

Investments in Flexible Production Capacity

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ABSTRACT

We examine the technology and capacity choice problems of a multi-output firm facing stochastic demands. The firm can produce by installing output-specific capital, or, at greater cost, flexible capital that can be used to produce different outputs. Investment is irreversible, i.e., the firm cannot disinvest. The firm must decide how much of each type of capital to install, knowing it can add more capital in the future as demand evolves. We show how an investment rule can be derived that maximizes the firm's market value, and accounts for irreversibility. We also address the analogous problem for a multi-input firm that faces stochastically evolving factor costs, and can install input-specific or flexible capital.

1. Introduction.

Consider a firm that produces two products, with (possibly interdependent) demands that vary stochastically over time. It can produce these products in one of two ways: by installing and utilizing certain amounts of output-specific capital, or by installing - at greater cost - a flexible type of capital that can be used to produce either product. Investments in all three types of capital are irreversible, i.e., the firm cannot disinvest, so the expenditures are sunk costs. The firm must decide how much of each type of capital - flexible or output-specific - to install, in order to maximize its market value.

Or, consider a firm that produces one product, using either of two alternative factor inputs whose costs vary stochastically over time. The firm can produce this product in two ways: by irreversibly investing in input-specific capital, or by installing a more costly flexible type of capital that allows the use of either input. Again, the firm must decide how much of each type of capital to install to maximize its market value.

These problems arise in part because of new manufacturing technologies. Automobile companies, for example, produce both four- and six-cylinder engines. The demands for the engines are interdependent, and vary stochastically over time in response to unpredictable changes in gasoline prices, GNP, interest rates, and tastes. In the past, a firm such as GM could invest in capacity specific to four-cylinder engines, and/or capacity specific to six-cylinder engines. New technologies allow the same production line to turn out either type of engine. Given the uncertainty over future demands, the more flexible capacity has an obvious advantage. But it is also more costly. The firm must decide whether the additional cost is justified, and how much of each type of capacity to install.

Or, consider an electric utility planning new generating capacity. The utility can build a coal-or oil-burning plant. If future coal and oil prices were known the choice would be straightforward. But future coal and oil prices are not known, and it is very costly to convert a coal-burning plant into an oil-burning one, or vice-versa. A third alternative is to build - at greater cost - a plant designed at the outset to burn either coal or oil. Which type of plant should be built, and how much capacity should be installed?

This paper develops a framework to address these problems. It yields an investment rule that maximizes the firm's market value, and accounts for the irreversibility of investment, and the opportunity cost that this implies.¹ As in Bertola (1987) and Pindyck (1988), we focus on incremental investment decisions; the firm must decide how much of each type of capacity to install, knowing that it can add more later should demands increase, but also knowing that investment expenditures are sunk costs.²

The value of flexibility in plant design was first examined by Fuss and McFadden (1978). More recently, Fine and Freund (1986) studied investments in output-flexible capacity, using a quadratic programming model in which investment occurs in the first period, before demands are known, and production in a second period. Their two-period framework provides insight into the value of flexibility and choice of technology. However, it does not account for the irreversibility of investment, it requires product demands to be independent, and the investment rule it yields does not necessarily maximize the firm's market value.

To understand the implications of irreversibility, consider a firm that must decide how much capacity to install to produce a single output, the demand for which fluctuates stochastically. The firm's capacity choice is

optimal when the present value of the expected cash flow from a marginal unit of capacity equals the total cost of the unit. This total cost includes the purchase and installation cost, plus the opportunity cost of exercising the option to buy the unit. An analysis of capacity choice therefore involves two steps.

First, the value of an extra unit of capacity must be determined, and in a way that accounts for the fact that if demand falls, the firm can choose not to utilize the unit. Second, the value of the option to invest in this unit must be determined (it will depend in part on the value of the unit itself), together with the decision rule for exercising the option. This decision rule is the solution to the optimal capacity problem. It maximizes the net value of the firm, which has two components: the value of installed capacity net of its cost, and the value of the firm's options to install more capacity in the future.

Now suppose the firm produces two products with interdependent demands that fluctuate stochastically. If it uses product-specific capital, it must decide how much of each type to purchase. Capacity choice requires the valuation of a marginal unit of each type of capital (which may now depend on how much of the other type is installed), the valuation of the options to invest in marginal units of each type, and the rule for exercising the options. That rule maximizes the firm's net value: the total value of installed capacity of both types net of costs, plus the value of the firm's options to add capacity in the future. Or, the firm could install flexible capacity, again choosing an amount to maximize its net value (the value of the capacity less its cost, plus the value of the firm's options to add capacity later). The optimal choice of technology then boils down to an ex ante comparison of the firm's net value under each alternative.

The next section discusses the solution to this investment problem for a firm that produces two outputs. Section 3 shows how the firm's value-maximizing choice of technology and capacity can be found for a class of models with linear demand functions and Leontief production technologies. Numerical solutions for a specific example are presented in Section 4. Section 5 discusses the analogous problem of investing in input-flexible capacity. Section 6 concludes and mentions some of the limitations of our approach.

2. Optimal Irreversible Investment and the Choice of Technology.

The value of a firm is the value of its installed capacity plus the value of its options to add capacity in the future. But installed capacity also represents a set of options. Each unit of capacity gives the firm options to produce at every point over the lifetime of the unit, and can be valued accordingly.³ Hence the firm's capacity choice problem can be reduced to one of option valuation. This is spelled out in more detail below, first for a firm that invests in one type of capital to produce a single output, and then for a firm that produces two outputs and must choose between output-specific and output-flexible capital.

A. The Single-Output Firm.

Consider a firm facing a demand curve that shifts stochastically, so that future demands are uncertain. Let θ denote the demand shift parameter, with $\partial Q(P, \theta) / \partial \theta > 0$. Suppose the firm can install units of capital one at a time, at a sunk cost k per unit, whenever it wishes. If K is the amount of capital in place, the value of the firm, W , is given by:

$$W = V(K; \theta) + F(K; \theta) \tag{1}$$

$V(K; \theta)$ is the value of the firm's capital in place, and $F(K; \theta)$ is the value

of its "growth options," i.e., the present value of any additional profits that might result should the firm add more capital in the future, less the present value of the cost of that capital. Note that $F(K;\theta)$ exceeds the present value of the expected flow of net profits from anticipated future investments, because the firm is not committed to any investment path.

Units of capital are installed sequentially, and we can number them in the order they are installed. Suppose units 1 through n have been installed so far. Then, suppressing θ , we can rewrite (1) by summing the value of each installed unit and the values of the options to install further units:

$$W = \Delta V(0) + \Delta V(1) + \Delta V(2) \dots + \Delta V(n-1) + \Delta F(n) + \Delta F(n+1) + \dots \quad (2)$$

$\Delta V(j)$ is the value of the $j+1$ st unit of capital, i.e. the present value of the expected flow of incremental profits generated by unit $j+1$. Of course the firm need not utilize this (or any other) unit of capital. It has an option to utilize it at each point during its lifetime, and $\Delta V(j)$ is equal to the value of these options. Section 3 shows how $\Delta V(j)$ can be calculated.

The firm must decide whether to add more capital. With n units in place, $\Delta F(n)$ is the value of the option to buy one more unit, i.e. unit $n+1$, at any time in the future. If the firm exercises this option, it pays k and receives an asset worth $\Delta V(n)$. The firm also gives up $\Delta F(n)$, because once exercised, the option is dead - whether or not the firm later buys more capital, it has now paid for unit $n+1$, and cannot disinvest. Hence $\Delta F(n)$ is also a cost of investing in this unit. The full cost of investing is thus $k + \Delta F(n)$, which must be compared to the benefit $\Delta V(n)$.

Once the firm buys unit $n+1$, it must decide when to exercise its option, worth $\Delta F(n+1)$, to buy unit $n+2$, which is worth $\Delta V(n+1)$. And so on. These options must be exercised sequentially, so the total value of the firm's options to grow is $F(n) = \sum_{j=n}^{\infty} \Delta F(j)$.

Letting these units become infinitesimally small, eq. (2) becomes:

$$W = \int_0^K \Delta V(\nu; \theta) d\nu + \int_K^\infty \Delta F(\nu; \theta) d\nu \quad (3)$$

The optimal capital stock K^* maximizes the firm's net value, $W - kK^*$. Using (3), this implies the following optimality condition that must hold whenever the firm is investing:

$$\Delta V(K^*; \theta) = k + \Delta F(K^*; \theta) \quad (4)$$

Thus the firm should invest until the value of a marginal unit of capital, $\Delta V(K; \theta)$, equals its total cost: the purchase and installation cost, k , plus the opportunity cost $\Delta F(K; \theta)$ of irreversibly exercising the option to invest in the unit, rather than waiting and keeping that option alive.⁴ (Later, should θ fall, the firm might find that its capital stock K^* is larger than it would like. It will invest further only when θ rises enough so that (4) is satisfied by a $K > K^*$.) The firm's investment problem can therefore be solved in two steps: First, determine $\Delta V(K; \theta)$ and $\Delta F(K; \theta)$, and second, use (4) to determine the optimal capacity $K^*(\theta)$.⁵

B. The Multi-Output Firm.

Now consider a firm that produces n outputs, with demand functions $Q_i(P_1, \dots, P_n, \theta_i)$, $i = 1, \dots, n$, where $\theta_1, \dots, \theta_n$ are shift parameters that follow (possibly correlated) stochastic processes. Suppose the firm uses flexible capital, which costs k_f per unit, and can produce all n outputs. Then, if the current capital stock is K_f , the value of the firm is:

$$W_f = \int_0^{K_f} \Delta V_f(\nu; \theta_1, \dots, \theta_n) d\nu + \int_{K_f}^\infty \Delta F_f(\nu; \theta_1, \dots, \theta_n) d\nu \quad (5)$$

Here, ΔV_f is the value of an incremental unit of flexible capacity, and ΔF_f is the value of the firm's option to invest in this incremental unit, given a capacity K_f in place and the current values of $\theta_1, \dots, \theta_n$.

The firm must choose a quantity of capital K_f^* to maximize its net value $W_f - k_f K_f^*$. Using (5), this implies the following optimality condition:

$$\Delta V_f(K_f^*; \theta_1, \dots, \theta_n) = k_f + \Delta F_f(K_f^*; \theta_1, \dots, \theta_n) \quad (6)$$

Once again, the firm invests until the value of a marginal unit of capital is equal to its purchase and installation cost plus the opportunity cost of exercising the option to invest.

Suppose instead that each of the firm's n outputs are produced with a specific type of capital, and capital of type i can be installed at a sunk cost of k_i per unit, with $k_i < k_f$ for all i , and $\sum_i k_i > k_f$. If the firm has quantities of capital K_1, \dots, K_n in place, its value is:

$$W = \sum_{i=1}^n V_i(K_1, \dots, K_n; \theta_1, \dots, \theta_n) + \sum_{i=1}^n F_i(K_1, \dots, K_n; \theta_1, \dots, \theta_n) \quad (7)$$

where V_i is the value of the capital K_i , given that the firm also has quantities K_1, K_2, \dots of the other types of capital, and given the current values of $\theta_1, \dots, \theta_n$. Likewise, F_i is the value of the firm's options to add more capital of type i in the future.

Let $\Delta V_i(K_1, \dots, K_n; \theta_1, \dots, \theta_n)$ denote the value of an incremental unit of capital of type i , given quantities of capital K_1, \dots, K_n in place, and let $\Delta F_i(K_1, \dots, K_n, \theta_1, \dots, \theta_n)$ denote the value of the firm's option to install one more unit of capital of type i . Then we can rewrite (7) as:

$$W = \sum_{i=1}^n \int_0^{K_i} \Delta V_i(K_1, \dots, K_{i-1}, \nu_i, \dots, K_n; \theta_1, \dots, \theta_n) d\nu_i + \sum_{i=1}^n \int_{K_i}^{\infty} \Delta F_i(K_1, \dots, K_{i-1}, \nu_i, \dots, K_n; \theta_1, \dots, \theta_n) d\nu_i \quad (8)$$

The firm must choose quantities of capital K_1^*, \dots, K_n^* to maximize its net value $W - \sum k_i K_i^*$. Using (8), this implies the following optimality condition that holds whenever the firm is investing in capital of type i :

$$\Delta V_i(K_1, \dots, K_i^*, \dots, K_n; \theta_1, \dots, \theta_n) = k_i + \Delta F_i(K_1, \dots, K_i^*, \dots, K_n; \theta_1, \dots, \theta_n) \quad (9)$$

When the firm chooses its initial capital stocks, it must simultaneously solve the set of equations (9) for each K_i^* , $i = 1, \dots, n$. Later, shifts in demand may result in the firm holding excess amounts of some types of capital, although it is still investing in other types.

Now suppose that given the current demand states $\theta_1, \dots, \theta_n$, the firm must decide which technology - flexible or output-specific - to invest in, and how much capacity to install.⁶ This can be solved as follows. First, calculate the functions $\Delta V_i(K_1, \dots, K_n; \theta_1, \dots, \theta_n)$ and $\Delta F_i(K_1, \dots, K_n; \theta_1, \dots, \theta_n)$, $i = 1, \dots, n$, and the functions $\Delta V_f(K_f; \theta_1, \dots, \theta_n)$ and $\Delta F_f(K_f; \theta_1, \dots, \theta_n)$. Second, use eqn. (7) to obtain the optimal amount of flexible capacity K_f^* , and eqn. (9) to obtain the optimal amounts of output-specific capital K_1^*, \dots, K_n^* . Finally, use eqns. (6) and (8) to determine the firm's market value for each technology. The optimal technology maximizes this value.

Note that the ΔV_i , ΔF_i , ΔV_f , and ΔF_f must be determined subject to a value-maximizing operating strategy. In other words, these functions are to be calculated under the assumption that the firm produces and invests optimally. We show how this can be done in the next two sections.

3. A Two-Output Model.

Consider a firm facing the following demand functions for its outputs:

$$P_1 = g_1(\theta_1) - \gamma_{11}Q_1 + \gamma_{12}Q_2 \quad (10a)$$

$$P_2 = g_2(\theta_2) + \gamma_{21}Q_1 - \gamma_{22}Q_2 \quad (10b)$$

with $\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} > 0$. Here g_1 and g_2 are arbitrary functions of θ_1 and θ_2 respectively, which in turn evolve according to the stochastic processes:

$$d\theta_i = \alpha_i \theta_i dt + \sigma_i \theta_i dz_i, \quad i = 1, 2 \quad (11)$$

where dz_i is the increment of a Weiner process, and $E(dz_1 dz_2) = \rho dt$. Thus

future values of θ_1 and θ_2 are jointly lognormally distributed with variances that grow linearly with the time horizon.

We will assume that stochastic changes in demand are spanned by existing assets, i.e., there are assets or dynamic portfolios of assets whose prices are perfectly correlated with θ_1 and θ_2 . (This is equivalent to saying that markets are sufficiently complete that the firm's decision to invest or produce does not affect the opportunity set available to investors.) With this assumption we can determine the investment rule that maximizes the firm's market value, and the investment problem reduces to one of contingent claim valuation. This lets us avoid making arbitrary assumptions about risk preferences or discount rates. If spanning does not hold, dynamic programming can still be used to maximize the present value of the firm's expected flow of profits, using an arbitrary discount rate.⁷ (But note that in such cases there is no theory for determining the correct discount rate; the CAPM, for example, would not hold.)

Let x_i be the price of an asset or dynamic portfolio of assets perfectly correlated with θ_i , and denote by ρ_{im} the correlation of x_i with the market portfolio. Then x_i evolves as:

$$dx_i = \mu_i x_i dt + \sigma_i x_i dz_i,$$

and by the CAPM, its expected return is $\mu_i = r + \phi \rho_{im} \sigma_i$, where ϕ is the market price of risk. We will assume that α_i , the expected percentage rate of change of θ_i , is less than μ_i , $i = 1, 2$. (If this were not the case, no capacity would ever be installed. Whatever the current levels of θ_1 and θ_2 , firms would be better off waiting and simply holding the option to install capacity in the future.) Denote by δ_i the difference $\mu_i - \alpha_i$.

The firm's cost and production constraints are as follows: (i) Units of flexible capital can be bought at a price k_f each, and each unit

provides the capacity to produce one unit of either type of output per time period, so $Q_1 + Q_2 \leq K_f$. Alternatively, units of output-specific capital can be bought at prices k_1 and k_2 , with $k_1 < k_f$, $k_2 < k_f$, and $k_1 + k_2 > k_f$. Each unit provides the capacity to produce one unit of the corresponding output, so $Q_1 \leq K_1$ and $Q_2 \leq K_2$. (ii) The firm has zero operating costs. (iii) Starting with no capacity, at $t = 0$ the firm must decide which technology to adopt and how much initial capacity to install. Later it may add more capacity, depending on how demand evolves. (iv) Capacity can be installed instantly, and capital in place does not depreciate. (v) Investment is completely irreversible - the firm cannot disinvest.

To solve the firm's investment problem we determine the value of a marginal unit of each type of capital, the value of the option to invest in that marginal unit, and the optimal rule for exercising this option. To determine which technology the firm should adopt, we calculate the net value of the firm for each.

A. The Value of a Marginal Unit of Capital.

First, suppose the firm is utilizing flexible capital. What is the value, $\Delta V_f(K_f)$, of an incremental unit of this capital, given that K_f is already in place? Denote by $\Delta \pi_f(K_f)$ the flow of profit that this incremental unit generates, and let $\gamma = \gamma_{12} + \gamma_{21}$.

Using eqns. (10a) and (10b) and solving for the firm's profit-maximizing output levels, we show in the Appendix that the profit that an incremental unit of capital generates at a future time t is given by the following nonlinear function of θ_{1t} , θ_{2t} , and K_f :⁸

$$\Delta \pi_{ft} = \max [0, f_1(\theta_{1t}), f_2(\theta_{2t}), f_3(\theta_{1t}, \theta_{2t})] \quad (12)$$

where $f_1(\theta_1) = g_1(\theta_1) - 2\gamma_{11}K_f$,

$$f_2(\theta_2) = g_2(\theta_2) - 2\gamma_{22}K_f,$$

and
$$f_3(\theta_1, \theta_2) = \frac{(2\gamma_{22} + \gamma)g_1(\theta_1) + 2(\gamma_{11} + \gamma)g_2(\theta_2) - (4\gamma_{11}\gamma_{22} - \gamma^2)K_f}{2(\gamma_{11} + \gamma_{22} + \gamma)}$$

Thus $\Delta V_f(K_f)$ can be written as:

$$\Delta V_f(K_f; \theta_1, \theta_2) = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \Delta \pi_{ft}(K_f; \theta_{1t}, \theta_{2t}) \phi(\theta_{1t}, \theta_{2t} | \theta_1, \theta_2) d\theta_{1t} d\theta_{2t} e^{-\mu t} dt \quad (13)$$

where $\phi(\)$ is the joint density function for θ_{1t} and θ_{2t} given their current values θ_1 and θ_2 , and μ is the risk-adjusted discount rate.

Eqn. (13) is clearly difficult to evaluate, and the discount rate μ might not be known. We will show below that $\Delta V_f(K_f)$ is more easily obtained by making use of the spanning assumption. First, however, we turn to the case of a firm that invests in output-specific capital.

We must determine the values $\Delta V_1(K_1, K_2)$ and $\Delta V_2(K_1, K_2)$ of an incremental unit of each type of capital, given that the firm currently has K_1 and K_2 in place, and given θ_1 and θ_2 . Denote by $\Delta \pi_i(K_1, K_2)$ the profit that an incremental unit of capital of type i generates. Using eqns. (10a) and (10b) and solving for the firm's profit-maximizing output levels, it can be shown (see the Appendix) that the profit that an incremental unit of capital of type i generates at a future time t is given by:⁹

$$\Delta \pi_{it} = \begin{cases} \max [0, \min \{ \max [f_{i1}(\theta_{it}), f_{i2}(\theta_{it}, \theta_{jt})], f_{i3}(\theta_{it}) \}], & \gamma < 0 \\ \max [0, \max \{ \min (f_{i1}(\theta_{it}), f_{i2}(\theta_{it}, \theta_{jt})), f_{i3}(\theta_{it}) \}], & \gamma > 0 \end{cases} \quad (14)$$

where $f_{i1}(\theta_i) = g_i(\theta_i) - 2\gamma_{ii}K_i + \gamma K_j,$

$$f_{i2}(\theta_i, \theta_j) = g_i(\theta_i) - 2\gamma_{ii}K_i + \gamma [g_j(\theta_j) + \gamma K_i] / 2\gamma_{jj},$$

and $f_{i3}(\theta_i) = g_i(\theta_i) - 2\gamma_{ii}K_i,$

for $i, j = 1, 2, i \neq j$. Then $\Delta V_i(K_1, K_2)$ can be written as in eqn. (13), but with $\Delta \pi_{ft}(K_f; \theta_1, \theta_2)$ replaced by $\Delta \pi_{it}(K_1, K_2; \theta_1, \theta_2)$. Again, rather than try to evaluate the integral directly, we make use of the spanning assumption.

In the Appendix we show that spanning implies that ΔV_i , the value of a marginal unit of capital of any type ($i = 1, 2, \text{ or } f$), must satisfy:

$$(1/2)\sigma_1^2\theta_1^2\Delta V_{i,11} + (1/2)\sigma_2^2\theta_2^2\Delta V_{i,22} + \rho\sigma_1\sigma_2\theta_1\theta_2\Delta V_{i,12} + (r-\delta_1)\theta_1\Delta V_{i,1} + (r-\delta_2)\theta_2\Delta V_{i,2} + \Delta\pi_i(\theta_1,\theta_2) - r\Delta V_i = 0 \quad (15)$$

where $\Delta V_{i,11}$ denotes $\partial^2\Delta V_i/\partial\theta_1^2$, etc. This differential equation must be solved subject to a set of boundary conditions that depend on the functions $g_1(\theta_1)$ and $g_2(\theta_2)$. We discuss these boundary conditions and the solution of (15) in the context of a specific example in Section 4.

B. The Investment Decision and Choice of Technology.

ΔV_i is the payoff to the firm from exercising its option, worth ΔF_i , to invest in an incremental unit of capacity of type i . The Appendix also shows that ΔF_i must satisfy:

$$(1/2)\sigma_1^2\theta_1^2\Delta F_{i,11} + (1/2)\sigma_2^2\theta_2^2\Delta F_{i,22} + \rho\sigma_1\sigma_2\theta_1\theta_2\Delta F_{i,12} + (r-\delta_1)\theta_1\Delta F_{i,1} + (r-\delta_2)\theta_2\Delta F_{i,2} - r\Delta F_i = 0 \quad (16)$$

As discussed in the Appendix, the following boundary conditions apply:

$$\Delta F_i(\theta_1^*,\theta_2) = \Delta V_i(\theta_1^*,\theta_2) - k_i, \quad i = 1, f \quad (17a)$$

$$\Delta F_i(\theta_1,\theta_2^*) = \Delta V_i(\theta_1,\theta_2^*) - k_i, \quad i = 2, f \quad (17b)$$

$$\partial\Delta F_i(\theta_1^*,\theta_2)/\partial\theta_1 = \partial\Delta V_i(\theta_1^*,\theta_2)/\partial\theta_1, \quad i = 1, f \quad (17c)$$

$$\partial\Delta F_i(\theta_1,\theta_2^*)/\partial\theta_2 = \partial\Delta V_i(\theta_1,\theta_2^*)/\partial\theta_2, \quad i = 2, f \quad (17d)$$

Here, θ_1^* and θ_2^* are critical values of θ_1 and θ_2 at which it is optimal to exercise the investment option (i.e., pay k_i and receive a unit of capital of type i , worth ΔV_i). For example, a firm investing in flexible capacity that has K_f in place should add another unit if $\theta_1 > \theta_1^*$ or if $\theta_2 > \theta_2^*$. (Note that ΔF_f and ΔV_f both depend on K_f .) Hence the solution to eqns. (16) and (17) is the optimal investment rule: if the firm invests in output-specific capital, (16) and (17) imply a $\theta_1^*(K_1, K_2)$ and $\theta_2^*(K_1, K_2)$,

or alternatively a $K_1^*(\theta_1, \theta_2)$ and $K_2^*(\theta_1, \theta_2)$, the optimal initial capacity levels. For flexible capital, (16) and (17) imply an optimal initial capacity $K_f^*(\theta_1, \theta_2)$.

The solution of (16) and (17) gives the optimal investment rule for a particular technology. To determine which technology is optimal, we must find the value of the firm for each. For flexible capital, the value is given by:

$$W_f = \int_0^{K_f^*} \Delta V_f(\nu; \theta_1, \theta_2) d\nu + \int_{K_f^*}^{\infty} \Delta F_f(\nu; \theta_1, \theta_2) d\nu \quad (18)$$

where ΔV_f is the solution to (15), and K_f^* and ΔF_f are jointly determined as the solution to (16) and (17) given the current state of demand θ_1 and θ_2 . For output-specific capital, the value of the firm is:

$$W = \int_0^{K_1^*} \Delta V_1(\nu, K_2^*; \theta_1, \theta_2) d\nu + \int_0^{K_2^*} \Delta V_2(K_1^*, \nu; \theta_1, \theta_2) d\nu + \int_{K_1^*}^{\infty} \Delta F_1(\nu, K_2^*; \theta_1, \theta_2) d\nu + \int_{K_2^*}^{\infty} \Delta F_2(K_1^*, \nu; \theta_1, \theta_2) d\nu \quad (19)$$

where ΔV_1 and ΔV_2 are likewise solutions to (15), and K_1^* , K_2^* , ΔF_1 and ΔF_2 are solutions to (16) and (17) given θ_1 and θ_2 .

Eqns. (15) and (16) are elliptic partial differential equations, and in general would have to be solved using numerical methods. In the next section we go through the steps outlined above for a simpler example that can be solved analytically.

4. An Example.

Our model can be simplified considerably, while retaining its basic economic structure, by reducing the number of stochastic state variables to one. We will assume as before that the firm produces two outputs with zero operating costs, but that the demand functions are given by:

$$P_1 = -\log\theta - \gamma_{11}Q_1 + \gamma_{12}Q_2 \quad (20a)$$

$$P_2 = \log\theta + \gamma_{21}Q_1 - \gamma_{22}Q_2 \quad (20b)$$

with $d\theta = \alpha\theta dt + \sigma\theta dz$ (so that the stochastic component of demand is normally distributed). We will assume that the two outputs are substitutes, so that $\gamma = \gamma_{12} + \gamma_{21} < 0$. (Q_1 and Q_2 might be the demands for large and small automobile engines, and $\theta(t)$ an index of gasoline prices.) To clarify the solution, we also make the demands symmetric: $\gamma_{11} = \gamma_{22} = b$. (In this case, γ drops out of the solution.)

A. Flexible Capital.

We first find the optimal investment rule and market value for a firm using flexible capital. Using (12), the profit generated by an incremental unit of capital, given K_f already in place, is:¹⁰

$$\Delta\pi_{ft} = \begin{cases} -\log\theta - 2bK_f ; & \log\theta \leq -2bK_f \\ 0 & ; \quad -2bK_f \leq \log\theta \leq 2bK_f \\ \log\theta - 2bK_f ; & \log\theta \geq 2bK_f \end{cases} \quad (21)$$

The value of this incremental unit of capital, $\Delta V_f(K_f; \theta)$, must satisfy (15):

$$(1/2)\sigma^2\theta^2\Delta V_{f,\theta\theta} + (r-\delta)\theta\Delta V_{f,\theta} - r\Delta V_f + \Delta\pi_f(\theta) = 0 \quad (22)$$

where $\Delta V_{f,\theta\theta}$ denotes $\partial^2\Delta V_f/\partial\theta^2$, etc. The boundary conditions are:

$$\lim_{\theta \rightarrow 0} \Delta V_f(\theta) + (\log\theta)/r + 2bK_f/r + (r-\delta-\sigma^2/2)/r^2 = 0 \quad (23a)$$

$$\lim_{\theta \rightarrow \infty} \Delta V_f(\theta) - (\log\theta)/r + 2bK_f/r - (r-\delta-\sigma^2/2)/r^2 = 0 \quad (23b)$$

and both ΔV_f and $\Delta V_{f,\theta}$ continuous in θ . Condition (23a) says that for θ close to zero, the firm can expect to use this unit of capital to produce good 1 and only good 1 for the indefinite future, and $\Delta V_f(\theta)$ is the corresponding present value of the expected stream of marginal profit.¹¹ Similarly, (23b) says that for θ very large, the firm can expect to use the

capital to produce good 2 and only good 2 for the indefinite future.

The reader can verify that the solution to (22) and its boundary conditions is:

$$\Delta V_f(\theta) = \begin{cases} A\theta^{\beta_1} - \frac{\log\theta}{r} - \frac{(r-\delta-\sigma^2/2)}{r^2} - 2bK_f/r ; & \theta \leq e^{-2bK_f} \\ B\theta^{\beta_1} + C\theta^{\beta_2} & ; e^{-2bK_f} \leq \theta \leq e^{2bK_f} \\ D\theta^{\beta_2} + \frac{\log\theta}{r} + \frac{(r-\delta-\sigma^2/2)}{r^2} - 2bK_f/r ; & \theta \geq e^{2bK_f} \end{cases} \quad (24)$$

$$\text{where: } \beta_1 = -\frac{(r-\delta-\sigma^2/2)}{\sigma^2} + \frac{1}{\sigma^2}[(r-\delta-\sigma^2/2)^2 + 2r\sigma^2]^{1/2} > 1$$

$$\beta_2 = -\frac{(r-\delta-\sigma^2/2)}{\sigma^2} - \frac{1}{\sigma^2}[(r-\delta-\sigma^2/2)^2 + 2r\sigma^2]^{1/2} < 0$$

$$A = \xi_1(e^{2bK_f\beta_1} + e^{-2bK_f\beta_1})$$

$$B = \xi_1 e^{-2bK_f\beta_1}$$

$$C = \xi_2 e^{-2bK_f\beta_2}$$

$$D = \xi_2(e^{-2bK_f\beta_2} + e^{2bK_f\beta_2})$$

$$\xi_1 = \frac{1 - \beta_2(r-\delta-\sigma^2/2)/r}{r(\beta_1 - \beta_2)}$$

$$\text{and } \xi_2 = \frac{1 - \beta_1(r-\delta-\sigma^2/2)/r}{r(\beta_1 - \beta_2)}$$

Figure 1 shows ΔV_f as a function of $\log\theta$ for $b = 1$, $K_f = .75$, $r = .04$, and $\sigma = 0, .2, \text{ and } .4$.¹² We let $\delta = r - \sigma^2/2$, making the expected rate of change of θ zero, so that $\Delta V_f(\theta)$ is symmetric around $\log\theta = 0$. When $\sigma = 0$, A, B, C, and D in eqn. (24) become zero, so $\Delta V_f = -\log\theta/r - 2bK_f/r$ if $\theta \leq e^{-2bK_f}$, $\Delta V_f = \log\theta/r - 2bK_f/r$ if $\theta \geq e^{2bK_f}$, and $\Delta V_f = 0$ otherwise. Note that for our choice of parameter values, ΔV_f is then greater than zero only if $\log\theta$ exceeds 1.5 in magnitude. But if $\sigma > 0$, $\Delta V_f > 0$ for all values of $\log\theta$, because of the possibility that θ will rise or fall in the future.

Given the value $\Delta V_f(K_f; \theta)$ of an incremental unit of flexible capital, we can determine the value $\Delta F_f(K_f; \theta)$ of the firm's option to invest in this unit. ΔF_f must satisfy the following differential equation:

$$(1/2)\sigma^2\theta^2\Delta F_{f,\theta\theta} + (r-\delta)\theta\Delta F_{f,\theta} - r\Delta F_f = 0 \quad (25)$$

with boundary conditions:

$$\Delta F_f(\underline{\theta}^*) = \Delta V_f(\underline{\theta}^*) - k_f \quad (26a)$$

$$\Delta F_f(\bar{\theta}^*) = \Delta V_f(\bar{\theta}^*) - k_f \quad (26b)$$

$$\Delta F_{f,\theta}(\underline{\theta}^*) = \Delta V_{f,\theta}(\underline{\theta}^*) \quad (26c)$$

$$\Delta F_{f,\theta}(\bar{\theta}^*) = \Delta V_{f,\theta}(\bar{\theta}^*) \quad (26d)$$

Here $\underline{\theta}^*$ and $\bar{\theta}^*$ are the lower and upper critical points, i.e., the firm should add a unit of capital if θ falls below $\underline{\theta}^*$ or rises above $\bar{\theta}^*$.

The solution to (25) is:

$$\Delta F_f(\theta) = a_1\theta^{\beta_1} + a_2\theta^{\beta_2} \quad (27)$$

The critical values $\underline{\theta}^*$ and $\bar{\theta}^*$, as well as a_1 and a_2 , are found by substituting (24) for ΔV_f and (27) for ΔF_f into (26a-d) and solving numerically. A solution is shown in Figure 2, for a cost of capital $k_f = 12$, $\sigma = .2$, and K_f , r , and δ as before. Note that if $\underline{\theta}^* < \theta < \bar{\theta}^*$, the total cost of investing in the incremental unit of capital, $\Delta F_f(\theta) + k_f$, exceeds the value of the unit, $\Delta V_f(\theta)$, and so the firm should not invest. Also, recall that a_1 , a_2 , $\underline{\theta}^*$, and $\bar{\theta}^*$ are all functions of K_f . As K_f increases, $\underline{\theta}^*$ falls and $\bar{\theta}^*$ rises. Thus, if the current value of θ is less than $\underline{\theta}^*$ or greater than $\bar{\theta}^*$, the firm will add capacity up to the point that θ just equals one of these critical values. Given this optimal capacity K_f^* , the value of the firm can then be found from eqn. (18).

B. Output-Specific Capital.

The optimal investment rule for output-specific capital is found in the same way. Using (14) with $\gamma < 0$, the profits from incremental units of

each type of capital, given K_1 and K_2 in place, are respectively:¹³

$$\Delta\pi_{1t} = \begin{cases} -\log\theta - 2bK_1 & ; \quad \log\theta \leq -2bK_1 \\ 0 & ; \quad \log\theta \geq -2bK_1 \end{cases} \quad (28)$$

$$\text{and } \Delta\pi_{2t} = \begin{cases} 0 & ; \quad \log\theta \leq 2bK_2 \\ \log\theta - 2bK_2 & ; \quad \log\theta \geq 2bK_2 \end{cases} \quad (29)$$

The value of an incremental unit of capital of type i , $\Delta V_i(K_i; \theta)$, $i = 1, 2$, must satisfy the following differential equation:

$$(1/2)\sigma^2\theta^2\Delta V_{i,\theta\theta} + (r-\delta)\theta\Delta V_{i,\theta} - r\Delta V_i + \Delta\pi_i(\theta) = 0 \quad (30)$$

The boundary conditions are:

$$\lim_{\theta \rightarrow 0} \Delta V_1(\theta) = -(\log\theta)/r - 2bK_1/r - (r-\delta-\sigma^2/2)/r^2 \quad (31a)$$

$$\lim_{\theta \rightarrow \infty} \Delta V_2(\theta) = (\log\theta)/r - 2bK_2/r + (r-\delta-\sigma^2/2)/r^2 \quad (31b)$$

and ΔV_i and $\Delta V_{i,\theta}$ continuous in θ . The solutions to these equations are:

$$\Delta V_1(\theta) = \begin{cases} A_1\theta^{\beta_1} - \frac{\log\theta}{r} - \frac{(r-\delta-\sigma^2/2)}{r^2} - 2bK_1/r & ; \quad \theta \geq e^{-2bK_1} \\ B_1\theta^{\beta_2} & ; \quad \theta \leq e^{2bK_1} \end{cases} \quad (32)$$

$$\Delta V_2(\theta) = \begin{cases} A_2\theta^{\beta_1} & ; \quad \theta \leq e^{2bK_2} \\ B_2\theta^{\beta_2} + \frac{\log\theta}{r} + \frac{(r-\delta-\sigma^2/2)}{r^2} - 2bK_2/r & ; \quad \theta \geq e^{2bK_2} \end{cases} \quad (33)$$

where $A_1 = \xi_2 e^{2bK_1\beta_1}$, $B_1 = \xi_1 e^{2bK_1\beta_2}$, $A_2 = \xi_2 e^{-2bK_2\beta_1}$, $B_2 = \xi_1 e^{-2bK_2\beta_2}$, and β_1 , β_2 , ξ_1 , and ξ_2 are defined as above.

Figure 3 shows ΔV_1 and ΔV_2 plotted against $\log\theta$ for $K_1 = K_2 = .75$, and again, $b = 1$, $r = .04$, $\sigma = 0, .2, \text{ and } .4$, and $\delta = r - \sigma^2/2$. As with the case of flexible capital, if $\sigma = 0$ and $-1.5 < \log\theta < 1.5$, an extra unit of capital would never be used, and has no value. For $\sigma > 0$, an extra unit of capital of either type might be used in the future, and has positive value

for all values of $\log\theta$. Note that ΔV_1 and ΔV_2 have the form of a call option, and increase with σ . Indeed each is the value of an infinite number of (European) call options to produce at every point in the future.

Given $\Delta V_1(K_1, K_2; \theta)$ and $\Delta V_2(K_1, K_2; \theta)$, $\Delta F_1(K_1, K_2; \theta)$ and $\Delta F_2(K_1, K_2; \theta)$ are found by solving:

$$(1/2)\sigma^2\theta^2\Delta F_{i,\theta\theta} + (r-\delta)\theta\Delta F_{i,\theta} - r\Delta F_i = 0, \quad i = 1, 2 \quad (34)$$

with boundary conditions:

$$\Delta F_1(\underline{\theta}^*) = \Delta V_1(\underline{\theta}^*) - k_1 \quad (35a)$$

$$\Delta F_{1,\theta}(\underline{\theta}^*) = \Delta V_{1,\theta}(\underline{\theta}^*) \quad (35b)$$

$$\Delta F_2(\bar{\theta}^*) = \Delta V_2(\bar{\theta}^*) - k_2 \quad (35c)$$

$$\Delta F_{2,\theta}(\bar{\theta}^*) = \Delta V_{2,\theta}(\bar{\theta}^*) \quad (35d)$$

$$\lim_{\theta \rightarrow \infty} \Delta F_1(\theta) = 0 \quad (35e)$$

$$\lim_{\theta \rightarrow 0} \Delta F_2(\theta) = 0 \quad (35f)$$

The solutions to (34) and boundary conditions (35e) and (35f) are:

$$\Delta F_1(\theta) = m_1\theta^{\beta_2} \quad (36a)$$

and $\Delta F_2(\theta) = m_2\theta^{\beta_1} \quad (36b)$

Note that $\underline{\theta}^*$ and $\bar{\theta}^*$ are again the critical values of θ ; the firm should add a unit of capital of type 1 if θ falls below $\underline{\theta}^*$ and add a unit of capital of type 2 if θ rises above $\bar{\theta}^*$. After substituting in (32), (33) and (36), eqns. (35a-d) can be solved simultaneously for $\underline{\theta}^*$, $\bar{\theta}^*$, m_1 , and m_2 .

A solution is shown in Figure 4 for costs of capital $k_1 = k_2 = 10$, and $\sigma = .2$. The critical values of $\log\theta$ are ± 2.35 . For $\log\theta$ inside this range, the value of a unit of either type of capital is less than the total cost of investing in the unit, so the firm does not invest. Again, $\underline{\theta}^*$, $\bar{\theta}^*$, m_1 , and m_2 are all functions of K_1 and K_2 ; as K_1 (K_2) increases, m_1 and $\underline{\theta}^*$ fall (m_2 falls and $\bar{\theta}^*$ rises). Thus given the current value of θ , (35a-d) can be used to find the firm's optimal initial capital stocks K_1^* and K_2^* . Then, given K_1^*

and K_2^* , eqn. (19) can be used to find the value of the firm.¹⁴

C. The Choice of Technology.

The ex ante choice of technology requires comparing the net value of the firm using flexible versus output-specific capital. This comparison will depend on the parameters σ , δ , and r , the capital costs k_1 , k_2 , and k_f , as well as the current state of demand, i.e., the value of θ .

Table 1 shows the net value of the firm and its components for various values of σ and θ , for flexible and nonflexible capital. Note that if $\sigma = 0$, the firm observes θ and installs as much capital as it will ever need, and the value of its options to grow (F_f in the flexible case, $F_1 + F_2$ in the nonflexible) is zero. The total value of the firm is then the same for either technology, so the firm will use the cheaper nonflexible capital. (In the nonflexible case, $K_1^* = 0$ for all combinations of σ and θ shown, but F_1 , the value of the option to install capital of type 1, is positive for $\sigma > 0$.) For both technologies, as σ increases, the amount of capital that the firm initially installs falls; although the value of each incremental unit of capital rises with σ , the value of the option to invest in the unit (an opportunity cost) rises even more. For large σ , much of the firm's value comes from its options to grow; for $\sigma = .4$ and $\log\theta = 1.5$, these options account for more than half of total value, with either technology.

In the example in Table 1, flexible capital makes the net value of the firm higher only when σ is .4. (It is misleading to compare total values. With equal amounts of installed capacity, a firm using the flexible technology will always have a higher total value. But flexible capital is more expensive, and, as Table 1 shows, the amounts of installed capacity differ in the two cases.) Figure 5 shows how the choice of technology depends on relative capital costs for $\sigma = 0$, .2, and .4, and $\log\theta = 2.5$.

(There, k_1 and k_2 are fixed at 10, and the optimal amount of flexible capacity and corresponding net value of the firm are calculated as k_f is varied between 10 and 15.) When $\sigma = .2$, the ratio of net values exceeds 1 only when k_f/k_2 is less than about 1.07.

These results illustrate how a value-maximizing choice of technology and capacity can be calculated, and how they depend on various parameters. One should not infer that the net benefit of flexible capital is low; our example is based on a specific production technology and specific demand functions, and our solutions apply to a limited range of parameter values.

5. Investments in Input-Flexible Capacity.

The analogous investment problem that arises with input-flexible capacity can be treated in the same way. To see this, consider a firm facing the following non-stochastic demand curve for its single output:

$$P = a - bQ \quad (37)$$

Suppose the firm must use, in addition to capital, one of two variable inputs whose costs, c_1 and c_2 , vary stochastically:

$$dc_i = \alpha_i c_i dt + \sigma_i c_i dz_i, \quad i = 1, 2 \quad (38)$$

with $E(dz_1 dz_2) = \rho dt$, and (assuming spanning), μ_i is the expected return on an asset or portfolio perfectly correlated with dz_i , and $\delta_i = \mu_i - \alpha_i$.

The firm can (irreversibly) purchase and install input-flexible capacity at a cost k_f per unit, or input-specific capacity at a (lower) cost k_1 or k_2 . Each unit of capacity allows the firm to produce one unit of output using one unit of the corresponding input.

This technology and capacity choice problem can be solved using the approach of Section 3. The profit generated by an incremental unit of flexible capacity at time t is given by:

$$\Delta\pi_t(K_f) = \max [0, a - 2bK_f - \min (c_{1t}, c_{2t})] \quad (39)$$

For an incremental unit of input-specific capacity of type 1, the profit is:

$$\Delta\pi_1(K_1, K_2) = \min \{ \max [0, a - 2bK_1 - c_{1t}], \max [0, c_{2t} - c_{1t}, a - 2b(K_1 + K_2) - c_{1t}] \} \quad (40)$$

(Similarly for $\Delta\pi_2$.) ΔV_i , $i = 1, 2$, and f , again satisfies eqn. (15), with boundary conditions derived from (39) and (40), and ΔF_i satisfies eqn. (16) and boundary conditions (17a) - (17d). The solutions of these equations give the optimal capacity levels, and (18) and (19) can be used to find the value of the firm for each technology.

In general, a solution requires numerical methods. However, the problem is much simpler if only one input cost is stochastic, and the other is constant. (This would apply, say, to an electric utility choosing among a coal-fired plant, an oil-fired plant, or a plant that can burn either fuel - coal prices fluctuate little compared to oil prices.) An analytical solution can then be found similar to the one presented in Section 4.

6. Conclusions.

The NPV rule, "Invest when the value of a unit of capital exceeds its purchase and installation cost," is not optimal when investment is irreversible, because it ignores the opportunity cost of exercising, or "killing," the option to invest at any time in the future. Likewise, the rule, "Choose that technology (flexible or nonflexible) that maximizes the present value of the firm's cash flows," is not optimal for the same reason.

We have shown how the value-maximizing choice of technology and capacity can be found in a way that is consistent with the irreversibility of investment, the fact that capacity in place need not always be utilized, and the existence of a competitive capital market. First, the value of an

incremental unit of capacity of each type is determined. Second, the value of the firm's option to invest in this unit is determined, together with the optimal exercise rule. The latter yields the firm's optimal initial capacity, and the corresponding net value of the firm can be calculated. The choice of technology can then be made by comparing ex ante net values.

Our numerical example suggests that irreversibility and uncertainty can have a substantial effect on the amount of capacity the firm initially installs; note from Table 1 that K^* falls rapidly as σ is increased, for both technologies. This is consistent with recent studies of irreversible investment (see the references in Footnotes 2 and 5), but some restrictive assumptions may have exaggerated this effect. For example, by assuming the firm can incrementally invest, we have ignored the lumpiness of investment. We have also ignored depreciation (if capital becomes obsolete rapidly, the opportunity cost of investing will be small). And, as mentioned earlier, our numerical results apply to a simplified model and a limited range of parameter values. This also limits the generality of our finding that flexible capital is the preferred choice only if its cost premium is low.

Other caveats deserve mention. We ignored scale economies, which could make cost increase with the number of products the firm produces, creating an incentive to produce only one output (and use nonflexible capital). Except for capital costs (and constant average variable costs), only demands affect the output mix in our model. (For a model that shows implications of scale economies, see de Groote (1987).) And we ignore strategic aspects of flexibility. As Vives (1986) and others have shown, flexibility can have a negative value in a small numbers environment because with it the firm is less able to commit itself to a particular output level or product mix.

APPENDIX

A. Marginal Profit Functions.

Here we derive the marginal profit functions $\Delta\pi_f(K_f; \theta_1, \theta_2)$, $\Delta\pi_1(K_1, K_2; \theta_1, \theta_2)$, and $\Delta\pi_2(K_1, K_2; \theta_1, \theta_2)$. First consider flexible capital. The total profit function is:

$$\pi = g_1(\theta_1)Q_1 + g_2(\theta_2)Q_2 - \gamma_{11}Q_1^2 - \gamma_{22}Q_2^2 + \gamma Q_1Q_2 \quad (\text{A.1})$$

where $\gamma = \gamma_{12} + \gamma_{21}$. This must be maximized subject to $Q_1 > 0$, $Q_2 > 0$, and $Q_1 + Q_2 \leq K_f$. Let Q_1^* and Q_2^* be the quantities that maximize π_f . If $Q_1^* + Q_2^* < K_f$, $\Delta\pi_f = 0$. Suppose $Q_1^* + Q_2^* = K_f$. Substitute $Q_2 = K_f - Q_1$ into (A.1), differentiate with respect to Q_1 , and set equal to zero, yielding:

$$Q_1(\theta_1, \theta_2) = \frac{g_1(\theta_1) - g_2(\theta_2) + (2\gamma_{22} + \gamma)K_f}{2(\gamma_{11} + \gamma_{22} + \gamma)} \quad (\text{A.2})$$

If $Q_1(\theta_1, \theta_2) > K_f$, then $Q_1^* = K_f$, $Q_2^* = 0$, and $\pi = g_1(\theta_1)K_f - \gamma_{11}K_f^2$, so $\Delta\pi_f = f_1(\theta_1) = g_1(\theta_1) - 2\gamma_{11}K_f$. If $Q_1(\theta_1, \theta_2) < 0$, then $Q_1^* = 0$, $Q_2^* = K_f$, and $\pi = g_2(\theta_2)K_f - \gamma_{22}K_f^2$, so $\Delta\pi_f = f_2(\theta_2) = g_2(\theta_2) - 2\gamma_{22}K_f$. If $0 < Q_1(\theta_1, \theta_2) < K_f$, then $Q_1^* = Q_1(\theta_1, \theta_2)$, and $Q_2^* = K_f - Q_1^*$. Substituting these values of Q_1^* and Q_2^* into (A.1) and differentiating with respect to K_f gives:

$$\Delta\pi_f = f_3(\theta_1, \theta_2) = \frac{(2\gamma_{22} + \gamma)g_1(\theta_1) + 2(\gamma_{11} + \gamma)g_2(\theta_2) - (4\gamma_{11}\gamma_{22} - \gamma^2)K_f}{2(\gamma_{11} + \gamma_{22} + \gamma)}$$

Hence we can write $\Delta\pi_f$ compactly as eqn. (12).

In the case of output-specific capital, the profit function (A.1) must be maximized subject to $0 \leq Q_1 \leq K_1$, and $0 \leq Q_2 \leq K_2$. By the Kuhn-Tucker Theorem, there exist $\lambda_1, \lambda_2 \geq 0$ such that Q_1^* and Q_2^* satisfy the constraints, and (i) $\lambda_1 \geq g_1(\theta_1) - 2\gamma_{11}Q_1^* + \gamma Q_2^*$ and $\lambda_2 \geq g_2(\theta_2) + \gamma Q_1^* - 2\gamma_{22}Q_2^*$; (ii) $\lambda_1(K_1 - Q_1^*) = \lambda_2(K_2 - Q_2^*) = 0$; (iii) $Q_1^*[g_1(\theta_1) - 2\gamma_{11}Q_1^* + \gamma Q_2^* - \lambda_1] - Q_2^*[g_2(\theta_2) + \gamma Q_1^* - 2\gamma_{22}Q_2^* - \lambda_2] = 0$. Note that $\lambda_1 = \Delta\pi_1$ and $\lambda_2 = \Delta\pi_2$. Because of symmetry, we only consider $\Delta\pi_2$.

If $Q_2^* < K_2$, $\Delta\pi_2 = 0$. If $Q_2^* = K_2$, there are three possibilities. (i) If $Q_1^* = K_1$, then the K-T conditions imply that $g_1(\theta_1) - 2\gamma_{11}K_1 + \gamma K_2 \geq 0$, $g_2(\theta_2) + \gamma K_1 - 2\gamma_{22}K_2 \geq 0$, and $\Delta\pi_2 = f_{21}(\theta_2) = g_2(\theta_2) + \gamma K_1 - 2\gamma_{22}K_2$. (ii) If $0 < Q_1^* < K_1$, then the K-T conditions imply that $g_1(\theta_1) - 2\gamma_{11}K_1 + \gamma K_2 < 0$, $g_1(\theta_1) + \gamma K_2 \geq 0$, $g_2(\theta_2) - 2\gamma_{22}K_2 + \gamma[g_1(\theta_1) + \gamma K_2]/2\gamma_{11} \geq 0$, and $\Delta\pi_2 = f_{22}(\theta_1, \theta_2) = g_2(\theta_2) - 2\gamma_{22}K_2 + \gamma[g_1(\theta_1) + \gamma K_2]/2\gamma_{11}$. (iii) If $Q_1^* = 0$, then $g_1(\theta_1) + \gamma K_2 \leq 0$, $g_2(\theta_2) - 2\gamma_{22}K_2 \geq 0$, and $\Delta\pi_2 = f_{23}(\theta_2) = g_2(\theta_2) - 2\gamma_{22}K_2$.

If $\gamma < 0$ (substitute products), the K-T conditions become: (i) If $Q_1^* = K_1$, then $f_{21} \geq f_{22}$, $f_{21} \geq 0$, and $\Delta\pi_2 = f_{21}$; (ii) if $0 < Q_1^* < K_1$, then $f_{22} \geq f_{21}$, $f_{21} \leq f_{23}$, $f_{22} \geq 0$, and $\Delta\pi_2 = f_{22}$; and (iii) if $Q_1^* = 0$, then $f_{22} \geq f_{23}$, $f_{23} \geq 0$, and $\Delta\pi_2 = f_{23}$. If $\gamma > 0$ (complements), the conditions instead become: (i) If $Q_1^* = K_1$, then $f_{21} \leq f_{22}$, $f_{21} \geq 0$, and $\Delta\pi_2 = f_{21}$; (ii) if $0 < Q_1^* < K_1$, then $f_{22} \leq f_{21}$, $f_{21} \geq f_{23}$, $f_{22} \geq 0$, and $\Delta\pi_2 = f_{22}$; and (iii) if $Q_1^* = 0$, then $f_{22} \leq f_{23}$, $f_{23} \geq 0$, and $\Delta\pi_2 = f_{23}$. Hence we can write $\Delta\pi_2$ (and $\Delta\pi_1$) compactly as eqn. (14).

B. Differential Equations for ΔV_i and ΔF_i .

To derive eqn. (15) for ΔV_i we value the marginal profit flow resulting from an incremental unit of capital of type i . Consider a portfolio that is long the rights to this profit flow (i.e., long ΔV_i), short $\Delta V_{i,1}$ units of θ_1 (or equivalently, $\theta_1 \Delta V_{i,1}/x_1$ units of x_1 , the asset or portfolio of assets perfectly correlated with θ_1), and short $\Delta V_{i,2}$ units of θ_2 (or equivalently, $\theta_2 \Delta V_{i,2}/x_2$ units of x_2). Because the expected rate of growth of θ_i is only $\alpha_i = \mu_i - \delta_i$, the short positions require a total payment of $\delta_1 \theta_1 \Delta V_{i,1} + \delta_2 \theta_2 \Delta V_{i,2}$ per unit time (or no rational investor would hold the corresponding long positions). The value of this portfolio is $\Phi = \Delta V_i - \theta_1 \Delta V_{i,1} - \theta_2 \Delta V_{i,2}$, and its instantaneous return is:

$$d\Phi = d\Delta V_i - \Delta V_{i,1} d\theta_1 - \Delta V_{i,2} d\theta_2 - \delta_1 \theta_1 \Delta V_{i,1} dt - \delta_2 \theta_2 \Delta V_{i,2} dt + \Delta\pi_i(\theta_1, \theta_2) dt$$

By Ito's Lemma, $d\Delta V_i = \Delta V_{i,1}d\theta_1 + \Delta V_{i,2}d\theta_2 + (1/2)\Delta V_{i,11}(d\theta_1)^2 + (1/2)\Delta V_{i,22}(d\theta_2)^2 + \Delta V_{i,12}d\theta_1d\theta_2$. Substitute eqn. (11) for $d\theta_1$ and $d\theta_2$ and observe that the return is riskless. Setting the return equal to $r\Phi dt = r\Delta V_i dt - \theta_1\Delta V_{i,1}dt - \theta_2\Delta V_{i,2}dt$ and rearranging yields eqn. (15).

Note that ΔV_i must be the solution to (15) even if the unit of capital did not exist or could not be included in a hedge portfolio. All that is needed is an asset or dynamic portfolio of assets (x_i) that replicates the stochastic dynamics of θ_i , $i = 1, 2$. As Merton (1977) has shown, one can replicate the value function with a portfolio consisting only of the assets x_1 and x_2 and risk-free bonds, and since the value of this portfolio will have the same dynamics as ΔV_i , the solution to (15), ΔV_i must be the value function to avoid dominance.

Finally, note that eqn. (15) can be obtained by dynamic programming. Consider the operating policy (produce a unit of output 1, produce a unit of output 2, or produce nothing) that maximizes the value of Φ of the above portfolio. Since $\Delta\pi_i$ is the maximum flow of profit that can be obtained from an incremental unit of capital of type i , the Bellman equation becomes:

$$r\Phi = \Delta\pi_i(\theta_1, \theta_2) - \delta_1\theta_1\Delta V_{i,1} - \delta_2\theta_2\Delta V_{i,2} + (1/dt)E_t d\Phi \quad (\text{B.1})$$

i.e., the competitive return $r\Phi$ has two components, the cash flow given by the first three terms on the RHS of (B.1), and the expected rate of capital gain. Expanding $d\Phi = d\Delta V_i - \Delta V_{i,1}d\theta_1 - \Delta V_{i,2}d\theta_2$, substituting into (B.1) and rearranging gives eqn. (15).

Eqn. (16) for ΔF_i can be derived in the same way. Conditions (17a) and (17b) define the boundary points θ_1^* and θ_2^* , and (17c) and (17d) are the continuity, or "smooth pasting" conditions. Other conditions will apply, depending on the demand functions. For example, since 0 is an absorbing barrier, the solution must satisfy $\Delta F_i(0,0) = \max [0, \Delta V_i(0,0) - k_i]$.

Table 1 - Value of the Firm*

A. Flexible Capacity

σ	$\log\theta$	K_f^*	$V_f(K_f^*; \theta)$	$F_f(K_f^*; \theta)$	Total Value	Net Value
0	1.5	0.50	12.5	0.0	12.5	6.5
	2.5	1.00	37.5	0.0	37.5	25.5
	3.5	1.50	75.0	0.0	75.0	57.0

0.1	1.5	0.35	10.5	1.5	12.0	7.8
	2.5	0.84	36.0	1.5	37.5	27.4
	3.5	1.34	74.6	1.5	76.1	60.0

0.2	1.5	0.27	9.2	5.1	14.3	11.1
	2.5	0.74	34.6	5.4	40.0	31.1
	3.5	1.25	73.3	5.4	78.7	63.7

0.4	1.5	0.22	10.1	18.8	28.9	26.3
	2.5	0.60	33.7	19.9	53.6	46.4
	3.5	1.08	72.5	20.0	92.5	79.5

B. Non-Flexible Capacity**

σ	$\log\theta$	K_2^*	$V_2(K_1^*, K_2^*; \theta)$	$F_2(K_1^*, K_2^*; \theta)$	$F_1(K_1^*, K_2^*; \theta)$	Total Value	Net Value
0	1.5	0.55	13.1	0.0	0.0	13.1	7.6
	2.5	1.05	38.1	0.0	0.0	38.1	27.6
	3.5	1.55	75.6	0.0	0.0	75.6	60.1

0.1	1.5	0.40	11.3	1.5	0.0	12.8	8.8
	2.5	0.90	37.0	1.5	0.0	38.5	29.5
	3.5	1.40	75.6	1.5	0.0	77.1	63.1

0.2	1.5	0.29	9.4	5.5	0.2	15.1	12.2
	2.5	0.79	35.0	5.5	0.0	40.5	32.6
	3.5	1.29	73.5	5.4	0.0	78.9	66.0

0.4	1.5	0.15	5.3	19.5	2.9	27.8	26.2
	2.5	0.65	31.0	19.3	1.4	51.7	45.2
	3.5	1.65	69.6	19.3	0.7	89.6	73.1

* $k_f = 12$, $k_1 = k_2 = 10$, $r = .04$, and $\delta = r - \sigma^2/2$. All of the solutions are symmetric around $\log\theta = 0$.

** In all cases shown, $K_1^* = 0$, so $V_1(K_1^*, K_2^*; \theta) = 0$.

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FOOTNOTES

1. When investment is irreversible and future demand or cost conditions are uncertain, an investment expenditure involves the exercising, or "killing," of an option - the option to productively invest at any time in the future. One gives up the possibility of waiting for new information that might affect the desirability or timing of the expenditure; one cannot disinvest should market conditions change adversely. As a result, the firm should invest in a unit of capital only when its value exceeds its purchase and installation cost by an amount equal to the value of keeping the option to invest alive - an opportunity cost of investing. McDonald and Siegel (1986) have shown that the value of this opportunity cost can be large, and investment rules that ignore it may be grossly in error.
2. Most of the literature on irreversible investment examines the decision to build a discrete project of some fixed size. See, for example, Baldwin (1982), Brennan and Schwartz (1985), McDonald and Siegel (1986), Majd and Pindyck (1987), and MacKie-Mason (1988).
3. This point and its implications are discussed in McDonald and Siegel (1985).
4. Note that $\Delta V(K)$ is not the marginal value of capital, as the term is used in marginal q theory. The marginal value of capital is the present value of the expected flow of profits throughout the future from whatever unit of capital is the marginal one, i.e.,
$$E \int_0^{\infty} [\partial \pi_t(K_t) / \partial K_t] e^{-\mu t} dt$$
where μ is the discount rate. This depends on the firm's capital stock, K_t , or its distribution at every future t , and its calculation can be difficult. Note that $\Delta V(K)$, the PV of the expected flow of incremental profits from the $K+1$ st unit of capital, is independent of how much capital the firm has in the future.
5. Pindyck (1988) solves this problem for a linear demand function and Leontief production technology, and θ a geometric random walk.
6. For simplicity, we only allow the firm to invest in a single technology. In general a firm might install a mixture of output-specific and flexible capital.
7. The spanning assumption will usually hold; most commodities are traded, often on both spot and futures markets, and the prices of manufactured goods are often correlated with the values of shares or portfolios of shares. In some cases, however, the assumption will not hold, e.g. a new product unrelated to any existing ones.
8. Suppose the capacity constraint is binding ($Q_1 + Q_2 = K_f$). Then the marginal profit of flexible capital is f_1 when $Q_2 = 0$ and $Q_1 = K_f$, f_2 when $Q_1 = 0$ and $Q_2 = K_f$, and f_3 otherwise.

9. The functions f_{i1} , f_{i2} , and f_{i3} are the marginal profit of capital of type i when $Q_i = K_i$, for $Q_j = K_j$, $0 < Q_j < K_j$, and $Q_j = 0$, $j \neq i$, respectively.
10. From (12), $\Delta\pi_{ft} = \max [0, f_1(\theta_t), f_2(\theta_t), f_3(\theta_t)]$, where $f_1(\theta) = -\log\theta - 2bK_f$, $f_2(\theta) = \log\theta - 2bK_f$, and $f_3(\theta) = -(4b^2 - \gamma^2)K_f/(4b + 2\gamma) < 0$. Since $f_3(\theta) < 0$ for all θ , this reduces to $\Delta\pi_{ft} = \max[0, f_1(\theta_t), f_2(\theta_t)]$, or equivalently (21).
11. As $\theta \rightarrow 0$, $\Delta\pi_{ft} \rightarrow f_1(\theta_t)$. Since $\theta_t = \theta_0 \exp[(\alpha - \sigma^2/2)t + \sigma z(t)]$, and $f_1(\theta_t) = -\log\theta_t - 2bK_f$,
- $$\Delta V_f \rightarrow - \int_0^\infty (\log\theta + 2bK_f)e^{-rt} dt - \int_0^\infty (\alpha - \sigma^2/2)te^{-\mu t} dt.$$
- K_f and the current value of θ are known, and are therefore discounted at the risk-free rate r , but the last term, which is stochastic, is discounted at the risk-adjusted rate μ . Also, $\alpha = \mu - \delta$.
12. The standard deviations of annual changes in the prices of commodities such as oil, natural gas, copper, and aluminum are in the range of 20 to 50 percent. For manufactured goods the numbers are lower (based on Producer Price Indices for 1948-87, they are 11 percent for cereal and bakery goods, 3 percent for electrical machinery, and 5 percent for photographic equipment). But variation in the sales of a product for one company will be much larger than variations in price for the entire industry. Thus a σ of .2 or .4 could be considered "typical."
13. Eq. (14) becomes $\Delta\pi_{1t} = \max [0, \min \{ \max [f_{11}(\theta_t), f_{12}(\theta_t)], f_{13}(\theta_t) \}]$, where $f_{11}(\theta) = -\log\theta - 2bK_1 + \gamma K_2$, $f_{12}(\theta) = -\log\theta - 2bK_1 + \gamma(\log\theta + \gamma K_1)/2b$, and $f_{13}(\theta) = -\log\theta - 2bK_1$. This reduces to (28).
14. K_1^* and K_2^* are both positive only if $\theta = \underline{\theta}(K_1^*, K_2^*) = \bar{\theta}(K_1^*, K_2^*)$. $K_1^* = 0$ if $\theta = \bar{\theta}(0, K_2^*) > \underline{\theta}(0, K_2^*)$, and $K_2^* = 0$ if $\theta = \underline{\theta}(K_1^*, 0) < \bar{\theta}(K_1^*, 0)$.

FIGURE 1
 VALUE OF AN INCREMENTAL UNIT OF FLEXIBLE CAPACITY
 ($K_f = .75, \sigma = 0, .2, .4$)

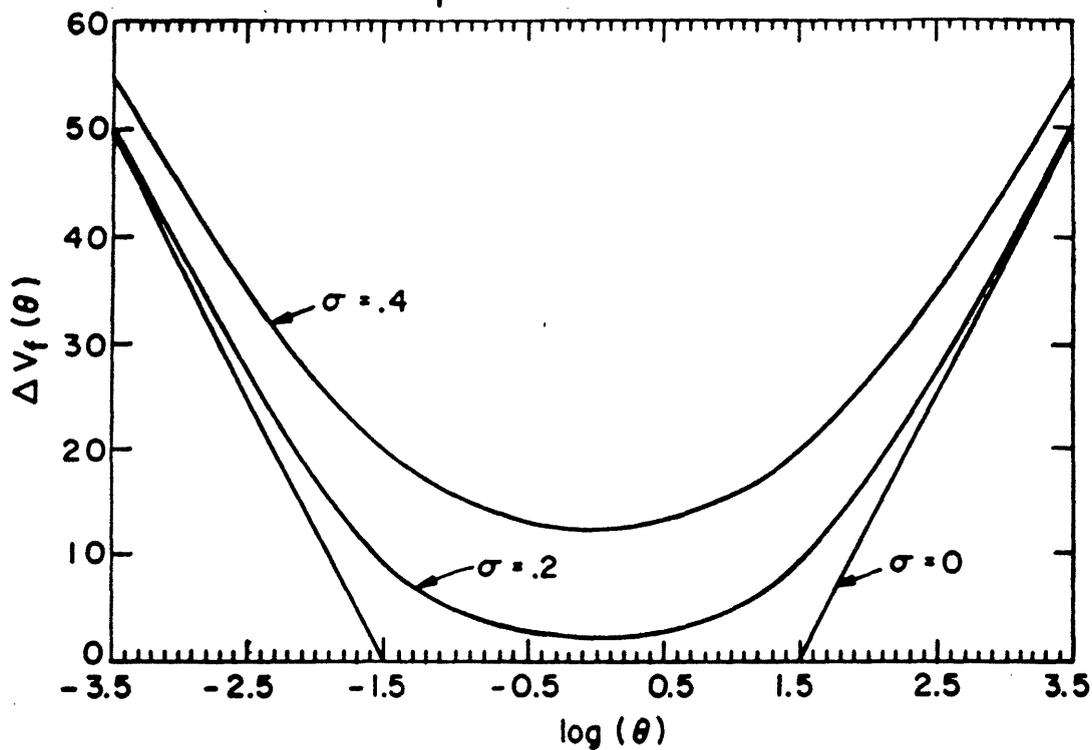


FIGURE 2
 OPTIMAL INVESTMENT RULE - FLEXIBLE CAPACITY
 ($K_f = .75, k_f = 12, \sigma = .2$)

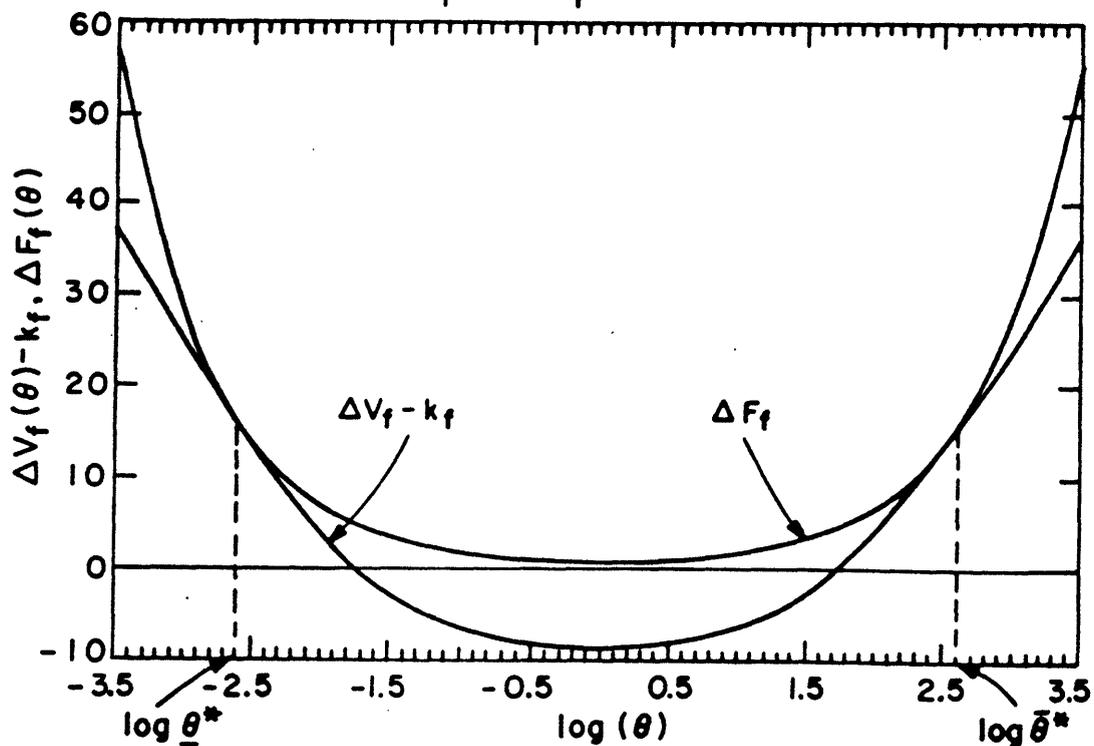


FIGURE 3
 VALUE OF AN INCREMENTAL UNIT OF OUTPUT-SPECIFIC CAPACITY
 ($K_1=K_2=.75, \sigma=0, .2, .4$)

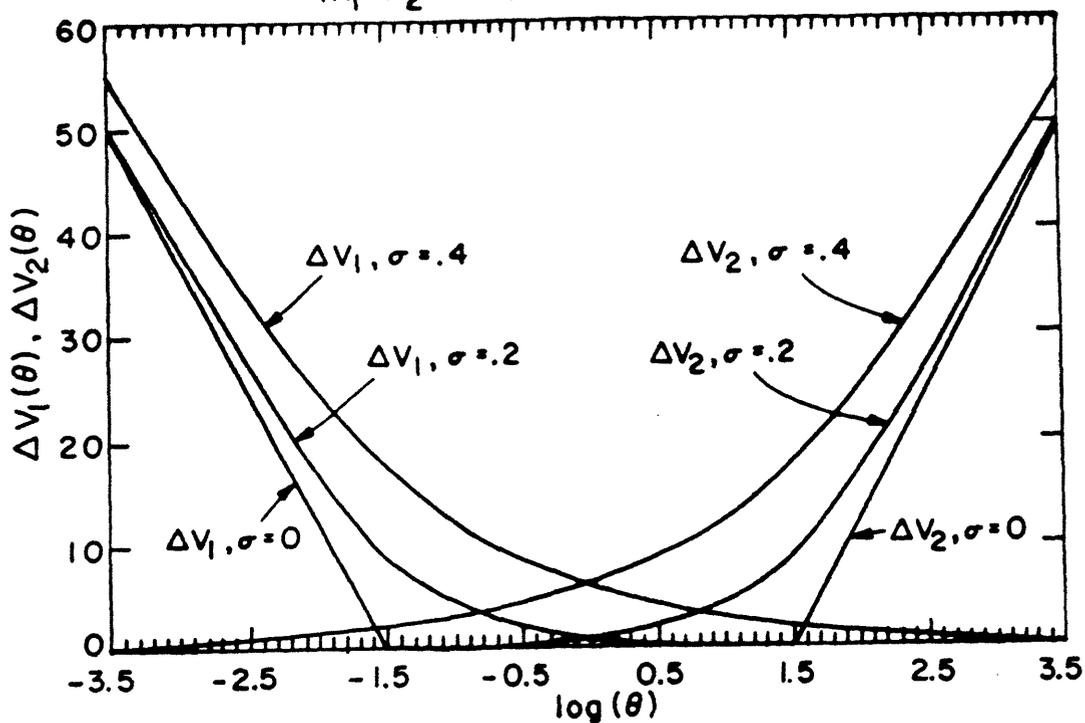


FIGURE 4
 OPTIMAL INVESTMENT RULE - NONFLEXIBLE CAPACITY
 ($K_1=K_2=.75, k_1=k_2=10, \sigma=.2$)

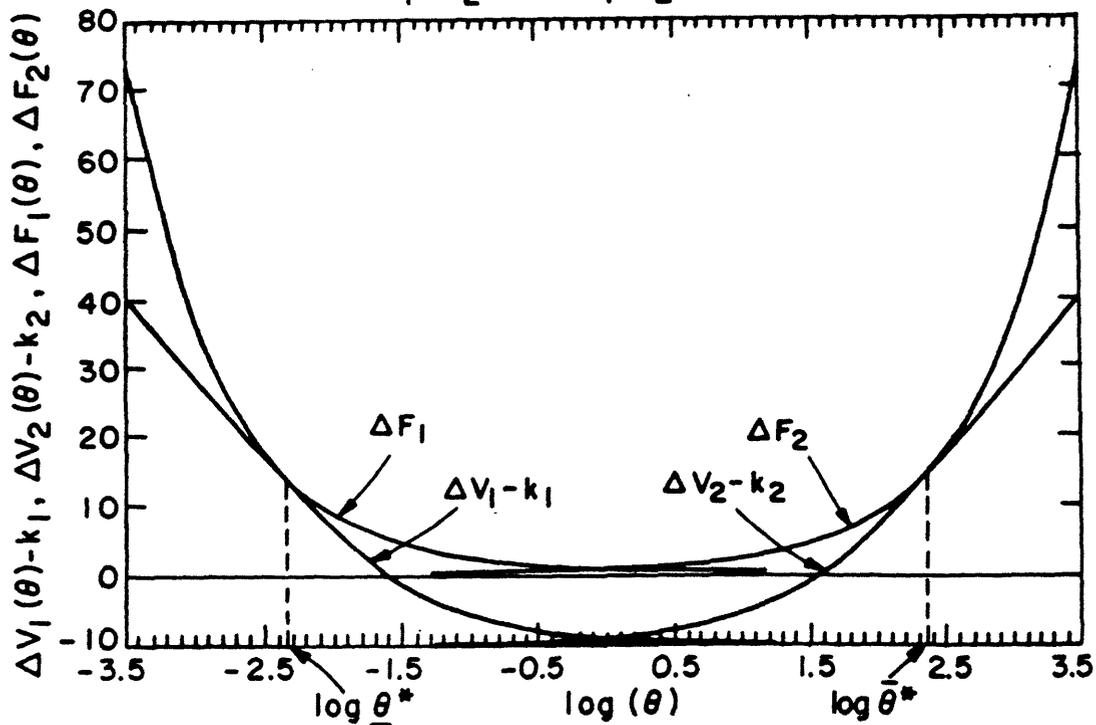


FIGURE 5
RATIO OF NET VALUES VS RATIO OF CAPITAL COSTS
($\log \theta = 2.5$)

