x \in S \mid A_{(j,k)}x = b_{(j,k)}\}$, for all $(j,k) \in \alpha_j$, for any $j=1, \ldots, p$. Then it can be shown that the requirements of theorem 2 are met. This being the case, the logic employed herein can be used to show that there exists a simplex $\sigma \in K$ such that $L(\sigma) \supseteq \alpha_j$ for some $j \in \{1, \ldots, p\}$. This latter statement is precisely theorem 2 of [7], and thus is a specific instance of theorem 2 of this paper.

The Sperner lemma, and its extension to the simplotope [7,17], does not appear to be a specific instance of theorems 1 or 2. Sperner's lemma can be derived from the Generalized Sperner lemma, see [6], but this derivation fails to carry over to the simplotope. In the last section of this paper, we present another combinatorial theorem on a bounded polyhedron, that specifies to Sperner's lemma on the simplex.


x_1^2
(2,1)
(1,2)
(2,1)
σ_2
σ_1
σ_4
σ_3
(1,2)
σ_5
(1,1)
(2,1)
x_1^1
χ
σ_6
(1,2)
(1,1)
(2,2)

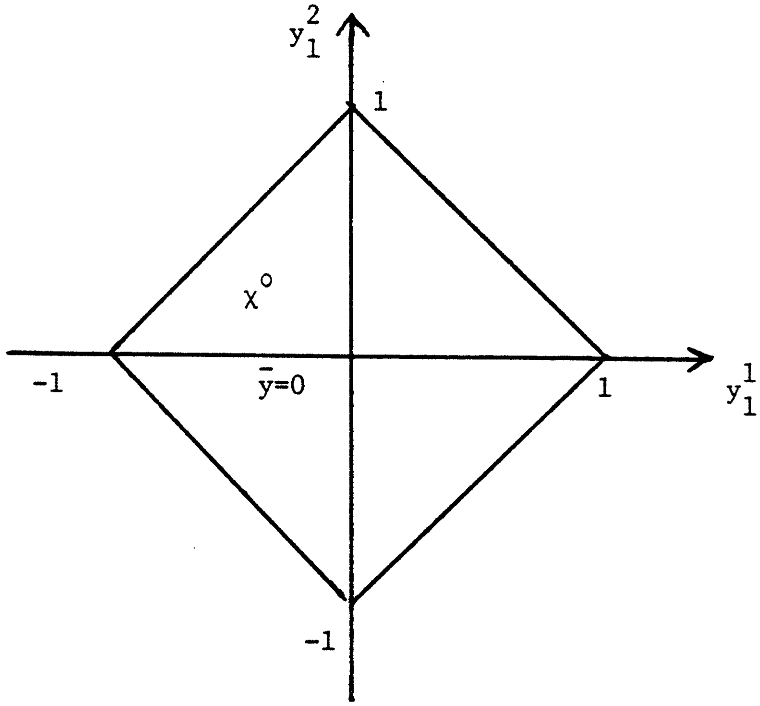

y_1^2
1
χ^o
-1
$\bar{y}=0$
1
y_1^1
-1

Figure 4

Relaxing the Assumptions of Theorem 1

Our final remarks of this section are concerned with relaxing the assumptions presented earlier. The assumption that χ is bounded is central to Brouwer's theorem, and to the counting arguments regarding endpoints of paths of simplices, as will be seen in the proof of theorem 1 in the next section. The assumptions that χ is solid and centered and scaled can be eliminated, but the definition of χ^o must then be changed. Let us first consider the case when $\chi = \{x \in R^n | Ax \leq b\}$ is solid but not centered and scaled. For any given $x^o \in \text{int } \chi$, $\chi' = \{x \in R^n | Ax \leq b - Ax^o\}$ is just a translation of χ by $-x^o$, and can alternatively be written as $\chi' = \{x \in R^n | \bar{A}x \leq e\}$, where $\bar{A}_i = A_i/(b_i - A_i x^o)$. χ' now is centered and scaled, and so the assertions of theorem 1 apply. In this case, the set $\chi^o = \{y \in R^n | y = \lambda\bar{A},\ e \cdot \lambda = 1,\ \lambda \geq 0\} = \{y \in R^n | y = \lambda A,\ \lambda \geq 0,\ \lambda \cdot (b - Ax^o) = 1\}$, and for $\alpha \subset M$, $S_\alpha = \{y \in R^n | y = \lambda_\alpha A_\alpha, \lambda_\alpha \geq 0\ \lambda_\alpha \cdot (b - Ax^o)_\alpha = 1\}$. Thus theorem 1 (and hence theorem 2) can be modified to include the case when χ is not centered and scaled.

Next, let us consider the case when χ is neither solid nor centered and scaled, and let k be the dimension of χ. Then in order to center and scale χ, a point $x^o \in \text{rel int } \chi$ can be found using, for example, the methodology in [8]. Once $x^o \in \text{rel int } \chi$ has been given, χ can be rewritten as $\chi = \{x \in R^n | Dx = d,\ Bx \leq b\}$, where $Dx^o = d$ and $s = b - Bx^o > 0$. Furthermore, by scaling the rows of (B,b), we can ensure that $s = b - Bx^o = e$. Let C be any matrix whose rows form an orthonormal basis for the subspace $N = \{x \in R^n | Dx = 0\}$, and let $\chi^o = \{y \in R^n | y = \mu D + \lambda B,\ s \cdot \lambda = 1,\ \lambda \geq 0\}$. Then the transformation $f(\cdot) = \chi \to R^k$

given by $f(x)=Cx-Cx^o$ maps χ onto $\chi'=\{z\in R^k | BC^Tz \leq s\}$, and the transformation $g(y)=Cy$ maps the set χ^o into the set $\chi'^o = \{v\in R^k | v=\lambda BC^T, \lambda \geq 0, \lambda\cdot s=1\}$. There is a one-to-one correspondence between each given point v in χ'^o and the subset $\{y\in R^n | y=C^Tv + D^T\mu$ for some $\mu\}$ of χ^o. The sets χ' and χ'^o conform to the conditions of theorem 1. For a given point $y\in\chi^o$, there is a unique point $v\in\chi'^o$ such that $y=C^Tv+D^T\mu$ for some μ. The point y in χ^o is called a regular point of χ^o if $y=C^Tv+D^T\mu$ and v is a regular point of χ'^o. Furthermore, for any $y\in\chi^o$, define $G_y=G_v$ where v is uniquely determined by the relation $y=C^Tv+D^T\mu$ for some μ.

This transformation, together with the above remarks on centering and scaling, can be used to prove the following extension of theorem 1:

Theorem 3. Let χ be a nonempty bounded polyhedron of dimension k in R^n that is nonredundant, of the form $\chi=\{x\in R^n | Dx=d, Bx \leq b\}$. Let x^o be a given point in rel int χ, and let $s=b-Bx^o > 0$ be given. Let T be a finite triangulation of χ and let K be the pseudomanifold corresponding to T. Let $L(\cdot):K^o\rightarrow M$, where $M=\{1, ..., m\}$ indexes the rows of B, and let $C(\sigma)=\{i\in M | B_ix=b_i$ for all $x\in\sigma\}$ for each $\sigma\in T$. Let χ^o be defined as in the remarks above. Then:

(i) If y is a regular point of χ^o, there exists an odd number of simplices $\sigma\in T$ with the property that $(L(\sigma) \cup C(\sigma)) \in G_y$, and hence at least one.

(ii) If $y \in$ rel int χ^o, then there exists at least one simplex $\sigma\in T$ with the property that $(L(\sigma) \cup C(\sigma)) \in G_y$. [X]

Theorem 3 obviously implies theorem 1 as a special case. The above remarks outline how to prove theorem 3 as a consequence of theorem 1, using the transformation $f(\cdot)$. Theorem 3 is the most general combinatorial theorem we will consider. The theorem still retains the nonredundancy assumption. This assumption is retained for convenience. Because redundant constraints do not contribute to either the geometric or combinatorial properties of a polyhedron, the fact they are assumed away does not detract from the generality of the results.

4. A Combinatorial Proof of Theorem 1

This section contains a combinatorial proof of theorem 1. The ideas behind the proof derive from relatively straightforward concepts that are easy to follow in two dimensions. In higher dimensions, they become more encumbered due to the possible presence of degeneracy in χ. Hence, in order to motivate the proof along more intuitive lines, we start by showing an example of the proof in two dimensions. We then proceed to the more general case.

Example of proof in two dimensions

Let χ and χ^o be as shown in figure 1, let T and $L(\cdot)$ be as shown in Figure 2, and let K be the pseudomanifold corresponding to T. Define $\bar{K}$ to be the pseudomanifold consisting of simplices $\sigma \in K$ "joined" with the indices of $C(\sigma)$, i.e.

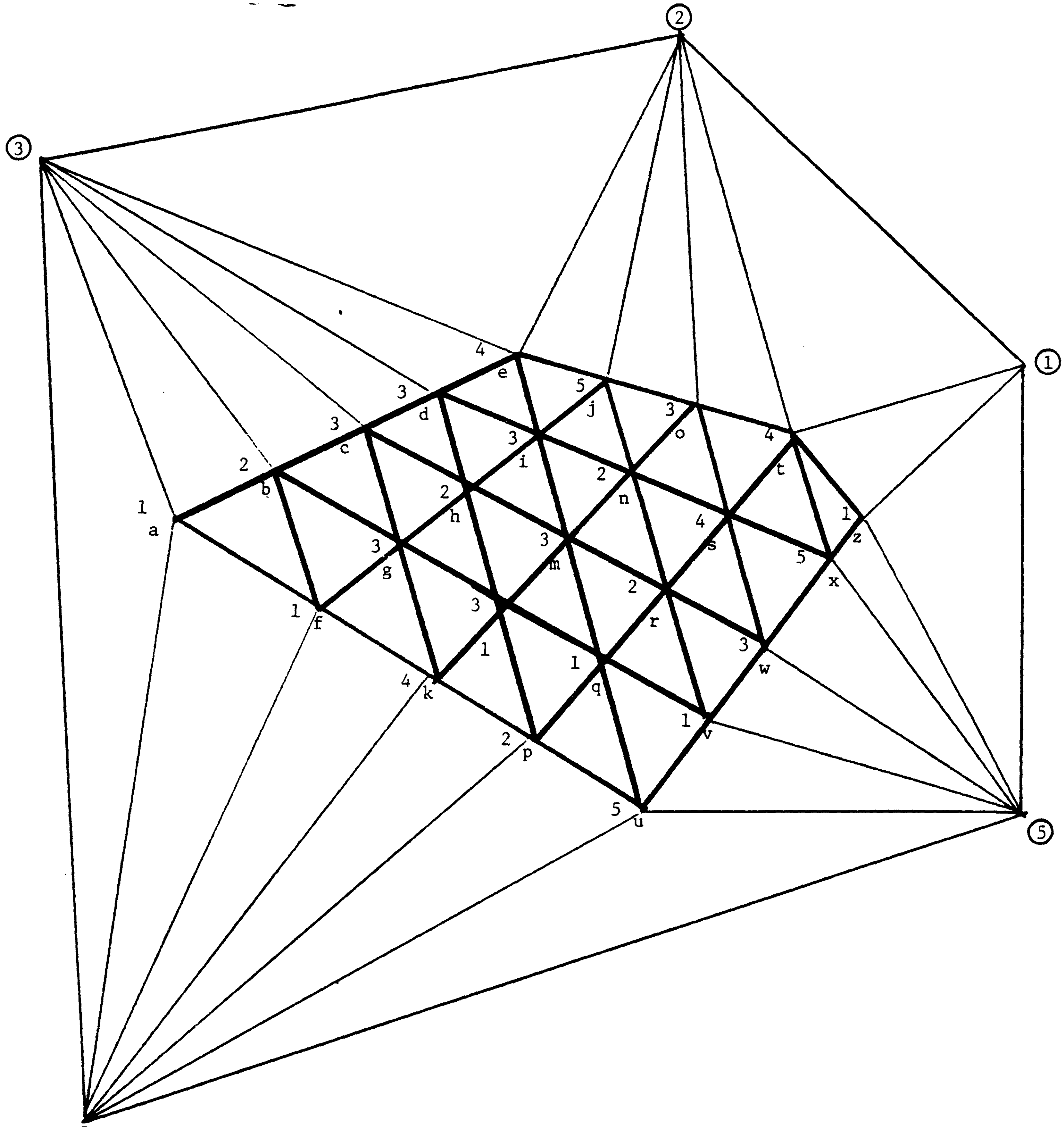
$$\bar{K} = \{\bar{\sigma} | \bar{\sigma} \neq \Phi,\ \bar{\sigma} \subseteq (\sigma \cup C(\sigma)),\ \sigma \in K\}, \text{ and}$$

$$\bar{K}^o = K^o \cup \{1, \ldots, m\} = K^o \cup M.$$

The construction of $\bar{K}$ is shown in Fgure 5. Note that

$$\partial \bar{K} = \{\beta | \beta = C(x) \text{ for some } x \in K\}.$$

For each $i \in M$, extend $L(\cdot): K^o \rightarrow M$ to $L(\cdot): \bar{K}^o \rightarrow M$ by the association $L(i)=i$ for $i \in M$. For each $y \in \chi^o$, let $\#G_y$ denote the number of simplices $\bar{\sigma} \in \bar{K}$ with the property that $L(\bar{\sigma}) \in G_y$. In order to prove theorem 1, it suffices to show that $\#G_y$ is odd for all regular points $y \in \chi^o$. Now let $\beta \subset M = \{1, \ldots, 5\}$ with $|\beta| = n = 2$. Let $R_\beta = \{\beta \cup \{j\},\ j \in M,\ j \notin \beta\}$. For example, for $\beta = \{1,3\}$, $R_\beta = \{\{1,2,3\}, (1,3,4\}, (1,3,5\}\}$.



The pseudomanifold $\bar{K}$

Figure 5

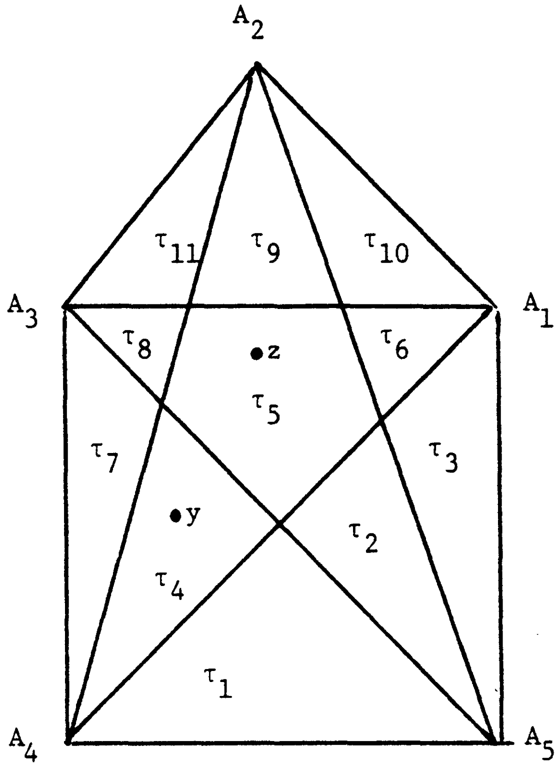
Let $\#R_\beta$ be the number of simplices $\bar{\sigma}\in\bar{K}$ with the property that $L(\bar{\sigma}) \in R_\beta$, and let q_β be the number of simplices $\bar{\sigma}\in\partial\bar{K}$ with the property that $L(\bar{\sigma}) = \beta$. A parity argument, first introduced by Kuhn [16], and later used by Gould and Tolle [11], states that the parity of $\#R_\beta$ and the parity of q_β is the same for any given β, with $|\beta|=n$. This implies, in particular, that

(i) if $\beta \in \partial\bar{K}$, $|\beta|=2$, then $\#R_\beta$ is odd, and

(ii) if $\beta \notin \partial\bar{K}$, $|\beta|=2$, then $\#R_\beta$ is even.

The first statement follows from the fact that if $\beta \in\partial\bar{K}$, then $L(\beta)=\beta$, and there is no other simplex $\bar{\sigma} \in \partial\bar{K}$ with $L(\bar{\sigma})=\beta$ (if so, then $\bar{\sigma}=L(\bar{\sigma}) = \beta$, a contradiction). Thus $q_\beta = 1$, an odd number, whereby $\#R_\beta$ is odd. As an example, let $\beta=\{4,5\}$. Note that $\beta \in\partial\bar{K}$. There are five simplices $\bar{\sigma}\in\bar{K}$ with $L(\bar{\sigma}) \in R_\beta$, namely (4,p,u}, (x,t,z}, {w,x,s}, {e,j,2}, and {e,j,i}, an odd number. The second statement follows from the fact that if $\beta \notin \partial\bar{K}$, there can be no simplices $\bar{\sigma} \in \partial K$ with $L(\bar{\sigma}) = \beta$ (for if so, then $\bar{\sigma} = L(\bar{\sigma}) = \beta$, a contradiction). Thus $q_\beta=0$, an even number, and hence $\#R_\beta$ is an even number. As an example, let $\beta = \{1,4\}$, and hence $\beta \notin \partial\bar{K}$. There are four simplices $\bar{\sigma}\in\bar{K}$ with $L(\bar{\sigma}) \in R_\beta$, namely {a,3,4}, {f,g,k}, {x,t,z}, and {t,1,2}.

Now consider the set X^o, now shown in Figure 6, subdivided into the eleven regions τ_k, k=1,...,11. For any $y \in \text{int}\ \tau_1$, y is a regular point of X^o. Also, for any $y \in \tau_1$, $G_y = \{\{1,4,5\}, \{2,4,5\}, \{3,4,5\}\}$, i.e. $G_y = R_\beta$, where $\beta=\{4,5\}$. Because $\beta \in \partial\bar{K}$, by (i) above, $\#R_\beta$ is odd, whereby $\#G_y$ is odd, because $R_\beta = G_y$. This proves



The subdivided cell $\overset{\circ}{X}$.

Figure 6

theorem 1 for all $y \in \operatorname{int} \tau_1$. For $y \in \operatorname{int} \tau_1$, those simplices $\bar{\sigma} \in \bar{K}$ for which $L(\bar{\sigma}) \in G_y$ are $\{p,u,4\}$, $\{s,x,w\}$, $\{x,t,z\}$, $\{e,i,j\}$, and $\{e,j,2\}$. The main fact that has been used is that all $y \in \operatorname{int} \tau_1$ are "sufficiently close" to the face $<A_4,A_5>$ so that $y \in \operatorname{int}$

$$\bigcap_{j \neq 4,5} \operatorname{conv} <A_4,A_5,A_j>, \text{ whereby } G_y = \{\{4,5,j\} \mid j \neq 4,\ j \neq 5,\ j \in M\},$$

i.e. $G_y = R_{\{4,5\}}$.

We next will show that if y and z are in the interior of adjacent regions τ_i and τ_j of χ^o, that the parity of $\#G_y$ equals the parity of $\#G_z$. Since the parity of $\#G_y$ is odd for $y \in \operatorname{int} \tau_1$, then this will mean that the parity of $\#G_y$ is odd for $y \in \operatorname{int} \tau_k$, $k=2,\ldots,11$, proving assertion (i) of theorem 1. Assertion (ii) follows from a closure argument.

Therefore, consider any two adjacent regions τ_i and τ_j, in χ^o, for example τ_4 and τ_5. For any $y \in \operatorname{int} \tau_4$ and $z \in \operatorname{int} \tau_5$,
$G_y = \{\{1,2,4\}, \{2,4,5\}, \{2,3,4\}, \{1,3,4\}, \{3,4,5\}$, and
$G_z = \{\{1,2,4\}, \{2,4,5\}, \{2,3,4\}, \{1,3,4\}, \{1,3,5\}, \{2,3,5\}\}$. Note that
$G_y \Delta G_z = \{\{1,3,5\}, \{2,3,5\}, \{3,4,5\}\} = R_{\{3,5\}}$.
Furthermore, the face of $\tau_4 \cap \tau_5$ that separates τ_4 from τ_5 is generated by the line segment $<A_3,A_5>$. It is no coincidence that the set $\{3,5\}$ appears in each of the last two statements. Every simplex $<A_3,A_5,A_j>$, $j \notin \{3,5\}$ contains either τ_4 or τ_5 but not both. This shows that $R_{\{3,5\}} \subset G_y \Delta G_z$. But because the line segment $<A_3,A_5>$ is the unique line segment separating τ_4 from τ_5, then any α that lies in $G_y \Delta G_z$ must contain $\{3,5\}$, i.e. $\alpha \supseteq \{3,5\}$, whereby $G_y \Delta G_z \subseteq R_{\{3,5\}}$. Thus $G_y \Delta G_z = R_{\{3,5\}}$. For any collection D

of subsets of M, let #D denote the number of simplicies $\bar{\sigma}\in\bar{K}$ such that $L(\bar{\sigma}) \in D$. Note that

$$G_y = \overline{(G_y \backslash G_z)} \cup (G_y \cap G_z), \text{ whereby}$$

$$\#G_y = \#(G_y \backslash G_z) + \#(G_y \cap G_z),$$

because these two sets are disjoint. Similarly, we have:

$$\#G_z = \#(G_z \backslash G_y) + \#(G_y \cap G_z).$$

We obtain:

$$\begin{aligned}\#G_y - \#G_z &= \#(G_y \backslash G_z) - \#(G_z \backslash G_y)\\ &= \#(G_y \backslash G_z) + \#(G_z \backslash G_y) - 2\#(G_z \backslash G_y)\\ &= \#(G_y \Delta G_z) - 2\#(G_z \backslash G_y)\\ &= \#R_{\{3,5\}} - 2\#(G_z \backslash G_y).\end{aligned}$$

However, $\#R_{\{3,5\}}$ is even, because $\{3,5\} \notin \partial\bar{K}$. Therefore $\#G_y - \#G_z$ is even, i.e. $\#G_y$ and $\#G_z$ have the same parity. This completes the proof of Theorem 1 for the example of figures 1 and 2.

The important facts leading to the proof that $\#G_y$ and $\#G_z$ have the same parity if y and z are interior to adjacent regions τ_i and τ_j of χ^o are as follows: If τ_i and τ_j are adjacent, there is a unique index set β such that the (n-1)-simplex $S_\beta = \{y\in R^n | y=\lambda_\beta A_\beta, \lambda_\beta \geq 0, e\cdot\lambda_\beta=1\}$, separates τ_i from τ_j. Furthermore, S_β cannot lie on $\partial\chi^o$, whereby $\beta \notin \partial\bar{K}$. Finally, $G_y \Delta G_z = R_\beta$. Therefore $\#G_y - \#G_z = \#R_\beta - 2\#(G_y \backslash G_z)$, which is an even number.

<u>Proof of Theorem 1</u>

Let χ, T, $L(\cdot)$, and K be as given in theorem 1. χ is said to be <u>centrally regular</u> if y=0 does not lie in the affine hull of n or fewer rows of A. Note that if χ is centrally regular, then y=0 must be a regular point of χ^o. The remark below allows us to assume, without loss of generality, that χ is centrally regular.

Remark 1. If theorem 1 is true when the polyhedron χ is centrally regular, then theorem 1 is true independent of χ being centrally regular.

Proof: Suppose χ is not centrally regular, but that χ satisfies the hypotheses of theorem 1. Then χ^o is solid, and almost every point $y \in \chi^o$ does not lie in the affine hull of any set of n or fewer rows of A. Let $\bar{y}$ be any such point χ^o. Then $\bar{y}$ must be a regular point in χ^o.

Now let $\chi' = \{x' \in R^n | (A=e\circ\bar{y})x' \leq b\}$, $\chi'^o = \{y' \in R^n | y' = \lambda(A-e\circ\bar{y}), \lambda \geq 0, \lambda\cdot e=1\}$. From the projective transformation lemma, the mapping $g(\cdot):\chi \to \chi'$ given by $g(x) = x/(1-\bar{y}\cdot x)$ maps χ onto χ', and maps T to the triangulation T'and K to the pseudomanifold K'. For each $v' \in K'^o$, define $L'(v') = L(g^{-1}(v'))$. Notice that $\chi'^o = \chi^o - \bar{y}$, and that 0 is a regular point in χ'^o. Therefore, because the system χ', T', $L'(\cdot)$, and K' is combinatorially equivalent to our original system and by hypothesis theorem 1 is true for this new system, theorem 1 is true for the original system. [X]

We therefore will assume for the remainder of this section that χ is centrally regular.

χ is said to be nondegenerate if $A_\alpha x=b_\alpha$ has no solution $x \in \chi$ when $|\alpha| \geq n+1$ and $\alpha \subset M$. In order to prove theorem 1, we will first assume that χ is nondegenerate. This assumption will be relaxed subsequently. The remark below lists some of the properties that are consequences of the property of nondegeneracy in conjunction with the assumption that χ is centrally regular.

Remark 2. If χ satisfies the assumptions of theorem 1 and χ is nondegenerate and centrally regular, then:

(a) every extreme point of χ meets exactly n facets of χ.

(b) every facet of χ^o is an (n-1)-simplex S_β, where $\beta = C(x)$ for some extreme point of χ, and for every extreme point $x \in \chi$, S_β is a facet of χ^o, where $\beta = C(x)$.

(c) for every $\alpha \subset M$ with $|\alpha| = n+1$, S_α is an n-simplex.

(d) for every $\beta \subset M$ with $|\beta| = n$, S_β is an (n-1)-simplex, whose affine hull contains no other rows A_i, $i \in M \backslash \beta$.

Proof: (a) is a direct consequence of the definition of nondegeneracy and the fact that an extreme point x of χ must meet at least n facets of χ, because the dimension of χ is n.

To prove (b), let F be a facet of χ^o. Then there exists a supporting hyperplane H of χ^o such that $H \cap \chi^o = F$. This hyperplane can be written as $H = \{y | y^T x = \theta\}$ for a particular (x, θ). Hence $F = S_\beta$, where $\beta = \{i | A_i x = \theta\}$ and $A_i x < \theta$ for $i \in M \backslash \beta$. Because $0 \in \text{int } \chi^o$, θ must be positive and we can assume $\theta = 1$. Therefore $x \in \chi$, and $A_\beta x = b_\beta$, whereby $|\beta| \leq n$, since χ is nondegenerate. But since F is an (n-1)-cell, we must have $|\beta| \geq n$, and hence $|\beta| = n$, and F is an (n-1)-simplex. Conversely, let x be an extreme point of χ and let $\beta = C(x)$. Then $|\beta| = n$, from (a), and the hyperplane $\{y \in R^n | y^T x = 1\}$ supports χ^o and contains S_β, since $A_i x = 1$ for each $i \in \beta$. Thus S_β is a face of χ^o. Since A_β must have linearly independent rows, the rows of A_β are affinely independent, whereby S_β is an (n-1)-simplex.

To prove (c), let $\alpha \subset M$, $|\alpha| = n+1$. If the matrix

$$Z = \begin{bmatrix} A_\alpha^T \\ \\ e_\alpha^T \end{bmatrix}$$

does not have rank n+1, then there exists $(x,\theta) \neq (0,0)$ such that $A_\alpha x = e_\alpha \theta$, a contradiction. Therefore Z has full rank, whereby the set S_α is an n-simplex.

For (d), let $\beta \subset M$, $|\beta| = n$, then $\alpha = \beta \cup \{i\}$ satisfies (c) for any $i \in M\backslash\beta$, whereby S_β is an (n-1)-simplex and A_i does not lie in the affine hull of S_β. [X]

Our next task is to construct the extended pseudomanifold $\bar{K}$, defined by

$$\bar{K}^o = K^o \cup M$$

$$\bar{K} = \{\bar{\sigma} \in \bar{K}^o | \bar{\sigma} \neq \phi,\ \bar{\sigma} \subset (\sigma \cup C(\sigma)) \text{ for some } \sigma \in K\}.$$

This construction is illustrated in figure 5. We have the following lemma:

<u>Lemma 1</u>. If χ is nondegenerate, $\bar{K}$ is an n-pseudomanifold, and $\partial\bar{K} = \{\beta \subset M | \beta \subseteq C(x) \text{ for some } x \in \chi,\ \beta \neq \phi\}$. β is an (n-1)-simplex in $\partial\bar{K}$ if and only if S_β is a facet (and an (n-1)-simplex) of χ^o.

<u>PROOF</u>: χ is nondegenerate, and so by Corrollary A1 of the appendix, $\bar{K}$ and $\partial\bar{K}$ are as stated. The second statement of the lemma follows from part (b) of remark 2. [X]

The construction of $\bar{K}$ will be generalized later in this section to include degenerate bounded polyhedra as well. This construction resembles the construction of an antiprism in Broadie [1], but is combinatorial in nature and so does not depend on the geometric projection property used in his work.

We now extend $L(\cdot):K^o \rightarrow M$ to $L(\cdot):\bar{K}^o \rightarrow M$, by defining $L(i)=i$ for $i \in M$.

For each $\beta \subset M$, $|\beta|=n$, define $R_\beta = \{\beta \cup \{j\} | j \in M\backslash\beta\}$. For any collection D of subsets of M, let #D denote the number of simplices

$\bar{\sigma} \in \bar{K}$ with the property that $L(\bar{\sigma}) \in D$. We have the followihng result:

<u>Lemma 2</u>: Let $\bar{\beta} \subset \bar{M}$ with $|\beta|=n$.

(a) If $\beta \in \partial\bar{K}$, then $\#R_\beta$ is odd, and

(b) If $\beta \notin \partial\bar{K}$, then $\#R_\beta$ is even.

<u>PROOF</u>: Let q_β be the number of simplices $\bar{\sigma} \in \partial\bar{K}$ with the property that $L(\bar{\sigma}) = \beta$. A parity argument, first introduced by Kuhn [16] and later used by Gould and Tolle [11], states that the parity of $\#R_\beta$ and the parity of q_β is the same for any $\beta \subset M$ with $|\beta|=n$. If $\beta \in \partial\bar{K}$ and $|\beta|=n$, then $L(\beta)=\beta$, and $q_\beta=1$, whereby $\#R_\beta$ is odd. If $\beta \notin \partial\bar{K}$, then $q_\beta=0$, and hence $\#R_\beta$ is even. [X]

Consider a regular point $y \in \chi^o$. Then $|\alpha| = n+1$ for every $\alpha \in G_y$. Furthermore, because y is regular, $y \in \operatorname{int} S_\alpha$, for every $\alpha \in G_y$, and thus $y \in \operatorname{int} \bigcap_{\alpha \in G_y} S_\alpha$, whereby $\bigcap_{\alpha \in G_y} S_\alpha$ is an n-cell, and every element of the interior of this cell is a regular point in χ^o. If y,z are two regular points in χ^o, then define the relationship $y \approx z$ if $G_y=G_z$. The relation $\approx$ is an equivalence relation on the regular points of χ^o. Furthermore, because G_y can take on only a finite number of values, this equivalence relation divides the regular points y of χ^o into p mutually disjoint sets of the form $\operatorname{int} (\bigcap_{\alpha \in G_{y_k}} S_\alpha)$, for p distinct values of y, $y = y_1, \ldots, y_p$.

For $k=1,\ldots,p$, define $\tau_k = \bigcap_{\alpha \in G_{y_k}} S_\alpha$, and each τ_k is thus an n-cell in χ^o, and $(\operatorname{int} \tau_k) \cap \tau_i = \phi$ for all $i \neq k$. Furthermore, a limiting argument easily demonstrates that $\bigcup_{k=1}^{p} \tau_k = \chi^o$. Figure 6

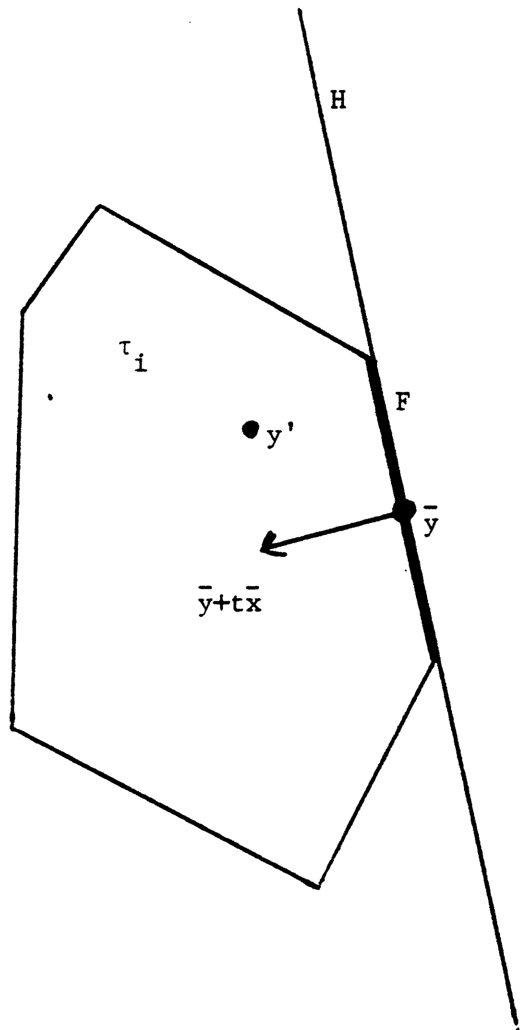


Figure 7

illustrates the above remarks. It is our aim to prove that the collection $\bar{M} = \{\tau_1,\ldots,\tau_k\}$ constitutes a PL subdivision of χ^o (see Eaves [4]).

<u>Lemma 3</u>. Let χ satisfy the assumptions of theorem 1 and suppose χ is nondegenerate. Let $\bar{M} = \{\tau_1,\ldots,\tau_p\}$. Then $(\bar{M}, \chi^o)$ is a subdivided n-manifold. Furthermore, if y and z lie in adjacent n-cells of χ^o, then $G_y \Delta G_z = R_\beta$ for some $\beta \subset M$, $|\beta|=n$, and $\beta \not\subset \partial\bar{K}$.

<u>PROOF</u>: In order to show that $(\bar{M},\chi^o)$ is a subdivided n-manifold, it suffices to show that if F is a facet of an n-cell τ_i, then either $F \subset \partial\chi^o$, or F is a facet of an n-cell τ_j, $j\neq i$. Let τ_i be given, let F be a facet of τ_i, and suppose that $F \not\subset \partial\chi^o$. Let $\bar{y} \in$ rel int F, and let y' be a given element of int τ_i. Then $\tau_i = \bigcap_{\alpha\in G_{y'}} S_\alpha$. Because $\bar{y} \in F$, there exists $\alpha \in G_{y'}$ with the property that $\bar{y} \in \partial S_\alpha$. Because $\bar{y} \in$ rel int F, there exists a unique subset $\beta \subset \alpha$, $|\beta|=n$, with the property that $\bar{y} \in S_\beta$, and $\bar{y} \in$ rel int S_β. Thus $\bar{y} = \bar{\lambda}_\beta A_\beta$, for some $\bar{\lambda}_\beta > 0$, $e_\beta\cdot\bar{\lambda}_\beta = 1$, and $\bar{\lambda}_\beta$ is uniquely determined. Let H be the unique hyperplane in R^n containing S_β (see Figure 7); H can be represented as $H = \{y\in R^n | y\cdot\bar{x} = \bar{\theta}\}$ for some $(\bar{x},\bar{\theta}) \neq (0,0)$ and unique up to scalar multiple. Because $S_\beta \subset H$, $A_\beta\bar{x} = e_\beta\bar{\theta}$. Because H is a supporting hyperplane of τ_i, then without loss of generality, we can assume that $y\cdot\bar{x} \geq \bar{\theta}$ for all $y\in\tau_i$. Because $\bar{y} \in$ rel int S_β, and $\bar{y} \in$ rel int F, $\bar{y} + t\bar{x} \in$ int τ_i for all $t > 0$ and sufficiently small. From remark 1(d), $A_j \notin H$ for any $j\in M\backslash\beta$, i.e. $A_j\bar{x} \neq \bar{\theta}$ for any $j\in M\backslash\beta$. Let $P = \{j\in M | A_j\bar{x} > \bar{\theta}\}$ and $N = \{j\in M | A_j\bar{x} < \bar{\theta}\}$. Then, P, N, and β are disjoint subsets of M. Furthermore, because we can assume that χ is centrally regular, the set $\{i | A_i\bar{x} = \bar{\theta}\}$ can contain at most n elements, whereby

$\beta = \{i | A_i\bar{x} = \bar{\theta}\}$ and $M = P \cup N \cup \beta$. For all $t > 0$ and sufficiently small, $\bar{y} + t\bar{x} \in \text{ int } S_{\beta \cup \{j\}}$ for each $j \in P$. Thus for $y = \bar{y} + t\bar{x}$, $G_y \supseteq (\beta \cup \{j\})$ for $t > 0$ and sufficiently small for $j \in P$. But $\bar{y} + t\bar{x} \in \tau_i$ for all $t > 0$ and sufficiently small. Thus, since $G_y = G_{y'}$ for all $t > 0$ and sufficiently small, $(\beta \cup \{j\}) \in G_{y'}$. Also, it is easy to verify $S_{\beta \cup \{j\}} \cap \tau_i = F$ for all $j \in N$, whereby $(\beta \cup \{j\}) \notin G_{y'}$ for $j \in N$. Now consider $y = \bar{y} - t\bar{x}$ for $t > 0$ and sufficiently small. Then $\bar{y} - t\bar{x} \in S_{\beta \cup \{j\}}$ for all $j \in N$, whereby $(\beta \cup \{j\}) \in G_y$ for $\bar{y} = y - \bar{t}x$ and $t > 0$ and sufficiently small. Furthermore, since $y \in \text{int } S_\alpha$ for all $\alpha \in G_{y'}$ such that $\beta \not\subset \alpha$, $(\bar{y} - t\bar{x}) \in S_\alpha$ for all $\alpha \in G_{y'}$, $\beta \not\subset \alpha$, and $t > 0$ and sufficiently small. Also, for $t > 0$ and sufficiently small, $(\bar{y} - t\bar{x}) \notin S_{\beta \cup \{j\}}$ for any $j \in P$. Therefore,

$$(G_{y'} \backslash (\bigcup_{j \in P} \beta \cup \{j\}) \cup (\bigcup_{j \in N} \beta \cup \{j\})) \subset G_y, \text{ for } \bar{y} = y - t\bar{x} \text{ and } t > 0 \text{ and}$$

sufficiently small. Suppose that $\alpha \in G_y$ for $y = \bar{y} - t\bar{x}$ and $t > 0$ and sufficiently small. If $\beta \not\subset \alpha$, then $\bar{y} \in S_\alpha$. If $\bar{y} \in \partial S_\alpha$, and $\beta \not\subset \alpha$, then $\bar{y} \in S_\gamma$ for some $\gamma \subset \alpha$, $\gamma \neq \beta$, and hence $\bar{y}$ is not in the relative interior of F. Thus $\bar{y} \in \text{int } S_\alpha$, whereby $\bar{y} + t\bar{x} \in \text{int } S_\alpha$ for all $t > 0$ and sufficiently small. and hence $\alpha \in G_{y'}$. Thus we have for all $t > 0$ and sufficiently small:

$$G_y = G_{y'} \setminus (\bigcup_{j \in P} \beta \cup \{j\}) \cup (\bigcup_{j \in N} \beta \cup \{j\}), \qquad (*)$$

for $y = \bar{y} - t\bar{x}$, and y is therefore a regular point of χ°. Let τ_k be the unique n-cell of $(\bar{M}, \chi^\circ)$ containing $y = \bar{y} + t\bar{x}$ for $t > 0$ and sufficiently small. There thus exists $t > 0$ such that $\hat{y} = \bar{y} - t\bar{x}$ is a regular point of χ°, and $\tau_k = \bigcap_{\alpha \in G_{\hat{y}}} S_\alpha$. Thus $\tau_k \cap H = \tau_i \cap H$, and so F is a facet of τ_k. and from (*), we have

$$G_{y'} \Delta G_y = \bigcup_{j \in P \cup N} (\beta \cup \{j\}) = \bigcup_{j \in M \setminus \beta} (\beta \cup \{j\}) = R_\beta .$$

Finally, note that β cannot be an element of $\partial \bar{K}$, for otherwise S_β would be a facet of χ^o (by lemma 1) and H would be a supporting hyperplane for χ^o, which cannot be true. [X]

The last intermediate result we will need to prove theorem 1 under nondegeneracy is:

<u>**Lemma 4**</u>. If χ satisfies the assumptions of theorem 1 and χ is nondegenerate, then for each $\beta \in \partial \bar{K}$ with $|\beta| = n$, $\bigcap_{j \in M \setminus \beta} S_{\beta \cup \{j\}} = \tau_k$ for some $k \in \{1, \ldots, p\}$.

<u>PROOF</u>: Let $\beta \in \partial \bar{K}$, $|\beta| = n$. Thus S_β is a facet (and is an (n-1)-simplex) of χ^o, by lemma 1. By remark 2 (b), there is an extreme point $\bar{x}$ of χ such that $\beta = C(\bar{x})$. Thus $A_\beta \bar{x} = e_\beta$ and $A_j \bar{x} < 1$ for all $j \in M \setminus \beta$. Let $\bar{y} \in \text{rel int } S_\beta$, and let $\tau = \bigcap_{j \in M \setminus \beta} S_{\beta \cup \{j\}}$. Then $(\bar{y} - t\bar{x}) \in S_{\beta \cup \{j\}}$ for all $t > 0$ and sufficiently small, whereby τ is an n-cell and $(\bar{y} - t\bar{x}) \in \text{int } \tau$ for all $t > 0$ and sufficiently small. Suppose $(\bar{y} - t\bar{x}) \in S_\alpha$ for all $t > 0$ and sufficiently small, and $|\alpha| = n+1$. If $\alpha \neq \beta \cup \{j\}$ for some $\{j\} \in M \setminus \beta$, there exists $i \in \beta$ such that $i \notin \alpha$, whereby, $\bar{y} \notin S_\alpha$. But $(\bar{y} - t\bar{x}) \in S_\alpha$ for all $t > 0$ and sufficiently small, a contradiction. Thus for $y = \bar{y} - t\bar{x}$, and $t > 0$ and sufficiently small, $G_y = \bigcup_{j \in M \setminus \beta} \beta \cup \{j\}$, and y is a regular point of χ^o, whereby $\tau = \tau_k$ for some $k \in \{1, \ldots, p\}$. [X]

We now have:

<u>Proof of theorem 1 when χ is nondegenerate:</u>

Let $\beta \in \partial \bar{K}$ with $|\beta|=n$. Then by lemma 4, $\bigcap_{j \in M \setminus \beta} S_{\beta \cup \{j\}} = \tau_k$ for some $k \in \{1,\ldots,p\}$. Therefore for every $y \in \text{int } \tau_k$, $G_y = \bigcap_{j \in M \setminus \beta} \beta \cup \{j\} = R_\beta$.

By lemma 2, $\#R_\beta$ is odd, and hence $\#G_y$ is odd. For any two adjacent n-n-cells τ_i, τ_j, $G_y \Delta G_z = R_\beta$ for some $\beta \notin \partial K$, $|\beta|=n$, for any $y \in \text{int } \tau_i$, $z \in \text{int } \tau_j$, by lemma 3. We have:

$G_y = (G_y \setminus G_z) \cup (G_y \cap G_z)$, whereby

$\#G_y = \#(G_y \setminus G_z) + \#(G_y \cap G_z)$, because these two sets are disjoint.

Similarly, we have $\#G_z = \#(G_z \setminus G_y) + \#(G_y \cap G_z)$.

$$
\begin{aligned}
\text{Therefore, } \#G_y - \#G_z &= \#(G_y \setminus G_z) - \#(G_z \setminus G_y) \\
&= \#G_y \setminus G_z) + \#(G_z \setminus G_y) - 2\ \#(G_z \setminus G_y) \\
&= \#(G_y \Delta G_z) - 2\ \#(G_z \setminus G_y) \\
&= \#R_\beta - 2\ \#(G_z \setminus G_y).
\end{aligned}
$$

However, $\#R_\beta$ is even, by lemma 2. Thus $\#G_y$ and $\#G_z$ have the same parity. Therefore, for any two adjacent n-cells τ_i and τ_j, $\#G_y$ and $\#G_z$ have the same parity for all $y \in \text{int } \tau_i$, $z \in \text{int } \tau_j$. Furthermore, for at least one τ_k, $\#G_y$ is odd, by choosing $\tau_k = \bigcap_{j \in M \setminus \beta} S_{\beta U\{j\}}$ where $\beta \in \partial \bar{K}$. Thus $\#G_y$ must be odd for all $y \in \text{int } \tau_k$ for all k, because $\bigcup_{k=1}^{p} \tau_k = \chi^o$ is a connected set. But $y \in \text{int } \tau_k$ for some k if and only if y is a regular point of χ^o. Thus $\#G_y$ is odd for all regular points of χ^o.

Therefore, if y is regular, there exists an odd number of simplices $\bar{\sigma} \in \bar{K}$ such that $L(\bar{\sigma}) \in G_y$. Each simplex $\bar{\sigma}$ is of the form $\sigma \cup C(\sigma)$ where $\sigma \in K$, and $L(\sigma) \cup C(\sigma) = L(\bar{\sigma})$ for all $\bar{\sigma} \in \bar{K}$. Thus there exists an odd number of simplices $\sigma \in K$ with the property $L(\sigma) \cup C(\sigma) \in G_y$. This proves assertion (i) of theorem 1. Assertion (ii) follows from an elementary closure argument. [X]

The proof of theorem 1 when χ is nondegenerate has depended critically on being able to create a pseudomanifold $\bar{K}$ whose boundary bears a combinatorial equivalence to the boundary of χ^o. This combinatorial equivalence is driven by the fact that every row A_i of A is an extreme point of χ^o, every facet of χ^o is an (n-1)-simplex, and that χ is nondegenerate. These observations suggest a more general combinatorial result, whose development will aid in proving theorem 1 for the more general (degenerate or nondegenerate) case.

Let Z be a polyhedron of the form $Z = \{z \in R^n | z=\lambda E,\ \lambda \geq 0,\ \lambda \cdot e=1\}$. Let $M = \{1,\ldots,m\}$ index the rows of E. For each $\alpha \subset M$, define $T_\alpha = \{z \in R^n | z=\lambda_\alpha E_\alpha,\ \lambda_\alpha \geq 0,\ \lambda_\alpha \cdot e_\alpha=1\}$. A point $z \in Z$ is said to be a regular point of Z if $z \notin T_\alpha$ for any $\alpha \subset M$ with $|\alpha| \leq n$. For every $z \in Z$, we define $G_z = \{\alpha \subset M | z \in T_\alpha\}$. Z is said to be <u>special</u> if

(i) every row of E is an extreme point of Z,

(ii) $z=0 \in$ int Z and $z=0$ does not lie in the affine hull ot T_α for any α with $|\alpha| \leq n$, and

(iii) no hyperplane H in R^n meets more than n extreme points of Z.

Let $\bar{K}$ be a finite pseudomanifold with vertex set $\bar{K}^o \supseteq M$. We say that $\bar{K}$ <u>agrees</u> with Z if

(i) $\beta \in \partial\bar{K}$ implies $\beta \subset M$ and T_β is a face of Z, and

(ii) if $\beta \subset M$ and T_β is a face of Z, then $\beta \in \partial\bar{K}$.

We have the following result:

Lemma 5. If Z is special, $\bar{K}$ agrees with Z, and $L(\cdot):\bar{K}^o \to M$ is a given labelling such that $L(i)=i$ for each $i \in M$, and z is a regular point in Z, then there are an odd number of simplices $\bar{\sigma} \in \bar{K}$ with the property that $L(\bar{\sigma}) \in G_z$.

PROOF: If Z is special, define $\chi = \{x \in R^n \mid Ex \le e\}$. Then χ is solid and centered and scaled, and because Z is special, χ is nonredundant bounded, nondegenerate, and centrally regular. In this case, we have $\chi^o = Z$, where $\chi^o = \{y \in R^n \mid y = \lambda E,\ \lambda \ge 0,\ e \cdot \lambda = 1\}$. Therefore Remark 2 pertains. Furthermore, lemma 2 is valid, because the proof of lemma 2 only depends on the fact that $\bar{K}$ agrees with χ^o, and not on how $\bar{K}$ was constructed from K and T. Finally, lemmas 3 and 4 hold true. Therefore, if z is a regular point of Z, i.e. z is a regular point of χ^o, there exists an odd number of simplices $\bar{\sigma} \in \bar{K}$ with the property that $L(\bar{\sigma}) \in G_z$. [X]

(The statement of lemma 5 can be regarded as a combinatorial version of the No Retraction Theorem (see Hirsch [13], e.g.). To see this, suppose Z is as given in the lemma, and let T be a triangulation of Z that does not refine any facet of Z. Then the set of vertices of T consist of $\bar{K}^o = \{E_i \mid i \in M\} \cup K^o$, where K^o are vertices of T in the interior of Z. Any simplex $\sigma \in T$ can be written in the form $\sigma = \langle v^o, \ldots, v^k, E_{i_1}, \ldots, E_{i_p} \rangle$, where each $v^i \in K^o$, and each E_i is a row of E. If we let $\bar{\sigma} = \{v^o, \ldots, v^k, i_i, \ldots, i_p\}$, then the collection $\bar{K}$ of all such $\bar{\sigma}$ is a pseudomanifold that agrees with Z. If $L(\cdot)$ is a labelling $\bar{K}^o \to M$, then for each $v \in K^o$, let $f(v) = E_{L(v)}$, and for each E_i, let $f(E_i) = E_i$. Then if $f(\cdot)$ is extended to a PL function $f(\cdot)$ maps Z into Z continuously and leaves the boundary fixed. According to lemma 5, for each regular point $z \in Z$, there exists an odd number of simplices $\bar{\sigma} \in \bar{K}$ such that $L(\bar{\sigma}) \in G_Z$. But this means that $z=f(x)$ for at least one $x \in \sigma$, where σ is the real simplex corresponding to $\bar{\sigma}$. Thus $f(\cdot)$ maps Z onto Z, proving the No Retraction Theorem.)

When χ is degenerate, then the preceding proof of theorem 1 is not valid. In particular, if χ is degenerate, the construction of $\bar{K}$ does not result in an n-pseudomanifold. (To see this, suppose χ is degenerate, and let $\bar{x}$ be an extreme point of χ for which $|C(x)| > n$. Then, since $\{\bar{x}\} \in K$, $\{\bar{x}\} \cup C(x) \in \bar{K}$, but this set contains at least n+2 elements, and so is not an n-simplex in $\bar{K}$.)

The typical method for side-stepping degeneracy is to perturb the constant coefficients of the constraints of χ by a vector of infinitesimals. In our case, however, such a perturbation of χ has adverse consequences. The perturbation will alter the combinatorial properties of $\partial\chi$, which is undesirable in a combinatorial analysis such as this. Also, if T is a triangulation of χ, it is unclear how to amend T so that the amended version is a triangulation of the perturbed χ. In any case, the combinatorial structure of T may change, which again is undesirable.

The usual perturbation of χ is performed by changing each right-hand side coefficient b_i to $b_i + \varepsilon^i$. We will instead perturb χ^o, by using this same construction in a dual form. Our first task, however, is to repair $\bar{K}$. We proceed as follows. A subset $\alpha \subset M$ is said to be <u>consistent</u> if there exists $x \in \chi$ with the property that $C(x)=\alpha$. For any matrix D or vector d, let (D) or (d) denote the number of leading zero columns or components of D or d, respectively. Let $B=[b,I]$. If $\alpha \subset M$ is consistent, a subset

$\beta \subseteq \alpha$ is said to be a basis for α if $AX+Y=B$ has a solution $\bar{X}$, $\bar{Y}$ with $\bar{Y}$ 0, $(\bar{Y}_\alpha) \geq 1$, $\bar{Y}_\beta = 0$, i.e., $(\bar{Y}_\beta) = m+1$, and $|\beta| = \text{rank } A_\alpha$.

Instead of constructing the pseudomanifold $\bar{K}$ by joining each simplex $\bar{\sigma}$ of K with its carrier set $C(\bar{\sigma})$, we now construct $\bar{K}$ by joining each $\bar{\sigma}$ of K with every subset β of its carrier set $C(\bar{\sigma})$ that forms a basis for this carrier set. We obtain the following theorem:

Theorem 4. Let χ be a solid, bounded, and nonredundant polyhedron. Let T be a triangulation of χ and let K be the pseudomanifold corresponding to T. Let $\bar{K}^o=K^o \cup M$, and define $\bar{K} = \{\bar{\sigma} \subseteq \bar{K}^o | \bar{\sigma} \neq \phi, \bar{\sigma} = \sigma \cup \beta$, where $\sigma \in K$ and β is a basis for $C(\sigma)\}$. Then $\bar{K}$ is an n-pseudomanifold, and $\partial\bar{K} = \{\beta \subset M | \beta \neq \phi$, and β is a basis for $\alpha = C(x)$ for some $x \in \chi\}$. [X].

The proof of this theorem is rather laborious, and so is relegated to the appendix. The boundary elements of $\bar{K}$ correspond in a natural way to subsets of faces of χ^o, in a manner that we will soon see. Theorem 4 thus gives a constructive procedure for triangulating the boundary of χ^o. The procedure of joining simplices σ of χ with bases $\beta \subset M$ is similar to the construction of an antiprism, see Broadie [1]. This construction of $\bar{K}$ is also closely related to the construction of a primal-dual pair of subdivided manifolds, as in Kojima and Yamamoto [14], although $\bar{K}$ is combinatorial while the primal-dual pair of manifolds is not.

Our next task is to perturb the n-cell χ^{o}. Define $A_i^{\varepsilon} = A_i/(1+\varepsilon^i)$ for $i \in M$, and define A^{ε} to be the matrix whose i^{th} row is A_i^{ε}. Define $\chi^{o\varepsilon} = \{y \in R^n | y = \bar{\lambda} \overline{A^{\varepsilon}}, \lambda \geq 0, e \cdot \lambda = 1\}$ and $S_{\alpha}^{\varepsilon} = \{y \in R^n | y = \lambda_{\alpha} A_{\alpha}^{\varepsilon}, \lambda_{\alpha} \geq 0, e_{\alpha} \cdot \lambda_{\alpha} = 1\}$.

<u>**Lemma 6**</u>. If $\alpha \subset M$ and $|\alpha| \geq n+1$, then $A_{\alpha}^{\varepsilon} x = \theta e_{\alpha}$ has no nontrivial solution for all sufficiently small positive ε.

<u>PROOF</u>: We will actually prove a stronger statement, that if $\alpha \subset M$ and $|\alpha| = n+1$, then $A_{\alpha}^{\varepsilon} x = \theta e_{\alpha}$ can only have a solution for at most n values of ε. The proof is by contradiction. Therefore let $\varepsilon_1, \ldots, \varepsilon_{n+1}$ be n distinct values of ε for which $A_{\alpha}^{\varepsilon} x = \theta e_{\alpha}$ has a nontrivial solution. If $\theta \neq 0$ in all of these solutions, then by rescaling, we can assume that $A_{\alpha}^{\varepsilon} x = e_{\alpha}$ has a solution for all $\varepsilon = \varepsilon_1, \ldots, \varepsilon_{n+1}$. Therefore $A_{\alpha} x = B[\varepsilon]$ has a solution for $\varepsilon = \varepsilon_1, \ldots, e_{n+1}$ where $B = [e, I]$ and $[\varepsilon] = (1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^m)$. Let Q be the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \varepsilon_1^1 & \varepsilon_2^1 & \cdots & \varepsilon_{n+1}^1 \\ \vdots & \vdots & & \vdots \\ \varepsilon_1^m & \varepsilon_2^m & \cdots & \varepsilon_{n+1}^m \end{bmatrix}$$

Then there exists a solution X to $A_{\alpha} X = B_{\alpha} Q$. But, since $m > n$, an induction argument establishes that the rank of Q is n+1, as is the rank of B_{α}. However, the rank of A_{α} is at most n, whence $A_{\alpha} X = B_{\alpha} Q$ cannot have a solution, because the rank of $B_{\alpha} Q$ is also n+1.

It only remains to show that if $A_{\alpha}^{\varepsilon} x = \theta e_{\alpha}$ has a nontrivial solution, it has a solution with $\theta \neq 0$. Suppose that $A_{\alpha}^{\varepsilon} x = \theta e_{\alpha}$ has a nontrivial solution $(x, \theta) = (\bar{x}, 0)$, and suppose that there is no solution to $A_\alpha^\varepsilon x=e_\alpha$. Then there exists λ_α with the property that $\lambda_\alpha A_\alpha^\varepsilon=0$, $\lambda_\alpha \cdot e_\alpha=1$, i.e. the zero vector is an element of the affine hull of S_α^ε. Denoting this affine hull by H^ε, we also have $A_\alpha^\varepsilon \bar{x}=0$, whereby H^ε has dimension at most n-1. Thus there exists a subset β of α such that $|\beta| \leq n$ and the affine hull of S_α^ε is spanned by affine combinations of the rows A_i^ε, $i\varepsilon\beta$. Therefore H^ε is the affine hull of S_β^ε. But $\bar{y}=0 \in H^\varepsilon$, and hence $\bar{y}=0$ lies in the affine hull of n or fewer rows of S_β^ε, and hence of S_β. This contradicts the assumption that χ is centrally regular, and the proof is now complete. [X]

We also have:

<u>Lemma 7</u>. For all sufficiently small positive ε,

(a) $\chi^{o\varepsilon}$ is special,

(b) $\chi^{o\varepsilon}$ agrees with $\bar{K}$, and

(c) any regular point y of χ^o is a regular point of $\chi^{o\varepsilon}$, and if y is a regular point of χ^o, $y\in S_\alpha$ if and only if $y\in S_\alpha^\varepsilon$.

<u>PROOF</u>: Let $\chi^\varepsilon=\{x\in R^n | A^\varepsilon x \leq b\} = \{x\in R^n | Ax \leq B[\varepsilon]\}$, where $B=[e,I]$, and $[\varepsilon]=(1,\varepsilon,\varepsilon^2,\ldots,\varepsilon^m)$. Because χ^ε is nondegenerate for all sufficiently small positive ε, the faces of $\chi^{o\varepsilon}$ are all simplices. Furthermore, since χ is nonredundant, the rows A_i of A are all extreme points of χ^o, whereby the rows A_i^ε of A^ε are all extreme points of $\chi^{o\varepsilon}$. Furthermore, because $\bar{y}=0 \in \text{int } \chi^o$, $\bar{y}=0 \in \text{int } \chi^{o\varepsilon}$ for all sufficiently small positive ε. Finally, because the conditions of lemma 6 are met, whenever $\alpha \subseteq M$ and $|\alpha| \geq n+1$, $A_\alpha^\varepsilon x=\theta e_\alpha$ has no nontrivial solution for all $\varepsilon > 0$ and sufficiently small. Thus for all sufficiently small positive ε, no hyperplane meets more than n extreme points of $\chi^{o\varepsilon}$. Thus $\chi^{o\varepsilon}$ is special, proving (a). Part (b) follows from the fact that $\beta\in\partial\bar{K}$ if and only if AX+W=B has a solution

with $W_\beta=0$ and and $W_{M\setminus\beta} \succ 0$; if and only if $Ax+w=B[\varepsilon]$ has a solution with $w_\beta=0$ and $w_{M\setminus\beta} > 0$ for all sufficiently small positive ε; if and only if $\{y\in R^n | y=\lambda_\beta A_\beta,\ \lambda_\beta \geq 0,\ \lambda_\beta \cdot B_\beta[\varepsilon]=1\}$ is a face of $\{y\in R^n | y=\lambda A,\ \lambda \geq 0,\ \lambda B[\varepsilon]=1\}$ for all sufficiently small positive ε; if and only if $\{y | y=\lambda_\beta A_\beta^\varepsilon,\ \lambda_\beta \geq 0,\ \lambda_\beta \cdot e=1\}$ is a face of $\{y\in R^n | y=\lambda A^\varepsilon,\ \lambda \geq 0,\ \lambda \cdot e=1\}$, i.e. if and only if S_β^ε is a face of $\chi^{o\varepsilon}$. For part (c), note that $\bar{y}$ is a regular point of χ^o if and only if y meets no S_β for $\beta \subseteq M$, $|\beta|=n$, and so $\bar{y}$ meets no S_β^ε, $\beta \subseteq M$, $|\beta|=n$, for all sufficiently small positive ε. [X]

Our last intermediary result is:

<u>Lemma 8</u>. Let χ, T, K, and $L(\cdot)$ satisfy the assumptions of theorem 1, and let y be a regular point of χ^o. Then there is a one to one correspondence between simplices $\bar{\sigma} \in \bar{K}$ that satisfy $L(\bar{\sigma}) \in G_y$, and simplices $\sigma \in K$ that satisfy $(L(\sigma) \cup C(\sigma)) \in G_y$.

<u>PROOF</u>: Let $\bar{\sigma} \in \bar{K}$ have the property that $L(\bar{\sigma}) \in G_y$. Then $\bar{\sigma}$ is of the form $\bar{\sigma}=\sigma \cup \beta$ where $\sigma\in K$ and $\beta \subseteq C(\sigma)$, and because $|\beta| \leq n$ and $|L(\bar{\sigma})|=n+1$, $|\sigma| \geq 1$, whereby $\sigma\neq\phi$. Now $L(\bar{\sigma}) \in G_y$ if and only if $(L(\sigma) \cup \beta) \in G_y$. Because $\beta \subseteq C(\sigma)$, this means $L(\sigma) \cup C(\sigma) \in G_y$. We thus must show that if $\bar{\sigma}_1=(\sigma_1 \cup \beta_1) \in G_y$ and $\sigma_2 = (\sigma_2 \cup \beta_2) \in G_y$ and $\bar{\sigma}_1 \neq \bar{\sigma}_2$, then we cannot have $\sigma_1=\sigma_2$. Suppose $\sigma=\sigma_1=\sigma_2$ and $\beta_1\neq\beta_2$. Then $\beta_1 \subset C(\sigma)$ and $\beta_2 \subset C(\sigma)$. Let $L=L(\sigma)$. Then there exists $\lambda_1, \lambda_2, u_1, u_2 \geq 0$. such that:

$$\lambda_1 A_L + u_1 A_{\beta_1} = y \qquad (1)$$

$$\lambda_2 A_L + u_2 A_{\beta_2} = y \qquad (2)$$

Because β_2 is a basis for $\alpha = C(\sigma)$ i=1,2, $|\beta_1|=|\beta_2|$ and there exists a sqaure matrix π such that $A_{\beta_2} = \pi A_{\beta_1}$, and π must be nonsingular, whereby $A_{\beta_1} = \pi^{-1}A_{\beta_2}$. If $t=\dim\sigma$, then $|L| \leq t+1$ and $|\beta_i|=n-t$, $i=1,2$. Because y is a regular point, $|L|=t+1$, and $|\overline{L\cup\beta_i}|=n+1$. Furthermore, $\lambda_i > 0$, $u_i > 0$, $i=1,2$, for otherwise y is not regular. Combining (1) and (2) above, we obtain

$$(\lambda_1-\lambda_2)\ A_L + (u_1-u_2\pi)\ A_{\beta_1}= 0 \qquad (3)$$

If $\lambda_1-\lambda_2 \neq 0$, or $u_1-u_2\pi \neq 0$, then we could use (3) to reduce the number of positive components in (1) or (2), violating the fact that y is a regular point. Thus $\lambda_1=\lambda_2$ and $u_1=u_2\pi$, and $u_1\pi^{-1}=u_2$.

Furthermore, because β_i is a basis for $\alpha = C(\sigma)$ and each $\beta_i \subseteq \alpha$, there must exist $\bar{X}_1$, $\bar{X}_2$, $\bar{Y}_1$, $\bar{Y}_2$ such that

$$A_{\beta_1}\bar{X}_1 = B_{\beta_1}$$

$$A_{\beta_2}\bar{X}_1 + \bar{Y}_1 = B_{\beta_2},\ \bar{Y}_1 \succeq 0,\ \bar{Y}_1 \neq 0.$$

$$A_{\beta_2}\bar{X}_2 = B_{\beta_2}$$

$$A_{\beta_1}\bar{X}_2 + \bar{Y}_2 = B_{\beta_1},\ \bar{Y}_2 \succeq 0,\ \bar{Y}_1 \neq 0.$$

Therefore $B_{\beta_1}= A_{\beta_1}\ \bar{X}_1 = \pi^{-1}A_{\beta_2}\ \bar{X}_1 = \pi^{-1}B_{\beta_2}- \pi^{-1}\bar{Y}_1 = \pi^{-1}A_{\beta_2}\ \bar{X}_2 - \pi^{-1}Y_1 = A_{\beta_1}\ \bar{X}_2 - \pi^{1}\bar{Y}_1 = B_{\beta_1}- \bar{Y}_2 - \pi^{-1}\bar{Y}_1$, and hence $\bar{Y}_2 = -\pi^{-1}\bar{Y}_1$. Because u_1, $u_2 > 0$, $0 \prec u_1\bar{Y}_2 = u_2\pi\bar{Y}_2 = -u_2\bar{Y}_1 \prec 0$, a contradiction. Thus $\beta_1=\beta_2$.

Next, suppose $\sigma \in K$ and $(L(\sigma) \cup C(\sigma)) \in G_y$. Let $\alpha = C(\sigma)$, and let $\delta = L(\sigma)$. We need to find a basis β for α with the property that $(\delta \cup \alpha) \in G_y$. Because $(\delta \cup \alpha) \in G_y$, there exists $\bar{\lambda}_\delta \geq 0$, $\bar{\lambda}_\alpha \geq 0$ with the property that $\bar{\lambda}_\delta A_\delta + \bar{\lambda}_\alpha A_\alpha = y$, and $e_\delta\bar{\lambda}_\delta + e_\alpha\bar{\lambda}_\alpha = 1$. Furthermore, there exists $\bar{x} \in \chi$ with the property that $C(\bar{x}) = C(\sigma) = \alpha$, i.e. $A_\alpha\bar{x} = b_\alpha$. Let $B = [b,I]$ and let $\bar{z}=\bar{\lambda}_\alpha A_\alpha$ and consider the following lexico-linear programs:

$$\left.\begin{array}{l}\text{lex max } \bar{z}\cdot X\\ \text{subject to } A_\alpha X+Y = B_\alpha\\ \qquad\qquad Y_\alpha \succeq 0\end{array}\right\} P \qquad \left.\begin{array}{l}\text{lex min } \lambda_\alpha \cdot B_\alpha\\ \text{subject to } \lambda_\alpha A_\alpha = \bar{z}\\ \qquad\qquad \lambda_\alpha \geq 0\end{array}\right\} D$$

Note that $X = (\bar{x};0)$, $Y = (0;I)$ is feasible for P and $\lambda_\alpha = \bar{\lambda}_\alpha$ is feasible for D, whereby there exists basic optimal solutions (X^*, Y^*), λ^*_α to P and D, respectively, with basis $\beta \subset \alpha$. Therefore, $\bar{\sigma} = (\sigma \cup \beta) \in \bar{K}$. Let v^* be the optimal value of P and D. Then because $X = (\bar{x};0)$, $Y = (0;I)$ is feasible for P and $\lambda_\alpha=\bar{\lambda}_\alpha$ is feasible for D, $v^*_1 \geq \bar{z}\cdot\bar{x} = \bar{\lambda}_\alpha A_\alpha \bar{x} = \bar{\lambda}_\alpha b_\alpha \geq v^*_1$, whereby $v^*_1 = \bar{\lambda}_\alpha e_\alpha$. Thus β is a basis for α, and $\bar{\lambda}_\delta A_\delta + \lambda^*_\beta A_\beta = \bar{\lambda}_\delta A_\delta + \bar{\lambda}_\alpha A_\alpha = y$, and $\bar{\lambda}_\delta e_\delta + \lambda^*_\beta e_\beta = \bar{\lambda}_\delta e_\delta + \bar{\lambda}_\alpha e_\alpha = 1$, and so $L(\bar{\sigma}) = (L(\sigma) \cup \beta) \in G_y$. Thus for every $\sigma \in K$ with $(L(\sigma) \cup C(\sigma)) \in G_y$, there exists $\beta \subset M$ with $\bar{\sigma} = (\sigma \cup \beta) \in \bar{K}$ and $L(\bar{\sigma}) \in G_y$. This completes the proof. [X]

<u>Proof of Theorem 1</u>. Let χ, T, K, and $L(\cdot)$ satisfy the assumptions of theorem 1. Then for all sufficiently positive ε, $\chi^{o\varepsilon}$ agrees with $\bar{K}$, by lemma 7. Therefore, by lemma 5, if z^ε is a regular point of $\chi^{o\varepsilon}$, $\#G_{z^\varepsilon}$ is odd. If y is a regular point of χ^o, then y is a regular point of $\chi^{o\varepsilon}$ for all sufficiently small positive ε, and hence $\#G_y$ is odd. Thus there are an odd number of simplices of $\bar{\sigma}\in\bar{K}$ with the property that $L(\bar{\sigma}) \in G_y$. By lemma 8, there are an odd number of simplices $\sigma\in K$ with the property that $(L(\sigma) \cup C(\sigma)) \in G_y$. [X]

5. A Combinatorial Theorem on a Bounded Polyhedron that Generalizes Sperner's lemma

Theorems ~~1 and~~ 2 have been shown to generalize combinatorial results on the simplex and simplotope that have unrestricted labels and dual-proper labels, respectively. In this section, we present a theorem that generalizes the results on the simplex and simplotope for proper labels, including Sperner's lemma [23].

Let χ, T, and $L(\cdot)$ satisfy the assumptions of theorem 1, and let $\chi^o = \{y \in R^n | y=\lambda A,\ \lambda \geq 0,\ \lambda\cdot e=1\}$. For any $y \in \text{int } \chi^o$, let $D_y = \{(\alpha,\beta) \in M\times M \mid \lambda_\beta A_\beta - \lambda_\alpha A_\alpha = y$ has a solution λ_β, λ_α such that $\lambda_\beta \geq 0$, $\lambda_\alpha \geq 0$, and $e_\alpha \cdot \lambda_\alpha + e_\beta \cdot \lambda_\beta = 1\}$. We have:

Theorem 5. Let $\chi = \{x \in R^n | Ax \leq b\}$ be bounded, solid, nonredundant, and centered and scaled. Let T be a triangulation of χ, let K be the pseudomanifold corresponding to T, and let $L(\cdot): K^o \to M$ be a labelling function. Then if $y \in \text{int } \chi^o$, there exists at least one simplex $\sigma \in K$ with the property that $(L(\sigma)),\ C(\sigma)) \in D_y$.

PROOF: Let χ, T, $L(\cdot)$, and K be given as in theorem 5. Let $\bar{y} \in \text{int } \chi^o$ be given, and define χ' and T' as in the projective transformation lemma, let K' be the pseudomanifold corresponding to T', and define $L(v') = L(g^{-1}(v'))$ for $v' \in K^{o\prime}$, where $g(\cdot)$ is as defined in the projective transformation lemma. For each $v' \in K^{o\prime}$, define $h'(v') = A_{L'(v')} + \bar{y}$, and extend $h(\cdot)$ in a PL manner over all of χ'. Define $f(x') = \arg\min_{z' \in X'} \|z'-x'+h(x')\|_2$, where $\|.\|_2$ denotes the Euclidean norm.

Because $h'(\cdot)$ is continuous, $f'(\cdot)$ is continuous and so contains a fixed point $\bar{x}'$. Let $\bar{\sigma}'$ be the smallest simplex σ' in T' that contains $\bar{x}'$, and let $\alpha = L(\bar{\sigma}')$, $\beta = C(\bar{\sigma}')$. Let $\bar{\sigma} = g^{-1}(\bar{\sigma}')$. Then $\alpha = L(\bar{\sigma})$ and $\beta = C(\bar{\sigma})$. The Karush-Kuhn-Tucker conditions state that

$\bar{x}'-\bar{x}'+h(\bar{x}') = \bar{\lambda}_\beta(A-eo\bar{y})_\beta$ for some $\bar{\lambda}_\beta \geq 0$. Furthermore, $h(\bar{x}') = -\bar{\lambda}_\alpha A_\alpha - \bar{y}$ for some particular $\bar{\lambda}_\alpha \geq 0$, $e_\alpha \cdot \bar{\lambda}_\alpha = 1$. Therefore, $\bar{\lambda}_\beta A_\beta - \bar{\lambda}_\alpha A_\alpha = (e \cdot \bar{\lambda}_\beta + e\bar{\lambda}_\alpha)\bar{y}$. After normalizing the vectors $\bar{\lambda}_\beta$ and $\bar{\lambda}_\alpha$ so that the sum of the component of both vectors is one, we see that $(\alpha,\beta) = (L(\bar{\sigma}), C(\bar{\sigma})) \in D_y$. [X]

The proof of theorem 1 using Brouwer's theorem, presented in Section 2, derives from the existence of an outward normal of the function h. The existence of an inward normal of $h(\cdot)$ is equivalent to the existence of a fixed point of $f(\cdot)$, see Eaves [3]. When $y=0$, the function $h'(\cdot)$ in the proof above is just $-h(\cdot)$ and the existence of an inward normal of $h(\cdot)$ is the same as the existence of an outward normal of $h'(\cdot)$.

To show that Sperner's lemma derives from theorem 5, let S^n, A^n be defined as in section 2, let T be a triangulation of S^n, K be the pseudomanifold corresponding to T, and $L(\cdot):K^o \to M$ be a labelling function, where $M = \{1,...,n+1\}$. $L(\cdot)$ is said to be <u>proper</u> if for each $v \in K^o$, $L(v)$ is the index of an element of $M \setminus C(v)$, i.e. $L(v)$ is the index of a nonbinding contraint of v, for $v \in K^o$. For $\chi = S^n$, the set $\chi^o = \{y \in R^n | y=\lambda A^n,\ \lambda \geq 0,\ e \cdot \lambda = 1\}$ is an n-simplex that contains the origin, and so $y=0 \in \text{int } \chi^o$. The conditions of theorem 5 are met, and so there exists a simplex $\sigma \in K$ with the property that $(L(\sigma), C(\sigma)) \in D_y$ for $y=0$. Let $\alpha = L(\sigma)$, $\beta = C(\sigma)$; then there exists λ_α, λ_β such that $\lambda_\beta A^n_\beta = \lambda_\alpha A^n_\alpha$, $\lambda_\beta \geq 0$, $\lambda_\alpha \geq 0$, $e_\beta \cdot \lambda_\beta + e_\alpha \cdot \lambda_\alpha = 1$. Because $L(\cdot)$ is proper $\alpha \cap \beta = \phi$. Note that for any $i, j \in M$, $i \neq j$, $A_i \cdot A_j \leq 0$. Thus $A^n_\beta (A^n_\alpha)^T \leq 0$ and so $0 \geq \lambda_\beta A^n_\beta (A^n_\alpha)^T \lambda_\alpha = (\lambda_\alpha A^n_\alpha)(\lambda_\alpha A^n_\alpha)^T \geq 0$ whereby $\lambda_\alpha A^n_\alpha = 0$, thus $\alpha = M = \{1,...,n+1\}$, and so $L(\sigma) = \{1,...,n+1\}$. This is precisely Sperner's lemma, without the oddness assertion.

The logic used above can also be used to prove theorem 3 of [7] (see also van der Laan and Talman [17]), which generalizes Sperner's lemma to the simplotope.

Theorem 5 does not contain an assertion of the oddness of the number of simplices under consideration. The basic constructs used to prove theorem 1 combinatorially do not appear to carry over directly to the case of theorem 5. It is an open question whether there exists a combinatorial proof of theorem 5 which asserts the existence of an odd number of simplices $\sigma \in K$ for which $(L(\sigma), C(\sigma)) \in D_y$, when y is regular.

Appendix A. A Pseudomanifold Extension Theorem

Let $\chi = \{x \in R \mid Ax \leq b\}$ be bounded, solid, and nonredundant, let T be a triangulation of χ, and let K be the pseudomanifold corresponding to T. Let $M = \{1,\ldots,m\}$ be the set of constraint row indices. We wish to construct an n-pseudomanifold $\bar{K}$, where each n-simplex $\bar{\sigma}$ of $\bar{K}$ consists of a simplex σ of K together with a subset β of $C(\sigma)$, the carrier indices of σ.

In order to construct $\bar{K}$ for arbitrary polyhedra, we need to work with the lexicographic system $AX + Y = B$, $Y \succ 0$, where $B = [b,I]$. For a given vector v, define (v) to be the number of leading zeroes of v. For a matrix V, let (V) denote the number of leading zero columns of V. An index set $\alpha \subset M$ is said to be consistent if there exists $x \in \chi$ with $C(x)=\alpha$. A subset $\beta \subset \alpha$ is said to be a basis for α if there exists $\bar{X}$, $\bar{Y}$ such that $A\bar{X} + \bar{Y} = B$, $\bar{Y} \succ 0$, $(\bar{Y}_\alpha) \geq 1$, and $(\bar{Y}_\beta) = m+1$, and $|\beta| = \text{rank}\,(A_\alpha)$. Let us construct $\bar{K}$ as follows:

let $\bar{K}^{\circ} = K^{\circ} \cup M$, where $M = \{1,\ldots,m\}$, and

let $\bar{K} = \{\delta \subset \sigma \cup \beta \mid \delta \neq \phi,\ \sigma \in K,\ \alpha = C(\sigma)$, and β is a basis for $\alpha\}$.

Our aim is to prove:

Theorem 4. Let χ be solid, bounded, and nonredundant. Let T be a triangulation of χ and let K be the pseudomanifold corresponding to T. Let $\bar{K}^{\circ} = K^{\circ} \cup M$, and define $\bar{K} = \{\bar{\sigma} \subset \bar{K}^{\circ} \mid \bar{\sigma} \neq \phi,\ \bar{\sigma} \subset \sigma \cup \beta$, where $\sigma \in K$ and β is a basis for $C(\sigma)\}$. Then $\bar{K}$ is an n-pseudomanifold, and $\partial\bar{K} = \{\beta \subset M \mid \beta \neq \phi$, and β is a basis for some α, where α is consistent$\}$.

When χ is nondegenerate, we have:

Corollary A1. Let $\chi = \{x \in R^n | Ax \leq b\}$ be a solid, bounded, nonredundant, and nondegenerate polyhedron. Let T be a triangulation of χ, and let K be the pseudomanifold corresponding to T. Let $\vec{\bar{K}}^{\circ} = K^{\circ} \cup M$, and define $\bar{K} = \{\bar{\sigma} \subset \vec{\bar{K}}^{\circ} \mid \bar{\sigma} \neq \phi, \bar{\sigma} \subset \sigma \cup \beta$, where $\sigma \in K$ and $\beta = C(\sigma)\}$. Then $\bar{K}$ is an n-pseudomanifold, and $\partial\bar{K} = \{\beta \subset M | \beta \neq \phi, \beta = C(x)$ for some $x \in \chi\}$.

Proof of Corollary A1 from Theorem 4. If χ is nondegenerate, then if α is consistent, the rows of A_α are linearly independent, the only basis β for α is $\beta=\alpha$, and the result then follows. [X]

In order to prove theorem 4, we proceed as follows. Throughout, it is assumed that χ is solid, bounded, and nonredundant.

Proposition A1. If α is consistent, then there exists X^*, Y^* such that $AX^* + Y^* = \bar{B}$, $\overline{Y^*} \succ 0$, and $(Y^*_\alpha) \geq 1$, i.e., the first column of Y_α is zero.

PROOF: Consider the lexico-linear program:

$$\left.\begin{aligned} \text{lex min } Z &= e_\alpha Y \\ \text{s.t. } AX + Y &= B \\ Y &\succeq 0 \end{aligned}\right\} P ,$$

where e_α is the vector with $(e_\alpha)_j$ equal to one if $j \in \alpha$ and equal to zero otherwise. Because α is consistent, there exists $\bar{x}$, $\bar{y}$ such that $A\bar{x} + \bar{y} = b$, $\bar{y} \geq 0$, and $\bar{y}_\alpha = 0$. Then define $\bar{X} = [\bar{x},0]$, $\bar{Y} = [\bar{y},I]$, and note that $\bar{X}$, $\bar{Y}$ is feasible for P. Furthermore, since $e_\alpha Y \succeq 0$ for any feasible X, Y, the above has an optimal solution, X^*, Y^*. If $(e_\alpha Y^*)_1 > 0$, then there exists $\pi^* \leq e_\alpha$ such that $\pi^* A = 0$, $(\pi^* B)_1 > 0$, by duality. Thus $0 < (\pi^* B)_1 = \pi^* b = \pi^* A\bar{x} + \pi^* \bar{y} \leq e_\alpha \bar{y} = 0$, a contradiction. Thus $(e_\alpha Y^*)_1 = 0$, whereby $(Y^*_\alpha) \geq 1$. ☒

Lemma A1. Let $\bar{X}$, $\bar{Y}$ satisfy $A\bar{X} + \bar{Y} = B$, $\bar{Y} \succ 0$, and let $\alpha = \{i \mid \bar{Y}_{i1} = 0\}$, $\beta = \{i \mid \bar{Y}_i = 0\}$. Then there exists $\beta' \supset \beta$ such that β' is a basis for α.

PROOF: First note that $|\beta| = \text{rank}(A_\beta)$. To see this, observe that $A_\beta \bar{X} = B_\beta$, whereby since $\text{rank}(B_\beta) = |\beta|$, $\text{rank}(A_\beta) \geq |\beta|$, but since $\text{rank}(A_\beta) \leq |\beta|$, we must have $\text{rank}(A_\beta) = |\beta|$. Therefore $|\beta| = \text{rank}(A_\beta) \leq \text{rank}(A_\alpha)$. And if $\beta' \supset \beta$, then $|\beta'| = \text{rank}(A_{\beta'}) \leq \text{rank}(A_\alpha)$.

Let $c = \text{rank}(A_\alpha) - |\beta|$. If $c = 0$, then let $\beta' = \beta$, and the lemma is proved. Suppose the lemma is true for $\text{rank}(A_\alpha) - |\beta| = 0,\ldots, c-1$, and consider the case $\text{rank}(A_\alpha) - |\beta| = c$. Let $\delta = \alpha\backslash\beta$, and $\tau = M\backslash\alpha$, i.e., $\tau = \{i \mid \bar{Y}_{i1} > 0\}$. Because $c > 0$, there exists $\hat{j} \in \delta$ with $A_{\hat{j}}$ independent of the rows of A_β . There thus exists $d \in \mathbb{R}^n$ such that $A_\beta d = 0$, and $A_{\hat{j}} d = 1$. Now, let

$$v = \operatorname*{lex\,min}_{A_i d > 0} \left\{ \frac{\overline{Y}_i}{A_i d} \right\} = \frac{\overline{Y}_{\hat{i}}}{A_{\hat{i}} d} \quad \text{for some } \hat{i} \in M .$$

We note that $(v) = (\overline{Y}_{\hat{i}}) \geq 1$, and $\hat{i} \notin \beta$, whereby $v \neq 0$. Let $X' = \overline{X} + d \circ v$, $Y' = B - AX'$, whereby $Y_i' = \overline{Y}_i - \frac{A_i d}{A_{\hat{i}} d} \overline{Y}_{\hat{i}}$. It then follows that $Y_\beta' = \overline{Y}_\beta = 0$, $Y_{\hat{i}}' = 0$, and $(Y_\alpha') \geq 1$, $(Y_\tau') = 0$, and $Y' \succeq 0$. Upon setting $\overline{\beta} = \beta \cup \{\hat{i}\}$, we have that $\operatorname{rank}(A_\alpha) - |\overline{\beta}| = c - 1$. We thus have reduced the problem to one where $\operatorname{rank}(A_\alpha) - |\overline{\beta}| < c$, which by induction, means that there exists β' with $\beta' \supset \beta$, and β' is a basis for α . ⊠

<u>Lemma A2</u>. Let α be consistent and β be a basis for α . Let $k \in \beta$. Let $\beta \setminus k \cup j$ be a basis for α , with $j \neq k$. Then the choice of j is unique.

<u>PROOF:</u> Because β is a basis for α , there exists $\overline{X}, \overline{Y}$ such that $A\overline{X} + \overline{Y} = B$, $\overline{Y} \succeq 0$, $\alpha = \{i \mid \overline{Y}_{i1} = 0\}$, $\beta = \{i \mid \overline{Y}_i = 0\}$, and $|\beta| = \operatorname{rank}(A_\beta) = \operatorname{rank}(A_\alpha)$. Now suppose there are two such $j \neq k$ such that $\beta \setminus k \cup j$ is a basis for α . Then, by reordering if necessary assume that $j = 1$ and $j = 2$. Then there exist X^1, Y^1, X^2, Y^2, such that:

$$AX^\ell + Y^\ell = B , Y^\ell \succeq 0 , \alpha = \{i \mid Y^\ell_{i1} = 0\}, \beta \setminus k \cup \ell = \{i \mid Y^\ell_i = 0\}, \ell = 1,2 .$$

Define $D^\ell = A(\overline{X} - X^\ell) = Y^\ell - \overline{Y}$, $\ell = 1,2$. Then we have:

$$D^\ell_i = 0 \quad , \quad i \in \beta \setminus k \quad , \quad \ell = 1,2 ,$$

$$D^\ell_\ell = -\overline{Y}_\ell \prec 0 \qquad , \quad \ell = 1,2 ,$$

and

$$D^\ell_k = Y^\ell_k \succ 0 \qquad \ell = 1,2 .$$

Also, because $\operatorname{rank}(A_\beta) = \operatorname{rank}(A_\alpha)$, and $|\beta| = \operatorname{rank}(A_\beta)$, there exists a unique matrix Π such that $\Pi A_\beta = A_\alpha$. For any $i \in \alpha$, we have $D_i^{\ell} = A_i(\bar{X}-X^{\ell}) = \Pi_i A_\beta(\bar{X}-X^{\ell}) = \Pi_{ik} D_k^{\ell}$. If we then define $\pi_i = \Pi_{ik}$ for all $i \in \alpha$, then we have $D_i^{\ell} = \pi_i D_k^{\ell}$ for all $i \in \alpha$.

Because $D_\ell^{\ell} \prec 0$ and $D_k^{\ell} \succ 0$, we must have $\pi_\ell < 0$, $\ell = 1,2$. Also, because $D_k^{\ell} = \frac{1}{\pi_\ell} D_\ell^{\ell}$, we can write $D_i^{\ell} = \pi_i D_k^{\ell} = \frac{\pi_i}{\pi_\ell} D_\ell^{\ell}$, $\ell = 1,2$.

Now, $Y_2^1 = \bar{Y}_2 + D_2^1 = \bar{Y}_2 + \frac{\pi_2}{\pi_1} D_1^1 = \bar{Y}_2 - \frac{\pi_2}{\pi_1}\bar{Y}_1 \succeq 0$. Thus, $\frac{\bar{Y}_1}{\pi_1} \succeq \frac{\bar{Y}_2}{\pi_2}$.

Using a parallel argument with Y^2, we obtain $\frac{\bar{Y}_2}{\pi_2} \succeq \frac{\bar{Y}_1}{\pi_1}$, whereby $\frac{\bar{Y}_2}{\pi_2} = \frac{\bar{Y}_1}{\pi_1}$, whereby $Y_2^1 = Y_1^2 = Y_1^1 = Y_2^2 = 0$.

Thus $A_{(\beta \setminus k \cup \{1\} \cup \{2\})} X^1 = B_{\beta \setminus k \cup \{1\} \cup \{2\}}$. But since $1,2 \in \alpha$, the matrix $A_{\beta \setminus k \cup \{1\} \cup \{2\}}$ has rank equal to $|\beta| = \operatorname{rank}(A_\alpha)$, but $\operatorname{rank}(B_{\beta \setminus k \cup \{1\} \cup \{2\}}) = |\beta| + 1$, a contradiction. Thus j is uniquely determined. $\boxtimes$

<u>Lemma A3</u> If α is consistent and β is a basis for α, and $i \in \beta$, then exactly one of the following statements are true:

i) there exists $j \in \alpha$, $j \neq i$ such that $\beta \setminus i \cup j$ is a basis for α, or

ii) $(\beta \setminus i)$ is a basis for some $\alpha' \subset \alpha$, where α' is consistent.

PROOF: Let d be chosen so that $A_\beta d = -e^i$, where e^i is the $i^{\underline{th}}$ unit vector. There exists $\overline{X}, \overline{Y}$ such that $A\overline{X} + \overline{Y} = B$, $\overline{Y} \succcurlyeq 0$, $\alpha = \{k | \overline{Y}_{k1} = 0\}$, $\beta = \{k | \overline{Y}_k = 0\}$. Upon examining $A_\alpha d$, we can either have $A_\alpha d \not\leq 0$ or $A_\alpha d \leq 0$. We have two cases:

Case 1 $A_\alpha d \not\leq 0$. Let $v = \underset{A_k d > 0}{\text{lex min}} \left\{ \frac{\overline{Y}_k}{A_k d} \right\}$, and we have $v = \frac{\overline{Y}_j}{A_j d}$ for some j. Furthermore, since $A_\alpha d \not\leq 0$, $j \in \alpha$. We must have $(v) = (\overline{Y}_j) \geq 1$. Let $X' = \overline{X} + d \circ v$, $Y' = B - AX'$. Then $Y' \succcurlyeq 0$, and $Y_k' = 0$ for all $k \in \beta \setminus i \cup j$. Also, $(Y_\alpha') \geq 1$, and $\{k | Y_k' = 0\} = \beta \setminus i \cup j$. Thus the conditions of (i) are satisfied.

Case 2 $A_\alpha d \leq 0$. Let $v = \underset{A_k d > 0}{\text{lex min}} \left\{ \frac{\overline{Y}_k}{A_k d} \right\}$. (This set is not empty, since otherwise χ is unbounded.) Then $v = \frac{\overline{Y}_j}{A_j d}$ for some $j \notin \alpha$, and hence $(v) = (\overline{Y}_j) = 0$. Let $X' = \overline{X} + \frac{1}{2} d \circ v$, $Y' = B - AX'$. Then for $k \in M$, $Y_k' = \overline{Y}_k - \frac{1}{2} \frac{A_k d}{A_j d} \overline{Y}_j$, and $Y' \succcurlyeq 0$. For $k \in \beta \setminus i$, $Y_k' = 0$. Also, $Y_i' = \frac{\overline{Y}_j}{2A_j d} \succ 0$, and $(Y_i') = 0$, i.e., $Y_{i1}' > 0$. Likewise, for $k \notin \alpha$, $(Y_k') = 0$, i.e., $Y_{k1}' > 0$. Define $\beta' = \beta \setminus i$, $\alpha' = \{i | Y_{i1}' = 0\}$. Then $\alpha' \subset \alpha$. Since β is a basis for α, there is a unique Π such that $\Pi A_\beta = A_\alpha$. Now for each $k \in \alpha'$, $Y_k' = \overline{Y}_k - \frac{A_k d}{A_j d} \overline{Y}_j$, whereby $(Y_k') \geq 1$ implies $A_k d = 0$. Likewise $k \in \alpha'$ if and only if $A_k d = 0$. Now $A_k d = \Pi A_\beta d = -\Pi_{ki} = 0$ if and only if $k \in \alpha'$. Thus each A_k is a linear combination of the rows of $A_{\beta \setminus i}$. Thus β' is a basis for α'.

It only remains to show that (i) and (ii) cannot take place simultaneously. If (i) holds, then there exists $\overline{X}, \overline{Y}$, such that $A\overline{X} + \overline{Y} = B$, $\overline{Y} \succeq 0$, $\beta = \{k | \overline{Y}_k = 0\}$, $\alpha = \{k | \overline{Y}_{k1} = 0\}$, and X', Y' with $AX' + Y' = B$, $Y' \succeq 0$, $\beta \setminus i \cup j = \{k | Y'_k = 0\}$, $\alpha = \{k | Y'_{k1} = 0\}$. Defining $D = A(\overline{X} - X')$, we have $D = Y' - Y$. Let Π be the unique solution to $\Pi A_\beta = A_\alpha$. Then $D_k = A_k(\overline{X} - X') = \Pi_k A_\beta(\overline{X} - X') = \Pi_{ki} D_i$, for $k \in \alpha$, because $D_{\beta \setminus i} = Y'_{\beta \setminus i} - \overline{Y}_{\beta \setminus i} = 0 - 0 = 0$. Now, $D_i = Y'_i \succ 0$, and $D_j = -\overline{Y}_j \prec 0$, whereby since $D_j = \Pi_{ji} D_i$, we must have $\Pi_{ji} < 0$.

If (ii) also holds, then there exists X', Y' with $AX' + Y' = B$, $Y' \succeq 0$, $\beta \setminus i = \{k | Y'_k = 0\}$, $\alpha' = \{k | Y'_{k1} = 0\}$, $\alpha' \subset \alpha$. Again letting $D = A(\overline{X} - X') = Y' - \overline{Y}$, we have $D_k = \Pi_{ki} D_i$ and $D_i \succ 0$, for $k \in \alpha$. Now, since $i \notin \alpha'$ (otherwise A_β would not have independent rows) we must have $D_{i1} > 0$, whereby since $D_k = \Pi_{ki} D_i$ for all $k \in \alpha$, and $Y'_k = \overline{Y}_k + D_k = \overline{Y}_k + \Pi_{ki} D_i$, we have $\Pi_{ki} = 0$ for all $k \in \alpha'$, $\Pi_{ki} > 0$ for all $k \in \alpha \setminus \alpha'$. Thus $\{k \in \alpha | \Pi_{ki} < 0\} = \phi$, contradicting the fact that $\Pi_{ji} < 0$ from above. Thus exactly one of (i) and (ii) holds. ⊠

<u>Lemma</u> A4 If $\alpha' \supset \alpha$ and α' and α are consistent, and $\text{rank}(A_{\alpha'}) = \text{rank}(A_\alpha) + 1$, and β is a basis for α, then there exists $\beta' \supset \beta$ that is a basis for α'. Furthermore, β' is uniquely determined.

<u>PROOF:</u> Let $\overline{X}, \overline{Y}$ satisfy $A\overline{X} + \overline{Y} = B$, $\overline{Y} \succeq 0$, $\alpha = \{i | \overline{Y}_{i1} = 0\}$, $\beta = \{i | \overline{Y}_i = 0\}$. Then some element of $A_{\alpha' \setminus \alpha}$ is independent of the rows of A_α, whereby some element of $A_{\alpha' \setminus \alpha}$ is independent of the rows of A_β. Let $j \in \alpha' \setminus \alpha$ be given, and let d be any vector such that $A_\beta d = 0$, $A_j d = 1$. Then let

$$v = \operatorname*{lex\,min}_{A_i d > 0} \left\{ \frac{\overline{Y}_i}{A_i d} \right\} = \frac{\overline{Y}_k}{A_k d} \quad \text{for some} \quad k \in \alpha' \setminus \alpha .$$ Then upon setting

$X' = X + \underline{d} \circ \underline{v}$, $Y' = B - AX'$, we have $Y_j' = \overline{Y}_j - \frac{A_j d}{A_k d}\overline{Y}_k$, $Y_j' \succeq 0$ for all j, $Y'_{\beta \cup k} = 0$. Now $\beta \cup k \subset \alpha'$, and $\text{rank}(A_{\beta \cup k}) = \text{rank}(A_{\alpha'})$, whereby $\beta' = \beta \cup k$ is a basis for α'.

Now suppose that $\beta^1 \neq \beta^2$ are both bases for α' that contain β. For ease of notation, suppose $\beta^1 = \beta \cup \{1\}$, $\beta^2 = \beta \cup \{2\}$, where $\{1,2\} \subset \alpha' \setminus \alpha$. Then consider β^1. $\beta^1 \setminus \{1\} \cup \{2\}$ is a basis for α', whereby by lemma 3, there does not exist $i \in \beta$ such that $\beta^1 \setminus \{i\}$ is a basis for some $\alpha \subset \alpha'$. But $\beta^1 \setminus \{1\}$ is a basis for $\alpha \subset \alpha'$, a contradiction. Thus β^1 is uniquely determined. ☒

With lemmas A1, A2, A3, and A4 as preparation, we can now prove Theorem 4.

<u>PROOF of Theorem 4:</u> Let $\delta \subset \overline{\sigma} \cup \beta$ be a simplex of $\overline{K}$, where $\overline{\sigma} \in K$, $\alpha = C(\overline{\sigma})$, and β is a basis for α. Then if $k = \text{rank}(A_\alpha)$, there exists an (n-k)-simplex $\overline{\tau} \in K$ with $\overline{\tau} \supset \overline{\sigma}$ such that $C(\overline{\tau}) = C(\overline{\sigma}) = \alpha$. Thus, $\delta \subset \overline{\sigma} \cup \beta \subset \overline{\tau} \cup \beta$, and $\overline{\tau} \cup \beta \in \overline{K}$, and $|\overline{\tau} \cup \beta| = n - k + 1 + k = n + 1$. Thus every simplex of $\overline{K}$ is contained in an n-simplex of $\overline{K}$.

Now let $v \in K^\circ$. Then $\{v\} \in K$, and let $\alpha = C(v)$. By proposition A1, there exists $\overline{X}$, $\overline{Y}$ such that $A\overline{X} + \overline{Y} = B$, $\overline{Y} \succeq 0$, and $(Y_\alpha) \geq 1$. By lemma A1, with $\beta = \{i | \overline{Y}_i = 0\}$, there exists $\beta' \supset \beta$ such that β' is a basis for α. Then $|\beta'| = \text{rank}(A_{\beta'}) = \text{rank}(A_\alpha)$. Now there exists $\sigma \in K$ such that $v \in \sigma$, $C(\sigma) = \alpha$, and σ is an (n-k)-simplex, where $k = \text{rank}(A_{\beta'}) = |\beta'|$. Thus, $\sigma \cup \beta'$ is an n-simplex of $\overline{K}$, and $\{v\} \subset \sigma \subset \overline{\sigma} \cup \beta'$. Thus every vertex of K is a zero simplex of $\overline{K}$.

Likewise let $i \in M$. Then, since χ has no redundant equations, there exists $x \in \chi$ with $Ax + y = b$, $y \geq 0$, $y_i = 0$, $y_j > 0$ for $j \neq i$. There thus exists X, Y with $AX + Y = B$, $Y \succeq 0$, $\{i\} = \{j | Y_{j1} = 0\}$, by proposition A1, and by lemma A1, we may assume $Y_i = 0$, $Y_j \neq 0$, for $j \neq i$. Let σ be the smallest simplex of T that contains x. Then $C(\sigma) = \{i\}$, and σ is an (n-1)-simplex; thus $|\sigma| = n$. Therefore $\sigma \cup \{i\} \in \bar{K}$, whereby $\{i\}$ is a zero-simplex of $\bar{K}$. Thus $\bar{K}^{\circ}$ is precisely the set of zero-simplices of $\bar{K}$.

Now let $\bar{\sigma} \cup \beta$ be an n-simplex of $\bar{K}$. Thus $C(\bar{\sigma}) = \alpha$ and β is a basis for α, and $\mathrm{rank}(A_\beta) = \mathrm{rank}(A_\alpha) = |\beta|$. Now let $s \in \bar{\sigma} \cup \beta$ and consider an (n-1)-simplex $\bar{\sigma} \cup \beta \setminus s \cup t$, where $t \neq s$. It is our aim to show that the choice of t is unique. Regarding s, we have either $s \in \bar{\sigma}$, i.e., $s = v$ for some $v \in K^{\circ}$, or $s \in \beta$, i.e., $s = i \in M$ for some i.

Case 1 $s = v \in \bar{\sigma}$. Then $\bar{\sigma} \in K_\alpha$, a k-pseudomanifold where $K_\alpha = \{\bar{\sigma} \in K \mid C(\bar{\sigma}) \supset \alpha\}$, and where $n-k = \mathrm{rank}(A_\alpha)$, and $\bar{\sigma}$ is a k-simplex. If $\bar{\sigma} \setminus v \notin \partial K_\alpha$, there exists a unique $v' \in K_\alpha^{\circ}$, $v' \neq v$, such that $\bar{\sigma} \setminus v \cup v'$ is a k-simplex in K_α. Thus $t = v'$ and t is uniquely determined. If $\bar{\sigma} \setminus v \in \partial K_\alpha$, then there exists $\alpha' \supset \alpha$ such that $\bar{\sigma} \setminus v \in K_{\alpha'}$, and $K_{\alpha'}$ is a (k-1)-pseudomanifold, and $\mathrm{rank}(A_{\alpha'}) = n-k+1$. By lemma A4, there exists a unique $j \in M$ such that $\beta' = \beta \cup \{j\}$ is a basis for α'. Thus $t = j$ and is uniquely determined.

Case 2 $s = i \in \beta$. From lemma A3, either there exists $j \in \alpha$, $j \neq i$, such that $\beta \setminus i \cup j$ is a basis for α and j is uniquely determined according to lemma A2, or $\beta \setminus i$ is a basis for some $\alpha' \subset \alpha$, where α' is consistent. In the former case, $t = j$. In the latter case, let τ be the unique simplex in $K_{\alpha'}$ that contains $\bar{\sigma} \in K_\alpha$. Then $\bar{\tau} = \bar{\sigma} \cup \{v\}$ for a unique $v \in K^{\circ}$. Thus $t = v$ and t is uniquely determined.

Because t can never attain more than one value, $\bar{K}$ is an n-pseudomanifold. Furthermore, the only instance where t cannot exist occurs when $\bar{\sigma}$ is an extreme point of χ, and $|\beta| = n$, and $s = v$ where $\sigma = \{v\}$. Thus $\partial \bar{K} = \{\delta \subset M \mid \delta \neq \phi,\ \delta \subset \beta,\ |\beta| = n$, and β is a basis for some α, where α is consistent$\}$. ☒

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