Atiyah-Bott Theory for Orbifolds and Dedekind Sums

by

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ABSTRACT

This paper shows how to deduce the reciprocity laws of Dedekind and Rademacher, as well as n-dimensional generalizations of these, from the Atiyah-Bott formula, by applying it to appropriate elliptic complexes on a "twisted" projective space. This twisted projective space is obtained by taking the quotient of $\mathbf{C}^n - 0$ by the action

$$\rho(\omega)(z_1, \ldots, z_n) = (\omega^{q_1} z_1, \ldots, \omega^{q_n} z_n), \quad \omega \in \mathbf{C}^*, q_i \in \mathbf{Z}^+,$$

where the q_i's are mutually prime. Since this is not a manifold, it is necessary to adapt Atiyah-Bott to the setting of orbifolds.

Thesis Supervisor: Victor Guillemin
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Introduction

Let Y be the "twisted" projective space obtained by taking the quotient of $\mathbf{C}^n - 0$ by the action

$$\rho(\omega)(z_1, \ldots, z_n) = (\omega^{q_1} z_1, \ldots, \omega^{q_n} z_n), \quad \omega \in \mathbf{C}^*, q_i \in \mathbf{Z}^+,$$

where the q_i's are mutually prime. We will show in this paper how to deduce the reciprocity laws of Dedekind and Rademacher, as well as n-dimensional generalizations of these formulas, from the Atiyah-Bott formula by applying it to appropriate elliptic complexes on Y. Since the twisted projective space, Y, is not a manifold, this will require our adapting Atiyah-Bott to the setting of orbifolds. The version of Atiyah-Bott needed for our purposes is described in section 1 and the number theoretic applications of it, mentioned above, are discussed in section 2.

1 Fixed point formula for orbifolds

1.1 The case of good orbifolds

Let X be a compact complex manifold of complex dimension n, G a finite group acting on X with action $\tau : G \times X \to X$. The quotient space $Y = X/G$ is consequently a good orbifold.

Define the Dolbeault cohomology of Y to be the G-invariant cohomology of X, $H^i(Y) = H^i_G(X)$, where $H^i_G(X)$ are the G-invariant subspaces of $H^i(Y), i = 1, \ldots, n$.

A holomorphic G-equivariant function $f : X \to X$ induces a quotient map $\check{f} : Y \to Y$ and $f^\sharp : H^i_G(X) \to H^i_G(X)$ from the pull-back on G-invariant forms.

We will define the Lefschetz number of the mapping $\check{f}$ to be

$$L(\check{f}) = \sum_{i=1}^{n}(-1)^i trace\ (f^\sharp : H^i_G(X) \to H^i_G(X)).$$

We will need the following elementary result:

Theorem. 1.1 *Let V be a vector space and $\rho : G \to Aut(V)$ a representation of a finite group G on V. If $L : V \to V$ is a G-equivariant linear map, then*

$$trace\ (L : V_G \to V_G) = \frac{1}{|G|}\sum_{g\in G} trace\ (\rho_g \circ L) : V \to V),$$

where V_G is the space of G-fixed vectors in V.

By the above Theorem 1.1, we have

$$L(\check{f}) = \sum_{i=1}^{n}\frac{1}{|G|}\sum_{g\in G} trace\ ((\tau_g \circ f)^\sharp : H^i(X) \to H^i(X)).$$

Supposing, in addition, that $\check{f} : Y \to Y$ has only non-degenerate isolated fixed points, or equivalently, that $\tau_g \circ f$ has only non-degenerate isolated fixed points for all $g \in G$, we can compute

$$\sum_{i=1}^{n}(-1)^i trace((\tau_g \circ f)^\sharp : H^i(X) \to H^i(X)) = \sum_{\{p|(\tau_g \circ f)(p)=p\}} sgn\, det\,(1-d(\tau_g \circ f)_p) \tag{1}$$

by the standard Lefschetz fixed point theorem [GP].

Theorem. 1.2 *Under the above conditions we have:*

$$L(\check{f}) = \frac{1}{|G|}\sum_{g\in G}\sum_{\{p|(\tau_g \circ f)(p)=p\}} sgn\ det\ (1-d(\tau_g \circ f)_p).$$

1.2 The case of general orbifolds

In order to write formula (1) in a form which makes sense for general orbifolds Y that are not globally quotients of the form X/G (X a manifold, G a finite group), let us determine the actual contribution of a fixed point q of $\check{f} : Y \to Y$. Still assuming $Y = X/G$, let $p_1, p_2, \ldots, p_k$ be the pre-images of q in X. Replacing, if necessary, f by $\tau_g \circ f$ for some $g \in G$, we can assume $f(p_1) = p_1$. Let G_{p_i} be the stabilizer group of p_i in G. Thus, the contribution of q to the Lefschetz number is:

$$\frac{1}{|G|}\sum_{i=1}^{k}\sum_{\{g\in G|(\tau_g \circ f)(p_i)=p_i\}} sgn\ det\ (1-d(\tau_g \circ f)_{p_i})$$

or

$$\frac{1}{|G|}\sum_{i=1}^{k}\sum_{g\in G_{p_i}} sgn\ det\ (1-d(\tau_g \circ f)_{p_i})$$

since f is G-equivariant. In fact,

$$\sum_{g\in G_{p_i}} sgn\ det\ (1-d(\tau_g \circ f)_{p_i}) = \sum_{g\in G_{p_1}} sgn\ det\ (1-d(\tau_g \circ f)_{p_1}$$

because the G_{p_i} are conjugate and f is G-equivariant, i.e.

$$d(\tau_{g_i} \circ \tau_g \circ \tau_{g_i}^{-1} \circ f)_{\tau_{g_i}(p)} = d(\tau_g \circ f)_p.$$

Therefore, the contribution of q to $L(\check{f})$ is

$$\frac{1}{|G|} \cdot k \cdot \sum_{g \in G_{p_1}} sgn\ det\ (1 - d(\tau_g \circ f)_{p_1})$$

or

$$\frac{1}{|G_{p_1}|} \sum_{g \in G_{p_1}} sgn\ det\ (1 - d(\tau_g \circ f)_{p_1}).$$

This motivates the following result (which we will give a proof of elsewhere):
Let $\check{f} : Y \rightarrow Y$ be a holomorphic function from a compact complex orbifold Y to itself, having only non-degenerate isolated fixed points $q_1, \ldots, q_m$. Define the Lefschetz number of $\check{f}$ to be

$$L(\check{f}) = \sum_{i=1}^{n} (-1)^i trace\ (\check{f}^\sharp : H^i(Y) \rightarrow H^i(Y)).$$

Taking orbifold charts around each of the q_i's, for a neighborhood Y_i of q_i, there are :

X and G such that $Y_i = X/G$,

a pre-image p_i of q_i,

an isotropy group G_i, and

a locally well-defined lift f_i of f.

Claim: We have

$$L(\check{f}) = \sum_{i=1}^{m} \frac{1}{|G_i|} \sum_{g \in G_i} sgn\ det\ (1 - d(\tau_g \circ f_i)_{p_i})$$

reducing again a global topological invariant to a finite number of local differential computations.

Remark: If $L \rightarrow G$ is a G-invariant holomorphic line bundle and $H^i(X, L)$ the cohomology groups obtained by tensoring the Dolbeault complex with L, we can compute the alternating sum of the traces of $f^\sharp$ on $H^i_G(X, L)$ by a sum over the fixed points of $\check{f} : Y \rightarrow Y$ of the terms

$$\frac{1}{|G_i|} \sum_{g \in G_i} \frac{trace\ (\tau_g \circ f_i : L_{p_i} \rightarrow L_{p_i})}{det\ (1 - d(\tau_g \circ f_i)_{p_i})} \qquad \text{[AB]}.$$

2 Application to a twisted projective space

2.1 General formula

Take Y to be the orbifold obtained by dividing $\mathbf{C}^n - 0$ by the group $\mathbf{C}^*$ where $\mathbf{C}^*$ acts by

$$\rho(\omega)(z_1, \ldots, z_n) = (\omega^{q_1} z_1, \ldots, \omega^{q_n} z_n), \quad q_i \in \mathbf{Z}^+.$$

Assuming $q_1, \ldots, q_n$ mutually prime, the orbifold Y is non-singular except at the n points:

$$[1, 0, \ldots, 0],\ [0, 1, 0, \ldots, 0],\ \ldots,\ [0, \ldots, 0, 1]$$

which have stabilizers $\mathbf{Z}/q_1, \ldots, \mathbf{Z}/q_n$, respectively, and thus may be singular. (Notice that when $q_i = 1$, the corresponding point is non-singular.)

The standard diagonal action of S^1 on $\mathbf{C}^n - 0$,

$$f_t(z_1, \ldots, z_n) = (e^{2\pi i t} z_1, \ldots, e^{2\pi i t} z_n)$$

induces an action $\check{f}_t$ on the orbifold Y (since it commutes with ρ). As long as $t \neq 0$, its fixed points are only

$$[1, 0, \ldots, 0],\ [0, 1, 0, \ldots, 0],\ \ldots,\ [0, \ldots, 0, 1].$$

Consider the holomorphic line bundle L over Y associated with the representation

$$\gamma : \mathbf{C}^* \to Aut(\mathbf{C}), \quad \gamma(\omega)c = \omega^d c,$$

i.e., $L = [(\mathbf{C}^n - 0) \times \mathbf{C}]/\{[z, \gamma(\omega)c] \sim [\rho(\omega)z, c], \omega \in \mathbf{C}^*\}$.

In order for L to be well-defined on Y, the condition

$$q_i | d, i = 1, \ldots, n, \quad \text{or equivalently,} \quad q_1 \cdots q_n | d$$

is required. We will write $d = l \cdot q_1 \cdots q_n$.

PROOF. The projection of the hyperplane $z_n = 1$ of $\mathbf{C}^n - 0$ on Y contains only one of the singular points, namely $[0,\ldots,0,1]$. The subgroup of $\mathbf{C}^*$ which fixes this cross-section is the group of q_n roots of unity that acts as $\rho(\omega)(z_1,\ldots,z_{n-1},1) = (\omega^{q_1}z_1,\ldots,\omega^{q_{n-1}}z_{n-1},1)$, while $\gamma(\omega)c = \omega^d c$ on the fiber of L. We have

$$\begin{array}{ccc} [(0,\ldots,0,1),\gamma(\omega)c] & \sim & [\rho(\omega)(0,\ldots,0,1),c] \\ \| & & \| \\ [(0,\ldots,0,1),\omega^d c] & & [(0,\ldots,0,1),c] \end{array}$$

hence, in order for L to be well-defined at $[0,\ldots,0,1]$ we need $q_n | d$.
Similarly for the other singular points. Q.E.D.

On the cross-section $z_n = 1$, $\tau_q = \rho(e^{2\pi i \frac{q}{q_n}})$ acts by

$$\tau_q(z_1,\ldots,z_{n-1},1) = (e^{2\pi i \frac{q q_1}{q_n}} z_1,\ldots,e^{2\pi i \frac{q q_{n-1}}{q_n}} z_{n-1},1),$$

whereas

$$f_t(z_1,\ldots,z_{n-1},1) = (e^{2\pi i t}z_1,\ldots,e^{2\pi i t}z_{n-1},e^{2\pi i t}) \sim (e^{2\pi i t(1-\frac{q_1}{q_n})}z_1,\ldots,e^{2\pi i t(1-\frac{q_{n-1}}{q_n})}z_{n-1},1).$$

We define an action of S^1 on L induced by letting S^1 act trivially on the second factor of $(\mathbf{C}^n - 0) \times \mathbf{C}$:

$$e^{2\pi i t}[(0,\ldots,0,1),c] = [(0,\ldots,0,e^{2\pi i t}),c] \sim [(0,\ldots,0,1),e^{2\pi i t \frac{d}{q_n}}c],$$

so the action of $e^{2\pi i t} \in S^1$ on the fiber of L above $[0,\ldots,0,1]$ is given by multiplication by $e^{2\pi i t \frac{d}{q_n}}$.

Interpreting these results in terms of the lift to the smooth $\mathbf{C}^{n-1}$ covering of this cross-section (which roughly amounts to ignoring the last coordinate z_n when it's 1), we conclude that

$$\tau_q \circ (f_t)_n = (f_t)_n = \text{ multiplication by } e^{2\pi i t \frac{d}{q_n}} : L_{(0,\ldots,0)} \to L_{(0,\ldots,0)},$$

$$d(\tau_q \circ (f_t)_n)_{(0,\ldots,0)} = diag(e^{2\pi i \frac{qq_1}{q_n}}, \ldots, e^{2\pi i \frac{qq_{n-1}}{q_n}}) \cdot diag(e^{2\pi i t(1-\frac{q_1}{q_n})} z_1, \ldots, e^{2\pi i t(1-\frac{q_{n-1}}{q_n})} z_{n-1}).$$

Summing over the q_n-th roots of unity, $\omega = e^{2\pi i \frac{q}{q_n}}$, $q = 0, 1, \ldots, q_n - 1$, the contribution of $[0, \ldots, 0, 1]$ to the Lefschetz number of $\check{f}_t$ is

$$\frac{1}{q_n} \sum_{q=0}^{q_n-1} \frac{e^{2\pi i t \frac{d}{q_n}}}{\prod_{m \neq n}(1 - e^{2\pi i(1-\frac{q_m}{q_n})t} \cdot e^{2\pi i \frac{qq_m}{q_n}})}.$$

Similar computations yield similar results for the other fixed points. Adding all these contributions up we finally get for the global Lefschetz number of $\check{f}_t$:

$$L(\check{f}_t) = \sum_{r=1}^{n} \frac{1}{q_r} \sum_{q=0}^{q_r-1} \frac{e^{2\pi i t \frac{d}{q_r}}}{\prod_{m \neq r}(1 - e^{2\pi i(1-\frac{q_m}{q_r})t} \cdot e^{2\pi i \frac{qq_m}{q_r}})}. \tag{2}$$

On the other hand, the Lefschetz number of $\check{f}_t$ was defined to be

$$L(\check{f}_t) = \sum_{i=1}^{n-1} (-1)^i trace(\check{f}_t^{\sharp} : H^i(Y, L) \rightarrow H^i(Y, L)).$$

We assume $H^i(Y, L) = 0$ for $i > 0$. As for $H^0(Y, L)$ this is the global holomorphic sections of L and these are just the monomials on $\mathbf{C}^n$ $z_1^{m_1} \cdots z_n^{m_n}$ which transform under the action of $\mathbf{C}^*$ according to the law

$$(\omega^{q_1} z_1)^{m_1} \cdots (\omega^{q_n} z_n)^{m_n} = \omega^d z_1^{m_1} \cdots z_n^{m_n}$$

and hence $q_1 m_1 + \ldots + q_n m_n = d$. Thus the dimension of $H^0(Y, L)$ is the number # of integer lattice points $(m_1, \ldots, m_n)$ satisfying $q_1 m_1 + \ldots + q_n m_n = d$, $m_1, \ldots, m_n \geq 0$. We will compute this dimension in the next section by studying the limit of (2) as $t \rightarrow 0$.

2.2 The limit case

Although our formula doesn't hold for $t = 0$ since $\check{f}_0$ leaves all points fixed, we can compute its limit as $t \rightarrow 0$. Notice that the dimension of $H^0(Y, L)$ is independent of t.

So, when $t \to 0$,

$$\# = \lim_{t\to 0}\sum_{r=1}^{n}\frac{1}{q_r}\sum_{q=0}^{q_r-1}\frac{e^{2\pi i t\frac{d}{q_r}}}{\prod_{m\neq r}(1-e^{2\pi i(1-\frac{q_m}{q_r})t}\cdot e^{2\pi i\frac{qq_m}{q_r}})}$$

$$= \sum_{r=1}^{n}\frac{1}{q_r}\sum_{q=1}^{q_r-1}\frac{1}{\prod_{m\neq r}(1-e^{2\pi i\frac{qq_m}{q_r}})} + \lim_{t\to 0}\sum_{r=1}^{n}\frac{1}{q_r}\frac{e^{2\pi i t\frac{d}{q_r}}}{\prod_{m\neq r}(1-e^{2\pi i(1-\frac{q_m}{q_r})t})} \qquad (3)$$

where the last limit can be computed writing a Laurent series for each summand:

$$\frac{a_{n-1,r}}{t^{n-1}} + \ldots + \frac{a_{1,r}}{t} + a_{0,r} + \ldots.$$

As $t \to 0$ the sums of the negative terms in these series must cancel and we end up with

$$\sum_{r=1}^{n}\left(a_{0,r} + \frac{1}{q_r}\sum_{q=1}^{q_r-1}\frac{1}{\prod_{m\neq r}(1-e^{2\pi i\frac{qq_m}{q_r}})}\right)$$

as the number of non-negative integral solutions of the equation

$$q_1m_1 + \ldots + q_nm_n = d.$$

Now we can write

$$\sum_{r=1}^{n}\frac{1}{q_r}\sum_{q=1}^{q_r-1}\frac{1}{\prod_{m\neq r}(1-e^{2\pi i\frac{qq_m}{q_r}})} = \sum_{r=1}^{n}\frac{1}{q_r}\sum_{\eta^{q_r}=1,\eta\neq 1}\frac{1}{\prod_{m\neq r}(1-\eta^{q_m})}$$

and relate this to generalized Dedekind sums according to [HZ] (see section 2.4).

2.3 The case $n = 3$ and reciprocity laws

For $n = 3$ (the first interesting case), formula (3) reads:

$$\#\{(m_1,m_2,m_3)\in \mathbf{Z}^3 | q_1m_1+q_2m_2+q_3m_3 = d, m_1,m_2,m_3 \geq 0\} =$$

$$\underbrace{\sum_{r=1}^{3}\frac{1}{q_r}\sum_{q=1}^{q_r-1}\frac{1}{\prod_{m\neq r}(1-e^{2\pi i\frac{qq_m}{q_r}})}}_{A} + \underbrace{\lim_{t\to 0}\sum_{r=1}^{3}\frac{1}{q_r}\frac{e^{2\pi i t\frac{d}{q_r}}}{\prod_{m\neq r}(1-e^{2\pi i(1-\frac{q_m}{q_r})t})}}_{B}.$$

Let's deal with each of these terms A and B in turn.

A:

We can write

$$A = \sum_{r=1}^{3} \frac{1}{q_r} \sum_{\eta^{q_r}=1, \eta\neq 1} \frac{1}{\prod_{m\neq r}(1-\eta^{q_m})}.$$

Setting $q_3 \equiv k_1 q_2 \bmod q_1$, $q_1 \equiv k_2 q_3 \bmod q_2$, $q_2 \equiv k_3 q_1 \bmod q_3$, we find

$$A = \sum_{r=1}^{3} \frac{1}{q_r} \sum_{\eta^{q_r}=1, \eta\neq 1} \frac{1}{(1-\eta)(1-\eta^{k_r})} = \sum_{r=1}^{3} \left(\frac{q_r - 1}{4q_r} - s(k_r, q_r) \right)$$

by the definition of $s(k_r, q_r)$ according to [RG, p.15]. But by the Rademacher reciprocity law [HZ, p.96],

$$\sum_{r=1}^{3} s(k_r, q_r) = \frac{1}{12} \cdot \frac{q_1^2 + q_2^2 + q_3^2}{q_1 q_2 q_3} - \frac{1}{4}.$$

When $q_3 = 1$ we can take $k_1 = q_2, k_2 = q_1, k_3 = 0$ and the formula reduces to

$$\sum_{r=1}^{3} s(k_r, q_r) = s(q_2, q_1) + s(q_1, q_2) = \frac{1}{12} \cdot \left(\frac{q_1}{q_2} + \frac{1}{q_1 q_2} + \frac{q_2}{q_1} \right) - \frac{1}{4}$$

which is just the Dedekind reciprocity law [RG].

B:

Each summand in B is of the form $\frac{1}{q_r} \cdot \frac{e^{\omega t}}{(1-e^{\omega_1 t})(1-e^{\omega_2 t})}$ for which the constant term in the Laurent expansion is

$$a_0 = \frac{1}{q_r} \left(\frac{1}{4} - \frac{1}{2}\frac{\omega}{\omega_1} - \frac{1}{2}\frac{\omega}{\omega_2} + \frac{1}{2}\frac{\omega^2}{\omega_1 \omega_2} + \frac{1}{12}\frac{\omega_1}{\omega_2} + \frac{1}{12}\frac{\omega_2}{\omega_1} \right).$$

Therefore,

$$B = \sum_{r=1}^{3} a_{0,r} = \frac{1}{4} \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \right) + \frac{l}{2}(q_1 + q_2 + q_3) + \frac{l^2}{2} q_1 q_2 q_3 + \frac{1}{12} \frac{q_1^2 + q_2^2 + q_3^2}{q_1 q_2 q_3}.$$

Next we should compute the left-hand side to see if it agrees. We have

$$\#\{(m_1, m_2, m_3) \in \mathbf{Z}^3 | q_1 m_1 + q_2 m_2 + q_3 m_3 = d = l q_1 q_2 q_3, m_1, m_2, m_3 \geq 0\}$$

$$= \sum_{m_3=0}^{lq_1q_2} \#\{(m_1,m_2) \in \mathbf{Z}^2 | q_1m_1 + q_2m_2 = (lq_1q_2 - m_3)q_3, m_1, m_2 \geq 0\}$$

$$= \sum_{m_3=0}^{lq_1q_2} \left\{ \left[\frac{(lq_1q_2 - m_3)q_3}{q_1q_2} \right] + 1 - \varepsilon(m_3) \right\},$$

where

$[x]$ denotes the greatest integer not exceeding x,

$\varepsilon(m_3) = 0$ or 1, with $\varepsilon(m_3) = 0$ whenever m_3 is a multiple of q_1 or q_2, and

$\varepsilon(m_3) + \varepsilon(lq_1q_2 - m_3) = 1$ when m_3 is neither a multiple of q_1, nor of q_2.

Therefore,

$$\sum_{m_3=0}^{lq_1q_2} \varepsilon(m_3) = \frac{1}{2}\#\{\text{integers in } [0, lq_1q_2] \text{ neither multiples of } q_1, \text{ nor of } q_2\} = \frac{l(q_1-1)(q_2-1)}{2}.$$

Also, since

$$\sum_{k=1}^{p-1} \left[-\frac{kq}{p} \right] = -\frac{(p-1)(q+1)}{2} \quad \text{for } p, q \text{ mutually prime,}$$

we get

$$\sum_{m_3=0}^{lq_1q_2} \left[\frac{(lq_1q_2 - m_3)q_3}{q_1q_2} \right] = \frac{l^2}{2} \cdot q_1q_2q_3 + \frac{l}{2}(q_3 - q_1q_2 + 1).$$

We conclude that

$$\#\{(m_1,m_2,m_3) \in \mathbf{Z}^3 | q_1m_1 + q_2m_2 + q_3m_3 = d = lq_1q_2q_3, m_1, m_2, m_3 \geq 0\}$$

$$= \frac{l^2}{2} \cdot q_1q_2q_3 + \frac{l}{2}(q_1 + q_2 + q_3) + 1$$

and, hence, in this case (3) is equivalent to Rademacher reciprocity law.

2.4 Generalized Dedekind sums

When $l = 0$, i.e., $d = 0$ and the line bundle L is trivial, formula (3) reduces to

$$1 = \sum_{r=1}^{n} \frac{1}{q_r} \sum_{q=1}^{q_r-1} \frac{1}{\prod_{m \neq r}(1 - e^{2\pi i \frac{qq_m}{q_r}})} + \lim_{t \to 0} \sum_{r=1}^{n} \frac{1}{q_r} \frac{1}{\prod_{m \neq r}(1 - e^{2\pi i (1 - \frac{q_m}{q_r})t})}$$

$$\Leftrightarrow \sum_{r=1}^{n} \frac{1}{q_r} \sum_{\eta^{q_r}=1,\eta\neq 1} \frac{1}{\prod_{m\neq r}(1-\eta^{q_m})} = 1 - \lim_{t\to 0} \sum_{r=1}^{n} \frac{1}{q_r} \frac{1}{\prod_{m\neq r}(1-e^{2\pi i(1-\frac{q_m}{q_r})t})} \tag{4}$$

where the last limit can be evaluated by the Laurent series argument. Letting

$$\delta_n(q_r; q_i, i\neq r) = \sum_{\eta^{q_r}=1,\eta\neq 1} \frac{1}{\prod_{m\neq r}(1-\eta^{q_m})}$$

$$\alpha_n(q_1,\ldots,q_n) = \sum_{r=1}^{n} \frac{1}{q_r} \delta_n(q_r; q_i, i\neq r),$$

when $n = 2,3,4,5$ we explicitly find the following generalized reciprocity laws.

$$\begin{array}{lll}
 & & \alpha_n(q_1,\ldots,q_n) \\
n=2 & : & 1-\frac{1}{2}\left(\frac{1}{q_1}+\frac{1}{q_2}\right) \\
n=3 & : & 1-\frac{1}{4}\left(\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}\right)-\frac{1}{12}\frac{q_1^2+q_2^2+q_3^2}{q_1q_2q_3} \\
n=4 & : & 1-\frac{1}{8}\left(\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}\right) \\
 & & -\frac{1}{24}\left(\frac{q_1+q_2}{q_3q_4}+\frac{q_1+q_3}{q_2q_4}+\frac{q_1+q_4}{q_2q_3}+\frac{q_2+q_3}{q_1q_4}+\frac{q_2+q_4}{q_1q_3}+\frac{q_3+q_4}{q_1q_2}\right) \\
n=5 & : & 1-\frac{1}{16}\sum\frac{1}{q_i}-\frac{1}{48}\cdot\frac{1}{q_1q_2q_3q_4q_5}\sum_{i\neq j<k\neq i} q_i^2q_jq_k \\
 & & -\frac{1}{144}\cdot\frac{1}{q_1q_2q_3q_4q_5}\sum_{i<j} q_i^2q_j^2+\frac{1}{720}\cdot\frac{1}{q_1q_2q_3q_4q_5}\sum q_i^4
\end{array}$$

Remark: It is possible to write the limit term (corresponding to B in section 2.3) in terms of Bernoulli numbers B_n defined by

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t-1},$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{3},\ldots$$

This is why the coefficients in the final expressions for the α_n's resemble products of Bernoulli numbers.

On the other hand, [HZ, p.100–101] gives results for generalized Dedekind sums of type δ_n for n odd, namely

$$\sum_{r=1}^{n}\frac{1}{q_r}\sum_{k=1}^{q_r-1}\prod_{m\neq r}\cot\frac{\pi k q_m}{q_r} = 1-\frac{l_{n-1}(q_1,\ldots,q_n)}{q_1\cdots q_n} \tag{5}$$

where l_{n-1} is a certain polynomial in n variables which is symmetric in its variables, even in each variable, and homogeneous of degree $n-1$. Formula (5) is related to the previous δ_n's and α_n's by

$$\begin{aligned}
\sum_{r=1}^{n}\frac{1}{q_r}\sum_{k=1}^{q_r-1}\prod_{m\neq r}\cot\frac{\pi k q_m}{q_r} &= \sum_{r=1}^{n}\frac{1}{q_r}\sum_{\eta^{q_r}=1,\eta\neq 1}\prod_{m\neq r}\frac{\eta^{q_m}+1}{\eta^{q_m}-1}\\
&= \sum_{r=1}^{n}\frac{1}{q_r}\sum_{\eta^{q_r}=1,\eta\neq 1}\sum_{j=0}^{n-1}\sum_{I\subseteq\{1\ldots n\}\setminus r,\#I=j}\frac{(-2)^j}{\prod_{i\in I}(1-\eta^{q_i})}\\
&= \sum_{j=0}^{n-1}(-2)^j\sum_{r=1}^{n}\frac{1}{q_r}\sum_{I\subseteq\{1\ldots n\}\setminus r,\#I=j}\delta_{j+1}(q_r;q_i,i\in I)\\
&= \sum_{j=1}^{n}(-2)^{j-1}\sum_{J\subseteq\{1\ldots n\},\#J=j}\alpha_j(q_i,i\in J).
\end{aligned}$$

When $n=3,5$

$$l_2(q_1,q_2,q_3)=\frac{1}{3}\sum_{i=1}^{3}q_i^2 \qquad l_4(q_1,q_2,q_3,q_4,q_5)=\frac{1}{18}\left(\sum_{i=1}^{5}q_i^2\right)^2-\frac{7}{90}\sum_{i=1}^{5}q_i^4$$

and it is easily seen that (5) is in agreement with our results for α_n, $n=2,3,4,5$. In some sense (4) extends (5) to the case of n even.

2.5 Counting lattice points

Considering again a general line bundle (i.e., arbitrary d, or l), we see that formula (3) provides an expression for the number $\# = \#_n(q_1,\ldots,q_n)$ of integer lattice points $(m_1,\ldots,m_n)$ satisfying $q_1m_1+\ldots+q_nm_n=d$, $m_1,\ldots,m_n\geq 0$, namely

$$\#_n(q_1,\ldots,q_n)=\underbrace{\sum_{r=1}^{n}\frac{1}{q_r}\sum_{q=1}^{q_r-1}\frac{1}{\prod_{m\neq r}(1-e^{2\pi i\frac{qq_m}{q_r}})}}_{A_n}+\underbrace{\lim_{t\to 0}\sum_{r=1}^{n}\frac{1}{q_r}\frac{e^{2\pi i t\frac{d}{q_r}}}{\prod_{m\neq r}(1-e^{2\pi i(1-\frac{q_m}{q_r})t})}}_{B_n}. \tag{6}$$

As in the case $d = l = 0$ (see formula (4)),

$$A_n = \sum_{r=1}^{n} \frac{1}{q_r} \sum_{q=1}^{q_r - 1} \frac{1}{\prod_{m \neq r}(1 - e^{2\pi i \frac{q q_m}{q_r}})} = 1 - \lim_{t \to 0} \sum_{r=1}^{n} \frac{1}{q_r} \frac{1}{\prod_{m \neq r}(1 - e^{2\pi i (1 - \frac{q_m}{q_r})t})},$$

and thus both A_n and B_n can be computed from the Laurent series argument. For $n \leq 5$ we get the following results.

$$\begin{aligned}
\#_1 &= 1 \\
\#_2 &= l + 1 \\
\#_3 &= \frac{l^2}{2} q_1 q_2 q_3 + \frac{l}{2}(q_1 + q_2 + q_3) + 1 \\
\#_4 &= \frac{l^3}{6}(q_1 q_2 q_3 q_4)^2 + \frac{l^2}{4} q_1 q_2 q_3 q_4 (q_1 + q_2 + q_3 + q_4) \\
&\quad + \frac{l}{12}(q_1^2 + q_2^2 + q_3^2 + q_4^2 + 3q_1q_2 + 3q_1q_3 + 3q_1q_4 + 3q_2q_3 + 3q_2q_4 + 3q_3q_4) + 1 \\
\#_5 &= \frac{l^4}{24}\left(\prod q_i\right)^3 + \frac{l^3}{12}\left(\prod q_i\right)^2 \left(\sum q_i\right) + \frac{l^2}{24}\left(\prod q_i\right)\left(\sum q_i^2 + 3 \cdot \sum_{i<j} q_i q_j\right) \\
&\quad + \frac{l}{24}\left(\sum_{i \neq j} q_i^2 q_j + 3 \cdot \sum_{i<j<k} q_i q_j q_k\right) + 1
\end{aligned}$$

Working out $\#_n(q_1, \ldots, q_n)$ directly for each n, by decomposing into sums generalizing the procedure in section 2.2, e.g.

$$\#_4 = \sum_{x=0}^{l q_1 q_2 q_3 q_4} \#\{q_1 m_1 + q_2 m_2 = x\} \cdot \#\{q_3 m_3 + q_4 m_4 = l q_1 q_2 q_3 q_4 - x\},$$

and equating similar powers of l in (6), we can gradually find many other interesting formulas.

We conclude that it is easy to deduce Number Theory results from Atiyah-Bott adapted for orbifolds, by applying it to specific examples.

References

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